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Optimal pairs of function spaces for weighted Hardy operators

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Názov práce: Optimálne páry priestorov funkcií pre váhove Hardyho operátory
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Abstrakt: Zameriame sa na určitý váhový Hardyho operátor so spojitou kvazikonkávnou váhou, definovaný na Banachových priestoroch funkcií, v ktorých má každá funkcia rovnakú normu ako jej prerovnanie. V teórii priestorov funkcií majú tieto operátory široké využitie. V predchádzajúcom výskume bolo dokázané, že platí ekvivalencia medzi ohraničenosťou niektorých z týchto operátorov a sobolevovskými vnoreniami. Nech je náš Hardyho operátor ohraničený z priestoru X do priestoru Y. Táto práca sa venuje hľadaniu takej dvojice priestorov X a Y, ktorá je optimálna. Zmienená optimalita by pri ďalšom výzkume mala viesť k optimalite v určitých sobolevovských vnoreniach. Našim druhým cieľom je štúdium supremálnych operátorov, ktoré tiež úzko súvisia s touto tématikou, a odvodenie niektorých ich základných vlastností.

Kľúčové slová: optimalita, váhový operátor Hardyovho typu, supremálny operátor

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Abstrakt: We focus on a certain weighted Hardy operator, with a continuous, quasiconcave weight, defined on a rearrangement-invariant Banach function spaces. The operators of Hardy type are of great use to the theory of function spaces. The mentioned operator is a more general version of the Hardy operator, whose boundedness was shown to be equivalent to a Sobolev-type embedding inequality. This thesis is concerned with the proof of existence of domain and range spaces of our Hardy operator that are optimal. This optimality should lead to the optimality in the Sobolev-type embedding equalities. Our another aim is to study supremum operators, which are also closely related to this issue, and establish some of their basic properties.

Keywords: optimality, weighted Hardy operator, supremum operator

Contents

Introduction

The first task of this thesis is to study a certain weighted Hardy operator and the optimality of function spaces on which it acts as a bounded operator. The second task is to study two operators involving supremum and establish some of their properties.

The structure of this thesis is as follows. The first chapter contains all definitions and results which are required in the next chapters. First, we will define a nonincreasing rearrangement, which will play a crucial role throughout this paper, and some of its properties. Then, we will introduce the abstract theory of Banach function spaces, which are Banach spaces of measurable functions in which the norm is related to the underlying measure in an appropriate way. As the most important terms we can mention a rearrangement-invariant space (or r.i. space), the associate space and the associate norm. Next, we will define the dilation operator and also provide the reader with a list of the function spaces which will be mentioned further. At the end of the chapter we will mention a measure-preserving transformation and its relation to the equimeasurability of the functions and introduce some parts of interpolation theory. All proofs except that of the last theorem are omitted and the reader can find them together with more detailed information in the monograph [1], which is also the main source of information used in this paper.

In the second chapter we will focus on the weighted Hardy operator

$$H_\varphi f(t) := \int_t^R \varphi(s) f(s) \frac{ds}{s}, \quad t \in \ [0, R],$$

where $R \in (0, \infty)$, φ is a continuous quasiconcave function and f is measurable and finite a.e. on $[0, R]$. In the case when $\varphi(t) = t^{\frac{m}{n}}$, $m \in \mathbb{Z}_+, 1 \leq m \leq n-1$, we get the operator $H_{n/m}$, which was introduced in [2]. As we can see from the main result of [2] the boundedness of the operator $H_{n/m}$ is equivalent to a certain Sobolev embedding, which proves the great importance of the operators of such type. In this special case also the problem of the optimality of the particular Sobolev embedding is reduced to the problem of finding an optimal pair of the r.i. spaces X_D and X_R, such that the operator $H_{n/m}$ is bounded into X_R from X_D. By the optimality we mean that X_D can not be replaced by a larger r.i. space and X_R can not be replaced by any smaller r.i. space. In this chapter we find such optimal pairs of r.i. spaces for the operator H_φ, that is, in a more general case. In our approach we start with an arbitrary r.i. space X and then construct the r.i. space X_R, which is the smallest r.i. space into which is H_φ bounded from X. Having X_R we find the r.i. space X_D, which is the largest r.i. space from which is H_φ bounded into X_R. Finally, we show that then the pair (X_D, X_R) is already optimal for the operator H_φ.

In the third chapter we will study the operators

$$(T_\varphi f)(t) := \frac{1}{\varphi(t)} \sup_{t \leq s < R} \varphi(s) f^*(s)$$

and

$$(S_\varphi f)(t) := \frac{\varphi(t)}{t} \sup_{0 < s \leq t} \frac{s}{\varphi(s)} f^*(s), \quad f \text{ measurable}, t \in [0, R]$$

which belong to the group of the so-called Hardy-type operators involving suprema. These operators have proved themselves to be very useful in various research projects during last years. In the centre of our interest is the search for optimal pairs of r.i. spaces for which Sobolev-type embedding holds, in [2], where these supremum operators played quite an important role. The usage of the supremum operators have two advantages. We can use them in order to make some quantity monotone and then, without any difficulty, get rid of them thanks to their boundedness on certain function spaces. The third chapter is concerned with the basic mapping properties of the operators T_φ and S_φ. We will also study what is the relations between $S\varphi f$, $S\varphi f^{**}$ and $(S\varphi f)^{**}$ and between $T\varphi f^{**}$ and $(T\varphi f)^{**}$.

Chapter 1
Preliminaries

In this chapter we will introduce all facts which are necessary in the further chapters.

Suppose I is the finite closed interval $[0, R]$, where $R \in (0, \infty)$, and λ is the Lebesgue measure on $\mathbb{R}$. Let $\mathcal{M}(I)$ denote the collection of all extended real-valued, λ-measurable functions on I, $\mathcal{M}_0(I)$ the class of functions in $\mathcal{M}(I)$ that are finite λ-a.e. and $\mathcal{P}(I)$ the class of nonnegative functions in $\mathcal{M}_0(I)$. As usual, any two functions coinciding λ-a.e. will be identified. The space $\mathcal{M}_0(I)$ with the topology of convergence in measure on sets of finite measure is a metrizable topological vector space.

Definition 1.1. Given $f \in \mathcal{M}_0(I)$, we define its *nonincreasing rearrangement* f^* on $(0, R]$ by

$$f^*(t) := \inf \{s > 0 : \lambda_f(s) \leq t\}, \quad t \in (0, R],$$

where *the distribution function* λ_f of a function f in $\mathcal{M}_0(I)$ is defined by

$$\lambda_f(s) := \lambda \{x \in I : |f(x)| > s\}, \quad s \in [0, \infty).$$

Definition 1.2. Functions f and g are said to be *equimeasurable* (denoted by $f \sim g$) if they have the same distribution function.

Proposition 1.1. *Suppose f, g and f_n, $(n = 1, 2, \cdots)$, belong to $\mathcal{P}(I)$ and let C be any real number. The nonincreasing rearrangement f^* is a nonnegative, nonincreasing, right-continuous function on I. Furthermore,*

$$\begin{aligned}
&|g| \leq |f| \ \textit{a.e.} \quad \Rightarrow \quad g^* \leq f^*;\\
&(Cf)^* = |C| f^*;\\
&(f+g)^*(s+t) \leq f^*(s) + g^*(t), \quad 0 < s+t < R;\\
&|f| \leq \liminf_{n\to\infty} |f_n| \ \textit{a.e.} \quad \Rightarrow \quad f^* \leq \liminf_{n\to\infty} f_n^*;
\end{aligned}$$

in particular,

$$|f_n| \uparrow |f| \ \textit{a.e.} \quad \Rightarrow \quad f_n^* \uparrow f^*,$$

where the symbol $\uparrow$ denotes monotone pointwise convergence of functions;

$$\begin{aligned}
&f \textit{ and } f^* \textit{ are equimeasurable};\\
&(|f|^p)^* = (f^*)^p, \quad p \in (0, \infty).
\end{aligned}$$

Proof. See [1, Chapter 2, Proposition 1.7]. □

While the nonincreasing rearrangement does not necessarily preserve sums or products of functions, there are some inequalities that govern these processes. The starting point is an elementary inequality due to G.H. Hardy and J.E. Littlewood.

Theorem 1.2 (Hardy & Littlewood)**.** *Let f and g belong to $\mathcal{P}(I)$, then*

$$\int_0^R f(t)g(t)\,dt \leq \int_0^R f^*(t)g^*(t)\,dt,$$

and in the special case when g belongs to $\mathcal{P}(I)$ and E is an arbitrary measurable subset of I,

$$\int_E g(x)\,dx \leq \int_0^{\lambda(E)} g^*(s)\,ds, \quad t \in I. \tag{1.1}$$

Proof. See [1, Chapter 2, Lemma 2.1 and Theorem 2.2]. □

In what follows we will use the maximal function

$$f^{**}(t) := t^{-1} \int_0^t f^*(s)\,ds, \quad t \in I,$$

which has some useful properties. The operator $f \mapsto f^{**}$ is sublinear, namely,

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t), \quad f, g \in \mathcal{M}_0(I), t \in I; \tag{1.2}$$

on the other hand, the operation of the nonincreasing rearrangement is not sublinear.

Example 1.3. The symbol χ_A denotes the *indicator function* of a set $A \subset I$, i.e. the function $\chi_A : I \to \{0,1\}$, defined by

$$\chi_A(t) = \begin{cases} 1 & \text{for } t \in A, \\ 0 & \text{for } t \in I \backslash A. \end{cases}$$

Then, a counterexample to the sublinearity of the nonincreasing rearrangement is obtained by

$$\left(\chi_{[0,\frac{R}{2})} + \chi_{[\frac{R}{2},R]}\right)^* \left(\frac{3}{4}R\right) > \left(\chi_{[0,\frac{R}{2})}\right)^* \left(\frac{3}{4}R\right) + \left(\chi_{[\frac{R}{2},R]}\right)^* \left(\frac{3}{4}R\right).$$

Let us introduce the abstract theory of Banach function spaces. The reader can find more detailed information in [1, Chapter 1].

Definition 1.3. A function ϱ on $\mathcal{P}(I)$ is called a *rearrangement-invariant Banach function norm (r.i. norm)* if, for all f, g, f_n in $\mathcal{P}(I)$, and for all constants $c \geq 0$, the following properties hold:

(P1) $\varrho(f) \geq 0$ with $\varrho(f) = 0$ if and only if $f = 0$ a.e. on $\mathcal{P}(I)$;

(P2) $\varrho(cf) = c\varrho(f)$, $c \geq 0$;

(P3) $\varrho(g+f) \leq \varrho(g) + \varrho(f)$;

(P4) $0 \leq f_n \uparrow f$ a.e. implies $\varrho(f_n) \uparrow \varrho(f)$;

(P5) $\varrho(\chi_I) < \infty$;

(P6) $\int_0^R f(t)\,dt \le C\varrho(f)$ for some constant $C \in (0,\infty)$, independent of f;

(P7) $\varrho(f) = \varrho(f^*)$.

When we omit (P7), we get just *Banach function norm* or shortly a *function norm.*
We use the symbol $\lesssim$ or $\gtrsim$ in situations similar to the one in (P6). It indicates that the quantity on the left side is smaller than that on the right side multiplied by certain constant which does not depend on any function or any other appropriate parameters involved.

Definition 1.4. Let ϱ be an r.i. norm. The collection $X(I) = X(\varrho)(I)$ of all functions in $\mathcal{M}_0(I)$ for which $\varrho(|f|) < \infty$ is called a *rearrangement-invariant space (r.i. space).* For each $f \in X(I)$, define

$$\|f\|_{X(I)} = \varrho(|f|).$$

If ϱ is just a function norm, the space $X(I)$ is called a *Banach function space* or shortly a *function space.* Instead of the notation $X(I)$ we use the abbreviated version X, so if it is not specified, a Banach function space is over the measure space (I,λ).

The basic example of an r.i. norm is *the Lebesgue norm*

$$\|f\|_{L^p} := \left(\int_0^R |f(t)|^p\,dt\right)^{1/p} = \left(\int_0^R |f^*(t)|^p\,dt\right)^{1/p}$$

for $f \in \mathcal{M}_0(I)$, $p \in [1,\infty)$, with

$$\|f\|_{L^\infty} := \operatorname*{ess\,sup}_{0<t<R} |f(t)| = \lim_{t\to 0_+} f^*(t).$$

These norms generate *the Lebesgue spaces*

$$L^p := \{f \in \mathcal{M}_0(I) : \|f\|_{L^p} < \infty\}.$$

Definition 1.5. Let X, Y be normed linear spaces. Then the space X is said to be *embedded* in the space Y, denoted by $X \hookrightarrow Y$, if $X \subset Y$ and the identity mapping $\mathrm{id} : X \to Y$ is continuous, i.e. for all $x \in X$:

$$\|x\|_Y \gtrsim \|x\|_X\,.$$

Theorem 1.4. *Let ϱ be a function norm and let $X = X(\varrho)$ and $\|\cdot\|_X$ be as in Definition 1.4. Then under the natural vector space operations, $(X, \|\cdot\|_X)$ is a normed linear space for which the inclusions*

$$S \subset X \hookrightarrow \mathcal{M}_0$$

hold, where S is the set of λ-simple functions on I. In particular, if $f_n \to f$ in X, then $f_n \to f$ in measure on sets of finite measure, and hence some subsequence converges pointwise λ-a.e. to f.

Proof. See [1, Chapter 1, Theorem 1.4]. □

Definition 1.6. Let ϱ be a function norm, then its *associate norm* ϱ' is defined by

$$\varrho'(g) = \sup\left\{\int_0^R f(x)g(x)\,dx; \varrho(f) \le 1\right\}, \quad g \in \mathcal{P}(I).$$

Theorem 1.5. *Let ϱ be a function norm. Then the associate norm ϱ' is itself a function norm. Moreover, if ϱ is the r.i. norm, then ϱ' is also the r.i. norm.*

Proof. See [1, Chapter 1, Theorem 2.2]. □

Definition 1.7. Let ϱ be a function norm and let $X = X(\varrho)$ be the Banach function space determined by ϱ. Let ϱ' be the associate norm of ϱ. The Banach function space $X(\varrho')$ determined by ϱ' is called the *associate space* of X and is denoted by X'. It follows from the definition of the associate norm that the norm of a function g in the associate space X' is given by

$$\|g\|_{X'} = \sup\left\{\int_0^R |f(x)g(x)|\,dx; \|f\|_X \le 1\right\}.$$

Remark 1.6. Let g be a positive nonincreasing function in X'. Notice that according to Theorem 1.2 we have

$$\int_0^R g(t)h(t)\,dt \le \int_0^R g(t)h^*(t)\,dt,$$

so we can write

$$\|g\|_{X'} = \sup\left\{\int_0^R f^*(x)g(x)\,dx; \|f^*\|_X \le 1\right\}.$$

Theorem 1.7 (Hölder's inequality)**.** *Let X be a Banach function space with associate space X'. If f belongs to X and g to X', then fg is integrable and*

$$\int_0^R |f(x)g(x)|\,dx \le \|f\|_X\,\|g\|_{X'}. \tag{1.3}$$

Proof. See [1, Chapter 1, Theorem 2.4]. □

Theorem 1.8. *Every Banach function space X coincides with its second associate space X''. In other words, a function f belongs to X if and only if it belongs to X'', and in that case*

$$\|f\|_X = \|f\|_{X''}.$$

Proof. See [1, Chapter 1, Theorem 2.7]. □

Lemma 1.9. *The norm of a function g in the associate space X' is given by*

$$\|g\|_{X'} = \sup\left\{\left|\int_0^R f(x)g(x)\,dx\right|; \|f\|_X \le 1\right\}.$$

Proof. See [1, Chapter 1, Lemma 2.8]. □

Proposition 1.10. *If X and Y are Banach function spaces and X is embedded in Y, then Y' is embedded in X'.*

Proof. See [1, Chapter 1, Proposition 2.10]. □

Definition 1.8. If f and g belong to $\mathcal{P}(I)$, we write $f \prec g$ if $f^{**} \leq g^{**}$, that is, if

$$\int_0^t f^*(s)\,ds \leq \int_0^t g^*(s)\,ds, \quad \text{for all } t \in I.$$

The relation $\prec$ is called *the Hardy-Littlewood-Pólya relation.*

Proposition 1.11 (Hardy's lemma)**.** *Let ξ_1 and ξ_2 be nonnegative measurable functions on $[0,\infty)$ and suppose*

$$\int_0^t \xi_1(s)\,ds \leq \int_0^t \xi_2(s)\,ds$$

for all $t \in [0,\infty)$. Let η be any nonnegative nonincreasing function on $[0,\infty)$. Then

$$\int_0^\infty \xi_1(s)\eta(s)\,ds \leq \int_0^\infty \xi_2(s)\eta(s)\,ds.$$

Proof. See [1, Chapter 2, Proposition 3.6]. □

Using these results we get the following relation between the Hardy-Littlewood-Pólya relation and the relation between the r.i. norms of two functions.

Theorem 1.12. *Let X be a r.i. space. Suppose f_1 belongs to $\mathcal{M}_0(I)$ and f_2 belongs to X. If $f_1 \prec f_2$, then f_1 belongs to X and $\|f_1\|_X \leq \|f_2\|_X$.*

Proof. There is a more general case proved in [1, Chapter 2, Theorem 4.6]. □

Definition 1.9. The *dilation operator* E_s, $s \in (0,\infty)$, given at $f \in \mathcal{P}(I), t \in I$, is defined by

$$(E_s f)(t) := \begin{cases} f(t/s) & \text{if } t \in [0,s]; \\ 0 & \text{if } t \in (s,R]. \end{cases}$$

Remark 1.13. The operator E_s is bounded on any r.i. space X for every fixed $s \in (0,\infty)$ as we can see in [1, Chapter 3, Proposition 5.11]. Let us denote by $h_X(s)$ the norm of E_s on X.

Proposition 1.14. *The function h_X is nondecreasing and submultiplicative on $(0,\infty)$, and it satisfies $h_X(1) = 1$ and*

$$h_X(t) \leq \max\{1,t\}, \quad t \in (0,\infty).$$

Proof. See [1, Chapter 3, Proposition 5.11]. □

Remark 1.15. Notice that the dilation operator E_s preserves equimeasurability. We have

$$\lambda_{E_s f} = s\lambda_f = s\lambda_g = \lambda_{E_s g}, \quad \text{for } f, g \in \mathcal{M}_0(I) \text{ equimeasurable.}$$

We shall now introduce some function spaces which will appear throughout the paper.

Definition 1.10. We denote $L^p(\theta)$, $\Gamma^p(\theta)$ and M_θ the spaces generated by the norms

$$\|f\|_{L^p(\theta)} = \left(\int_0^R |f(t)|^p \theta(t)\,dt\right)^{1/p};$$
$$\|f\|_{\Gamma^p(\theta)} = \left(\int_0^R f^{**}(t)^p \theta(t)\,dt\right)^{1/p};$$
$$\|f\|_{M_\theta} = \sup_{0<t<R} \theta(t) f^{**}(t),$$

for $f \in \mathcal{M}_0(I)$ and $\theta \in \mathcal{P}(I)$.

The process of rearranging the finite sequence could be described by the permutation. We are working with measurable functions defined on I, so we need something more general. The notion of permutation is in our case replaced by that of measure-preserving transformation [1, Chapter 2, Section 7] or [4].

Definition 1.11. Let (R_1, μ_1) and (R_2, μ_2) be totally σ-finite measure spaces. A mapping σ from R_1 into R_2 is said to be a *measure-preserving transformation* if, whenever E is a μ_2-measurable subset of R_2, the set $\sigma^{-1}E = \{x \in R_1 : \sigma(x) \in E\}$ is a μ_1-measurable subset of R_1 and

$$\mu_1(\sigma^{-1}E) = \mu_2(E).$$

A measure-preserving transformation provides us with a connection between two equimeasurable functions as we can see in the following result.

Theorem 1.16. *Let $\sigma : R_1 \to R_2$ be a measure-preserving transformation. If f_2 is a nonnegative μ_2-measurable function on R_2, then the function $f_1 = f_2 \circ \sigma$ is a nonnegative μ_1-measurable function on R_1 and f_1, f_2 are equimeasurable.*

Proof. See [1, Chapter 2, Proposition 7.2]. □

The equimeasurability and measure-preserving transformation are connected also in an opposite way.

Theorem 1.17. *Let (S, μ) be a finite nonatomic measure space and let f be a nonnegative μ-measurable function on R. Then there is a measure-preserving transformation $\sigma_f : S \to (0, \mu(S))$ such that $f = f^* \circ \sigma_f$ μ-a.e.*

Proof. See [1, Chapter 2, Theorem 7.5]. □

Remark 1.18. The measure space (I, λ) is finite and nonatomic. It follows from Theorem 1.17 that for two equimeasurable functions f and g there is a measure-preserving transformation σ such that $f = g \circ \sigma$.

Finally, we introduce some parts of interpolation theory, all from [1, Chapters 3 and 5], which will be required in the following sections.

Definition 1.12. A pair (X_0, X_1) of Banach function spaces X_0 and X_1 is called a *compatible couple* if there is some Hausdorff topological vector space, say $\mathcal{H}$, into which each of X_0 and X_1 is continuously embedded.

According to Theorem 1.4 each of the spaces L^p, $L^p(\theta)$, $\Gamma^p(\theta)$, M_θ is for any weight θ and any $p \in [1, \infty]$ continuously embedded in the topological vector space $\mathcal{M}_0$ of λ-measurable functions that are finite λ-a.e. Since metrizable topological vector space is also Hausdorff, we obtain that each pair of mentioned spaces forms a compatible couple.

Definition 1.13. Let (X_0, X_1) be a compatible couple of Banach function spaces, with corresponding Hausdorff space $\mathcal{H}$. Let $X_0 + X_1$ denote the *sum* of X_0 and X_1, that is, the set of elements x in $\mathcal{H}$ that are representable in the form $x = x_0 + x_1$ for some x_0 in X_0 and x_1 in X_1. For each x in $X_0 + X_1$, set

$$\|x\|_{X_0+X_1} = \inf\left\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1\right\}, \tag{1.4}$$

where the infimum extends over all representations $x = x_0 + x_1$ of x where $x_0 \in X_0$ and $x_1 \in X_1$.

Notice that $X_0 + X_1$ and its norm do not depend on the choice of Hausdorff space $\mathcal{H}$ associated with the couple (X_0, X_1).

Theorem 1.19. *If (X_0, X_1) is a compatible couple of Banach function spaces, then $X_0 + X_1$ is Banach function space under the norm (1.4).*

Proof. See [1, Chapter 3, Theorem 1.3]. □

Definition 1.14. Let (X_0, X_1) be a compatible couple of Banach function spaces. The *K-functional* is defined for each $f \in X_0 + X_1$ and $t > 0$ by

$$K(f, t; X_0, X_1) = \inf\left\{\|f_0\|_{X_0} + t\,\|f_1\|_{X_1} : f = f_0 + f_1\right\},$$

where the infimum extends over all representations $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$.

Definition 1.15. Let X be an r.i. space. For each t belonging to I, let E be a subset of I with $\lambda(E) = t$ and let

$$\varphi_X(t) = \|\chi_E\|_X .$$

The function φ_X so defined is called the *fundamental function* of X.

Remark 1.20. Observe that the particular choice of set E with $\lambda(E) = t$ is immaterial since if F is any subset of I with $\lambda(F) = t$, then χ_E and χ_F are equimeasurable and so $\|\chi_E\|_X = \|\chi_F\|_X$ because of rearrangement-invariance of X. Hence, φ_X is well defined.

Proposition 1.21. *Let X be an r.i. space. Then the fundamental function φ_X of X satisfies:*

$$\varphi_X \text{ is nondecreasing on } I;\ \varphi_X(t) = 0 \text{ if and only if } t = 0;$$

$$\varphi_X(t)/t \text{ is nonincreasing on } (0, R];$$

$$\varphi_X \text{ is continuous, except perhaps at the origin.}$$

Proof. See [1, Chapter 2, Corollary 5.3]. □

The K-functional for a pair which consists of an arbitrary r.i. space X and L^∞ can be equivalently expressed by the norm of X in the following way.

Theorem 1.22. *Let X be an r.i. space and suppose f belongs to $\mathcal{P}(I)$. Then*

$$K(f, t; X, L^\infty) \approx \left\| \chi_{\left(0, \varphi_X^{-1}(t)\right)} f^* \right\|_X,$$

for all $t \in \varphi_X(I)$.

Remark 1.23. The fundamental function φ_X do not have to be strictly monotonous, so φ_X^{-1} is the generalized inverse of φ_X, defined as

$$\varphi_X^{-1}(t) = \sup\left\{x \in I; \varphi_X(x) = t\right\}, \quad t \in \varphi_X(I).$$

Notice that the following statements hold:

$$\varphi_X\left(\varphi_X^{-1}(t)\right) = t, \quad t \in \varphi_X(I)$$

and

$$\varphi_X^{-1}\left(\varphi_X(t)\right) \geq t, \quad t \in I.$$

Hence,

$$t = \left\| \chi_{\left(0, \varphi_X^{-1}(t)\right)} \right\|_X, \quad t \in \varphi_X(I). \tag{1.5}$$

Proof of Theorem 1.22. Fix f in $\mathcal{P}(I)$ and $t > 0$ and let α_t denote the K-functional on the left side.

In the first step we show that

$$\left\| \chi_{\left(0, \varphi_X^{-1}(t)\right)} f^* \right\|_X \leq \alpha_t. \tag{1.6}$$

We can assume that f belongs to $X + L^\infty = X$. Otherwise the infimum α_t would be infinite and there would be nothing to prove. The spaces X and $X + L^\infty$ are indeed the same because we have $L^\infty \hookrightarrow X$, which follows from (P6) in Definition 1.3 and Proposition 1.10. Assume that $f = g + h$ with g in X and h in L^∞. Then

$$\begin{aligned}
\left\| \chi_{\left(0, \varphi_X^{-1}(t)\right)} f^* \right\|_X &\leq \left\| \chi_{\left(0, \varphi_X^{-1}(t)\right)} g^* \right\|_X + \left\| \chi_{\left(0, \varphi_X^{-1}(t)\right)} h^* \right\|_X \\
&\leq \|g^*\|_X + \|h\|_{L^\infty} \left\| \chi_{\left(0, \varphi_X^{-1}(t)\right)} \right\|_X \\
&= \|g\|_X + \varphi_X\left(\varphi_X^{-1}(t)\right) \|h\|_{L^\infty} = \|g\|_X + t \|h\|_{L^\infty}.
\end{aligned}$$

The first inequality holds due to Proposition 1.11. Indeed, we know that for $u, v \in \mathcal{P}(I)$ it holds

$$\int_0^t (u+v)^*(s)\,ds \le \int_0^t u^*(s)\,ds + \int_0^t v^*(s)\,ds, \quad t \in I,$$

so

$$\int_0^R (u+v)^*(s)w^*(s)\,ds \le \int_0^R u^*(s)w^*(s)\,ds + \int_0^R v^*(s)w^*(s)\,ds,$$

where $w \in \mathcal{M}_0(I)$.

Hence, using Remark 1.6, we get

$$\begin{aligned}
\left\|\chi_{(0,a)}(u+v)^*\right\|_X &= \sup\left\{\int_0^a (u+v)^*(x)w^*(x)\,dx; \|w^*\|_{X'} \le 1\right\}\\
&\le \sup\left\{\int_0^a u^*(x)w^*(x)\,dx + \int_0^a v^*(x)w^*(x)\,dx; \|w^*\|_{X'} \le 1\right\}\\
&\le \sup\left\{\int_0^a u^*(x)w^*(x)\,dx; \|w^*\|_{X'} \le 1\right\}\\
&\qquad\qquad + \sup\left\{\int_0^a v^*(x)w^*(x)\,dx; \|w^*\|_{X'} \le 1\right\}\\
&= \left\|\chi_{(0,a)}u^*\right\|_X + \left\|\chi_{(0,a)}v^*\right\|_X,
\end{aligned}$$

for any $a \in I$. Taking the infimum over all possible representations $f = g + h$, we obtain (1.6).

Next, we will prove

$$\alpha_t \le 2\left\|\chi_{\left(0,\varphi_X^{-1}(t)\right)} f^*\right\|_X.$$

It will suffice to find g in X and h in L^∞ such that $f = g + h$ and

$$\|g\|_X + t\,\|h\|_{L^\infty} \le 2\left\|\chi_{\left(0,\varphi_X^{-1}(t)\right)} f^*\right\|_X. \tag{1.7}$$

Fix and let $\left\|\chi_{\left(0,\varphi_X^{-1}(t)\right)} f^*\right\|_X$ be finite. Then for each subset E of I with $\lambda(E) = \varphi_X^{-1}(t)$ we have

$$\|\chi_E f\|_X \le \left\|\chi_{\left(0,\varphi_X^{-1}(t)\right)} f^*\right\|_X < \infty,$$

so the function

$$g(x) = \max\{f(x) - f^*\left(\varphi_X^{-1}(t)\right), 0\}$$

belongs to X, because $\lambda\{x : f(x) > f^*\left(\varphi_X^{-1}(t)\right)\} \le \varphi_X^{-1}(t)$, and the function

$$h(x) = \min\{f(x), f^*\left(\varphi_X^{-1}(t)\right)\}$$

belongs to L^∞ with the L^∞-norm at most $f^*\left(\varphi_X^{-1}(t)\right)$. Hence, using (1.5)

$$\begin{aligned}
\|g\|_X + t\,\|h\|_{L^\infty} &= \|g^*\|_X + \left\|\chi_{\left(0,\varphi_X^{-1}(t)\right)}\right\|_X \|h\|_{L^\infty} \\
&\leq \left\|\chi_{\left(0,\varphi_X^{-1}(t)\right)} f^*\right\|_X + \left\|\chi_{\left(0,\varphi_X^{-1}(t)\right)} f^*\left(\varphi_X^{-1}(t)\right)\right\|_X \\
&\leq 2\left\|\chi_{\left(0,\varphi_X^{-1}(t)\right)} f^*\right\|_X .
\end{aligned}$$

Since $f = g + h$, the proof is complete.

□

Chapter 2
A weighted Hardy operator

In the present chapter we will introduce a weighted Hardy operator H_φ and its dual operator H'_φ for which we will find some mapping properties and also construct an optimal pair of r.i. spaces X_D and X_R such that

$$H_\varphi : X_D \longrightarrow X_R.$$

Definition 2.1. Let φ be a nonnegative, strictly increasing function defined on the interval I. If

(i) $\varphi(0) = 0$;

(ii) $\varphi(t)/t$ is nonincreasing on $(0, R]$,

then φ is said to be *quasiconcave*.

Definition 2.2. Let φ be a continuous, quasiconcave function on I. Then we define the *weighted Hardy operators* H_φ and H'_φ by

$$(H_\varphi g)(t) := \int_t^R \varphi(s) g(s) \frac{ds}{s},$$

$$(H'_\varphi g)(t) := \frac{\varphi(t)}{t} \int_0^t g(s)\, ds,$$

where $g \in \mathcal{P}(I)$, $t \in I$.

Remark 2.1. There is a certain duality between H'_φ and H_φ. Indeed, using the Fubini theorem, we get for $f, g \in \mathcal{P}(I)$

$$\begin{aligned}\int_0^R f(t) H'_\varphi g(t)\, dt &= \int_0^R f(t) \frac{\varphi(t)}{t} \int_0^t g(s)\, ds\, dt = \int_0^R \int_0^t f(t) \frac{\varphi(t)}{t} g(s)\, ds\, dt \\ &= \int_0^R \int_s^R f(t) \frac{\varphi(t)}{t} g(s)\, dt\, ds = \int_0^R g(s) \int_s^R f(t) \frac{\varphi(t)}{t}\, dt\, ds \\ &= \int_0^R g(s) H_\varphi f(s)\, ds.\end{aligned}$$

Applying this, we obtain the following simple result.

Lemma 2.2. *The operator H'_φ is dual to the operator H_φ in a sense that, for X, Y two r.i. spaces and for $f, g \in \mathcal{P}(I)$*

$$||H_\varphi f||_Y \lesssim ||f||_X \quad \textit{if and only if} \quad ||H'_\varphi g||_{X'} \lesssim ||g||_{Y'}.$$

Proof. Let $f, g \in \mathcal{P}(I)$. We know that the norm in the associate space is given by

$$||g||_{X'} = \sup\left\{\left|\int_0^R f(t)g(t)\,dt\right| : f \in X, ||f||_X \le 1\right\}.$$

We use the duality from Remark 2.1 and the Hőlder inequality (1.3) to prove the second statement from the first one:

$$\begin{aligned} ||H'_\varphi g||_{X'} &= \sup\left\{\left|\int_0^R f(t)H'_\varphi g(t)\,dt\right| : f \in X, ||f||_X \le 1\right\} \\ &= \sup\left\{\left|\int_0^R H_\varphi f(t)g(t)\,dt\right| : f \in X, ||f||_X \le 1\right\} \\ &\le \sup\{||g||_{Y'}||H_\varphi f||_Y : f \in X, ||f||_X \le 1\} \\ &\lesssim \sup\{||g||_{Y'}||f||_X : f \in X, ||f||_X \le 1\} = ||g||_{Y'}. \end{aligned}$$

Proof of the reverse implication is similar. □

Lemma 2.3. *Let φ be defined as above and, in addition, suppose $t/\varphi(t)$ is absolutely continuous on I. Then,*

$$H_\varphi : L^1 \to L^1\left(\frac{d}{dt}\frac{t}{\varphi(t)}\right) \quad \text{and} \quad H_\varphi : L^1\left(\frac{\varphi(t)}{t}\right) \to L_\infty$$

and

$$H'_\varphi : L^1 \to M_{\frac{1}{(\varphi(t)/t)^{**}}} \quad \text{and} \quad H'_\varphi : M_\varphi \to L^\infty.$$

Also

$$H'_\varphi : L^1 \to M^b_{\frac{1}{(\varphi(t)/t)^{**}}},$$

where X^b represents the closure of the bounded functions in an r.i. space X of functions in $\mathcal{P}(I)$.

Remark 2.4. By the symbol $\frac{d}{dt}\frac{t}{\varphi(t)}$ we mean the derivative of $\frac{t}{\varphi(t)}$, hence the class of functions which are defined a.e. on I such that

$$\int_0^s \frac{d}{dt}\frac{t}{\varphi(t)}\,dt = \frac{s}{\varphi(s)}.$$

Let us remind from the basic course of calculus that the absolute continuity of the function $t/\varphi(t)$ is equivalent with the existence of an integrable representative of this class, so the absolute continuity of $t/\varphi(t)$ is exactly what we need to guarantee the existence of the space $L^1\left(\frac{d}{dt}\frac{t}{\varphi(t)}\right)$.

Proof of Lemma 2.3. All assertions except the last one are easy to prove, using just the Fubini theorem. Assuming f belongs to L^1 we get

$$\begin{aligned}\|H_\varphi f\|_{L^1\left(\frac{d}{dt}\frac{t}{\varphi(t)}\right)} &\le \int_0^R \int_t^R \varphi(s)|f(s)|\,\frac{ds}{s}\left(\frac{d}{dt}\frac{t}{\varphi(t)}\right)dt\\ &= \int_0^R\int_0^s \left(\frac{d}{dt}\frac{t}{\varphi(t)}\right)dt\frac{\varphi(s)}{s}|f(s)|\,ds = \int_0^R |f(s)|\,ds < \infty\end{aligned}$$

and

$$\|H_\varphi f\|_{L^\infty} = \sup_{0<t<R}\left|\int_t^R \varphi(s)f(s)\,\frac{ds}{s}\right| \le \int_0^R \varphi(s)\frac{|f(s)|}{s}\,ds = \|f\|_{L^1\left(\frac{\varphi(t)}{t}\right)}\,.$$

As for the operator H'_φ, we have

$$\begin{aligned}\left\|H'_\varphi f\right\|_{M_{\frac{1}{(\varphi(t)/t)^{**}}}} &= \sup_{0<t<R}\frac{1}{(\varphi(s)/s)^{**}(t)}\left(\frac{\varphi(s)}{s}\int_0^s f(z)\,dz\right)^{**}(t)\\ &\le \sup_{0<t<R}\frac{1}{(\varphi(s)/s)^{**}(t)}\left(\frac{\varphi(s)}{s}\int_0^R f(z)\,dz\right)^{**}(t)\\ &= \left|\int_0^R f(z)\,dz\right| \le \|f\|_{L^1} < \infty,\end{aligned}$$

and

$$\begin{aligned}\left\|H'_\varphi f\right\|_{L^\infty} = \sup_{0<t<R}\left|\frac{\varphi(t)}{t}\int_0^t f(s)\,ds\right| &\le \sup_{0<t<R}\frac{\varphi(t)}{t}\int_0^t |f(s)|\,ds\\ \le \sup_{0<t<R}\frac{\varphi(t)}{t}\int_0^t f^*(s)\,ds = \|f\|_{M_\varphi} < \infty.\end{aligned}$$

To prove the last assertion, we suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of bounded functions converging to f in L^1. Then the functions $H'_\varphi f_n$ are also bounded and

$$|(H'_\varphi f_n)(t)| \le \frac{\varphi(t)}{t}\int_0^t |f_n(s)|\,ds \le \varphi(R)\,\|f_n\|_{L^\infty}\,,\quad t\in I,$$

while

$$\begin{aligned}|(H'_\varphi f)(t) - (H'_\varphi f_n)(t)| &\le \frac{\varphi(t)}{t}\int_0^t |f(s)-f_n(s)|\,ds\\ &\le \frac{\varphi(t)}{t}\,\|f-f_n\|_{L^1}\,,\quad t\in I,\end{aligned}$$

so

$$(H'_\varphi f - H'_\varphi f_n)^{**}(t) \le \left(\frac{\varphi(t)}{t}\right)^{**}\|f-f_n\|_{L^1}\,,\quad t\in I,$$

which means

$$\left\|H'_\varphi f - H'_\varphi f_n\right\|_{M_{\frac{1}{(\varphi(s)/s)^{**}(t)}}} \le \|f-f_n\|_{L^1} \to 0,\ \text{as } n\to\infty.$$

□

In what follows we will try to determine the optimal pair of the r.i. spaces X_D and X_R, such that the operator H_φ is bounded into X_R from X_D.

Definition 2.3. Let X_D and X_R be r.i. spaces and suppose T is an operator defined on X_D such that

$$T : X_D \to X_R.$$

We say that X_D and X_R form an *optimal pair* for T if

$$T : Y \to X_R \quad \text{implies} \quad Y \hookrightarrow X_D,$$

and

$$T : X_D \to Z \quad \text{implies} \quad X_R \hookrightarrow Z,$$

when Y and Z are r.i. spaces.

Remark 2.5. In other words, r.i. spaces X_D and X_R form an optimal pair for the operator T if X_D can not be replaced in $T : X_D \to X_R$ by any essentially larger r.i. space and X_R can not be replaced in $T : X_D \to X_R$ by any essentially smaller r.i. space preserving the boundedness of T.

First, we start with the domain X and construct the optimal range X_R. Second, we construct the optimal domain X_D for X_R. Finally, we obtain the optimality of this pair for the operator H_φ.

In view of Proposition 1.10, we define the r.i. space determined by the norm associated with the norm $\|\cdot\|_X$ using the operator H'_φ. It will be convenient to use the nonincreasing rearrangement of a function. So the next simple statement will be of use.

Lemma 2.6. *Let A, B be Banach function spaces. If B is an r.i. space, then the following two statements are equivalent:*

(i) $||H'_\varphi g||_A \lesssim ||g||_B, \quad g \in \mathcal{P}(I),$

(ii) $||H'_\varphi g^*||_A \lesssim ||g||_B, \quad g \in \mathcal{P}(I).$

Proof. In an r.i. space B, one always has $||g^*||_B = ||g||_B$. Thus, the second statement is the reduction of the first one, considering just decreasing functions, so we have *(ii)* from *(i)*. Conversely, it follows from (1.1) that

$$\left\| \frac{\varphi(t)}{t} \int_0^t g(s)\, ds \right\|_A \le \left\| \frac{\varphi(t)}{t} \int_0^t g^*(s)\, ds \right\|_A \lesssim ||g||_B, \quad g \in \mathcal{P}(I).$$

□

Theorem 2.7. *Let X be an r.i. space. Defining the functional $\|\cdot\|_{X'[H'_\varphi]}$ by*

$$\|g\|_{X'[H'_\varphi]} := \left\| H'_\varphi g^* \right\|_{X'} = \left\| \frac{\varphi(t)}{t} \int_0^t g^*(s)\, ds \right\|_{X'} = \|\varphi g^{**}\|_{X'}$$

for all $g \in \mathcal{P}(I)$*, we get an r.i. norm and* $\left(X'[H'_\varphi]\right)'$*, the associate space of* $X'[H'_\varphi]$*, is the optimal range space for* X *with respect to* H_φ*, in the sense that every r.i. space* Z *satisfying*

$$\|H_\varphi f\|_Z \lesssim \|f\|_X\,, \quad f \in \mathcal{P}(I); \tag{2.1}$$

is essentially larger than $\left(X'[H'_\varphi]\right)'$*, which means*

$$\|g\|_Z \lesssim \|g\|_{\left(X'[H'_\varphi]\right)'}\,, \quad g \in \mathcal{P}(I).$$

Proof. We first show that $\|\cdot\|_{X'[H'_\varphi]}$ is indeed an r.i. norm.

Properties (P1) and (P2) are satisfied due to the linearity of the integral, the monotonicity of φ and the properties of the norm $\|\cdot\|_X$.

The triangle inequality (P3) holds because of the subadditivity of the maximal operator $f \mapsto f^{**}$, since

$$\|f+g\|_{X'[H'_\varphi]} = \|\varphi(f+g)^{**}\|_{X'} \le \|\varphi f^{**} + \varphi g^{**}\|_{X'} \le \|\varphi f^{**}\|_{X'} + \|\varphi g^{**}\|_{X'}\,.$$

If $f_n \uparrow f$ a.e., then $f_n^* \uparrow f^*$. Using the monotone convergence theorem, we obtain the pointwise convergence $H'_\varphi f_n^* \uparrow H'_\varphi f^*$, which implies $\|f_n\|_{X'[H'_\varphi]} \uparrow \|f\|_{X'[H'_\varphi]}$ and we have (P4).

The function φ is bounded because it is a continuous function defined on the closed interval I. In the sequel, we use the boundedness of φ:

$$\|\chi_I\|_{X'[H'_\varphi]} = \|\varphi(\chi_I)^{**}\|_{X'} \le \varphi(R)\,\|\chi_I\|_{X'} < \infty.$$

Property (P7) is obvious, because our functional is defined using nonincreasing rearrangement.

To prove (P6), we use property (P6) for the norm $\|\cdot\|_{X'}$ and the Fubini theorem. For some positive C which does not depend on the choice of f, one has

$$\int_0^R \varphi(t) f^{**}(t)\,dt \le C\,\|\varphi f^{**}\|_{X'}\,.$$

We can express the left side in another way as

$$\int_0^R \varphi(t) f^{**}(t)\,dt = \int_0^R \frac{\varphi(t)}{t} \int_0^t f^*(s)\,ds\,dt = \int_0^R f^*(s) \int_s^R \frac{\varphi(t)}{t}\,dt\,ds.$$

The function $\frac{1}{t}\varphi(t)$ is positive on interval $(0,R]$, so the function $\phi(s) = \int_s^R \frac{\varphi(t)}{t}\,dt$ is decreasing, continuous, positive on $[0,R]$ and with $\phi(R) = 0$. It remains to find a constant K such that

$$\int_0^R f(t)\,dt = \int_0^R f^*(t)\,dt \le K \int_0^R f^*(t)\phi(t)\,dt.$$

We can assume that $\phi(0) > 1$, otherwise we would just multiply the right side of previous inequality with a suitable constant. Due to the intermediate value property of continuous functions, there is $t' \in (0, R)$ such that $\phi(t') = 1$. With $K = \frac{R}{t'} = 1 + \frac{R-t'}{t'}$ and with the monotonicity of ϕ, we are quickly done:

$$\begin{aligned}\int_0^R f^*(t)\,dt &= \int_0^{t'} f^*(t)\,dt + \int_{t'}^R f^*(t)\,dt \le \int_0^{t'} f^*(t)\phi(t)\,dt + f^*(t')(R-t') \\ &\le \int_0^R f^*(t)\phi(t)\,dt + \frac{R-t'}{t'}\int_0^{t'} f^*(t')\,dt \\ &\le \int_0^R f^*(t)\phi(t)\,dt + \frac{R-t'}{t'}\int_0^R f^*(t)\phi(t)\,dt \\ &= \frac{R}{t'}\int_0^R f^*(t)\phi(t)\,dt.\end{aligned}$$

Indeed, we have the third inequality because $\phi(t) > 1$ and $f^*(t) > f^*(t')$ on $(0, t')$.

The optimality of $\left(X'[H'_\varphi]\right)'$ follows from its definition. If Z is an r.i. space such that

$$\|H_\varphi f\|_Z \lesssim \|f\|_X\,, \quad f \in \mathcal{P}(I);$$

we have also

$$\left\|H'_\varphi g\right\|_{X'} \lesssim \|g\|_{Z'}\,,$$

using Lemma 2.2. Hence,

$$\|g\|_{X'[H'_\varphi]} = \|\varphi g^{**}\|_{X'} \lesssim \|g^*\|_{Z'} = \|g\|_{Z'}$$

and using duality one more time, we are done. □

Example 2.8. Let us find the optimal range of operator H_φ for the Lebesgue space $L^p([0,1])$, where $p \in (1,\infty)$. Then we have $(L^p)'([0,1]) = L^{p'}([0,1])$, where $\frac{1}{p} + \frac{1}{p'} = 1$, so the norm, which generates the space associated to the desired optimal range space is the following

$$\|g\|_{\bar{Y}'} = \|\varphi g^{**}\|_{L^{p'}([0,1])}\,.$$

In other words, we can express our norm as $\|g\|_{Y'} = \|g\|_{\Gamma^{p'}(\varphi^{p'})([0,1])}$. We see that the desired optimal range is the space $\left(\Gamma^{p'}(\varphi^{p'})\right)'([0,1])$, which was described for example in [3] by

$$\|g\|_{\left(\Gamma^{p'}(\varphi^{p'})\right)'([0,1])} = \int_0^1 \frac{g^{**}(t)^p}{\left(\int_t^\infty \left(\frac{\varphi(s)}{s}\right)^{p'} ds\right)^{p-1}} \frac{dt}{t}.$$

It remains to find the optimal domain Z_D for a given r.i. space Z with respect to H_φ. Since $H_\varphi : L^1 \to L^1\left(\frac{d}{dt}\frac{t}{\varphi(t)}\right)$, we should assume the space $L^1\left(\frac{d}{dt}\frac{t}{\varphi(t)}\right)$ is embedded in the space Z.

Theorem 2.9. *Suppose $t/\varphi(t)$ is an absolutely continuous function on $(0,R]$ and $\varphi(t)/t$ is integrable function on $(0,R]$. Let Z be an r.i. space, such that*

$$Z \hookrightarrow L^1\left(\frac{d}{dt}\frac{t}{\varphi(t)}\right).$$

Defining the functional $\|\varrho\|_{sZ[H_\varphi]}$ by

$$\|g\|_{sZ[H_\varphi]} := \sup_{h\sim g} \|H_\varphi h\|_Z = \sup_{h\sim g} \left\| \int_t^R \frac{\varphi(s)}{s} h(s)\, ds \right\|_Z, \quad h,g \in \mathcal{P}(I),$$

we get an r.i. norm on $\mathcal{P}(I)$, which is essentially the smallest r.i. norm satisfying

$$\|H_\varphi f\|_Z \lesssim \|f\|_{sZ[H_\varphi]}, \quad f \in \mathcal{P}(I). \tag{2.2}$$

Proof. We get the optimality of $sZ[H_\varphi]$ and (2.2) immediately from its definition. Indeed, supposing the norm generating an r.i. space Y satisfies

$$\|H_\varphi h\|_Z \lesssim \|h\|_Y, \quad h \in \mathcal{P}(I),$$

we get

$$\|f\|_{sZ[H_\varphi]} = \sup_{h\sim f} \|H_\varphi h\|_Z \lesssim \sup_{h\sim f} \|h\|_Y = \|f\|_Y, \quad f \in \mathcal{P}(I).$$

As in the previous case properties (P1) and (P2) are obvious.

We use Remark 1.18 which asserts the existence of a measure-preserving transformation for every pair of equimeasurable functions such that the first function is a composition of the second one and this transformation. If $h \sim f+g$, there exists a measure-preserving transformation σ_1 such that $h = (f+g)\circ\sigma_1$. We define the functions $h_f = f\circ\sigma_1$ and $h_g = g\circ\sigma_1$, which are equimeasurable with f and g due to Theorem 1.16 and $h = h_f + h_g$.

Now, (P3) follows from

$$\begin{aligned}\|f+g\|_{sZ[H_\varphi]} &= \sup_{h_f+h_g\sim f+g} \|H_\varphi(h_f+h_g)\|_Z \\ &\le \sup_{h_f+h_g\sim f+g} \left[\|H_\varphi h_f\|_Z + \|H_\varphi h_g\|_Z\right] \le \|f\|_{sZ[H_\varphi]} + \|g\|_{sZ[H_\varphi]}.\end{aligned}$$

To prove (P4), let us suppose $f_n \uparrow f$. Again, $h \sim f_n$ implies the existence of a measure-preserving transformation σ_2 such that $h = f_n\circ\sigma_2 \le f_{n+1}\circ\sigma_2 = k$, where $k \sim f_{n+1}$. Hence

$$\|f_n\|_{sZ[H_\varphi]} = \sup_{h\sim f_n} \|H_\varphi h\|_Z \le \sup_{k\sim f_{n+1}} \|H_\varphi k\|_Z = \|f_{n+1}\|_{sZ[H_\varphi]}.$$

Next, if σ_3 is a measure-preserving transformation such that $h = f\circ\sigma_3$, then $h_n := f_n\circ\sigma_3 \uparrow f\circ\sigma_3 = h$. So $\varrho(H_\varphi h_n) \uparrow \varrho(H_\varphi h)$, which shows $\tau_\varrho(f_n) \uparrow \tau_\varrho(f)$.

For (P5) and (P6) we use also the integrability of $\varphi(t)/t$:

$$\|\chi_I\|_{sZ[H_\varphi]} = \|H_\varphi \chi_I\|_Z = \left\| \int_t^R \frac{\varphi(s)}{s}\, ds \right\|_Z \leq \|\chi_I\|_Z \int_0^R \frac{\varphi(s)}{s}\, ds < \infty,$$

and

$$\begin{aligned}\|f\|_{sZ[H_\varphi]} &\geq \left\| \int_t^R f(s) \frac{\varphi(s)}{s}\, ds \right\|_Z \gtrsim \left\| \int_t^R f(s) \frac{\varphi(s)}{s}\, ds \right\|_{L^1\left(\frac{d}{dt}\frac{t}{\varphi(t)}\right)} \\ &= \int_0^R f(t)\, dt, \quad f \in \mathcal{P}(I).\end{aligned}$$

Finally, for every pair of equimeasurable functions $f, g \in \mathcal{P}(I)$ we have the equality $\|f\|_{sZ[H_\varphi]} = \|g\|_{sZ[H_\varphi]}$, so (P7) holds. □

Combining Theorems 2.7 and 2.9, we get an optimal pair of r.i. function norms with desired properties. Let us summarize these results in the following theorem.

Theorem 2.10. *Suppose $t/\varphi(t)$ is an absolutely continuous function on $(0, R]$ and $\varphi(t)/t$ is integrable function on $(0, R]$ and let X be an r.i. space. Then r.i. spaces X_R and X_D generated by*

$$\|f\|_{X_R} := \|f\|_{\left(X'[H'_\varphi]\right)'}, \tag{2.3}$$

and

$$\|f\|_{X_D} := \|f\|_{sX_R[H_\varphi]}, \quad f \in \mathcal{P}(I), \tag{2.4}$$

form an optimal pair for the operator H_φ.

Proof. According to (P6) in Definition 1.3, each r.i. space is a subset of L^1, so using $H_\varphi : L^1 \to L^1\left(\frac{d}{dt}\frac{t}{\varphi(t)}\right)$ from Lemma 2.3 together with $H_\varphi : X \to X_R$, we obtain that X_R is embedded in $L^1\left(\frac{d}{dt}\frac{t}{\varphi(t)}\right)$. Hence, we can use Theorem 2.9 and we get the optimality of the domain X_D in $H_\varphi : X_D \to X_R$.

Assume that an r.i. space Z satisfies $H_\varphi : X_D \to Z$. Since the domain X_D is optimal in $H_\varphi : X_D \to X_R$ and $H_\varphi : X \to X_R$ we obtain $X \hookrightarrow X_D$. Hence, we get $H_\varphi : X \to Z$ which implies $X_R \hookrightarrow Z$, because the r.i. space X_R is optimal in $H_\varphi : X \to X_R$ which follows from Theorem 2.7. □

Chapter 3
Supremum operators

Definition 3.1. Let φ be a continuous, quasiconcave function. Then, we define the *supremum operators* T_φ and S_φ by

$$(T_\varphi f)(t) := \frac{1}{\varphi(t)} \sup_{t \le s < R} \varphi(s) f^*(s),$$

and

$$(S_\varphi f)(t) := \frac{\varphi(t)}{t} \sup_{0 < s \le t} \frac{s}{\varphi(s)} f^*(s) \quad f \in \mathcal{P}(I), t \in I.$$

Definition 3.2. We say the function φ satisfies the *$B1$-condition* if

$$\int_0^t \frac{1}{\varphi(s)}\, ds \lesssim \frac{t}{\varphi(t)}, \quad t \in I,$$

and the *$B2$-condition* if

$$\int_0^t \frac{\varphi(s)}{s}\, ds \lesssim \varphi(t), \quad t \in I.$$

We can easily show that the supremum operators T_φ and S_φ have a property similar to the sublinearity.

Proposition 3.1. *The supremum operators T_φ and S_φ satisfy*

$$\left(T_\varphi[f+g]\right)(t) \le 2\left(T_\varphi f + T_\varphi g\right)\left(\frac{t}{2}\right), \tag{3.1}$$

and

$$\left(S_\varphi[f+g]\right)(t) \le 2\left(S_\varphi f + S_\varphi g\right)\left(\frac{t}{2}\right), \quad f, g \in \mathcal{P}(I), t \in I. \tag{3.2}$$

Proof. Using the inequality $(f+g)^*(s+t) \le f^*(s) + g^*(t)$ from Proposition 1.1 together with the sublinearity of the supremum, we get

$$\begin{aligned}
\left(T_\varphi[f+g]\right)(t) &= \frac{1}{\varphi(t)} \sup_{t \le s < R} \varphi(s)(f+g)^*(s) \\
&\le \frac{1}{\varphi(t)} \sup_{t \le s < R} \varphi(s) \left[f^*\left(\frac{s}{2}\right) + g^*\left(\frac{s}{2}\right)\right] \\
&\le \sup_{t \le s < R} \frac{\varphi(s)}{\varphi(t)} f^*\left(\frac{s}{2}\right) + \sup_{t \le s < R} \frac{\varphi(s)}{\varphi(t)} g^*\left(\frac{s}{2}\right).
\end{aligned}$$

The function φ is quasiconcave, so we have $\frac{\varphi\left(\frac{s}{2}\right)}{\frac{s}{2}} \geq \frac{\varphi(s)}{s}$, which implies $\varphi(s) \leq 2\varphi\left(\frac{s}{2}\right)$ and $\frac{\varphi(s)}{\varphi(t)} \leq 2\frac{\varphi\left(\frac{s}{2}\right)}{\varphi\left(\frac{t}{2}\right)}$, because φ is strictly increasing. Hence,

$$\begin{aligned}
&\sup_{t\leq s<R} \frac{\varphi(s)}{\varphi(t)} f^*\left(\frac{s}{2}\right) + \sup_{t\leq s<R} \frac{\varphi(s)}{\varphi(t)} g^*\left(\frac{s}{2}\right) \\
&\qquad \leq 2\frac{1}{\varphi\left(\frac{t}{2}\right)} \left(\sup_{t\leq s<R} \varphi\left(\frac{s}{2}\right) f^*\left(\frac{s}{2}\right) + \sup_{t\leq s<R} \varphi\left(\frac{s}{2}\right) g^*\left(\frac{s}{2}\right) \right) \\
&\qquad \leq 2\frac{1}{\varphi\left(\frac{t}{2}\right)} \left(\sup_{\frac{t}{2}\leq s<R} \varphi(s) f^*(s) + \sup_{\frac{t}{2}\leq s<R} \varphi(s) g^*(s) \right) \\
&\qquad = 2\left(T_\varphi f + T_\varphi g\right)\left(\frac{t}{2}\right).
\end{aligned}$$

The second inequality can be proved by the same method, namely,

$$\begin{aligned}
\left(S_\varphi[f+g]\right)(t) &= \frac{\varphi(t)}{t} \sup_{0<s\leq t} \frac{s}{\varphi(s)} (f+g)^*(s) \\
&\leq \sup_{0<s\leq t} \frac{s\varphi(t)}{t\varphi(s)} f^*\left(\frac{s}{2}\right) + \sup_{0<s\leq t} \frac{s\varphi(t)}{t\varphi(s)} g^*\left(\frac{s}{2}\right) \\
&\leq 2\frac{\varphi\left(\frac{t}{2}\right)}{\frac{t}{2}} \left(\sup_{0<s\leq t} \frac{\frac{s}{2}}{\varphi\left(\frac{s}{2}\right)} f^*\left(\frac{s}{2}\right) + \sup_{0<s\leq t} \frac{\frac{s}{2}}{\varphi\left(\frac{s}{2}\right)} g^*\left(\frac{s}{2}\right) \right) \\
&\leq 2\left(S_\varphi f + S_\varphi g\right)\left(\frac{t}{2}\right).
\end{aligned}$$

□

Lemma 3.2. *Let φ satisfy the $B1$-condition, then the operator T_φ satisfies*

$$T_\varphi : L^1 \to L^1 \quad \textit{and} \quad T_\varphi : M_\varphi \to M_\varphi. \tag{3.3}$$

Next, let φ satisfy the $B2$-condition, then the operator S_φ satisfies

$$S_\varphi : M_{\frac{t}{\varphi(t)}} \to M_{\frac{t}{\varphi(t)}} \quad \textit{and} \quad S_\varphi : L^\infty \to L^\infty. \tag{3.4}$$

Proof. The set of all continuous functions with support in I, $C_0(I)$, is dense in $L^1(I)$. Hence, it suffices to verify

$$\|T_\varphi f\|_{L^1} \lesssim \|f\|_{L^1}$$

for $f \in C_0(I)$. In this case, we can extend the domain of f^* on I by

$$f^*(0) := \lim_{t\to 0_+} f^*(t).$$

Assuming $f \not\equiv 0$, define

$$(Qf)(t) := \sup_{t\leq s<R} \varphi(s) f^*(s), \quad t \in I,$$

and set $A := \{k \in \mathbb{Z}_+ : (Qf)\left(2^{-k}R\right) > (Qf)\left(2^{1-k}R\right)\}$. Then, A is nonempty, because if $\max_{0<t<R} \varphi(t)f^*(t)$ is attained in $\left[2^{-k}R, 2^{1-k}R\right)$, then $k \in A$. Given $k \in A$ we define

$$t_k = \begin{cases} 0 & \text{if } (Qf)(t) = (Qf)\left(2^{-k}R\right), \quad t \in \left(0, 2^{-k}R\right]; \\ \min\{2^{-j}R; (Qf)\left(2^{-j}R\right) = (Qf)\left(2^{-k}R\right)\} & \text{otherwise.} \end{cases}$$

Hence,

$$(Qf)(t) = (Qf)\left(2^{-k}R\right), \quad k \in A, t \in \left[t_k, 2^{-k}R\right].$$

By the definition of A, $\sup_{2^{-k}R \le s < 1} \varphi(s)f^*(s)$ is attained in $\left[2^{-k}R, 2^{1-k}R\right)$ when $k \in A$. Thus, for every $k \in A$ and $t \in \left[t_k, 2^{1-k}R\right)$, together with the monotocity of $(Qf)(t)$ we get

$$\begin{aligned} (Qf)(t) \le (Qf)\left(2^{-k}R\right) &= \sup_{2^{-k}R \le s < 2^{1-k}R} \varphi(s)f^*(s) \\ &\le \varphi\left(2^{1-k}R\right) f^*\left(2^{-k}R\right). \end{aligned} \tag{3.5}$$

It is obvious that $\bigcup_{k\in A} \left[t_k, 2^{1-k}R\right) \supseteq (0, R)$. So, using all this, the monotocity of f^* and the $B1$-condition for φ we have

$$\begin{aligned} \|T_\varphi f\|_{L^1} &= \int_0^R \frac{(Qf)(t)}{\varphi(t)}\, dt \le \sum_{k\in A} \int_{t_k}^{2^{1-k}R} \frac{(Qf)(t)}{\varphi(t)}\, dt \\ &\le \sum_{k\in A} \varphi\left(2^{1-k}R\right) f^*\left(2^{-k}R\right) \int_0^{2^{1-k}R} \frac{1}{\varphi(t)}\, dt \\ &\lesssim \sum_{k\in A} 2^{1-k}Rf^*\left(2^{-k}R\right) = 4\sum_{k\in A} \int_{2^{-1-k}R}^{2^{-k}R} dt f^*\left(2^{-k}R\right) \\ &\le 4\sum_{k\in A} \int_{2^{-1-k}R}^{2^{-k}R} f^*(t)\, dt \le 4\,\|f\|_{L^1}. \end{aligned}$$

Next, $T_\varphi f$ is nonincreasing for all $f \in \mathcal{P}(I)$, so $(T_\varphi f)^{**}(t) = \frac{1}{t}\int_0^t (T_\varphi f)(s)\, ds$. Then, for every $f \in M_\varphi(I)$ we have

$$\begin{aligned} \|T_\varphi f\|_{M_\varphi} &= \sup_{0<t<R} \{\varphi(t)(T_\varphi f)^{**}(t)\} \\ &= \sup_{0<t<R} \left\{ \frac{\varphi(t)}{t} \int_0^t \frac{1}{\varphi(s)} \sup_{s\le r<R} \varphi(r)f^*(r)\, ds \right\} \\ &\le \sup_{0<t<R} \left\{ \frac{\varphi(t)}{t} \int_0^t \frac{1}{\varphi(s)} \sup_{0<r<R} \varphi(r)f^{**}(r)\, ds \right\} \\ &= \|f\|_{M_\varphi} \sup_{0<t<R} \left\{ \frac{\varphi(t)}{t} \int_0^t \frac{1}{\varphi(s)}\, ds \right\}, \end{aligned}$$

but the last supremum is finite because of the $B1$-condition for φ.

Next, suppose f belongs to L^∞. We have

$$\begin{aligned}\|S_\varphi f\|_{L^\infty} &= \operatorname*{ess\,sup}_{0<t<R} \frac{\varphi(t)}{t} \sup_{0<s\le t} \frac{s}{\varphi(s)} f^*(s) \\ &\le \|f\|_{L^\infty} \operatorname*{ess\,sup}_{0<t<R} \sup_{0<s\le t} \frac{\varphi(t)}{t} \frac{s}{\varphi(s)} = \|f\|_{L^\infty},\end{aligned}$$

because $\frac{s}{\varphi(s)}$ is nondecreasing, so the supremum is attained in t.

Finally, we will observe that, for every $g \in \mathcal{P}(I)$ and $t \in I$,

$$\sup_{0<s\le t} \frac{s}{\varphi(s)} g^*(s) \approx \sup_{0<s\le t} \frac{s}{\varphi(s)} g^{**}(s). \tag{3.6}$$

Indeed, setting

$$M = \sup_{0<s\le t} \frac{s}{\varphi(s)} g^*(s),$$

we get

$$g^*(s) \le M \frac{\varphi(s)}{s}, \quad s \in (0,t].$$

Using $B2$-condition for φ, we obtain

$$g^{**}(y) = \frac{1}{y}\int_0^y g^*(s)\,ds \le \frac{M}{y}\int_0^y \frac{\varphi(s)}{s}\,ds \lesssim \frac{M}{y}\varphi(y), \quad y \in (0,t],$$

and, therefore,

$$\sup_{0<y\le t} \frac{y}{\varphi(y)} g^{**}(y) \lesssim M, \quad t \in I.$$

The second inequality is obvious since $g^* \le g^{**}$.

Applying (3.6), we get

$$\begin{aligned}\|S_\varphi f\|_{M_{\frac{t}{\varphi(t)}}} &= \sup_{0<t\le R} \frac{t}{\varphi(t)} \left(\frac{\varphi(y)}{y} \sup_{0<s<y} \frac{s}{\varphi(s)} f^*(s) \right)^{**}(t) \\ &\lesssim \sup_{0<t\le R} \frac{t}{\varphi(t)} \left(\frac{\varphi(y)}{y} \sup_{0<s<y} \frac{s}{\varphi(s)} f^*(s) \right)^{*}(t) \\ &\le \|f\|_{M_{\frac{t}{\varphi(t)}}} \sup_{0<t<R} \frac{t}{\varphi(t)} \left(\frac{\varphi(y)}{y} \right)^*(t) = \|f\|_{M_{\frac{t}{\varphi(t)}}},\end{aligned}$$

using again that $\frac{\varphi(t)}{t}$ is nonincreasing. □

In the following part of this section we introduce some further properties of the operators S_φ and T_φ.

Lemma 3.3. *Let* (X_0, X_1) *and* (Y_0, Y_1) *be a compatible couples of Banach function spaces. Then, for every operator* T *satisfying* $T : X_0 \to Y_0$, $T : X_1 \to Y_1$ *and*

$$(T[f+g])(t) \leq 2(Tf + Tg)\left(\frac{t}{2}\right), \quad f, g \in \mathcal{P}(I), t \in I,$$

we have

$$K(Tf, t; Y_0, Y_1) \lesssim K(f, t; X_0, X_1), \quad f \in X_0 + X_1, t > 0.$$

Proof. It follows from the embeddings that we have two positive constants C_i, $i = 0, 1$, such that

$$\|Tf\|_{Y_i} \leq C_i \|f\|_{X_i}, \quad f \in \mathcal{P}(I), i = 0, 1.$$

Fix $f \in X_0 + X_1$ and assume that $f = f_0 + f_1$ where $f_i \in X_i, i = 0, 1$. Then, by decomposing Tf in an appropriate way, we get

$$Tf = T[f_0 + f_1] \leq 2E_2Tf_0 + 2E_2Tf_1,$$

so $Tf - 2E_2Tf_0 \leq 2E_2Tf_1$. Note that, according to Proposition 1.14, we have $\|E_2f\|_X \leq 2\|f\|_X$, because

$$\frac{1}{\|f\|_X}\|E_2f\|_X = \left\|\frac{1}{\|f\|_X}E_2f\right\|_X = \left\|E_2\frac{f}{\|f\|_X}\right\|_X \leq 2, \quad f \in \mathcal{P}(I). \tag{3.7}$$

Applying this and the sublinearity of T, we get

$$\|2E_2Tf_0\|_{Y_0} \leq 4\|Tf_0\|_{Y_0} \leq 4C_0\|f_0\|_{X_0},$$

and similarly

$$\|Tf - 2E_2Tf_0\|_{Y_1} \leq \|2E_2Tf_1\|_{Y_1} \leq 4C_1\|f_1\|_{X_1}.$$

Finally,

$$\begin{aligned} K(Tf, t; Y_0, Y_1) &\leq \|2E_2Tf_0\|_{Y_0} + t\|Tf - 2E_2Tf_0\|_{Y_1} \\ &\leq 4\max\{C_0, C_1\}\left(\|f_0\|_{X_0} + t\|f_1\|_{X_1}\right). \end{aligned}$$

Taking the infimum over all possible representations $f = f_0 + f_1$, we obtain the required inequality. □

Theorem 3.4. *Suppose that* T *is an operator defined on* $M_{\frac{t}{\varphi(t)}}$ *satisfying*

$$(T[f+g])(t) \leq 2(Tf + Tg)\left(\frac{t}{2}\right), \quad f, g \in \mathcal{P}(I), t \in I,$$

as well as

$$T : M_{\frac{t}{\varphi(t)}} \to M_{\frac{t}{\varphi(t)}} \quad \text{and} \quad T : L^\infty \to L^\infty. \tag{3.8}$$

Then,

$$(Tf)^{**}(t) \lesssim (S_\varphi f^{**})(t) \lesssim (S_\varphi)(t), \quad f \in \mathcal{P}(I), t \in I. \tag{3.9}$$

Moreover, if φ satisfies B2-condition, we have

$$(S_\varphi f)^{**}(t) \approx (S_\varphi f^{**})(t) \approx (S_\varphi f)(t), \quad f \in \mathcal{P}(I), t \in I. \tag{3.10}$$

Remark 3.5. We denote $\frac{t}{\varphi(t)}$ by $\psi(t)$. Notice that every nonincreasing $g \in \mathcal{P}(I)$ satisfies

$$\sup_{0<s\le t} \psi(s) g^{**}(s) = \sup_{0<s<R} \left(\chi_{(0,t)} g\right)^{**}(s)\psi(s), \quad t \in (0,R]. \tag{3.11}$$

Indeed, since

$$\left(\chi_{(0,t)} g\right)^{**}(s)\psi(s) = \frac{1}{s}\int_0^s \left(\chi_{(0,t)} g\right)^{*}(y)\,dy \frac{s}{\varphi(s)} = \int_0^t \left(\chi_{(0,t)} g\right)^{*}(y)\,dy \frac{1}{\varphi(s)}$$

when $s \in [t,R)$ and $\frac{1}{\varphi(s)}$ is strictly decreasing, we have that

$$\sup_{0<s<R} \left(\chi_{(0,t)} g\right)^{**}(s)\psi(s) = \sup_{0<s\le t} \left(\chi_{(0,t)} g\right)^{**}(s)\psi(s).$$

This yields (3.11) because

$$\left(\chi_{(0,t)} g\right)^{**}(s) = g^{**}(s), \quad s \in (0,t].$$

Proof of Theorem 3.4. Using (3.8) and Lemma 3.3 we get the following K-functional inequality

$$K\left(Tf, t; M_\psi, L^\infty\right) \lesssim K\left(f, t; M_\psi, L^\infty\right), \quad t \in I.$$

We may assume $C > 1$ and replace t with $\psi(t)$ in this inequality, and we get

$$K\left(Tf, \psi(t); M_\psi, L^\infty\right) \lesssim K\left(f, \psi(t); M_\psi, L^\infty\right), \quad t \in I.$$

Then, together with Theorem 1.22, the equality (3.11) in the case when $g = (Tf)^*$ and the fact that the fundamental function of M_ψ is

$$\varphi_{M_\psi}(t) = \psi(t),$$

we get

$$\begin{aligned}
\sup_{0<s\le t} \psi(s)(Tf)^{**}(s) &= \left\|\chi_{(0,t)}(Tf)^*\right\|_{M_\psi} \le \left\|\chi_{\left(0,\varphi^{-1}_{M_\psi}(\psi(t))\right)}(Tf)^*\right\|_{M_\psi} \\
&\approx K\left(Tf, \psi(t); M_\psi, L^\infty\right) \lesssim K\left(f, \psi(t); M_\psi, L^\infty\right) \\
&\approx \left\|\chi_{\left(0,\varphi^{-1}_{M_\psi}(\psi(t))\right)} f^*\right\|_{M_\psi} \\
&= \sup_{0<s\le \varphi^{-1}_{M_\psi}(\psi(t))} \psi(s) f^{**}(s), \quad t \in I.
\end{aligned}$$

Hence,

$$\psi(t)(Tf)^{**}(t) \lesssim \sup_{0<s\le\varphi_{M_\psi}^{-1}(\psi(t))} \psi(s)f^{**}(s),$$

or

$$\begin{aligned}(Tf)^{**}(t) &\lesssim \frac{1}{\psi(t)} \frac{\varphi_{M_\psi}^{-1}(\psi(t))}{\varphi\left(\varphi_{M_\psi}^{-1}(\psi(t))\right)} (S_\varphi f^{**})\left(\varphi_{M_\psi}^{-1}(\psi(t))\right) \\ &= (S_\varphi f^{**})\left(\varphi_{M_\psi}^{-1}(\psi(t))\right) \le (S_\varphi f^{**})(t),\end{aligned}$$

since $S_\varphi f^{**}$ is nonincreasing and we have Remark 1.23. The first part of (3.9) is proved and the second part follows readily from (3.6).

Now, in view of (3.2) and (3.4) the operator S_φ satisfies all the assumptions necessary to replace T by S_φ, so (3.9) also holds when $T = S_\varphi$.

Finally, it follows from (3.6) and the monotocity of $S_\varphi f^{**}$ that

$$(S_\varphi f^{**})(t) \le (S_\varphi f^{**})^{**}(t) \approx (S_\varphi f)^{**}(t), \quad t \in I,$$

so we have (3.10). □

Theorem 3.6. *Let T_φ be the supremum operator defined in Definition 3.1 and let φ satisfy the $B1$-condition. Then, for all $f \in \mathcal{P}(I)$ and $t \in I$,*

$$(T_\varphi f)^{**}(t) \lesssim (T_\varphi f^{**})(t). \tag{3.12}$$

Proof. For an arbitrary pair of $f \in \mathcal{P}(I)$ and $t \in I$, set

$$f_t(s) = \min[f(s), f^*(t)]$$

and

$$f^t(s) = \max[f(s) - f^*(t), 0], \quad s \in I.$$

Then

$$\begin{aligned}&f(s) = f_t(s) + f^t(s),\\ &f^*(s) = f_t^*(s) + (f^t)^*(s), \quad s \in I,\end{aligned}$$

where

$$\begin{aligned}&f_t^*(s) = \min[f^*(s), f^*(t)],\\ &(f^t)^*(s) = \max[f^*(s) - f^*(t), 0] = (f^*(s) - f^*(t))\chi_{(0,t)}, \quad s \in I.\end{aligned}$$

Since $(T_\varphi f)(t)$ is nonincreasing in t

$$\begin{aligned}(T_\varphi f)^{**}(t) &= \frac{1}{t}\int_0^t \frac{1}{\varphi(s)} \sup_{s\le r<R} \varphi(r)f^*(r)\,ds \\ &\le \frac{1}{t}\int_0^t \frac{1}{\varphi(s)} \sup_{s\le r<R} \varphi(r)f_t^*(r)\,ds \\ &\qquad + \frac{1}{t}\int_0^t \frac{1}{\varphi(s)} \sup_{s\le r<R} \varphi(r)(f^t)^*(r)\,ds \\ &= \mathrm{I} + \mathrm{II}.\end{aligned}$$

Next, using the $B1$-condition for φ, we get

$$\begin{aligned}
\mathrm{I} &= \frac{1}{t}\int_0^t \frac{1}{\varphi(s)} \sup_{s\le r<R} \varphi(r)\min[f^*(r), f^*(t)]\,ds \\
&= \frac{1}{t}\int_0^t \frac{1}{\varphi(s)} \max\left[\sup_{t\le r<R}\varphi(r)f^*(r), \sup_{s\le r<t}\varphi(r)f^*(t)\right] ds \\
&= \frac{1}{t}\int_0^t \frac{1}{\varphi(s)} \sup_{t\le r<R}\varphi(r)f^*(r)\,ds = \frac{1}{t}\int_0^t \frac{1}{\varphi(s)}\,ds \sup_{t\le r<R}\varphi(r)f^*(r) \\
&\lesssim \frac{1}{\varphi(t)}\sup_{t\le r<R}\varphi(r)f^*(r) = (T_\varphi)(t)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{II} &= \frac{1}{t}\int_0^t \frac{1}{\varphi(s)} \sup_{s\le r<R}\varphi(r)(f^t)^*(r)\,ds = \frac{1}{t}\int_0^t (T_\varphi f^t)(s)\,ds \\
&\le \frac{1}{t}\int_0^R (T_\varphi f^t)(s)\,ds.
\end{aligned}$$

We use $T_\varphi : L^1 \to L^1$,

$$\frac{1}{t}\int_0^R (T_\varphi f^t)(s)\,ds \lesssim \frac{1}{t}\int_0^R (f^t)^*(s)\,ds = \frac{1}{t}\int_0^t [f^*(s) - f^*(t)]\,ds \lesssim f^{**}(t).$$

Combining these two results and

$$f^{**}(t) = \sup_{t\le s<R} f^{**}(s) = \frac{1}{\varphi(t)}\varphi(t)\sup_{t\le s<R} f^{**}(s) \le \frac{1}{\varphi(t)}\sup_{t\le s<R}\varphi(s)f^{**}(s),$$

we obtain

$$(T_\varphi f)^{**}(t) \lesssim [f^{**}(t) + (T_\varphi)(t)] \lesssim (T_\varphi f^{**})(t), \qquad f\in\mathcal{P}(I), \quad t\in I. \tag{3.13}$$

□

Now we are in a position to state and prove the main result of this section.

Theorem 3.7. *Let φ satisfy the $B2$-condition. Then there exists $C>0$ such that, for any r.i. space X and f in $\mathcal{P}(I)$,*

$$\left\|\sup_{t\le s<R}\varphi(s)f^{**}(s)\right\|_X \le C\,\|\varphi(t)f^{**}(t)\|_X\,. \tag{3.14}$$

Proof. According to Theorem 1.12 and (3.7) we have $g \prec h$ implies $\|g\|_X \le C\,\|h\|_X$ and $\|E_2 f\|_X \le 2\,\|f\|_X$. Thus, it suffices to find $C>0$, independent of $f\in\mathcal{P}(I)$ and t on some right neighbourhood of 0, say, $t\in\left(0, \frac{R}{3}\right)$, for which

$$\int_0^t \sup_{s\le z<R}\varphi(z)f^{**}(z)\,ds \le C\int_0^t (\varphi(z)f^{**}(z))^*\left(\frac{s}{2}\right)\,ds. \tag{3.15}$$

Fix $f \in \mathcal{P}(I)$ and $t \in \left(0, \frac{R}{3}\right)$ and take f_t and f^t from the proof of previous Theorem. We use the decomposition of f^{**} on f_t^{**} and $(f^t)^{**}$, so we have

$$\begin{aligned}\sup_{s\le z<R} \varphi(z) f^{**}(z) &= \sup_{s\le z<R} \varphi(z)[f_t^{**}(z) + (f^t)^{**}(z)] \\ &\le \sup_{s\le z<R} \varphi(z) f_t^{**}(z) + \sup_{s\le z<R} \varphi(z)(f^t)^{**}(z).\end{aligned}$$

Now, we estimate each of these summands. Firstly, we rewrite the functions f_t^{**} and $(f^t)^{**}$ to see what are their forms on $(0,t)$ and (t,R):

$$f_t^{**}(z) = \begin{cases} f^*(t), & 0 < z < t; \\ \frac{t}{z} f^*(t) + \frac{1}{z}\int_t^z f^*(y)\,dy, & t \le z < R; \end{cases}$$

$$(f^t)^{**}(z) = \begin{cases} f^{**}(z) - f^*(t), & 0 < z < t; \\ \frac{t}{z}[f^{**}(t) - f^*(t)], & t \le z < R. \end{cases}$$

Hence, for $0 < s \le t$, using the monotonicity of $\varphi(t)/t$ we get

$$\begin{aligned}\sup_{s\le z<R} \varphi(z)(f^t)^{**}(z) &= \max\Bigg[\sup_{s\le z<t} \varphi(z)[f^{**}(z) - f^*(t)], \\ &\qquad \sup_{t\le z<R} \varphi(z)\frac{t}{z}\left[f^{**}(t) - f^*(t)\right]\Bigg] \\ &\le \max\left[\sup_{s\le z<t} \varphi(z) f^{**}(z), \sup_{t\le z<R} \frac{\varphi(z)}{z} t f^{**}(t)\right] \\ &\le \sup_{s\le z\le t} \varphi(z) f^{**}(z),\end{aligned}$$

so using once more time the monotonicity of $\varphi(t)/t$ and the $B2$-condition for φ, we have

$$\begin{aligned}\int_0^t \sup_{s\le z<R} \varphi(z)(f^t)^{**}(z)\,ds &\le \int_0^t \sup_{s\le z\le t} \frac{\varphi(z)}{z} \int_0^z f^*(y)\,dy\,ds \\ &\le \left(\int_0^t f^*(y)\,dy\right)\left(\int_0^t \sup_{s\le z\le t} \frac{\varphi(z)}{z}\,ds\right) \\ &= \left(\int_0^t f^*(y)\,dy\right)\left(\int_0^t \frac{\varphi(s)}{s}\,ds\right) \\ &\le C\varphi(t)\int_0^t f^*(s)\,ds.\end{aligned} \tag{3.16}$$

The monotonicity of g^* implies $\int_0^t g^* \ge \int_t^{2t} g$ and clearly $g\left(\frac{\cdot}{2}\right)^*(s) = g^*\left(\frac{s}{2}\right)$. Since $\varphi(t)/t$ is nonincreasing,

$$\frac{\varphi(2t)}{\varphi(t)} \le 2. \tag{3.17}$$

All this together with the monotonicity of f^{**} asserts

$$\begin{aligned}\varphi(t)\int_0^t f^*(s)\,ds &\le 2\,\varphi\left(\frac{t}{2}\right)tf^{**}(t) = 2\int_t^{2t}\varphi\left(\frac{t}{2}\right)f^{**}\left(\frac{2t}{2}\right)\,ds\\ &\le 2\int_t^{2t}\varphi\left(\frac{s}{2}\right)f^{**}\left(\frac{s}{2}\right)\,ds \le 2\int_0^t(\varphi(z)f^{**}(z))^*\left(\frac{s}{2}\right)\,ds,\end{aligned}$$

which together with (3.16) yields

$$\int_0^t \sup_{s\le z<R}\varphi(z)(f^t)^{**}(z)\,ds \le 2C\int_0^t(\varphi(z)f^{**}(z))^*\left(\frac{s}{2}\right)\,ds.$$

Now, we estimate the second summand, that is,

$$\int_0^t \sup_{s\le z<R}\varphi(z)f_t^{**}(z)\,ds \le C\int_0^t(\varphi(z)f^{**}(z))^*\left(\frac{s}{2}\right)\,ds.$$

For an arbitrary $s\in(0,t)$ we have

$$\begin{aligned}\sup_{s\le z<R}\varphi(z)f_t^{**}(z) &= \max\left[\sup_{s\le z<t}\varphi(z)f_t^{**}(z),\ \sup_{t\le z<R}\varphi(z)f_t^{**}(z)\right]\\ &\le \max\left[\varphi(t)f^*(t),\ \sup_{t\le z<R}\left[\frac{\varphi(z)}{z}tf^*(t)+\frac{\varphi(z)}{z}\int_t^z f^*(y)\,dy\right]\right]\\ &\le \max\left[\varphi(t)f^*(t),\varphi(t)f^*(t)+\sup_{t\le z<R}\frac{\varphi(z)}{z}\int_t^z f^*(y)\,dy\right]\\ &= \varphi(t)f^*(t)+\sup_{t\le z<R}\frac{\varphi(z)}{z}\int_t^z f^*(y)\,dy\\ &\le 3\sup_{t\le z<R}\frac{\varphi(z)}{z}\int_t^z f^*(y)\,dy,\end{aligned}$$

since φ is nondecreasing, $\frac{\varphi(t)}{t}$ is nonincreasing and

$$\begin{aligned}\varphi(t)f^*(t) &\le 2\frac{\varphi(2t)}{2t}tf^*(t) = 2\frac{\varphi(2t)}{2t}\int_t^{2t}f^*(t)\,dy\\ &\le 2\frac{\varphi(2t)}{2t}\int_t^{2t}f^*\left(\frac{y}{2}\right)\,dy \le 2\sup_{t\le z<R}\frac{\varphi(z)}{z}\int_t^z f^*(y)\,dy.\end{aligned}$$

Thus,

$$\int_0^t \sup_{s\le z<R}\varphi(z)f_t^{**}(z)\,ds \le 3t\sup_{t\le z<R}\frac{\varphi(z)}{z}\int_t^z f^*(y)\,dy.$$

It will suffice to prove that

$$\sup_{t\le z<R}\frac{\varphi(z)}{z}\int_t^z f^*\left(\frac{y}{2}\right)\,dy \le C\frac{\varphi(s)}{s}\int_0^s f^*\left(\frac{y}{2}\right)\,dy$$

on a set E of measure at least t, (for all $s \in E$), since using this inequality, the monotonicity of f^*, (3.17) and (1.1), we get

$$\begin{aligned} 3t \sup_{t\le z<R} \frac{\varphi(z)}{z} \int_t^z f^*(y)\,dy &\le 3\int_E \sup_{t\le z<R} \frac{\varphi(z)}{z} \int_t^z f^*(y)\,dy\,ds \\ &\le 3\int_E \sup_{t\le z<R} \frac{\varphi(z)}{z} \int_t^z f^*\left(\frac{y}{2}\right) dy\,ds \\ &\le 3C\int_E \frac{\varphi(s)}{s} \int_0^s f^*\left(\frac{y}{2}\right) dy\,ds \\ &\le 6C\int_E \varphi\left(\frac{s}{2}\right) f^{**}\left(\frac{s}{2}\right) ds \le C' \int_0^t (\varphi(z) f^{**}(z))^* \left(\frac{s}{2}\right) ds. \end{aligned}$$

Next suppose $z_0 \in (t, R]$ is such that

$$\sup_{t\le z<R} \frac{\varphi(z)}{z} \int_t^z f^*\left(\frac{y}{2}\right) dy = \frac{\varphi(z_0)}{z_0} \int_t^{z_0} f^*\left(\frac{y}{2}\right) dy.$$

This supremum is attained in $(t, R]$ because the function inside is positive, continuous on $[t, R]$ and vanishing for $z = t$. We consider two cases.

Let $z_0 \in (t, 2t)$. Since f^* is nonincreasing,

$$\int_0^{z_0} f^*\left(\frac{y}{2}\right) dy \ge 2\int_t^{z_0} f^*\left(\frac{y}{2}\right) dy.$$

Thus, for $s \in (z_0, z_0 + t)$, we have

$$\begin{aligned} \frac{\varphi(s)}{s} \int_0^s f^*\left(\frac{y}{2}\right) dy &\ge \frac{\varphi(z_0+t)}{z_0+t} \int_0^{z_0} f^*\left(\frac{y}{2}\right) dy \\ &\ge \frac{\varphi(2z_0)}{2z_0} 2\int_t^{z_0} f^*\left(\frac{y}{2}\right) dy \ge \frac{\varphi(z_0)}{z_0} \int_t^{z_0} f^*\left(\frac{y}{2}\right) dy \\ &= \sup_{t\le z<R} \frac{\varphi(z)}{z} \int_t^z f^*\left(\frac{y}{2}\right) dy, \end{aligned}$$

since $\varphi(t)/t$ is nonincreasing and $\varphi(t)$ is increasing. In this case, we set $E = (z_0, z_0+t)$.

Let $z_0 \in [2t, R]$. Similarly, for $s \in (z_0 - t, z_0)$, we have

$$\begin{aligned} \frac{\varphi(s)}{s} \int_0^s f^*\left(\frac{y}{2}\right) dy &\ge \frac{\varphi(z_0)}{z_0} \int_0^{z_0-t} f^*\left(\frac{y}{2}\right) dy \\ &\ge \frac{1}{2}\frac{\varphi(z_0)}{z_0} \int_0^{z_0} f^*\left(\frac{y}{2}\right) dy \ge \frac{1}{2}\frac{\varphi(z_0)}{z_0} \int_t^{z_0} f^*\left(\frac{y}{2}\right) dy \\ &= \frac{1}{2} \sup_{t\le z<R} \frac{\varphi(z)}{z} \int_t^z f^*\left(\frac{y}{2}\right) dy. \end{aligned}$$

So, in this case it suffices to set $E = (z_0 - t, z_0)$. Choosing maximal constant among all those which we have used, each inequality holds and the proof is complete. □

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