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Analysis of attractors for generalized Newtonian fluids in 3d domains

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I declare that I wrote this master thesis independently, using only the cited sources, literature and other professional sources.

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Prague, July 27, 2011 Josef Žabenský

Název práce: Analýza atraktorů zobecněných Newtonovských tekutin v 3d oblastech
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Abstrakt: Zkoumáme systém nelineárních parciálních diferenciálních rovnic, konkrétně tzv. model Ladyženské, ve třech prostorových dimenzích. Ukážeme, že po přidání perturbace vyššího řádu tento model vykazuje podstatně lepší analyzovatelnost, obzvláště díky relativně snadno dokazatelné diferencovatelnosti řešení podle počáteční podmínky. Díky tomuto faktu budeme na rozdíl od původního modelu oprávněni aplikovat metodu ljapunovských exponentů k odhadu fraktální dimenze exponenciálního atraktoru. Než ovšem dosáhneme tohoto výsledku, bude nutné obvyklými metodami dokázat existenci a jednoznačnost řešení, zlepšenou regularitu a především existenci kompaktní invariantní množiny pro celý systém.

Klíčová slova: Dynamický systém, atraktor, dimenze atraktoru, kompaktnost, diferencovatelnost podle počáteční podmínky.

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Abstract: We investigate a system of nonlinear partial differential equations, specifically the so-called Ladyzhenskaya model, in three spatial dimensions. It will be shown that after inclusion of a perturbation of a higher order, the model exhibits a considerably better behavior, in particular it will become quite straightforward to prove differentiability of solutions with respect to the initial condition. Due to this fact we may consequently employ the method of Lyapunov exponents to estimate the fractal dimension of the exponential attractor. First, however, it will be necessary to show existence and uniqueness of solutions, improved regularity and existence of a compact invariant set for the entire system.

Keywords: Dynamical system, attractor, dimension of the attractor, compactness, differentiability with respect to the initial condition.

Contents

Introduction

Keeping rigorous definitions for a later time (see Definition 2.1 below), we analyze the following perturbed incompressible fluid model of Ladyzhenskaya:

$$\left.\begin{aligned} \partial_t \boldsymbol{u} - \varepsilon\Delta^3 \boldsymbol{u} - \operatorname{div}_x \mathbb{S}(\mathbb{D}\boldsymbol{u}) + \operatorname{div}_x(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla_x p = \boldsymbol{f} \\ \operatorname{div}_x \boldsymbol{u} = 0 \end{aligned}\right. \quad \text{in } (0,T)\times\Omega, \tag{1}$$

with a sufficiently smooth domain $\Omega \subset \mathbb{R}^3$. The equation is accompanied with the boundary condition

$$\boldsymbol{u}|_{\partial\Omega} = 0 \qquad \text{in } (0,T)$$

and the initial condition

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in } \Omega.$$

The stress tensor $\mathbb{S}$ is considered in the form

$$\mathbb{S}(\mathbb{X}) = \nu_0\mathbb{X} + \nu_1 |\mathbb{X}|^{r-2}\,\mathbb{X}, \qquad \mathbb{X}\in\mathbb{R}^{3\times 3},$$

with constant viscosities $\nu_0, \nu_1 > 0$ and a parameter r that will be expounded on later.

The main aim of the present thesis is to verify that the perturbed model (i.e. $\varepsilon > 0$) admits an exponential attractor whose qualities (certain regularity, rate of attraction and, most importantly, the dimension estimate) are independent of $\varepsilon > 0$. A closer look at the proofs reveals that the induced ε-regularity is only needed in justification of a certain technical tool (which is the differentiability of the solution operator), while the core estimate is meaningful and independent of it.

Unfortunately, it still remains an open (and possibly difficult) problem to assess whether these estimates remain valid also for $\varepsilon = 0$. If so, it would be a significant improvement of the estimates delivered in [1] (and then assimilated into [6]), where a different approach (the method of trajectories) was used.

From the very beginning it should also be emphasized that the only estimates' quality we are interested in is their dependency on ε or a lack thereof. Their exact form in terms of data is otherwise nearly always neglected. From this point of view we do count on a superior nature of the method we use here and the estimates it delivers, as opposed to those derived by means of the method of trajectories. The reason for doing so is to make the presentation as lucid as possible. The only instance of breaking this habit will occur in Remark 4.6.

Let us conclude this brief introduction with a few bibliographical remarks. Although the problem studied in this thesis is new (to the best of our knowledge), the techniques we adopt are by no means original in the literature. First of all, the so-called Ladyzhenskaya model of incompressible fluid – i.e. the system (1), or more precisely (2.1), with $\varepsilon = 0$, was first suggested by Ladyzhenskaya [8], who also provided the basic theory. Our presentation in section 2.4, including higher regularity with respect to time, follows mostly the ideas of chapter 5 in [9].

The attractor part (chapters 3 and 4) is based on the general presentation given in chapter 2 of [6]. But, once again, these techniques and concepts are nowadays classical. Exponential attractors date back at least to the survey book [4]. Lyapunov exponents are treated extensively also in [10] or [12]. Regarding the construction of exponential attractors using Lyapunov exponents, see [3].

1. Preliminaries

Before we delve deep into the productive part of the work, we have to first present few topics and tools used to varying degrees later.

1.1 Symbolism in use

The following notation is for the most part not elucidated elsewhere and as such it is usually taken for granted.

c_i	a constant dependent only on $\boldsymbol{f}$, Ω, r, T, ν_0, ν_1 and $\boldsymbol{u}_0$
$\boldsymbol{u}$, $\boldsymbol{v}$, etc.	$\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{v} = (v_1, v_2, v_3)$, etc.
u', $\partial_t u$, $\frac{\mathrm{d}}{\mathrm{d}t}u$	the generalized time derivative introduced in Definition 1.4
$\mathbb{D}$	the symmetric gradient, $1/2(\nabla_x + \nabla_x^T)$
$\Omega \in \mathcal{C}^3(\mathbb{R}^3)$	$\Omega \in \mathcal{C}^3$ and $\Omega \subset \mathbb{R}^3$ bounded and open
$\mathcal{V}$, H, V_r, V^3	the function spaces from Definition 1.2
X^*	the (topological) dual space to X
$\mathcal{D}'(\Omega)$	the space of distributions on Ω
$\mathcal{L}(X)$	the space of all continuous linear operators on X
$\langle \cdot\,, \cdot \rangle_{X^*,X}$	duality between X^* and X
$(\cdot\,, \cdot)$	the scalar product on $L^2(\Omega)$, $L^2(\Omega, \mathbb{R}^3)$ or $L^2(\Omega, \mathbb{R}^{3\times 3})$
$(\cdot\,, \cdot)_X$	the scalar product on X
$\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$	the scalar product on $W_0^{3,2}(\Omega)$ defined in Note 1.13
$\vert\vert\vert \cdot \vert\vert\vert$	the norm on $W_0^{3,2}(\Omega)$ defined in Note 1.13
δ_{ij}	Kronecker's delta
$\mathrm{span}\{\ldots\}$	linear hull
$\mathbb{N}$	the set of positive integers
$\mathbb{R}$	the set of real numbers
$\mathbb{X} : \mathbb{Y}$	$\sum_{i,j=1}^d x_{ij} y_{ij}$, $\quad \mathbb{X} = \{x_{ij}\}_{i,j=1}^d$, $\mathbb{Y} = \{y_{ij}\}_{i,j=1}^d$
$\boldsymbol{a} \otimes \boldsymbol{b}$	$\{a_i b_j\}_{i,j=1}^d$, $\boldsymbol{a} = (a_1, \ldots, a_d)$, $\boldsymbol{b} = (b_1, \ldots, b_d)$
$\hookrightarrow$	continuous embedding
$\overset{d}{\hookrightarrow}$	dense and continuous embedding
$\hookrightarrow\hookrightarrow$	compact embedding
$\mathrm{supp}\ \phi$	the support of ϕ
$B(0, \delta)$	$\{x \in X;\ \Vert x \Vert < \delta\}$, where X is a space in question
$A + B$	$\{a + b;\ a \in A,\ b \in B\}$
r'	$r/(r-1)$ (or a different letter as the case may be)
$A_i \nearrow B$	monotone pointwise convergence of the sets $\{A_i\}$ a.e. to B
$\vert A \vert$	the (correspondingly-dimensional) Lebesgue measure of A
$\mathbb{I}$	the identity $\mathbb{R}^{3\times 3\times 3\times 3}$-tensor
$\{a_j\}$	an infinite sequence of real numbers $\{a_j\}_{j=1}^\infty$
$l^2(\mathbb{R})$	$\Big\{\{a_i\};\ \forall i \in \mathbb{N}\colon a_i \in \mathbb{R},\ \sum_{i=1}^\infty \vert a_i \vert^2 < \infty\Big\}$
$A^\perp$	the orthogonal complement of A
$\rightrightarrows$	uniform convergence

Note 1.1 On many occasions, we will make a tacit use of the following elementary equality holding for any $d \geq 2$:

$$\frac{1}{2}\left(\mathbb{X}+\mathbb{X}^T\right):\mathbb{Y}=\frac{1}{2}\left(\mathbb{X}+\mathbb{X}^T\right):\frac{1}{2}\left(\mathbb{Y}+\mathbb{Y}^T\right),\qquad \mathbb{X},\mathbb{Y}\in\mathbb{R}^{d\times d}. \tag{1.1}$$

1.2 Divergence-free functions

As we will most often treat functions with zero divergence, the first supplementary step consists of defining the appropriate function spaces. For more information on this topic consult e.g. [13].

Definition 1.2 The following spaces will be used extensively throughout this work:

$$\begin{aligned}\mathcal{V} &= \left\{\boldsymbol{\varphi};\ \boldsymbol{\varphi}\in\mathcal{C}_c^\infty(\Omega,\mathbb{R}^3),\ \operatorname{div}_x\boldsymbol{\varphi}=0\right\},\\ H &= \text{closure of } \mathcal{V} \text{ in } L^2(\Omega,\mathbb{R}^3),\\ V_p &= \text{closure of } \mathcal{V} \text{ in } W^{1,p}(\Omega,\mathbb{R}^3),\\ V^3 &= \text{closure of } \mathcal{V} \text{ in } W^{3,2}(\Omega,\mathbb{R}^3).\end{aligned}$$

H, V_p and V^3 are considered with the topology of the corresponding closure.

The next theorem will be applied on such a regular basis that its invocation will not even be explicitly mentioned in the text.

Theorem 1.3 (**Green's theorem**, [13], Theorem 1.1.2)
If $\Omega\in\mathcal{C}^2(\mathbb{R}^3)$, the following formula is true for all $\boldsymbol{u}\in W_0^{1,2}(\Omega,\mathbb{R}^{3\times3})$ and $v\in W^{1,2}(\Omega,\mathbb{R}^3)$, or $\boldsymbol{u}\in W_0^{1,2}(\Omega,\mathbb{R}^3)$ and $v\in W^{1,2}(\Omega)$, respectively:

$$(\boldsymbol{u},\nabla_x v)=-(\operatorname{div}_x\boldsymbol{u},v).$$

1.3 Vector-valued functions

In this auxiliary section, we are not going to introduce Bochner spaces from scratch but merely mention several results that will play a major role in what is to come. For a self-contained introduction into the theory of vector-valued functions, refer to e.g. [14] (a quick initiation) or [7] (a thorough treatment).

In what follows, consider $T>0$.

Definition 1.4 Let V be a Hilbert space and X be a Banach space satisfying $X \overset{d}{\hookrightarrow} V$. Let $u\in L^1_{\mathrm{loc}}(0,T;X)$. We define its *generalized time derivative*, denoted by u', $\partial_t u$ or $\frac{\mathrm{d}}{\mathrm{d}t}u$, as a distribution on $\mathcal{C}_c^\infty(0,T;X)$ given by the formula

$$\langle u',v\rangle=-\int_0^T(u(t),v'(t))_V\,dt,\qquad v\in\mathcal{C}_c^\infty(0,T;X).$$

Lemma 1.5 ([14], Lemma 2.2.3)
Let X satisfy the conditions from the above definition and u be a distribution on $\mathcal{C}_c^\infty(0,T;X)$ with $u'\in L^p(0,T;X)$, $1\leq p\leq\infty$. Then $u\in\mathcal{C}([0,T];X)$ (modulo a representative) and the following representation takes place:

$$u(t)=u(0)+\int_0^t u'(s)\,ds,\qquad t\in(0,T).$$

Lemma 1.6 *Assume X is a Banach space and $1 < p < \infty$. The space $(L^p(0,T;X))^*$ is isometrically isomorphic to $L^{p'}(0,T;X^*)$ and for every $\chi \in (L^p(0,T;X))^*$ the corresponding $u \in L^{p'}(0,T;X^*)$ satisfies*

$$\langle \chi, v \rangle_{\Lambda^*,\Lambda} = \int_0^T \langle u(t), v(t) \rangle_{X^*,X} \, dt \tag{1.2}$$

for every $v \in \Lambda = L^p(0,T;X)$.

Remark 1.7 In view of the previous lemma, the two isomorphic spaces will always be identified here. In this fashion, replacing χ with u in (1.2) would be legitimate. Hence we also observe $u = 0$ in Λ^* if and only if $u(t) = 0$ in X^* a.e. in $(0,T)$.

Theorem 1.8 (**Aubin-Simon lemma**, [11], Corollary 4)
Assume X, Y, Z are Banach spaces related via $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$. Let $1 \le p, q < \infty$ and $F \subset L^p(0,T;X)$ satisfy

$$\sup_{u \in F} \|u\|_{L^p(0,T;X)} < \infty,$$
$$\sup_{u \in F} \|\partial_t u\|_{L^q(0,T;Z)} < \infty.$$

Then F is relatively compact in $L^p(0,T;Y)$.

Theorem 1.9 ([5], Corollary 2.1)
Let V be a separable Hilbert space and $u_n : [0,T] \to V$, $n \in \mathbb{N}$, be a sequence of strongly measurable functions (i.e. a pointwise limit of simple functions) satisfying

(i) $\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \text{ess} \|u_n(t)\|_V < \infty$;

(ii) *There exists a dense subset $F \subset V$ such that the functions $(u_n(t), \phi)_V : [0,T] \to \mathbb{R}$, $n \in \mathbb{N}$, are equicontinuous for every $\phi \in F$.*

Then a function $u \in \mathcal{C}_w([0,T];V)$ exists and a subsequence of $\{u_n\}$ (without loss of generality the original one) fulfilling

$$u_n \to u \qquad \text{in } \mathcal{C}_w([0,T];V).$$

That is to say $(u_n(t), \phi)_H \rightrightarrows (u(t), \phi)_H$ on $[0,T]$ for every $\phi \in V$.

Theorem 1.10 (a proof completely analogous to [13], Lemma 3.1.2)
Let V be a Hilbert space and X, Y be Banach spaces mutually injected as $X \hookrightarrow Y \hookrightarrow V$. Suppose $1 < p \le q < \infty$ and $u \in L^p(0,T;X) \cap L^q(0,T;Y)$ with its generalized time derivative $u' \in L^{p'}(0,T;X^*) + L^{q'}(0,T;Y^*)$. Then the following properties are in effect:

(i) *$u \in \mathcal{C}([0,T];V)$ up to a representative;*

(ii) *For any $0 \le t_1 < t_2 \le T$ one has*

$$\int_{t_1}^{t_2} \langle u'(s), u(s) \rangle \, ds = \frac{1}{2} \|u(t_2)\|_V^2 - \frac{1}{2} \|u(t_1)\|_V^2 ,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $X^* + Y^*$ and $X \cap Y$.

Lemma 1.11 *Let V be a separable Hilbert space, X a Banach space such that $V \overset{d}{\hookrightarrow} X$ and $1 < p < q < \infty$. Then functions of the form $\sum_{i=1}^n \alpha_i(t)\beta_i$, $\alpha_i(t) \in L^q(0,T)$, $\beta_i \in V$, are dense in $L^p(0,T;V) \cap L^q(0,T;X)$.*

Proof. We will show that functions of the prescribed form are dense in $L^q(0,T;V)$, which is clearly dense in $L^p(0,T;V) \cap L^q(0,T;X)$ due to density of V in X.

Denote $(\cdot\,,\cdot)_V$ the scalar product on V, $\|\cdot\|_V$ its induced norm and let $\{\beta_i\}$ be an orthonormal basis in V. Take and fix $v \in L^q(0,T;V)$. Since $\sum_{i=1}^n (v(t),\beta_i)_V \beta_i \to v(t)$, $n \to \infty$, in V for a.e. $t \in (0,T)$, we construct the temporal part as $\alpha_i(t) = (v(t),\beta_i)_V$. Cauchy-Schwartz inequality implies $\alpha_i(t) \in L^q(0,T)$. Invoking Bessel's inequality, $\|\sum_{i=1}^n \alpha_i(t)\beta_i\|_V \le \|v(t)\|_V$ a.e. in $(0,T)$ and the proof of density is thus finished using Lebesgue's dominated convergence theorem. ■

1.4 Existence of a basis

In order to successfully proceed with the Galerkin approximation scheme later on, it is crucial to have a basis of a suitable function space. This section is devoted to proving existence of a set of functions that form an orthonormal basis in H and an orthogonal basis in V^3 for $\Omega \in \mathcal{C}^3(\mathbb{R}^3)$.

To begin with, we report a result with crucial significance throughout the work.

Theorem 1.12 ([10], Proposition 6.18, 6.19)
If $\Omega \in \mathcal{C}^3(\mathbb{R}^3)$, there are constants K_1, K_2 satisfying for any $f \in W_0^{3,2}(\Omega)$ the inequality

$$K_1 \|\nabla_x \Delta f\|_2 \le \|f\|_{3,2} \le K_2 \|\nabla_x \Delta f\|_2 .$$

Note 1.13 In light of this theorem, we are allowed to define an equivalent norm $|||\cdot|||$ on $W_0^{3,2}(\Omega)$ (and naturally also on V^3) induced by the scalar product

$$\langle\langle f, g\rangle\rangle = (\nabla_x \Delta f, \nabla_x \Delta g), \qquad f, g \in W_0^{3,2}(\Omega).$$

Considering all reference to $W_0^{3,2}(\Omega)$ in this work will take place with the use of $|||\cdot|||$, we will treat $W_0^{3,2}(\Omega)$ as though implicitly equipped with this norm.

Customarily enough, the said basis will be sought via Hilbert-Schmidt theorem. To pave the way for its invocation, we study the problem of finding $\boldsymbol{u} \in V^3$ that for a fixed $\boldsymbol{f} \in H$ and any $\boldsymbol{v} \in V^3$ satisfies

$$\langle\langle \boldsymbol{u}, \boldsymbol{v}\rangle\rangle = (\boldsymbol{f}, \boldsymbol{v}) . \tag{1.3}$$

Thanks to Note 1.13 and imbedding $H \hookrightarrow \left(V^3\right)^*$, we can handle the problem easily via Riesz representation theorem to obtain a unique such $\boldsymbol{u} \in V^3$ with

$$|||\boldsymbol{u}||| \le K \|\boldsymbol{f}\|_H .$$

The constant K does not depend on $\boldsymbol{f}$ or $\boldsymbol{u}$. Now define the solution operator

$$\Gamma : H \longrightarrow V^3 \hookrightarrow\hookrightarrow H,$$
$$\boldsymbol{f} \longmapsto \boldsymbol{u}$$

where $\boldsymbol{u}$ is the solution to (1.3) corresponding to $\boldsymbol{f}$. Thus we obtained a compact, linear operator, which is self-adjoint too, since for $\Gamma\boldsymbol{f} = \boldsymbol{u}$, $\Gamma\boldsymbol{g} = \boldsymbol{v}$, an easy computation yields

$$(\Gamma\boldsymbol{f}, \boldsymbol{g}) = (\boldsymbol{u}, \boldsymbol{g}) = \langle\langle \boldsymbol{u}, \boldsymbol{v}\rangle\rangle = (\boldsymbol{f}, \boldsymbol{v}) = (\boldsymbol{f}, \Gamma\boldsymbol{g}) \qquad \text{for any } \boldsymbol{f}, \boldsymbol{g} \in H.$$

It is obvious that $\mathcal{V} \subset \Gamma(H)$. Invoking Hilbert-Schmidt theorem thus guarantees existence of $\{\boldsymbol{w}_j\} \subset V^3$, a set of eigenfunctions of Γ and an orthonormal system in H that is complete due to density of $\mathcal{V}$ in H.

We also observe

$$\langle\!\langle \lambda_j \boldsymbol{w}_j, \boldsymbol{w}_i \rangle\!\rangle = \langle\!\langle \boldsymbol{\Gamma} \boldsymbol{w}_j, \boldsymbol{w}_i \rangle\!\rangle = (\boldsymbol{w}_j, \boldsymbol{w}_i) = \delta_{ij}, \qquad i, j \in \mathbb{N},$$

where λ_j are the corresponding eigenvalues. Note that although $\lambda_j \to 0_+$ modulo rearrangement, the sequence remains non-zero. Finally, let $\boldsymbol{\varphi} \in V^3$ be such that

$$\langle\!\langle \boldsymbol{w}_j, \boldsymbol{\varphi} \rangle\!\rangle = 0 \qquad \text{for all } j.$$

Then $\boldsymbol{\varphi} = 0$ because

$$0 = \lambda_j \langle\!\langle \boldsymbol{w}_j, \boldsymbol{\varphi} \rangle\!\rangle = \langle\!\langle \boldsymbol{\Gamma} \boldsymbol{w}_j, \boldsymbol{\varphi} \rangle\!\rangle = (\boldsymbol{w}_j, \boldsymbol{\varphi}) \qquad \text{for all } j.$$

Hence we see that $\{\boldsymbol{w}_j\}$ is also a complete orthogonal system in V^3 with

$$|||\boldsymbol{w}_j||| = \lambda_j^{-1/2} \longrightarrow \infty, \qquad j \longrightarrow \infty.$$

The last few words of this part will touch the L^2-orthogonal projections to this basis. For $\boldsymbol{u} \in H$ and $n \in \mathbb{N}$ introduce

$$\mathrm{P}_n \boldsymbol{u} = \sum_{i=1}^{n} (\boldsymbol{u}, \boldsymbol{w}_i) \boldsymbol{w}_i.$$

As $\{\boldsymbol{w}_j\}$ is an orthonormal basis in H, we immediately obtain the bound

$$\|\mathrm{P}_n\|_{\mathcal{L}(H)} \leq 1. \tag{1.4}$$

Now take $\boldsymbol{v} \in V^3$. Parseval's identity asserts

$$\sum_{i=1}^{\infty} \left\langle\!\!\left\langle \boldsymbol{v}, \lambda_i^{1/2} \boldsymbol{w}_i \right\rangle\!\!\right\rangle^2 = |||\boldsymbol{v}|||^2 .$$

Therefrom we observe

$$|||\, \mathrm{P}_n \boldsymbol{v}|||^2 = \sum_{i=1}^{n} (\boldsymbol{w}_k, \boldsymbol{v})^2 \, |||\boldsymbol{w}_k|||^2 = \sum_{i=1}^{n} \lambda_i^2 \, \langle\!\langle \boldsymbol{w}_k, \boldsymbol{v} \rangle\!\rangle^2 \, \lambda_i^{-1} \leq |||\boldsymbol{v}|||^2 .$$

Thus we have deduced another bound, this time

$$\|\mathrm{P}_n\|_{\mathcal{L}(V^3)} \leq 1. \tag{1.5}$$

2. Solution

Definition 2.1 Let $\Omega \in \mathcal{C}^3(\mathbb{R}^3)$, $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^3)$, $\varepsilon > 0$, $r \geq 5/2$, $T > 0$ and $\boldsymbol{u}_0 \in H$. Denote

$$\Upsilon = L^r\left(0,T;V_r\right) \cap L^2\left(0,T;V^3\right).$$

We will call $\boldsymbol{u}$ a weak solution to the problem

$$\left.\begin{aligned} \partial_t \boldsymbol{u} - \varepsilon\Delta^3\boldsymbol{u} - \operatorname{div}_x \mathbb{S}(\mathbb{D}\boldsymbol{u}) + \operatorname{div}_x(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla_x p = \boldsymbol{f} \\ \operatorname{div}_x \boldsymbol{u} = 0 \end{aligned}\right\} \quad \textit{in } (0,T)\times\Omega,$$
$$\left.\begin{aligned} \boldsymbol{u} = \Delta\boldsymbol{u} = \mathbf{0} \\ \frac{\partial \boldsymbol{u}}{\partial \mathbf{n}} = \mathbf{0} \end{aligned}\right\} \quad \textit{on } (0,T)\times\partial\Omega, \tag{2.1}$$
$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \textit{on } \{t=0\}\times\Omega,$$

if $\boldsymbol{u} \in \Upsilon$, $\partial_t\boldsymbol{u} \in \Upsilon^*$ and for any $\boldsymbol{v} \in \Upsilon$ the following integral identity holds true:

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{v}\rangle_{\Upsilon^*,\Upsilon} + \int_0^T\!\!\int_\Omega \varepsilon\, \nabla_x\,\Delta\boldsymbol{u} : \nabla_x\,\Delta\boldsymbol{v} + \mathbb{S}(\mathbb{D}\boldsymbol{u}) : \nabla_x\boldsymbol{v} - (\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla_x\boldsymbol{v}\; dx\,dt = \int_0^T\!\!\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\; dx\,dt. \tag{2.2}$$

We also want $\boldsymbol{u}$ to attain the initial condition in the form

$$\lim_{t\to 0_+} \|\boldsymbol{u}_0 - \boldsymbol{u}(t,\cdot)\|_2 = 0.$$

Note 2.2 Although the pressure term p seemingly disappeared in (2.2), it can be reconstructed from the weak solution. Generally, we will not spent more time discussing pressure than absolutely necessary in this work. For more information concerning this topic, we refer the reader to [13], p. 180.

Remark 2.3 Thanks to Theorem 1.10, our definition of a solution guarantees it has a representative satisfying $\boldsymbol{u} \in \mathcal{C}([0,T];H)$. Firstly, this fact renders the requirement for attainment of the initial condition meaningful. We could even demand only $\boldsymbol{u}_0(x) = \boldsymbol{u}(0,x)$ a.e. in Ω. Secondly, the definition has therefore eschewed a redundant requirement $\boldsymbol{u} \in L^\infty(0,T;H)$. In the existence theorem we would, however, be able to prove boundedness of $\|\boldsymbol{u}(t)\|_2$ in time easily even without knowledge of continuity.

Note 2.4 With so many data in the Definition 2.1, shedding some light on a measure of their changeability throughout the work is in place. The entities Ω and $\boldsymbol{f}$ remain static permanently and no further discussion concerning these entities will be ever done. The number ε is also given, yet we will often toy with the idea of what would happen should it tend to zero. The number r is fixed as well, although the entire interval whence it is picked, i.e. $[5/2,\infty)$, must always be taken into account. The initial condition and terminal time, on the other hand, will change often in the later chapters so as to suit our immediate desires.

Theorem 2.5 *There exists a unique weak solution to (2.1). Furthermore, the solution enjoys a higher local regularity*

$$\boldsymbol{u} \in L^\infty\left(\delta,T;V_r\right) \cap L^\infty\left(\delta,T;V^3\right),$$
$$\partial_t\boldsymbol{u} \in L^2\left(\delta,T;H\right)$$

for any $\delta > 0$. We may set $\delta = 0$ provided $\boldsymbol{u}_0 \in V^3$.

Remark 2.6 In the limit case $\varepsilon = 0$, this result holds as well (see [6], [9]), including continuity of the solution in H. The definition of Υ would then consist only of $L^r(0, T; V_r)$ and the only boundary condition would be $\boldsymbol{u}|_\Omega = 0$. In addition, we would have to part with $\boldsymbol{u}$ belonging into $L^\infty(\delta, T; V^3)$.

Remark 2.7 The impending proof shall be lengthy and arduous, at times almost unnecessarily. The motives for going into such details are to use the improved regularity brought about by $\varepsilon\Delta^3\boldsymbol{u}$ as scarcely as possible. Even in the statement alone, wanting to rely strongly on the ε-term, we could have taken $r \geq 12/5$. In such a case, however, the proof of uniqueness would rest completely on the ε-term, which would obviously have a negative effect if we later wanted $\varepsilon \to 0_+$.

In a similar manner, the improved regularity is presented seemingly superfluously, having $V^3 \hookrightarrow V_r$. We will be again very careful not to make use of ε-induced regularity gratuitously.

2.1 Galerkin approximation

Here and in what follows, let the notation from section 1.4 be in force. Following the standard procedure, the sought-after solution will be constructed as a limit of a sequence of functions that solve a certain approximative problem. In this case, we first try to find a function $\boldsymbol{u}_n(t,x)$ built with the help of $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ as

$$\boldsymbol{u}_n(t,x) = \sum_{k=1}^{n} d_k^n(t)\boldsymbol{w}_k(x).$$

What to require of $d_k^n(t)$? Imagine we want $\boldsymbol{u}_n$ to solve

$$\partial_t\boldsymbol{u}_n(t) - \varepsilon\Delta^3\boldsymbol{u}_n(t) - \mathrm{P}_n\mathrm{div}_x\,\mathbb{S}\left(\mathbb{D}\boldsymbol{u}_n(t)\right) + \mathrm{P}_n\mathrm{div}_x\left(\boldsymbol{u}_n(t)\otimes\boldsymbol{u}_n(t)\right) = \mathrm{P}_n\boldsymbol{f} \qquad (2.3)$$

as an equation in Υ^*, more specifically

$$(\partial_t\boldsymbol{u}_n(t), \boldsymbol{v}) + \varepsilon\left\langle\!\left\langle \boldsymbol{u}_n(t), \boldsymbol{v}\right\rangle\!\right\rangle + \left(\mathbb{S}\left(\mathbb{D}\boldsymbol{u}_n(t)\right), \nabla_x\,\mathrm{P}_n\boldsymbol{v}\right) - \left(\boldsymbol{u}_n(t)\otimes\boldsymbol{u}_n(t), \nabla_x\,\mathrm{P}_n\boldsymbol{v}\right)$$
$$= (\boldsymbol{f}, \mathrm{P}_n\boldsymbol{v}) \quad (2.4)$$

holds for any $\boldsymbol{v} \in V^3$ a.e. in $(0,T)$. Applying (2.4) to $\boldsymbol{w}_k$, we deduce immediately the equations for $d_k^n(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,d_k^n + \varepsilon\lambda_k^{-1}\,d_k^n + \left(\mathbb{S}(\mathbb{D}\boldsymbol{u}_n) - \boldsymbol{u}_n\otimes\boldsymbol{u}_n, \nabla_x\boldsymbol{w}_k\right) = f_k, \qquad k = 1, \dots, n, \qquad (2.5)$$

where $f_k = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}_k\,dx$. We complete the setting with the initial condition

$$d_k^n(0) = \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{w}_k\,dx, \qquad k = 1, \dots, n. \qquad (2.6)$$

Note that $\left(\mathbb{S}(\mathbb{D}\boldsymbol{u}_n) - \boldsymbol{u}_n\otimes\boldsymbol{u}_n, \nabla_x\boldsymbol{w}_k\right)$ is a continuous function in $d_1^n, \dots, d_n^n$. Hence we are allowed to employ the classical theory of ordinary differential equations to guarantee existence of a continuously differentiable $(d_1^n, \dots, d_n^n)$ defined on some short time interval $(0, T_n^*)$ and satisfying (2.5) and (2.6).

2.2 Apriori estimates of $\boldsymbol{u}_n$

To show that $T_n^* = T$, we have to verify $d_1^n, \dots, d_n^n$ remain bounded. From the definition of $\boldsymbol{u}_n$, the task is tantamount to checking boundedness of $\|\boldsymbol{u}_n(t)\|_2$ on $(0,T)$. Apart from that, the desired passing to the limit later necessitates bounds on $\boldsymbol{u}_n$ in various spaces, independently of n. This section adresses deduction of all these uniform estimates. Where no confusion threatens, the time argument will be omitted for the sake of a more transparent notation.

First, let us rewrite (2.5) equivalently as:

$$(\partial_t \boldsymbol{u}_n, \boldsymbol{w}_k) + \varepsilon \langle\!\langle \boldsymbol{u}_n, \boldsymbol{w}_k \rangle\!\rangle + (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n) - \boldsymbol{u}_n \otimes \boldsymbol{u}_n, \nabla_x \boldsymbol{w}_k) = (\boldsymbol{f}, \boldsymbol{w}_k), \quad k = 1, \dots, n.$$

Now multiply corresponding equations by d_k^n and sum over k to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_n\|_2^2 + \varepsilon |\!|\!| \boldsymbol{u}_n |\!|\!|^2 + (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \mathbb{D}\boldsymbol{u}_n) - (\boldsymbol{u}_n \otimes \boldsymbol{u}_n, \nabla_x \boldsymbol{u}_n) = (\boldsymbol{f}, \boldsymbol{u}_n). \tag{2.7}$$

This is the first time we actually used (1.1). Before approaching to estimates, terms in (2.7) must be slightly refined:

- The definition of $\mathbb{S}$ (page 1) and Korn's inequality (6.1) yield

$$(\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \mathbb{D}\boldsymbol{u}_n) \geq c_1 \left(\|\mathbb{D}\boldsymbol{u}_n\|_2^2 + \|\mathbb{D}\boldsymbol{u}_n\|_r^r\right) \geq c_2 \left(\|\nabla_x \boldsymbol{u}_n\|_2^2 + \|\nabla_x \boldsymbol{u}_n\|_r^r\right).$$

- Next, remember $\operatorname{div}_x \boldsymbol{u}_n = 0$ and $\boldsymbol{u}_n|_{\partial\Omega} = \boldsymbol{0}$. A straightforward computation yields

$$(\boldsymbol{v} \otimes \boldsymbol{u}) : \nabla_x \boldsymbol{v} = \frac{1}{2}\boldsymbol{u} \cdot \nabla_x |\boldsymbol{v}|^2. \tag{2.8}$$

  By virtue of these three properties and Green's theorem 1.3, we have disclosed $(\boldsymbol{u}_n \otimes \boldsymbol{u}_n, \nabla_x \boldsymbol{u}_n) = 0$.

- The term $(\boldsymbol{f}, \boldsymbol{u}_n)$ is easily estimated with the help of Young's and Poincaré's inequality.

$$(\boldsymbol{f}, \boldsymbol{u}_n) \leq \frac{c_2}{2}\|\nabla_x \boldsymbol{u}_n\|_2^2 + c_3 \|\boldsymbol{f}\|_2^2.$$

Inserting these observations into (2.7) indicates a promising outcome

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_n\|_2^2 + \varepsilon |\!|\!| \boldsymbol{u}_n |\!|\!|^2 + c_2 \|\nabla_x \boldsymbol{u}_n\|_r^r \leq c_3 \|\boldsymbol{f}\|_2^2.$$

Choose $0 < t \leq T$ and integrate this inequality over $(0,t)$ to finally obtain

$$\begin{aligned} \sup_{t\in[0,T]} \|\boldsymbol{u}_n(t)\|_2^2 + \varepsilon \|\boldsymbol{u}_n\|_{L^2(0,T;V^3)}^2 + c_2 \|\nabla_x \boldsymbol{u}_n\|_{L^r(0,T;V_r)}^r & \\ \leq c_4 \left(\|\boldsymbol{f}\|_2^2 + \|\boldsymbol{u}_n(0)\|_2^2\right) & \\ \leq c_4 \left(\|\boldsymbol{f}\|_2^2 + \|\boldsymbol{u}(0)\|_2^2\right). & \end{aligned} \tag{2.9}$$

The last inequality is due to $\boldsymbol{u}_n(0) = \mathrm{P}_n \boldsymbol{u}_0$ and (1.4).

Note 2.8 Remark that with the bound on the norm of $\varepsilon^{1/2}\boldsymbol{u}_n$ in $L^2\left(0,T;V^3\right)$ there is no chance controlling the $L^2\left(0,T;V^3\right)$-norm of mere $\boldsymbol{u}$, should we ever want ε to approach zero. In this fashion, if we denote $\boldsymbol{u}_\varepsilon$ the solution (as yet undiscovered) corresponding to a given value of ε, we could somewhat oddly write

$$\boldsymbol{u}_\varepsilon \in \frac{1}{\varepsilon}L^2\left(0,T;V^3\right), \tag{2.10}$$

whereby we mean there exists $c > 0$ such that

$$\sup_{\varepsilon>0} \varepsilon \left\| \boldsymbol{u}_\varepsilon \right\|_{L^2(0,T;V^3)}^2 \le c.$$

2.3 Apriori estimates of $\partial_t \boldsymbol{u}_n$

Time derivative in our setting can be bounded very easily. At this moment, we will not estimate its norm in Υ^*. A bound in Ψ^*, where

$$\Psi = L^r\left(0, T; V^3\right),$$

will be sufficient for passing to the limit $n \to \infty$.

Note the fact that with (2.5) at our disposal, we are allowed to pair the equation with $\mathrm{P}_n \boldsymbol{\varphi}$ of an arbitrary $\boldsymbol{\varphi} \in V^3$. Therefore, we have for any $\boldsymbol{v} \in \Psi$

$$\begin{aligned} &\int_0^T (\partial_t \boldsymbol{u}_n, \mathrm{P}_n \boldsymbol{v})\, dt \\ &= \int_0^T \Big((\boldsymbol{f}, \mathrm{P}_n \boldsymbol{v}) - \varepsilon \left\langle\!\left\langle \boldsymbol{u}_n, \mathrm{P}_n \boldsymbol{v} \right\rangle\!\right\rangle - (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \nabla_x \mathrm{P}_n \boldsymbol{v}) - (\boldsymbol{u}_n \otimes \boldsymbol{u}_n, \nabla_x \mathrm{P}_n \boldsymbol{v}) \Big)\, dt. \end{aligned} \tag{2.11}$$

Considering orthogonality of $\{\boldsymbol{w}_j\}$ in H and V^3, we have both

$$\int_0^T (\partial_t \boldsymbol{u}_n, \mathrm{P}_n \boldsymbol{v})\, dt = \int_0^T (\partial_t \boldsymbol{u}_n, \boldsymbol{v})\, dt \qquad \text{and} \qquad \int_0^T \left\langle\!\left\langle \boldsymbol{u}_n, \mathrm{P}_n \boldsymbol{v} \right\rangle\!\right\rangle dt = \int_0^T \left\langle\!\left\langle \boldsymbol{u}_n, \boldsymbol{v} \right\rangle\!\right\rangle dt.$$

We are now prepared to plunge ourselves into estimates of the terms constituting (2.11). In the following, we will make good, yet silent, use of classical Sobolev embeddings.

- By Hölder's inequality and (1.4)

$$\left| \int_0^T (\boldsymbol{f}, \mathrm{P}_n \boldsymbol{v})\, dt \right| \le c_5 \left\| \boldsymbol{f} \right\|_2 \left\| \boldsymbol{v} \right\|_\Psi.$$

- By means of Hölder's inequality itself

$$\left| \int_0^T \varepsilon \left\langle\!\left\langle \boldsymbol{u}_n, \boldsymbol{v} \right\rangle\!\right\rangle dt \right| \le \varepsilon \left\| \boldsymbol{u}_n \right\|_{L^2(0,T;V^3)} \left\| \boldsymbol{v} \right\|_\Psi.$$

  Recall here that $\varepsilon^{1/2} \left\| \boldsymbol{u}_n \right\|_{L^2(0,T;V^3)}$ is bounded, so there is no harm relying on the V^3-regularity at all.

- Dealing with $\int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \nabla_x \mathrm{P}_n \boldsymbol{v})\, dt$ is similarly simple due to V^3-continuity of P_n given in (1.5):

$$\begin{aligned} &\left| \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \nabla_x \mathrm{P}_n \boldsymbol{v})\, dt \right| \\ &\qquad \le \int_0^T \Big(\nu_0 \left\| \mathbb{D}\boldsymbol{u}_n \right\|_2 \left\| \nabla_x \mathrm{P}_n \boldsymbol{v} \right\|_2 + \nu_1 \left\| \mathbb{D}\boldsymbol{u}_n \right\|_r^{r-1} \left\| \nabla_x \mathrm{P}_n \boldsymbol{v} \right\|_r \Big)\, dt \\ &\qquad \le c_6 \left\| \boldsymbol{u}_n \right\|_{L^r(0,T;V_r)} \left\| \mathrm{P}_n \boldsymbol{v} \right\|_{L^r(0,T;V_r)} \\ &\qquad \le c_7 \left\| \boldsymbol{u}_n \right\|_{L^r(0,T;V_r)} \left\| \boldsymbol{v} \right\|_\Psi. \end{aligned} \tag{2.12}$$

- Even the convective term is estimated painlessly. We will investigate the worst case, i.e. $r < 3$. Note that $6/(5r-6) \leq 1$:

$$\begin{aligned}\left|\int_0^T (\boldsymbol{u}_n \otimes \boldsymbol{u}_n, \nabla_x \mathrm{P}_n \boldsymbol{v})\ dt\right| &\leq \int_0^T \|\boldsymbol{u}_n\|_{\frac{2r}{r-1}}^2 \|\nabla_x \mathrm{P}_n \boldsymbol{v}\|_r\ dt \\ &\leq c_8 \int_0^T \|\boldsymbol{u}_n\|_2^{\frac{10r-18}{5r-6}} \|\boldsymbol{u}_n\|_{1,r}^{\frac{6}{5r-6}} \|\nabla_x \mathrm{P}_n \boldsymbol{v}\|_r\ dt \\ &\leq c_9 \int_0^T \Big(1 + \|\boldsymbol{u}_n\|_{1,r}^{r-1}\Big) \|\nabla_x \mathrm{P}_n \boldsymbol{v}\|_r\ dt \qquad (2.13) \\ &\leq c_{10} \|\boldsymbol{v}\|_\Psi .\end{aligned}$$

Everything being taken into account and utilizing (2.9), we have obtained:

$$\sup_{n\in\mathbb{N}} \|\partial_t \boldsymbol{u}_n\|_{\Psi^*} < \infty. \qquad (2.14)$$

Before moving on to new estimates, let us write down one more observation stemming from the calculations above:

$$\sup_{n\in\mathbb{N}} \left(\|\mathrm{P}_n \mathrm{div}_x\, \mathbb{S}(\mathbb{D}\boldsymbol{u}_n)\|_{\Psi^*} + \|\mathrm{P}_n \mathrm{div}_x (\boldsymbol{u}_n \otimes \boldsymbol{u}_n)\|_{\Psi^*} \right) < \infty. \qquad (2.15)$$

2.4 Apriori estimates for improved regularity

A great advantage of the equation (2.3) is that, unlike the original problem (2.2), it can be paired with (or "tested" by) the time derivative. Given that $\partial_t \boldsymbol{u}_n(t) \in V^3$ for all $t \in [0, T]$ by definition, it is evidently an admissible function (in the sense of testability). On account of this fact, we will include into the approximation scheme also uniform bounds in

$$L^\infty(\delta, T; V_r) \cap \frac{1}{\varepsilon} L^\infty\left(\delta, T; V^3\right)$$

for $\boldsymbol{u}_n$ (see the eccentric notation introduced in (2.10)) and in

$$L^2(\delta, T; H)$$

for $\partial_t \boldsymbol{u}_n(t)$, where $\delta > 0$ is arbitrarily small.

Before the estimates commence, we first introduce $\mathscr{S}$, the potential of $\mathbb{S}$:

$$\begin{aligned}\mathscr{S} : \mathbb{R}^{3\times 3} &\longrightarrow \mathbb{R} \\ \mathbb{X} &\longmapsto \frac{\nu_0}{2} |\mathbb{X}|^2 + \frac{\nu_1}{r} |\mathbb{X}|^r .\end{aligned}$$

Clearly $\nabla \mathscr{S}(\mathbb{X}) = \mathbb{S}(\mathbb{X})$. For the sake of avoiding visual confusion, we will replace $\nabla \mathscr{S}$ with $\partial_{\mathbb{X}} \mathscr{S}$, so that we have $\partial_{\mathbb{X}} \mathscr{S}(\mathbb{D}\boldsymbol{u}) = \mathbb{S}(\mathbb{D}\boldsymbol{u})$.

Having been got acquainted with $\mathscr{S}$, we are prepared to begin with an approach whose gist originates from [9]. As it has already been hinted, we apply (2.11) with $\boldsymbol{v} = \partial_t \boldsymbol{u}_n$. Remember $\mathrm{P}_n \partial_t \boldsymbol{u}_n = \partial_t \boldsymbol{u}_n$.

$$\|\partial_t \boldsymbol{u}_n\|_2^2 + \varepsilon \langle\!\langle \boldsymbol{u}_n, \partial_t \boldsymbol{u}_n \rangle\!\rangle + (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \mathbb{D}\partial_t \boldsymbol{u}_n) - (\boldsymbol{u}_n \otimes \boldsymbol{u}_n, \nabla_x \partial_t \boldsymbol{u}_n) = (\boldsymbol{f}, \partial_t \boldsymbol{u}_n). \qquad (2.16)$$

Defining an auxiliary quantity

$$Y_n(t) = 1 + \frac{\varepsilon}{2} |||\boldsymbol{u}_n(t)|||^2 + \int_\Omega \mathscr{S}(\mathbb{D}\boldsymbol{u}_n(t))\, dx, \qquad t \in (0, T), \qquad (2.17)$$

we observe $Y_n(t) \geq 1$ and

$$Y_n' = \varepsilon \langle\!\langle \boldsymbol{u}_n, \partial_t \boldsymbol{u}_n \rangle\!\rangle + (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \mathbb{D}\partial_t \boldsymbol{u}_n) \,.$$

Since Korn's and Hölder's inequality imply

$$c_{11} Y_n(t) \leq 1 + \varepsilon |\!|\!| \boldsymbol{u}_n(t) |\!|\!|^2 + \|\nabla_x \boldsymbol{u}_n(t)\|_r^r \leq c_{12} Y_n(t), \tag{2.18}$$

boundedness of Y_n ensures boundedness of $\varepsilon^{1/2} |\!|\!| \boldsymbol{u}_n |\!|\!|$ and $\|\nabla_x \boldsymbol{u}_n\|_r$ in turn. We will pursue the former, i.e. bounding Y_n. Note here two crucial observations:

(i) Thanks to (2.9) we have

$$\sup_{n\in\mathbb{N}} \|Y_n\|_{L^1(0,T)} < \infty. \tag{2.19}$$

(ii) The same apriori estimates warrant for any $\delta > 0$ and $n \in \mathbb{N}$ existence of some $t_n \in [0, \delta]$ such that

$$\sup_{n\in\mathbb{N}} Y_n(t_n) = Y_0 < \infty. \tag{2.20}$$

Y_0 may, obviously, worsen with diminishing δ. Note also that in the event of $u_0 \in V^3$, all t_n may be set zero due to (1.5).

The next step will be reached by two simple estimates:

- Due to Hölder's and Young's inequality:

$$|(\boldsymbol{f}, \partial_t \boldsymbol{u}_n)| \leq \frac{1}{4} \|\partial_t \boldsymbol{u}_n\|_2^2 + \|\boldsymbol{f}\|_2^2 \,. \tag{2.21}$$

- Observing

$$\operatorname{div}_x(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla_x)\boldsymbol{v} + (\operatorname{div}_x \boldsymbol{u})\boldsymbol{v},$$

with $\operatorname{div}_x \boldsymbol{u}_n = 0$ we have

$$\begin{aligned}(\boldsymbol{u}_n \otimes \boldsymbol{u}_n, \nabla_x \partial_t \boldsymbol{u}_n) &= -\left(\operatorname{div}_x(\boldsymbol{u}_n \otimes \boldsymbol{u}_n), \partial_t \boldsymbol{u}_n\right) \\ &= -\left((\boldsymbol{u}_n \cdot \nabla_x)\boldsymbol{u}_n, \partial_t \boldsymbol{u}_n\right) - \left((\operatorname{div}_x \boldsymbol{u}_n)\boldsymbol{u}_n, \partial_t \boldsymbol{u}_n\right) \\ &= -\left(\partial_t \boldsymbol{u}_n \otimes \boldsymbol{u}_n, \nabla_x \boldsymbol{u}_n\right).\end{aligned}$$

Like in the previous point, we estimate

$$|(\partial_t \boldsymbol{u}_n \otimes \boldsymbol{u}_n, \nabla_x \boldsymbol{u}_n)| \leq \frac{1}{4} \|\partial_t \boldsymbol{u}_n\|_2^2 + \|\boldsymbol{u}_n\|_{\frac{2r}{r-2}}^2 \|\nabla_x \boldsymbol{u}_n\|_r^2 \,. \tag{2.22}$$

Inserting (2.17), (2.21) and (2.22) into (2.16) produces

$$\frac{1}{2} \|\partial_t \boldsymbol{u}_n\|_2^2 + Y_n' \leq \|\boldsymbol{u}_n\|_{\frac{2r}{r-2}}^2 \|\nabla_x \boldsymbol{u}_n\|_r^2 + \|\boldsymbol{f}\|_2^2 \,. \tag{2.23}$$

The major obstacle here is, of course, the term $\|\boldsymbol{u}_n\|_{2r/(r-2)}^2$. Supposing we wanted to be unadventurous, we could resort to its estimating by means of $|\!|\!| \boldsymbol{u}_n |\!|\!|$ but, again, the resultant bound would undesirably worsen with fading ε.

The course of action will fork into two branches, depending on the value of r. Both follow the very same lines, although due to certain subtleties, they have to stand separately.

$r \geq 3$: Sobolev embedding $V_r \hookrightarrow L^{\frac{2r}{r-2}}(\Omega, \mathbb{R}^3)$ and Poincaré's inequality yield

$$\|\boldsymbol{u}_n\|^2_{\frac{2r}{r-2}} \|\nabla_x \boldsymbol{u}_n\|^2_r \leq c_{13} \|\nabla_x \boldsymbol{u}_n\|^4_r \,, \tag{2.24}$$

and inserting this result into (2.23) and invoking (2.18) gives

$$Y_n' \leq c_{14} Y_n^{\frac{4}{r}} + \|\boldsymbol{f}\|_2^2 \,.$$

If $4/r \leq 1$ then $Y_n^{\frac{4}{r}} \leq Y_n$. Otherwise multiply the inequality by $Y^{1-\frac{4}{r}}$ and bear in mind this quantity is bounded from above by 1. In either case we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_n^\beta \leq c_{15} Y_n + c_{16} \|\boldsymbol{f}\|_2^2 \,, \tag{2.25}$$

where $\beta = 1$ in the former case and $\beta = 2 - 4/r$ in the latter. Come what may, $\beta \in (0, 1]$. Next integrate (2.25) from t_n to t, while recollecting (2.19) and (2.20):

$$\sup_{t \in (0,T)} Y_n^\beta(t) \leq Y_0 + \int_0^T \Big(c_{17} Y_n + c_{18} \|\boldsymbol{f}\|_2^2 \Big) \, dt \leq c_{19}, \tag{2.26}$$

where c_{19} does not depend on n, wherefore the supremum may be taken also over all $n \in \mathbb{N}$. That amounts to bounding $\|\boldsymbol{u}_n\|_{V_r}$ and $\varepsilon |\!|\!| \boldsymbol{u}_n |\!|\!|^2$. The remaining bound on $\partial_t \boldsymbol{u}_n$ then follows from integration of (2.23) between δ and T, while applying (2.18), (2.24) and (2.26).

$r < 3$: This option suggests to invoke

$$V_r \hookrightarrow L^{\frac{3r}{3-r}}(\Omega).$$

The fact that $r \geq 5/2$ also guarantees

$$\frac{3r}{3-r} \geq \frac{2r}{r-2}.$$

The weaker assumption on r from Remark 2.7 actually stemmed from this condition. Using usual interpolation and (2.9), we obtain

$$\|\boldsymbol{u}_n\|^2_{\frac{2r}{r-2}} \leq c_{20} \|\boldsymbol{u}_n\|_2^{1-\alpha} \|\nabla_x \boldsymbol{u}_n\|_r^\alpha \leq c_{21} \|\nabla_x \boldsymbol{u}_n\|_r^\alpha \,,$$

where $\alpha = \frac{6}{5r-6} \in (0, 1)$. Like before, hence we infer

$$Y_n' \leq c_{22} \|\nabla_x \boldsymbol{u}_n\|_r^{\frac{10r}{5r-6}} + \|\boldsymbol{f}\|_2^2 \leq c_{23} Y_n^{\frac{10}{5r-6}} + \|\boldsymbol{f}\|_2^2 \,.$$

To conclude, multiply the inequality by $Y^{\frac{5r-16}{5r-6}}$, which is again less or equal to 1 thanks to $-1 < \frac{5r-16}{5r-6} < 0$.

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_n^{1+\frac{5r-16}{5r-6}} \leq c_{24} Y_n + c_{25} \|\boldsymbol{f}\|_2^2 \,.$$

The rest would be reasoned in the same manner as in the preceding variant.

2.5 Passing to limit

Now it is time to finally utilize all the uniform estimates we have deduced so far. Firstly, reflexivity of Υ and (2.9) guarantee existence of $\boldsymbol{u} \in \Upsilon$ such that a subsequence of $\{\boldsymbol{u}_n\}$, without loss of generality the original sequence, fulfills

$$\boldsymbol{u}_n \longrightarrow \boldsymbol{u} \qquad \text{weakly in } \Upsilon. \tag{2.27}$$

Secondly, invoking Banach-Alaoglu theorem and the bounds (2.14) and (2.15), we are presented with $\boldsymbol{v}$, F, $G \in \Psi^*$ satisfying (modulo subsequence again)

$$\left.\begin{aligned} \partial_t \boldsymbol{u}_n &\longrightarrow \boldsymbol{v} \\ \mathrm{P}_n \mathrm{div}_x\, \mathbb{S}(\mathbb{D}\boldsymbol{u}_n) &\longrightarrow F \\ \mathrm{P}_n \mathrm{div}_x (\boldsymbol{u}_n \otimes \boldsymbol{u}_n) &\longrightarrow G \end{aligned}\right. \qquad \text{weakly-}^* \text{ in } \Psi^*.$$

Thirdly, with the help of the same theorem, the promised higher regularity holds true as well:

$$\begin{aligned} &\boldsymbol{u} \in L^\infty(\delta, T; V_r) \cap L^\infty\left(\delta, T; V^3\right), \\ &\boldsymbol{v} \in L^2(\delta, T; H) \end{aligned}$$

for any $\delta > 0$, with $\delta = 0$ provided $\boldsymbol{u}_0 \in V^3$.

It is not immediately obvious that $\partial_t \boldsymbol{u} = \boldsymbol{v}$. Nevertheless, showing it is rather simple. Let $\boldsymbol{\varphi} \in \mathcal{C}_c^\infty(0, T; \mathcal{V})$. Then

$$\langle \partial_t \boldsymbol{u}_n, \boldsymbol{\varphi} \rangle_{\Psi^*, \Psi} \longrightarrow \langle \boldsymbol{v}, \boldsymbol{\varphi} \rangle_{\Psi^*, \Psi}$$

by weak-* convergence and at the same time by definition and (2.27)

$$\langle \partial_t \boldsymbol{u}_n, \boldsymbol{\varphi} \rangle_{\Psi^*, \Psi} = -\int_0^T (\boldsymbol{u}_n, \partial_t \boldsymbol{\varphi})\, dt \longrightarrow -\int_0^T (\boldsymbol{u}, \partial_t \boldsymbol{\varphi})\, dt = \langle \partial_t \boldsymbol{u}, \boldsymbol{\varphi} \rangle_{\Psi^*, \Psi}.$$

Hence $\partial_t \boldsymbol{u} = \boldsymbol{v}$ as elements of Ψ^*.

By reason of $\Upsilon \hookrightarrow L^2(0, T; V_r)$ and $\Psi^* \hookrightarrow L^{r'}(0, T; (V^3)^*)$, we may apply Aubin-Simon lemma 1.8 with the space triplet $V_r \hookrightarrow\hookrightarrow H \hookrightarrow (V^3)^*$ and suppose

$$\begin{aligned} \boldsymbol{u}_n &\longrightarrow \boldsymbol{u} \qquad \text{in } L^r(0, T; H), \\ \boldsymbol{u}_n &\longrightarrow \boldsymbol{u} \qquad \text{a.e. in } (0, T) \times \Omega. \end{aligned} \tag{2.28}$$

Convergence on the part of $\boldsymbol{f}$ is clear since $\int_0^T (\mathrm{P}_n \boldsymbol{f}, \boldsymbol{v})\, dt \to \int_0^T (\boldsymbol{f}, \boldsymbol{v})\, dt$ for any $\boldsymbol{v} \in \Upsilon \supset \Psi$ owing to $\mathrm{P}_n \boldsymbol{v} \to \boldsymbol{v}$ in $L^2(\Omega)$. Finally, $\int_0^T \langle\langle \boldsymbol{u}_n, \boldsymbol{v} \rangle\rangle\, dt \to \int_0^T \langle\langle \boldsymbol{u}, \boldsymbol{v} \rangle\rangle\, dt$ for any $\boldsymbol{v} \in \Upsilon$ by reason of $\int_0^T \langle\langle \cdot\,, \boldsymbol{v} \rangle\rangle\, dt \in \Upsilon^*$. On the whole, we have reached an equality holding for any $\boldsymbol{v} \in \Psi$:

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{v} \rangle_{\Psi^*, \Psi} + \int_0^T \varepsilon \langle\langle \boldsymbol{u}, \boldsymbol{v} \rangle\rangle\, dt - \langle F, \boldsymbol{v} \rangle_{\Psi^*, \Psi} + \langle G, \boldsymbol{v} \rangle_{\Psi^*, \Psi} = \int_0^T (\boldsymbol{f}, \boldsymbol{v})\, dt.$$

Next we will show $\partial_t \boldsymbol{u} \in \Upsilon^*$, for which it evidently suffices to establish the same for F and G. As far as G is concerned, recall (2.13) to readily deduce

$$\sup_{n \in \mathbb{N}} \|\mathrm{div}_x (\boldsymbol{u}_n \otimes \boldsymbol{u}_n)\|_{(L^r(0,T;V_r))^*} < \infty. \tag{2.29}$$

We may hence suppose $\mathrm{div}_x(\boldsymbol{u}_n \otimes \boldsymbol{u}_n) \to \widetilde{G}$ weakly-* in $(L^r(0, T; V_r))^*$. Since $\widetilde{G} = G$ on a dense set (see Lemma 1.11) of $\Upsilon \overset{d}{\hookrightarrow} L^r(0, T; V_r)$, we have $G \in (L^r(0, T; V_r))^*$. The argument with F would be quite analogous with recollection of (2.12). Accordingly, $\partial_t \boldsymbol{u} \in \Upsilon^*$ and $\boldsymbol{u} \in \mathcal{C}([0,T];H)$ by Theorem 1.10.

There is apparantly still much to prove before existence can be triumphantly announced. The main task is to show

$$\begin{aligned} \langle F, \boldsymbol{v}\rangle_{\Upsilon^*,\Upsilon} &= -\int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}), \nabla_x \boldsymbol{v})\ dt \\ \langle G, \boldsymbol{v}\rangle_{\Upsilon^*,\Upsilon} &= -\int_0^T (\boldsymbol{u}\otimes\boldsymbol{u}, \nabla_x \boldsymbol{v})\ dt \end{aligned} \qquad \boldsymbol{v} \in \Upsilon.$$

Before getting down to verifying this, however, we will first have a look at one more kind of convergence that might be required of $\boldsymbol{u}_n$. Recalling Theorem 1.9 with $V = H$, we know that its first assumption is satisfied. As for the second, let the countable dense set be linear combinations of $\boldsymbol{w}_j$ with rational coefficients. For the sake of more comprehensible notation, the spatial argument will be shamefully omitted. Let $0 \le t_1 < t_2 \le T$:

$$\begin{aligned} |(\boldsymbol{u}_n(t_2), \boldsymbol{w}_j) - (\boldsymbol{u}_n(t_1), \boldsymbol{w}_j)| &\le \int_{t_1}^{t_2} |(\partial_t \boldsymbol{u}_n(s), \boldsymbol{w}_j)|\, ds \le \lambda_j^{-\frac{1}{2}} \int_{t_1}^{t_2} \|\partial_t \boldsymbol{u}_n(s)\|_{(V^3)^*}\, ds \\ &\le c_{26}\, \|\partial_t \boldsymbol{u}_n\|_{\Psi^*}\, |t_2 - t_1|^{\frac{1}{r}} \le c_{27} |t_2 - t_1|^{\frac{1}{r}}. \end{aligned}$$

We exploited (2.14) in the last inequality. The estimate may possibly worsen with increasing j but it does not bother us. For a fixed j the real-valued functions $(\boldsymbol{u}_n(t), \boldsymbol{w}_j)$ are equicontinuous on $(0,T)$. Consequently, above and over what we have told about $\boldsymbol{u}$, we may also assume

$$(\boldsymbol{u}_n(t), \boldsymbol{v}) \rightrightarrows (\boldsymbol{u}(t), \boldsymbol{v}) \ \text{ on } [0,T] \tag{2.30}$$

for any $\boldsymbol{v} \in H$.

2.6 Attainment of the initial condition

As even this point will turn out quite helpful in a short time, we present it before convergence of nonlinearities too.

Since $\boldsymbol{u} \in \mathcal{C}([0,T];H)$, the task reduces to showing $\boldsymbol{u}(0) = \boldsymbol{u}_0$ a.e. in Ω. From (2.30) we know $(\boldsymbol{u}_n(0), \boldsymbol{u}(0) - \boldsymbol{u}_0) \to (\boldsymbol{u}(0), \boldsymbol{u}(0) - \boldsymbol{u}_0)$, $n \to \infty$. At the same time, $\boldsymbol{u}_n(0) = \mathrm{P}_n \boldsymbol{u}_0 \to \boldsymbol{u}_0$ in H, $n \to \infty$. Therefore $(\boldsymbol{u}_0, \boldsymbol{u}(0) - \boldsymbol{u}_0) = (\boldsymbol{u}(0), \boldsymbol{u}(0) - \boldsymbol{u}_0)$, which is actually tantamount to $\boldsymbol{u}(0) = \boldsymbol{u}_0$ a.e. in Ω.

2.7 Convergence of the convective term

In order to show $\langle G, \boldsymbol{v}\rangle_{\Upsilon^*,\Upsilon} = -\int_0^T (\boldsymbol{u}\otimes\boldsymbol{u}, \nabla_x \boldsymbol{v})\, dt$, $\boldsymbol{v} \in \Upsilon$, we will avail ourselves of verifying

$$\int_0^T (\boldsymbol{u}_n \otimes \boldsymbol{u}_n, \nabla_x \boldsymbol{v})\ dt \longrightarrow \int_0^T (\boldsymbol{u}\otimes\boldsymbol{u}, \nabla_x \boldsymbol{v})\ dt,\ n \longrightarrow \infty, \qquad \text{for any } \boldsymbol{v} \in \Upsilon. \tag{2.31}$$

According to the discussion below (2.29), this is sufficient. We will be so bold as to pair with $\boldsymbol{v} \in L^r(0,T;V_r)$. Given that $(\boldsymbol{u}_n \otimes \boldsymbol{u}_n) \to (\boldsymbol{u}\otimes\boldsymbol{u})$ a.e. in $(0,T)\times\Omega$, utilizing a weak version of dominated convergence theorem, i.e. Lemma 6.2, presents itself. We only need to prove $\sup_n \|\boldsymbol{u}_n\|_{L^{2r'}((0,T)\times\Omega)} < \infty$. Like once before in (2.13), assume without loss of generality $r < 3$, for that represents the scenario with worse regularity.

$$\begin{aligned}\int_0^T\!\!\int_\Omega |\boldsymbol{u}_n|^{\frac{2r}{r-1}}\,dx\,dt &\le c_{28}\int_0^T \|\boldsymbol{u}_n\|_2^{\frac{5r-9}{5r-6}\cdot\frac{2r}{r-1}}\,\|\boldsymbol{u}_n\|_{1,r}^{\frac{3}{5r-6}\cdot\frac{2r}{r-1}}\,dt\\ &\le c_{29}\int_0^T\Big(1+\|\boldsymbol{u}_n\|_{1,r}^r\Big)\,dt \le c_{30}\end{aligned}$$

due to apriori estimates and $\frac{6}{(5r-6)(r-1)} \le 1$. Hence the convergence is assured by Lemma 6.2.

2.8 Convergence of stress tensor: a cunning variant

Our usual scruples about summoning V^3-regularity notwithstanding, it would be a shame to not present a streamlined proof of $\langle F, \boldsymbol{v}\rangle_{\Upsilon^*,\Upsilon} = -\int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}), \nabla_x \boldsymbol{v})\,dt$, $\boldsymbol{v} \in \Upsilon$, which is possible only due to the ε-term. Like in the previous section, we will concentrate foremost on showing

$$\int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \mathbb{D}\boldsymbol{v})\,dt \longrightarrow \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}), \mathbb{D}\boldsymbol{v})\,dt \qquad \text{for any } \boldsymbol{v} \in \Upsilon. \tag{2.32}$$

Enrichment with P_n would be analogous to what follows (2.29), like above. A sine qua non here will be Lemma 6.2 again.

On account of $V^3 \hookrightarrow\hookrightarrow V_r$, Aubin-Simon lemma actually lets us assume

$$\begin{aligned}\boldsymbol{u}_n &\longrightarrow \boldsymbol{u} &&\text{in } L^2(0,T;V_r),\\ \nabla_x\boldsymbol{u}_n &\longrightarrow \nabla_x\boldsymbol{u} &&\text{a.e. in } (0,T)\times\Omega,\end{aligned}$$

and therefore

$$\mathbb{S}(\mathbb{D}\boldsymbol{u}_n) \longrightarrow \mathbb{S}(\mathbb{D}\boldsymbol{u}) \qquad \text{a.e. in } (0,T)\times\Omega.$$

Since for any $\varphi \in L^r((0,T)\times\Omega, \mathbb{R}^{3\times3})$ we have (see 2.13 for greater clarity)

$$\begin{aligned}\left|\int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \varphi)\,dt\right| &\le c_{31}\,\|\boldsymbol{u}_n\|_{L^r(0,T;V_r)}\,\|\varphi\|_{L^r((0,T)\times\Omega,\mathbb{R}^{3\times3})}\\ &\le c_{32}\,\|\varphi\|_{L^r((0,T)\times\Omega,\mathbb{R}^{3\times3})},\end{aligned} \tag{2.33}$$

we obtain from Lemma 6.2

$$\int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \varphi)\,dt \longrightarrow \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}), \varphi)\,dt \qquad \text{for any } \varphi \in L^r((0,T)\times\Omega, \mathbb{R}^{3\times3}). \tag{2.34}$$

Bringing into play common knowledge $L^r((0,T)\times\Omega,\mathbb{R}^{3\times3}) = L^r(0,T;L^r(\Omega,\mathbb{R}^{3\times3}))$, (2.32) holds true as a special case of (2.34).

2.9 Convergence of stress tensor: an ε-independent variant

When longing for independence from ε, we must have recourse to what is known as the Minty's trick. To begin with, recall (2.33), which allows us to write $\operatorname{div}_x F$ instead of F (so that now, in fact, $F \in L^{r'}(0,T;L^{r'}(\Omega,\mathbb{R}^{3\times3}))$). As $\boldsymbol{u}$ has become an admissible test function, let us test with it the equation we have nurtured so far. We cannot do without Theorem 1.10 anymore.

$$\frac{1}{2}\|\boldsymbol{u}(T)\|_2^2 - \frac{1}{2}\|\boldsymbol{u}(0)\|_2^2 + \varepsilon\int_0^T |||\boldsymbol{u}|||^2\,dt + \langle F, \nabla_x \boldsymbol{u}\rangle_{\Upsilon^*,\Upsilon} - \int_0^T (\boldsymbol{u}\otimes\boldsymbol{u}, \nabla_x\boldsymbol{u})\,dt = \int_0^T \boldsymbol{f}\cdot\boldsymbol{u}\,dt. \quad (2.35)$$

The last term on the left is zero by (2.8) and $\langle F, \nabla_x \boldsymbol{u}\rangle_{\Upsilon^*,\Upsilon} = \langle F, \mathbb{D}\boldsymbol{u}\rangle_{\Upsilon^*,\Upsilon}$ from symmetry of $\mathbb{S}(\mathbb{D}\boldsymbol{u}_n)$. At the same time, integration of (2.7) yields

$$\frac{1}{2}\|\boldsymbol{u}_n(T)\|_2^2 - \frac{1}{2}\|\boldsymbol{u}_n(0)\|_2^2 + \varepsilon\int_0^T |||\boldsymbol{u}_n|||^2\,dt + \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \mathbb{D}\boldsymbol{u}_n)\,dt = \int_0^T \boldsymbol{f}\cdot\boldsymbol{u}_n\,dt. \quad (2.36)$$

A computation with $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{3\times 3}$ and $\mathbb{Z}_s = \mathbb{Y} + s(\mathbb{X}-\mathbb{Y})$ reveals

$$\begin{aligned}\mathbb{S}(\mathbb{X}) - \mathbb{S}(\mathbb{Y}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\mathbb{S}(\mathbb{Z}_s)\,ds \\ &= \left(\int_0^1 \left(\nu_0 + \nu_1|\mathbb{Z}_s|^{r-2}\right)\mathbb{I}\otimes\mathbb{I} + \nu_1(r-2)|\mathbb{Z}_s|^{r-4}\mathbb{Z}_s\otimes\mathbb{Z}_s\,ds\right)(\mathbb{X}-\mathbb{Y}).\end{aligned} \quad (2.37)$$

And so, taking $\boldsymbol{v}\in\Upsilon$ arbitrary,

$$\begin{aligned}0 &\le \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n) - \mathbb{S}(\mathbb{D}\boldsymbol{v}), \mathbb{D}\boldsymbol{u}_n - \mathbb{D}\boldsymbol{v})\,dt \\ &= \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \mathbb{D}\boldsymbol{u}_n)\,dt - \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \mathbb{D}\boldsymbol{v})\,dt - \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{v}), \mathbb{D}\boldsymbol{u}_n - \mathbb{D}\boldsymbol{v})\,dt.\end{aligned}$$

Next, insert (2.36):

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{u}_n(T)\|_2^2 - \frac{1}{2}\|\boldsymbol{u}_n(0)\|_2^2 + \varepsilon\int_0^T |||\boldsymbol{u}_n|||^2\,dt \\ &\qquad \le \int_0^T \boldsymbol{f}\cdot\boldsymbol{u}_n\,dt - \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{u}_n), \mathbb{D}\boldsymbol{v})\,dt - \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{v}), \mathbb{D}\boldsymbol{u}_n - \mathbb{D}\boldsymbol{v})\,dt.\end{aligned}$$

Let $n\to\infty$. From (2.30) it follows $\|\boldsymbol{u}(T)\|_2 \le \liminf_n \|\boldsymbol{u}_n(T)\|_2$ and likewise with $\int_0^T |||\boldsymbol{u}_n|||^2\,dt$. Using further strong convergence of the initial conditions and weak convergence in the rest, we obtain

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{u}(T)\|_2^2 - \frac{1}{2}\|\boldsymbol{u}(0)\|_2^2 + \varepsilon\int_0^T |||\boldsymbol{u}|||^2\,dt \\ &\qquad \le \int_0^T \boldsymbol{f}\cdot\boldsymbol{u}\,dt - \langle F, \mathbb{D}\boldsymbol{v}\rangle_{\Upsilon^*,\Upsilon} - \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{v}), \mathbb{D}\boldsymbol{u} - \mathbb{D}\boldsymbol{v})\,dt.\end{aligned}$$

Finally, compare the inequality with (2.35):

$$0 \le \langle F, \mathbb{D}\boldsymbol{u}\rangle_{\Upsilon^*,\Upsilon} - \langle F, \mathbb{D}\boldsymbol{v}\rangle_{\Upsilon^*,\Upsilon} - \int_0^T (\mathbb{S}(\mathbb{D}\boldsymbol{v}), \mathbb{D}\boldsymbol{u} - \mathbb{D}\boldsymbol{v})\,dt.$$

In other words

$$0 \le \langle F - \mathbb{S}(\mathbb{D}\boldsymbol{v}), \mathbb{D}\boldsymbol{u} - \mathbb{D}\boldsymbol{v}\rangle_{\Upsilon^*,\Upsilon}.$$

Setting $\boldsymbol{v} = \boldsymbol{u} - \mu\boldsymbol{\varphi}$, $\mu > 0$, $\boldsymbol{\varphi} \in \Upsilon$ arbitrary, the inequality takes on the form

$$0 \le \mu \langle F - \mathbb{S}(\mathbb{D}\boldsymbol{u} - \mu \mathbb{D}\boldsymbol{\varphi}), \mathbb{D}\boldsymbol{\varphi} \rangle_{\Upsilon^*,\Upsilon}.$$

The proof is finished with division by μ and sending it to zero. As $\boldsymbol{\varphi}$ was arbitrary, we have $F = \mathbb{S}(\mathbb{D}\boldsymbol{u})$ (in the sense of Υ^*). Notice that we would have done no harm picking test functions from $L^r(0,T;V_r)$ instead of only Υ as the higher regularity conferred in the latter space was not used.

2.10 Uniqueness

The last step of our grand theorem consists of verifying that there cannot exist any other solution save $\boldsymbol{u}$. The property of possible solutions to serve as testing functions in the weak formulation (2.2) will prove indispensable here.

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two weak solutions and $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$. We will insert $\boldsymbol{u}$ and $\boldsymbol{v}$ into (2.2), while tested by $\boldsymbol{w}$. After subtraction of the resultant identities and recollection of Theorem 1.10, we obtain an equality holding for a.e. $t \in (0,T)$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}\|_2^2 + \varepsilon |||\boldsymbol{w}|||^2 + (\mathbb{S}(\mathbb{D}\boldsymbol{u}) - \mathbb{S}(\mathbb{D}\boldsymbol{v}), \mathbb{D}\boldsymbol{u} - \mathbb{D}\boldsymbol{v}) = (\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{v}\otimes\boldsymbol{v}, \nabla_x \boldsymbol{w}). \tag{2.38}$$

Combining (2.37) with Korn's inequality, we infer

$$(\mathbb{S}(\mathbb{D}\boldsymbol{u}) - \mathbb{S}(\mathbb{D}\boldsymbol{v}), \mathbb{D}\boldsymbol{u} - \mathbb{D}\boldsymbol{v}) \ge c_{33}\|\nabla_x \boldsymbol{w}\|_2^2.$$

We may carry on inserting this into (2.38):

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}\|_2^2 + \varepsilon |||\boldsymbol{w}|||^2 + c_{33}\|\nabla_x \boldsymbol{w}\|_2^2 &\le (\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{v}\otimes\boldsymbol{v}, \nabla_x \boldsymbol{w}) \\ &= (\boldsymbol{u}\otimes\boldsymbol{w} + \boldsymbol{w}\otimes\boldsymbol{v}, \nabla_x \boldsymbol{w}) \qquad (2.39) \\ &= (\boldsymbol{u}\otimes\boldsymbol{w}, \nabla_x \boldsymbol{w}), \end{aligned}$$

where we used (2.8) in the last equality. Next, Green's theorem and acute observation divulge

$$(\boldsymbol{u}\otimes\boldsymbol{w}, \nabla_x\boldsymbol{w}) = -(\mathrm{div}_x(\boldsymbol{u}\otimes\boldsymbol{w}), \boldsymbol{w}) = -((\boldsymbol{w}\cdot\nabla_x)\boldsymbol{u}, \boldsymbol{w}) = -(\boldsymbol{w}\otimes\boldsymbol{w}, \nabla_x\boldsymbol{u}). \tag{2.40}$$

Not even here will we want to break what has become a tradition, i.e. avoiding unneeded use of V^3-regularity. Therefore, the best we can require of $\nabla_x\boldsymbol{u}$ is belonging into $L^r(\Omega, \mathbb{R}^{3\times 3})$ and the following Hölder's inequality will take that into account.

$$|(\boldsymbol{w}\otimes\boldsymbol{w}, \nabla_x\boldsymbol{u})| \le \|\boldsymbol{w}\|_{\frac{2r}{r-1}}^2 \|\nabla_x\boldsymbol{u}\|_r.$$

Employing usual interpolation, continuous embedding, Poincaré's and Young's inequalities yield

$$\begin{aligned} \|\boldsymbol{w}\|_{\frac{2r}{r-1}}^2 \|\nabla_x\boldsymbol{u}\|_r \le \|\boldsymbol{w}\|_2^{\frac{2r-3}{r}} \|\boldsymbol{w}\|_6^{\frac{3}{r}} \|\nabla_x\boldsymbol{u}\|_r &\le c_{34}\|\boldsymbol{w}\|_2^{\frac{2r-3}{r}} \|\nabla_x\boldsymbol{w}\|_2^{\frac{3}{r}} \|\nabla_x\boldsymbol{u}\|_r \\ &\le c_{33}\|\nabla_x\boldsymbol{w}\|_2^2 + c_{35}\|\boldsymbol{w}\|_2^2 \|\nabla_x\boldsymbol{u}\|_r^{\frac{2r}{2r-3}}. \end{aligned}$$

After insertion of this result into (2.39) we finally get

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{w}\|_2^2 + \varepsilon |||\boldsymbol{w}|||^2 \leq c_{36} \|\boldsymbol{w}\|_2^2 \|\nabla_x \boldsymbol{u}\|_r^{\frac{2r}{2r-3}} . \tag{2.41}$$

We will find this form with $\varepsilon |||\boldsymbol{w}|||^2$ useful in the future. Because $r \geq 5/2$ guarantees time integrability of $\|\nabla_x \boldsymbol{u}\|_r^{\frac{2r}{2r-3}}$, the proof is finished using Gronwall's inequality:

$$\|\boldsymbol{w}(t)\|_2^2 \leq \|\boldsymbol{w}(0)\|_2^2 \exp\left(c_{36} \int_0^t 1 + \|\nabla_x \boldsymbol{u}(s)\|_r^r \, ds \right) . \tag{2.42}$$

2.11 Recapitulation

The existence theorem has thus been successfully proven. Along the way we tried persistently to keep out of calling down ε-induced regularity of the solution. In fact, under closer look it would now be very easy to prove existence and uniqueness for the case $\varepsilon = 0$ (see Remark 2.6). Without the requirement of uniqueness, we could have taken $r \geq 12/5$, and if we did not want solution to possess higher regularity guaranteed in statement of the existence theorem, even $r \geq 11/5$ would be adequate. For a more general treatment of the problem without any ε-term, see [9], chapter 5.

3. Solution semigroup

In this chapter, we will focus on the notions of dynamical systems and attractors. Still, at heart remains the model whose solution we constructed in the previous chapter. As such, new definitions will serve only as starters for proofs, which are to show that our model possesses the given property. The exposition follows closely [6]. In reality, the ε-term, by which our models differs from one investigated in the book, is very well behaved in terms of its not interfering with any of the beneficial properties manifested by the model lacking any ε. The list of presented notions tries by no means to be exhaustive, it offers only the minimum necessary for upcoming procedures. For getting a better insight to the theory of dynamical systems, consult e.g. the cited source or [10].

3.1 Dynamical systems

Definition 3.1 Let X be a Banach space or its closed subset. We call the family of operators $\mathcal{S}(t) : X \to X$, $t \geq 0$, a *semigroup* provided

(i) $\mathcal{S}(0)x = x$,

(ii) $\mathcal{S}(t+s)x = \mathcal{S}(t)\mathcal{S}(s)x \qquad$ for any $x \in X$ and $s, t \geq 0$.

If, in addition, the mapping $\mathcal{S}(\cdot)\cdot : [0,\infty) \times X \to X$ is continuous, we call the couple $(\mathcal{S}(t), X)$ a *dynamical system*.

Note that due to uniqueness of solutions for our model, it gives rise to a so-called *solution semigroup* on H defined as $\mathcal{S}(t)\boldsymbol{u}_0 = \boldsymbol{u}(t)$, where $\boldsymbol{u}$ is the unique solution with the initial condition $\boldsymbol{u}_0$. In addition, this solution semigroup happens to be also a dynamical system. Indeed, if $t_n \to t$ and $\boldsymbol{u}_0^n \to \boldsymbol{u}_0$ in H, we have

$$\|\mathcal{S}(t_n)\boldsymbol{u}_0^n - \mathcal{S}(t)\boldsymbol{u}_0\|_2 \leq \|\mathcal{S}(t_n)\boldsymbol{u}_0^n - \mathcal{S}(t_n)\boldsymbol{u}_0\|_2 + \|\mathcal{S}(t_n)\boldsymbol{u}_0 - \mathcal{S}(t)\boldsymbol{u}_0\|_2 .$$

The first term tends to zero by (2.42) and $\nabla_x \boldsymbol{u} \in L^r((0,T)\times\Omega)$, where $\boldsymbol{u}$ is the solution with the initial condition $\boldsymbol{u}_0$. The second follows from continuity of $\mathcal{S}(\cdot)\boldsymbol{u}_0$ on $[0,T]$.

Definition 3.2 Let $\{\mathcal{S}(t), t \geq 0\}$ be a semigroup on X. We term a set $B \subset X$

- *positively invariant*, if $\mathcal{S}(t)B \subset B$ for all $t \geq 0$. If inclusion may be replaced with the set equality for all $t \geq 0$ then B is called *invariant*.
- *uniformly absorbing*, if for every bounded $E \subset X$ there is a time $t^* \geq 0$ such that $\mathcal{S}(t)E \subset B$ for all $t \geq t^*$.

A fundamental concept of the entire work is that of a global and exponential attractor.

Definition 3.3 We say a dynamical system $(\mathcal{S}(t), X)$ admits the *global attractor*, if there is $\mathcal{A} \subset X$ that

(i) is compact in X;

(ii) is invariant;

(iii) attracts bounded sets of X, i.e. for every $\delta > 0$ and every bounded $B \subset X$ there is $t^* \geq 0$ such that $\mathcal{S}(t)B \subset \mathcal{A} + B(0, \delta)$ for all $t \geq t^*$.

Notice that the definite article before global attractor is in place, for its properties bestow the quality of uniqueness upon it. Before moving on to yet another sort of attractor, we have to first define what is known as fractal dimension.

Definition 3.4 Let A be a relatively compact subset of X. The *fractal dimension* of A is defined as

$$\dim_f^X A = \limsup_{\delta \to 0_+} \frac{\log N_X(A, \delta)}{-\log \delta},$$

where $N_X(A, \delta)$ denotes the smallest number of balls with radius δ necessary for covering A.

Lastly we introduce a so-called exponential attractor. In comparison with the global attractor it may be "larger" but, on the other hand, we have control over its rate of attraction. First and foremost, nonetheless, this kind we will treat here. It can be shown that existence of an exponential attractor entails that of the global attractor ([6], Theorem 2.1).

Definition 3.5 We say a dynamical system $(\mathcal{S}(t), X)$ admits an *exponential attractor*, assuming there exists $\mathcal{M} \subset X$ that

(i) is compact in X;

(ii) is positively invariant;

(iii) attracts bounded sets of X exponentially, i.e. for every $B \subset X$ there is $C > 0$ such that $\mathcal{S}(t)B \subset \mathcal{M} + B(0, Ce^{-\gamma t})$ for all $t \geq 0$, where $\gamma = \gamma(\mathcal{M})$ is a constant independent of B;

(iv) has a finite fractal dimension.

3.2 Absorbing set

From now on, let $\{\mathcal{S}(t)\}_{t \geq 0} = (\mathcal{S}(t), H)$ be the solution semigroup arising from the system (2.1). Like before and in most cases at the later time, we will write $\boldsymbol{u}(t)$ instead of $\boldsymbol{u}(t, \cdot)$.

Theorem 3.6 *There exists a compact set $\mathcal{B} \subset H$ that is positively invariant and uniformly absorbing for $\{\mathcal{S}(t)\}_{t \geq 0}$. Furthermore, for solutions $\boldsymbol{u}$ with initial conditions in $\mathcal{B}$ the following estimates hold true: for a fixed $T > 0$:*

(i) $\sup_{\boldsymbol{u} \in \mathcal{B}} \left(\varepsilon |||\boldsymbol{u}|||^2 + \|\nabla_x \boldsymbol{u}\|_r \right) \leq c_1;$

(ii) $\int_0^T \|\partial_t \boldsymbol{u}(t)\|_2^2 \, dt \leq c_2.$

The bounding constant c_2 is uniform for all solutions starting from $\mathcal{B}$.

Proof. Recalling definition of (2.1) and Theorem 1.10, we test with the solution itself to obtain an inequality holding for a.e. $t \in [0,T]$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(t)\|_2^2 + \varepsilon|||\boldsymbol{u}(t)|||_2^2 + \nu_0\|\mathbb{D}\boldsymbol{u}(t)\|_2^2 + \nu_1\|\mathbb{D}\boldsymbol{u}(t)\|_r^r \leq \|\boldsymbol{f}\|_2\|\boldsymbol{u}(t)\|_2 . \tag{3.1}$$

By Korn's and Poincaré's inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(t)\|_2^2 + c_3\|\boldsymbol{u}(t)\|_2^2 \leq 2\|\boldsymbol{f}\|_2\|\boldsymbol{u}(t)\|_2$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(t)\|_2^2 \leq \|\boldsymbol{u}(t)\|_2\left(2\|\boldsymbol{f}\|_2 - c_3\|\boldsymbol{u}(t)\|_2\right).$$

Hence, setting $c_0 = 1 + 2c_3^{-1}\|\boldsymbol{f}\|_2$ we have a positively invariant and uniformly absorbing set $B_0 = B(0, c_0)$ (as a ball in H). The sought-after set will be defined as

$$\mathcal{B} = \overline{\bigcup_{t\geq 1}\mathcal{S}(t)B_0}^H .$$

The rest of the proof will take verification of the set's announced properties. Uniform absorption is checked effortlessly. As for positive invariance, the set $\mathcal{B}' = \bigcup_{t\geq 1}\mathcal{S}(t)B_0$ fulfills the property obviously. Furthermore, $\mathcal{S}(t) : B_0 \to B_0$ is continuous for every $t > 0$. This result is due to (2.9), (2.42) and weak lower semicontinuity of the norm. In other words $\int_0^T\|\nabla_x\boldsymbol{u}(t)\|_r^r\,dt$ is bounded for L^2-bounded initial conditions. Subsequently, we infer

$$\mathcal{S}(t)\mathcal{B} = \mathcal{S}(t)\overline{\mathcal{B}'} \subset \overline{\mathcal{S}(t)\mathcal{B}'} \subset \overline{\mathcal{B}'} = \mathcal{B},$$

with all closures taken in H.

Properties (i) and (ii) will first be investigated only on $\mathcal{B}'$. Right from its definition and local higher regularity of solutions, we know that every solution starting in $\mathcal{B}'$ belongs to $L^\infty(0,T;V_r) \cap L^\infty\left(0,T;V^3\right)$ with time derivative in $L^2(0,T;H)$. Since this regularity was in fact deduced from L^2-norm of the initial condition (not mentioning T and other static data, see Note 2.4), we obtain a uniform bound in these spaces for all solutions originating in $\mathcal{B}'$.

Now let $\boldsymbol{u}(t)$ be a solution with the initial value in $\mathcal{B}$. There are $\boldsymbol{u}_n(0) \in \mathcal{B}'$ converging to $\boldsymbol{u}(0)$ in H and from (2.42) we even know that the corresponding solutions $\boldsymbol{u}_n$ converge to $\boldsymbol{u}$ strongly in $\mathcal{C}([0,T];H)$. As $\boldsymbol{u}_n$ clearly satisfies (i) and (ii), so does $\boldsymbol{u}$ by Banach-Alaoglu theorem and weak lower semicontinuity of the norm.

Finally, $\mathcal{B}$ is compact from $V_r \hookrightarrow\hookrightarrow H$ and (i). ■

Note 3.7 The definition of B_0 conspicuously warrants for any $\boldsymbol{u}_0$ existence of a time $t^* = t^*(\|\boldsymbol{u}_0\|_2)$ such that $\mathcal{S}(t)\boldsymbol{u}_0 \in \mathcal{B}$ for every $t \geq t^*$. Therefore, $\mathcal{B}$ seems like a promising candidate for an exponential attractor. This, however, we cannot determine for there is no telling if $\dim_f^H\mathcal{B} < \infty$. Expectedly enough, the sought-after exponential attractor will in the end turn out to be a subset of $\mathcal{B}$.

We conclude this section with a few simple observations about further properties of $\mathcal{B}$ indispensable during our forthcoming advance.

Corollary 3.8 *Let $\boldsymbol{u}, \boldsymbol{v}$ be two solutions starting from $\mathcal{B}$, let $T > 0$ and $t, t_1, t_2 \in [0,T]$. Then*

(i) $\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|_2 \leq c_4\|\boldsymbol{u}(0) - \boldsymbol{v}(0)\|_2$;

(ii) $\|\boldsymbol{u}(t_1) - \boldsymbol{u}(t_2)\|_2 \le c_5 \, |t_1 - t_2|^{1/2}$;

(iii) $\varepsilon \int_0^T |||\boldsymbol{u}(t) - \boldsymbol{v}(t)|||^2 \, dt \le c_6 \, \|\boldsymbol{u}(0) - \boldsymbol{v}(0)\|_2^2$.

Again, the bounding constants are uniform for all solutions starting in $\mathcal{B}$.

Proof. (i) is immediately obtained from (2.42) and Theorem 3.6 (i).

Recalling that we have set up the relation (2.41), it is now due integrating from 0 to T. Invoking Theorem 3.6 (i) once again, we reveal

$$\begin{aligned}\varepsilon \int_0^T |||\boldsymbol{w}(t)|||^2 \; dt &\le c_7 \, \|\boldsymbol{w}(0)\|_2^2 + c_8 \int_0^T \|\boldsymbol{w}(t)\|_2^2 \, \|\nabla_x \boldsymbol{u}(t)\|_r^{\frac{2r}{2r-3}} \, dt \\ &\le \|\boldsymbol{w}(0)\|_2^2 \left(c_7 + c_9 \int_0^T \exp \left(c_{10} \int_0^T 1 + \|\nabla_x \boldsymbol{u}(s)\|_r^r \; ds \right) dt \right) \\ &\le c_{11} \, \|\boldsymbol{w}(0)\|_2^2 .\end{aligned}$$

In the second inequality we exploited (2.42) as well as boundedness of $\|\nabla_x \boldsymbol{u}(t)\|_r$ in time. Hence (iii) is proved.

In order to verify (ii), we make use of Theorem 3.6 (ii) and also Theorem 1.5. Suppose $t_2 > t_1$:

$$\boldsymbol{u}(t_2) = \boldsymbol{u}(t_1) + \int_{t_1}^{t_2} \partial_t \boldsymbol{u}(s) \, ds,$$

and thence

$$\|\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)\|_2 \le \int_{t_1}^{t_2} \|\partial_t \boldsymbol{u}(s)\|_2 \, ds \le |t_2 - t_1|^{1/2} \left(\int_0^T \|\partial_t \boldsymbol{u}(s)\|_2^2 \, ds \right)^{1/2} \le c_{12} \, |t_2 - t_1|^{1/2} .$$

■

3.3 Differentiability

Definition 3.9 Let W be a (not necessarily open) subset of a Banach space X. We say the mapping $T : W \to X$ is *uniformly Fréchet differentiable* (on W) if there exists a family of compact, linear operators $\{L_x : X \to X;\, x \in W\}$ such that

$$\lim_{\delta \to 0_+} \sup_{\substack{x,y \in W \\ 0 < \|x-y\|_X < \delta}} \frac{\|T(y) - T(x) - L_x(y-x)\|_X}{\|y - x\|_X} = 0.$$

Notice that due to the abandoned requirement for W being open, the operators L_x may not be uniquely determined. This is the price to be paid for working on a compact set $\mathcal{B}$. We will apply the definition to seek an appropriate kind of a derivative of $\mathcal{S}(t) : \mathcal{B} \to \mathcal{B}$, which will, in the final chapter, impose bounds on the fractal dimension of the exponential attractor. Now that we know what it should fulfill, we have yet to offer a candidate, for which to prove the quality of being a uniform Fréchet derivative of $\mathcal{S}(t)$.

We will now heuristically deduce Gâteaux differential of the equation (2.1) with respect to the initial condition in the direction $\boldsymbol{\xi} \in H$. Less vaguely, fix $\boldsymbol{u}_0 \in H$ and rewrite (2.1) as

$$\left.\begin{aligned} \partial_t \boldsymbol{u}(t,\boldsymbol{x},\boldsymbol{u}_0) &= \mathcal{F}\left(\boldsymbol{u}(t,\boldsymbol{x},\boldsymbol{u}_0),\boldsymbol{x}\right) \\ \operatorname{div}_x \boldsymbol{u}(t,\boldsymbol{x},\boldsymbol{u}_0) &= 0 \end{aligned}\right\} \quad \textit{in } (0,T)\times\Omega,$$
$$+ \text{ boundary conditions.}$$

Next, fix $\boldsymbol{\xi} \in H$ and denote

$$\mathcal{U}(t) = \lim_{\delta\to 0} \frac{\boldsymbol{u}(t,\boldsymbol{x},\boldsymbol{u}_0+\delta\boldsymbol{\xi}) - \boldsymbol{u}(t,\boldsymbol{x},\boldsymbol{u}_0)}{\delta}.$$

Assuming that all we do is correct and meaningful, $\mathcal{U}(t)$ is expected to satisfy a so-called first variation equation:

$$\begin{aligned} &\left.\begin{aligned} \partial_t \mathcal{U} &= \partial_{\boldsymbol{u}}\mathcal{F}(\boldsymbol{u},\boldsymbol{x})\mathcal{U} \\ \operatorname{div}_x \mathcal{U} &= 0 \end{aligned}\right\} && \textit{in } (0,T)\times\Omega, \\ &\mathcal{U} = \frac{\partial\mathcal{U}}{\partial\mathbf{n}} = \Delta\mathcal{U} = \mathbf{0} && \textit{on } (0,T)\times\partial\Omega, \\ &\mathcal{U}(0) = \boldsymbol{\xi} && \textit{in } \Omega. \end{aligned} \tag{3.2}$$

Expanding $\mathcal{F}(\boldsymbol{u},\boldsymbol{x})$ as

$$\mathcal{F}(\boldsymbol{u},\boldsymbol{x}) = \varepsilon\Delta^3\boldsymbol{u} + \operatorname{div}_x \mathbb{S}(\mathbb{D}\boldsymbol{u}) - \operatorname{div}_x(\boldsymbol{u}\otimes\boldsymbol{u}) - \nabla_x p + \boldsymbol{f}$$

reveals what $\partial_{\boldsymbol{u}}\mathcal{F}(\boldsymbol{u},\boldsymbol{x})\mathcal{U}$ looks like, namely

$$\partial_{\boldsymbol{u}}\mathcal{F}(\boldsymbol{u},\boldsymbol{x})\mathcal{U} = \varepsilon\Delta^3\mathcal{U} + \operatorname{div}_x(\partial_{\mathbb{D}}\mathbb{S}(\mathbb{D}\boldsymbol{u})\mathbb{D}\mathcal{U}) - \operatorname{div}_x(\mathcal{U}\otimes\boldsymbol{u}) - \operatorname{div}_x(\boldsymbol{u}\otimes\mathcal{U}) - \nabla_x q, \tag{3.3}$$

where

$$\begin{aligned} \partial_{\mathbb{D}}\mathbb{S} : \mathbb{R}^{3\times3} &\longrightarrow \mathbb{R}^{3\times3\times3\times3} \\ \mathbb{X} &\longmapsto \left(\nu_0 + \nu_1|\mathbb{X}|^{r-2}\right)\mathbb{I}\otimes\mathbb{I} + \nu_1(r-2)|\mathbb{X}|^{r-4}\mathbb{X}\otimes\mathbb{X}. \end{aligned}$$

Before this informal intermezzo on the first variation equation has been brought to its end, two remarks are in place. Firstly, the pressure term did not vanish like external forces $\boldsymbol{f}$. Even though it is explicitly a function (a distribution in fact) of the temporal and spatial variables only, in reality, it is constructed from the weak solution itself. Therefore, pressure is actually dependent on $\boldsymbol{u}$ and by this reason it left a descendant in (3.3). Like in the original equation, we will deal with the term $\nabla_x q$ in so far as by saying that it disappears in the weak formulation but can be recreated from the weak solution $\mathcal{U}$. Keep in mind this disappearance takes place only when testing by zero-divergence functions.

Secondly it will soon be useful to know how $\partial_{\mathbb{D}}\mathbb{S}(\mathbb{D}\boldsymbol{u})$ act:

$$\begin{aligned} \partial_{\mathbb{D}}\mathbb{S}(\mathbb{D}\boldsymbol{u})\mathbb{D}\mathcal{U} &= \left(\nu_0 + \nu_1\left|\mathbb{D}\boldsymbol{u}\right|^{r-2}\right)\cdot\mathbb{D}\mathcal{U} + \nu_1(r-2)\left|\mathbb{D}\boldsymbol{u}\right|^{r-4}\left(\mathbb{D}\boldsymbol{u}\otimes\mathbb{D}\boldsymbol{u}\right)\cdot\mathbb{D}\mathcal{U}, \\ \partial_{\mathbb{D}}\mathbb{S}(\mathbb{D}\boldsymbol{u})\mathbb{D}\mathcal{U} : \mathbb{D}\mathcal{U} &= \left(\nu_0 + \nu_1\left|\mathbb{D}\boldsymbol{u}\right|^{r-2}\right)\left|\mathbb{D}\mathcal{U}\right|^2 + \nu_1(r-2)\left|\mathbb{D}\boldsymbol{u}\right|^{r-4}\left(\mathbb{D}\boldsymbol{u} : \mathbb{D}\mathcal{U}\right)^2. \end{aligned} \tag{3.4}$$

Thus we have a linear problem (3.2) to solve. Denoting $\mathcal{U}$ its solution, we will subsequently show that $\mathcal{U}(t)$ is a uniform Fréchet differential of $\mathcal{S}(t)$.

Definition 3.10 We call the problem (3.2) the *linearized equation on a neighbourhood of* $\boldsymbol{u}$.

Although we should name it rather linearization of Ladyzhenskaya model with a higher order perturbation, given that we investigate no other equation here, our simplified designation ought not to be a source of any perplexion.

Definition 3.11 Let $\boldsymbol{u}$ be a fixed weak solution to (2.1) and $\boldsymbol{\xi} \in H$. A function $\mathcal{U}$ will be termed a weak solution to the linearized equation in a neighbourhood of $\boldsymbol{u}$ (3.2), if

$$\mathcal{U} \in L^2(0,T;V^3), \qquad \partial_t \mathcal{U} \in L^2(0,T;(V^3)^*)$$

and for every $\boldsymbol{v} \in L^2(0,T;V^3)$ the following identity is satisfied:

$$\int_0^T \langle \partial_t \mathcal{U}, \boldsymbol{v} \rangle_{(V^3)^*,V^3} + \varepsilon \, \langle\!\langle \mathcal{U}, \boldsymbol{v} \rangle\!\rangle + (\partial_{\mathbb{D}} \mathbb{S}(\mathbb{D}\boldsymbol{u}) \mathbb{D}\mathcal{U}, \mathbb{D}\boldsymbol{v}) - (\boldsymbol{u} \otimes \mathcal{U} + \mathcal{U} \otimes \boldsymbol{u}, \nabla_x \boldsymbol{v}) \, dt = 0. \tag{3.5}$$

The initial condition is attained in the form

$$\boldsymbol{\xi} = \mathcal{U}(0) \qquad \text{a.e. in } \Omega,$$

which is sensible as $\mathcal{U} \in \mathcal{C}([0,T];H)$ by Theorem 1.10.

Theorem 3.12 *For any* $\boldsymbol{\xi} \in H$ *there exists a unique weak solution to the linearized equation in a neighbourhood of* $\boldsymbol{u}$*, where* $\boldsymbol{u}$ *starts from* $\mathcal{B}$*.*

Proof. The proof might follow the very lines of Theorem 2.5, only with some considerable simplification such as linearity of all terms and boundedness of $|\mathbb{D}\boldsymbol{u}|$ due to Theorem 3.6 (i). Ergo, we dare to skip it. ■

Remark 3.13 It is easy to see directly from the weak formulation (3.5) that apriori estimates analogous to sections 2.2 and 2.3 are again accompanied by deleterious effect of $\varepsilon \to 0_+$ on the norm of $\mathcal{U}$ and $\partial_t \mathcal{U}$ in their native spaces. To add salt to injury, we are not even able to secure bounds in less regular, yet still practical spaces such as $L^2(0,T;V_2)$, liberated from ε. The reason is the necessity to control $|\mathbb{D}\boldsymbol{u}|$ somehow, which was done by means of an ε-dominated estimate.

Remark 3.14 Continuing in the tracks of analogy, it is worth reminding that for any index set I the set

$$\left\{ \boldsymbol{\xi}_\alpha \in H;\, \alpha \in I,\, \sup_\alpha \|\boldsymbol{\xi}_\alpha\|_2 < \infty \right\}$$

gives rise to a family of weak solutions to the linearized problem $\{(\mathcal{U}_\alpha, \partial_t \mathcal{U}_\alpha);\, \alpha \in I,\, \mathcal{U}_\alpha(0) = \boldsymbol{\xi}_\alpha\}$, which is bounded in $L^2(0,T;V^3) \times L^2(0,T;(V^3)^*)$. Consequently, by Aubin-Simon lemma the set of solutions

$$A = \{\mathcal{U}_\alpha; \alpha \in I, \mathcal{U}_\alpha(0) = \boldsymbol{\xi}_\alpha\}$$

is precompact in $L^2(0,T;H)$.

Now, due to temporal continuity of solutions in H, it makes sense to ask whether $\{\mathcal{U}_\alpha(T);\, \mathcal{U}_\alpha \in A\}$ is precompact in H. Positive answer would be guaranteed if the mapping $\mathcal{T} : A \to H$, defined as $\mathcal{T}(\mathcal{U}_\alpha) = \mathcal{U}_\alpha(T)$, was Lipschitz continuous, which we will show presently. With regard to later applications and tantalising simplification, we will require $\boldsymbol{u}$ to originate from $\mathcal{B}$.

Generally speaking, this is only a brief glance into what is known as the method of trajectories (see [6], chapter 2).

Lemma 3.15 *Let $\boldsymbol{u}$ starts in $\mathcal{B}$. The mapping $\mathcal{T}$ is then Lipschitz continuous.*

Proof. Let $\mathcal{U}, \mathcal{V} \in A$, and set $\mathcal{W} = \mathcal{U} - \mathcal{V}$. We will show $\|\mathcal{W}(T)\|_2 \leq c_{13} \|\mathcal{W}\|_{L^2(0,T;H)}$. The bounding constant is of course universal for entire A. Subtracting the equalities for $\mathcal{U}$ and $\mathcal{V}$ while tested by $\mathcal{W}$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{W}\|_2^2 + \varepsilon|||\mathcal{W}|||_2^2 + (\partial_{\mathbb{D}}\mathbb{S}(\mathbb{D}\boldsymbol{u})\mathbb{D}\mathcal{W}, \mathbb{D}\mathcal{W}) = (\boldsymbol{u}\otimes\mathcal{W} + \mathcal{W}\otimes\boldsymbol{u}, \nabla_x\mathcal{W}) \\ = -(\mathcal{W}\otimes\mathcal{W}, \nabla_x\boldsymbol{u})$$

a.e. in $(0,T)$. The last equality is thanks to (2.8) and (2.40). Invoking Theorem 3.6 (i) and (3.4), we continue:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{W}\|_2^2 \leq c_{14}\|\mathcal{W}\|_2^2.$$

By Gronwall inequality, for any $0 < s < T$:

$$\|\mathcal{W}(T)\|_2^2 \leq c_{15}\|\mathcal{W}(s)\|_2^2.$$

Finally, integrate over $(0,T)$:

$$\|\mathcal{W}(T)\|_2 \leq c_{13}\|\mathcal{W}\|_{L^2(0,T;H)}.$$

■

Uniqueness of solutions to the linearized problem again lets us introduce a solution semigroup $\{\mathcal{L}_{\boldsymbol{u}}(t)\}_{t\geq 0}$ on H defined as $\mathcal{L}_{\boldsymbol{u}}(t)\boldsymbol{\xi} = \mathcal{U}(t)$, where $\mathcal{U}$ is the unique weak solution to the linearized equation on a neighbourhood of $\boldsymbol{u}$ with the initial condition $\boldsymbol{\xi}$. With this candidate for the pursued uniform Fréchet derivative of $\mathcal{S}(t)$, most effort will be made to prove that it really is one. Be forewarned that $\varepsilon > 0$ will be crucial here and the result cannot be shown for $\varepsilon = 0$ in the presented manner.

Theorem 3.16 *$\mathcal{L}_{\boldsymbol{u}}(t)$ is a uniform Fréchet derivative of $\mathcal{S}(t)$ on $\mathcal{B}$ for a.e. $t > 0$. In other words*

$$\lim_{\delta\to 0_+} \sup_{\substack{\boldsymbol{u}_0,\boldsymbol{v}_0\in\mathcal{B} \\ 0<\|\boldsymbol{u}_0-\boldsymbol{v}_0\|_H<\delta}} \frac{\|\mathcal{S}(t)\boldsymbol{v}_0 - \mathcal{S}(t)\boldsymbol{u}_0 - \mathcal{L}_{\boldsymbol{u}}(t)(\boldsymbol{v}_0-\boldsymbol{u}_0)\|_H}{\|\boldsymbol{v}_0-\boldsymbol{u}_0\|_H} = 0,$$

where $\boldsymbol{u}_0 = \boldsymbol{u}(0)$.

Proof. $\mathcal{L}_{\boldsymbol{u}}(t)$ is evidently a linear operator in H. Compactness was actually corroborrated in Remark 3.14 as there is no problem to set $T = t$.

Fix $\delta > 0$ and $\boldsymbol{v}_0 \in \mathcal{B}$ such that $0 < \|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_H < \delta$. Denote

$$\begin{aligned}\boldsymbol{v}(t) &= \mathcal{S}(t)\boldsymbol{v}_0,\\ \boldsymbol{w}(t) &= \boldsymbol{v}(t) - \boldsymbol{u}(t),\\ \mathcal{U}(t) &= \mathcal{L}_{\boldsymbol{u}}(t)(\boldsymbol{v}_0 - \boldsymbol{u}_0),\\ \boldsymbol{\eta}(t) &= \boldsymbol{w}(t) - \mathcal{U}(t).\end{aligned}$$

We will find $\beta > 0$ such that $\|\boldsymbol{\eta}\|_2 \leq c_{16}\|\boldsymbol{w}(0)\|_2^{1+\beta}$, where β and c_{16} do not depend on the initial condition (or any other undesired quantity with the exception of ε, for that matter). The statement will thus be proved. We may assume without loss of generality that $t \in (0,T)$. As $\boldsymbol{\eta}(s) \in V^3$ a.e. in (0,T), we will make use of the pointwise alternative of a weak solution in equations for $\boldsymbol{v}$, $\boldsymbol{u}$ and $\mathcal{U}$:

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{\eta} \rangle_{(V^3)^*, V^3} + \varepsilon \langle\!\langle \boldsymbol{v}, \boldsymbol{\eta} \rangle\!\rangle + (\mathbb{S}(\mathbb{D}\boldsymbol{v}), \mathbb{D}\boldsymbol{\eta}) = (\boldsymbol{v} \otimes \boldsymbol{v}, \nabla_x \boldsymbol{\eta}) + (f, \boldsymbol{\eta}),$$

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{\eta} \rangle_{(V^3)^*, V^3} + \varepsilon \langle\!\langle \boldsymbol{u}, \boldsymbol{\eta} \rangle\!\rangle + (\mathbb{S}(\mathbb{D}\boldsymbol{u}), \mathbb{D}\boldsymbol{\eta}) = (\boldsymbol{u} \otimes \boldsymbol{u}, \nabla_x \boldsymbol{\eta}) + (f, \boldsymbol{\eta}),$$

$$\langle \partial_t \mathcal{U}, \boldsymbol{\eta} \rangle_{(V^3)^*, V^3} + \varepsilon \langle\!\langle \mathcal{U}, \boldsymbol{\eta} \rangle\!\rangle + (\partial_{\mathbb{D}} \mathbb{S}(\mathbb{D}\boldsymbol{u}) \mathbb{D}\mathcal{U}, \mathbb{D}\boldsymbol{\eta}) = (\boldsymbol{u} \otimes \mathcal{U} + \mathcal{U} \otimes \boldsymbol{u}, \nabla_x \boldsymbol{\eta})$$

almost everywhere in $(0,T)$. Next we subtract the second and the third equation from the first one and also invoke Theorem 1.10 to handle the time derivative:

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{\eta}\|_2^2 + \varepsilon |\!|\!| \boldsymbol{\eta} |\!|\!| + (\mathbb{S}(\mathbb{D}\boldsymbol{v}) - \mathbb{S}(\mathbb{D}\boldsymbol{u}) - \partial_{\mathbb{D}} \mathbb{S}(\mathbb{D}\boldsymbol{u}) \mathbb{D}\mathcal{U}, \mathbb{D}\boldsymbol{\eta})$$
$$= (\boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{u} \otimes \mathcal{U} - \mathcal{U} \otimes \boldsymbol{u}, \nabla_x \boldsymbol{\eta}). \quad (3.6)$$

We now have to foxily estimate the nonlinear terms

$$\begin{aligned} I &= (\mathbb{S}(\mathbb{D}\boldsymbol{v}) - \mathbb{S}(\mathbb{D}\boldsymbol{u}) - \partial_{\mathbb{D}} \mathbb{S}(\mathbb{D}\boldsymbol{u}) \mathbb{D}\mathcal{U}, \mathbb{D}\boldsymbol{\eta}), \\ II &= (\boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{u} \otimes \mathcal{U} - \mathcal{U} \otimes \boldsymbol{u}, \nabla_x \boldsymbol{\eta}). \end{aligned} \quad (3.7)$$

With help of the mean-value formula, we may write

$$\mathbb{S}(\mathbb{D}\boldsymbol{v}) - \mathbb{S}(\mathbb{D}\boldsymbol{u}) = \partial_{\mathbb{D}} \mathbb{S}(\mathbb{D}\boldsymbol{u} + \theta(\mathbb{D}\boldsymbol{v} - \mathbb{D}\boldsymbol{u}))(\mathbb{D}\boldsymbol{v} - \mathbb{D}\boldsymbol{u}), \qquad \theta : (0,T) \times \Omega \to (0,1),$$

and combining with $\mathcal{U} = \boldsymbol{w} - \boldsymbol{\eta}$, the term I will hence be treated as

$$I = \underbrace{\int_\Omega \partial_{\mathbb{D}} \mathbb{S}(\mathbb{D}\boldsymbol{u}) \mathbb{D}\boldsymbol{\eta} : \mathbb{D}\boldsymbol{\eta} \, dx}_{I_1} + \underbrace{\int_\Omega \left(\partial_{\mathbb{D}} \mathbb{S}(\mathbb{D}\boldsymbol{u} + \theta \mathbb{D}\boldsymbol{w}) - \partial_{\mathbb{D}} \mathbb{S}(\mathbb{D}\boldsymbol{u}) \right) \mathbb{D}\boldsymbol{w} : \mathbb{D}\boldsymbol{\eta} \, dx}_{I_2}.$$

Korn's inequality and (3.4) imply

$$I_1 \geq c_{17} \|\nabla_x \boldsymbol{\eta}\|_2^2. \quad (3.8)$$

As for I_2, a computation with $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{3\times 3}$ yields

$$\partial_{\mathbb{D}} \mathbb{S}(\mathbb{X}) - \partial_{\mathbb{D}} \mathbb{S}(\mathbb{Y}) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \partial_{\mathbb{D}} \mathbb{S}(\mathbb{Y} + s(\mathbb{X} - \mathbb{Y})) \, ds.$$

Since $V^3 \hookrightarrow\hookrightarrow V_\infty$ and $|\partial_{\mathbb{D}\mathbb{D}} \mathbb{S}(\mathbb{X})| \approx |\mathbb{X}|^{r-3}$, invoking Theorem 3.6 (i) enables to bound I_2 as:

$$I_2 \leq c_{18} \, \varepsilon^{(3-r)/2} \int_\Omega |\mathbb{D}\boldsymbol{w}|^2 \, |\mathbb{D}\boldsymbol{\eta}| \, dx \leq \frac{c_{17}}{2} \|\nabla_x \boldsymbol{\eta}\|_2^2 + c_{19} \, \varepsilon^{3-r} \|\nabla_x \boldsymbol{w}\|_4^4. \quad (3.9)$$

Let us now draw our attention to II. First, a slight readjustment is in place:

$$\boldsymbol{v} \otimes \boldsymbol{v} = \boldsymbol{w} \otimes \boldsymbol{w} + \boldsymbol{w} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{w} + \boldsymbol{u} \otimes \boldsymbol{u},$$

whence

$$\boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{u} \otimes \mathcal{U} - \mathcal{U} \otimes \boldsymbol{u} = \boldsymbol{w} \otimes \boldsymbol{w} + \boldsymbol{\eta} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\eta}.$$

This form will be most convenient for the upcoming estimates. We are going to use (2.8), Theorem 3.6 (i) and Poincaré's inequality.

$$\begin{aligned} II &= (\boldsymbol{w} \otimes \boldsymbol{w} + \boldsymbol{\eta} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\eta}, \nabla_x \boldsymbol{\eta}) \\ &\leq \int_\Omega \Big(|\boldsymbol{w}|^2 \, |\nabla_x \boldsymbol{\eta}| + |\boldsymbol{u}| \, |\boldsymbol{\eta}| \, |\nabla_x \boldsymbol{\eta}| \Big) \, dx \\ &\leq \|\boldsymbol{w}\|_4^2 \, \|\nabla_x \boldsymbol{\eta}\|_2 + c_{20} \, \|\boldsymbol{\eta}\|_2 \, \|\nabla_x \boldsymbol{\eta}\|_2 \\ &\leq \frac{c_{17}}{2} \, \|\nabla_x \boldsymbol{\eta}\|_2^2 + c_{21} \, \|\nabla_x \boldsymbol{w}\|_4^4 + c_{22} \, \|\boldsymbol{\eta}\|_2^2 \, . \end{aligned} \tag{3.10}$$

Putting back together (3.6)–(3.10), we have obtained

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \, \|\boldsymbol{\eta}\|_2^2 + c_{17} \, \|\nabla_x \boldsymbol{\eta}\|_2^2 \leq c_{17} \, \|\nabla_x \boldsymbol{\eta}\|_2^2 + c_{22} \, \|\boldsymbol{\eta}\|_2^2 + c_{23} \, \varepsilon^{3-r} \, \|\nabla_x \boldsymbol{w}\|_4^4 \, ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \, \|\boldsymbol{\eta}\|_2^2 = c_{24} \, \|\boldsymbol{\eta}\|_2^2 + c_{25} \, \varepsilon^{3-r} \, \|\nabla_x \boldsymbol{w}\|_4^4 \, .$$

The final series of steps begins with Gronwall's inequality, while keeping in mind $\boldsymbol{\eta}(0) = \mathbf{0}$. The remainder rests on Theorems 3.6 (i) and Corollary 3.8 (i) and (iii):

$$\begin{aligned} \sup_{t \in [0,T]} \|\boldsymbol{\eta}(t)\|_2^2 &\leq c_{26} \, \varepsilon^{3-r} \int_0^T \|\nabla_x \boldsymbol{w}(t)\|_4^4 \, dt \\ &\leq c_{26} \, \varepsilon^{3-r} \operatorname*{sup\,ess}_{t \in [0,T]} \|\nabla_x \boldsymbol{w}(t)\|_2^2 \int_0^T \|\nabla_x \boldsymbol{w}(t)\|_\infty^2 \, dt \\ &\leq c_{27} \, \varepsilon^{3-r} \operatorname*{sup\,ess}_{t \in [0,T]} \|\nabla_x \boldsymbol{w}(t)\|_2^2 \int_0^T |||\boldsymbol{w}(t)|||_2^2 \, dt \\ &\leq c_{28} \, \varepsilon^{2-r} \|\boldsymbol{w}(0)\|_2^2 \operatorname*{sup\,ess}_{t \in [0,T]} \|\boldsymbol{w}(t)\|_2 \, \|\Delta \boldsymbol{w}(t)\|_2 \\ &\leq c_{29} \varepsilon^{(3-2r)/2} \|\boldsymbol{w}(0)\|_2^3 \, . \end{aligned}$$

In the third and the fifth inequality we recalled Sobolev's imbeddings of V^3 and in the fourth, the interpolation $\|\nabla_x \boldsymbol{w}\|_2^2 \leq \|\boldsymbol{w}\|_2 \, \|\Delta \boldsymbol{w}\|_2$. We have thus discovered $\beta = 1/2$. Note, however, that $\varepsilon^{(3-2r)/2} \to \infty$ for $\varepsilon \to 0_+$. ■

4. Bounding the dimension

In the spirit of the existence theorem from chapter Solution, we have to first swerve from the straight path and regale ourselves with a little basic information pertaining to yet another function basis.

4.1 Eigenvalues of the Stokes operator

In section 1.4 we constructed an orthonormal basis in H, which, in addition, was also an orthogonal basis in V^3. In a while we will make use of a similar set of functions, forming an orthonormal basis in H and an orthogonal basis in V_2. The procedure how to find it would be completely analogous to that in the referred section, hence we will mention only the starting point ultimately spawning the desired functions. The problem posited is to find $\boldsymbol{u} \in V_2$ that for a fixed $\boldsymbol{f} \in H$ and any $\boldsymbol{v} \in V_2$ satisfies

$$(\nabla_x \boldsymbol{u}, \nabla_x \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})\,.$$

Recall that $(\nabla_x \cdot\,, \nabla_x \cdot)$ is a scalar product on $W_0^{1,2}(\Omega)$, inducing there a norm equivalent to the original one and as such we may regard $W_0^{1,2}(\Omega)$ as being innately equipped with this "gradient norm". Therefore, employing the very same reasoning as in pursuit of $\{\boldsymbol{w}_j\}$, we acquire a set $\{\boldsymbol{m}_j\}$ with both desired properties and $\{\kappa_j\}$, tending again monotonously to infinity and satisfying $(\kappa_j^{-1} \nabla_x \boldsymbol{m}_j, \nabla_x \boldsymbol{v}) = (\boldsymbol{m}_j, \boldsymbol{v})$ for any $j \in \mathbb{N}$ and $\boldsymbol{v} \in V_2$. To nip confusion in the bud, κ_j corresponds to λ_j^{-1} from the above-mentioned section.

Unlike our $\{\lambda_j\}$, elements of $\{\kappa_j\}$, known as eigenvalues of the Stokes operator, are well-documented in terms of their properties (see [2], Chapter 4). We will need two of them, one elementary and the other via a citation only.

Lemma 4.1 *The principal eigenvalue of the Stokes operator, κ_1, satisfies the following two properties:*

(i) $\kappa_1 = \min \left\{ \|\nabla_x \boldsymbol{v}\|_2^2;\ \boldsymbol{v} \in V_2, \|\boldsymbol{v}\|_2 = 1 \right\}$;

(ii) *κ_1 is inversely proportional to $|\Omega|$. In other words, κ_1 can be made arbitrarily large by means of shrinking Ω.*

Proof. Let $\boldsymbol{v} \in V_2$ such that $\|\boldsymbol{v}\|_2 = 1$. Since $\{\boldsymbol{m}_j\}$ is an orthogonal basis in H, one has

$$\sum_{j=1}^{\infty} (\boldsymbol{m}_j, \boldsymbol{v})^2 = 1. \tag{4.1}$$

Similarly, $\{\kappa_j^{-1/2} \boldsymbol{m}_j\}$ is an orthonormal basis in V_2 with the L^2-gradient norm. Combined with $(\kappa_j^{-1} \nabla_x \boldsymbol{m}_j, \nabla_x \boldsymbol{v}) = (\boldsymbol{m}_j, \boldsymbol{v})$ for every $j \in \mathbb{N}$ and (4.1), we have

$$\|\nabla_x \boldsymbol{v}\|_2^2 = \sum_{j=1}^{\infty} (\kappa_j^{-1/2} \nabla_x \boldsymbol{m}_j, \nabla_x \boldsymbol{v})^2 = \sum_{j=1}^{\infty} \kappa_j (\boldsymbol{m}_j, \boldsymbol{v})^2 \geq \kappa_1.$$

We reach the equality setting $\boldsymbol{v} = \boldsymbol{m}_1$, whence (i) is proved. To conclude, we observe

$$\kappa_1 = \|\nabla_x \boldsymbol{m}_1\|_2^2 \geq K \|\boldsymbol{m}_1\|_6^2 \geq K |\Omega|^{-2/3} \|\boldsymbol{m}_1\|_2^2 = K |\Omega|^{-2/3}\,,$$

where $K^{1/2}$ is the constant from the Sobolev embedding $W_0^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, which is scale invariant. Thus (ii) is verified too. ■

Note 4.2 In accordance with (2.2), replacing $\boldsymbol{u}(t,x)$ on Ω with $\boldsymbol{u}(t,\alpha x)$ on $\alpha^{-1}\Omega$ for $\alpha > 0$ will not have any impinge on qualitative properties of the system. Subsequently we may, without loss of generality, assume Ω is small enough so that to ensure $\kappa_1 \geq 1$.

Lemma 4.3 ([2], Theorem 4.11)
There exists a scale invariant constant K_3 such that for every $j \in \mathbb{N}$ one has

$$\frac{\kappa_j}{\kappa_1} \geq K_3\, j^{2/3}.$$

We only remark that in a general dimension n, the estimate reads $\frac{\kappa_j}{\kappa_1} \geq K_n\, j^{2/n}$.

4.2 Method of Lyapunov exponents

At long last we have reached the point of our utmost interest. Before we state the main theorem whose assumptions for our model will be verified in the subsequent section, some light should be cast on what the concept is all about. Since there is no point in copying other books by chapter and verse, the introduction will only be motivational and as such, bereft of rigor. What will matter is the said justification later on. Let an interested reader consult [6] or other sources given in the bibliographical remarks on the first page for getting a deeper understanding of the fuelling mechanisms.

The method of Lyapunov exponents is a powerful tool that, if accompanied by a bit of more theory, provides us with a way to existence of an exponential attractor as well as with bounds of its fractal dimension. Its core lies in a relatively simple observation about compact linear operators on Hilbert spaces. Such mappings image the unit ball onto an ellipsoid with semiaxes shrinking to zero and such that the preimages of its semiaxes are orthonormal. Hence we are able to deduce two things. Firstly, n-dimensional cubes will be mapped on sets with well-known upper bounds of their n-dimensional volume. Intuitively it is quite clear that the volume cannot be greater than the product of the lengths of the n longest semiaxes of the ellipsoid. The fact is also mathematically provable, indeed. Secondly, an ellipsoid with semiaxes shrinking to zero can be covered with a well estimated number of balls, whose diamater is uniformly bounded from below by an arbitrarily small number. Therefore, so can be the image of the unit ball under the mapping in question.

Now, if we have a uniformly Fréchet differentiable function on a subset of a Hilbert space, then it behaves locally almost like a compact linear operator. In this way one is capable of reaching the same two observations locally, which is enough due to the uniform nature of differentiability we have. Finally, let the function be further generalized into a semigroup, not unlike the dynamical system of ours. Then also the corresponding mappings assuring differentiability are in effect time-dependent. If it happens to be proven that they image cubes from certain dimension higher on onto sufficiently quickly dwindling sets, we could expect the discussed semigroup assumes over time only a limitedly dimensional subset of its original domain. It really is the case and the quality of quick dwindling is checked by a direct generalization of Liouville's formula that likewise, although precisely, expresses time evolution of volume.

Having absorbed this cursory tutorial whose full-fledged content constitutes section 2.5 of [6], we can restore the mathematical precision by citing a couple of key results from the same source.

The setting for the main theorem is as follows. We investigate an abstract evolutionary problem on a Hilbert space X:

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \mathcal{F}(u),$$
$$u(0) = u_0 \in W, \tag{4.2}$$

with W a bounded and closed subset of X. Let the corresponding solution semigroup $(\mathcal{S}(t), W)$ be well-defined, i.e. there is a global, unique solution for every $u_0 \in W$ that remains in W for all $t \geq 0$. In addition, suppose there exists $a \in (0, 1]$ such that for every $T > 0$ we can find constants $K_1, K_2 > 0$ satisfying

$$\|\mathcal{S}(t_1)u_0 - \mathcal{S}(t_2)v_0\|_X \leq K_1 \|u_0 - v_0\|_X + K_2 |t_1 - t_2|^a \tag{4.3}$$

for all $u_0, v_0 \in W$ and $t_1, t_2 \in [0, T]$.

Next, assume the first variation equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U} = \frac{\mathrm{d}}{\mathrm{d}u}\mathcal{F}(u)\mathcal{U},$$
$$\mathcal{U}(0) = \xi \in X,$$

gives for any u, a solution to the original problem, rise to a correctly defined semigroup $(\mathcal{L}_u(t), X)$. We will furthermore require $\mathcal{U}(t) \in Y$ for almost every t, where Y is a Banach space densely and continuously embedded into X, and $\frac{\mathrm{d}}{\mathrm{d}u}\mathcal{F}(u(t))\phi \in Y^*$ for any $\phi \in Y$ at almost every t.

Finally, we presuppose that $\mathcal{L}_u(t)$ is a uniform Fréchet derivative of $\mathcal{S}(t)$ on W for almost every $t > 0$.

With the setting established, it is time to state the theorem ([6], Lemma 2.8 and Theorem 2.9).

Theorem 4.4 *Let there exist numbers $A, B, \sigma, T > 0$ such that the inequality*

$$\frac{1}{t}\int_0^t \sup_{\{\phi_i\}} \sum_{i=1}^k \left\langle \frac{\mathrm{d}}{\mathrm{d}u}\mathcal{F}(u(s))\phi_i, \phi_i \right\rangle_{Y^*,Y} ds \leq B - Ak^\sigma \tag{4.4}$$

holds true for any $t \geq T$, $k \in N$ and any solution u of (4.2). The supremum is taken over all families $\{\phi_i\} \subset Y$ that are orthonormal in X. Let $m \in N$ be such that

$$m \geq \left(\frac{2B}{A}\right)^{1/\sigma}.$$

Then the dynamical system $(\mathcal{S}(t), W)$ admits an exponential attractor $\mathcal{M}$ satisfying

$$\dim_f^X \mathcal{M} \leq \frac{3m+1}{a}.$$

The rate of attraction γ (see Definition 3.5 (iii)) is bounded from below as follows:

$$\gamma \geq \frac{\log 48}{\max\left\{T, \frac{m}{B}\log 192\right\}}.$$

Note 4.5 As it might not be apparent at the very first sight, notice that quantities m and γ are related proportionately, that is to say, the smaller m we take, the smaller the lower bound on γ is. In other words, an attractor with a lower dimension is expected to attract solutions at a slower rate and vice versa.

4.3 Application to our model

This section will be devoted to verifying that our model (2.1) meets the assumptions of Theorem 4.4. Despite the theorem's relation only to the dynamical system $(\mathcal{S}(t), \mathcal{B})$, we may extend the result to $(\mathcal{S}(t), H)$ owing to Note 3.7.

Mind that the model satisfies the setting completely with $\mathcal{B} = W \subset X = H$ and $Y = V^3$. Condition (4.3) is guaranteed by Corollary 3.8. Some effort will have to be made to show (4.4).

We only reiterate that we are not interested in precise bounds on the attractor dimension here, i.e. A, B and σ, in terms of equation data (Ω, ν_0, r, etc.). After all, it is not even possible due to our universal c_i constants. What will matter, nevertheless, is the fact that the bounds will not depend on ε, thence the dimension of our model's attractor is not affected in the process $\varepsilon \to 0_+$. Indeed, it does not imply the attractor keeps its dimension even for $\varepsilon = 0$.

Now then, let us move on to the proof of (4.4).

Proof. Recalling (3.3), for any $\{\boldsymbol{v}_i\}_{i=1}^k \in V^3$, $\{\boldsymbol{v}_i\}_{i=1}^k$ orthonormal in H, we express

$$\frac{1}{t}\int_0^t \sum_{i=1}^k \left\langle \frac{\mathrm{d}}{\mathrm{d}u}\mathcal{F}(u(s))\boldsymbol{v}_i, \boldsymbol{v}_i \right\rangle_{(V^3)^*,V^3} ds$$
$$= \frac{1}{t}\int_0^t \sum_{i=1}^k \Big(-\varepsilon |||\boldsymbol{v}_i|||^2 - (\partial_{\mathbb{D}}\mathbb{S}(\mathbb{D}\boldsymbol{u})\mathbb{D}\boldsymbol{v}_i, \mathbb{D}\boldsymbol{v}_i) + (\boldsymbol{u}\otimes\boldsymbol{v}_i + \boldsymbol{v}_i\otimes\boldsymbol{u}, \nabla_x \boldsymbol{v}_i) \Big)\, ds.$$

We have to estimate the terms under the sum:

- From (2.8) we traditionally observe
$$(\boldsymbol{v}_i \otimes \boldsymbol{u}, \nabla_x \boldsymbol{v}_i) = 0. \tag{4.5}$$
- In like fashion, we have already seen usage of Korn's inequality and (3.4) to deduce
$$-\sum_{i=1}^k (\partial_{\mathbb{D}}\mathbb{S}(\mathbb{D}\boldsymbol{u})\mathbb{D}\boldsymbol{v}_i, \mathbb{D}\boldsymbol{v}_i) \le -\sum_{i=1}^k c_0 \|\nabla_x \boldsymbol{v}_i\|_2^2 . \tag{4.6}$$
- The term $(\boldsymbol{u}\otimes\boldsymbol{v}_i, \nabla_x\boldsymbol{v}_i)$ will utilize usual interpolation, Hölder's, Poincaré's and Young's inequality and $\|\boldsymbol{v}_i\|_2 = 1$:
$$\begin{aligned}\sum_{i=1}^k (\boldsymbol{u}\otimes\boldsymbol{v}_i, \nabla_x\boldsymbol{v}_i) &\le \|\boldsymbol{u}\|_{15} \sum_{i=1}^k \|\boldsymbol{v}_i\|_{\frac{30}{13}} \|\nabla_x \boldsymbol{v}_i\|_2 \\ &\le c_1 \|\nabla_x \boldsymbol{u}\|_{\frac{5}{2}} \sum_{i=1}^k \|\nabla_x \boldsymbol{v}_i\|_2^{6/5} \\ &\le c_1 \|\nabla_x \boldsymbol{u}\|_{\frac{5}{2}} k^{2/5} \left(\sum_{i=1}^k \|\nabla_x \boldsymbol{v}_i\|_2^2 \right)^{3/5} \\ &\le c_2 \|\nabla_x \boldsymbol{u}\|_{\frac{5}{2}}^{5/2} k + \frac{c_0}{2} \sum_{i=1}^k \|\nabla_x \boldsymbol{v}_i\|_2^2 .\end{aligned} \tag{4.7}$$

Recalling section 4.1 and the entities introduced therein, namely$\{\boldsymbol{m}_j\}$ and $\{\kappa_j\}$, for any $i \in \{1, \dots, k\}$ we have

$$\|\boldsymbol{v}_i\|_2^2 = \sum_{j=1}^{\infty} (\boldsymbol{v}_i, \boldsymbol{m}_j)^2,$$

$$\|\nabla_x \boldsymbol{v}_i\|_2^2 = \sum_{j=1}^{\infty} (\nabla_x \boldsymbol{v}_i, \kappa_j^{-1/2} \nabla_x \boldsymbol{m}_j)^2 = \sum_{j=1}^{\infty} \kappa_j (\boldsymbol{v}_i, \boldsymbol{m}_j)^2.$$

Since $\{\kappa_j\}$ is a non-decreasing sequence and $\{(\boldsymbol{v}_i, \boldsymbol{m}_j)\}_{j=1}^{\infty}$, $i \in \{1, \dots, k\}$, constitute an orthonormal set in $l^2(\mathbb{R})$, we invoke algebraic Lemma 6.4 to further infer

$$\sum_{i=1}^{k} \|\nabla_x \boldsymbol{v}_i\|_2^2 = \sum_{i=1}^{k} \sum_{j=1}^{\infty} \kappa_j (\boldsymbol{v}_i, \boldsymbol{m}_j)^2 \geq \sum_{i=1}^{k} \kappa_i. \tag{4.8}$$

Next we apply $\kappa_1 \geq 1$ (see Note 4.2) and Lemma 4.3 to finalize this estimate as

$$\sum_{i=1}^{k} \kappa_i \geq \kappa_1^k \sum_{i=1}^{k} \frac{\kappa_i}{\kappa_1} \geq c_3 \sum_{i=1}^{k} i^{2/3} \geq c_4 k^{5/3}. \tag{4.9}$$

The last inequality is due to the integral approximation $\sum_{i=1}^{k} i^{2/3} \geq \int_0^k s^{2/3}\, ds$.

Combining (4.5)–(4.9) and invoking Theorem 3.6 (i) yields

$$\begin{aligned} \frac{1}{t} \int_0^t \sum_{i=1}^{k} \left\langle \frac{\mathrm{d}}{\mathrm{d}u} \mathcal{F}(u(s)) \boldsymbol{v}_i, \boldsymbol{v}_i \right\rangle_{(V^3)^*, V^3} ds &\leq \frac{1}{t} \int_0^t c_2 \, \|\nabla_x \boldsymbol{u}\|_{\frac{5}{2}}^{5/2} k - c_5 k^{5/3} \, ds \\ &\leq c_6 k - c_5 k^{5/3} \\ &\leq B - A k^{5/3}, \end{aligned} \tag{4.10}$$

which was to be proven. ■

Remark 4.6 With the primary task finished, we may yet try deducing a more explicit form of A and B in terms of the viscosities ν_0 and ν_1 to corroborate the strength of Theorem 4.4. We only remind that these two numbers are usually very small and so their inverse values may be not quite amiable.

First, remembering the proof of Theorem 3.6, the diameter of $\mathcal{B}$ was of magnitude $\sim \nu_0^{-1}$, where the symbol "$\sim$" purifies estimates from other data save ν_0 and ν_1. Integrating the inequality (3.1) would reveal

$$\frac{1}{t} \int_0^t \|\nabla_x \boldsymbol{u}(s)\|_r^r \, ds \sim \nu_0^{-1} \nu_1^{-1}$$

for t large enough, with the right-hand side unaffected by t. Now, the estimate (4.6) could replace c_3 with ν_0, which would make c_5 from (4.7) $\sim \nu_0^{-1}$. Finally, inserting all into (4.10), the integral term is $\sim \nu_0^{-2} \nu_1^{-1}$ and an easy computation divulges an option $B \sim \left(\nu_0^2 \nu_1\right)^{-5/2}$, which is naturally brought about by $k^{5/3}$, and A unaffected by viscosities altogether. All in all, the best estimates of dimension are $\sim \left(\nu_0^2 \nu_1\right)^{-1/6}$.

Do not be deceived by an apparent absence of reliance on r. The worst possibility we used for the estimates, i.e. $r = 5/2$, projected itself into $\sigma = 5/3$ and as such into the bound of the fractal dimension as well.

5. Epilogue

And so it came to pass that we managed to assure conditions for invoking the principal Theorem 4.4. Even in spite of doing so without any dependency on the omnipresent ε, some work has yet definitely to be done to show that the primordial model (i.e. case $\varepsilon = 0$) enjoys an exponential attractor with an equally bounded fractal dimension. A deterring example, suggesting that something could conceivably go wrong, is a sequence of dilating lines in a plane. These obviously keep their fractal dimension bounded but the limit set is not compact all the same.

A different matter is that the primordial model does possess an exponential attractor (see [6], chapter 7), albeit its fractal dimension can be bounded only via less efficient (though not less elaborate) manners such as the method of trajectories. The fact of superior nature of the principle we used was taken wickedly for granted (with the exception of Remark 4.6). It took place in a complete ignorance of bounding constants c_i. We entirely dismissed their precise form in terms of data hidden therein for the sake of a maximally lucid presentation.

Should we want to keep a complete track of the constants, and consequently the fractal dimension and the rate of attraction of the exponential attractor, we would have naturally also wished for the best estimates in every step one could imagine. This way it would transpire that not all of our estimates (if any) were the most efficient available. Cases in point are the oft-used consideration of only the worst case $r = 5/2$ or the chain of inequalities (4.7) which might be pushed to a far higher level quantitatively by means of what is known as Lieb-Thirring inequality ([6], Theorem 9.14). All these isolated instances to explain would be to dive deeper than we can go.

We have witnessed that differentiability was the only point where the original model failed to keep up with the perturbed model in pursuit of the method of Lyapunov exponents. On the other hand, in that aspect it failed utterly and incorrigibly. Not only were we unable to prove that a certain mapping was a uniform Fréchet differential for the solution semigroup of the limit problem, but we did not even have an appropriate candidate for a derivative since the first variation equation became obstinate to beget any solution at all. The key question is, nonetheless, how harmful such a property is when investigating an attractor and if there is any reason why a hardly differentiable solution semigroup should not possess an attractor closely similar to those of its smooth relatives.

This work should pave the way for more serious attempts to use the higher level perturbations for deduction of the attractor dimension bounds for the original Ladyzhenskaya model. In fact, that should have formed the heart of this work and what has been really proved ought to have been only a derivative, yet indispensable, extra. The coveted aim has not been reached up to the present moment, although the work is underway, daunting the task as it might appear.

6. Appendix

This chapter contains several unrelated results whose inclusion elsewhere would result in a disruption of the continuous narrative.

6.1 Varied results

Theorem 6.1 (**Korn's inequality**, [9], Theorem 1.10)
*Let $1 < p < \infty$, $d \geq 2$ and $\Omega \in \mathcal{C}^{0,1}(\mathbb{R}^d)$. Then there is a constant K (a so-called **Korn constant** of Ω) such that for any $\boldsymbol{u} \in W_0^{1,p}(\Omega, \mathbb{R}^d)$ the inequality*

$$\|\boldsymbol{u}\|_{1,p} \leq K \|\mathbb{D}\boldsymbol{u}\|_p$$

is fulfilled.

Theorem 6.2 (**A weak version of dominated convergence theorem**, [10], Lemma 8.3.)

Let $d_1, d_2 \in \mathbb{N}$, $\mathcal{O} \in \mathbb{R}^{d_1}$ bounded, $1 < p < \infty$ and $z_n, z \in L^p(\mathcal{O}, \mathbb{R}^{d_2})$ such that $z_n \to z$ a.e. in $\mathcal{O}$ and $\sup_n \|z_n\|_p < \infty$. Then $z_n \to z$ weakly in $L^p(\mathcal{O}, \mathbb{R}^{d_2})$.

Proof. Fix $k \in \mathbb{N}$ and define $E_k = \{x \in \mathcal{O};\ |z_n(x) - z(x)| \leq 1 \text{ for all } n \geq k\}$. From the assumption on convergence, we have $E_k \nearrow \mathcal{O}$.

Next define $A_k = \{\varphi \in L^{p'}(\mathcal{O}, \mathbb{R}^{d_2});\ \operatorname{supp} \varphi \subset E_k\}$ and $\mathcal{A} = \bigcup_k A_k$. Then $\mathcal{A}$ is dense in $L^{p'}(\mathcal{O}, \mathbb{R}^{d_2})$ and for any $\varphi \in \mathcal{A}$ we observe $\int_{\mathcal{O}} (z_n - z)\cdot\varphi \, dx \to 0$, $n \to \infty$. Indeed so, $\varphi \in A_K$ and for $n \geq K$ holds $|(z_n - z)\cdot\varphi| \leq |\varphi|$ by definition, wherefore we may apply the classical dominated convergence theorem. For an arbitrary function from $L^{p'}(\mathcal{O}, \mathbb{R}^{d_2})$, the result is obtained by approximation and $\sup_n \|z_n\|_p < \infty$. ■

6.2 Algebraic lemmas

Lemma 6.3 *Let $m \in \mathbb{N}$ and $\{a_1, \dots, a_m\}$ be an orthonormal set in $l^2(\mathbb{R})$, $a_i = \{a_i^j\}$. Then for any $j \in \mathbb{N}$ the inequality $\sum_{i=1}^m |a_i^j|^2 \leq 1$ holds true.*

Proof. Without loss of generality assume $j = 1$. If we take an arbitrary $n \in \mathbb{N}$ and $\{b_{ij}\}_{i,j=1}^n = B \in \mathbb{R}^{n \times n}$ an orthonormal matrix, then right from the definition it follows $\sum_{i=1}^k |b_{i1}|^2 \leq 1$ for any $k \leq n$. This observation would serve as a proof in case each element of $\{a_1, \dots, a_m\}$ contained only finitely many non-zero elements.

Otherwise, the above argument will provide an inspiration. Proceeding by contradiction, suppose $\sum_{i=1}^m |a_i^1|^2 = \alpha > 1$. Take $\epsilon > 0$ small enough to ensure $1 > \epsilon\, m$. From continuity of the scalar product in $l^2(\mathbb{R})$, there exists $n \in \mathbb{N}$ such that for any $a_i, a_{i_1}, a_{i_2} \in \{a_1, \dots, a_m\}$, $a_{i_1} \neq a_{i_2}$, we have

$$\left| \sum_{j=1}^N a_{i_1}^j a_{i_2}^j \right| < \epsilon \qquad \text{and} \qquad 1 - \sum_{j=1}^N |a_i^j|^2 < \epsilon \tag{6.1}$$

for every $N \geq n$. Assuming at the same time n is large so much so that vectors

$(a_i^1, \ldots, a_i^n)$, $1 \le i \le m$, form a linearly independent set in $\mathbb{R}^n$, we define a matrix

$$A = \begin{pmatrix} a_1^1 & a_1^2 & a_1^3 & \ldots & a_1^n \\ a_2^1 & a_2^2 & a_2^3 & \ldots & a_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_m^1 & a_m^2 & a_m^3 & \ldots & a_m^n \\ b_1^1 & b_1^2 & b_1^3 & \ldots & b_1^n \\ b_2^1 & b_2^2 & b_2^3 & \ldots & b_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-m}^1 & b_{n-m}^2 & b_{n-m}^3 & \ldots & b_{n-m}^n \end{pmatrix} = \begin{pmatrix} \widetilde{a}_1 \\ \widetilde{a}_2 \\ \vdots \\ \widetilde{a}_m \\ \widetilde{b}_1 \\ \widetilde{b}_2 \\ \vdots \\ \widetilde{b}_{n-m} \end{pmatrix}$$

where $\{\widetilde{b}_1, \ldots, \widetilde{b}_{n-m}\}$ is an orthonormal basis of $\{\widetilde{a}_1, \ldots, \widetilde{a}_m\}^\perp$ in $\mathbb{R}^n$. Note that A is regular and, hence, so is A^T. An adventurous computation yields

$$AA^T = \begin{pmatrix} \tau_1 & \epsilon_{12} & \ldots & \epsilon_{1m} & 0 & \ldots & 0 \\ \epsilon_{21} & \tau_2 & \ldots & \epsilon_{2m} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \epsilon_{m1} & \epsilon_{m2} & \ldots & \tau_m & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix}$$

where $|1 - \tau_i| < \epsilon$ and $|\epsilon_{ij}| < \epsilon$ for all $1 \le i, j \le m$ according to (6.1). By this reason, denoting $Z = AA^T - I$ and considering the usual operator norm for matrices $n \times n$, one infers $\|Z\| < \epsilon\, m$. The symbol I stands here for the $n \times n$ identity matrix. Regularity of A^T further implies $A^T Z (A^T)^{-1} = A^T A - I$ and we will try to bound the left-hand side in terms of ϵ. We observe

$$\left\|A^T x\right\|^2 = x^T A A^T x = x^T (Z + I) x \begin{cases} \le (1 + \epsilon\, m) \left\|x\right\|^2 \\ \ge (1 - \epsilon\, m) \left\|x\right\|^2 \end{cases}$$

and

$$\left\|A^T Z (A^T)^{-1}\right\| \le \epsilon\, m \frac{1 + \epsilon\, m}{1 - \epsilon\, m}$$

accordingly. Inasmuch as $\sum_{i=1}^m |a_i^1|^2 \le (A^T A)_{11}$, where $(A^T A)_{11}$ is the first row/column element of $A^T A$, we reach a contradiction setting ϵ so small that

$$\epsilon\, m \frac{1 + \epsilon\, m}{1 - \epsilon\, m} < \alpha - 1.$$

Notice that we never needed the value of the dimension n, so its possible increase does not trouble us. ■

Lemma 6.4 *Let $m \in \mathbb{N}$, $\{a_1, \ldots, a_m\}$ be an orthonormal set in $l^2(\mathbb{R})$ and $\{\gamma_j\}$ be a non-decreasing sequence of real numbers. Then, with the notation $a_i = \{a_i^j\}$, we have*

$$\gamma_1 + \ldots + \gamma_m \le \sum_{j=1}^{\infty} \gamma_j \left(|a_1^j|^2 + \ldots + |a_m^j|^2 \right).$$

Proof. Elementary estimations yield:

$$\begin{aligned}
&\sum_{j=1}^{\infty} \gamma_j \left(|a_1^j|^2 + \ldots + |a_m^j|^2\right) \\
&\quad \geq \sum_{j=1}^{m-1} \gamma_j \left(|a_1^j|^2 + \ldots + |a_m^j|^2\right) + \gamma_m \left(\left(1 - \sum_{j=1}^{m-1} |a_1^j|^2\right) + \ldots + \left(1 - \sum_{j=1}^{m-1} |a_m^j|^2\right)\right) \\
&\quad = m\gamma_m + (\gamma_1 - \gamma_m) \sum_{i=1}^{m} |a_i^1|^2 + \ldots + (\gamma_{m-1} - \gamma_m) \sum_{i=1}^{m} |a_i^{m-1}|^2 \\
&\quad \geq m\gamma_m + (\gamma_1 - \gamma_m) + \ldots + (\gamma_{m-1} - \gamma_m) = \gamma_1 + \ldots + \gamma_m,
\end{aligned}$$

where Lemma 6.3 was utilized in the last inequality. ■

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