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Algebraická a kripkovská sémantika substrukturálních logik

Algebraic and Kripke semantics of substructural logics

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Abstract

This thesis is about the distributive full Lambek calculus, i.e., intuicionistic logic without the structural rules of exchange, contraction and weakening and particularly about the two semantics of this logic, one of which is algebraic, the other one is a Kripke semantic. The two semantics are treated in separate chapters and some results about them are shown, for example the disjunction property is proven by amalgamation of Kripke models. The core of this thesis is nevertheless the relation of these two semantics, since it is interesting to study what do they have in common and how can they actually differ, both being a semantics of the same logic. We show how to translate frames to algebras and algebras to frames, and, moreover, we extend such translation to morphisms, thus constructing two functors between the two categories.

Key words:distributive FL logic, distributive full Lambek calculus, structural rules, distributive residuated lattice, Kripke frames, frame morphisms, category, functor

Abstrakt

Tato práce se zabývá distributivním Lambekovým kalkulem, tedy intuicionistickou logikou bez pravidel záměny, kontrakce a oslabení, a především dvěma různými sémantikami této logiky, totiž sémantikou algebraickou a kripkovskou. Tyto dvě sémantiky jsou nejdříve pojednávány ve zvláštních kapitolách a jsou prezentovány některé výsledky, které se jich týkají, např. se ukáže vlastnostnost disjunkce pomocí sloučení dvou Kripkovských modelů. Jádrem práce je nicméně především vztah těchto dvou sémantik, protože je zajímavé porovnávat, co mají společného, a čím se vůbec mohou lišit, když jsou obě sémantikami téže logiky. Bude ukázán překlad kripkovských rámců na algebry a algeber na rámce a dále bude tento překlad rožšířen i na morfismy, čímž budou zkonstruovány dva funktory mezi oběma kategoriemi.

Klíčová slova:distributivní logika FL, distributivní Lambekův kalkul, strukturální pravidla, distributivní residuované svazy, kripkovské rámce, morfismy mezi rámci, kategorie, funktor

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Introduction

The aim of this thesis is to study the algebraic and Kripke semantics of substructural logics, mostly of the distributive FL logic (as opposed to the FL logic in general, in the logic we are about to study the distributivity between $\wedge$ and $\vee$ holds), which is the intuicionistic logic without the rules of exchange, contraction and weakening. The thesis begins by explaining the structural rules we are talking about and presenting a sequent variant of the distributive full Lambek calculus (originally, Lambek worked only with implication fragment, other connectives were added later, therefore the adjective full and the abbreviation FL). This calucus first apperared in [L] and a very comprehesive presentation can be found in [O]. This calculus won't be studied here very much and it is presented mainly because it might help to introduce the distributive FL logic more comprehensively and also to some degree to trace the historical development, because the syntactic view preceeded the semantic one. In this calculus the constant 0 will be introduced, but it won't be used neither in the chapter about the algebraic semantics, nor in the chapter about Kripke semantics, because it doesn't really play any role in the semantic topics this thesis is interested about. The connective of negation is also omitted in this treatise, but it might be interesting to study the possibilities of introducing it.

In the chapter 2 about algebraic semantics we will be working with distributive residuated lattices and present some results about them. The semantics based on these algebraic structures will be presented and some particular properties of the connectives will be proven from the algebraic point of view. It will be also shown in what algebras do the structural rules hold. The semantics and most of the results about it are from [alg], including some basic properties of residuated maps in general.

The chapter 3 about Kripke semantics (which will be presented in a form only slightly differing from the one to be found in [Res]) will work with the concepts of Kripke frame and Krike model and will present the definitions of validity of a formula or entailment between formulae. It will be also shown how to capture the equivalence between two frames. This will be done using the concept of bisimulation, which was a topic of my bachelor thesis([bac]). In that thesis, however, it was used for different logics, but it is straightforward to use in the present case as well. A bisimulation relation can be used to show that some structural properties of Kripke frames cannot be defined by a formula, but I am not examining this matter in the current thesis. The structural rules will be treated from the point of view of this semantic and particularly its relation to the more simple Kripke semantics of intuicionistic logic. This should clarify why does the Kripke semantic of distributive FL logic work with the ternary relation R and with the truth set T. Particular attention is payed to the disjunction property of the distributive FL logic, which is proven using the Kripke semantics (an algebraic proof can be found in [dis]), after recalling the well known proof for the case of intuicionistic logic.

The core of this thesis will be a comparison of the two semantics from the point of view of the category theory (most of the necessary categorical background is drawn from [cat]). A pair of functors will be presented, the first of which assigns to each Kripke frame a distributive residuated lattice and to each frame morphism an algebraic morphism. The second functor will on the other hand assign to each distributive residuated lattice a Kripke frame and to each algebraic morphism a frame morphism. In order to obtain these functors, some more algebraic theory about filters and ideals will be needed, which I took mostly from [alg2] (though I am generalizing some of the results, particularly the theorem about extending a filter into a prime filter) and two important lemmas are taken from [art].

The conclusion of the thesis is open as it presents a way how a categorical adjunctions between the two categories might be obtained. Yet it remains unsure, whether there is an adjunction between the two categories we work with. A half of what might be a proof is presented with the other one missing.

Chapter 1

Substructural logics

1.1 What makes substructural logics substructural?

Since the topic of this thesis will be the comparison of algebraic and Kripke semantics of substructural logics, we shall begin by introducing substructural logics themselves. The characteristic features of those is that they as opposed to classical or intuicionistic logics do not pressupose the so called structural rules. Various rules can be called structural, but there are three essential ones that are the most considered, exchange, contraction and weakening. They all specify some changes which can be applied to a structure of a collection of formulae which we use as assumptions (a very good explanation of how does the omission of structural rules lead to the development of the logic we will work with can be found in [O]).

So the exchange rule allows to shift the order of the formulae we use as assumtion.This can be formalised in a Gentzen calculus as follows

$$\frac{\Gamma,\, \varphi,\, \psi,\, \Sigma \Rightarrow \chi}{\Gamma,\, \psi,\, \varphi,\, \Sigma \Rightarrow \chi}\, E$$

The contraction rule allows to contract a multiple use of a single formula, which may be formalised as follows

$$\frac{\Gamma,\, \varphi,\, \varphi\ \Sigma \Rightarrow \chi}{\Gamma,\, \varphi,\, \Sigma \Rightarrow \chi}\, C$$

Finally the weaking rule allows to add a new formula to the collection of assumptions we use for the inference and so to weaken the proof

$$\frac{\Gamma,\, \varphi,\, \Sigma \Rightarrow \chi}{\Gamma,\, \varphi,\, \psi,\, \Sigma \Rightarrow \chi}\, W$$

All the tree rules can also be introduced in a Hilbertian manner by assuming the following axiomatic schemes for all formulae. For exchange we have

$$\varphi \& \psi \vdash \psi \& \varphi$$

The exchange rule does in fact say that the connective & is commutative. Now the contraction rule can be stated as:

$$\varphi \vdash \varphi \& \varphi$$

Finally the Hilbertian statement of the weakening rule has the following form:

$$\varphi \& \psi \vdash \varphi$$

as well as

$$\varphi \& \psi \vdash \psi$$

Actually, these are two different rules, one might be called a left weakening, while the second one a right weakening. Clearly enough, with the exchange rule both the weakening rules would be equivalent, since we could interchange the order of φ and ψ. It is also notable that the Hilbertian variant interchanges the formula which appears in the proof first with the second one, as opposed to the Gentzen form we are using here.

Now, there are various other structural rules and substructural logics based on ommiting them and it is hard to say which logic is basic among them. Our logic, the distributive FL logic, is a logic which can be seen as an intuicionistic logic without the mentioned structural rules of exchange, contraction and weakening. This logic has got the following language

$$\{\wedge, \vee, \&, \rightarrow, \leftarrow, 1, 0, \top, \bot\},$$

where the last four symbols are constants and the other ones are all binary.

The two implications and the two types of conjuntion are due to the omission of the three structural rules. The new conjunction of the form & can be understood as a connective which unites the formulae in the antecedent of a sequent into one single formula as the classical conjunction in the classical and intuicionnistic logics does. Omitting the structural rules is also the reason for having two kinds of truth constants.

Formula in this logic is defined in a classical inductive manner, so we start by fixing a countable set of propositional atoms At and then every propositional atom and each of the four propositional constants is a formula and when two formulae are connected by one of the connectives, then they form a formula, too.

1.2 The distributive Full Lambek calculus

Though this thesis will be treating rather semantic than syntax, we shall nevertheless introduce the distributive full Lambek calculus, since both the semantics are equivalent in respect to it (the calculus is complete and sound with respect to both the semantics). Here I am using the definitions from [O], where much more about the calculus can be found.

This logic has got all the connectives (but we won't introduce negation here) and constants, which is why it is called Full Lambek, since it was first developed only as an implication fragment. It is notable that implication has motivated the development of many non classical logics, since its classical treatment had been felt as problematic by many logicians and philosophers.

There are various equivalent calculi, which capture the Full Lambek logic. The one introduced here is a sequent calculus with a few peculiar properties differing from the variant for classical logic.

In classical logics the sequent of the form

$$\Gamma \Rightarrow \Delta$$

is typically interpreted as a pair of two finite sets of formulae, the first one of which could be actually taken for a big conjunction of all the formulae contained, while the second of one as their disjunction. Here instead the the second part of the pair is allowed to contain maximally one formula (the calculus which is introduced here is one with single conclusion sequents), so we should write either

$$\Gamma \Rightarrow \varphi$$

for a sequent with one formula on its right or

$$\Gamma \Rightarrow$$

in case the right side is empty.

What regards the first part of a sequent, this time it is not a finite set, but a finite sequence, which means two differences. First, it matters, what the order of the formulae is, so it not allowed to shift them. Thus the calculus omits the exchange rule. The other difference is that also matters how many times each formula is included, so when for example in one sequent the formula φ is included once and in another one twice, then these are not the same two sequents. This makes for the refusal of the contraction rule.

It should be noted that the calculus which will be presented now is using the cut rule, though it is equivalent to a cut free calculus, which is nevertheless more complicated. This cut free calculus can be found in [K].

Now we are about to see the rules for introducing the connectives. The classical connectives behave in a similar manner as in the classical or intuicionistic logics. There are two rules for each connective, depending on what side of the sequent are we introducing them.

Capital Greek letters will be used to denote sequences of formulae, while single formulae will be denoted by Greek lower case. To begin, we present the right rule for conjunction:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi} \Rightarrow \wedge$$

The left rule then is:

$$\frac{\Gamma, \varphi, \Sigma \Rightarrow \chi}{\Gamma, \varphi \wedge \psi, \Sigma \Rightarrow \chi} \Leftarrow \wedge \qquad \qquad \frac{\Gamma, \psi, \Sigma \Rightarrow \chi}{\Gamma, \varphi \wedge \psi, \Sigma \Rightarrow \chi} \Rightarrow \wedge$$

This means, that in both cases we can introduce $\wedge$ on the left side of the sequent.

Just as little surpring are the rules for disjunction. So the right is:

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \Rightarrow \vee \qquad \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \Rightarrow \vee$$

And the left one:

$$\frac{\Gamma, \varphi, \Sigma \Rightarrow \chi \qquad \Gamma, \psi, \Sigma \Rightarrow \chi}{\Gamma, \varphi \vee \psi, \Sigma \Rightarrow \chi} \Leftarrow \vee$$

Now we are going to see the less familiar connectives, which this calculus features. We begin with the two implications. The one of the classical form $\rightarrow$ has this left rule:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Pi, \psi, \Sigma \Rightarrow \chi}{\Pi, \Gamma, \varphi \rightarrow \psi, \Sigma \Rightarrow \chi} \Leftarrow \rightarrow$$

The right rule, then, is such:

$$\frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi} \Rightarrow \rightarrow$$

The other implication, written as $\leftarrow$, has got the following rules. The right one

$$\frac{\Gamma \Rightarrow \varphi \qquad \Pi, \psi, \Sigma \Rightarrow \chi}{\Pi, \psi \leftarrow \varphi, \Gamma, \Sigma \Rightarrow \chi} \Rightarrow \leftarrow$$

Though it is very similar to the rule for $\rightarrow$, it should be noticed that the sequence Γ does stand on different position in the bottom sequent. The left rules is the following one:

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \psi \leftarrow \varphi} \Leftarrow \leftarrow$$

The last connective to be treated is &, which might be called a strong conjunction (while $\wedge$, on the other hand, might be called a weak conjunction). The right rule is this one:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma \Rightarrow \psi}{\Gamma, \Sigma \Rightarrow \varphi \,\&\, \psi} \Rightarrow \&$$

The left one is then:

$$\frac{\Gamma, \varphi, \psi, \Delta \Rightarrow \chi}{\Gamma, \varphi \,\&\, \psi, \Delta \Rightarrow \chi} \Leftarrow \&$$

This time it is the upper sequent, which shifts the order of formulae, as opposed to the rule for $\rightarrow$.

There is one rule which is not about introducing a connective, but about introducing a constant. It is a left rule for the constant 1 and has gor the following form:

$$\frac{\Gamma,\ \Delta \Rightarrow\ \varphi}{\Gamma,\ 1,\ \Delta \Rightarrow\ \varphi} \Leftarrow 1$$

To be able to present what is a proof of a sequent in this calculus, we have to present the axioms, too. Together with the rules just presented, they will form the desired definition. First we present an axiomatic scheme, which is particularly prominent in the proofs of this calculus:

$$\varphi \Rightarrow \varphi$$

The other scheme is the one which actually makes this calculus a distributive FL calculus, as opposed to FL calculus:

$$\varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

Then there are four axioms (the first two are actually axiomatic schemes again), which specify how do the truth constants work. First we specify the stronger ones. So for the constant $\top$:

$$\Gamma \Rightarrow\ \top$$

Then for $\bot$:

$$\bot \Rightarrow\ \varphi$$

Then there are the two, in a sense, weaker constants. So for 1:

$$\Rightarrow 1$$

And finally for 0:

$$0 \Rightarrow$$

Now we can finish the presentation of the calculus by defining what a proof is.

Definition 1.2.1. *A proof of a sequent of the form $\Gamma \Rightarrow \varphi$ in the Fl sequent calculus is a tree of sequents, with $\Gamma \Rightarrow \varphi$ as its root. All the leafs are axioms of the calculus and every node of tree which is not a leaf is derived from the one preceding it by one of the rules of the calculus. We say that the formula φ is provable in the calculus if there is a proof of the sequent $\Rightarrow \varphi$.*

Thus we have seen the basic substructural logic FL from the proof theoretic point of view. But since it is semantics we are concerned with here, let's see how the two alternative approaches, namely the algebraic and the Kripke one, look like. Both these semantics are equivalent to the calculus of the distributive FL logic (which is an extension of the calculus just presented), which means that exactly those formulae, which can be proved in this calculus, are valid in the two semantics. Thus those semantics can describe the logics, too, and it is natural to ask, how else can they be related to each other (particularly from the point of view of the categoty theory) and how much can they differ from each other, having so much in common. The completeness, however, won't be proved in this thesis, only stated.

It should be noted that the symbols & and $\vee$ will be used with two different meanings in this thesis. They will either denote the connective of the distributive FL logic (which means that these symbols will be used as a part of the calculus and of the two semantics we shall see) or they will be used as metaconnectives. No confusion should occur, but to clarify this difference, extra free spaces will be used when these symbols denote the metaconnectives. The same holds for $\vee$. In addition to that, we will sometimes have to use a metaimplication, which will be denoted as $\Rightarrow$ or sometimes as $\Leftarrow$ (as opposed to $\rightarrow$ and $\leftarrow$). Metaequivalece will be denoted as $\iff$, metanegation as $\neg$.

Chapter 2

Algebraic semantics

As there are various substructural logics based on ommiting various substructural rules, we will present only a hint of all the richness. The algebras we will study and which can semanthically capture the distributive FL logic (and some other substructural logics) are the so called distributive residuated lattices and even here we will present only a very humble introduction. The definitions are mostly inspired by [alg]. A few steps will be needed to define such algebras.

The language of such algebras will correspond to one of the substructural logics mentioned in the above section. Thus it will contain the following functional symbols

$$\wedge, \vee, *, \backslash, /, 1, \top, \bot$$

In the calculus we have presented one more constant, namely 0. But we won't use it here, because it is immaterial for the purposes of this thesis.

First we will need to specify the operators $\wedge$ and $\vee$. They will form a lattice.

Definition 2.0.2. *Algebra with the operators $\wedge$ and $\vee$ is lattice if it satisfies the following conditions*

(i) *both operators are commutative*

(ii) *both operators are associative*

(iii) $a \wedge (a \vee b) = a$

(iv) $a \vee (a \wedge b) = a$

We will be mostly using an alternative definition with an partial order $\leq$.

Definition 2.0.3. *A partially ordered set A ordered by $\leq$ is a lattice if the two following conditions hold*

(i) *for each pair of elements $a, b \in A$ there is the least upper bound, denoted as $a \vee b$*

(*ii*) *for each pair of elements* $a, b \in A$ *there is the greatest lower bound, denoted as* $a \wedge b$

Adding $\leq$ to the by

$$a \leq b \iff a = a \wedge b \iff b = a \vee b$$

the two definitions can be easily shown to be equivalent. It should be noted that sometimes it might be useful to write $b \geq a$ instead of $a \leq b$.

Now let's specify the behaviour of the connective * and of the constant 1. They will behave as in a monoid.

Definition 2.0.4. *An algebra the language of which contains the symbols * and 1 is called a monoid when it satisfies the following conditions*

(*i*) $\forall x \ (x * 1 = 1 * x = x)$

(*ii*) $\forall x \forall y \forall z \ ((x * y) * z = x * (y * z))$

Having introduced introduced monoids and lattices we are prepared to define residuated lattices by combining the two former concepts and adding a property which will make those lattices residuated.

Definition 2.0.5. *An algebra* $\mathbb{A}$ *the language of which contains the symbols* $\wedge$, $\vee$, $*$, $\backslash$, $/$, $\top$, $\bot$ *and* 1 *is called a distributive residuated lattice if satisfies the following conditions:*

(*i*) $(P, \wedge, \vee)$ *is a lattice*

(*ii*) $(P, *, 1)$ *is a monoid*

(*iii*) $\forall x, \forall y, \forall z \ (y \leq x \backslash z \iff x * y \leq z \iff x \leq y/z)$

(*iv*) *for each three elements* a, b *and* $c \in A$ *we have*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

(*v*) *for each three elements* a, b *and* $c \in A$ *we have*

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

(*vi*) $\top$ *is the greatest element, which means that for all* $a \in \mathbb{A}$

$$a \leq \top$$

(*vii*) $\bot$ *is the least element, which means that for all* $a \in \mathbb{A}$

$$\bot \leq a$$

The condition (iii) is what makes the lattice residuated. We can say that $/$ and $\backslash$ are residuums of $*$. So there are two residuated pairs of maps, the first pair is $f = - * y$ (so a unary map which for a given x assigns $f(x) = x * y$) and $g = y/-$ (which to a given z assigns $g(z) = y/z$), the second pair is $h = x * -$ (assigning $h(y) = x * y$ to each y) and $i = x\backslash-$ (assigning $i(z) = x\backslash z$ to each z). Though obsolete at first glance this condition causes the operation $*$ to have important properties such as monotonicity and consequently distributivity in respect to $\wedge$ and $\vee$, though in fact not a complete distributivity. We shall see the relatively straightforward proofs of these.

Theorem 2.0.6. *When two maps $f : A \longrightarrow B$ and $g : B \longrightarrow A$ form a residuated pair (which means that for any $a \in A$ and $b \in B$ we have $f(a) \leq b$ if and only if $a \leq g(b)$, where A and B are partially ordered sets) then the following holds:*

(i) *$g \circ f$ is expanding (that is to say that for every x we have $x \leq g(f(x))$)*

(ii) *$f \circ g$ is contracting (that is to say that for every x we have $f(g(x)) \leq x$)*

(iii) *both f and g are monotone (that is to say that for every pair x and y either from A or from B we have that if $x \leq y$, then $f(x) \leq f(y)$ or $g(x) \leq g(y)$)*

Proof. We can proof the first two items together adding the third one separately

(i, ii) Let's start with $f(x) \leq f(x)$ and $g(x) \leq g(x)$, respectively. Thus we get by the item (iii) in the definition 2.0.5 that $x \leq g(f(x))$ and $f(g(x)) \leq x$.

(iii) When we have $x \leq y$, then by the item (i) we also get $y \leq g(f(y))$ and thus also $x \leq g(f(y))$. Using (iii) of the definition 2.0.5 once again we get $f(x) \leq f(y)$. The proof of the monotonicity of the second map proceeds analogously.

□

In the sequel we will find the following fact about residuated pairs of maps useful as well.

Lemma 2.0.7. *When two maps $f : A \longrightarrow B$ and $g : B \longrightarrow A$ form a residuated pair, then the following relations between them hold*

(i) $g(b) = max\{a \in A;\ f(a) \leq b\}$

(ii) $f(a) = min\{b \in B;\ a \leq g(b)\}$

Proof. For (i) we have to show both that g(b) is an upper bound of the mentioned set of a's and that is its element. Obviously it is an upper bound, since $f(a) \leq b$ implies $a \leq g(b)$. It is an element of the set, because $f(g(b)) \leq b$ which follows from the contractivity of $f \circ g$. The proof of (ii) is entirely analogous. □

Having proven the monotonicity of * we can now also demostrate its distributivity with respect to $\wedge$ and $\vee$.

Lemma 2.0.8. *When two maps $f : A \longrightarrow B$ and $g : B \longrightarrow A$ form a residuated pair, then the following four facts hold:*

(i) $f(a \vee b) = f(a) \vee f(b)$

(ii) $f(a \wedge b) \leq f(a) \wedge f(b)$

(iii) $g(a \wedge b) = g(a) \wedge g(b)$

(iv) $g(a) \vee g(b) \leq g(a \vee b)$

Proof. (i) To obtain this item, note that $f(a \vee b)$ is obviously an upper bound of $f(a)$ and $f(b)$ by monotonicity of f. Let's have then any upper bound c of $f(a)$ and $f(b)$. By residuation we obtain both $a \leq g(c)$ and $b \leq g(c)$, so consequently $a \vee b \leq g(c)$. Using residuation once again we finally get $f(a \vee b) \leq c$.

(ii) Proving $f(a \wedge b) \leq f(a) \wedge f(b)$, consider that both $f(a)$ $f(b)$ have to be above $f(a \wedge b)$ by monotonicity. Since $f(a) \wedge f(b)$ is their greatest lower bound, $f(a \wedge b) \leq f(a) \wedge f(b)$ clearly holds.

(iii) Similarly to the first item, we know that $g(a \wedge b)$ is a lower bound of both $g(a)$ and $g(b)$ because g is monotone. Now, let's consider any lower bound c of $g(a)$ and $g(b)$. We know that $f(c) \leq a$ and $f(c) \leq b$ and so $f(c) \leq a \wedge b$, as well. Finally, we get $c \leq g(a \wedge b)$, which means that $g(a \wedge b)$ is the greatest lower bound of $g(a)$ and $g(b)$.

(iv) Obviously both $g(a) \leq g(a \vee b)$ and $(b) \leq g(a \vee b)$ hold because g is monotone. $g(a) \vee g(b)$ is the least upper bound of $g(a)$ and $g(b)$ and therefore $g(a) \vee g(b) \leq g(a \vee b)$. □

Now we were proving the last two lemmas and the theorem preceding them for any two maps which form a residuated pair, yet we are actually working with two pairs of residuated maps, as was already mentioned. What regards monotonicity we have actually proven that for the operation $*$ we have $a * b \leq x * y$ for $a \leq x$ and $b \leq y$. Similarly we have proven $a \backslash b \leq a \backslash c$ for $b \leq c$ and also $a/b \leq a/c$. Now we can consider the operations $f = -/z$(assigning $f(y) = y/z$ to each y) and $g = -\backslash z$(assigning $f(y) = y \backslash z$ to each y). We will now show that these operations are antitone, which will be useful for other proofs in following chapters.

Lemma 2.0.9. *Let there be a distributive lattice $\mathbb{A}$ and three of its elements $x, y, z \in \mathbb{A}$. Then the following two facts hold*

(i) $x \leq y \;\Rightarrow\; y/z \leq x/z$

(ii) $x \leq y \;\Rightarrow\; y \backslash z \leq x \backslash z$

Proof.

(i) Because of the reflexivity of $\leq$ we know that $y/z \leq y/z$, which is equivalent to $(y/z)*y \leq z$ (recall the item (iii) of the definition 2.0.5). Now we can use to monotonicity of $*$ to obtain $(y/z)*x \leq (y/z)*y$, which together gives us $(y/z)*x \leq z$, a fact which is equivalent with the desired $y/z \leq x/z$.

(ii) Analogously to the previous item.

□

It will be useful later on to have the following fact, which relates / and \ to $\wedge$ and $\vee$.

Lemma 2.0.10. *Let there be a distributive lattice $\mathbb{A}$ and three of its elements $x, y, z \in \mathbb{A}$. Then the following two facts holds*

(i) $(x/z) \wedge (y/z) = (x \vee y)/z$

(ii) $(x \backslash z) \wedge (y \backslash z) = (x \vee y) \backslash z$

Proof. We will prove only the item (i), the other one is analogous. In the previous lemma it was demostrated that $/z$ is antitone and therefore $(x \vee y)/z$ is cleary a lower bound of both x/z and y/z. We want to show that it is also the greatest lower bound. Let's have any a such that $a \leq x/z$ and $a \leq y/z$. We can then infer that $a * x \leq z$ and $a * y \leq z$. We then also know that $(a * x) \vee (a * y) \leq z$, since $\vee$ gives the least upper bound. Now we can use the distributivity of $*$ with respect to $\vee$ to get $a * (x \vee y) \leq z$ and finally $a \leq (x \vee y)/z$, which is what we wanted to prove.

□

Now it will be shown how to make sense of the familiar logical notions of validity of a formula as well the notions of tautology and contradiction when working with residuated lattices and in fact with any algebras. Let's begin with defining an evaluation in an algebra.

Definition 2.0.11. *Let $\mathbb{A}$ be an algebra. Then we call a map $e : Fle \to A$ an $\mathbb{A}$-evaluation (with Fle being the set of all formulae) if it satisfies the following:*

(i) $e(\varphi \wedge \psi) = e(\varphi) \wedge e(\psi)$

(ii) $e(\varphi \vee \psi) = e(\varphi) \vee e(\psi)$

(iii) $e(\varphi \& \psi) = e(\varphi) * e(\psi)$

(iv) $e(\varphi \to \psi) = e(\varphi)/e(\psi)$

(v) $e(\psi \leftarrow \varphi) = e(\varphi) \backslash e(\psi)$

Having determined what an evaluation means we proceed to the notions of a formula being satisfied by a given evaluation a to the notion of logical entailment.

Definition 2.0.12. *Let $\mathbb{A}$ be an algebra. Then we say that a given formula φ is satisfied in this algebra by an $\mathbb{A}$-evalution e when $e(\varphi) \geq 1$. A formula is valid in an algebra if it is satisfied by all of its evaluations.*

The algebras we are working with do not allow us to be as specific as to demand e(φ) to equal 1, since we can have such algebras in which 1 is not the greates element.

A valid formula in a given algebra $\mathbb{A}$ can be called a tautology with respect to this algebra. We can also generalize this notion speaking of a tautology with respect to a given class of algebras. Thus we obtain e.g. a notion of a tautology with respect to all distributive residuated lattices (distributive FL algebras). Such formulae are exactly those which can be proven in distributive FL-calculus defined in section 1.

Definition 2.0.13. *We say that the formulae $\varphi_1 \ldots \varphi_n$ entail the formula ψ and we denote it as $\varphi_1 \ldots \varphi_n \models_A \psi$ if for all $\mathbb{A}$-evalutions e we have*

$$e(\varphi_1) * \ldots * e(\varphi_n) \leq e(\psi)$$

Thus we can formulate the completeness of this semantics with respect to the Fl calculus. A proof can be found in [alg].

Theorem 2.0.14. *For each finite sequence of formulae $\varphi_1, \ldots, \varphi_n$ and each formula ψ the following relation*

$$e(\varphi_1) * \ldots * e(\varphi_n) \leq e(\psi)$$

is true for every evaluation in every distributive residuated lattice if and only if the sequent

$$\varphi_1, \ldots, \varphi_n \Rightarrow \psi$$

has got a proof in the distributive FL calculus.

Now we shall see how to introduce the three structural rules into this kind of semantics.

Definition 2.0.15. *Let $\mathbb{A}$ be a distributive lattice*

We say that rule of exchange holds in $\mathbb{A}$ if

$$\varphi * \psi \models_A \psi * \varphi$$

holds for any formulae φ and ψ.

We say that the rule of contraction holds in $\mathbb{A}$ if

$$\varphi \models_A *\varphi$$

holds for any formula φ.

And finally we say that the rule of left weakening holds if

$$\varphi \& \psi \models_A \psi$$

holds. And the rule of right weakening holds if

$$\varphi \& \psi \models_A \varphi$$

holds.

Introducing the three mentioned structural rules into an algebra is relatively straightforward.

Definition 2.0.16. *A residuated lattice* $\mathbb{A}$ *is called*

(*i*) *commutative if* $a * b = b * a$ *holds*

(*ii*) *contractive if* $a \leq a * a$ *holds*

(*iii*) *a lattice with left weakening if* $a * b \leq b$ *holds*

(*iv*) *a lattice with right weakening if* $a * b \leq a$ *holds*

The following lemma is an immediate consequence of the definitions 2.0.12 and 2.0.16 .

Lemma 2.0.17. *When* $\mathbb{A}$ *is a distributive residuated lattice, then the structural rules of exchange, contraction and left and right weakening hold in it if and only if it satisfies the conditions from the last definition* (*i*), (*ii*), (*iii*) *and* (*iv*), *of the last definition, respectively.*

The rule of weakening can be also obtained by the equality of the truth constants.

Lemma 2.0.18. *Let* $\mathbb{A}$ *be a distributive residuated lattice. Then if* $\top = 1$ *holds in* $\mathbb{A}$, *the rule of both left and right weakening hold in* $\mathbb{A}$, *too.*

Proof. Let's suppose that $\top = 1$ holds and let's have a formula $\varphi \& \psi$. Since 1 behaves as in a monoid, we know that

$$1 * a = a = a * 1$$

holds for each a. We also know that $a \leq \top$ holds. And since $*$ is monotonous, we also know that

$$a * b \leq a * 1$$

Thus we finally have that

$$a * b \leq a$$

and analogously for $a * b \leq b$. □

After having shown how an algebraic approach can semantically express the substructural logics, let's now have a look at how the Kripke approach does the same in order to show somewhat later in what manner do those approaches correspond.

Chapter 3

Kripke frames

Just as with modal logics or the intuicionistic logic one has to begin by defining Kripke frames to be able to define Kripke models. The frames our logics will expand into models will be a little bit more complicated in their structure in order to be able to express the possible omission of substructural rules. It should be noted that in this paper an orderered n-tuple of elements $a_1,\ a_2,\ \ldots a_n$ is written as $< a_1, a_2, \ldots a_n >$. Before introducing this Kripke semantic, it should be remarked that I am mostly inspired by [Res]. An alternative type of Kripke semantic can be found in [alt].

Definition 3.0.19. *A Kripke frame F is an ordered set of four sets*

$$\mathbb{F} =< P, \leq, T, R >$$

of which P is a non-empty set of points. The following conditions hold:

(i) The binary relation $\leq \subseteq P^2$ is a partial order, which means that it is reflexive, transitive and antisymmetric. Reflexivity means

$$\forall w \in P\ (w \leq w),$$

transitivity means

$$\forall u \forall v \forall w \in P\ (u \leq v\ \&\ v \leq w\ \Rightarrow\ u \leq w),$$

and being antisymmetric means

$$\forall v \forall w \in P\ (v \leq w\ \&\ v \neq w\ \Rightarrow\ w \nleq v)$$

(ii) the set $T \subseteq P$ is an upper set with respect to $\leq$

(iii) The ternary relation $R \subseteq P^3$ satisfies the following three conditions, relating it to T and $\leq$s:

(a) $x \leq y \iff \exists z \in T(R(zxy)) \iff \exists z \in T(R(xzy))$

(b) $R(xyz)$ & $x^{'} \leq x$ & $y^{'} \leq y$ & $z \leq z^{'} \;\Rightarrow\; R(x^{'}y^{'}z^{'})$

(c) $\exists u(R(xyu)\ \&R(uzv)) \iff \exists w(R(yzw)\ \&\ R(xwv))$

A model is obtained from a frame just by determining where do the atomic formulae hold.

Definition 3.0.20. *Let* $\mathbb{F}$=*(P, ≤, T, R) be a Kripke frame and let* $V : At \longrightarrow Up(P)$ *be a map from assigns to each atomic formula a an upper subset of* A*. The the set*

$$\mathbb{M} =< \mathbb{F}, V >$$

is called a Kripke model over the frame $\mathbb{F}$*.*

Already equipped with the notion of Kripke model, let's explain inductively when a formula is valid in a given model.

Definition 3.0.21. *Let* $\mathbb{M}$ *be a model and* φ *a given formula. Then we say that it is valid in a point* $a \in P$ *of* $\mathbb{M}$ *(abrreviated as* $\mathbb{M}, a \Vdash \varphi$*) if and only if it satisfies the following conditions:*

(i) *If* φ *is an atom, then* $a \in V(\varphi)$

(ii) *If* $\varphi = \psi \wedge \chi$*, then* $\mathbb{M}, a \Vdash \psi$ *and* $\mathbb{M}, a \vdash \chi$*.*

(iii) *If* $\varphi = \psi \vee \chi$*, then* $\mathbb{M}, a \Vdash \psi$ *or* $\mathbb{M}, a \Vdash \chi$*.*

(iv) *If* $\varphi = \psi \& \chi$*, then* $\exists b, c \in A(R(bca)$ *and* $\mathbb{M}, b \Vdash \psi$ *and* $\mathbb{M}, c \Vdash \chi$*)*

(v) *If* $\varphi = \psi \rightarrow \chi$*, then* $\forall b, c \in A$ *(if* $R(abc)$ *and* $\mathbb{M}, b \Vdash \psi$*, then* $\mathbb{M}, c \Vdash \chi$*)*

(vi) *If* $\varphi = \chi \leftarrow \psi$*, then* $\forall b, c \in A$ *(if* $R(bac)$ *and* $\mathbb{M}, b \Vdash \psi$*, then* $\mathbb{M}, c \Vdash \chi$*)*

(vii) *If* $\varphi = 1$*, then* $a \in T$*.*

(viii) *If* $\varphi = \top$*, then* $a \in P$ *(so* φ *is valid in every point of the model).*

(ix) *If* $\varphi = \bot$*, then* $a \notin P$ *(so* φ *is not valid in any point of the model).*

We can generalize the notion of validity in a given point of a model to that of validity in the whole model and then even in all models over given frame(or we can simply say validity in a given frame).

Definition 3.0.22. *We say that a formula is valid in model* $\mathbb{M}$ *if it is valid in all the points from the set* T*. We say that a formula is valid on frame* $\mathbb{F}$ *if it is valid in all the models over this frame. We say that a formula is valid for Fl logic if it is valid in all the Kripke frames.*

To get more familiar with Kripke semantics, let's have a glance at a few properties of such models. First we will show the persistency of validity of a formula.

Lemma 3.0.23. *Let φ be a formula, $\mathbb{M}$ a Kripke model and a and b a pair of its point. Then*

$$a \leq b \;\&\; \mathbb{M}, a \Vdash \varphi \;\Rightarrow\; \mathbb{M}, b \Vdash \varphi$$

Proof. We proceed by induction on the complexity of φ. When it is an atom, then we get the persistence by definition of a valuation map $V : At \longrightarrow Up(A)$. The cases of $\varphi = \psi \wedge \chi$ and $\varphi = \psi \vee \chi$ are both mere routine, so let's look more closely just at the proof for $\varphi = \psi \& \chi$ and $\varphi = \psi \to \chi$ with $\varphi = \chi \leftarrow \psi$ being analogous.

(*i*) If $a \leq b$, $\varphi = \psi \& \chi$ and $\mathbb{M}, a \Vdash \varphi$, then there are some worlds c and d such that $R(cda)$ and $\mathbb{M}, c \Vdash \psi$ and $\mathbb{M}, d \Vdash \chi$. Then since $c \leq c$, $d \leq d$ and $a \leq b$, we can just use the item $(i), b$ of the definition 3.0.19 and get that $R(cdb)$ which means that $\mathbb{M}, b \Vdash \varphi$, as we desired.

(*ii*) If $a \leq b$, $\varphi = \psi \to \chi$ and $\mathbb{M}, a \Vdash \varphi$, the let's examine any worlds c and d such that $R(bcd)$ and $\mathbb{M}, c \Vdash \psi$. Then because of reflexivity of $\leq$ and the item $(iii), b$ of the definition 3.0.19 we get $R(acd)$ and, consequently we obtain the desired $\mathbb{M}, d \Vdash \chi$.

(*iii*) The point for $\varphi = \chi \leftarrow \psi$ is entirely analogous to the previous one, so we once again use the reflexivity of $\leq$ and the item $(iii), b$ of the definition 3.0.19 to finish the proof of persistency.

Now we will see a relation between two models which ensures that exactly that same formulae are valid in the points in that relation. This relation, called bisimulation, is the topic of my bachelor thesis ([bac]).

Definition 3.0.24. *Let $\mathbb{M}$ and $\mathbb{N}$ be two kripkean models and let there be a relation $B \subseteq M \times N$. Then such a relation is called a bisimulation if the following conditions hold:*

(*i*) *For each atomic formula φ and for each $a \in M$ and $u \in N$ such that aBu we have*

$$\mathbb{M}, a \Vdash \varphi \iff \mathbb{N}, u \Vdash \varphi.$$

This condition is usually called atomic harmony

(*ii*) $\forall a \forall b \forall c \in M \forall u \in N \; (R(abc) \;\&\; aBu \Rightarrow \exists v \exists w \in N \; (bBv \;\&\; cBw \;\&\; R(uvw)))$

(*iii*) $\forall a \forall b \forall c \in M \forall u \in N \; (R(bac) \;\&\; aBu \;\Rightarrow\; \exists v \exists w \in N \; (bBv \;\&\; cBw \;\&\; R(vuw)))$

(*iv*) $\forall a \forall b \forall c \in M \forall u \in N \; (R(bca) \;\&\; aBu \;\Rightarrow\; \exists v \exists w \in N \; (bBv \;\&\; cBw \;\&\; R(vwu)))$

(*v*) $\forall u \forall v \forall w \in N \forall a \in M \; (R(uvw) \;\&\; aBu \;\Rightarrow\; \exists b \exists c \in M \; (bBv \;\&\; cBw \;\&\; R(abc)))$

(vi) $\forall u \forall v \forall w \in N \forall a \in M \ (R(vuw) \ \& \ aBu \ \Rightarrow \ \exists b \exists c \in M \ (bBv \ \& \ cBw \ \& \ R(bac)))$

(vii) $\forall u \forall v \forall w \in N \forall a \in M \ (R(vwu) \ \& \ aBu \ \Rightarrow \ \exists b \exists c \in M \ (bBv \ \& \ cBw \ \& \ R(bca)))$

The existence of a bisimulation connecting two points of two models is interesting because it guarantees an equivalence with respect to formulae valid in these points, which will be demonstrated now.

Theorem 3.0.25. *Let there be two Kripke models $\mathbb{M}$ and $\mathbb{N}$ and B a bisimulation relation between them and two point $a \in M$ and $u \in N$ such that aBu. Then for every formula φ the following holds:*

$$\mathbb{M}, a \Vdash \varphi \iff \mathbb{N}, u \Vdash \varphi.$$

Proof. We shall proceed by induction on the form of φ.

- (i) For φ being an atomic formula, we just use the condition of atomic harmony for B.
- (ii) For $\varphi = \psi \wedge \chi$, we consider both ψ and χ being satisfied either at a or u and use the induction (since they are both simpler than φ) on them to show that they are both valid in the corresponding point in the relation B.
- (iii) For $\varphi = \psi \vee \chi$, we proceed similarly to the previous item, this time considering that at least one of ψ and χ is satisfied and by induction ensure that it is satisfied in the corresponding point, too.
- (iv) For $\varphi = \psi \& \chi$, we have to start considering the relation R. Let's have it that $\mathbb{M}, a \Vdash \psi \& \chi$. Now, this means that there are some points b and c in M, such that $R(bca)$ and $\mathbb{M}, b \Vdash \psi$ and $\mathbb{M}, c \Vdash \chi$. Since a is in the relation B with u, we also know that there are some points v and w such that $R(vwu)$ and bBv and cBw. Since ψ and χ are simpler in their form than φ, we can by induction infer that $\mathbb{N}, v \Vdash \psi$ and $\mathbb{N}, w \Vdash \chi$, which is what we wanted to prove. Starting from $\mathbb{N}, u \Vdash \psi \& \chi$, the proof is entirely analogous.
- (v) Let's proceed to the case of $\varphi = \psi \rightarrow \chi$ and suppose that $\mathbb{M}, a \Vdash \psi \rightarrow \chi$. Let's take any two points v and w in N such that $R(uvw)$ and $\mathbb{N}, v \Vdash \psi$. Since aBu, we have some points b and c in M such that $R(abc)$ and bBv and cBw. By induction we can infer that $\mathbb{M}, b \Vdash \psi$ and since we suppose that $\psi \rightarrow \chi$ is valid at a, we can infer that $\mathbb{M}, c \Vdash \chi$ and then by cBw and induction we obtain the desired $\mathbb{N}, w \Vdash \chi$. Again, starting from $\mathbb{N}, u \Vdash \psi \rightarrow \chi$, the proof proceeds analogously.
- (vi) Similar to the previous item.

□

In my bachelor thesis I tried to give a hint of how useful such a relation can be thanks the property we have just shown, since it enables us to see how two models can be very different in their structure and yet be equivalent with regards to some logic.

As well as in the algebraic case we shall define the relation of entailment.

Definition 3.0.26. *Let φ and ψ be a pair of formulae. We say that φ entails ψ if for every Kripke model $\mathbb{M}$ and a point $a \in A$ the following holds:*

$$\mathbb{M}, a \Vdash \varphi \ \Rightarrow \ \mathbb{M}, a \Vdash \psi.$$

With this definition we can formulate the equivalence of these semantics with the distributive FL calculus. The proof of this completeness theorem can be found in [Res].

Theorem 3.0.27. *Let there be formulae $\varphi_1 \dots \varphi_n$ and ψ. Then the formula $\varphi_1 \& \dots \& \varphi_n$ entails the formula ψ if and only if the sequent*

$$\varphi_1, \ \dots, \ \varphi_n \Rightarrow \psi$$

has got a proof in the distributive FL calculus.

As in the algebraic case, we are not going to prove this completeness theorem here.

Having seen how to introduce the structural laws into the algebraic semantics in the lemma 2.0.17 , let's now look at them in the Kripke semantics. First we introduce, how to understand the structural rules in this semantics.

Definition 3.0.28. *Let $\mathbb{M}$ be a Kripke model. We say that the rule of exchange holds in this model if any formula of the form $\varphi\&\psi$ entails $\psi\&\varphi$.*

We say that the rule of contraction holds if any formula φ entails $\varphi\&\varphi$.

And we say that the rule of left and right weakening holds if any formula of the form $\varphi\&\psi$ entails φ or ψ, respectively.

Now it will be demonstrated how can the structure of a Kripke frame ensure that the structural rules hold.

Definition 3.0.29. *We call a Kripke frame*

- (i) *commutative if for each three points a, b and c if $R(abc)$, then $R(bac)$.*
- (ii) *contractive if for each point a we have $R(aaa)$.*
- (iii) *a frame with left weakening if for each three points a, b and c if $R(abc)$, then $b \leq c$.*
- (iv) *a frame with right weakening if for each three points a, b and c if $R(abc)$, then $a \leq c$.*

The properties of frames we have just defined correspond to the validity of substructural rules on given frame.

Lemma 3.0.30. *Let $\mathbb{F}$ be a Kripke frame. Then the following four equivalences hold:*

- (i) *$\mathbb{F}$ is commutative $\iff$ the exchange rule holds for the formulae valid in this frame.*
- (ii) *$\mathbb{F}$ is contractive $\iff$ the contraction rule holds for the formulae valid in this frame.*

(*iii*) $\mathbb{F}$ *is a frame with left weakening* $\iff$ *the left-weakening rule holds for the formulae valid in this frame.*

(*iv*) $\mathbb{F}$ *is a frame with right weakening* $\iff$ *the right-weakening rule holds for the formulae valid in this frame.*

Proof. (*i*) $\Rightarrow$:Let's take take a model $\mathbb{M}$ over $\mathbb{F}$. When $\mathbb{M}, a \Vdash \psi\&\chi$ then there are some b,c such, that $R(bca)$ and ψ holds in b and χ holds in c. Since $\mathbb{F}$ is commutative, we have $R(cba)$ and so $\mathbb{M}, a \Vdash \chi\&\psi$.

$\Leftarrow$:Let $\mathbb{F}$ not be commutative, so for some $a, b, c \in A$ we have $R(abc)$, but not $R(bac)$. Let's take two atomic formulae ψ and χ and evaluate the points a and b by them so that ψ holds in a but not in b, while χ holds in b, but not in a.

(*ii*) $\Rightarrow$:Let's take take a model $\mathbb{M}$ over $\mathbb{F}$. When $\mathbb{M}, a \Vdash \varphi$, then since we have $R(aaa)$, we get $\mathbb{M}, a \Vdash \varphi\&\varphi$.

$\Leftarrow$:Let's have a point a such that $R(aaa)$ and further let's take such a valuation, which to an atomic formula ψ assigns an upper set of point u such that $a \leq u$. This also means that ψ is satisfied in a. Now we know that there can't be such two points b and c that $R(bca)$ and $a \leq b$ and $a \leq c$, because this would lead to $R(aaa)$, the fact we suppose not to hold. Thus there are no points b and c such that $R(bca)$, with ψ being satisfied in both b and c.

(*iii*) $\Rightarrow$: When $\mathbb{F}, a \Vdash \psi\&\chi$, then there are some b,c such, that $R(bca)$ and ψ holds in b and χ holds in c. We have $c \leq a$ and by using the persistency proved in lemma 3.0.23.

$\Leftarrow$:Let's take two atoms φ and ψ and three points $R(bca)$. Let the formula $\varphi\&\psi$ be valid in a, φ in b and ψ in c in such a model over the frame that ψ is valid in such points u that $c \leq u$. We know that ψ is valid in a which necesarrily means that $c \leq$.

(*iv*) Analogous to (*iii*).

□

As well as in the algebraic case, the equivalence of the two truth constants implies the weakening rule.

Lemma 3.0.31. *Each formula* φ *is valid in any point of any model if and only if the formula* $\varphi\&1$ *(and* $1\&\varphi$*) is.*

Proof. Let's have any point a of any Kripke model $\mathbb{F}$.

Since the relation $\leq$ is reflexive and because of its relation to R stated in the definition 3.0.19, we know that there is a point b in the truth set such that

$$R(aba)$$

So, if φ is satisfied at the point a, so is $\varphi\&1$ and $1\&\varphi$ (using $R(baa)$ for the latter formula).

If, on the other hand, $\varphi \& 1$ is satisfied at a, then we have the point $b \in T$ and a point $c \in P$ such that $R(cba)$. But this means that $c \leq a$ and we can finish the proof by using the persistency proven earlier in 3.0.23. □

Now we can prove the point about the truth constants.

Lemma 3.0.32. *Let $\mathbb{F}$ be a Kripke frame such that in every model above it, the constant* 1 *is equivalent to the constant $\top$ (which means it is satisfied at every point). Then the rule of weakening is valid in each of the models.*

Proof. Let's take any formula of the form $\varphi \& \psi$ and any point a of a given model $\mathbb{M}$ at which it is satisfied. We know that there are some points b and c in P such that $R(bca)$, while φ is valid at b and ψ is valid at c. But since $\top$ is satisfied at every point, so it is at c. And since 1 is equivalent to the other constant, then it is satisfied at c, as well. And we know $\varphi \& 1$ is equivalent to φ. This means that φ is satisfied at a, which is what we wanted.

The right weakening is proven analogously.

□

3.1 Relation to intuicionistic logic

3.1.1 Introducing R in intuicionistic semantics

It has been said that the logic we are working with is actually intuicionistic logic without the structural rules of exchange, contraction and both right and left weakening. Now we will in a certain way show that this is indeed the case. Let's begin by reminding ourselves how a Kripke frame for intuicionistic logic is defined.

Definition 3.1.1. *An ordered pair $F =< P, \leq >$ is a Kripke frame for intuicionistic logic if P is a nonempty set and $\leq$ is a partial order on P.*

A model is then defined as a pair of frame and evaluation, which is assigns to each atomic formula an upper subset of P. There is neither the connective &, nor $\leftarrow$ in this logic, actually there is no need for them, since & becomes equivalent to $\wedge$ and $\leftarrow$ becomes equivalent to $\rightarrow$. Formula $\psi \rightarrow \chi$ is satisfied at any $w \in P$ of a given model $\mathbb{M}$ under the following condition:

$$\mathbb{M}, w \Vdash \psi \rightarrow \chi \iff \forall v \in P\ (\leq \Upsilon \Rightarrow\ (\mathbb{M}, v \Vdash \psi\ \Rightarrow\ \mathbb{M}, v \Vdash \chi))$$

There is also just one type of constants, denoted as 1 and 0, the first one being satisfied at every point of P, the second one at none.

So, such a semantics is not in need of a ternary relation which is a part of the frames for the distributive FL logic, neither the truth set T. How can we introduce them in it?

Lemma 3.1.2. *Let's have a Kripke frame for intuicionistic logic $\mathbb{F} < P, \leq >$ such and let's define $T = P$ and R as follows:*

$$\forall u \forall v \forall w\ (R(uvw) \iff (u \leq w\ \&\ v \leq w))$$

Then $G =< P, \leq, T, R >$ is a Kripke frame for the distributive FL logic as defined in 3.0.19.

Proof. We have to prove all the items of the definition 3.0.19. The binary relation $\leq$ is a partial order by definition and it is clear that $T = P$ is an upper set.

Now we should prove the three items about the ternary relation R.

(a) Let's have x and $y \in P$ such that $x \leq y$. Then we can take $R(xxy)$, with $x \in T$. The other direction immediately follows from the definition of R in this frame.

(b) This item simply follows from the transitivity of $\leq$.

(c) Let's have some u such that $R(xyu)$ and $R(uzv)$. Then we can take v as the point we are looking for, since $R(yzv)$ and $R(xvv)$ follow from the reflexivity and transitivity of $\leq$.

If, on the other hand, we have some w such that $R(yzw)$ and $R(xwv)$, then we can use v once again for the same reasons.

□

Having done this, we can now show that such frames are exactly the frames of the definition 3.0.19 with the three structural rules and only one type of constants.

Theorem 3.1.3. *Let there be a frame $F =< P, \leq, T, R >$. Then this frame is an intuicionistic frame as defined in the previous lemma if and only if it is frame defined in 3.0.19, which is commutative, contractive, with both left and right weakening and is its truth set T is equal to P.*

Proof. We will prove both the implications.

($\Rightarrow$) Let F be an intuicionistic frame. We have already shown that it is a frame as defined in 3.0.19, so we only have to prove the three structural properties (the constants are the same by definition). That F is commutative follows directly from the definition of R, that it is contractive follows from the reflexivity of $\leq$ and both left and right weakening follow from directly from the definition of R as well.

($\Leftarrow$) We want to prove the equivalence

$$\forall u \forall v \forall w \ (R(uvw) \iff (u \leq w \ \& \ v \leq w))$$

The direction from the left to the right follows from the left and right weakening.

The other direction follows from the fact that F is contractive. Let's have some u and v such that $u \leq w$ and $v \leq w$. Since F is contractive, we know that $R(www)$. Then we can use the item (iii, b) to obtain the desired $R(uvw)$.

□

The proofs just presented, though quite simple, might hopefully give a hint of the relation between the intuicionistic logic and the distributive FL logic and how the structural rules are actually omitted in the Kripke semantics. It is now simple to show that the connectives & and $\wedge$ as well as $\rightarrow$ and $\leftarrow$ become equivalent in intuicionistic logic.

Lemma 3.1.4. *Let $\mathbb{F}$ be a Kripke frame according to the definition 3.0.19. Then for each model $\mathbb{M}$ over the frame, each point $w \in P_F$ and each two formulae ψ and χ the following holds:*

(*i*) *If $\mathbb{F}$ is commutative (recall the definition 3.0.29), then the following holds*

$$\mathbb{M}, w \Vdash \chi \leftarrow \psi \iff \mathbb{M}, w \Vdash \psi \rightarrow \chi$$

(*ii*) *If $\mathbb{F}$ is a frame with left and right weakening and if it is contractive then the following holds*

$$\mathbb{M}, w \Vdash \psi \& \chi \iff \mathbb{M}, w \Vdash \psi \wedge \chi$$

Proof. (*i*) With the commutativity the set of all pairs of points u and v such that $R(wuv)$ is exactly the same as the set of the pairs such that $R(uwv)$. This means that the conditions for the validity of the formula $\psi \rightarrow$ are exactly the same as for $\chi \leftarrow \psi$.

(*ii*) For the direction from the left to the right, consider the both the weakening rules and use the lemma 3.0.23. The two facts guarantee that both ψ and χ are valid at w.

The opposite direction is a simple consequence of the fact that $\mathbb{F}$ is contractive.

□

It is well known that the intuicionistic logic has got the so called disjunction property. We will now concentrate ourselves on that property a little bit and see how it works for the distributive FL logic.

3.2 Disjunction property

First let us remind of the property as it can be proven for the intuicionistic logic. To state this property, we first need to state what a valid formula for intuicionistic logic is.

Definition 3.2.1. *We say that a formula φ is valid in intuicionistic logic if it is valid in every point of every model over every frame.*

Now we can proceed to the property itself(for an interested reader it should be noted that this propery was first proven by Gödel in [G], but the first proof using Kripke semantics was presented by Smorynski in [S]).

Theorem 3.2.2. *Let there be a formula of the form $\psi \vee \chi$ that is valid in ituicionistic logic. Then at least one of the formulae ψ or χ is valid in intuicionistic logic as well.*

It should be noted that, when we nees to distinguish between between frames or models, we will denote P_F, $\leq_F$, T_F and R_F the set of points, the truth set, the partial order and the ternary relation R of the frame F, as opposed to such four sets of another frame. Similarly for the models over such frames, we will write P_M, $\leq_M$ and so on, including the valuation V_M.

Proof. The proof is contrapositive, which means that we begin with ψ and χ having counterexamples(i.e. some models with a point in which the given formula is not valid) and deduce from this that $\psi \vee \chi$ has got a counterexample as well.

Let's have then have a model $\mathbb{M}$ and its point u such that $\mathbb{M}, u \nVdash \psi$ and another model $\mathbb{N}$ and its point v such that $\mathbb{N}, v \nVdash \chi$. Let's then create a model $\mathbb{O}$ such that P_O is a disjunct union of P_M and P_N(which means that P_O is basically $P_M \cup P_N$, but if the models $\mathbb{M}$ and $\mathbb{N}$ happen to have points in common, then we rename the ones in $\mathbb{N}$), and we take $\leq_O = \leq_M \cup \leq_N$, $T_O = T_M \cup T_N$ and $R_O = R_M \cup R_N$(with the common points always renamed). We then add to the model O a new point w such that $w \leq u$ and $w \leq v$ and close $\leq$ in an obvious manner to become a partial order. We complete the construction defining the valuation V_O by letting no atomic formula to be satisfied at w.

In this model neither ψ is valid at u, nor is χ at v. And since it can be easily proven that the validity of each formula is persistent in intuicionist models (meaning that it is transferred upwards with respect to $\leq$), we can close that neither ψ nor χ is valid at w. $\square$

Now we shall see that the disjunction property holds for the distributive FL logic as well (there is also an algebraic proof which can be found in [dis]).

Theorem 3.2.3. *Let there be a formula of the form $\psi \vee \chi$ that is valid in FL logic. Then at least one of the formulae ψ or χ is valid in FL logic as well.*

Proof. Let's then have a model $\mathbb{M}$ and its point u such that $\mathbb{M}, u \nVdash \psi$ and another model $\mathbb{N}'$ and its point v such that $\mathbb{N}', v \nVdash \chi$ (both u and v are elements of the truth sets of their models). Let's make the two models disjunct by renaming as in the previous proof and make a union of the two disjunct models (so we can rename points in the model $\mathbb{N}'$, namely those which this model has got in common with the other one, to obtain a model $\mathbb{N}$ and take a valuation, which assigns to each atomic formula the disjunct union of points assigned to it in $\mathbb{M}$ and $\mathbb{N}$). This clearly is a model, let's denote it as $\mathbb{O} =< P, \leq, T, R, V >$. We will extend it in a few steps:

(*i*) Let's begin by expanding the truth set T into T' by adding a new point w to it (so we have also expanded P into P' by adding this new point).

(*ii*) Let's then expand the relation $\leq'$ by adding $w \leq x$ for each $x \in T'$(note that this in fact means that $\leq'$ is a partial order).

(iii) Finally we expand the ternary relation R into $R^{'}$ by adding $R(wxy)$ and $R(xwy)$ for each $x \in P^{'}$ and $y \in P^{'}$ such that $x \leq^{'} y$.

The evaluation of this extension is exactly the same as of the disjunct union of $\mathbb{O}$, which means that no atomic formulae are valid at w.

Now we shall show that such an extension, let's denote it as $\mathbb{Q} =< P^{'}, \leq^{'}, T^{'}, R^{'}, V^{'} >$, is indeed a model and that $\mathbb{Q}, w \Vdash \psi \vee \chi$ does not hold.

Lemma 3.2.4. *$\mathbb{Q}$ is a Kripke model.*

Proof. The valuation $V^{'}$ obviously assigns to each atomic formula an upper subset of $P^{'}$.

Clearly, even with w as its element, the truth set $T^{'}$ is still an upper set.

It is also obvious that $\leq^{'}$ is a partial order.

The item (iii, a) states that $x \leq^{'} y$ is the case if and only if there is some $a \in T^{'}$ and $b \in T^{'}$ such that $R^{'}(axy)$ and $R^{'}(xby)$. It is a clear consequnce of the construction of $\leq^{'}$ and R' that this item holds, since $\leq^{'}$ is expanded only by adding $w \leq a$ for all $a \in T^{'}$ and we know that $R(wwa)$ for such a case. And $R^{'}$ was expanded only by $R(xwy)$ and $R(wxy)$ for the cases of $x \leq^{'} y$.

That the item (iii, b) states that for any $R(xyz)$ if we have some $a \leq^{'} x$, $b \leq^{'} y$ and $z \leq^{'} c$, then we have $R(abc)$ as well. That this holds is also a simple consequence of the construction, since it reckons with the transitivity of $\leq^{'}$.

It remains to prove the item (iii, c), which means that there is an x sucht that $R^{'}(abx)$ and $R^{'}(xyz)$ if and only if there is a v such that $R^{'}(byv)$ and $R^{'}(avz)$. It is clear that this item holds in the model $\mathbb{O}$. Now, we have created the ternary relation $R^{'}$ only by adding by adding $R(xwy)$ and $R(wxy)$ for the case of $x \leq {'}y$. Now we shall check that none of the new elements of $R^{'}$ (new with respect to R) violates this condition. It is helpful to note that if $R^{'}(klw)$, then $k = w$ and $l = w$. This means that there is nothing to prove for the case of $w = z$ (since both the triplets are $R^{'}(www)$ for both the directions) and for the cases of $w = x$ and $w = v$ we can simplify the proof by supposing that $a = w$ and $b = w$ for the first case and $b = w$ and $y = w$ for the other one. Let's start by proving the direction from the left to the right of this point.

($\Rightarrow$)

(i) Let's consider the case of $R^{'}(www)$ and $R^{'}(wyz)$. Since $z \leq^{'} z$, we can deduce that $R^{'}(wzz)$ and we already know that $R^{'}(wyz)$, which means that z is the point we were looking for.

(ii) Let there be $R^{'}(aby)$ and $R^{'}(ywz)$ (because of the fact that w is in the truth set, we can infer that $y \leq^{'} z$). Then we know that $R^{'}(bwb)$ (since $b \leq^{'} b$). Finally we also know that $R^{'}(abz)$, because of the item (iii, b) of 3.0.19. This time b is the desired point.

(iii) Let there be two points a and x such that $R^{'}(wax)$ (this means that $a \leq^{'} x$) and also $R^{'}(xyz)$. Because of (iii, b) of 3.0.19 we know that $R^{'}(ayz)$ and we know that $R^{'}(wzz)$, as well. Thus z is the point we were looking for.

(iv) Let there be two points a and x such that $R^{'}(awx)$ (which means $a \leq^{'} x$) and also $R^{'}(xyz)$. We know that $R^{'}(wyy)$ and because of (iii, b) of 3.0.19 we also know that $R^{'}(ayz)$. This time y is the point we were looking for.

Now we shall check the opposite direction of (iii, c), as well.
$(\Leftarrow)$

(i) Let's start with $R^{'}(www)$ and $R^{'}(ywz)$. We know that $R^{'}(zwz)$ and together with $R^{'}(ywz)$ this means that z is the desired point.

(ii) Let there be $R^{'}(aby)$ and $R^{'}(wyz)$ (and therefore $y \leq^{'} z$). We know that $R^{'}(waa)$ and because of the item (iii, b) of the definition 3.0.19 we also know that $R^{'}(abz)$. Thus a is the desired point.

(iii) Let there be two points a and x such that $R^{'}(wax)$ (thus also $a \leq^{'} x$) and $R^{'}(yxz)$. We know that $R^{'}(ywy)$ and further we know that $R^{'}(yaz)$, since $a \leq^{'} x$. So we can take y as the point.

(iv) Let there be two points a and x such that $a \leq^{'} x$ and $R^{'}(awx)$ and $R^{'}(yxz)$. Since $a \leq^{'} x$, we know that $R^{'}(yaz)$ and we further know that $R^{'}(zwz)$. So, this time z is the point we were looking for.

□

Having proven that we have indeed constructed a model, we can now use it to finish the proof by showing that neither ψ nor χ is valid at the point $w \in P^{'}$. We will proceed more generally and prove that each formula φ is valid at a given point of the model $\mathbb{O}$ if and only if it is valid at that point in the model $\mathbb{Q}$. To state this fine, yet important difference clearly, we can write:

$$\forall p \in P \ (\mathbb{O}, p \Vdash \varphi \iff \mathbb{Q}, p \Vdash \varphi).$$

Now we shall prove this by induction on the complexity of the given formula φ. Doing the step for the connectives &, $\rightarrow$ and $\leftarrow$ we will be using the lemma 3.0.23.

(i) If φ is an atomic formula, then the case is clear, since p, as every point of the model $\mathbb{O}$ has got the same atomic formulae assigned to it by the evaluations V and $V^{'}$..

(ii) If $\varphi = \gamma \vee \delta$ is the case, then proving $\mathbb{Q}, p \Vdash \varphi$ from $\mathbb{O}, p \Vdash \varphi$, we have either $\mathbb{O}, p \Vdash \gamma$ or $\mathbb{O}, p \Vdash \delta$ and we use the induction assuption on one of the two formulae. Proving the opposite direction is the same.

(iii) For $\varphi = \gamma \wedge \delta$ a similar consideration applies.

(iv) Let's have $\varphi = \gamma \& \delta$. Let's begin with $\mathbb{O}, p \Vdash \gamma \& \delta$. We know that there are some points q and r in the model $\mathbb{O}$ such that $R(qrp)$ and further $\mathbb{O}, q \Vdash \gamma$ and $\mathbb{O}, r \Vdash \delta$. Now, since R' is an extension of R, we have $R'(qrp)$ and we can apply the induction assumption to obtain $\mathbb{Q}, q \Vdash \gamma$ and $\mathbb{Q}, r \Vdash \delta$, which together gives $\mathbb{Q}, p \Vdash \gamma \& \delta$, which is what we wanted to prove.

This time let's start with $\mathbb{O}, p \nVdash \gamma \& \delta$ and show that $\mathbb{Q}, p \nVdash \gamma \& \delta$. Let's take any points q and r of the model $\mathbb{Q}$ such that $R'(qrp)$. If both q and r are from the model $\mathbb{O}$, then we know that $R(qrp)$ and we can use the induction assumption to get $\mathbb{Q}, q \nVdash \gamma$ and $\mathbb{Q}, r \nVdash \delta$ from $\mathbb{O}, q \nVdash \gamma$ and $\mathbb{O}, r \nVdash \delta$.

Now let's consider the case of $r = w$ and q being a point of the model $\mathbb{O}$ (in case of $q = w$ and r being a point of $\mathbb{O}$ proceed in the same way). Since $w \in T'$, we know that $q \leq' p$. Thus we also have $q \leq p$ and therefore there is some $t \in T$ such that $R(qtp)$. Because we suppose that $\mathbb{O}, p \nVdash \gamma \& \delta$, we know that $\mathbb{O}, q \nVdash \gamma$ and $\mathbb{O}, t \nVdash \delta$. From the first fact we get by the induction assumption that $\mathbb{Q}, q \nVdash \gamma$, from the second one we get $\mathbb{Q}, t \nVdash \delta$. Because we have $t \in T'$, we also have $w \leq' t$ and so we can use the fact that $\mathbb{Q}$ is a model and the lemma 3.0.23 to obtain $\mathbb{Q}, w \nVdash \delta$, which together gives $\mathbb{Q}, p \nVdash \gamma \& \delta$.

Finally, we shall consider the case of $q = r = w$(then we know that $p \in T'$ because of the way the model $\mathbb{Q}$ was constructed). Since $p \leq p$, there is some $t \in T$ such that $R(ptp)$. We know $\mathbb{O}, t \nVdash \delta$ and therefore $\mathbb{Q}, t \nVdash \delta$. Now we use the fact that $\mathbb{Q}$ is a model and the lemma 3.0.23 to obtain $\mathbb{Q}, w \nVdash \delta$, which finishes the proof for the connective &.

(v) Now for the case of $\varphi = \gamma \to \delta$. Let's start with $\mathbb{O}, p \Vdash \gamma \to \delta$. Let's take any two points q and r of the model $\mathbb{Q}$ such that $R'(pqr)$ and $\mathbb{Q}, q \Vdash \gamma$. Now, if q is also a point of the model $\mathbb{O}$, then so is by the construction of the model $\mathbb{Q}$ the point r. This means that we have $\mathbb{O}, q \Vdash \gamma$ and therefore $\mathbb{O}, r \Vdash \delta$, which guarantees that $\mathbb{Q}, r \Vdash \delta$, which is what was to be proven.

Let's now consider the case of $q = w$ and $\mathbb{Q}, w \Vdash \gamma$. Thus we have $R'(pwr)$ and consequently $p \leq' r$. Because of the construction of $\mathbb{Q}$ we know that $r \in \mathbb{O}$ and further $p \leq r$. Therefore we have some $t \in T$ such that $R'(ptr)$. Now we can use the fact that $w \leq' t$ and lemma 3.0.23 to get $\mathbb{Q}, t \Vdash \gamma$ and by the induction assumption also $\mathbb{O}, t \Vdash \gamma$. This means that $\mathbb{O}, r \Vdash \delta$ and using the induction assuption we finally get $\mathbb{Q}, r \Vdash \delta$.

For the other direction let's start with $\mathbb{O}, p \nVdash \gamma \to \delta$. This means that there are some point q and r in the model $\mathbb{O}$ such that $\mathbb{O}, q \Vdash \gamma$ and $\mathbb{O}, r \nVdash \delta$. Now we can use the induction assuption to get $\mathbb{Q}, q \Vdash \gamma$ and $\mathbb{Q}, r \nVdash \delta$, which finishes the proof for the connective $\to$

(vi) The case of $\varphi = \delta \leftarrow \gamma$ is proven analogously to the previous one.

With this equivalence proven we can assert that $\mathbb{Q}, u \nVdash \psi$ and $\mathbb{Q}, v \nVdash \chi$. Using the fact that $\mathbb{Q}$ is a model and the lemma 3.0.23 once again we obtain that $\mathbb{Q}, w \nVdash \psi$ and $\mathbb{Q}, w \nVdash \chi$. Thus we have finally proven the desired $\mathbb{Q}, w \nVdash \psi \vee \chi$.

□

So we have proven the disjunction property for the distributive FL logic, which means for a logic without the structural rules of exchange, contraction and both left and right weakening. Nevertheless, it should be noted that the construction we have used implies that $R_{\mathbb{Q}}(www)$ for the added point w and also that $R_{\mathbb{Q}}(wqp)$ holds if and only if $R_{\mathbb{Q}}(qwp)$ does. So exactly the same proof would work also for the distributive FL logic with the rule of exchange, usually denoted as distributive FL_e,(which means it is the distributive FL logic over the frames such that $R(abc)$ holds if and only if $R(bac)$ does for the points of the frame) and also for the distributive FL logic with the contraction rule, usually denoted as distributive FL_c(so it is the distributive FL logic with such frames that $R(aaa)$ for each point). This means that the disjunction property holds for both these logics and also for the distributive FL logic with both the structural rules, usually denoted as distributive FL_{ec}.

Chapter 4

Transforming Kripke frames into residuated lattices and vice versa

In this chapter we will show how do the two types of semantics just introduced correspond to each other. We will have a construction that can make a residuated lattice out of a Kripke frame and another construction that can make the reverse. After introducing them we will show that they are actually functors in a categorical sense.

4.1 Transforming Kripke frames into residuated lattices

To obtain a residuated lattice from a given Kripke frame we will use a map $\mathbb{I}$ that will create a residuated lattice by taking upper subsets of the frame and defining the desired operations on them, some of them in a familiar manner known from the set theory.

Definition 4.1.1. *A map $\mathbb{I}$ defined on the class of all Kripke frames from the definition 3.0.19 is a frame transfer if for a given frame $\mathbb{F}$ we have*

$$\mathbb{I}(\mathbb{F}) = < Up(P), \subseteq, \cup, \cap, \backslash, /, *, T, P, \emptyset >$$

where $Up(A)$ a set of all upper subsets of P, $\cup$ and $\cap$ are union and intersection of sets, T is the truth set of the frame, P is the set of points and the remaining three operations are defined for any B,$C \in Up(P)$ as follows

(i) $B/C = \{a \in P;\ \forall b \forall c\ (b \in B\ \&\ R(abc)\ \Rightarrow\ c \in C)\}$

(ii) $B \backslash C = \{a \in P;\ \forall b \forall c\ (b \in B\ \&\ R(bac)\ \Rightarrow\ c \in C)\}$

(iii) $B * C = \{a \in P;\ \exists b \exists c\ (b \in B\ \&\ c \in C\ \&\ R(bca))\}$

Now let's check that such a map indeed gives us a distributive residuated lattice.

Theorem 4.1.2. *For any frame $\mathbb{F}$ its corresponding algebraic sturcture $\mathbb{I}(\mathbb{F})$ is a residuated lattice.*

Proof. We have to prove all the desired properties from the definition 2.0.5.

But even before that we have to prove that $\mathbb{I}(\mathbb{F})$ contains all the results of the defined operations, which is to say, that every operation gives an upper set. For two upper sets X and Y, which are elements of $\mathbb{I}(\mathbb{F})$, it is clear that $X \cup Y$ is an upper set, since $\cup$ yields a set of elements which are at least in one of the sets X or Y. Now, both these sets are upper sets and so anything above element of $X \cup Y$ with respect to $\leq$ is in at least in one of the two sets again. For $X \cap Y$, we use that both the sets X and Y are upper sets again. Proving the point for $X * Y$, consider consider the item (iii, b) of the definition 3.0.19. If there are points $a \in X$ and $b \in Y$ such that $R(abc)$ and a point $d \geq c$, then we know that $R(abd)$, as well. The part (iii, b) of the definition of a Kripke frame serve to prove to point for the operations $/$ and $\backslash$, as well.

First of all we will have to demonstrate that

$$< Up(P), \cup, \cap >$$

is a distributive lattice. Yet it is a trivial observation of set theory that for any set X, Y the set $X \cup Y$ is their least upper bound, while $X \cap Y$ is their greates lower bound with respect to the inclusion relation. The distributivity of union and intersection is also a trivial fact form the set theory.

Let's show that

$$< Up(P), *, T >$$

forms a monoid.

(i) To show that for any $B \in Up(P)$ we have $T * B = B * T = B$ we will use the item (iii, a) of the definition 3.0.19. Let's prove $T * B = B$ and begin with the $\supseteq$ part. Let's take any $b \in B$. We know that $b \leq b$ and therefore there is some $t \in T$ such that $R(tbb)$, therefore we have $b \in T * B$.

Now for the $\subseteq$ part. Let there be a given $c \in T * B$. We know that there is some $t \in T$ and some $b \in B$ such that $R(tbc)$. This means that $b \leq c$ and since B is an upper set, we have $c \in B$.

(ii) To show that for any$B, C, D \in Up(A)$ we have $(B * C) * D = B * (C * D)$, consider the item (iii, c) of the definition 3.0.19 and use its $\Rightarrow$ part for the proof of $\subseteq$ and its $\Leftarrow$ part for the proof of $\supseteq$.

Finally we shall prove the item (ii) of the definition 2.0.5, so we shall proof that $\backslash, /$ taken together and $*$ form a residuated pair. Let's show why

$$B * C \subseteq D \iff B \subseteq C/D$$

holds. We will examine both directions separately.

(⇒) We know that

$$\forall d\ (\exists b \exists c\ (b \in B\ \&\ c \in C\ \&\ R(bcd))\ \Rightarrow\ d \in D).$$

So if we take any $b \in B$ and any c and d such that $c \in C$ and $R(bcd)$, we can infer that $d \in D$, as we wanted.

(⇐) Here we know that

$$\forall b\ (b \in B\ \Rightarrow\ \forall c \forall d\ ((c \in C\ \&\ R(bcd))\ \Rightarrow\ d \in D)).$$

So let some $b \in B$ and $c \in C$ such that $R(bcd)$, exist. Then we can infer that $d \in D$, which is exactly what we wanted to demonstrate.

Showing that

$$B * C \subseteq D \iff C \subseteq B \backslash D$$

is entirely analogous.

Lastly it is obvious that P is the greatest element of the set of all upper subsets of the frame $\mathbb{F}$ and that $\emptyset$ is the least element.

Thus we have shown that by applying $\mathbb{I}$ to $\mathbb{F}$ we obtain a residuated lattice. □

Before examining further properties of the operation $\mathbb{I}$, let's see how to do such a transition the other way round.

4.2 Transforming residuated lattices into Kripke frames

As in the last section the transformation was led by the idea of upper sets, which can be in a way seen as a meaning of a given formula(set of the points where the formula is valid), here we will work with a kind of upper sets, too. Namely, with filters and for certain reasons those filters will have to be specifically prime filters. But before we proceed to the desired transformation itself, we shall introduce some necessary notions, beginning with that of filter.

Definition 4.2.1. *Let $\mathbb{A}$ be an algebra. then its subset $F \subseteq A$ is called a filter if it satisfies the following:*

(*i*) $\top \in F$

(*ii*) $\forall a \forall b\ (a \in F \text{ and } a \leq b, \text{ then } b \in F)$

(*iii*) $\forall a \forall b\ (a \in F \text{ and } b \in F, \text{ then } a \wedge b \in F)$

Adding one single condition we obtain the notion of a prime filter.

Definition 4.2.2. *We call a filter $F \subseteq A$ on algebra $\mathbb{A}$ a prime filter if it satisfies*

$$\forall a \forall b\ (a \vee b \in F\ \Rightarrow\ (a \in F\ \vee\ b \in F))$$

For the upcoming transformation of a residuated lattices into a Kripke frame we will need the lemma, which states how to obtain a prime filter which contains the elements we want to and omits the ones we want to. In order to prove it we will use Zorn's lemma, well known from the set theory, which is using the concepts of upper bound, chain and maximal element.

Definition 4.2.3. *Let there be a set X and $\leq$ (this time the symbol denotes any relation, not necessarily the one we know from the Kripke frames) a partial order $\leq \subseteq X^2$. Then we say that $Y \subseteq X$ is a chain if every two elements of Y are comparable, which means that the following condition holds:*

$$\forall x \forall y \in Y\ (x \leq y\ \vee\ y \leq x)$$

Then we also call $\leq$ a total order on X.
Let's further have $Y \subseteq X$. We say that an element $x \in X$ is:

(*i*) *an upper bound of Y if an only if for each element y of Y, the following holds:$y \leq x$.*

(*ii*) *is a maximal element of Y if and only if $x \in Y$ and, except x itself, there is no element $y \in Y$ such that $x \leq y$.*

Now the Zorn's lemma comes. It will be just stated here, an interested reader can find more about it in various publications about set theory.

Lemma 4.2.4. *Let there be a set X , $\leq$ a partial order $\leq \subseteq X^2$ such that every $Y \subseteq X$, which is a chain, has got an upper bound. Then there exist a maximal element $x \in X$.*

The lemma which will be presented now is a generalisation of one found in [alg2].

Lemma 4.2.5. *Let there be $B \subseteq A$ and $C \subseteq A$, two subsets of A such that*

$$\forall b_1 \in B \ldots \forall b_n \in B \forall c_1 \in C \ldots \forall c_m \in C\ \neg((b_1 \wedge \ldots \wedge b_n) \leq (c_1 \vee \ldots \vee c_m)),$$

for any finite n and m. Then there is a prime filter $U \subseteq A$ such that

$$B \subseteq U\ \&\ C \cap U = \emptyset.$$

Proof. We begin by a construction of a filter with the desired property. We can obtain it simply as(N denotes the set of all natural numbers):

$$F = \{a \in A;\ \exists n \in N \exists b_1 \in B \ldots \exists b_n \in B((b_1 \wedge \ldots \wedge b_n) \leq a)\}.$$

Thus we see that the set of all filters which contain the whole B and have common element with C is not empty. Now we shall consider the relation of inclusion $\subseteq$ on such filters.

Obviously, such a relation is a partial order. In addition to that, let's consider any chain G of such filters. This chain has got an upper bound, since we can consider the union of this chain $\bigcup G$. This union is even an element of this chain, since if there were some $b_1 \in B \ldots \exists b_n$ in it such that there would be some $c_1 \in C \ldots \forall c_m \in C$, which would break the condition together, then we can find all the elements $b_1 \in B, \ldots, b_n \in B$ in some filters of the chain and take the greatest one of them, which would yield a contradiction.

Now it is possible to apply the Zorn's lemma on the set of all the filters we are talking about, we obtain a maximal filter U satisfying our condition. We now show that U is a prime filter.

Consider any $u, v \notin U$. Then there must some $b_1, \ldots, b_n, b_{n+1}, \ldots b_m \in U$ and some $c_1, \ldots c_k, c_{k+1} \ldots c_l \in C$ such that $u \wedge b_1 \wedge \ldots \wedge b_n \leq c_1 \vee \ldots \vee c_k$ and $v \wedge b_{n+1} \wedge \ldots \wedge b_m \leq c_{k+1} \vee \ldots \ldots \vee c_l$. Since the operation of $\vee$ obtains the least upper bound, we can infer

$$c_1 \vee \ldots \vee c_k \vee c_{k+1} \vee \ldots \vee c_l \geq (u \wedge b_1 \wedge \ldots \wedge b_n) \vee (v \wedge b_{n+1} \wedge \ldots \wedge b_m).$$

And because the law of distributivity between $\vee$ and $\wedge$ holds, we also have

$$(u \wedge b_1 \wedge \ldots \wedge b_n) \vee (v \wedge b_{n+1} \wedge \ldots \wedge b_m) =$$

$$=(\mathrm{u} \vee v) \wedge (u \vee (b_{n+1} \wedge \ldots \wedge b_m)) \wedge (v \vee (b_1 \wedge \ldots \wedge b_n)) \wedge ((b_1 \wedge \ldots \wedge b_n) \vee (b_{n+1} \wedge \ldots \wedge b_m)).$$

Since the last three parts of the conjunction on the right side of the equation are in U, we can close that $u \vee v \notin U$, which is what was to be proven. □

The lemma just proven is a generalisation of a lemma which speaks just about two elements and not subsets of an algebra. Yet this lemma can be generalised even more.

Lemma 4.2.6. *Let there be n(some natural number) of pairs of subsets of a distributive residuated lattice and let all the pairs satisfy the condition B and C satisfied in the previous lemma. Then there is a sequence of prime filters $< P_1, ..., P_n >$ such that the filter P_i contains the first part of the pair number i, while omitting the other one.*

Proof. Extend all the first parts of the pairs to filters just as in the previous lemma and then consider the relation on n-tuples of filters such that one n-tuple $< P_1, ..., P_n >$ is under another one $< Q_1, ..., Q_n >$ if and only if for all $i \in 1, ..., n$, we have $P_i \subseteq Q_i$. This is a pre-order and just as in the previous lemma we can prove that each chain of such n-tuples has got an upper bound, so we can apply Zorn's lemma once again. □

But it will be a slightly different variant of the lemma yet which will prove so useful for transfering algebraic morphisms into frame morphisms. It will be actually three lemas. First we will need to define a specific ternary relation between upper subsets of a distributive residuated lattice.

Definition 4.2.7. *Let there be a distributive residuated lattice $\mathbb{A}$ and its three upper (with respect to $\leq$) subsets X, Y and Z. Then we introduce a ternary relation R such that $R(XYZ)$ if and only if the following condition holds:*

$$\forall a, b \ ((a \in X \ \& \ b \in Y) \ \Rightarrow \ a * b \in Z)$$

Now we shall see that the relation just defined might be introduced in other two equivalent manners.

Lemma 4.2.8. *Let there be a distributive residuated lattice $\mathbb{A}$ and its three upper subsets X, Y and Z. Then the following equivalences hold:*

$$R(XYZ) \iff \forall a \forall b\, (a/b \in X \;\&\; a \in Y \;\Rightarrow b \in Z) \iff \forall a \forall b\, (a \backslash b \in Y \;\&\; a \in X \;\Rightarrow b \in Z)$$

Proof. We will show the equivalence of the first two conditions, the equivalence of the first and the third one is analogous, which together gives us the remaining equivalence between the second and third condition as well.

($\Rightarrow$) Let the three upper sets satisfy the condition $R(XYZ)$. Let's also have $a/b \in X$ and $a \in Y$. Thanks to the fact that $R(XYZ)$ we know that $(a/b) * a \in Z$. Because of the lemma 2.0.7 we further know that $(a/b) * a \leq b$ and since Z is an upper set, we obtain the desired $b \in Z$.

($\Leftarrow$) Let the upper sets X, Y and Z satisfy the second of the three conditions which are being proven equivalent. Let's further have $a \in X$ and $b \in Y$. Because of the reflexivity of $\leq$ we have $a * b \leq a * b$, which is equivalent to $a \leq b/(a * b)$. Since X is an upper set, we have $b/(a * b) \in X$ and since the three upper sets satisfy the second of the conditions, we can close that $a * b \in Z$, which is what was to be proven.

□

With this proven we can proceed to the following lemma, which will be useful for the transfering of algebraic morphisms into frame morphisms.

Lemma 4.2.9. *Let there be two filters B and C and a prime filter H on a given distributive residuated lattice such that $R(BCH)$. Then there are prime filters $F \supseteq B$ and $G \supseteq C$ such that $R(FGH)$.*

Proof. Let's consider the following set of pairs of filters:

$$E = \{< F, G >;\ B \subseteq F \;\&\; C \subseteq G \;\&\; R(FGH)\}$$

Let's consider an order $\leq'$ on this set such that $< F, G > \leq' < F', G' >$ if and only if both $F \subseteq F'$ and $G \subseteq G'$. This is clearly a partial order. Now let's further consider any non-empty chain, which is a subset of this partial order, we can denote its elements as $< F_i, G_i >$ with i from some index set I. Now we can simply take the union of all the first parts of the pairs in the chain and the union of the second pair, let's denote them $\cup F_i$ and $\cup G_i$. We can see that $< \cup F_i, \cup G_i > \in E$, since if there were some $a \in \cup F_i$ and $b \in \cup G_i$ such that $a * b \notin H$, then there will be some F_i containing a and some G_j containing b. Now we can take $k = max\{i, j\}$ and since $< F_k, G_k > \in E$, we can infer that $a * b \in H$, which is a contradiction.

This means that each chain has got its maximal element and therefore we can apply Zorn's lemma to obtain the maximal element $< F, G >$ of E with respect to $\leq'$.

Now we shall see that both F and G are prime filters. We will prove this for F, the case of G is similar. Let's then have some $a \notin F$ and $b \notin F$ (thus we want to demonstrate that $a \vee b \notin F$). This means that any filter which extends F with respect to inclusion and which has got one of these two elements is not in E, so there is some $f_1 \in F$ and some $f_2 \in F$ such that there are elemets y_1 such that $f_1 \wedge a \leq y_1$ and y_2 such that $f_2 \wedge b \leq y_2$ and further there are some $g_1 \in G$ and $g_2 \in G$ such that $y_1 * g_1 \notin H$ and $y_2 * g_2 \notin H$. Now let's put $g = g_1 \wedge g_2$ and $f = f_1 \wedge f_2$. Clearly, since G and F are filters, we have $g \in G$ and $f \in F$ and further, since H is a filter and $*$ is monotone, we also have $(f \wedge a) * g \notin H$ and $(f \wedge b) * g \notin H$. And because H is also a prime filter, we also know that $((f \wedge a) * g) \vee ((f \wedge b) * g) \notin H$. Now we can use the distributivity between $*$ and $\vee$ and also between $\wedge$ and $\vee$. Thus we obtain

$$((f \wedge a) * g) \vee ((f \wedge b) * g) = ((f \wedge a) \vee (f \wedge b)) * g = (f \wedge (a \vee b)) * g$$

And therefore, if $a \vee b \in F$ were the case, then $((f \wedge a) * g) \vee f \wedge b) * g \in H$ would be also, which is a contradiction. □

For transfering algebraic morphisms into frame morphisms, we will also have to work with the concept of an ideal, which dual to that of a filter.

Definition 4.2.10. *Let there be a distributive residuated lattice $\mathbb{A}$ and its subset I. We call this subset an ideal if the following two conditions hold:*

- (i) $\forall a \forall b\ (a \in I\ \&\ b \leq a\ \Rightarrow\ b \in I)$
- (ii) $\forall a \forall b\ (a \in I\ \&\ b \in I\ \Rightarrow\ a \vee b \in I)$

And as in the case of filter, the definition can be more specified to obtain a prime ideal.

Definition 4.2.11. *Let there be a distributive residuated lattice $\mathbb{A}$ and its subset I. Then we call this subset a prime ideal if the following two conditions hold:*

- (i) *I is an ideal.*
- (ii) $\forall a \forall b\ (a \wedge b \in I\ \Rightarrow\ (a \in I\ \vee\ b \in I))$

Now we shall see that a complement of a prime ideal is a prime filter.

Lemma 4.2.12. *Let there be a distributive residuated lattice $\mathbb{A}$ and a prime ideal I on it. Let's further have have $I \neq \mathbb{A}$ Then the complement of I (denoted as $\overline{I}$) is a prime filter.*

Proof. We will first check that all the conditions for $\overline{I}$ being a filter hold.

So, let's have $a \in \overline{I}$ and $a \leq b$. Because $a \notin I$ and I is an ideal, we can infer that $b \in \overline{I}$.

Since $I \neq \mathbb{A}$, we can infer that $\top \in \overline{I}$.

Finally, if $a \in \overline{I}$ and $b \in \overline{I}$, we can infer that $a \wedge b \in \overline{I}$, since I is prime and thus cannot have $a \wedge b$ without having neither a, nor b as its element.

Now we shall see that $\overline{I}$ is a prime filter. Let's then have $a \in I$ and $b \in I$. Because I is an ideal, we can close $a \vee b \in I$, which completes the proof. □

Now we shall see two lemmas which, just as the lemma 4.2.9 will be useful for tranfering algebraic morphisms into frame morphisms. Both are from [art].

Lemma 4.2.13. *Let's have a distributive residuated lattice* $\mathbb{A}$ *and* F *a prime filter,* B *a filter and* C *an ideal on it such that* $R(FB\overline{C})$*. Then there is is prime filter* $G \supseteq B$ *and a prime ideal* $I \supseteq C$ *such that* $R(FG\overline{I})$ *still holds.*

Proof. Let's consider the following set of pairs of a filter and an ideal:

$$E = \{< G, I >;\ B \subseteq G \ \&\ C \subseteq I \ \&\ R(FG\overline{I})\}$$

Let's now consider the relation $\leq'$ on this set such that $< G, I > \leq' < G', I' >$ if an only if both $G \subseteq G'$ and $I \subseteq I'$. This is obviously a partial order and since every non-empty chain of such pairs has got a maximal element(proceed similarly as in lemma 4.2.9, which means consider the pair of union of all the filters of the chain and the union of all the ideals of the chain), we can apply Zorn's lemma and obtain the maximal element $< G, I >$ of E.

Now we shall see that G is a prime filter and I is a prime ideal. Let's start with G. We will use the lemma 4.2.8, which enables us to work with the relation R defined by the operation $/$ instead of $*$.

So, let's have two elements $a \notin G$ and $b \notin G$. This means that there are some $g_1 \in G$ and $g_2 \in G$ such that there are y_1 and y_2 such that $g_1 \wedge a \leq y_1$ and $g_2 \wedge b \leq y_2$ and finally there are some $y_1/i_1 \in F$ and $y_2/i_2 \in F$ such that $i_1 \in I$ and $i_2 \in I$. Let's put $g = g_1 \wedge g_2$ and $i = i_1 \vee i_2$. Obviously, $g \in G$ and $i \in I$. Further we can use that $/$ is antitone-monotone, as proven in the theorem 2.0.6 and in the lemma 2.0.9, and get both $(g \wedge a)/i \in F$ and $(g \wedge b)/i \in F$. Since F is a filter, we have $((g \wedge a)/i) \wedge (g \wedge b)/i \in F$ and using the lemma 2.0.10, we finally have that $((g \wedge a) \vee (g \wedge b))/i \in F$. Using distributivity between $\wedge$ and $\vee$, we further proceed to $(g \wedge (a \vee b))/i \in F$. Now, if $a \vee b \in G$ were the case, then $(g \wedge (a \vee b) \in F$ would also be and consequently $i \in \overline{I}$. But we have already seen that $i \in I$, so we have got a contradiction.

We shall now finish the proof by showing that the ideal I is prime. Let's have $a \notin I$ and $b \notin I$ and we will show that $a \wedge b \notin I$. Thus we have $i_1 \in I$ and $i_2 \in I$ and y_1 and y_2 such that $y_1 \leq a \vee i_1$ and $y_2 \leq b \vee i_2$ such that there some $g_1 \in G$ and $g_2 \in G$ with $g_1/y_1 \in F$ and $g_2/y_2 \in F$. Let's put $g = g_1 \wedge g_2 \in G$ and $i = i_1 \vee i_2 \in I$. Because the operation $/$ is antitone-monotone, we have $g/(a \vee i) \in F$ and $g/(b \vee i) \in F$. Thus we also have $(g/(a \vee i)) \wedge (g/(b \vee i)) \in F$ and using the lemma 2.0.8(distributivity of $/i$ with respect to $\wedge$), we get $g/((a \vee i) \wedge (b \vee i)) \in F$. Finally we use the distributivity between $\wedge$ and $\vee$, which gives us $g/((a \wedge b) \vee i)) \in F$. Now, if $a \wedge b$ were an element of I, then so would be $(a \wedge b) \vee i$, which would contradict the fact that $R(FG\overline{I})$.

□

One more lemma will be needed for transforming algeraic morphisms into frame morphisms.

Lemma 4.2.14. *Let's have a distributive residuated lattice* $\mathbb{A}$ *and* B *a filter,* G *a prime filter and* C *an ideal on it such that* $R(BG\overline{C})$*. Then there is is prime filter* $F \supseteq B$ *and a prime ideal* $I \supseteq C$ *such that* $R(FG\overline{I})$ *still holds.*

Proof. We won't do the proof in much detail, because it is quite similar to the previous one, only that this time it is working with the operation $\backslash$ instead of $/$. Thus we begin with a construction analogous to the one in the previous lemma, finding the maximal pair of a filter F and an ideal I such that $R(FG\overline{I})$. It remains to show that F is a prime filter and I is a prime ideal.

Let's start with F and some $a \notin F$ and $b \notin F$. Proceeding similarly to the previous lemma, we obtain some $f \in F$ and $i \in I$ such that $(f \wedge a)\backslash i \in G$ and $(f \wedge b)\backslash i \in G$. Now we know that $((f \wedge a) \vee (f \wedge b))\backslash i \in G$ and thanks to the distributivity also $(f \wedge (a \vee b))\backslash i \in G$. Thus if $a \vee b$ were in F, then so would be $f \wedge (a \vee b)$ and we would get a contradiction with $R(FG\overline{I})$.

Let's show that I is prime. Let's have some $a \notin I$ and $b \notin I$. Thus we will have some $i \in I$ and some $f \in F$ such that $f\backslash(a \vee i) \in G$ and $f\backslash(b \vee i) \in G$. Thus we can go on with $(f\backslash(a \vee i)) \wedge (f\backslash(b \vee i)) \in G$ and consequently also $f\backslash((a \vee i) \wedge (b \vee i)) \in G$ and by distributivity $f\backslash((a \wedge b) \vee i) \in G$. Now, if $a \wedge b \in I$ were the case, then so would be $(a \wedge b) \vee i) \in I$, which would contradict the fact that $R(FG\overline{I})$. Thus we have should that I is a prime ideal. □

After proving some facts which will be useful for transforming algebraic morphism into frame morphism, we shall now get back to the transforming distributive residuated lattices into Kripke frames and introduce a map which will do this.

Definition 4.2.15. *A map* $\mathbb{J}$

$$\mathbb{J}(\mathbb{A}) =< PF(\mathbb{A}), PF1, \subseteq, R >$$

with the class of all residuated lattices as its domain is said to transform residuated lattices if it satisfies the following conditions:

- (i) *$PF(\mathbb{A})$ is the set of all prime filters on $\mathbb{A}$ and $PF1$ is a set of all prime filters on $\mathbb{A}$, which contain 1 as their element.*
- (ii) *$\subseteq$ is the inclusion relation on $PF(\mathbb{A})$.*
- (iii) *The relation R is the relation introduced in the definition 4.2.7.*

Now we shall approach the proof that the map from the definition 4.2.15 does indeed transfer residuated lattices into Kripke frames.

Theorem 4.2.16. *For any residuated lattice $\mathbb{A}$ its picture $\mathbb{J}(\mathbb{A})$ is a Kripke frame as defined in 3.0.19.*

Proof. We have to prove that $PF1$, $\subseteq$ and R have the necessary properties. The point for $PF1$ is obvious, since $\subseteq$ is an inclusion relation. As such it is also obviously a partial order, so we have to prove just the items regarding the relation R, which are all stated as parts of (iii). Let's handle them one after another with any prime filers B, C, D and E on $\mathbb{A}$.

(a) $(\Leftarrow)$: We have a prime filter U such that $U \in PF1$ and $R(BUC)$. We want to demonstrate that $B \subseteq C$. Let's consider any $b \in B$. Since we have that $b * 1 = b$ and $b * 1 \in C$, we also have that $b \in C$, which means that B is a subset of c.

$(\Rightarrow)$:We begin with $B \subseteq C$ and we want to find a prime filter U such that $1 \in U$ and $R(BUC)$. This means that we want not only 1 to be an element of U, but also none of

$$N = \{c;\ \exists b \in B(b * c \notin C)\}$$

to be an element of U. Let's take any finite subset $\{c_1, \ldots, c_n\}$ of C and the element $c_1 \vee \ldots \vee c_n \ \in \mathbb{A}$, abbreviated as $\bigvee c$.

We will prove that

$$1 \leq \bigvee c.$$

is not true. Let's argue by contradiction. We know that each c_i has got its b_i such that $b_i \in B$ and $b_i * c_i \notin C$. Since $B \subseteq C$, we also know that $b_i \in C$ and thus writing b (with a meaning analogous to $\bigvee c$), we have $b \in C$. Since $*$ is monotonous and since we have the lemma 2.0.8, we can infer that

$$1 * \bigwedge b = \bigwedge b \leq (c_1 * (\bigwedge b)) \vee \ldots \vee ((c_n * (\bigwedge b)).$$

Thus the right side of $\leq$ is also in C. Then, since C is a prime filter, for some $i \in \{1 \ldots n\}$ we have $c_i * (\bigwedge b) \in C$. With C being an upper set we also have $c_i * b_i \in C$, but this is a contradiction.

Now we can just apply the lemma 4.2.5.

(b) This item is clearly a consequence of some basic properties of the inclusion relation.

(c) $(\Rightarrow)$: Let's suppose there is a prime filter F such that $R(BCF)$ and $R(FDE)$. We are going to show that there exists a prime filter G such that $R(BGE)$ and $R(CDG)$. This means that we want G to contain every $c * d$'s for any elements of C and D. So

$$\{c * d;\ c \in C \ \& \ d \in D\} \subseteq G.$$

We also want that

$$(G \cap \{g;\ \exists b \in B(b * g \notin E)\}) = \emptyset.$$

We will apply the lemma 4.2.5 once again, but first we have have to prove that no $(c_1 * d_1) \wedge \ldots \wedge (c_n * d_n)$ is under some $g_1 \vee \ldots \vee g_m$, which we abbreviate as

$$\bigwedge (c * d) \not\leq \bigvee g.$$

Let's suppose the opposite for the sake of contradiction and have such $\bigwedge(c * d)$ and $\bigvee g$. For each we g_i we have a corresponding $b_i \in B$ such that $b_i * c_i \notin E$. Now we have

$$(\bigwedge b) * (\bigwedge (c * d)) \leq (\bigwedge b) * (\bigvee g).$$

By a multiple use of the lemma 2.0.8 and the by associativity of $*$, we first obtain that $(\bigwedge b) * (\bigwedge(c * d)) \in E$ and consequently, we get $(\bigwedge b) * (\bigvee g) \in E$. Now, using 2.0.8 again and the fact that E is a prime filter, we obtain that for some b_i and g_i

$$b_i * g_i \in E,$$

which is a contradiction.

The $(\Leftarrow)$ direction is done in a similar way.

$\square$

Chapter 5

Transfering algebraic homomorfisms into frame-morphisms and vice versa

In the previous chapter we have examined the transitions between Kripke frames and distributive residuated lattices, which from the point of view of category theory means transforming objects of two distinct categories. Now we are going to proceed by looking at similar transformations between what the category theory calls arrows, which in our two considered categories means algebraic morphisms and frame-morphisms. Let's begin with some general introduction to the category theory. The notions from category theory which we will use can be found in [cat], where an interested reader can find more about them.

Definition 5.0.17. *Let's have an ordered class*

$$C = < O, Ar, \circ, 1 >$$

where $Ar = \{Ar_i;\ Ar_i \subseteq O^2\}$, $\circ$ *is a binary operation on* $\bigcup Ar$ *(*$f \circ g$ *defined for such elements* f *and* g *of* $\bigcup Ar$ *that the codomain, the second part, of the pair of* f *is the same as the domain, the first part, of* g*) and* 1 *is a unary operation which assigns to an element of* O *and element of* Ar*. We call such a class a category if it satisfies the following conditions:*

(i) For any f, g *and* $h \in \bigcup Ar$,

$$f \circ (g \circ h) = (f \circ g) \circ h$$

(ii) For any $o \in O$ *and any* $f \in \bigcup Ar$ *such that* $1(o) \circ f$ *exists we have*

$$1(o) \circ f = f$$

and for every $g \in \bigcup Ar$ *such that* $g \circ 1(o)$ *exists we have:*

$$g \circ 1(o) = g$$

Now we shall consider two categories, namely those of Kripke frames and frame morphism between them, and of distributive residuated lattices and algebraic morphisms between them. We will show how to transform the two kinds of morphishms in each other, which, together with the results obtained in the previous chapter, allows to define two functors between the two categories. It should be mentioned that categories could be either small, which means that both its objects and arrows are sets, or big, which means at least one of them is a proper class. The two categories we will work with here are both big.

5.0.1 The two categories

Let's begin with a category of distributive residuated lattices and their morphisms. First we must define what a morphism is.

Definition 5.0.18. *A map $f : A \longrightarrow B$ between two residuated lattices is a morphism if it satisfies the following:*

(*i*) *We have $f(1_{\mathbb{A}}) = 1_{\mathbb{B}}$ and similarly for $\top$ and $\bot$.*

(*ii*) *For any a and $b \in A$ we have*

$$f(a) * f(b) = f(a * b)$$

(*iii*) *For any a and $b \in A$ we have*

$$f(a) \vee f(b) = f(a \vee b)$$

(*iv*) *For any a and $b \in A$ we have*

$$f(a) \wedge f(b) = f(a \wedge b)$$

(*v*) *For any a and $b \in A$ we have*

$$f(a) \backslash f(b) = f(a \backslash b)$$

(*vi*) *For any a and $b \in A$ we have*

$$f(a)/f(b) = f(a/b)$$

Now we will introduce the category based on the concepts we have.

Definition 5.0.19. *The category of category of distributive residuated lattices and their morphisms is the following class*

$$RLM =< RL, M, \circ, 1 >,$$

where RL is the class of all residuated lattices, M is a class of morphisms among them, $\circ$ is a composition of morphisms and 1 assigns to each residuated lattice the identity map on it.

It is obvious that we have indeed defined a category.

Lemma 5.0.20. *RLM is a category according to the definition 5.0.17.*

Proof. Obvious. □

To get the second of our categories, let's first define the frame-morphisms. The definition we are about to present can be found in [CEJ]

Definition 5.0.21. *Let $\mathbb{F}$ and $\mathbb{G}$ be two Kripke frames according to the definition 3.0.19. Then a map $f : \mathbb{F} \longrightarrow \mathbb{G}$ is called a frame-morphism if it satisfies the following conditions:*

(*i*) $f^{-1}[T_{\mathbb{G}}] = T_{\mathbb{F}}$

(*ii*) $\forall a \forall b \forall c \in \mathbb{F}\ (R_{\mathbb{F}}(abc)\ \Rightarrow\ R_{\mathbb{G}}(f(a)f(b)f(c)))$

(*iii*) $\forall a \forall b \forall c \in \mathbb{G} \forall k \in \mathbb{F}\ (R_{\mathbb{G}}(abc)\ \&\ f(k) = a \Rightarrow \exists l \exists m (f(l \geq b\ \&\ f(m) \leq c\ \&\ R_{\mathbb{F}}(klm))$

(*iv*) $\forall a \forall b \forall c \in \mathbb{G} \forall k \in \mathbb{F}\ (R_{\mathbb{G}}(bac)\ \&\ f(k) = a\ \Rightarrow\ \exists l \exists m\ (f(l) \geq b\ \&\ f(m) \leq c\ \&\ R_{\mathbb{F}}(lkm))$

(*v*) $\forall a \forall b \forall c \in \mathbb{G} \forall k \in \mathbb{F}\ (R_{\mathbb{G}}(bca)\ \&\ f(k) = a \Rightarrow \exists l \exists m\ (f(l) \geq b\ \&\ f(m) \geq c\ \&\ R_{\mathbb{F}}(lmk))$

So (*ii*) is a forward looking condition, while the remaining three are the backward looking conditions.

It should be noted that a more strict definition of frame morphisms, where $\leq$ is replaced with $=$, is usually used. It can be found in [Gb].

The following lemma relates the concept of frame morphism to the relation $\leq$.

Lemma 5.0.22. *Let $\mathbb{F}$ and $\mathbb{G}$ be two Kripke frames according to the definition 3.0.19 and $f : \mathbb{F} \to \mathbb{G}$ be a frame-morphism. Then the following holds:*

(*i*) $\forall a \forall b \in \mathbb{F}(a \leq_{\mathbb{F}} b\ \Rightarrow\ f(a) \leq_{\mathbb{G}} f(b))$

(*ii*) $\forall a \forall b \in G \forall c \in \mathbb{F}\ (a \leq_{\mathbb{G}} b\ \&\ f(c) = a\ \Rightarrow\ \exists d \in \mathbb{F}(c \leq_{\mathbb{F}} d\ \&\ f(d) \leq_{\mathbb{G}} b))$

(*iii*) $\forall a \forall b \in G \forall c \in \mathbb{F}\ (b \leq_{\mathbb{G}} a\ \&\ f(c) = a\ \Rightarrow\ \exists d \in \mathbb{F}(d \leq_{\mathbb{F}} c\ \&\ f(d) \geq_{\mathbb{G}} b))$

Proof. Just consider the definition 5.0.21 and the relation between $\leq$ and R, as stated in the definition 3.0.19. For two points a and b in a frame we have that

$$a \leq b \iff \exists t \in T(R(atb)).$$

□

The concept of a frame-morphism may be seen as a weakening of the concept of bisimulation, which was presented in the definition 3.0.24. It is nevertheless still enought for the equivalency provided by bisimulation.

Theorem 5.0.23. *Let there be two Kripke models $\mathbb{M}$ and $\mathbb{N}$ and $f : M \longrightarrow N$ such a frame-morphism that for any $a \in \mathbb{M}$ we have that exactly the same atomic formulae hold in a and in $f(a)$ Then for every formula φ and any $a \in \mathbb{M}$ and $u \in \mathbb{N}$ the following holds:*

$$\mathbb{M}, a \Vdash \varphi \iff \mathbb{N}, u \Vdash \varphi.$$

Proof. The proof is mainly done by the induction on the complexity of φ just as for the theorem 3.0.25 for the bisimulation relation. We will only demonstrate the parts which slightly differ.

Let's suppose that $\varphi = \psi \& \chi$ and $\mathbb{N}, u \Vdash \varphi$. we know that there are some v and w in the model $\mathbb{N}$ such that $R_{\mathbb{N}}(vwu)$, $\mathbb{N}, v \Vdash \psi$ and $\mathbb{N}, w \Vdash \chi$. Now we know that in the model $\mathbb{M}$ there are some points b and c such that $R_{\mathbb{N}}(bca)$ and $u \leq_{\mathbb{N}} f(b)$ and $w \leq_{\mathbb{N}} f(c)$. Using the persistency proven in 3.0.23, we obtain $\mathbb{N}, f(b) \Vdash \psi$ and $\mathbb{N}, f(c) \Vdash \chi$. And finally, using the induction assuption, we get $\mathbb{M}, b \Vdash \psi$ and $\mathbb{M}, c \Vdash \chi$, which means $\mathbb{M}, a \Vdash \psi \& \chi$.

There is also a little more consideration required for the proof of $\mathbb{N}, u \Vdash \varphi$ from $\mathbb{M}, a \Vdash \varphi$ for the case of $\varphi = \psi \rightarrow \chi$ (the case of $\varphi = \psi \leftarrow \chi$ is analogous). So, let's take random two points v and w of the model $\mathbb{N}$ such that $R_{\mathbb{N}}(uvw)$ and $\mathbb{N}, v \Vdash \psi$. Now, we know that there are some points b and c in the other model such that $R_{\mathbb{M}}(abc)$, $v \leq_{\mathbb{N}} f(b)$ and $f(c) \leq_{\mathbb{N}} w$. Now we know that $\mathbb{N}, f(b) \Vdash \psi$ and thus $\mathbb{M}, b \Vdash \psi$. Because we pressupose $\mathbb{M}, a \Vdash \psi \rightarrow \chi$, we know that $\mathbb{M}, c \Vdash \chi$. Using the induction assuption, we obtain $\mathbb{N}, f(c) \Vdash \chi$ and because $f(c) \leq_{\mathbb{N}} w$, we finally get $\mathbb{N}, w \Vdash \chi$, which means $\mathbb{N}, u \Vdash \varphi$.

□

With this notion of morphism we have obtained yet another category.

Definition 5.0.24. *The category of Kripke frames and their morphisms is the following class*

$$FRB = < Fr, B, \circ, 1 >,$$

where Fr is the class of all Kripke frames as defined in 3.0.19, B is a class of frame-morphisms, $\circ$ is a composition of them and 1 assigns to each Kripke frame the identity map on it (which is clearly a frame-morphism itself).

It is obvious that we have indeed defined a category.

Lemma 5.0.25. *FRB is a category according to the definition 5.0.17.*

Proof. Obvious. □

Being equipped with all the necessary definitions, we shall proceed to show the transition between the two kinds of morphisms.

5.1 Transforming frame-morphisms to algebraic morphisms

Now we are about to see how to transform any frame-morphism into an algebraic morphism so that we will also preserve the map composition and the identity morphism. We will use the map $\mathbb{I}$ from the definition 4.1.1.

Definition 5.1.1. *A map K which has the class of all frame-morphisms as its domain is a transformation of frame-morphisms if for any two Kr. frames $\mathbb{F}$ and $\mathbb{G}$ and $f : \mathbb{F} \longrightarrow \mathbb{G}$ a frame-morphism among them we have*

$$K(f) : \mathbb{I}(\mathbb{G}) \longrightarrow \mathbb{I}(\mathbb{F})$$

And for any upper set $U \in \mathbb{I}(\mathbb{G})$ we have

$$K(f)(U) = \{w \in \mathbb{F};\ f(w) \in U\} = f^{-1}[U]$$

Now we will show that such a map does indeed transform a frame morphism into an algebraic morphism, but one little fact has to be demostrated before doing that.

Lemma 5.1.2. *For any frame-morphism $f : \mathbb{F} \longrightarrow \mathbb{G}$ and an upper set $U \subseteq G$ the pre-image of this upper set $f^{-1}[U]$ is an upper set too.*

Proof. Let's have any v and $w \in F$ such that $v \leq w$ and $f(v) \in U$. Then we have $f(v) \leq f(w)$ and consequently $f(w) \in U$, which is what we wanted, since it means that $w \in f^{-1}[U]$. □

Theorem 5.1.3. *For any frame morphism $f : \mathbb{F} \longrightarrow \mathbb{G}$ its value $K(f)$ is an algebraic morphism.*

Proof. This proof will be rather complex, we will have to show that $K(f)$ behaves well for all the operations as stated in the definition 5.0.18. It is clear, though, that $K(f)(1_{I(\mathbb{G})})$ is exactly $(1_{I(\mathbb{F})})$, which follows from the item (i) of the definition 5.0.21. Let's now prove the case for all the binary operations.

(i) We want to show that for any upper sets U and $V \in I((G))$ we have

$$K(f)(U \cap V) = K(f)(U) \cap K(f)(V).$$

But both the inclusions $\subseteq$ and $\supseteq$ are quite obvious.

(ii) The case for the operation $\cup$ is practically the same.

(iii) Let's now consider the operation $*$, which means proving

$$K(f)(U * V) = K(f)(U) * K(f)(V).$$

Let's first do $\subseteq$. In fact, according to the definition 4.1.1, $c \in K(f)(U * V)$ means that there are some $u \in U$ and $v \in V$ such that $R(uvf(c))$. Then, according to the definition 5.0.21, we obtain that there are some a and b such that $R(abc)$, while $f(a) \geq u$ and $f(b) \geq v$. Since we are working with upper sets, it also means that $f(a) \in U$ and $f(b) \in V$, which is what we wanted, since it means that $c \in K(f)(U) * K(f)(V)$.

For $\supseteq$ we use just the fact that since f is a frame morphism, we can infer $R(f(a)f(b)f(c))$ from $R(abc)$.

(*iv*) Now we are proving

$$K(f)(U/V) = K(f)(U)/K(f)(V).$$

Let's start with $\supseteq$ this time. Consider any $u \in K(f)(U)/K(f)(V)$. Now we have to show that $f(u) \in U/V$. Let's consider any a and b such that $R(f(u)ab)$ and $a \in U$. Using the definition 5.0.21 once again we obtain that there are some v and w such that $R(uvw)$, $f(v) \geq a$ and $f(w) \leq b$. Since U is an upper set, we have that $v \in K(f)(U)$. Because $u \in K(f)(U)/K(f)(V)$, we also know that $w \in K(f)(V)$. And since V is an upper set as well, we can close that $b \in V$, which is what was to be proven.

In the case of $\subseteq$, we have some $u \in K(f)(U/V)$. Let's consider any points v and w such that $v \in K(f)(U)$ and $R(uvw)$. By the definition of frame morphism we have that $R(f(u)f(v)f(w))$ and since $f(v) \in U$, we also have that $f(w) \in V$ and finally that $w \in K(f)(V)$, which is what we wanted.

(*v*) The last thing is to prove that

$$K(f)(U \backslash V) = K(f)(U) \backslash K(f)(V).$$

But that is entirely analogous to the previous item.

□

It is also straightforward to see that K behaves well with respect to identity maps and map composition.

Lemma 5.1.4. *Let $f : \mathbb{F} \longrightarrow \mathbb{G}$ and $g : \mathbb{G} \longrightarrow \mathbb{H}$ be two frame-morphisms. Then:*

$$K(f \circ g) = K(g) \circ K(f).$$

In addition to that for any identity map $f : \mathbb{F} \longrightarrow \mathbb{F}$ on a Kripke frame $K(f)$ is an identity map on the corresponding residuated lattice $I(\mathbb{F})$

Proof. Obvious. □

Now we shall proceed to the opposite transition and prove similar facts about it.

5.2 Transforming algebraic morphisms into frame-morphisms

Definition 5.2.1. *A map L which has the class of all algebraic morphisms between distributive residuated lattices as its domain is a transformation of algebraic morphisms if for any two distributive residuated lattices $\mathbb{A}$ and $\mathbb{B}$ and $f : \mathbb{A} \longrightarrow \mathbb{B}$ an algebraic morphism among them we have*

$$L(f) : \mathbb{J}(\mathbb{B}) \longrightarrow \mathbb{J}(\mathbb{A}),$$

(the map $\mathbb{J}$ was defined in 4.2.15) and for any prime filter $P \in \mathbb{J}(\mathbb{B})$ we have

$$L(f)(P) = \{a \in \mathbb{A};\ f(a) \in P\} = f^{-1}[P]$$

Just as we had to prove that $K(f)$ of a frame-morphism did indeed assign to an upper set an upper set, we now have to prove that L of an algebraic morphism does indeed assign to a prime filter a prime filter.

Lemma 5.2.2. *For any algebraic morphism $f : \mathbb{A} \longrightarrow \mathbb{B}$ and a prime filter $P \subseteq B$ the pre-image of this prime filter $f^{-1}[P]$ is a prime filter too.*

Proof. We have to check three conditions about $f^{-1}[P]$ - that it contains $\top$, that it is an upper set, that it is closed on $\wedge$ and that it is prime.

- (i) It is obvious that $\top \in f^{-1}[P]$, since $\top \in P$.
- (ii) Let's have any p and q such that $p \in f^{-1}[P]$ and $p \leq q$. Then $f(p) \leq f(q)$ and $f(q)$ is in P, since it is an upper set.
- (iii) Let's have any p and q such that both are elements of $f^{-1}[P]$. Since f is an algebraic morphism we know that
$$f(p) \wedge f(q) = f(p \wedge q),$$
which means that $f(p \wedge q) \in P$, which is what we wanted to prove.
- (iv) Let's have any $p \vee q \in f^{-1}[P]$. Once again we use that f is an algebraic morphism, so we know that
$$f(p) \vee f(q) = f(p \vee q).$$

Since P is a prime filter, we know that at least one of $f(p)$ and $f(q)$ is an element of P and consequently at least one of p and q is an element of $f^{-1}[P]$. □

Now we are about to show that L is indeed transforming algebraic morphisms into frame-morphisms.

Theorem 5.2.3. *For any algebraic morphism $f : \mathbb{A} \longrightarrow \mathbb{B}$ its value $L(f)$ is a frame-morphism.*

Proof. We have to prove all the items of the definition 5.0.21. Let's have an arbitrary algebraic morphism $f : \mathbb{A} \longrightarrow \mathbb{B}$.

(i) Since f is an algebraic morphism we have

$$f(1_{\mathbb{A}}) = 1_{\mathbb{B}},$$

which guarantees the condition which is to be proven.

(ii) Let's have three prime filters X, Y and $Z \in J(\mathbb{B})$ such that $R(XYZ)$. We want to show that $L(f)(X)$, $L(f)(Y)$ and $L(f)(Z)$ are in the relation R defined in 4.2.7. Let's have then any $a \in L(f)(X)$ and $b \in L(f)(Y)$. Since f is an algebraic morphism we have

$$f(a) * f(b) = f(a * b),$$

which means that $a * b \in L(f)(Z)$, which means that the three prime filters are in the relation R, as we wanted.

(iii) Let's have three prime filters X, Y and $Z \in J(\mathbb{A})$ such that $R(XYZ)$ and let's also have a prime filter $S \in J(\mathbb{B})$ such that $L(f)(S) = X$. We want to find some prime filters T and U such that $L(f)(T) \supseteq Y$, $L(f)(U) \subseteq Z$ and also $R(STU)$. We know that $\overline{Z}$ is a prime ideal. Let's consider $f[\overline{Z}]$ and close it downwards. Thus we get an ideal, let's denote it as I' (This will be an ideal because of the behaviour of f as an algebraic morphism to $\vee$. As we can extend any subset of a distributive residuated lattice into a filter, so we can extend it into an ideal by adding all the elements which are under some D, where D is a finite subset of the set we are extending into an ideal). Let's similarly consider $f[Y]$ and close it upwards, obtaining thus a filter (this will be a filter because of the behaviour of f as an algebraic morphism to $\wedge$), which we shall denote as $T^{'}$. Now if we consider some $t/u \in S$ such that $t \in T^{'}$, then we will show that $u \in \overline{I'}$. Let's for the sake of contradiction suppose that $u \in I^{'}$. Thus there is some $f(a) \leq t$ such that $a \in Y$ and some $f(i)$ such that $u \leq f(i)$. Because the operation $/$ is antitone-monotone, we obtain $f(a)/f(i) \in S$ and since f is an algebraic morphism, we can deduce from this that $f(a/i) \in S$, as well. Thus we know that $(a/i) \in X$, then $a \in Y$ and finally that $i \in \overline{Z}$, which is a contradiction with the fact that $R(XYZ)$. Thus we have shown that $R(ST^{'}U')$.

So now we can use the lemma 4.2.13 and obtain a prime filter $T \supseteq T^{'}$ and a prime ideal $I \supseteq I^{'}$ such that $R(ST\overline{I})$. We obviously have that $L(f)(T) \supseteq Y$. Further we have $I \supseteq f[\overline{Z}]$. From this we can infer that $\overline{I} \subseteq \overline{f[\overline{Z}]}$. Let's now consider any $f(z) \in \overline{I}$. We know that $f(z) \in \overline{f[\overline{Z}]}$, which means $f(z) \notin f[\overline{Z}]$. Therefore $z \in Z$. So we have proven $L(f)(\overline{I}) \subseteq Z$ and thus we may put $U = \overline{I}$, which is the prime filter we were looking for.

(iv) Let's have three prime filters X, Y and $Z \in J(\mathbb{A})$ such that $R(XYZ)$ and let's also have a prime filter $T \in J(\mathbb{B})$ such that $L(f)(T) = Z$. Let' proceed as in the previous item and obtain a filter $S^{'} \supseteq f[X]$ and an ideal $I^{'} \supseteq f[\overline{Z}]$ such that $R(S^{'}T\overline{I^{'}})$(this time we check this condtion working with the operation $\backslash$ and not with $/$ as in the previous item).

Now we use the lemma 4.2.14 to extend S' into a prime filter S and I' into a prime ideal I such that $R(ST\overline{I})$. We finally take a prime filter $U = \overline{I}$, which finishes the proof.

(*v*) Let's have three prime filters X, Y and $Z \in J(\mathbb{A})$ such that $R(XYZ)$ and let's also have a prime filter $U \in J(\mathbb{B})$ such that $L(f)(U) = Z$. Let's consider $f[X]$ and $f[Y]$. Obviously, for each $x \in f[X]$ and each $y \in f[Y]$ we have $x * y \in U$, because f is an algebraic morphism. This still holds even if we extend $f[X]$ and $f[Y]$ to filters (let's denote them as S' and T') by closing them upwards with the respect to $\leq$. Thus we have a prime filter U and filters S' and T' such that $R(S'T'U)$. Now we use the lemma 4.2.9 and get a prime filter $S \supseteq S'$ and $T \supseteq T'$ such that $R(STU)$. From the construction it is obvious that $L(f)[S] \supseteq X$ and $L(f)[T] \supseteq Y$ and thus we have finished the proof.

□

Now we can also see that L behaves well with respect to identity maps and to map composition.

Lemma 5.2.4. *Let $f : \mathbb{A} \longrightarrow \mathbb{B}$ and $g : \mathbb{B} \longrightarrow \mathbb{C}$ be two algebraic morphisms. Then:*

$$L(f \circ g) = L(g) \circ L(f).$$

In addition to that for any identity map $f : \mathbb{A} \longrightarrow \mathbb{A}$ on a residuated lattice $L(f)$ is an identity map on the corresponding Kripke frame $\mathbb{J}((A))$.

Proof. Obvious. □

Now we shall put the just proven facts into the context of the category theory. We have in fact constructed two functors between the two categories we are working with.

Definition 5.2.5. *Let there be two categories $C =< O_C, Ar_C, \circ_C, 1_C >$ and $D =< O_D, Ar_D, \circ_D, 1_D >$. Then we say that the map $G : O_C \cup Ar_C \longrightarrow O_D \cup Ar_D$ is a functor from the category C to category D if the following conditions hold*

(*i*) $\forall o \in O_C \ (G(o) \in O_D)$

(*ii*) $\forall f \in Ar_C \ (G(f) \in Ar_D)$

(*iii*) *For each arrow $f : o_1 \longrightarrow o_2$, which is an element of Ar_C, the corresponding arrow $G(f)$ is between the values of of o_1 and o_2, so*

$$G(f) : G(o_1) \longrightarrow G(o_2)$$

.

(*iv*) $\forall o \in O_C \ (G(1_C(o) = 1_D(G(o)))$

(*v*) $\forall f \forall g \in Ar_C\ (G(f \circ g) = G(f) \circ G(g))$

Now, for every category we can consider its corresponding opposite category.

Let there be a category $C =< O, Ar, \circ, 1 >$. Then C^{op} is a category with exactly the same objects, so with the same O and also with the same 1. The arrows of C^{op}, denoted as Ar^{op}, are among the reversed pairs of objects. So, if there is an arrow $f : o_1 \longrightarrow o_2$ in C, then there is an arrow $f^{op} : o_2 \longrightarrow o_1$ in C^{op} and there are no other arrows in Ar^{op}.

Now we can state what functors we have actually constructed between the categories RLM and FRB.

Theorem 5.2.6. *Let there be a map* $G : RLM \longrightarrow FRB^{op}$ *such that*

(*i*) $\forall \mathbb{A} \in RL\ (G(\mathbb{A}) = J(\mathbb{A}))$

(*ii*) $\forall f \in M\ (G(f) = L(f))$

Then G *is a functor*

Proof. For the item (*iii*) of the definition of the functor, note that the map L was constructed as assigning to a given algebraic morphism $f : \mathbb{A} \longrightarrow \mathbb{B}$ a frame morphism $L(f) : J(\mathbb{B}) \longrightarrow J(\mathbb{A})$. All the other items are obvious. □

And there is also an opposite functor.

Theorem 5.2.7. *Let there be a map* $H : FRB \longrightarrow RLM^{op}$ *such that*

(*i*) $\forall \mathbb{F} \in Fr\ (G(\mathbb{F}) = I(\mathbb{F}))$

(*ii*) $\forall f \in B\ (G(f) = K(f))$

Then H *is a functor*

Proof. For the item (*iii*) of the definition of the functor, note that the map K was constructed as assigning to a given frame morphism $f : \mathbb{F} \longrightarrow \mathbb{G}$ an algebraic morphism $K(f) : I(\mathbb{G}) \longrightarrow I(\mathbb{F})$. All the other items are obvious. □

Chapter 6

Adjunction? A few remarks

We have seen how a distributive residuated lattice can be transferred to a Kripke frame and vice versa and we have also seen how to transfer morphism between the two structures.

It is natural to ask, what would a residuated lattice look like if we transfer it to a Kripke frame and then back. Or, on the other hand, what would a Kripke frame look like after such a double transition. What relations does this new distributive lattice or Kripke frame have to the original one?

In case of the distributive residuated lattice the following theorem gives a hint of the relation.

Theorem 6.0.8. *Let $\mathbb{A}$ be a distributive residuated lattice. Then there is an algebraic morphism f such that*

$$f : \mathbb{A} \longrightarrow I(J(\mathbb{A})).$$

Proof. Let's consider a map f, which assigns to an element $a \in \mathbb{A}$ a set of all prime filters which contain it as an element (obviously, such a set is an upper set). Let's show that such a map is the map the existence of which is claimed by the theorem. We will have to prove that it preserves all the operations.

(i) To show that $f(a \wedge b) = f(a) \cap f(b)$, consider that $a \wedge b \leq a$ and $a \wedge b \leq b$ for $\subseteq$ and for $\supseteq$, consider that each filter is closed on $\wedge$.

(ii) To show that $f(a \vee b) = f(a) \cup f(b)$, consider what the definition 4.2.2 says about prime filters and for $\supseteq$ consider that $a \leq a \vee b$ and $b \leq a \vee b$.

(iii) To prove the part $\supseteq$ of $f(a * b) = f(a) * f(b)$ is very simple. For the other inclusion, for a prime filter $Q \in f(a * b)$ we need to find two prime filters O and P such that $R(OPQ)$. Such a pair of prime filters can be obtained exactly in the same manner as in the part ((vii)) of the prove of theorem 5.2.3.

(iv) This time it is the $\subseteq$ part of $f(a/b) = f(a)/f(b)$, which is obvious. The other part is done by contraposition. Let's have a prime filter O such, that it does not contain a/b as its element. Now let's take a the smallest filter extension of a (which means a set containing every $d \geq a$) and the smallest ideal extension of b (which means a set containing every $e \leq b$). Now we can just proceed as in the lemma 4.2.13.

(v) Analogously to the previous.

□

So far I haven't succeeded at proving something similar for the Kripke frames. I have tried to use the map which to a given frame $\mathbb{F}$ would assign to each of its points all the upper subsets of P_F, which contain this point. A set of such upper sets is indeed a prime filter, but I haven't found a way to prove the last three conditions of the definition 5.0.21. Thus it remains open whether for a given frame $\mathbb{F}$ there is a frame morphism $f : \mathbb{F} \longrightarrow J(I(\mathbb{F}))$.

Proving an existence of such a morphism would have interesting consequences, since this would enable to establish an adjunction between the two categories we are working with. This is then an open question of my thesis and an impulse for further research.

Conclusion

Thus the two semantics of the distributive FL logic were presented and some steps were done to show how do they relate to each other from the point of view of the category theory. But more can be done in this field, as was mentioned in the last chapter.

Besides this it might be interesting to study how can the constant 0 be useful for both the semantics and how can the connective of negation be introduced (probably using this constant) and what can be proven about it.

It would be also interesting to study further properties of the extensions of the distributive FL logic and their semantics, for example to study the relevance logic. With the conective of negation the disjunction property might lead to another result, namely to the admissibility of disjunctive syllogism in some of the substructural logics (this syllogism, which from the validity of $\varphi \vee \psi$ and $\neg\varphi$ deduces validity ψ is valid for the case of classical logic but not the intuicionistic logic, in the relevance logic it is not valid, but admissible). An interested reader can find more about disjunctive syllogism in substructural logics in [Res2].

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