1. Overview of the Thesis

Synovial fluid is a polymeric liquid which generally behaves as a viscoelastic fluid due to the presence of polysaccharide molecules called hyaluronan. The Thesis under review is concerned with biochemical and mechanical processes in the synovial fluid, particularly with the modelling, analysis and computational simulation of the synovial fluid.

The Thesis is structured as follows. Chapter 1 [pp. 1–4] is of introductory nature. It motivates the study undertaken in the Thesis regarding the mathematical modelling of the synovial fluid. It describes the aims of the Thesis and gives a summary of the main results. At the end of the Introduction, the Candidate also indicates some relevant open problems that merit further study. These include: fluid-structure interaction of the membrane and the synovial fluid, as well as the whole joint system; computer simulations in a real geometry; modelling of nonlinear viscoelastic responses; mathematical modelling of pathological synovial fluids; the mathematical analysis of the complete time-dependent system; mathematical analysis and development of reliable numerical methods for the viscoelastic models.

Chapter 2 of the Thesis [pp. 5–14] is devoted to the biology of the synovial fluid, the biology of joints, the synovial joint system, and diseases of joints.

Chapter 3 [pp. 15–22] introduces the basic concepts of generalized viscosity and viscoelasticity from the point of view of continuum mechanics. It gives a description of the response of viscous and viscoelastic materials in the context of the synovial fluid.

Chapter 4 [pp. 23–30] is focussed on the rheology of the synovial fluid; it describes the hyaluronan, the viscoelastic properties of hyaluronan solutions, bulk flow properties of hyaluronan solutions, and the rheology of pathological synovial fluid. The chapter summarizes the various rheological descriptions and available experimental data of previous studies, and the nature of responses of the synovial fluid under different test conditions, for both normal and pathological synovial fluid.
Chapter 5 [pp. 31–38] is concerned with the modelling of viscous responses. It formulates the constitutive equation, describes the viscosity model, and gives a detailed description of the fitting procedure for the identification of the model parameters. The chapter closes with a discussion of the fitted results. The chapter establishes a new generalized power-law fluid model for the viscosity of the synovial fluid in which the power-law exponent depends on the concentration, thereby characterizing the concentration dependence of the shear-thinning behaviour of the synovial fluid.

Chapter 6 [pp. 39–52] is devoted to the modelling of viscoelastic responses, including a discussion of the available experimental data, such as small deformation experiments for a hyaluronan solution. Section 6.1 reviews linear viscoelastic models for the synovial fluid, including the Maxwell and Oldroyd-B models. The section also presents mechanical analogues of these models in terms of a spring–dashpot formalism. The overall aim here is to use the theory of linear viscoelasticity for the modelling of the synovial fluid in the case of small-amplitude oscillatory tests. The Maxwell and Oldroyd-B models are generalized by admitting concentration-dependent material parameters. A fitting procedure is used for finding suitable values of all free parameters in the models. The conclusion of the chapter is that, even for small deformations, the synovial fluid behaves in a more complex, nonlinear way, than can be captured by the Maxwell and Oldroyd-B models. Nevertheless the Candidate is able to obtain qualitatively satisfactory results for the concentrations in the physiological range.

Chapter 7 [pp. 53–74] focuses on the formulation of a proposed new model, the governing equations and their mathematical analysis. The mathematical model is stated under the assumptions that the synovial fluid can be approximated by a homogeneous single constituent fluid, that the synovial fluid is incompressible, the hyaluronan influences the mechanical responses of the synovial fluid through its concentration, that concentration appears in the material parameters of the model for synovial fluid, and that pathological cases are excluded. The equations of flow for the synovial fluid [eqs. (7.1)–(7.3)] are stated as the balance laws for a generalized viscous fluid with concentration, as a coupled system involving the generalized Navier–Stokes equations and a convection-diffusion equation with a nonlinear diffusion term. The corresponding stationary problem for the model is also stated with a variable power-law exponent, and the existence of a weak solution is proved by using the theory of monotone operators in the framework of generalized Sobolev spaces. A key step in the argument is the proof of Hölder continuity of the concentration, which ensures Hölder continuity of the variable exponent, and which is in turn a crucial assumption for ensuring the density of smooth functions in generalized Sobolev spaces, the generalizations of embedding theorems and Korn’s inequality. To the best of my knowledge, Theorem 7.6 on p.62 of the Thesis is the first result concerning such a system. The proof is based on using a Galerkin approximation of the sequence in conjunction with a weak compactness argument. The theory of monotone operators is applied in order to identify the limits in the viscous term in the generalized Navier–Stokes equation. The Hölder continuity of the concentration is stated in Theorem 7.7 and a technical proof, based on Morrey’s lemma, is presented on pp. 69–73.
Chapter 8 [pp. 75–102] is concerned with the construction of finite element methods for the approximate solution of the system of partial differential equations stated in Chapter 7. The temporal discretization is based on using the \( \theta \)-scheme. The nonlinear equations are solved by using a quasi-Newton method. Since the equations under consideration are transport-dominated (because the Péclet number corresponding to the synovial fluid is large), a stabilized finite element method has to be used to avoiding spurious oscillations in the numerical solution exhibited by a standard continuous Galerkin finite element method. Three numerical stabilization techniques are considered: the streamline upwind Petrov–Galerkin (SUPG), the continuous interior penalty method, the Galerkin least-squares method, and the continuous interior penalty method. In Section 8.9 the Candidate compares the results of the various numerical experiments obtained by the three numerical methods in the case of the concentration equation. In Section 8.10 the finite element methods proposed are extended to the viscoelastic model problem [eqs. (8.71)–(8.74)].

Chapter 9 [pp. 103–120] is devoted to computational simulations of synovial fluid flow in two dimensions. The Candidate compares the different stabilization methods introduced and explores the differences between the various models for generalized viscosity introduced in Chapter 5. Two different geometries are considered to investigate the influence of the ratio of length and width of the domain on the solution.

The final chapter, Chapter 10, [pp. 121–138] is concerned with the modelling of synovial membranes. In this chapter the Candidate proposes a phenomenological model for the flow and transport process of dilute solutions, as well as for the synovial fluid, in domains that are separated by a leaky semipermeable membrane. Transmission conditions for the flow are formulated and the solute concentration across the membrane, taking into account the property of the membrane to partly reject the solute, the accumulation of rejected solute at the membrane, and the influence of the solute concentration on the volume flow, known as the osmotic effect. The model is solved numerically in the case of a two-dimensional domain consisting of two subdomains separated by a rigid fixed interface representing the membrane. The numerical results for different values of the material parameters and different computational settings are compared.

2. Assessment and Recommendation

This 154-page Thesis, including a bibliography with 130 entries, is an exceptionally broad contribution to the subject of mathematical modelling, analysis and computational simulation of synovial fluids.

The presentation is clear, demonstrating complete command of the subject, and the exposition is scholarly and lucid. In the Thesis the Candidate demonstrates a wide range of expertise spanning biological, biophysical, physical, mathematical and computational questions. The theoretical results in Chapter 7 and the numerical experiments in Chapters 9 and 10 are particularly impressive.
The Thesis *Biochemical and mechanical processes in synovial fluid: modeling, analysis and computational simulations* by Petra Pustějovská represents an excellent original contribution to the mathematical theory and numerical simulation of synovial fluids. I have no hesitation in strongly recommending the award of a Ph.D. degree to Petra Pustějovská.

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