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DOCTORAL THESIS

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Collections of compact sets in descriptive set theory

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Abstract: Tato práce se skládá ze tří článků ([16],[15],[14]).

V kapitole 2 se zabýváme souvislostmi mezi složitostí dané funkce f z polského prostoru X do polského prostoru Y a složitostí množiny $C(f) = \{K \in \mathcal{K}(X);\ f\restriction_K \text{ je spojitá}\}$, kde symbol $\mathcal{K}(X)$ označuje prostor všech kompaktních podmnožin prostoru X opatřený Vietorisovou topologii. Dokážeme, že jestliže $C(f)$ je analytická, pak f je borelovská. Za předpokladu $\mathbf{\Delta}^1_2$-determinovanosti ukážeme, že f je borelovská právě tehdy když $C(f)$ je koanalytická. Předkládáme též podobné výsledky pro projektivní třídy.

V kapitole 3 pokračujeme ve zkoumání systému $C(f)$ a taktéž studujeme restrikci tohoto systému na konvergentní posloupnosti($\widetilde{C}(f)$). Ukážeme, že systém $\widetilde{C}(f)$ je borelovský právě tehdy když f je borelovská. Předkládáme též podobné výsledky pro projektivní třídy.

V kapitole 4 pojednáváme o H^N-množinách, které tvoří důležitou podtřídu třídy množin jednoznačnosti pro trigonometrické řady. Velikost těchto tříd je zkoumána pomocí systému měr zvanému polára, který měří nulou každou množinu patřící do daného systému. Hlavní výsledek této kapitoly je zodpovědět negativně otázku položenou Lyonsem [9], zda poláry tříd H^N-množin jsou stejné.
Klíčová slova: Deskriptivní teorie množin, kompaktní množina, spojitost, harmonická analýza, H^N-množiny, množiny jednoznačnosti.

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Abstract: This work consists of three articles ([16],[15],[14]).

In Chapter 2, we dissert on the connections between complexity of a function f from a Polish space X to a Polish space Y and complexity of the set $C(f) = \{K \in \mathcal{K}(X);\ f\restriction_K \text{ is continuous}\}$, where $\mathcal{K}(X)$ denotes the space of all compact subsets of X equipped with the Vietoris topology. We prove that if $C(f)$ is analytic, then f is Borel; and assuming $\mathbf{\Delta}^1_2$-Determinacy we show that f is Borel if and only if $C(f)$ is coanalytic. Similar results for projective classes are also presented.

In Chapter 3, we continue in our investigation of collection $C(f)$ and also study its restriction on convergent sequences ($\widetilde{C}(f)$). We prove that $\widetilde{C}(f)$ is

Borel if and only if f is Borel. Similar results for projective classes are also presented.

The Chapter 4 disserts on H^N-sets, which form an important subclass of the class of sets of uniqueness for trigonometric series. We investigate the size of these classes which is reflected by the family of measures called polar which annihilate all the sets belonging to the given class. The main aim of this chapter is to answer in the negative the question stated by Lyons [9], whether the polars of the classes of H^N-sets are same.

Keywords: Descriptive set theory, compact sets, continuity, harmonical analysis, H^N-sets, sets of uniqueness.

Contents

Chapter 1

Introduction

This thesis deals with descriptive set theory and some important collections of sets that are studied in harmonic analysis. According to [5], descriptive set theory is the study of "definable sets" in Polish (i.e., separable completely metrizable) spaces. The task of the descriptive set theory is to classify these sets according to the complexity of their definitions.

Let X be a Polish space. We will work with the so-called *Hyperspace of Compact Sets*, which is the space of all compact subsets of X equipped with the Vietoris topology. We will denote such a space as $\mathcal{K}(X)$. We remind that Vietoris topology is generated by the sets of the form

$$\{K \in \mathcal{K}(X);\ K \subset U\},$$

$$\{K \in \mathcal{K}(X);\ K \cap U \neq \emptyset\},$$

for U open in X. Let $\rho \leq 1$ be a compatible complete metric on X and d_H be the corresponding Hausdorff metric. It is well known that d_H is compatible with the Vietoris topology.

Let us assume that we have a function f from one Polish space to another Polish space. We focus ourselves on investigation of connections between the descriptive properties of the function f and the complexity of the collection of compact sets, on which this function is continuous. We denote this collection as $C(f)$. This topic was studied first by F. Jordan in [2, 3]. Besides other results he showed that, if f is Borel, then f is a Baire class one function provided $C(f)$ is an $F_{\sigma\delta}$ subset of $\mathcal{K}(X)$. He also proved that if f is Borel and $C(f)$ is analytic, then $C(f)$ is Borel.

It is possible to show that there are nontrivial interactions between descriptive properties of collections of compact sets and their set structure. These general results play important role in different parts of analysis, mainly in the theory of exceptional sets in harmonic analysis.

One of the most important collections of compact sets (i.e. in the harmonic analysis) are sets of uniqueness. The idea of a set of uniqueness arose from the problem: If a function $f : [0,1] \to \mathbb{R}$ admits a trigonometric expansion $f(x) = \sum_n c_n e^{2\pi inx}$, is this expansion unique? A set $P \subset [0,1]$ is a *set of uniqueness* if every trigonometric series which converges to 0 outside P is identically 0. We denote collection of closed sets of uniqueness as U. It is very difficult to recognize whether some set is a set of uniqueness or not. So, there were developed some collections of sets which help us with this task, i.e. U_0-sets or H^N-sets which will be defined in Chapter 4. It is well known that

$$H^1 \subset H^2 \subset \cdots \subset U \subset U_0.$$

There are many results comparing the size of these collections. The size of these collections is reflected by the family of measures called polar which annihilate all the sets belonging to the given collections. There are some interesting results concerning this topic (see [4], [9]).

This thesis consists of three articles which coincide with individual chapters. We have only done a minimum changes in these articles. The reason for this is that we want to keep the former articles in the original form. The only main change is in the proof of Theorem 3.2.7. We hope that the explanation in the proof is now more clear and easy to understand.

Chapter 2 is based on the article [16] which is a joint work with the supervisor M. Zelený. We prove in this chapter that f is Borel provided $C(f)$ is analytic. This result allows us to improve several Jordan results from [2]. We also characterize Borel functions (under assumption of $\mathbf{\Delta}^1_2$-determinacy) by the descriptive quality of $C(f)$.

Chapter 3 is based on the article [15] which is the author's own result. In this chapter, we continue in our investigation of collection $C(f)$ and also investigate the collection

$$\widetilde{C}(f) := \{K \in \mathcal{K}(X);\ f\restriction_K \text{ is continuous and } K \text{ has exactly one limit point}\},$$

where $f : X \to Y$. We find a characterization of Borel functions using collection $\widetilde{C}(f)$. The advantage of this approach is that we do not need any determinacy axiom. We also show several connections between a Baire class of a function f and descriptive properties of $\widetilde{C}(f)$. We also study the property $C(f) = C(g)$. We show that the descriptive qualities of two functions satisfying this property are very similar.

Chapter 4 is based on the article [14] which is the author's own result. This chapter disserts on H^N-sets, which form an important subclass of the class of sets of uniqueness. Lyons in [9] asked a question about the size of collections of H^N-sets. He asked, whether the polar of H^N is equal to the polar of H^{N+1}. We answer this question negatively for arbitrary $N \in \mathbb{N}$.

Chapter 2

Compact sets of continuity for Borel functions

This chapter is based on the paper [16]. We investigate the connections between complexity of a function f from a Polish space X to a Polish space Y and complexity of the set $C(f) = \{K \in \mathcal{K}(X);\ f\restriction_K \text{ is continuous}\}$, where $\mathcal{K}(X)$ denotes the space of all compact subsets of X equipped with the Vietoris topology. We prove that if $C(f)$ is analytic, then f is Borel; and assuming $\mathbf{\Delta}^1_2$-Determinacy we show that f is Borel if and only if $C(f)$ is coanalytic. Similar results for projective classes are also presented.

2.1 Introduction

Let X be a Polish space. Denote the space of all compact subsets of X, which is equipped with the Vietoris topology, by $\mathcal{K}(X)$. At least since the important paper [7] it is well known that descriptive properties of families of compact sets and their set structure (like being σ-ideal or ideal) can interact in a nontrivial way (see [10] for a recent survey). These general results were applied in different parts of analysis, mainly in the theory of exceptional sets in harmonic analysis.

F. Jordan ([2, 3]) studies the following situation. Let f be a function from X to a Polish space Y and

$$C(f) := \{K \in \mathcal{K}(X);\ f\restriction_K \text{ is continuous}\}.$$

Jordan investigates relationships between descriptive properties of the function f and the ideal $C(f)$. Besides other results he showed that, if f is Borel, then f is a Baire class one function provided $C(f)$ is an $F_{\sigma\delta}$ subset of $\mathcal{K}(X)$. He also proved that if f is Borel and $C(f)$ is analytic, then $C(f)$ is Borel. In this note we show that the assumption of Borelness of f can be omitted by proving the following result.

Theorem 2.1.1. *Let X, Y be Polish spaces and $f : X \to Y$ be a function. If $C(f)$ is an analytic subset of $\mathcal{K}(X)$, then f is Borel.*

One can show that if f is Borel, then $C(f)$ is coanalytic (see Theorem 2.2.5(i)). This and Theorem 2.1.1 imply the next corollary giving a restriction on complexity of ideals of compact sets of the form $C(f)$.

Corollary 2.1.2. *Let X, Y be Polish spaces and $f : X \to Y$ be a function. If the ideal $C(f)$ is an analytic subset of $\mathcal{K}(X)$, then $C(f)$ is Borel.*

Compare this result with those of [7], which are of a similar nature.

Further we show that, assuming $\boldsymbol{\Delta}^1_2$-determinacy, Borelness of f can be actually characterized by descriptive properties of $C(f)$.

Theorem 2.1.3. $(\mathrm{Det}(\boldsymbol{\Delta}^1_2))$ *Let X, Y be Polish spaces and $f : X \to Y$ be a function. Then f is Borel if and only if $C(f)$ is a coanalytic subset of $\mathcal{K}(X)$.*

Remark 2.1.4. *We do not know whether the assumption on determinacy of $\boldsymbol{\Delta}^1_2$ games can be omitted in Theorem 2.1.3.*

The next section contains more detailed versions of our results. We formulate them also for projective classes. Proofs are given in the last section. Throughout the paper we follow the notation used in [5], where one can also find all needed definitions.

2.2 Results

A connection between complexities of $C(f)$ and $\operatorname{graph} f$ is established by the following result.

Theorem 2.2.1. *Let X, Y be Polish spaces, $f : X \to Y$ be a function and Γ be a class of subsets of Polish spaces which is closed under Borel preimages. If $C(f) \in \Gamma$, then $\mathcal{K}(\operatorname{graph} f) \in \Gamma$ and $\operatorname{graph} f \in \Gamma$.*

The next corollary immediately follows from Theorem 2.2.1. Note that it is easy to see that f is $\boldsymbol{\Delta}^1_n$-measurable if and only if $\operatorname{graph} f \in \boldsymbol{\Sigma}^1_n$.

Corollary 2.2.2. *Let X, Y be Polish spaces, $n \geq 1$, and $f : X \to Y$ be a function. If $C(f)$ is $\boldsymbol{\Pi}^1_n$ ($\boldsymbol{\Sigma}^1_n$ respectively), then $\operatorname{graph} f$ is $\boldsymbol{\Pi}^1_n$ ($\boldsymbol{\Sigma}^1_n$ respectively). In the latter case f is $\boldsymbol{\Delta}^1_n$-measurable.*

More general forms of Theorem 2.1.1 and Corollary 2.1.2 read as follows. Theorem 2.2.3 was proved by Jordan [2] using an additional assumption that f is Borel.

Theorem 2.2.3. *Let X, Y be Polish spaces and $f : X \to Y$ be a function. Then the following are equivalent:*

(i) *$C(f)$ is Borel,*

(ii) *$C(f)$ is analytic,*

(iii) *f has G_δ graph.*

Theorem 2.2.4. *Let X, Y be Polish spaces and $f : X \to Y$ be a function. If $C(f)$ is $\mathbf{\Sigma}^1_n$, then $C(f)$ is $\mathbf{\Delta}^1_n$.*

The following theorem provides a characterization of $\mathbf{\Delta}^1_n$-measurable functions (the assertions (i) and (ii)) and of functions having $\mathbf{\Pi}^1_n$ graph ((iii) and (iv)) assuming $\mathrm{Det}(\mathbf{\Delta}^1_{n+1})$.

Theorem 2.2.5. *Let X, Y be Polish spaces, $n \geq 1$, and $f : X \to Y$ be a function.*

(i) *If f is $\mathbf{\Delta}^1_n$-measurable, then $C(f)$ is $\mathbf{\Pi}^1_n$.*

(ii) $(\mathrm{Det}(\mathbf{\Delta}^1_{n+1}))$ *If $C(f)$ is $\mathbf{\Pi}^1_n$, then f is $\mathbf{\Delta}^1_n$-measurable.*

(iii) *If f has $\mathbf{\Pi}^1_n$ graph, then $C(f)$ is $\mathbf{\Delta}^1_{n+1}$.*

(iv) $(\mathrm{Det}(\mathbf{\Delta}^1_{n+1}))$ *If $C(f)$ is $\mathbf{\Delta}^1_{n+1}$, then f has $\mathbf{\Pi}^1_n$ graph.*

Corollary 2.2.2 and Theorem 2.2.5(i) give a ZFC result on projective functions, i.e., on functions which are $\mathbf{\Delta}^1_n$-measurable for some $n \geq 1$.

Corollary 2.2.6. *Let X, Y be Polish spaces and $f : X \to Y$ be a function. Then f is a projective function if and only if $C(f)$ is projective.*

2.3 Proofs

2.3.1 Notation

Let X and Y be Polish spaces and $f : X \to Y$ be a continuous function. Then the function $\widehat{f} : \mathcal{K}(X) \to \mathcal{K}(Y)$ is defined by $\widehat{f}(K) = f(K)$. If $A \subset X$, then $\mathcal{K}(A)$ stands for the set of all compact subsets of A. The symbols π_X and π_Y denote the projections from $X \times Y$ to X and to Y respectively. If $x \in X$, then $\mathcal{U}(x)$ denotes the family of all open neighborhoods of x.

2.3.2 Proof of Theorem 2.2.1

Lemma 2.3.1. *Let X, Y be Polish spaces and $f : X \to Y$ be a function. Then $C(f) = \widehat{\pi_X}(\mathcal{K}(\operatorname{graph} f))$.*

Proof. Define $\Psi : X \to X \times Y$ by $\Psi(x) = (x, f(x))$. Let $K \in C(f)$. Then Ψ is continuous on K. Consequently, $\Psi(K) \subset \operatorname{graph} f$ is compact. So, $K \in \widehat{\pi_X}(\mathcal{K}(\operatorname{graph} f))$.

Let $K \in \widehat{\pi_X}(\mathcal{K}(\operatorname{graph} f))$ be arbitrary. Then $\operatorname{graph}(f\restriction_K)$ is compact and $f\restriction_K$ is clearly continuous. □

Lemma 2.3.2. *Let X be a Polish space and D be a countable dense subset of X. Then there exists a Borel function $\Phi : \mathcal{K}(X) \to \mathcal{K}(X)$ such that for every $K \in \mathcal{K}(X)$*

- $K \subset \Phi(K) \subset K \cup D$,
- $\overline{\Phi(K) \cap D} = \Phi(K)$.

Proof. Let ρ be a compatible complete metric on X with $\rho \leq 1$. Let d_H be the corresponding Hausdorff metric on $\mathcal{K}(X)$. Let $(F_i)_{i\in\omega}$ be a sequence of all finite subsets of D. For every $n \in \omega$ we define an auxiliary function $\varphi_n : \mathcal{K}(X) \to \mathcal{K}(X)$ as follows. The value $\varphi_n(K)$ equals F_j where $j \in \omega$ is the smallest number with $d_H(K, F_j) < \frac{1}{n+1}$. Since $\{F_i;\ i \in \omega\}$ is dense in $\mathcal{K}(X)$, the definition is correct and $\lim_n \varphi_n(K) = K$ for every $K \in \mathcal{K}(X)$. Thus $K \cup \bigcup_{n\in\omega} \varphi_n(K) \in \mathcal{K}(X)$.

We define $\Phi : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$\Phi(K) = K \cup \bigcup_{n\in\omega} \varphi_n(K).$$

Let $V \subset X$ be open. Since φ_n is obviously Borel for each $n \in \omega$ and $\Phi(K) \cap V \neq \emptyset$ if and only if

$$K \cap V \neq \emptyset \text{ or } \exists n \in \omega :\ \varphi_n(K) \cap V \neq \emptyset,$$

we see that the set $\{K \in \mathcal{K}(X);\ \Phi(K) \cap V \neq \emptyset\}$ is Borel. Now it is easy to infer that Φ is a Borel function.

Let $K \in \mathcal{K}(X)$. Then $K \subset \Phi(K) \subset K \cup D$ by definition. Further, we have

$$\Phi(K) \subset \overline{\bigcup_{n\in\omega} \varphi_n(K)} \subset \overline{\Phi(K) \cap D} \subset \Phi(K)$$

and we are done. □

Proof of Theorem 2.2.1. Find a set $D \subset \operatorname{graph} f$ which is countable and dense in $\operatorname{graph} f$. Let $\Phi : \mathcal{K}(\overline{D}) \to \mathcal{K}(\overline{D})$ be the function from Lemma 2.3.2, where X is replaced by $\overline{D}$. We show

$$\left(\widehat{\pi_X} \circ \Phi\right)^{-1}(C(f)) = \mathcal{K}(\operatorname{graph} f). \tag{2.1}$$

Let $K \in \mathcal{K}(\operatorname{graph} f)$ be arbitrary. Since $\Phi(K) \subset K \cup D \subset \operatorname{graph} f$, we have $\Phi(K) \in \mathcal{K}(\operatorname{graph} f)$. By Lemma 2.3.1, $\widehat{\pi_X}(\Phi(K)) \in C(f)$ and therefore $K \in \left(\widehat{\pi_X} \circ \Phi\right)^{-1}(C(f))$.

Let $K \in (\widehat{\pi_X} \circ \Phi)^{-1}(C(f))$. The graph of $f \restriction_{\pi_X(\Phi(K))}$ is compact and $\Phi(K) \cap D \subset \operatorname{graph}(f \restriction_{\pi_X(\Phi(K))})$. Then

$$K \subset \Phi(K) = \overline{\Phi(K) \cap D} \subset \overline{\operatorname{graph}(f \restriction_{\pi_X(\Phi(K))})} = \operatorname{graph}(f \restriction_{\pi_X(\Phi(K))}) \subset \operatorname{graph} f.$$

Thus, $K \in \mathcal{K}(\operatorname{graph} f)$ and (2.1) is proved.

Let $C(f) \in \Gamma$. The formula (2.1) implies that $\mathcal{K}(\operatorname{graph} f) \in \Gamma$. Further, the function $S : X \to \mathcal{K}(X)$ defined by $x \mapsto \{x\}$ is continuous. Then we have $\operatorname{graph} f = S^{-1}(\mathcal{K}(\operatorname{graph} f))$. This implies $\operatorname{graph} f \in \Gamma$. □

2.3.3 Proof of Theorem 2.2.3

(iii) ⇒ (i) According to the well-known fact that $\mathcal{K}(A)$ is G_δ provided that A is a G_δ set, we have that $\mathcal{K}(\operatorname{graph} f)$ is G_δ. The function $\widehat{\pi_X} \restriction_{\mathcal{K}(\operatorname{graph} f)}$ is injective and by Lemma 2.3.1 we have $C(f) = \widehat{\pi_X}(\mathcal{K}(\operatorname{graph} f))$. Thus, $C(f)$ is Borel.

(ii) ⇒ (iii) The set $\mathcal{K}(\operatorname{graph} f)$ is clearly a σ-ideal and it is analytic by Theorem 2.2.1. Theorem 11 of [7, Section 1] says that each analytic σ-ideal is in fact G_δ. Thus, $\mathcal{K}(\operatorname{graph} f)$ is G_δ and, consequently, $\operatorname{graph} f$ is G_δ as well.

(i) ⇒ (ii) This implication is trivial.

2.3.4 Proof of Theorem 2.2.4

Assume that $C(f)$ is $\mathbf{\Sigma}^1_n$. Then Corollary 2.2.2 gives that f is $\mathbf{\Delta}^1_n$-measurable. Now the fact that $\mathcal{K}(A)$ is $\mathbf{\Pi}^1_n$ provided A is $\mathbf{\Pi}^1_n$ and the next lemma give that $C(f)$ is $\mathbf{\Pi}^1_n$. Thus, $C(f)$ is $\mathbf{\Delta}^1_n$.

The next lemma is inspired by [3].

Lemma 2.3.3. *Let X, Y be Polish spaces, $n \geq 1$, and $f : X \to Y$ be a function. If f is $\mathbf{\Delta}^1_n$-measurable, then there exist sets $H^i_l \in \mathbf{\Delta}^1_n(X)$, $i, l \in \omega$, such that $C(f) = \bigcap_{l \in \omega} \bigcup_{i \in \omega} \mathcal{K}(H^i_l)$.*

Proof. Let ρ and β be compatible complete metrics on X and Y respectively. Set $A^x = \pi_Y(A \cap (\{x\} \times Y))$ for $x \in X$ and $A \subset X \times Y$. Let $\mathcal{V}$ and $\mathcal{U}$ be countable bases for X and Y respectively containing only closed balls. Let $\mathfrak{W}$ be the set of all finite collections $\mathcal{Z}$ of sets of the form $B_1 \times B_2$ with $B_1 \in \mathcal{V}$, $B_2 \in \mathcal{U}$. Further we define $\mathfrak{W}_l$, $l \in \omega$, as the system of all $\mathcal{Z} \in \mathfrak{W}$ such that $\operatorname{diam}_\beta((\bigcup \mathcal{Z})^x) < \frac{1}{l+1}$ whenever $x \in \pi_X(\bigcup \mathcal{Z})$. Let $\mathfrak{W}_l = \{\mathcal{Z}^i_l;\ i \in \omega\}$. Set $H^i_l = \pi_X((\bigcup \mathcal{Z}^i_l) \cap \operatorname{graph} f)$. We have that H^i_l is $\mathbf{\Delta}^1_n$ by $\mathbf{\Delta}^1_n$-measurability of f. Set $\mathcal{T} = \bigcap_{l \in \omega} \bigcup_{i \in \omega} \mathcal{K}(H^i_l)$. We prove $\mathcal{T} = C(f)$.

Let $K \in \mathcal{T}$, $x \in K$, and $\varepsilon > 0$ be arbitrary. Then there exist $l \in \omega$ and $\mathcal{Z} \in \mathfrak{W}_l$ such that $\frac{1}{l+1} < \varepsilon$ and $K \subset \pi_X((\bigcup \mathcal{Z}) \cap \operatorname{graph} f)$. Set $\mathcal{P} = \{B_1 \times B_2 \in \mathcal{Z};\ x \notin B_1\}$. Since $\mathcal{Z}$ is finite, one can find $\delta > 0$ with $B_\rho(x, \delta) \cap \pi_X(\bigcup \mathcal{P}) = \emptyset$. Let $\tilde{x} \in B_\rho(x, \delta) \cap K$ be arbitrary. Then $f(\tilde{x}) \in \pi_Y(\bigcup(\mathcal{Z} \setminus \mathcal{P})) = (\bigcup \mathcal{Z})^x$. Thus, $\beta(f(x), f(\tilde{x})) < \varepsilon$ since $\operatorname{diam}(\bigcup \mathcal{Z})^x < \frac{1}{l+1} < \varepsilon$. This gives $K \in C(f)$.

Let $K \in C(f)$ and $l \in \omega$ be arbitrary. Then there exists $m \in \omega$ such that for every $x, \tilde{x} \in K$ with $\rho(x, \tilde{x}) < \frac{2}{m}$ we have $\beta(f(x), f(\tilde{x})) < \frac{1}{4l+4}$. Then for every $x \in K$ there exist $B_{1,x} \in \mathcal{V}$ and $B_{2,x} \in \mathcal{U}$ such that

(a) x is in the interior of $B_{1,x}$,

(b) $\operatorname{diam}_\rho B_{1,x} < \frac{1}{m}$,

(c) $f(B_{1,x} \cap K) \subset B_{2,x}$, and

(d) $\operatorname{diam}_\beta B_{2,x} < \frac{1}{4l+4}$.

Clearly, the union of interiors of $B_{1,x}$, $x \in K$, covers K. Since K is compact, we can find finitely many points $x_1, \dots, x_m$ in K such that the balls $B_{1,x_1}, \dots, B_{1,x_m}$ cover K. Set

$$\mathcal{Z} = \{B_{1,x_s} \times B_{2,x_s};\ s = 1, \dots, m\}.$$

Let $\hat{x} \in \pi_X(\bigcup \mathcal{Z})$ and $y, z \in (\bigcup \mathcal{Z})^{\hat{x}}$. Then there exist $1 \leq i, j \leq m$ such that $y \in B_{2,x_i}$, $z \in B_{2,x_j}$, and $\hat{x} \in B_{1,x_i} \cap B_{1,x_j}$. This yields $\rho(x_i, x_j) \leq \frac{2}{m}$ and, consequently, $\beta(f(x_i), f(x_j)) < \frac{1}{4l+4}$. Using (c) and (d) we also have $\beta(f(x_i), y) \leq \frac{1}{4l+4}$ and $\beta(f(x_j), z) \leq \frac{1}{4l+4}$. Therefore, $\beta(y, z) < \frac{3}{4l+4}$ and, consequently, $\operatorname{diam}_\beta (\bigcup \mathcal{Z})^{\hat{x}} < \frac{1}{l+1}$. This implies $\mathcal{Z} \in \mathfrak{W}_l$. Using (c) we get that $\mathcal{Z}$ covers $\operatorname{graph}(f\restriction_K)$. Thus, we have $K \in \mathcal{T}$. □

2.3.5 Proof of Theorem 2.2.5

Lemma 2.3.4. *Let X, Y be Polish spaces, $n \geq 1$, and $f : X \to Y$ be a function. If f is not $\mathbf{\Delta}^1_n$-measurable then there exist $x \in X$ and $U \in \mathcal{U}(f(x))$ such that, for every $\tilde{U} \in \mathcal{U}(f(x))$ and $V \in \mathcal{U}(x)$, the set $f^{-1}(\tilde{U}) \cap V$ cannot be separated from the set $f^{-1}(Y \setminus U) \cap V$ by a $\mathbf{\Pi}^1_n$ subset of V.*

Proof. Let $\mathcal{V}$ and $\mathcal{U}$ be countable open bases of X and Y respectively. Since f is not $\mathbf{\Delta}^1_n$-measurable one can find an open set $W \subset Y$ such that $f^{-1}(W)$ is not in $\mathbf{\Pi}^1_n$. For $\tilde{U} \in \mathcal{U}$ and $V \in \mathcal{V}$, let $G(\tilde{U}, V)$ be a $\mathbf{\Pi}^1_n$ subset of V separating $f^{-1}(\tilde{U}) \cap V$ from $f^{-1}(Y \setminus W) \cap V$, if such a set exists, otherwise set $G(\tilde{U}, V) := \emptyset$.

Suppose towards a contradiction that the desired x and U do not exist. Then $f^{-1}(W) = \bigcup\{G(\tilde{U}, V);\ \tilde{U} \in \mathcal{U}, V \in \mathcal{V}\}$. Since $\mathcal{U}$ and $\mathcal{V}$ are countable, we have that $f^{-1}(W)$ is $\mathbf{\Pi}^1_n$, a contradiction. □

Lemma 2.3.5. ($\mathrm{Det}(\mathbf{\Delta}^1_{n+1})$) *Let X be a Polish space, $n \geq 1$, $A, B \in \mathbf{\Delta}^1_{n+1}(X)$, and $A \cap B = \emptyset$. If there is no $\mathbf{\Pi}^1_n$ set separating A from B, then there is a compact set $C \subset A \cup B$ such that $C \cap A$ is $\mathbf{\Sigma}^1_n$-hard. In particular, if $D \subset X$ is $\mathbf{\Delta}^1_{n+1} \setminus \mathbf{\Pi}^1_n$, then D is $\mathbf{\Sigma}^1_n$-hard.*

Proof. First assume that $X = \omega^\omega$. Let $Q \subset 2^\omega$ be $\mathbf{\Pi}^1_n$-complete. Consider the separation game $SG(Q; B, A)$ as in [5, 21.F]. This game is determined by $\mathrm{Det}(\mathbf{\Delta}^1_{n+1})$. Since there is no $\mathbf{\Pi}^1_n$ set separating A from B, player I cannot have a winning strategy. So II has a winning strategy, which gives a compact set $C \subset A \cup B$ such that $C \cap A$ is $\mathbf{\Sigma}^1_n$-hard (see [5, 21.F]).

Now consider the general case. Let $\varphi : \omega^\omega \to X$ be a continuous surjection. Denote $\tilde{A} = \varphi^{-1}(A)$ and $\tilde{B} = \varphi^{-1}(B)$. The sets $\tilde{A}$, $\tilde{B}$ are $\mathbf{\Delta}^1_{n+1}$. Assume that $T \subset \omega^\omega$ is a $\mathbf{\Pi}^1_n$ set separating $\tilde{A}$ from $\tilde{B}$. Then $X \setminus \varphi(\omega^\omega \setminus T)$ is a $\mathbf{\Pi}^1_n$ set separating A from B, a contradiction. Thus, $\tilde{A}$ cannot be separated by a $\mathbf{\Pi}^1_n$ set from $\tilde{B}$. This means that there is a compact set $\tilde{C} \subset \tilde{A} \cup \tilde{B}$ such that $\tilde{C} \cap \tilde{A}$ is $\mathbf{\Sigma}^1_n$-hard. Setting $C := \varphi(\tilde{C})$ we are done. □

The next lemma is also inspired by [3].

Lemma 2.3.6. ($\mathrm{Det}(\mathbf{\Delta}^1_{n+1})$) *Let X, Y be Polish spaces, $n \geq 1$, and $f : X \to Y$ be a function. If f is not $\mathbf{\Delta}^1_n$-measurable and f is $\mathbf{\Delta}^1_{n+1}$-measurable, then $C(f)$ is $\mathbf{\Sigma}^1_n$-hard.*

Proof. Let ρ be a complete compatible metric on X and let β be a complete compatible metric on Y. By Lemma 2.3.4, there are $p \in X$, $U \in \mathcal{U}(f(p))$, and decreasing sequences (V_l) and (U_l) of open sets in X and Y respectively such that

- $\lim_l \operatorname{diam}_\rho V_l = 0$,
- $\lim_l \operatorname{diam}_\beta U_l = 0$

and, for every $l \in \omega$,

- $p \in V_l$, $f(p) \in U_l \subset U$,
- $A_l := f^{-1}(U_l) \cap V_l$ cannot be separated from the set $B_l := f^{-1}(Y \setminus U) \cap V_l$ by a $\mathbf{\Pi}^1_n$ subset of V_l.

Since $A_l, B_l \in \mathbf{\Delta}^1_{n+1}(X)$ and $A_l \cap B_l = \emptyset$, Lemma 2.3.5 guarantees that there is a compact set $K_l \subset A_l \cup B_l \subset V_l$ such that $D_l := K_l \cap A_l$ is $\mathbf{\Sigma}^1_n$-hard.

Set $K := \{p\} \cup \bigcup_{l\in\omega} K_l$. The set K is clearly compact, since $K_l \to \{p\}$. Let $h : \prod_{l\in\omega} K_l \to \mathcal{K}(K)$ be defined by

$$h(\sigma) = \{\sigma(l);\ l \in \omega\} \cup \{p\}.$$

It is easy to verify that h is well-defined, continuous, and that

$$T := h^{-1}(C(f\restriction_K)) = \{\sigma \in \prod_{l\in\omega} K_l;\ \exists i_0 \in \omega\ \forall i > i_0 :\ \sigma(i) \in D_i\}.$$

We show that T is $\mathbf{\Sigma}^1_n$-hard. Let B be a $\mathbf{\Sigma}^1_n$ subset of ω^ω. Find a continuous function $\varphi_l : \omega^\omega \to K_l$ such that $\varphi_l^{-1}(D_l) = B$. Let $\psi : \omega^\omega \to \prod_{l\in\omega} K_l$ be defined by $\psi(\nu)(l) = \varphi_l(\nu)$. It is easy to see that $\psi^{-1}(T) = B$ and ψ is continuous. Thus, $C(f\restriction_K)$ is $\mathbf{\Sigma}^1_n$-hard. So, $C(f)$ is $\mathbf{\Sigma}^1_n$-hard. □

Proof of Theorem 2.2.5. (i) Let f be $\mathbf{\Delta}^1_n$-measurable. By Lemma 2.3.3 there exist sets $H^i_l \in \mathbf{\Delta}^1_n(X)$ such that $C(f) = \bigcap_{l\in\omega} \bigcup_{i\in\omega} \mathcal{K}(H^i_l)$. Since $\mathcal{K}(H^i_l)$ is $\mathbf{\Pi}^1_n$, we get $C(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$.

(ii) Let $C(f)$ be $\mathbf{\Pi}^1_n$. By Corollary 2.2.2 the function f is $\mathbf{\Delta}^1_{n+1}$-measurable. Suppose f is not $\mathbf{\Delta}^1_n$-measurable. According to Lemma 2.3.6, $C(f)$ is $\mathbf{\Sigma}^1_n$-hard, a contradiction.

(iii) Let f have $\mathbf{\Pi}^1_n$ graph. Then $\mathcal{K}(\operatorname{graph} f)$ is $\mathbf{\Pi}^1_n$ in $\mathcal{K}(X \times Y)$. By Lemma 2.3.1 we have $C(f) = \widehat{\pi_X}(\mathcal{K}(\operatorname{graph} f))$. This gives that $C(f)$ is $\mathbf{\Sigma}^1_{n+1}$. Now Theorem 2.2.4 implies the desired conclusion.

(iv) Suppose that $C(f)$ is $\mathbf{\Delta}^1_{n+1}$ and $\operatorname{graph} f$ is not $\mathbf{\Pi}^1_n$. Then by Theorem 2.2.1, $\operatorname{graph} f$ is in $\mathbf{\Delta}^1_{n+1} \setminus \mathbf{\Pi}^1_n$. Using Lemma 2.3.5 we have that $\operatorname{graph} f$ is $\mathbf{\Sigma}^1_n$-hard. Using Lemma [10, Lemma 1.1] (cf. Lemma 1 in [7, Section 1]) we have that $\mathcal{K}(\operatorname{graph} f)$ is $\mathbf{\Pi}^1_{n+1}$-hard. On the other hand, $\mathcal{K}(\operatorname{graph} f)$ is $\mathbf{\Delta}^1_{n+1}$ by Theorem 2.2.1. This is a contradiction. □

Chapter 3

Countable compacta of continuity for projective functions

This chapter is based on the paper [15]. We investigate connections between complexity of a function f from a Polish space X to a Polish space Y and complexity of the set $\widetilde{C}(f) = \{K \in \mathcal{K}(X);\ f\restriction_K$ is continuous and K has exactly one limit point$\}$. We prove that $\widetilde{C}(f)$ is Borel if and only if f is Borel. Similar results for projective classes are also presented.

3.1 Introduction

Let X, Y be Polish spaces. In this chapter we investigate the collection

$$\widetilde{C}(f) := \{K \in \mathcal{K}(X);\ f\restriction_K \text{ is continuous and } K \text{ has exactly one limit point}\},$$

where $f : X \to Y$. Compacta with one limit point are in fact convergent sequences with limit point. So, we investigate continuity of functions on convergent sequences. But, by Heine theorem we have that continuity on convergent sequences characterizes continuity on general sets. We show that Borelness of f can be actually characterized by descriptive properties of $\widetilde{C}(f)$ assuming no determinacy axiom.

Theorem 3.1.1. *Let X, Y be Polish spaces and $f : X \to Y$ be a function. Then the following are equivalent:*

(i) $\widetilde{C}(f)$ is Borel,

(ii) $\widetilde{C}(f)$ is analytic,

(iii) f is Borel.

One can show that $\widetilde{C}(f)$ is coanalytic if and only if $C(f)$ is coanalytic (see Lemma 3.3.9). Thus, Theorems 2.1.3 and 3.1.1 imply the next corollary giving a restriction on complexity of sets of the form $\widetilde{C}(f)$.

Corollary 3.1.2. (Det($\mathbf{\Delta}^1_2$)) *Let X, Y be Polish spaces and $f : X \to Y$ be a function. Then $\widetilde{C}(f)$ is Borel if and only if $\widetilde{C}(f)$ is a coanalytic subset of $\mathcal{K}(X)$.*

We do not know whether the assumption on determinacy of $\mathbf{\Delta}^1_2$ games can be omitted in Corollary 3.1.2. In fact, the following two statements are equivalent in ZFC:

(i) $\widetilde{C}(f) \in \mathbf{\Delta}^1_n(\mathcal{K}(X))$ if and only if $\widetilde{C}(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$,

(ii) f is $\mathbf{\Delta}^1_n$-measurable if and only if $C(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$.

This equivalency simply follows from Theorem 3.2.1 and Lemma 3.3.9. So, if we proved Corollary 3.1.2 using a weaker axiom, then we would be able to prove Theorem 2.1.3 using the same axiom.

We also show several connections between a Baire class of a function f and descriptive properties of $\widetilde{C}(f)$.

We also study the property $C(f) = C(g)$. Clearly, $C(f) = C(g)$ if and only if $\widetilde{C}(f) = \widetilde{C}(g)$. We show that two Borel functions f and g with $C(f) = C(g)$ belong to the same Baire class. We also show that if f is Lebesgue measurable and $C(f) = C(g)$ then g is also Lebesgue measurable.

As for the notation and all needed definitions we refer to [5].

3.2 Results

Let us describe the main results of the paper. The following theorem provides a characterization of $\mathbf{\Delta}^1_n$-measurable functions. This is a generalization of Theorem 3.1.1 to projective classes, Theorem 3.1.1 is a special case of Theorem 3.2.1 for $n = 1$.

Theorem 3.2.1. *Let X, Y be Polish spaces, $n \in \mathbb{N}$ and $f : X \to Y$ be a function. Then the following are equivalent:*

(i) $\widetilde{C}(f) \in \mathbf{\Delta}^1_n(\mathcal{K}(X))$,

(ii) $\widetilde{C}(f) \in \mathbf{\Sigma}^1_n(\mathcal{K}(X))$,

(iii) f is $\mathbf{\Delta}^1_n$-measurable.

The next corollary is a more general version of Corollary 3.1.2.

Corollary 3.2.2. (Det($\mathbf{\Delta}^1_2$)) *Let X, Y be Polish spaces, $f : X \to Y$ be a function and $n \in \mathbb{N}$. Then $\widetilde{C}(f) \in \mathbf{\Delta}^1_n(\mathcal{K}(X))$ if and only if $\widetilde{C}(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$.*

Let X and Y be Polish spaces. The symbol $\mathcal{B}_\alpha(X, Y)$ stands for the set of all functions of Baire class α from X to Y. Now we define the classes $\mathcal{E}_\alpha$. These classes were studied by Jordan [3] who showed their interesting connections with Baire classes and collections $C(f)$.

Definition 3.2.3. *Let X, Y be Polish spaces, $2 \leq \alpha < \omega_1$. We define $\mathcal{E}_\alpha(X, Y)$ as a collection of functions f such that for all $x \in X$ and W open neighborhood of $f(x)$ there exist $G \in \mathbf{\Pi}^0_\beta(X)$ with $\beta < \alpha$ and open sets $U \subset X$, $V \subset Y$ such that $x \in f^{-1}(V) \cap U$, $f^{-1}(V) \cap U \subset G$ and $f(G) \subset W$.*

The following theorem shows us that if functions f and g satisfy $C(f) = C(g)$ then descriptive properties of f are very similar to descriptive properties of g.

Theorem 3.2.4. *Let X, Y be Polish spaces, $1 \leq \alpha < \omega_1$, $\mathcal{A}$ be a σ-algebra on X containing $\mathbf{\Delta}^1_1(X)$ and $f, g : X \to Y$ be functions with $C(f) = C(g)$.*

(i) If $\alpha \geq 2$, then we have $f \in \mathcal{E}_\alpha(X, Y)$ if and only if $g \in \mathcal{E}_\alpha(X, Y)$.

(ii) We have $f \in \mathcal{B}_\alpha(X, Y)$ if and only if $g \in \mathcal{B}_\alpha(X, Y)$.

(iii) We have f is $\mathcal{A}$-measurable if and only if g is $\mathcal{A}$-measurable.

Using (iii) from previous theorem we can simply prove the following two corollaries.

Corollary 3.2.5. *Let $n \in \mathbb{N}$, $f, g : \mathbb{R}^n \to \mathbb{R}$ be functions and $C(f) = C(g)$. Then f is Lebesgue measurable if and only if g is Lebesgue measurable.*

Corollary 3.2.6. *Let X, Y be Polish spaces, $f, g : X \to Y$ be functions and $C(f) = C(g)$. Then f is Baire measurable if and only if g is Baire measurable.*

Jordan [2] shows that $f \in \mathcal{B}_1$ implies that $C(f)$ is $\mathbf{\Pi}^0_4$. He also shows that $C(f)$ is Borel if and only if f has $\mathbf{\Pi}^0_2$ graph. Thus, there exists $f \in \mathcal{B}_2$ such that $C(f)$ is not Borel. So, we cannot find any upper bound on descriptive quality of $C(f)$ for higher Baire classes. This is why we use collection $\widetilde{C}(f)$ instead of $C(f)$ to study higher Baire classes.

Theorem 3.2.7. *Let X, Y be Polish spaces, $1 \leq \alpha < \omega_1$, and $f : X \to Y$ be a function.*

(i) If $\widetilde{C}(f) \in \mathbf{\Pi}^0_{\alpha+1}(\mathcal{K}(X))$ and $\alpha \geq 2$, then $f \in \mathcal{E}_\alpha(X, Y)$.

(ii) If $f \in \mathcal{E}_{\alpha+1}(X, Y)$, then $\widetilde{C}(f) \in \mathbf{\Pi}^0_{\alpha+4}(\mathcal{K}(X))$.

(iii) If $f \in \mathcal{E}_\alpha(X, Y)$ and α is a limit ordinal, then $\widetilde{C}(f) \in \mathbf{\Pi}^0_{\alpha+1}(\mathcal{K}(X))$.

Theorem 3.2.8. *Let X, Y be Polish spaces, $1 \leq \alpha < \omega_1$, and $f : X \to Y$ be a function.*

(i) If $\widetilde{C}(f) \in \mathbf{\Pi}^0_{\alpha+2}(\mathcal{K}(X))$, then $f \in \mathcal{B}_\alpha(X, Y)$.

(ii) If $f \in \mathcal{B}_\alpha(X, Y)$, then $\widetilde{C}(f) \in \mathbf{\Pi}^0_{\alpha+5}(\mathcal{K}(X))$.

3.3 Proofs

3.3.1 Notation

Let us recall the notation from Chapter 2 and add some new terms. Let X and Y be Polish spaces and $f : X \to Y$ be a continuous function. Then the function $\widehat{f} : \mathcal{K}(X) \to \mathcal{K}(Y)$ is defined by $\widehat{f}(K) = f(K)$. Let Γ be a collection of subsets of X. We define its dual collection $\check{\Gamma}$ by

$$\check{\Gamma} = \{A \subset X;\ X \setminus A \in \Gamma\}.$$

The symbol $\boldsymbol{d}_{CB}$ denotes the Cantor-Bendixson derivative. If $A \subset X$, then $\mathcal{K}(A)$ stands for the set of all compact subsets of A. We also define collections $\mathcal{O}(A)$, $\mathcal{S}^n(A)$, $\mathcal{S}^{<\omega}(A)$ by

$$\begin{aligned} \mathcal{O}(A) &= \{K \in \mathcal{K}(A);\ \mathrm{card}(\boldsymbol{d}_{CB}(K)) = 1\}, \\ \mathcal{S}^n(A) &= \{K \subset A;\ \mathrm{card}(K) \leq n\},\ n \in \mathbb{N}, \\ \mathcal{S}^{<\omega}(A) &= \{K \subset A;\ \mathrm{card}(K) < \omega\}. \end{aligned}$$

The symbols π_X and π_Y denote the projections from $X \times Y$ to X and to Y respectively. The symbol $\mathcal{N}$ denotes the Baire space $\mathbb{N}^{\mathbb{N}}$. If $x \in X$, then $\mathcal{U}(x)$ denotes the family of all open neighborhoods of x.

3.3.2 Proof of Theorem 3.2.1

Lemma 3.3.1. *Let X be a Polish space. Then $\mathcal{O}(X) \in \boldsymbol{\Pi}^0_3(\mathcal{K}(X))$.*

Proof. Set $M := \{\{x\};\ x \in X\}$. Clearly, M is closed in $\mathcal{K}(X)$ and $\mathcal{O}(X) = \boldsymbol{d}_{CB}^{-1}(M)$. Since $\boldsymbol{d}_{CB} : \mathcal{K}(X) \to \mathcal{K}(X)$ is a Baire class two function (see [5, 24.9]), we have $\mathcal{O}(X) \in \boldsymbol{\Pi}^0_3(\mathcal{K}(X))$. □

Lemma 3.3.2. *Let X be a Polish space, $n \in \mathbb{N}$, $3 \leq \alpha < \omega_1$ and $A \subset X$.*

(i) If $A \in \boldsymbol{\Pi}^0_\alpha(X)$, then $\mathcal{S}^{<\omega}(A) \in \boldsymbol{\Pi}^0_\alpha(\mathcal{K}(X))$.

(ii) If $A \in \boldsymbol{\Delta}^1_n(X)$, then $\mathcal{S}^{<\omega}(A) \in \boldsymbol{\Delta}^1_n(\mathcal{K}(X))$.

Proof. (i) Clearly, $\mathcal{S}^{<\omega}(X) = \bigcup_{n\in\mathbb{N}} \mathcal{S}^n(X)$ and $\mathcal{S}^n(X)$ is closed in $\mathcal{K}(X)$. So, we have $\mathcal{S}^{<\omega}(X) \in \boldsymbol{\Sigma}^0_2(\mathcal{K}(X))$. Let $F \subset X$ be closed. Then $\mathcal{S}^{<\omega}(F) = \mathcal{K}(F) \cap \mathcal{S}^{<\omega}(X)$. Thus we have $\mathcal{S}^{<\omega}(F) \in \boldsymbol{\Sigma}^0_2(\mathcal{K}(X))$. Let $B \in \boldsymbol{\Pi}^0_2(X)$. Then $\mathcal{S}^{<\omega}(B) = \mathcal{K}(B) \cap \mathcal{S}^{<\omega}(X)$. Thus we have $\mathcal{S}^{<\omega}(B) \in \boldsymbol{\Sigma}^0_3(\mathcal{K}(X))$, since $\mathcal{K}(B) \in \boldsymbol{\Pi}^0_2(\mathcal{K}(X))$. Let $A_i, i \in \mathbb{N}$, be arbitrary subsets of X. Clearly,

$$\begin{aligned} \mathcal{S}^{<\omega}\left(\bigcap_{i\in\mathbb{N}} A_i\right) &= \bigcap_{i\in\mathbb{N}} \mathcal{S}^{<\omega}(A_i), \\ \mathcal{S}^{<\omega}\left(\bigcup_{i\in\mathbb{N}} A_i\right) &= \bigcup_{k\in\mathbb{N}} \mathcal{S}^{<\omega}\left(\bigcup_{i\leq k} A_i\right). \end{aligned}$$

For $\alpha \geq 3$ and $A \in \mathbf{\Pi}^0_\alpha(X)$ we have by transfinite induction that $\mathcal{S}^{<\omega}(A) \in \mathbf{\Pi}^0_\alpha(\mathcal{K}(X))$.

(ii) Assume $A \in \mathbf{\Pi}^1_n(X)$. Since classes $\mathbf{\Pi}^1_n$ are closed under coprojection and

$$\mathcal{S}^{<\omega}(A) = \{K \in \mathcal{S}^{<\omega}(X);\ \forall x \in X:\ x \notin K \vee x \in A\}$$

we have $\mathcal{S}^{<\omega}(A) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$.

Assume $n = 1$. So, A is a Borel set. Then A is coanalytic. So, $\mathcal{S}^{<\omega}(A)$ is coanalytic. The set A is also analytic. So, there exists a closed set $B \subset X \times \mathcal{N}$ such that $A = \pi_X(B)$. Clearly, $\mathcal{S}^{<\omega}(A) = \widehat{\pi_X}(\mathcal{S}^{<\omega}(B))$. So, $\mathcal{S}^{<\omega}(A)$ is analytic. Thus, $\mathcal{S}^{<\omega}(A)$ is Borel.

Now assume $n > 1$. Then $A \in \mathbf{\Pi}^1_n(X)$. So, $\mathcal{S}^{<\omega}(A) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$. The set A also belongs to $\mathbf{\Sigma}^1_n(X)$. So, there exists a set $B \in \mathbf{\Pi}^1_{n-1}(X \times \mathcal{N})$ such that $A = \pi_X(B)$. Clearly, $\mathcal{S}^{<\omega}(A) = \widehat{\pi_X}(\mathcal{S}^{<\omega}(B))$. So, $\mathcal{S}^{<\omega}(A) \in \mathbf{\Sigma}^1_n(\mathcal{K}(X))$. Thus, $\mathcal{S}^{<\omega}(A) \in \mathbf{\Delta}^1_n(\mathcal{K}(X))$. □

Definition 3.3.3. *Let X be a Polish space, $\rho \leq 1$ be a compatible metric on X and $v \geq s > 0$. Define $\Phi : \mathcal{O}(X) \to X$ and $\Lambda_{v,s,\rho} : \mathcal{O}(X) \to \mathcal{K}(X)$ by*

$$\{\Phi(A)\} = \boldsymbol{d}_{CB}(A),$$
$$\Lambda_{v,s,\rho}(K) = K \cap P(\Phi(K), s, v),$$

where $P(x, s, v)$ is defined by

$$P(x, s, v) = \{y \in X;\ s \leq \rho(x, y) \leq v\},\ x \in X.$$

Lemma 3.3.4. *Let X be a Polish space. Let $\Phi : \mathcal{O}(X) \to X$ be as in Definition 3.3.3. Then Φ is a Baire class one function.*

Proof. Let U be an arbitrary open subset of X and $\rho \leq 1$ be a compatible metric on X. Set $F_n := \{x \in X;\ \operatorname{dist}(x, X \setminus U) \geq \frac{1}{n}\}$. Let $K \in \Phi^{-1}(U)$. Then $K \cap U$ is compact. So, $\operatorname{dist}(K \cap U, X \setminus U) > 0$. Thus we have

$$\Phi^{-1}(U) = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \{K \in \mathcal{O}(X);\ \operatorname{card}(K \cap (X \setminus F_n)) \leq i\}.$$

Clearly, $\{K \in \mathcal{O}(X);\ \operatorname{card}(K \cap (X \setminus F_n)) \leq i\}$ is closed. So, $\Phi^{-1}(U) \in \mathbf{\Sigma}^0_2(\mathcal{O}(X))$. □

Lemma 3.3.5. *Let X be a Polish space and ρ be a fixed compatible metric on X. Then $\Lambda_{v,s,\rho} \in \mathcal{B}_2(\mathcal{O}(X), \mathcal{K}(X))$.*

Proof. Let $\Phi : \mathcal{O}(X) \to X$ be as in Definition 3.3.3. By Lemma 3.3.4 Φ is a Baire one function from $\mathcal{O}(X) \to X$. Thus, it is enough to verify that $g : \mathcal{O}(X) \times X \to \mathcal{K}(X)$ defined by

$$g(K, x) = K \cap P(x, s, v)$$

is a Baire class one function. The sets of the form $\mathcal{K}(U)$ and $M_U := \{K \in \mathcal{K}(X);\ K \cap U \neq \emptyset\}$, where U is open in X, form a subbase of the Vietoris topology. Let $U \subset X$ be an arbitrary open set. There exist open sets $U_n \subset X$, $n \in \mathbb{N}$ such that $\bigcap_{n\in\mathbb{N}} U_n = X \setminus U$. Thus, $M_U = \mathcal{K}(X) \setminus \bigcap_{n\in\mathbb{N}} \mathcal{K}(U_n)$. So, it is enough to verify that $g^{-1}(\mathcal{K}(U))$ is open in $\mathcal{O}(X) \times X$.

Suppose towards a contradiction that there exist $(K_n, x_n) \notin g^{-1}(\mathcal{K}(U))$, $n \in \mathbb{N}$ and $(K, x) \in g^{-1}(\mathcal{K}(U))$ such that

$$(K_n, x_n) \to (K, x).$$

Thus for every $n \in \mathbb{N}$ there exists $z_n \in (K_n \cap P(x_n, s, v)) \setminus U$. Clearly, $K \cup \bigcup_{j\in\mathbb{N}} K_j$ is compact. Since $z_n \in K \cup \bigcup_{j\in\mathbb{N}} K_j$ there exist $z \in X$ and a subsequence $\{z_{n_i}\}$ converging to z. Thus, $z \in (K \cap P(x, s, v)) \setminus U$, a contradiction. □

Lemma 3.3.6. *Let X be a Polish space, $n \in \mathbb{N}$, $1 \le \alpha < \omega_1$ and $A \subset X$.*

(i) If $A \in \mathbf{\Pi}^0_\alpha(X)$, then $\mathcal{O}(A) \in \mathbf{\Pi}^0_{\alpha+2}(\mathcal{K}(X))$.

(ii) If $A \in \mathbf{\Delta}^1_n(X)$, then $\mathcal{O}(A) \in \mathbf{\Delta}^1_n(\mathcal{K}(X))$.

Proof. Let $\Lambda_{v,s,\rho}, \Phi$ be as in Definition 3.3.3. By Lemma 3.3.5 we have

$$\Lambda_{v,s,\rho} \in \mathcal{B}_2(\mathcal{O}(X), \mathcal{K}(X)).$$

By Lemma 3.3.4 we have that Φ is a Baire one function. Clearly,

$$\mathcal{O}(A) = \{K \in \mathcal{O}(X);\ \Phi(K) \in A \wedge \forall s, v \in \mathbb{Q}, v \ge s > 0 :\ \Lambda_{v,s,\rho}(K) \subset A\}.$$

Thus we have

$$\mathcal{O}(A) = \{K \in \mathcal{O}(X);\ \Phi(K) \in A \wedge \forall s, v \in \mathbb{Q}, v \ge s > 0 : \Lambda_{v,s,\rho}(K) \in \mathcal{S}^{<\omega}(A)\}.$$

So,

$$\mathcal{O}(A) = \Phi^{-1}(A) \cap \bigcap_{s,v\in\mathbb{Q}, v\ge s>0} \Lambda^{-1}_{v,s,\rho}(\mathcal{S}^{<\omega}(A)).$$

(i) Assume $\alpha \ge 3$. By Lemma 3.3.2(i) we have $\mathcal{S}^{<\omega}(A) \in \mathbf{\Pi}^0_\alpha(\mathcal{K}(X))$. Thus, we have $\mathcal{O}(A) \in \mathbf{\Pi}^0_{\alpha+2}(\mathcal{O}(X))$. Using Lemma 3.3.1 we are done.

On the other hand, if $\alpha < 3$ then $\mathcal{O}(A) = \mathcal{O}(X) \cap \mathcal{K}(A) \in \mathbf{\Pi}^0_3(\mathcal{K}(X))$.

(ii) By Lemma 3.3.2(ii) we have $\mathcal{S}^{<\omega}(A) \in \mathbf{\Delta}^1_n(\mathcal{K}(X))$. Since classes $\mathbf{\Delta}^1_n$ are closed under Borel preimages and countable intersections (see [5, Proposition 37.1]), we have $\mathcal{O}(A) \in \mathbf{\Delta}^1_n(\mathcal{O}(X))$. Using Lemma 3.3.1 we are done. □

Lemma 3.3.7. *Let X, Y be Polish spaces and $f : X \to Y$ be a function. Then*

$$\widetilde{C}(f) = \widehat{\pi_X}(\mathcal{O}(\mathrm{graph}(f))).$$

Proof. The proof is similar to Lemma 2.3.1. □

Lemma 3.3.8. *Let X be a Polish space and $M \subset X$ be a countable set. Then*

$$\mathcal{S} := \{K \in \mathcal{K}(X);\ \overline{K \cap M} = K\} \in \mathbf{\Pi}^0_3(\mathcal{K}(X)).$$

Proof. Let $\mathcal{W}$ be a countable open base of X. Then for $A, B \subset X$ we have

$$\overline{A} = \overline{B} \Leftrightarrow \forall V \in \mathcal{W} : (A \cap V = \emptyset \Leftrightarrow B \cap V = \emptyset).$$

Thus,

$$\mathcal{S} = \{K \in \mathcal{K}(X);\ \forall V \in \mathcal{W} : (K \cap V = \emptyset) \vee (\exists d \in V \cap M : d \in K)\}.$$

So, $\mathcal{S} \in \mathbf{\Pi}^0_3(\mathcal{K}(X))$ since M is countable.

□

Proof of Theorem 3.2.1. (iii) $\Rightarrow$ (i) According to [5, Exercise 37.3] we have that $\mathrm{graph}(f) \in \mathbf{\Delta}^1_n(X \times Y)$. By Lemma 3.3.6 we have $\mathcal{O}(\mathrm{graph}(f)) \in \mathbf{\Delta}^1_n(\mathcal{K}(X \times Y))$. By Lemma 3.3.7 we have $\widetilde{C}(f) = \widehat{\pi_X}(\mathcal{O}(\mathrm{graph}(f)))$. Thus $\widetilde{C}(f) \in \mathbf{\Sigma}^1_n(\mathcal{K}(X))$. By Theorem 2.2.5(i) we have that $C(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$. Since $\widetilde{C}(f) = C(f) \cap \mathcal{O}(X)$ we have $\widetilde{C}(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$. Thus, $\widetilde{C}(f) \in \mathbf{\Delta}^1_n(\mathcal{K}(X))$.

(i) $\Rightarrow$ (ii) This implication is trivial.

(ii) $\Rightarrow$ (iii) Let F be an arbitrary closed subset of Y. It is sufficient to prove that

$$f^{-1}(F) \in \mathbf{\Sigma}^1_n(X).$$

Find a set $D \subset X \times Y$ which is a countable and dense subset of

$$A := \mathrm{graph}(f) \cap (X \times F).$$

By Lemma 3.3.8 we have

$$\mathcal{V} := (\mathcal{S}^{<\omega}(X) \cup \widetilde{C}(f)) \cap \{K \in \mathcal{K}(X);\ \overline{K \cap \pi_X(D)} = K\} \in \ \mathbf{\Sigma}^1_n(\mathcal{K}(X)).$$

We prove that $f^{-1}(F) = \bigcup \mathcal{V}$, which implies that $f^{-1}(F) \in \mathbf{\Sigma}^1_n(X)$.

Let $x \in f^{-1}(F) = \pi_X(A)$. Since D is dense in A there exist $x_n \in \pi_X(D)$, $n \in \mathbb{N}$, such that $(x_n, f(x_n)) \to (x, f(x))$. Clearly, $\{x\} \cup \{x_n;\ n \in \mathbb{N}\} \in \mathcal{V}$. Thus, $x \in \bigcup \mathcal{V}$.

Let $K \in \mathcal{V}$. Then there exists $B \subset D$ such that $\mathrm{graph}(f) \cap (\pi_X)^{-1}(K) = \overline{B}$. So, $\mathrm{graph}(f) \cap (\pi_X)^{-1}(K) \subset A$. Thus, $K \subset \pi_X(A) = f^{-1}(F)$. □

3.3.3 Proof of Corollary 3.2.2

Lemma 3.3.9. *Let X, Y be Polish spaces, $f : X \to Y$ be a function and $n \in \mathbb{N}$. Then $C(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$ if and only if $\widetilde{C}(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$.*

Proof. Let $C(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$. Clearly, $\widetilde{C}(f) = C(f) \cap \mathcal{O}(X)$. By Lemma 3.3.1 we have $\widetilde{C}(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$.

Let $\widetilde{C}(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$. By Heine theorem we have

$$C(f) = \{K \in \mathcal{K}(X);\ \mathcal{O}(K) \subset \widetilde{C}(f)\}.$$

Thus we have

$$C(f) = \{K \in \mathcal{K}(X);\ \forall L \in \mathcal{K}(X) :\ (L \notin \mathcal{O}(K) \vee L \in \widetilde{C}(f))\}.$$

So, $C(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$. □

Proof of Corollary 3.2.2. Let $\widetilde{C}(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$. By Lemma 3.3.9 we have $C(f) \in \mathbf{\Pi}^1_n(\mathcal{K}(X))$. Using Theorem 2.2.5(ii) we have f is $\mathbf{\Delta}^1_n$-measurable. By Theorem 3.2.1 we have $\widetilde{C}(f) \in \mathbf{\Delta}^1_n(\mathcal{K}(X))$. The converse implication is trivial. □

3.3.4 Proof of Theorem 3.2.4

Definition 3.3.10. *Let X, Y be Polish spaces and $\Gamma \subset 2^X$. We define $\Omega_1(X)$, $\Omega_2(X)$, $\Omega(X)$, $\boldsymbol{M}_\Gamma(X, Y)$, $\boldsymbol{E}_\Gamma(X, Y)$ by*

(i) $\Omega_1(X) = \{\mathcal{V} \subset 2^X;\ \mathcal{V}$ *is closed under countable unions*$\}$,

(ii) $\Omega_2(X) = \{\mathcal{V} \subset 2^X;\ \mathcal{V}$ *is closed under finite intersections* $\wedge (\mathcal{V} \supset \mathbf{\Pi}^0_1(X) \vee \mathcal{V} \supset \mathbf{\Sigma}^0_1(X))\}$,

(iii) $\Omega(X) = \Omega_1(X) \cap \Omega_2(X)$,

(iv) $\boldsymbol{M}_\Gamma(X, Y) = \{f \in Y^X;\ f$ *is* Γ*-measurable*$\}$,

(v) $\boldsymbol{E}_\Gamma(X, Y)$ *denotes the family of all functions $f : X \to Y$ such that*

$$\forall x \in X \forall W \in \mathcal{U}(f(x)) \exists U \in \mathcal{U}(x) \exists V \in \mathcal{U}(f(x)) \exists G \in \Gamma :$$
$$G \supset f^{-1}(V) \cap U \text{ and } f(G) \subset W.$$

Recall that by Lebesgue-Hausdorff-Banach theorem ([5, Theorem 24.3]) we have

$$\mathcal{B}_\alpha(X, Y) = \boldsymbol{M}_{\mathbf{\Sigma}^0_{\alpha+1}(X)}(X, Y).$$

Lemma 3.3.11. *Let X, Y be Polish spaces and $\Gamma \in \Omega_1(X)$. Then we have* *(i)* $\boldsymbol{E}_{\check{\Gamma}}(X,Y) \supset \boldsymbol{M}_{\Gamma}(X,Y)$,

(ii) $\boldsymbol{E}_{\Gamma}(X,Y) = \boldsymbol{M}_{\Gamma}(X,Y)$.

Proof. (i) Let $f \in \boldsymbol{M}_{\Gamma}(X,Y)$, $x \in X$, and $W \in \mathcal{U}(f(x))$ be arbitrary. We find $V \in \mathcal{U}(f(x))$ such that $\overline{V} \subset W$. Set $G := f^{-1}(\overline{V})$ and $U := X$. Then $G \in \check{\Gamma}$, $G \supset f^{-1}(V) \cap U$ and $f(G) \subset W$. Thus we have $f \in \boldsymbol{E}_{\check{\Gamma}}(X,Y)$.

(ii) The inclusion $\boldsymbol{E}_{\Gamma}(X,Y) \supset \boldsymbol{M}_{\Gamma}(X,Y)$ can be proved similarly to (i) by setting $V := W$, $G := f^{-1}(V)$ and $U := X$.

Now we prove $\boldsymbol{E}_{\Gamma}(X,Y) \subset \boldsymbol{M}_{\Gamma}(X,Y)$. Let $f \in \boldsymbol{E}_{\Gamma}(X,Y)$ and an open set $W \subset Y$ be arbitrary. Let $\{U_n;\ n \in \mathbb{N}\}$ and $\{V_n;\ n \in \mathbb{N}\}$ be countable open bases of X and Y respectively. We set

$$\mathcal{M} := \{(n,m) \in \mathbb{N}^2;\ \exists G \in \Gamma : G \supset f^{-1}(V_m) \cap U_n \wedge f(G) \subset W\}.$$

For each $(n,m) \in \mathcal{M}$ we fix $G_{n,m} \in \Gamma$ satisfying $G_{n,m} \supset f^{-1}(V_m) \cap U_n$ and $f(G_{n,m}) \subset W$. Since $f \in \boldsymbol{E}_{\Gamma}(X,Y)$ we have that for all $x \in f^{-1}(W)$ there exists $(n_x, m_x) \in \mathcal{M}$ such that $U_{n_x} \in \mathcal{U}(x)$ and $V_{m_x} \in \mathcal{U}(f(x))$. Since $x \in G_{n_x,m_x} \subset f^{-1}(W)$ for all $x \in f^{-1}(W)$ we have $f^{-1}(W) = \bigcup_{x \in f^{-1}(W)} G_{n_x,m_x}$. Since $\Gamma \in \Omega_1$ and the set $\{G_{n_x,m_x};\ x \in f^{-1}(W)\}$ is countable, we have $f^{-1}(W) \in \Gamma$. □

Lemma 3.3.12. *Let X,Y be Polish spaces and $1 \le \alpha < \omega_1$. Then*

$$\mathcal{B}_\alpha(X,Y) \supset \mathcal{E}_{\alpha+1}(X,Y) \supset \mathcal{E}_\alpha(X,Y) \supset \bigcup_{\beta+1<\alpha} \mathcal{B}_\beta(X,Y).$$

Proof. Clearly, $\boldsymbol{\Sigma}^0_\beta(X) \in \Omega_1(X)$ for all $1 \le \beta < \omega_1$. So,

$$\begin{aligned}\mathcal{B}_\alpha(X,Y) &= \boldsymbol{M}_{\boldsymbol{\Sigma}^0_{\alpha+1}(X)}(X,Y) = \boldsymbol{E}_{\boldsymbol{\Sigma}^0_{\alpha+1}(X)}(X,Y) \qquad \text{(Lemma 3.3.11)}\\ &\supset \boldsymbol{E}_{\boldsymbol{\Pi}^0_{\alpha}(X)}(X,Y) = \mathcal{E}_{\alpha+1}(X,Y) \supset \mathcal{E}_\alpha(X,Y).\end{aligned}$$

Clearly,

$$\begin{aligned}\mathcal{E}_\alpha(X,Y) &= \boldsymbol{E}_{\bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta(X)}(X,Y) \supset \boldsymbol{E}_{\bigcup_{\beta+1<\alpha} \boldsymbol{\Pi}^0_{\beta+1}(X)}(X,Y)\\ &\supset \bigcup_{\beta+1<\alpha} \boldsymbol{E}_{\boldsymbol{\Pi}^0_{\beta+1}(X)}(X,Y).\end{aligned}$$

By Lemma 3.3.11(i) we have

$$\boldsymbol{E}_{\boldsymbol{\Pi}^0_{\beta+1}(X)}(X,Y) \supset \boldsymbol{M}_{\boldsymbol{\Sigma}^0_{\beta+1}(X)}(X,Y).$$

Thus,

$$\bigcup_{\beta+1<\alpha} \boldsymbol{E}_{\boldsymbol{\Pi}^0_{\beta+1}(X)}(X,Y) \supset \bigcup_{\beta+1<\alpha} \boldsymbol{M}_{\boldsymbol{\Sigma}^0_{\beta+1}(X)}(X,Y) = \bigcup_{\beta+1<\alpha} \mathcal{B}_\beta(X,Y).$$

□

Lemma 3.3.13. *Let X, Y be Polish spaces, $f, g : X \to Y$ be functions, and $C(f) \supset C(g)$. Then for all $x \in X$, $V \in \mathcal{U}(f(x))$ there exist $\widetilde{V} \in \mathcal{U}(g(x))$, $\widetilde{U} \in \mathcal{U}(x)$ such that*

$$f^{-1}(V) \supset g^{-1}(\widetilde{V}) \cap \widetilde{U}. \tag{3.1}$$

Proof. Let $x \in X$ and $V \in \mathcal{U}(f(x))$ be arbitrary. Let $\{U_n;\ n \in \mathbb{N}\}$ and $\{V_n;\ n \in \mathbb{N}\}$ be decreasing sequences of open sets, which form bases of neighborhoods of x and $g(x)$ respectively. Assume towards contradiction that there are no $\widetilde{U}$ and $\widetilde{V}$ satisfying (3.1). Thus, there is a sequence $\{x_n \in X;\ n \in \mathbb{N}\}$ such that

$$x_n \in \big(g^{-1}(V_n) \cap U_n\big) \setminus f^{-1}(V). \tag{3.2}$$

So, $x_n \to x$ and $g(x_n) \to g(x)$. Thus, $\{x_n;\ n \in \mathbb{N}\} \cup \{x\} \in C(g) \subset C(f)$. By (3.2) we have $\{x_n;\ n \in \mathbb{N}\} \cup \{x\} \notin C(f)$, a contradiction. □

Lemma 3.3.14. *Let X, Y be Polish spaces, $\Gamma \in \Omega_2(X)$, $f, g : X \to Y$ be functions and $C(f) = C(g)$. Then $f \in \boldsymbol{E}_\Gamma(X, Y)$ if and only if $g \in \boldsymbol{E}_\Gamma(X, Y)$.*

Proof. Assume that $f \in \boldsymbol{E}_\Gamma(X, Y)$ and $g \notin \boldsymbol{E}_\Gamma(X, Y)$. For every $x \in X$ let $\{U_n(x);\ n \in \mathbb{N}\}$, $\{V_n(x);\ n \in \mathbb{N}\}$ and $\{W_n(x);\ n \in \mathbb{N}\}$ be decreasing sequences of open sets, which form bases of neighborhoods of x, $f(x)$ and $g(x)$ respectively. Let $\{G^\gamma(x);\ \gamma \in \Gamma_n(x)\}$ be the family of all $G \in \Gamma$ satisfying $G \supset g^{-1}(W_n(x)) \cap U_n(x)$. Since $g \notin \boldsymbol{E}_\Gamma(X, Y)$ we have that there exist $x \in X$ and $W \in \mathcal{U}(g(x))$ such that for all $n \in \mathbb{N}$ and $\gamma \in \Gamma_n(x)$ there exists $x_n^\gamma \in G^\gamma(x)$ such that

$$g(x_n^\gamma) \notin W. \tag{3.3}$$

Since $f \in \boldsymbol{E}_\Gamma(X, Y)$ we have that for all $s \in \mathbb{N}$ there exist $\widetilde{U}^s \in \mathcal{U}(x)$, $\widetilde{V}^s \in \mathcal{U}(f(x))$ and $\widetilde{G}^s \in \Gamma$ such that $\widetilde{G}^s \supset f^{-1}(\widetilde{V}^s) \cap \widetilde{U}^s$ and $f(\widetilde{G}^s) \subset V_s(x)$. Set $U^s := U_s(x) \cap \widetilde{U}^s$ and $V^s := V_s(x) \cap \widetilde{V}^s$. If Γ contains open sets, then we set $G^s := U_s(x) \cap \widetilde{G}^s$. If Γ contains closed sets, then we set $G^s := \overline{U_s(x)} \cap \widetilde{G}^s$. Thus,

$$f^{-1}(V^s) \cap U^s \subset G^s \in \Gamma \tag{3.4}$$

and $f(G^s) \subset V_s(x)$. By Lemma 3.3.13 we have that for all $s \in \mathbb{N}$ there exists $m(s) \in \mathbb{N}$ such that

$$f^{-1}(V^s) \cap U^s \supset g^{-1}(W_{m(s)}(x)) \cap U_{m(s)}(x).$$

By (3.4) we have that there exists $\gamma_s \in \Gamma_{m(s)}(x)$ such that $G^s = G^{\gamma_s}(x)$. Set $y^s := x_{m(s)}^{\gamma_s}$ for $s \in \mathbb{N}$. Since $y_s \in G^s \subset f^{-1}(V_s(x))$ we have $y^s \to x$ and $f(y^s) \to f(x)$. Since $C(f) = C(g)$ we have $g(y^s) \to g(x)$. It contradicts (3.3). □

Lemma 3.3.15. *Let X, Y be Polish spaces, $\Gamma \in \Omega(X)$, $f, g : X \to Y$ be functions and $C(f) = C(g)$. Then $f \in \boldsymbol{M}_\Gamma(X, Y)$ if and only if $g \in \boldsymbol{M}_\Gamma(X, Y)$.*

Proof. This follows from Lemmas 3.3.11(ii) and 3.3.14. □

Proof of Theorem 3.2.4. (i) This follows from Lemma 3.3.14 and

$$\mathcal{E}_\alpha(X,Y) = \boldsymbol{E}_{\bigcup_{\beta<\alpha}\boldsymbol{\Pi}^0_\beta(X)}(X,Y),$$

$$\bigcup_{\beta<\alpha}\boldsymbol{\Pi}^0_\beta(X) \in \Omega_2(X).$$

Since $\boldsymbol{\Sigma}^0_{\alpha+1}(X), \mathcal{A} \in \Omega(X)$, (ii) and (iii) follow from Lemma 3.3.15.

□

3.3.5 Proof of Theorem 3.2.7

We fix some compatible complete metrics on X and Y. Let $S \subset X$ and $g : S \to Y$ be a function. We define $\operatorname{osc}(g) = \operatorname{diam}(\operatorname{rng}(g))$. The following Remark compare our definition of $\mathcal{E}_\alpha(X,Y)$ with Jordan's definition in [3].

Remark 3.3.16. *Let $2 \le \alpha < \omega_1$ and $f \in \mathcal{E}_\alpha(X,Y)$. Then for every $x \in X$ and $\epsilon > 0$ there exist sets $U \in \mathcal{U}(x)$, $V \in \mathcal{U}(f(x))$ and $G \in \bigcup_{\beta<\alpha}\boldsymbol{\Pi}^0_\beta(X)$ such that $f^{-1}(V) \cap U \subset G$ and $\operatorname{osc}(f\restriction_G) \le \epsilon$.*

Let us begin with the proof of the Theorem.

(i) This statement is similar to [3, Theorem 3]. We only replace $C(f)$ by $\widetilde{C}(f)$. Proof is also similar. In fact the only difference between the proof of this statement and the proof of [3, Theorem 3] lies in [3, Lemma 10]. For completeness of this work, we show here the changed version of this lemma.

Lemma 3.3.17. *Let X, Y be Polish spaces, $\alpha \ge 2$ be a successor ordinal and $f : X \to Y$ be a function. If $\widetilde{C}(f) \in \boldsymbol{\Pi}^0_{\alpha+1}(\mathcal{K}(X))$ then $f \in \mathcal{E}_\alpha$.*

Proof. Let $\beta + 1 = \alpha$. Suppose $f \notin \mathcal{E}_\alpha$. There is an $p \in X$ and $\epsilon > 0$ such that for every pair of open sets U and V such that $U \in \mathcal{U}(p)$ and $V \in \mathcal{U}(f(p))$ we have $\operatorname{osc}(f\restriction_G) \ge \epsilon$ for any $G \in \boldsymbol{\Pi}^0_\beta(X)$ containing $f^{-1}(V) \cap U$.

Let $\{V_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of open neighborhoods of $f(p)$ such that $\operatorname{diam}(V_n)$
$< 1/2^n$ for every $n \in \mathbb{N}$. Let $\{U_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of open neighborhoods of p such that $\operatorname{diam}(U_n) < 1/2^n$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ such that $1/2^{n-1} < \epsilon$. Since $\operatorname{osc}(f\restriction_G) \ge \epsilon$ for every $G \in \boldsymbol{\Pi}^0_\beta(X)$ containing $f^{-1}(V_n) \cap U_n$, we conclude that $f^{-1}(V_n) \cap U_n$ cannot be separated from the set $\overline{U_n} \setminus f^{-1}(B(f(p), \epsilon/2))$ by a $\boldsymbol{\Pi}^0_\beta$-subset of $\overline{U_n}$. Since $\widetilde{C}(f) \in \boldsymbol{\Pi}^0_{\alpha+1}(\mathcal{K}(X))$ we have by Theorem 3.2.1 that f is a Borel function. Thus, the non-separable sets are Borel in X. Now, [5, 22.13, 24.20] guarantees that there is a compact set $K_n \subset \overline{U_n}$ and a $\boldsymbol{\Sigma}^0_\beta$-complete $D_n \subset K_n$ such that $D_n = K_n \cap (f^{-1}(V_n) \cap U_n)$ and $K_n \setminus D_n = K_n \cap (\overline{U_n} \setminus f^{-1}(B(f(p), \epsilon/2)))$.

Let $K = \{p\} \cup \bigcup_{n\in\mathbb{N}} K_n$. Let $h : \prod_{n\in\mathbb{N}} K_n \to \mathcal{K}(K)$ be defined by $h(\sigma) = \{\sigma(n);\ n \in \mathbb{N}\} \cup \{p\}$. Notice that h is continuous and that $h^{-1}(\widetilde{C}(f\restriction_K))$ is precisely the set of all $\sigma \in \prod_{n\in\mathbb{N}} K_n$ such that $\sigma(n) \in D_n$ for all n sufficiently large. Since each D_n is $\mathbf{\Sigma}^0_\beta$-complete, [3, Proposition 9] guarantees that $h^{-1}(\widetilde{C}(f\restriction_K))$ is $\mathbf{\Sigma}^0_{\beta+2}$-complete. Thus, $\widetilde{C}(f\restriction_K)$ is $\mathbf{\Sigma}^0_{\beta+2}$-hard in $\mathcal{K}(K)$. Since $\mathcal{K}(K)$ is closed in $\mathcal{K}(X)$ we have $\widetilde{C}(f) \notin \mathbf{\Pi}^0_{\beta+2}(\mathcal{K}(X)) = \mathbf{\Pi}^0_{\alpha+1}(\mathcal{K}(X))$. □

A similar argument establishes the limit case (see [3, Proposition 8]).

(ii) The case $f \in \mathcal{E}_2$ immediately follows from [3, Theorem 7]. Thus we can assume that $\alpha > 1$. Following ideas of the proof of [3, Theorem 6] one can find sets $H^s_l \in \mathbf{\Pi}^0_\alpha(X)$, $s, l \in \mathbb{N}$, such that $\widetilde{C}(f) = \bigcap_{l\in\mathbb{N}} \bigcup_{s\in\mathbb{N}} \mathcal{O}(H^s_l)$. By Lemma 3.3.6(i) we have $\mathcal{O}(H^s_l) \in \mathbf{\Pi}^0_{\alpha+2}(\mathcal{K}(X))$. Consequently, $\widetilde{C}(f) \in \mathbf{\Pi}^0_{\alpha+4}(\mathcal{K}(X))$. For completeness of this work we show here the way of finding of these sets H^s_l. The following Lemma is inspired by [3, Lemma 26].

Lemma 3.3.18. *Let $\alpha > 2$, $f \in \mathcal{E}_\alpha(X,Y)$, $\epsilon > 0$ and $\mathcal{D}_1$, $\mathcal{D}_2$ be countable open bases for X and Y respectively. If $x \in A \subset X$ and $f\restriction_A$ is continuous then there exist $D_1 \in \mathcal{D}_1$, $D_2 \in \mathcal{D}_2$ and $G \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$ such that $x \in D_1 \cap f^{-1}(D_2)$, $f^{-1}(\overline{D_2}) \cap \overline{D_1} \subset G$, $\operatorname{osc}(f\restriction_G) \le \epsilon$ and $f\big(A \cap \overline{D_1}\big) \subset \overline{D_2}$.*

Proof. Since $f \in \mathcal{E}_\alpha$ there exist open sets $U \in \mathcal{U}(x)$, $V \in \mathcal{U}(f(x))$ and $G \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$ such that $f^{-1}(V) \cap U \subset G$ and $\operatorname{osc}(f\restriction_G) \le \epsilon$. Pick $D_1 \in \mathcal{D}_1$ and $D_2 \in \mathcal{D}_2$ so that $\overline{D_1} \subset U$, $\overline{D_2} \subset V$, $x \in D_1$ and $f(x) \in D_2$. Now $f^{-1}\big(\overline{D_2}\big) \cap \overline{D_1} \subset f^{-1}(V) \cap U \subset G$ and $\operatorname{osc}(f\restriction_G) \le \epsilon$. Since $f\restriction_A$ is continuous we may assume D_1 is small enough that $f\big(A \cap \overline{D_1}\big) \subset \overline{D_2}$. □

Let $\mathcal{A}$ be a collection of subsets of $X \times Y$. We define

$$M(\mathcal{A}) = \bigcup_{x\in X} \Big(\pi_X^{-1}(\{x\}) \cap \bigcap\{A \in \mathcal{A};\ x \in \pi_X(A)\}\Big).$$

Let $\epsilon > 0$ and $S \subset X$. We say that a function $f : S \to Y$ is ϵ-continuous if

$$\inf\{\operatorname{osc}(f\restriction_{S\cap B(x,\delta)};\ \delta > 0\} \le \epsilon$$

for all $x \in S$. It is easy to verify that f is continuous if and only if f is 0-continuous. The following Lemma is inspired by [3, Lemma 27].

Lemma 3.3.19. *Let $\alpha > 2$, $\epsilon > 0$ and $\mathcal{A}$ be a finite collection of closed boxes in $X \times Y$ such that for every $A \in \mathcal{A}$: $\operatorname{diam}\big(\pi_Y(A)\big) \le \epsilon/5$, there is $G_A \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$ such that $\pi_X(A \cap \operatorname{graph}(f)) \subset G_A$ and $\operatorname{osc}(f\restriction_{G_A}) \le \epsilon/5$. Then there is $G \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$ such that $\pi_X(M(\mathcal{A}) \cap \operatorname{graph}(f)) \subset G$ and $f\restriction_G$ is ϵ-continuous.*

Proof. Let $G_1 = \bigcup_{A\in\mathcal{A}}\big(G_A \cap \overline{\pi_X(A\cap \mathrm{graph}(f))}\big)$. Clearly, $G_1 \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$ and

$$\pi_X(M(\mathcal{A})\cap \mathrm{graph}(f)) \subset \bigcup_{A\in\mathcal{A}} \pi_X(A\cap \mathrm{graph}(f)) \subset G_1 \subset \bigcup_{A\in\mathcal{A}} \pi_X(A).$$

Let

$$G = G_1 \cap \pi_X(M(\mathcal{A})) = G_1 \setminus \Big(\bigcup\{\bigcap_{i=1}^{s} \pi_X(A_i);\ s\in\mathbb{N}\wedge A_i\in\mathcal{A}\wedge \bigcap_{i=1}^{s}\pi_Y(A_i)=\emptyset\}\Big).$$

Now $G \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$ and

$$\pi_X(M(\mathcal{A})\cap \mathrm{graph}(f)) \subset G \subset \pi_X(M(\mathcal{A})).$$

We now show that $f\restriction_G$ is ϵ-continuous. Let $x, x_n \in G$ be such that $x_n \to x$. Let $\{x_{b_n}\}$ and $\{x_{c_n}\}$ be subsequences of $\{x_n\}$ such that there exist $A, B, C \in \mathcal{A}$ such that

$$\begin{aligned} x &\in G_A \cap \overline{\pi_X(A\cap \mathrm{graph}(f))} \subset \pi_X(A),\\ \{x_{b_n}; n\in\mathbb{N}\} &\subset G_B \cap \overline{\pi_X(B\cap \mathrm{graph}(f))} \subset \pi_X(B),\\ \{x_{c_n}; n\in\mathbb{N}\} &\subset G_C \cap \overline{\pi_X(C\cap \mathrm{graph}(f))} \subset \pi_X(C). \end{aligned}$$

Since $\pi_X(B)$ and $\pi_X(C)$ are closed, we have $x \in \pi_X(B)\cap\pi_X(C)$. Since $x \in \pi_X(M(\mathcal{A}))$ we have $A\cap B\cap C \neq \emptyset$. Since

$$\mathrm{diam}(\pi_Y(A)\cup\pi_Y(B)\cup\pi_Y(C)) \le 2\epsilon/5 \quad\text{and}$$

$$\max\{\mathrm{osc}(f\restriction_{G_A}), \mathrm{osc}(f\restriction_{G_B}), \mathrm{osc}(f\restriction_{G_C})\} \le \epsilon/5$$

we have

$$\mathrm{diam}(f(\{x_{b_n};\ n\in\mathbb{N}\}\cup\{x_{c_n};\ n\in\mathbb{N}\}\cup\{x\})) \le 4\epsilon/5.$$

Thus, we simply get that $f\restriction_G$ is ϵ-continuous. □

The following Lemma is inspired by [3, Lemma 28].

Lemma 3.3.20. *Let $\alpha > 2$, $f \in \mathcal{E}_\alpha(X,Y)$. Then there exist sets $H_l^s \in \bigcup_{\beta<\alpha}\mathbf{\Pi}^0_\beta(X)$, $s, l\in\mathbb{N}$ such that $\widetilde{C}(f) = \bigcap_{l\in\mathbb{N}}\bigcup_{s\in\mathbb{N}} \mathcal{O}(H_l^s)$.*

Proof. Let $l\in\mathbb{N}$. Let $\mathcal{D}_1 = \{D_1^k;\ k\in\mathbb{N}\}$ and $\mathcal{D}_2 = \{D_2^k;\ k\in\mathbb{N}\}$ be countable bases for X and Y respectively. Let $\mathcal{W}$ be the collection of all finite collections $Z = \{W_1,\dots,W_m\}$ of sets of the form $\overline{D_1^i}\times\overline{D_2^j}$ for some $i,j\in\mathbb{N}$ satisfying

- $\mathrm{diam}(D_2^j) \le 1/(2^l\cdot 5)$ and
- there exists a $G \in \bigcup_{\beta<\alpha}\mathbf{\Pi}^0_\beta(X)$ such that $\pi_X\big(\mathrm{graph}(f)\cap\big(\overline{D_1^i}\times\overline{D_2^j}\big)\big) \subset G$ and
  $\mathrm{osc}(f\restriction_G) \le 1/(2^l\cdot 5)$.

Let $Z \in \mathcal{W}$. By Lemma 3.3.19, there is a $H_Z \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$ such that $\pi_X(M(Z) \cap \mathrm{graph}(f)) \subset H_Z$ and $f\restriction_{H_Z}$ is $1/2^l$-continuous. We enumerate collection $\{H_Z;\ Z \in \mathcal{W}\}$ as $\{H^s_l;\ s \in \mathbb{N}\}$. Clearly, $f\restriction_C$ is $1/2^l$-continuous for every $C \in \bigcup_{s\in\mathbb{N}} \mathcal{O}(H^s_l)$.

We set $\tau = \bigcap_{l\in\mathbb{N}} \bigcup_{s\in\mathbb{N}} \mathcal{O}(H^s_l)$. Since $C \in \tau$ implies that $f\restriction_C$ is $1/2^l$-continuous for all $l \in \mathbb{N}$ we have $\tau \subset \widetilde{C}(f)$.

Let $C \in \widetilde{C}(f)$ and $l \in \mathbb{N}$ be arbitrary. We will construct a finite collection $W = \{W_1, \dots, W_m\}$ of sets of the form $W_k = \overline{D^i_1} \times \overline{D^j_2}$ for some $i, j \in \mathbb{N}$ such that

(a) $\mathrm{graph}(f\restriction_C) \subset \bigcup W$,

(b) there is $G \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$ such that $\pi_X(\mathrm{graph}(f) \cap W_k) \subset G$ and $\mathrm{osc}(f\restriction_G) < 1/(2^l \cdot 5)$,

(c) $\mathrm{diam}(D^j_2) < 1/(2^l \cdot 5)$ and

(d) $\mathrm{graph}(f\restriction_{C\cap\pi_X(W_k)}) \subset W_k$.

By Lemma 3.3.18, for every $x \in C$ there exist $i(x), j(x) \in \mathbb{N}$ such that $D^{i(x)}_1 \in \mathcal{D}_1$, $D^{j(x)}_2 \in \mathcal{D}_2$ and $G \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$ such that $x \in D^{i(x)}_1 \cap f^{-1}(D^{j(x)}_2)$, $\mathrm{diam}(D^{j(x)}_2) < 1/(2^l \cdot 5)$, $f\left(\overline{D^{i(x)}_1} \cap C\right) \subset \overline{D^{j(x)}_2}$, $\pi_X\left(\mathrm{graph}(f) \cap \overline{D^{i(x)}_1} \times \overline{D^{j(x)}_2}\right) \subset G$ and $\mathrm{osc}(f\restriction_G) \leq 1/(2^l \cdot 5)$. Since $\mathrm{graph}(f\restriction_C)$ is compact we may find a finite subcover $W^* = \{W^*_1, \dots, W^*_m\}$ of $\{D^{i(x)}_1 \times D^{j(x)}_2;\ x \in C\}$. For every $1 \leq k \leq m$ we set $W_k = \overline{W^*_k}$. The collection $W = \{W_1, \dots, W_m\}$ clearly satisfies conditions (a), (b), (c) and (d).

By (b) and (c), $W \in \mathcal{W}$. Let $x \in C$. By (a), there is some $W_k \in W$ such that $(x, f(x)) \in W_k$. By (d), for any $W_o \in W$ if $x \in \pi_X(W_o)$ then $(x, f(x)) \in W_o$. Thus, $(x, f(x)) \in M(W) \cap \mathrm{graph}(f)$. So, $x \in \pi_X(M(W) \cap \mathrm{graph}(f))$. Therefore, $C \subset \pi_X(M(W) \cap \mathrm{graph}(f)) \subset H_W$. Thus, $\widetilde{C}(f) \subset \bigcup_{s\in\mathbb{N}} \mathcal{O}(H^s_l)$.

It now follows that $\widetilde{C}(f) \subset \tau$. □

(iii) Similarly as in the previous case, we find sets $H^i_l \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$, $i, l \in \mathbb{N}$, such that $\widetilde{C}(f) = \bigcap_{l\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \mathcal{O}(H^i_l)$. By Lemma 3.3.6(i) we have $\mathcal{O}(H^i_l) \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$ since α is limit. Consequently, $\widetilde{C}(f) \in \mathbf{\Pi}^0_{\alpha+1}(\mathcal{K}(X))$.

3.3.6 Proof of Theorem 3.2.8

(i) By Theorem 3.2.7(i) we have $f \in \mathcal{E}_{\alpha+1}(X, Y)$. By Lemma 3.3.12 we have $f \in \mathcal{B}_\alpha(X, Y)$.

(ii) By Lemma 3.3.12 we have $f \in \mathcal{E}_{\alpha+2}(X, Y)$. By Theorem 3.2.7(ii) we have $\widetilde{C}(f) \in \mathbf{\Pi}^0_{\alpha+5}(\mathcal{K}(X))$.

Chapter 4

The size of the classes of H^N sets

This chapter is based on the paper [14]. We investigate the size of H^N-sets which is reflected by the family of measures called polar which annihilate all the sets belonging to the given class. The main aim of this paper is to answer in the negative the question stated by Lyons [9], whether the polars of the classes of H^N-sets are same.

4.1 Introduction

Let M be a collection of closed subsets of $[0,1]$ and $\mathcal{M}([0,1])$ be the set of all Radon measures on the interval $[0,1]$. Then the *polar* $M^\perp \subset \mathcal{M}([0,1])$ is defined by

$$M^\perp = \{\nu \in \mathcal{M}([0,1]);\ \forall B \in M:\ \nu(B) = 0\}.$$

We say that $\mu \in \mathcal{M}([0,1])$ is *Rajchman* if $\lim_{|n|\to\infty} \widehat{\mu}(n) = 0$. The family of all Rajchman measures is denoted by $\mathcal{R}$. Let us recall that closed sets of *extended uniqueness* (U_0 sets) are those closed sets which are annihilated by every Rajchman measure. Thus by definition we have that $\mathcal{R} \subset U_0^\perp$. Lyons [8] showed that $\mathcal{R} = U_0^\perp$. Thus there appears a natural question, whether there are some other interesting classes $A \subset U_0$ with the property $A^\perp = \mathcal{R}$. More generally, one can consider two families of closed sets $A \subset B$ and may ask whether $B^\perp \subsetneq A^\perp$. If this is the case then B can be considered much larger than A. For example, Kaufman [4] proved that U_0 is in this sense larger than the family of closed sets of uniqueness U, i.e., $\mathcal{R} = U_0^\perp \subsetneq U^\perp$ (see [6] for the definition).

Rajchman [12] introduced an important subclass of U-sets, so called H-sets (or H^1-sets) (see the next section for the definition). Rajchman conjectured that every set of uniqueness is a countable union of H-sets. This was disproved by Pyatetskii-Shapiro in [11], where he also introduced the classes of H^N-sets. We have $H^N \subset H^{N+1} \subset U \subset U_0$ for every $N \in \mathbb{N}$ and for each N there is an H^{N+1}-set which cannot be written as a countable union of H^N-sets. Lyons [9] showed that $\mathcal{R} \subsetneq \left(\bigcup_{N\in\mathbb{N}} H^N\right)^\perp$. Thus, classes H^N are "small" in U_0 in the sense given above.

Lyons [9] posed a question whether $H^{N+1^\perp} = H^{N^\perp}$. The aim of this chapter is to prove the next theorem which answers Lyons' question in the negative for every $N \in \mathbb{N}$.

Theorem 4.1.1. *Let $N \in \mathbb{N}$. Then $H^{N+1^\perp} \neq H^{N^\perp}$.*

There also arises an open question, whether $\left(\bigcup_{N\in\mathbb{N}} H^N\right)^\perp \supsetneq U^\perp$.

4.2 Proof of Theorem 4.1.1

Notation 4.2.1.

(i) We denote the Lebesgue measure on $\mathbb{R}$ by λ and the number of elements of a finite set A by $\sharp A$.

(ii) The symbols $[x]$ and $\langle x\rangle$ stand for the integer part and fractional part of $x \in \mathbb{R}$ respectively, i.e., $\langle x\rangle = x - [x]$. Further, for $B \subset \mathbb{R}$ we denote $\langle B\rangle = \{\langle x\rangle;\ x \in B\}$.

(iii) The symbols $\mathbb{Q}$ and $\mathbb{Q}^*$ stand for the set of rational numbers and nonzero rational numbers respectively.

(iv) For $N \in \mathbb{N}$ and $\boldsymbol{a} \in \left(\mathbb{R}^N\right)^{\mathbb{N}}$, we employ the following coordinate notation $\boldsymbol{a} = \{a_j\}_{j\in\mathbb{N}}$ and $a_j = (a_j^1, \dots, a_j^N) \in \mathbb{R}^N$.

(v) By an open interval *$J \subset \mathbb{R}^N$ we mean any product of nonempty open intervals $J^i \subset \mathbb{R}$, $i = 1, \dots, N$.*

Definition 4.2.2. *Let $N \in \mathbb{N}$, $L \in \mathbb{R}$, and $P \subset \mathbb{R} \setminus \{0\}$.*

(i) A sequence of vectors $\boldsymbol{a} \in (\mathbb{R}^N)^{\mathbb{N}}$ is quasi independent *if for every nonzero $\lambda \in \mathbb{Z}^N$ we have $\lim_j |(\lambda, a_j)| = \infty$, where (u, v) denotes the scalar product of vectors $u, v \in \mathbb{R}^N$. The set of all quasi independent sequences of vectors from P^N is denoted by $\mathcal{Q}^N(P)$.*

(ii) A closed set $A \subset [0,1]$ is in $H^N(P)$ if there exist $\boldsymbol{a} \in \mathcal{Q}^N(P)$ and an open interval $J \subset [0,1]^N$ such that for every $x \in A$ and every $j \in \mathbb{N}$ we have $\langle xa_j\rangle = (\langle xa_j^1\rangle, \dots, \langle xa_j^N\rangle) \notin J$. We will write just H^N instead of $H^N(\mathbb{N})$. The elements of H^N are called H^N-sets.

(iii) A closed set $A \subset [0,1]$ is in $H_L^N(P)$ if there exist $\boldsymbol{a} \in \mathcal{Q}^N(P)$ and an open interval $J \subset [0,1]^N$ witnessing $A \in H^N(P)$ and satisfying

$$\left|\frac{a_j^{i+1}\lambda(J^i)}{a_j^i}\right| \geq L$$

for every $i \in \{1, \dots, N-1\}$ and $j \in \mathbb{N}$.

Remark 4.2.3. *(i) Let $N, M \in \mathbb{N}$, $N \leq M$, $L, K \in \mathbb{R}$, $L \leq K$, and $P \subset \mathbb{R} \setminus \{0\}$. Then we clearly have $H_K^N(P) \subset H_L^M(P)$, $H^N(P) = H_0^N(P)$, and $H_L^N(\mathbb{N}) \subset H_L^N(\mathbb{Q}^*)$. Further the families $H^N(P)$ and $H_L^N(P)$ are hereditary.*

(ii) In the role of P we will use $\mathbb{N}$ or $\mathbb{Q}^*$. Note that each set from $H^N(\mathbb{Q}^*)$ is a finite union of H^N-sets (see [1, pp. 919–921]). Consequently, $H^N(\mathbb{Q}^*)^\perp = H^{N\perp}$.

The proof of the main result is based on the following two results which will be proved in the next sections.

Lemma 4.2.4. *Let $N \in \mathbb{N}$. Then $H^{N+1\perp} \subsetneq H_{10}^N(\mathbb{Q}^*)^\perp$.*

Theorem 4.2.5. *Let $N, L \in \mathbb{N}$. Then $H_L^N(\mathbb{Q}^*) = H^N(\mathbb{Q}^*)$.*

Granting these results the proof goes as follows.

Proof of Theorem 4.1.1. Using Lemma 4.2.4, Theorem 4.2.5, and Remark 4.2.3 we get

$$H^{N+1\perp} \subsetneq H_{10}^N(\mathbb{Q}^*)^\perp = H^N(\mathbb{Q}^*)^\perp = H^{N\perp}.$$

□

4.3 Proof of Lemma 4.2.4

Throughout this section $N \in \mathbb{N}$ will be fixed. We will construct a measure $\mu \in H_{10}^N(\mathbb{Q}^*)^\perp \setminus H^{N+1\perp}$.

4.3.1 Construction of the measure μ.

Notation 4.3.1. *For $n \in \mathbb{N}$, $j = 1, \ldots, N+1$, and $I \subset [0,1]$ we fix the following:*

$$x_n^j = 2^{(N+1)n+j}((N+1)n+j)!, \qquad x_n = (x_n^1, \ldots, x_n^{N+1}). \tag{4.1}$$

Note that the numbers x_n^j are chosen so that $\frac{x_n^{j+1}}{2x_n^j}$ and $\frac{x_{n+1}^1}{2x_n^{N+1}}$ are natural numbers bigger than n.

$$P_n = \left\{x \in [0,1];\ \langle x \cdot x_i \rangle \notin (1/2, 1)^{N+1}, i = 1, \ldots, n\right\}, \tag{4.2}$$

$$\mathcal{P}_{n,j} = \left\{ \left[\frac{i-1}{2x_n^j}, \frac{i}{2x_n^j}\right] \subset [0,1];\ i \in \mathbb{N}, \left(\frac{i-1}{2x_n^j}, \frac{i}{2x_n^j}\right) \cap P_n \neq \emptyset \right\}, \tag{4.3}$$

$$\mathcal{P}_{n,j}^I = \{V \in \mathcal{P}_{n,j};\ V \subset I\}, \tag{4.4}$$

$$\|\mathcal{P}_{n,j}\| = 1/(2x_n^j). \tag{4.5}$$

Remark 4.3.2.

(i) If $V \in \mathcal{P}_{n,j}$, then $\|\mathcal{P}_{n,j}\| = \lambda(V)$.

(ii) Let $I, J \in \mathcal{P}_{n,N+1}$, $k > n$ and $j \leq N+1$. By definition we get $\sharp\,\mathcal{P}^I_{k,j} = \sharp\,\mathcal{P}^J_{k,j}$.

(iii) Let $I \in \mathcal{P}_{n-1,N+1}$ and $j \leq N+1$. By definition we get

$$\sharp\,\mathcal{P}^I_{n,N+1} \leq 2 \sum_{R \in \mathcal{P}^I_{n,j}} \sharp\,\mathcal{P}^R_{n,N+1}.$$

Lemma 4.3.3. *Let $n, s \in \mathbb{N}$, $s \leq N+1$, $I \in \mathcal{P}_{n-1,N+1}$ and let $Z \subset I$ be an interval with $\lambda(Z) \geq 9\|\mathcal{P}_{n,s}\|$. Then $\lambda(\bigcup \mathcal{P}^Z_{n,s}) \geq \frac{1}{3}\lambda(Z)$.*

Proof. Let $\mathcal{S}$ be the system of all intervals of the form $\left[\frac{i-1}{2x_n^s}, \frac{i}{2x_n^s}\right]$, $i \in \mathbb{N}$, which are contained in Z. Then we have $\lambda(Z) \leq (\sharp\,\mathcal{S} + 2)\|\mathcal{P}_{n,s}\|$. If intervals $\left[\frac{i-1}{2x_n^s}, \frac{i}{2x_n^s}\right]$ and $\left[\frac{i}{2x_n^s}, \frac{i+1}{2x_n^s}\right]$ are in $\mathcal{S}$, then at least one of them belongs to $\mathcal{P}^Z_{n,s}$ by (4.2), (4.3) and $Z \subset I \in \mathcal{P}_{n-1,N+1}$. Then we have

$$\begin{aligned}
\lambda\left(\bigcup \mathcal{P}^Z_{n,s}\right) &\geq \left[\frac{1}{2}\sharp\,\mathcal{S}\right]\|\mathcal{P}_{n,s}\| \geq \frac{1}{2}(\sharp\,\mathcal{S} - 1)\|\mathcal{P}_{n,s}\| \\
&= \frac{1}{2}(\sharp\,\mathcal{S} + 2)\|\mathcal{P}_{n,s}\| - \frac{3}{2}\|\mathcal{P}_{n,s}\| \geq \frac{1}{2}\lambda(Z) - \frac{3}{2}\cdot\frac{1}{9}\lambda(Z) = \frac{1}{3}\lambda(Z).
\end{aligned}$$

□

Construction 4.3.4. *We define a Radon measure μ_n, $n \in \mathbb{N}$, on the interval $[0,1]$ such that*

$$\mu_n(A) = \left(1 - 2^{-(N+1)}\right)^{-n} \lambda(A \cap P_n),$$

whenever $A \subset [0,1]$ is Borel.

Lemma 4.3.5.

(i) For each $n \in \mathbb{N}$, μ_n is a continuous probability measure.

(ii) If $Q = \left[\frac{i-1}{2x_n^{N+1}}, \frac{i}{2x_n^{N+1}}\right]$, where $n, i \in \mathbb{N}$, $i \leq 2x_n^{N+1}$, then

$$\mu_k(Q) = \mu_n(Q) = \begin{cases} \frac{1}{\sharp\,\mathcal{P}_{n,N+1}} & \text{for } Q \in \mathcal{P}_{n,N+1}, \\ 0 & \text{for } Q \notin \mathcal{P}_{n,N+1}, \end{cases}$$

for every $k \geq n$.

Proof. (i) Since μ_n is a restriction of a multiple of Lebesgue measure, μ_n is continuous. Since

$$\begin{aligned}
\lambda(P_1) &= 1 - 2^{-(N+1)}, \\
\lambda(P_{n+1}) &= \left(1 - 2^{-(N+1)}\right)\lambda(P_n), \qquad n \in \mathbb{N},
\end{aligned}$$

we get that μ_n is a probability.

(ii) Let $k \in \mathbb{N}$, $k \geq n$. If $Q \notin \mathcal{P}_{n,N+1}$, then $Q^0 \cap P_k \subset Q^0 \cap P_n = \emptyset$. Consequently, $\mu_k(Q) = \mu_n(Q) = 0$. Otherwise we have

$$\mu_k(Q) = \mu_k(Q \cap P_k) = \sum_{J \in \mathcal{P}^Q_{k,N+1}} \mu_k(J) = \sum_{J \in \mathcal{P}^Q_{k,N+1}} (1 - 2^{-(N+1)})^{-k} \lambda(J).$$

Using Remark 4.3.2(i) and (i) we infer

$$\begin{aligned}\mu_k(Q) &= \sum_{J \in \mathcal{P}^Q_{k,N+1}} (1 - 2^{-(N+1)})^{-k} \lambda(J) = \sum_{J \in \mathcal{P}^Q_{k,N+1}} \frac{1}{\sharp\, \mathcal{P}_{k,N+1}} \\ &= \frac{\sharp\, \mathcal{P}^Q_{k,N+1}}{\sharp\, \mathcal{P}_{k,N+1}} = \frac{\sharp\, \mathcal{P}^Q_{k,N+1}}{\sum_{V \in \mathcal{P}_{n,N+1}} \sharp\, \mathcal{P}^V_{k,N+1}}.\end{aligned}$$

By Remark 4.3.2(ii) we get $\mu_k(Q) = \frac{1}{\sharp\, \mathcal{P}_{n,N+1}}$ and the statement follows. □

Lemma 4.3.6. *The sequence $\{\mu_n\}$ w^*-converges to a continuous Radon measure μ.*

Proof. Let Q be any interval with endpoints $\frac{i-1}{2x_n^{N+1}}$, $\frac{i+1}{2x_n^{N+1}}$, where $n, i \in \mathbb{N}$, which is contained in $[0,1]$. By Lemma 4.3.5(ii) we have that $\lim_k \mu_k(Q)$ exists finite. Since each continuous function can be uniformly approximated by a function $\sum_{j=1}^m c_j \chi_{Q_j}$, where Q_j, $j = 1, \ldots, m$, are intervals of the above form, we get that $\lim_k \int f d\mu_k$ exists finite for each continuous function f on $[0,1]$. Thus $\{\mu_k\}$ w^*-converges to a Radon probability measure μ.

Let $x \in [0,1]$ and $\varepsilon > 0$. One can find an interval Q of the above form and corresponding n with x in the interior of Q considered with respect to $[0,1]$ and $\mu_n(Q) \leq 2/\sharp\, \mathcal{P}_{n,N+1} < \varepsilon$ by Lemma 4.3.5(ii). Then

$$\mu(\{x\}) \leq \mu(Q^0) \leq \liminf_k \mu_k(Q^0) \leq 2/\sharp\, \mathcal{P}_{n,N+1} < \varepsilon.$$

Thus μ is continuous. □

As a simple corollary of continuity of measure μ and Lemma 4.3.5(ii) we get the following.

Remark 4.3.7. *If $V \in \mathcal{P}_{n,N+1}$, then $\mu(V) = \frac{1}{\sharp\, \mathcal{P}_{n,N+1}}$.*

4.3.2 Verification of $\mu \notin {H^{N+1}}^{\perp}$

Lemma 4.3.8. *Set*

$$P = \left\{x \in [0,1];\ \forall i \in \mathbb{N}:\ \langle x \cdot x_i \rangle \notin (1/2, 1)^{N+1}\right\}.$$

Then P is an H^{N+1}-set and $\mu(P) = 1$.

Proof. We have $\{x_n\}_{n\in\mathbb{N}} \in \mathcal{Q}(\mathbb{N})$ by (4.1) and therefore $P \in H^{N+1}$. By (4.2) we have $P = \bigcap_{k\in\mathbb{N}} P_k$. Using the definition of μ and continuity of μ we get

$$\mu(P) = \lim_{k\to\infty} \mu(P_k) = \lim_{k\to\infty} \lim_{n\to\infty} \mu_n(P_k) = 1,$$

where the last equation follows from $\mu_n(P_k) = 1$ for all $n \geq k$. Indeed, μ_n is a probability, $\operatorname{supp}(\mu_n) = P_n$ and $P_n \subset P_k$. □

4.3.3 Verification of $\mu \in H^N_{10}(\mathbb{Q}^*)^\perp$

We will need the following notation.

Notation 4.3.9. *Let $\rho \in \mathbb{N}$, $I \subset [0,1]^\rho$ be an interval and $\boldsymbol{z} \in \mathcal{Q}^\rho(\mathbb{Q})$. Then we define*

$$H(\boldsymbol{z}, I) = \{x \in [0,1];\ \forall k \in \mathbb{N}:\ \langle x \cdot z_k\rangle \notin I\}.$$

Remark 4.3.10. *Let $\rho \in \mathbb{N}$.*

(i) If $A \in H^\rho(\mathbb{Q}^*)$ then there exist $\boldsymbol{z} \in \mathcal{Q}^\rho(\mathbb{Q}^*)$ and an open interval $W \subset [0,1]^\rho$ such that $A \subset H(\boldsymbol{z}, W)$.

(ii) If $I \subset J \subset [0,1]^\rho$ are open intervals and $\boldsymbol{r} \in \mathcal{Q}^\rho(\mathbb{Q})$, then $H(\boldsymbol{r}, J) \subset H(\boldsymbol{r}, I)$.

Now we fix an arbitrary $X \in H^N_{10}(\mathbb{Q}^*)$. We find an open interval $W = \prod_{j=1}^N W_j \subset [0,1]^N$ and $\boldsymbol{z} \in \mathcal{Q}^N(\mathbb{Q}^*)$ witnessing $X \in H^N_{10}(\mathbb{Q}^*)$. Thus, we have

$$\left|\frac{z_i^{j+1}\lambda(W_j)}{z_i^j}\right| \geq 10 \qquad \text{for every } i \in \mathbb{N}, j \in \{1,\dots,N-1\}. \tag{4.6}$$

We have $X \subset H(\boldsymbol{z}, W)$. We want to show that $\mu(X) = 0$, so it is sufficient to prove $\mu(H(\boldsymbol{z}, W)) = 0$. Denote $A = H(\boldsymbol{z}, W)$.

Let $0 \leq \sigma \leq \rho \leq N$ be integers. We set

$$A_{k,\sigma,\rho} = \{x \in [0,1]; \exists \sigma < j \leq \rho : \langle x \cdot z_k^j\rangle \notin W_j\}, \tag{4.7}$$

$$A_k = \{x \in [0,1]; \forall i \leq k : \langle x \cdot z_i\rangle \notin W\} = \bigcap_{i\leq k} A_{i,0,N}. \tag{4.8}$$

Further fix a constant $l \in \mathbb{N}$ such that

$$l > 100 \quad \text{and} \quad l > 1/\lambda(W_j), \quad j = 1,\dots,N. \tag{4.9}$$

Notation 4.3.11. *Let $n, k \in \mathbb{N}$ and $S, T \subset [0,1]$. We define*

$$\mathcal{B}_n(S,T) = \{V \in \mathcal{P}^S_{n,N+1};\ V \cap T \neq \emptyset\},$$

and if $\mathcal{P}^S_{n,N+1} \neq \emptyset$, then we set

$$\mu^S_{n,k} = \frac{\sharp\, \mathcal{B}_n(S, A_k)}{\sharp\, \mathcal{P}^S_{n,N+1}} \qquad \text{and} \qquad \mu_{n,k} = \mu^{[0,1]}_{n,k}.$$

Lemma 4.3.12.

(i) We have $\mu(A) = \inf\{\mu_{n,k};\ n, k \in \mathbb{N}\}$.

(ii) If $n, t, s, k \in \mathbb{N}$, $n \geq s \geq t$ *and* $I \in \mathcal{P}_{t,N+1}$ *then* $\mu^I_{n,k} \leq \mu^I_{s,k} \cdot \sup\{\mu^V_{n,k};\ V \in \mathcal{P}^I_{s,N+1}\}$.

Proof. (i) For every $k, n \in \mathbb{N}$ we have

$$A_k \cap P \subset A_k \cap P_n \subset \bigcup \mathcal{B}_n([0,1], A_k). \tag{4.10}$$

The set $A_k \cap P$ is closed and $\bigcap_{n=1}^{\infty} \bigcup \mathcal{B}_n([0,1], A_k) \subset P$. The diameters of intervals from $\mathcal{B}_n([0,1], A_k)$ tend uniformly to 0 if $n \to \infty$, so we get $A_k \cap P = \bigcap_{n=1}^{\infty} \bigcup \mathcal{B}_n([0,1], A_k)$. Using continuity of μ and Remark 4.3.7 we can conclude

$$\mu\left(\bigcup \mathcal{B}_n([0,1], A_k)\right) = \sum_{J \in \mathcal{B}_n([0,1], A_k)} \mu(J) = \frac{\sharp\, \mathcal{B}_n([0,1], A_k)}{\sharp\, \mathcal{P}_{n,N+1}} = \mu_{n,k}. \tag{4.11}$$

Since $A \cap P \subset \bigcup \mathcal{B}_n([0,1], A_k)$ by (4.10), we get $\mu(A) = \mu(A \cap P) \leq \mu_{n,k}$ by (4.11).

Since

$$A \cap P = \bigcap_{k=1}^{\infty} A_k \cap P = \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup \mathcal{B}_n([0,1], A_k),$$

we get $\mu(A) \geq \inf\{\mu_{n,k};\ n, k \in \mathbb{N}\}$.

(ii) It is easy to verify that

$$\begin{aligned} \mu^I_{n,k} = \frac{\sharp\, \mathcal{B}_n(I, A_k)}{\sharp\, \mathcal{P}^I_{n,N+1}} &= \sum_{V \in \mathcal{P}^I_{s,N+1}} \frac{\sharp\, \mathcal{B}_n(V, A_k)}{\sharp\, \mathcal{P}^I_{s,N+1} \cdot \sharp\, \mathcal{P}^V_{n,N+1}} \\ &= \frac{\sum_{V \in \mathcal{P}^I_{s,N+1}} \mu^V_{n,k}}{\sharp\, \mathcal{P}^I_{s,N+1}} = \frac{\sum_{V \in \mathcal{B}_s(I, A_k)} \mu^V_{n,k}}{\sharp\, \mathcal{P}^I_{s,N+1}}, \end{aligned}$$

where the last equality follows from the fact that $\mu^V_{n,k} = 0$ for all $V \in \mathcal{P}^I_{s,N+1} \backslash \mathcal{B}_s(I, A_k)$. Thus, we have

$$\mu^I_{n,k} \leq \sup\{\mu^V_{n,k};\ V \in \mathcal{P}^I_{s,N+1}\} \cdot \frac{\sharp\, \mathcal{B}_s(I, A_k)}{\sharp\, \mathcal{P}^I_{s,N+1}} = \mu^I_{s,k} \cdot \sup\{\mu^V_{n,k};\ V \in \mathcal{P}^I_{s,N+1}\}.$$

□

Let us assume that $k \in \mathbb{N}$ *is fixed in the following definition, Lemma 4.3.14 and Lemma 4.3.15.*

Let $j \in \{1, \dots, N\}$. We define an auxiliary family $\mathcal{K}_j$ as the family of all intervals of the form $\frac{1}{z_k^j}(W_j + m)$, $m \in \mathbb{Z}$. We denote $\|\mathcal{K}_j\| = \lambda(W_j)/|z_k^j|$. Let $S \subset [0, 1]$ be an interval. We inductively define

$$\mathcal{K}_{j,j}^S = \left\{K \cap S^0;\ K \in \mathcal{K}_j, \lambda(K \cap S) \geq \|\mathcal{K}_j\|/2\right\},$$
$$\mathcal{K}_{j,t}^S = \bigcup\{\mathcal{K}_{t,t}^Z;\ Z \in \mathcal{K}_{j,t-1}^S\}, \qquad j < t \leq N.$$

Remark 4.3.13. *The system $\mathcal{K}_{j,t}^S$ contains just intervals of the form $K \cap Z$, which have length greater than one half of the length of K, where $Z \in \mathcal{K}_{j,t-1}^S$ and $K \in \mathcal{K}_t$. By definition we get the following two properties.*

(i) For every $Z \in \mathcal{K}_{j,t}^S$ we have $\lambda(Z) \geq \frac{1}{2}\|\mathcal{K}_t\| = \frac{1}{2}\lambda(W_j)/|z_k^j|$.

(ii) If $S_1 \subset S_2 \subset [0, 1]$ are intervals and $K \in \mathcal{K}_{j,t}^{S_1}$, then there exists $\tilde{K} \in \mathcal{K}_{j,t}^{S_2}$ with $K \subset \tilde{K}$.

Lemma 4.3.14. *Let $0 < j \leq t \leq N$ and $S \subset [0, 1]$ be an interval with $\lambda(S) \geq 1/|z_k^j|$. Then $\lambda\left(\bigcup \mathcal{K}_{j,t}^S\right) \geq \lambda(S) \cdot (4l)^{j-t-1}$.*

Proof. If $1 \leq i \leq N$ and $J \subset [0, 1]$ is an interval with $\lambda(J) \geq 1/|z_k^i|$, then

$$\lambda\left(\bigcup \mathcal{K}_{i,i}^J\right) \geq \frac{1}{4}\lambda(J)\lambda(W_i). \tag{4.12}$$

Indeed, we have $\sharp\, \mathcal{K}_{i,i}^J \geq \left[\lambda(J)|z_k^i|\right] \geq \frac{1}{2}\lambda(J)|z_k^i|$ and using Remark 4.3.13(i) we get

$$\lambda\left(\bigcup \mathcal{K}_{i,i}^J\right) \geq \sharp\, \mathcal{K}_{i,i}^J \cdot \frac{\lambda(W_i)}{2|z_k^i|} \geq \frac{1}{4}\lambda(J)\lambda(W_i).$$

Using this for $J := S$, $i := j$ and (4.9) we get

$$\lambda\left(\bigcup \mathcal{K}_{j,j}^S\right) \geq \frac{\lambda(S)}{4l}. \tag{4.13}$$

Further we proceed by induction on t. We assume that the desired inequality holds for $t = i \geq j$. Let $Z \in \mathcal{K}_{j,i}^S$. By Remark 4.3.13(i) and (4.6) we have $\lambda(Z) \geq \frac{\lambda(W_i)}{2|z_k^i|} \geq \frac{5}{|z_k^{i+1}|} \geq \frac{1}{|z_k^{i+1}|}$. Thus we can use (4.12) for $J := Z$ and $i := i + 1$. We get

$$\lambda\left(\bigcup \mathcal{K}_{i+1,i+1}^Z\right) \geq \frac{1}{4}\lambda(Z)\lambda(W_{i+1})$$

and we infer

$$\lambda\left(\bigcup \mathcal{K}_{j,i+1}^S\right) = \sum_{Z \in \mathcal{K}_{j,i}^S} \lambda\left(\bigcup \mathcal{K}_{i+1,i+1}^Z\right) \geq \sum_{Z \in \mathcal{K}_{j,i}^S} \frac{1}{4}\lambda(Z)\lambda(W_{i+1})$$
$$= \lambda\left(\bigcup \mathcal{K}_{j,i}^S\right) \frac{\lambda(W_{i+1})}{4} \geq \lambda(S)(4l)^{j-i-2}.$$

The last inequality follows from induction hypothesis and (4.9) and we are done. □

Lemma 4.3.15. *Let $0 \le \sigma \le \rho \le N$ be integers, $n \in \mathbb{N}$, $I \in \mathcal{P}_{n-1,N+1}$ and $G \subset I$ be a closed interval. Suppose that the following conditions are satisfied*

$$n \ge l^2, \tag{4.14}$$

$$\frac{\lambda(G)}{\lambda(I)} > \frac{1}{\sqrt{n}}, \tag{4.15}$$

$$\lambda(G) \ge \frac{1}{|z_k^i|} \ge \|\mathcal{P}_{n,N+1}\|, \qquad \sigma < i \le \rho. \tag{4.16}$$

Let $\alpha \in (0,1]$. We define $\mathcal{D}^\alpha$ as the family of all intervals $V \in \mathcal{P}^G_{n,N+1}$ for which there exist a closed interval $O \subset V$ such that $\lambda(O) \ge \alpha\lambda(V)$ and $O \cap A_{k,\sigma,\rho} = \emptyset$.

(i) Then we have

$$\frac{\sharp\,\mathcal{D}^\tau}{\sharp\,\mathcal{P}^I_{n,N+1}} \ge \frac{\lambda(G)}{18\lambda(I)}(4l)^{\sigma-\rho}, \quad \textit{where}$$

$$\tau = \min\left\{1, \frac{\lambda(W_\rho)}{6 \cdot |z_k^\rho| \cdot \|\mathcal{P}_{n,N+1}\|}\right\}. \tag{4.17}$$

(ii) If $\sigma = \rho$, then we have $\frac{\sharp\,\mathcal{D}^1}{\sharp\,\mathcal{P}^I_{n,N+1}} \ge \frac{\lambda(G)}{6\lambda(I)}$.

(iii) If $\sigma = \rho$ and $G = I$, then we have $\sharp\,\mathcal{D}^1 = \sharp\,\mathcal{P}^I_{n,N+1}$.

Proof. (i) Since $\mathcal{D}^1 \subset \mathcal{D}^\tau$ the case $\sigma = \rho$ will follow from (ii). Here we will assume that $\sigma < \rho$. Let $\alpha \in (0,1]$, $S \subset [0,1]$ be an interval, $s \le N+1$ and $i, j \in \{\sigma, \dots, \rho\}, i < j$. Then we define

$$\mathcal{D}^{\alpha,S}_{i,j,s} = \left\{V \in \mathcal{P}^S_{n,s}; \exists K \in \mathcal{K}^S_{i+1,j}\ \exists O \subset K \cap V \text{ closed interval: } \lambda(O) \ge \alpha\lambda(V)\right\}.$$

Note that for $\alpha = 1$ we have by definition

$$\mathcal{D}^{1,S}_{i,j,s} = \left\{V \in \mathcal{P}^S_{n,s}; \exists K \in \mathcal{K}^S_{i+1,j} :\ V \subset K\right\} = \bigcup_{K \in \mathcal{K}^S_{i+1,j}} \mathcal{P}^K_{n,s}. \tag{4.18}$$

Clearly, $A_{k,\sigma,\rho} \cap \bigcup \mathcal{K}^G_{\sigma+1,\rho} = \emptyset$. Thus, we have

$$\mathcal{D}^\alpha \supset \mathcal{D}^{\alpha,G}_{\sigma,\rho,N+1}. \tag{4.19}$$

Claim 4.3.16. *Let $\sigma < j \le \rho$ and let $s < N+1$ be such that*

$$\frac{1}{\left|z_k^j\right|} \ge \|\mathcal{P}_{n,s}\|. \tag{4.20}$$

Then we have

$$\frac{\sharp\,\mathcal{D}^{1,G}_{\sigma,j,s+1}}{\sharp\,\mathcal{P}^I_{n,s+1}} \ge \frac{\lambda(G)}{3\lambda(I)}(4l)^{\sigma-j}.$$

Proof of Claim. By (4.9) we have $\|\mathcal{K}_j\| \geq \frac{1}{l\left|z_k^j\right|}$. Using (4.14) we get

$$\|\mathcal{K}_j\| \geq \frac{1}{\left|z_k^j\right|\sqrt{n}}.$$

Since $\frac{\|\mathcal{P}_{n,s}\|}{\|\mathcal{P}_{n,s+1}\|} \geq n$ we have

$$\|\mathcal{K}_j\| \geq \frac{\sqrt{n}\|\mathcal{P}_{n,s+1}\|}{\left|z_k^j\right|\|\mathcal{P}_{n,s}\|}.$$

By (4.20) we have

$$\|\mathcal{K}_j\| \geq \sqrt{n}\|\mathcal{P}_{n,s+1}\|.$$

Let $Z \in \mathcal{K}^G_{\sigma+1,j}$ be arbitrary. By (4.9), (4.14) and Remark 4.3.13(i) we have

$$\lambda(Z) \geq 50\|\mathcal{P}_{n,s+1}\|.$$

Using Lemma 4.3.3 we obtain

$$\lambda\left(\bigcup \mathcal{P}^Z_{n,s+1}\right) \geq \frac{\lambda(Z)}{3}. \tag{4.21}$$

Clearly,

$$\frac{\sharp\,\mathcal{D}^{1,G}_{\sigma,j,s+1}}{\sharp\,\mathcal{P}^I_{n,s+1}} = \frac{\lambda\left(\bigcup \mathcal{D}^{1,G}_{\sigma,j,s+1}\right)}{\lambda\left(\bigcup \mathcal{P}^I_{n,s+1}\right)} \geq \frac{\lambda\left(\bigcup \mathcal{D}^{1,G}_{\sigma,j,s+1}\right)}{\lambda(I)}. \tag{4.22}$$

By (4.18), (4.21), Lemma 4.3.14 and (4.16) we have

$$\begin{aligned}\lambda\left(\bigcup \mathcal{D}^{1,G}_{\sigma,j,s+1}\right) &= \sum_{Z\in\mathcal{K}^G_{\sigma+1,j}} \lambda\left(\bigcup \mathcal{P}^Z_{n,s+1}\right) \\ &\geq \frac{1}{3}\lambda\left(\bigcup \mathcal{K}^G_{\sigma+1,j}\right) \geq \frac{\lambda(G)}{3}(4l)^{\sigma-j}.\end{aligned} \tag{4.23}$$

Now Claim follows from (4.22) and (4.23). □

Claim 4.3.17. *Let $\sigma < j \leq \rho$. Then we have*

$$\sharp\,\mathcal{D}^{\tau,G}_{\sigma,\rho,N+1} \geq \begin{cases} \sum_{J\in\mathcal{P}^G_{n,j}} \sharp\,\mathcal{D}^{\tau,J}_{j-1,\rho,N+1} & \text{for } j = \sigma+1, \\ \sum_{J\in\mathcal{D}^{1,G}_{\sigma,j-1,j}} \sharp\,\mathcal{D}^{\tau,J}_{j-1,\rho,N+1} & \text{for } j > \sigma+1. \end{cases}$$

Proof of Claim. Clearly, $\mathcal{P}^G_{n,j} \supset \mathcal{D}^{1,G}_{\sigma,j-1,j}$ for all $j > \sigma+1$. If $J_1, J_2 \in \mathcal{P}^G_{n,j}$ and $J_1 \neq J_2$ then $\mathcal{D}^{\tau,J_1}_{j-1,\rho,N+1} \cap \mathcal{D}^{\tau,J_2}_{j-1,\rho,N+1} = \emptyset$. So, it is sufficient to prove that

$$\mathcal{D}^{\tau,G}_{\sigma,\rho,N+1} \supset \bigcup_{J\in\mathcal{P}^G_{n,\sigma+1}} \mathcal{D}^{\tau,J}_{\sigma,\rho,N+1} \tag{4.24}$$

and

$$\mathcal{D}^{\tau,G}_{\sigma,\rho,N+1} \supset \bigcup_{J\in\mathcal{D}^{1,G}_{\sigma,j-1,j}} \mathcal{D}^{\tau,J}_{j-1,\rho,N+1}, \quad j > \sigma + 1. \tag{4.25}$$

The formula (4.24) is obvious. Assume $j > \sigma + 1$. Let $J \in \mathcal{D}^{1,G}_{\sigma,j-1,j}$ and $V \in \mathcal{D}^{\tau,J}_{j-1,\rho,N+1}$ be arbitrary. Then there exists $K_1 \in \mathcal{K}^G_{\sigma+1,j-1}$ with $J \subset K_1$ and there exist $K_2 \in \mathcal{K}^J_{j,\rho}$ and a closed interval $O \subset V \cap K_2$ such that $\lambda(O) \geq \tau\lambda(V)$. By Remark 4.3.13(ii) there exists $\widetilde{K}_2 \in \mathcal{K}^{K_1}_{j,\rho}$ such that $\widetilde{K}_2 \supset K_2$. Thus, we have $\widetilde{K}_2 \in \mathcal{K}^G_{\sigma+1,\rho}$ and $O \subset V \cap \widetilde{K}_2$. Thus $V \in \mathcal{D}^{\tau,G}_{\sigma,\rho,N+1}$ and the formula (4.25) is proved. □

We distinguish two possibilities. We have either

(a) $1 \geq \|\mathcal{P}_{n,\rho}\| \cdot |z^\rho_k|$ or

(b) there exists $j \in \mathbb{N}$ such that $\sigma < j \leq \rho$, $1 < \|\mathcal{P}_{n,j}\| \cdot |z^j_k|$ and for every $\sigma < i < j$ we have $1 \geq \|\mathcal{P}_{n,i}\| \cdot |z^i_k|$.

(a) Since $\rho \leq N$, we have $1 \geq \|\mathcal{P}_{n,\rho}\| \cdot |z^\rho_k| \geq \|\mathcal{P}_{n,N}\| \cdot |z^\rho_k|$. By Claim 4.3.16 and (4.19) we have

$$\frac{\sharp\,\mathcal{D}^\tau}{\sharp\,\mathcal{P}^I_{n,N+1}} \geq \frac{\sharp\,\mathcal{D}^1}{\sharp\,\mathcal{P}^I_{n,N+1}} \geq \frac{\lambda(G)}{3\lambda(I)}(4l)^{\sigma-\rho}.$$

(b) Let $J \in \mathcal{P}^G_{n,j}$ and $Z \in \mathcal{K}^J_{j,\rho}$ be arbitrary. By $\lambda(J) = \|\mathcal{P}_{n,j}\| > 1/|z^j_k|$ and Lemma 4.3.14 we have

$$\lambda\left(\bigcup \mathcal{K}^J_{j,\rho}\right) \geq \lambda(J) \cdot (4l)^{j-\rho-1}. \tag{4.26}$$

By (4.17) and Remark 4.3.13(i) we have

$$\tau\|\mathcal{P}_{n,N+1}\| \leq \frac{\lambda(W_\rho)}{6\,|z^\rho_k|} \leq \frac{\lambda(Z)}{3}. \tag{4.27}$$

Since $J \in \mathcal{P}_{n,j}$ we have

$$\bigcup \mathcal{P}^J_{n,N+1} = J. \tag{4.28}$$

We find $v, s \geq 0$ such that $Z = (\frac{s}{v}, \frac{s+3}{v})$. If $V \in \mathcal{P}^J_{n,N+1}$ and $V \cap (\frac{s+1}{v}, \frac{s+2}{v}) \neq \emptyset$ then $V \subset Z$ or $\lambda(V \cap Z) > \frac{1}{3}\lambda(Z)$. In the first case the interval Z witnesses $V \in \mathcal{D}^{1,J}_{j-1,\rho,N+1} \subset \mathcal{D}^{\tau,J}_{j-1,\rho,N+1}$. In the second case there exists a closed interval $O \subset V \cap Z$ such that $\lambda(O) \geq \frac{1}{3}\lambda(Z)$. By (4.27) we have

$$\lambda(O) \geq \frac{1}{3}\lambda(Z) \geq \tau\|\mathcal{P}_{n,N+1}\| = \tau\lambda(V).$$

Consequently, $V \in \mathcal{D}^{\tau,J}_{j-1,\rho,N+1}$.

The considerations of the previous paragraph and (4.28) gives $\bigcup \mathcal{D}^{\tau,J}_{j-1,\rho,N+1} \supset (\frac{s+1}{v}, \frac{s+2}{v})$. Thus, we have

$$\frac{\lambda\left(\bigcup \mathcal{D}^{\tau,J}_{j-1,\rho,N+1}\right)}{\lambda\left(\bigcup \mathcal{K}^J_{j,\rho}\right)} \geq \frac{1}{3}.$$

Thus, by (4.26) and (4.28) we have

$$\frac{\lambda\left(\bigcup \mathcal{D}^{\tau,J}_{j-1,\rho,N+1}\right)}{\lambda\left(\bigcup \mathcal{P}^J_{n,N+1}\right)} \geq (4l)^{j-\rho-1} \cdot \frac{1}{3}.$$

So,

$$\frac{\sharp\, \mathcal{D}^{\tau,J}_{j-1,\rho,N+1}}{\sharp\, \mathcal{P}^J_{n,N+1}} \geq (4l)^{j-\rho-1} \cdot \frac{1}{3}. \tag{4.29}$$

We distinguish two cases.

(b1) Assume $j = \sigma + 1$. By (4.1) and (4.5) we have $\lambda(I) > n||\mathcal{P}_{n,1}||$. Using (4.14), (4.15), and $l > 100$ we get

$$\lambda(G) > \frac{\lambda(I)}{\sqrt{n}} > \sqrt{n}\|\mathcal{P}_{n,1}\| > 100 \cdot \|\mathcal{P}_{n,1}\| > 9 \cdot \|\mathcal{P}_{n,\sigma+1}\|. \tag{4.30}$$

Using Lemma 4.3.3 we obtain $\frac{\sharp\, \mathcal{P}^G_{n,\sigma+1}}{\sharp\, \mathcal{P}^I_{n,\sigma+1}} \geq \frac{\lambda(G)}{3\lambda(I)}$. By Claim 4.3.17, (4.19), Remark 4.3.2 and (4.29) we have

$$\frac{\sharp\, \mathcal{D}^\tau}{\sharp\, \mathcal{P}^I_{n,N+1}} \geq \frac{\sum_{J \in \mathcal{P}^G_{n,\sigma+1}} \sharp\, \mathcal{D}^{\tau,J}_{\sigma,\rho,N+1}}{2\sum_{R \in \mathcal{P}^I_{n,\sigma+1}} \sharp\, \mathcal{P}^R_{n,N+1}} \geq \frac{\sharp\, \mathcal{P}^G_{n,\sigma+1}}{\sharp\, \mathcal{P}^I_{n,\sigma+1}} \cdot (4l)^{\sigma-\rho} \cdot \frac{1}{6} \geq \frac{\lambda(G)}{18\lambda(I)} \cdot (4l)^{\sigma-\rho}.$$

(b2) Let $j > \sigma + 1$. By Claim 4.3.17, (4.19), Remark 4.3.2 and (4.29) we have

$$\frac{\sharp\, \mathcal{D}^\tau}{\sharp\, \mathcal{P}^I_{n,N+1}} \geq \frac{\sum_{J \in \mathcal{D}^{1,G}_{\sigma,j-1,j}} \sharp\, \mathcal{D}^{\tau,J}_{j-1,\rho,N+1}}{2\sum_{R \in \mathcal{P}^I_{n,j}} \sharp\, \mathcal{P}^R_{n,N+1}} \geq \frac{\sharp\, \mathcal{D}^{1,G}_{\sigma,j-1,j}}{\sharp\, \mathcal{P}^I_{n,j}} \cdot (4l)^{j-\rho-1} \cdot \frac{1}{6}.$$

By Claim 4.3.16 we get the desired inequality.

(ii) If $\sigma = \rho$ then $A_{k,\sigma,\rho} = \emptyset$. If $V \in \mathcal{P}^G_{n,N+1}$, we set $O = V$. So, $\lambda(O) = \lambda(V)$ and $O \cap A_{k,\sigma,\rho} = \emptyset$. So,

$$\mathcal{D}^1 = \mathcal{P}^G_{n,N+1}. \tag{4.31}$$

By Lemma 4.3.3 and (4.30) we have $\frac{\sharp \mathcal{P}^G_{n,1}}{\sharp \mathcal{P}^I_{n,1}} \geq \frac{\lambda(G)}{3\lambda(I)}$. Thus, by (4.31) and Remark 4.3.2 we have

$$\frac{\sharp \mathcal{D}^1}{\sharp \mathcal{P}^I_{n,N+1}} = \frac{\sharp \mathcal{P}^G_{n,N+1}}{\sharp \mathcal{P}^I_{n,N+1}} \geq \frac{\sum_{R \in \mathcal{P}^G_{n,1}} \sharp \mathcal{P}^R_{n,N+1}}{2 \sum_{R \in \mathcal{P}^I_{n,1}} \sharp \mathcal{P}^R_{n,N+1}} = \frac{\sharp \mathcal{P}^G_{n,1}}{2 \sharp \mathcal{P}^I_{n,1}} \geq \frac{\lambda(G)}{6\lambda(I)}.$$

(iii) We use (4.31) and $G = I$ and we are done. □

Lemma 4.3.18. *There exists $K > 0$ such that for every $n, k \in \mathbb{N}$ there exist $n(n,k) \in \mathbb{N}$ and $k(n,k) \in \mathbb{N}$ such that*

$$\mu_{n(n,k),k(n,k)} \leq (1 - K) \cdot \mu_{n,k}.$$

Proof. Fix $n, k \in \mathbb{N}$. We set $n_0 = \max\{n+1, 36l^2\}$. We will construct $k_0 > k$, $s \leq N$ and sequences $n_0 < n_1 < \cdots < n_s$ and $0 = v_0 < v_1 < \cdots < v_s = N$ such that

$$\forall 0 < i \leq s \; \forall v_{i-1} < j \leq v_i : \; \|\mathcal{P}_{n_i,N+1}\| \leq \frac{1}{\left|z^j_{k_0}\right|} < \|\mathcal{P}_{n_i-1,N+1}\|.$$

Since $\boldsymbol{z} \in \mathcal{Q}^N(\mathbb{Q}^*)$ and (4.6) we have $\lim |z^1_i| = \infty$ and $\left|z^{j+1}_i\right| \geq 10 \left|z^j_i\right|$ for every $i \in \mathbb{N}, j < N$. Thus, we can find $k_0 > k$ such that $\frac{1}{|z^1_{k_0}|} < \|\mathcal{P}_{n_0,N+1}\|$. We set $v_0 = 0$. Assume that we have already constructed $n_0, \ldots, n_i$ and $v_0, \ldots, v_i$ for some $i \geq 0$. If $v_i = N$ we set $s = i$ and we are done. If $v_i < N$ we find $n_{i+1} \in \mathbb{N}$ such that

$$\|\mathcal{P}_{n_{i+1},N+1}\| \leq \frac{1}{\left|z^{v_i+1}_{k_0}\right|} < \|\mathcal{P}_{n_{i+1}-1,N+1}\|.$$

Further we find the largest $v_{i+1} \in \{v_i + 1, \ldots, N\}$ such that

$$\frac{1}{\left|z^{v_{i+1}}_{k_0}\right|} \geq \|\mathcal{P}_{n_{i+1},N+1}\|$$

and we are done.

For $n_1 - 1 \leq t \leq n_s + 1$ we set

$$\tau_t = \begin{cases} \min\left\{1, \frac{\lambda(W_{v_i})}{6\left|z^{v_i}_{k_0}\right| \|\mathcal{P}_{n_i,N+1}\|}\right\}, & \text{if } t = n_i \text{ for some } i = 1, \ldots, s, \\ 1, & \text{otherwise.} \end{cases}$$

Clearly,

$$\tau_t \geq \frac{1}{6l} \geq \frac{1}{\sqrt{n_0}}. \tag{4.32}$$

Further we set

$$[\sigma_t, \rho_t] = \begin{cases} [v_{i-1}, v_i], & \text{if } t = n_i, \\ [v_{i-1}, v_{i-1}], & \text{if } n_{i-1} < t < n_i, \qquad i = 1, \dots, s. \\ [N, N], & \text{if } t = n_s + 1, \end{cases}$$

We set $n(n,k) := n_s + 1$, $k(n,k) := k_0$ and $K := 2^{-6N} 3^{-5N} l^{-3N}$. Since $k(n,k) \geq k$ we have $A_{k(n,k)} \subset A_k$. Thus we have $\mathcal{B}_n([0,1], A_{k(n,k)}) \subset \mathcal{B}_n([0,1], A_k)$ and

$$\mu_{n,k(n,k)} \leq \mu_{n,k}. \tag{4.33}$$

By Lemma 4.3.12(ii) and $n_1 - 1 \geq n$ we have

$$\mu_{n_1-1,k(n,k)} \leq \mu_{n,k(n,k)} \cdot \sup\{\mu^I_{n_1-1,k(n,k)};\ I \in \mathcal{P}_{n,N+1}\} \leq \mu_{n,k(n,k)}. \tag{4.34}$$

Using Lemma 4.3.12(ii) again we have

$$\mu_{n(n,k),k(n,k)} \leq \mu_{n_1-1,k(n,k)} \cdot \sup\{\mu^I_{n(n,k),k(n,k)};\ I \in \mathcal{P}_{n_1-1,N+1}\}. \tag{4.35}$$

Fix some $I \in \mathcal{P}_{n_1-1,N+1}$. By (4.33), (4.34) and (4.35), it is sufficient to prove that

$$\mu^I_{n(n,k),k(n,k)} \leq 1 - K. \tag{4.36}$$

Suppose that $n_1 \leq t \leq n_s + 1$ and G, J are closed intervals such that $G \subset J$ and

(i) $J \in \mathcal{P}^I_{t-1,N+1}$,

(ii) $\frac{\lambda(G)}{\lambda(J)} \geq \tau_{t-1}$,

(iii) if $t = n_i$ and $\sigma_t < j \leq \rho_t$, then $\lambda(G) \geq \frac{1}{|z^j_{k_0}|}$.

Since $\tau_{t-1} \geq \frac{1}{\sqrt{n_0}} \geq \frac{1}{\sqrt{t}}$ one can verify that J, G, t, k_0, σ_t and ρ_t satisfy the assumptions of Lemma 4.3.15 for I, G, n, k, σ and ρ. For such t, J, G we set

$$\mathcal{F}(t,J,G) = \{R \in \mathcal{P}^G_{t,N+1};\ \exists C \subset R \text{ closed interval} : \frac{\lambda(C)}{\lambda(R)} \geq \tau_t \wedge C \cap A_{k_0,\sigma_t,\rho_t} = \emptyset\} \tag{4.37}$$

and Lemma 4.3.15 gives

$$\frac{\sharp\,\mathcal{F}(t,J,G)}{\sharp\,\mathcal{P}^J_{t,N+1}} \geq \begin{cases} \frac{\lambda(G)}{18\lambda(J)}(4l)^{\sigma_t-\rho_t}, & t \in \mathcal{E}_1 := \{n_i;\ 1 \leq i \leq s\}, \\ \frac{\lambda(G)}{6\lambda(J)}, & t \in \mathcal{E}_2 := \{n_i + 1;\ 1 \leq i \leq s\} \setminus \mathcal{E}_1, \\ 1, & t \notin \mathcal{E}_1 \cup \mathcal{E}_2. \end{cases} \tag{4.38}$$

In particular, $\mathcal{F}(t, J, G)$ is nonempty. We set

$$K_t = \begin{cases} \frac{(4l)^{\sigma_t - \rho_t}}{108l}, & t \in \mathcal{E}_1, \\ \frac{1}{36l}, & t \in \mathcal{E}_2, \\ 1, & t \notin \mathcal{E}_1 \cup \mathcal{E}_2. \end{cases}$$

By (ii) and (4.32) we get $\frac{\lambda(G)}{\lambda(J)} \geq \tau_{t-1} > \frac{1}{6l}$. This and (4.38) imply

$$\frac{\sharp\, \mathcal{F}(t, J, G)}{\sharp\, \mathcal{P}^J_{t,N+1}} \geq K_t. \tag{4.39}$$

We will inductively define systems of intervals $\mathcal{F}_t$, $t = n_1, \dots, n_s + 1$, with the following properties.

- $\mathcal{F}_{n_1} = \mathcal{F}(n_1, I, I)$,
- $\mathcal{F}_{t+1} = \bigcup\{\mathcal{F}(t+1, J, C(J));\ J \in \mathcal{F}_t\}$,
- for each $J \in \mathcal{F}_t$ there exists a closed interval $C(J) \subset J$ such that
  - (a) $C(J) \cap A_{k_0, \sigma_t, \rho_t} = \emptyset$,
  - (b) $\frac{\lambda(C(J))}{\lambda(J)} \geq \tau_t$,
  - (c) if $t + 1 \in \mathcal{E}_1$ and $\sigma_{t+1} < j \leq \rho_{t+1}$, then $\lambda(C(J)) \geq \frac{1}{|z^j_{k_0}|}$.

Suppose that we have already defined $\mathcal{F}_t$, $t \leq n_s$, and corresponding intervals $C(J)$ for every $J \in \mathcal{F}_t$. We set

$$\mathcal{F}_{t+1} = \bigcup\{\mathcal{F}(t+1, J, C(J));\ J \in \mathcal{F}_t\}.$$

Observe that $\mathcal{F}_{t+1} \subset \mathcal{P}^I_{t+1,N+1}$ by (4.37). Fix $\tilde{J} \in \mathcal{F}_{t+1}$. Find $C(\tilde{J})$ satisfying (a) and (b) by (4.37). Now we verify the condition (c) for $C(\tilde{J})$. Let $t + 2 = n_i$ for some i and $\sigma_{t+2} < j \leq \rho_{t+2}$. Thus $\sigma_{t+2} = v_{i-1} < j$. If $t + 1 \neq n_{i-1}$, then $\tau_{t+1} = 1$ and $\lambda(C(\tilde{J})) = \lambda(\tilde{J}) = \|\mathcal{P}_{t+1,N+1}\| = \|\mathcal{P}_{n_i-1,N+1}\|$. Then by the definition of v_i's we get $\lambda(C(\tilde{J})) = \|\mathcal{P}_{n_i-1,N+1}\| > \frac{1}{|z^j_{k_0}|}$. If $t + 1 = n_{i-1} = n_i - 1$, then $\tau_{t+1} = \frac{\lambda(W_{v_{i-1}})}{6|z^{v_{i-1}}_{k_0}|\|\mathcal{P}_{n_{i-1},N+1}\|}$. Using (b) and (4.6) we get

$$\begin{aligned} \lambda(C(\tilde{J})) \geq \tau_{t+1}\lambda(\tilde{J}) = \tau_{t+1}\|\mathcal{P}_{n_i-1,N+1}\| &= \frac{\lambda(W_{v_{i-1}})}{6|z^{v_{i-1}}_{k_0}|} \\ &\geq \frac{10}{6|z^{v_{i-1}+1}_{k_0}|} > \frac{1}{|z^{v_{i-1}+1}_{k_0}|} \geq \frac{1}{|z^j_{k_0}|}. \end{aligned}$$

Thus we verified (c) and the construction of $\mathcal{F}_t$'s is finished.

Using (a) we have

$$\mathcal{F}_{n(n,k)} \subset \mathcal{P}_{n(n,k),N+1} \setminus \mathcal{B}_{n(n,k)}(I, A_{k(n,k)}),$$
$$\mu^I_{n(n,k),k(n,k)} = \frac{\sharp\, \mathcal{B}_{n(n,k)}(I, A_{k(n,k)})}{\sharp\, \mathcal{P}^I_{n(n,k),N+1}} \leq 1 - \frac{\sharp\, \mathcal{F}_{n(n,k)}}{\sharp\, \mathcal{P}^I_{n(n,k),N+1}}.$$

Finally, using (4.39) we estimate

$$\frac{\sharp\, \mathcal{F}_t}{\sharp\, \mathcal{P}^I_{t,N+1}} \geq \prod_{j=n_1}^{t} K_j, \qquad t = n_1, \dots, n_s + 1,$$

and this implies

$$\frac{\sharp\, \mathcal{F}_{n(n,k)}}{\sharp\, \mathcal{P}^I_{n(n,k),N+1}} \geq \prod_{i=n_1}^{n_s+1} K_i = \prod_{i \in \mathcal{E}_1} \frac{(4l)^{\sigma_i - \rho_i}}{108 l} \prod_{i \in \mathcal{E}_2} \frac{1}{36 l} \geq \frac{(4l)^{-N}}{(108 l)^N} \cdot \frac{1}{(36 l)^N} = K.$$

Thus we have (4.36). □

Now we can prove Lemma 4.2.4.

Proof of Lemma 4.2.4. Using Lemma 4.3.12(i) and Lemma 4.3.18 we get

$$\mu(A) = \inf\{\mu_{n,k};\ n, k \in \mathbb{N}\} = 0.$$

□

4.4 Proof of Theorem 4.2.5

Notation 4.4.1. *Let $N, n \in \mathbb{N}$, $\boldsymbol{a} \in \mathcal{Q}^N(\mathbb{R})$, $y \in \mathbb{R}$ and $J \subset \mathbb{R}, \mathcal{J} = \prod_{j=1}^N J^j \subset [0,1]^N$ be open sets. We set*

$$T(y, J) = \{x \in [0,1];\ \langle xy \rangle \in \langle J \rangle\},$$
$$H^n(\boldsymbol{a}, \mathcal{J}) = [0,1] \setminus \bigcap_{p=1}^{N} T(a^p_n, J^p).$$

Remind Notation 4.3.9 and Remark 4.3.10.

Remark 4.4.2. *Let $N \in \mathbb{N}, \boldsymbol{a} = \{a_j\} \in \mathcal{Q}^N(\mathbb{Q}^*)$, $\{j_k\}$ be an increasing sequence of integers and $\mathcal{J} \subset \mathcal{U} \subset [0,1]^N$ be open intervals. Then the following assertions hold.*

(i) $\{a_{j_k}\} \in \mathcal{Q}^N(\mathbb{Q}^*)$

(ii) $H(\boldsymbol{a}, \mathcal{U}) \subset H(\{a_{j_k}\}, \mathcal{U})$

(iii) $H(\boldsymbol{a},\mathcal{U}) = \bigcap_{n\in\mathbb{N}} H^n(\boldsymbol{a},\mathcal{U})$

(iv) *Let* $L \in \mathbb{Q}^{N\times N}$ *be a regular matrix. Then there exists an increasing sequence* $\{v_k\}$ *of positive integers such that* $\{L(a_{v_k})\} \in \mathcal{Q}^N(\mathbb{Q}^*)$.

(v) *Let* $y \in \mathbb{Q}^*$ *and* $J \subset \mathbb{R}$ *be an open interval. Then* $T(y, J) = \bigcup_{n\in\mathbb{Z}} \frac{1}{y}(J + n) \cap [0, 1]$.

(vi) *Let* $m \in \mathbb{Z} \setminus \{0\}$, $y \in \mathbb{Q}^*$ *and* $u, r \in \mathbb{R}$. *Then we have* $T(y, B(u, r)) \supset T\big(\frac{y}{m}, B\big(\frac{u}{m}, \frac{r}{|m|}\big)\big)$, *where the symbol* $B(x, s) = (x - s, x + s)$.

(vii) *Let* $y \in \mathbb{Q}^*$, $J \subset \mathbb{R}$ *and* $V \subset \langle J\rangle$ *be open intervals. Then* $T(y, J) \supset T(y, V)$.

We will use the following well known approximation theorem.

Lemma 4.4.3. *[13, Dirichlet's Theorem on Simultaneous Approximations] Let* $\alpha_1, \ldots, \alpha_n$ *be real numbers and* $Q > 1$ *be an integer. Then there exist integers* $q, p_1, \ldots, p_n$ *with* $1 \le q < Q^n$ *and* $|\alpha_i q - p_i| \le 1/Q$ *for all* $1 \le i \le n$.

Lemma 4.4.4. *Let* $N \in \mathbb{N}, \boldsymbol{a} \in \mathcal{Q}^N(\mathbb{Q}^*)$ *and* $\mathcal{U}_n = U^1 \times \cdots \times U^{N-1} \times U^N_n \subset [0, 1]^N$, $n \in \mathbb{N}$ *be open intervals. If there exists* $\alpha > 0$ *such that* $\lambda(U^N_n) \ge \alpha$ *for all* $n \in \mathbb{N}$ *then there exist an increasing sequence* $\{j_n\}$ *of positive integers and an open interval* $\mathcal{J} = U^1 \times \cdots \times U^{N-1} \times J^N \subset [0, 1]^N$ *such that*

(i) $\forall n \in \mathbb{N} : 4\lambda(J^N) \ge \lambda(U^N_{j_n})$,

(ii) $\bigcap_{n\in\mathbb{N}} H^n(\{a_j\},\mathcal{U}_n) \subset H(\{a_{j_n}\}, \mathcal{J})$.

Proof. Since $\inf\{\lambda(U^N_n);\ n \in \mathbb{N}\} \ge \alpha > 0$ there exists an increasing sequence v_n of positive integers such that

$$4 \inf\{\lambda(U^N_{v_n});\ n \in \mathbb{N}\} > 3 \sup\{\lambda(U^N_{v_n});\ n \in \mathbb{N}\}.$$

We find $l \in \mathbb{N}$ such that

$$\frac{2}{l} \le \inf\{\lambda(U^N_{v_n});\ n \in \mathbb{N}\} < \frac{3}{l}.$$

For all $j \in \mathbb{N}$ we find $b_j \in \mathbb{N}_0$ and an open interval $J^N_j = (\frac{b_j}{l}, \frac{b_j+1}{l})$ such that $J^N_j \subset U^N_{v_j}$. Since the set $\{J^N_j;\ j \in \mathbb{N}\}$ is finite there exists an increasing sequence $\{p_n\}$ of positive integers and an open interval J^N such that $J^N_{p_n} = J^N$ for all $n \in \mathbb{N}$. We set $\mathcal{J} = U^1 \times \cdots \times U^{N-1} \times J^N$ and $j_n = v_{p_n}$. Thus,

$$\bigcap_{n\in\mathbb{N}} H^n(\{a_j\},\mathcal{U}_n) \subset H(\{a_{j_n}\}, \mathcal{J}).$$

Clearly,

$$4\lambda(J^N) = \frac{4}{l} \ge \frac{4}{3} \inf\{\lambda(U^N_{v_n});\ n \in \mathbb{N}\} > \sup\{\lambda(U^N_{v_n});\ n \in \mathbb{N}\} \ge \lambda(U^N_{j_m})$$

for all $m \in \mathbb{N}$.

□

The following lemma was inspired by Zajíček [17].

Lemma 4.4.5. *Let $y, z \in \mathbb{Q}^*$, $y \neq z$, $U = B(u, r_1), V = B(v, r_2)$ be subsets of $[0,1]$ and $\delta \le \min\left\{\frac{\lambda(V)}{|y|}, \frac{\lambda(U)}{|z|}\right\}$. If $4|y| > 3|z|$ then*

$$T(y,V) \cap T(z,U) \supset T\left(z, B\left(u, \frac{|z|\delta}{4}\right)\right) \cap T\left(y-z, B\left(v-u, \frac{r_2}{4}\right)\right).$$

Proof. Since $|z|\delta/4 \le r_1$ we have $B\left(u, |z|\delta/4\right) \subset U$. Thus,

$$T(z,U) \supset T\left(z, B\big(u, |z|\delta/4\big)\right).$$

Let $x \in T\left(z, B\big(u, |z|\delta/4\big)\right) \cap T\left(y-z, B\big(v-u, r_2/4\big)\right)$. Then there exist $\xi \in B\left(0, r_2/4\right)$, $\mu \in B\left(0, |z|\delta/4\right)$ and $m, n \in \mathbb{Z}$ such that

$$\begin{aligned} x &= (\xi + v - u + n)\frac{1}{y-z}, \\ x &= (\mu + u + m)\frac{1}{z}. \end{aligned}$$

Thus, $x = (\xi+\mu+v+m+n)\frac{1}{y}$. Since $|\xi+\mu| \le \frac{r_2}{4} + \frac{|z|\delta}{4} < \frac{r_2}{4} + \frac{|y|\delta}{3} < \frac{r_2}{4} + \frac{2r_2}{3} < r_2$ we have $\xi + \mu + v \in V$. Thus, $x \in T(y, V)$.

□

Lemma 4.4.6. *Let $N \in \mathbb{N}, \boldsymbol{a} \in \mathcal{Q}^N(\mathbb{Q}^*)$, $\mathcal{U} = \prod_{i=1}^N U^i \subset [0,1]^N$ be an open interval, $L \in \mathbb{N}$ and $\delta_j = \min\left\{\frac{\lambda(U^i)}{|a_j^i|};\ i = 1, \ldots, N\right\}$. Then there exist a regular matrix $\mathcal{L} \in \mathbb{Q}^{N\times N}$, an increasing sequence $\{v_n\}$ of positive integers and an open interval $\mathcal{J} = \prod_{i=1}^N J^i \subset [0,1]^N$ such that*

(a) $\boldsymbol{x} := \{\mathcal{L}(a_{v_n})\} \in \mathcal{Q}^N(\mathbb{Q}^*)$,

(b) $H(\boldsymbol{a},\mathcal{U}) \subset H(\boldsymbol{x},\mathcal{J})$,

(c) $\forall n \in \mathbb{N}\ \forall i < N : \left|\frac{x_n^N \lambda(J^i)}{x_n^i}\right| \ge L$,

(d) $\frac{\lambda(J^N)}{|x_n^N|} \ge \delta_{v_n}/16$.

Proof. Going to a subsequence and permutate indexes if necessary, we can assume that $|a_n^i| < |a_n^{i+1}|$ for all $n \in \mathbb{N}$ and $i < N$. We find $Q \in \mathbb{N}$ such that $\frac{1}{Q} < \frac{\min\{\lambda(U^i);\ i=1,\ldots,N\}}{8L}$. By Lemma 4.4.3 for every $j \in \mathbb{N}$ there exist $q_j, p_j^1, \ldots, p_j^{N-1} \in \mathbb{Z}$ such that

$$\begin{aligned} 1 \le q_j &\le Q^{N-1}, \\ \left|q_j \frac{a_j^i}{a_j^N} - p_j^i\right| &\le \frac{1}{Q}, \qquad i = 1, \ldots, N-1. \end{aligned} \tag{4.40}$$

Since $\frac{|a_j^i|}{|a_j^N|} < 1$, we have $|p_j^i| \le Q^{N-1}$ for every $j \in \mathbb{N}$ and $i = 1, \dots, N-1$. Going to a subsequence if necessary, we can assume that there exist $q, p^1, \dots, p^{N-1}$ such that $q = q_j, p^i = p_j^i$ for every $j \in \mathbb{N}$. Clearly, there exists $0 \le s < N$ such that $p^i = 0$ if and only if $i \le s$. Denote by u^i the center of the interval U^i and set

$$y_j^i = \begin{cases} a_j^i & \text{for } i \le s, \\ \frac{a_j^i}{p^i} - \frac{a_j^N}{q} & \text{for } s < i < N, \\ \frac{a_j^N}{q} & \text{for } i = N, \end{cases} \qquad j \in \mathbb{N}.$$

Further we define

- $J^i = U^i$ for $i \le s$,
- $\widetilde{J}^i = B\left(\frac{u^i}{p^i} - \frac{u^N}{q}, \frac{\lambda(U^i)}{8|p^i|}\right)$ for $s < i < N$,
- $\widetilde{J}_j^N = B\left(\frac{u^N}{q}, \frac{\delta_j |y_j^N|}{4}\right)$ for $j \in \mathbb{N}$,
- $J_j^N = \widetilde{J}_j^N \cap (0,1)$.

Since $\frac{u^N}{q} \in (0,1)$ we have $\lambda(J_j^N) \ge \frac{1}{2}\lambda(\widetilde{J}_j^N)$. Going to a subsequences if necessary and using Remark 4.4.2(iv) we have that $\boldsymbol{y} := \left\{(y_j^1, \dots, y_j^N)\right\}_j$ is in $\mathcal{Q}^N(\mathbb{Q}^*)$. For each $s < i < N$ we find an open interval $J^i \subset [0,1]$ such that $\lambda(J^i) \ge \frac{\lambda(\widetilde{J}^i)}{2}$ and $J^i \subset \langle \widetilde{J}^i \rangle$. By Remark 4.4.2(vi) we have

$$\begin{aligned} T(a_j^i, U^i) &\supset T\left(\frac{a_j^i}{p^i}, B\left(\frac{u^i}{p^i}, \frac{\lambda(U^i)}{2|p^i|}\right)\right), \\ T(a_j^N, U^N) &\supset T\left(y_j^N, B\left(\frac{u^N}{q}, \frac{\lambda(U^N)}{2q}\right)\right). \end{aligned} \tag{4.41}$$

Since

$$\left| q\frac{a_j^i}{a_j^N} - p^i \right| \le \frac{1}{Q} < \frac{\min\{\lambda(U^i);\ i = 1, \dots, N\}}{8L} \le \frac{1}{8},$$

we have $4\left|\frac{a_j^i}{p^i}\right| \ge 3\left|y_j^N\right|$. Since $\frac{a_j^i}{p^i} - y_j^N = y_j^i$ and $\boldsymbol{y} \in \mathcal{Q}^N(\mathbb{Q}^*)$, we have $\frac{a_j^i}{p^i} \ne y_j^N$. By Lemma 4.4.5 we have

$$T\left(\frac{a_j^i}{p^i}, B\left(\frac{u^i}{p^i}, \frac{\lambda(U^i)}{2|p^i|}\right)\right) \cap T\left(y_j^N, B\left(\frac{u^N}{q}, \frac{\lambda(U^N)}{2q}\right)\right) \supset T(y_j^N, \widetilde{J}_j^N) \cap T(y_j^i, \widetilde{J}^i). \tag{4.42}$$

By Remark 4.4.2(vii) and our choice of sets J^i, J_j^N we have

$$T(y_j^N, \widetilde{J}_j^N) \cap T(y_j^i, \widetilde{J}^i) \supset T(y_j^N, J_j^N) \cap T(y_j^i, J^i). \tag{4.43}$$

By (4.41), (4.42) and (4.43) we have

$$H^n(\boldsymbol{a},\mathcal{U}) \subset H^n(\boldsymbol{y}, J^1 \times \cdots \times J^{N-1} \times J^N_n). \tag{4.44}$$

Observe that we have

$$\lambda(J^N_j) \geq \frac{1}{2}\lambda(\tilde{J}^N_j) = \frac{1}{4}\delta_j|y^N_j| = \frac{1}{4}\delta_j\frac{|a^N_j|}{q} \geq \frac{1}{4}\frac{\min\{\lambda(U^i); i=1,\dots,N\}}{|a^N_j|}\frac{|a^N_j|}{q}$$
$$= \frac{1}{4}\frac{\min\{\lambda(U^i); i=1,\dots,N\}}{q}.$$

Thus we can use Lemma 4.4.4 to get an open interval J^N and an increasing sequence v_n of positive integers such that

$$\bigcap_{n\in\mathbb{N}} H^n(\boldsymbol{y}, J^1 \times \cdots \times J^{N-1} \times J^N_n) \subset H(\{y_{v_n}\}, J^1 \times \cdots \times J^N), \tag{4.45}$$
$$4\lambda(J^N) \geq \lambda(J^N_{v_j}).$$

We set $x^i_n := y^i_{v_n}$ and $\mathcal{J} = J^1 \times \cdots \times J^N$. By the definition of $\boldsymbol{y}$ we simply get that $\mathcal{L}$ is triangle matrix without any zero element on diagonal. Thus we have (a). By (4.44) and (4.45) we get (b). Assume $i \leq s$. Since

$$\left|\frac{x^i_j}{x^N_j}\right| = \left|q\frac{a^i_{v_j}}{a^N_{v_j}} - p^i\right| \leq \frac{1}{Q} < \frac{\min\{\lambda(U^i);\ i=1,\dots,N\}}{8L}$$

we have

$$\left|\frac{x^N_j\lambda(J^i)}{x^i_j}\right| = \left|\frac{x^N_j\lambda(U^i)}{x^i_j}\right| \geq \left|\frac{x^N_j 8L}{x^i_j Q}\right| \geq 8L.$$

Let $s < i < N$. Since

$$\left|\frac{x^i_j p^i}{x^N_j}\right| = \left|q\frac{a^i_{v_j}}{a^N_{v_j}} - p^i\right| \leq \frac{1}{Q} < \frac{\min\{\lambda(U^i);\ i=1,\dots,N\}}{8L}$$

we have

$$\left|\frac{x^N_j\lambda(J^i)}{x^i_j}\right| \geq \left|\frac{x^N_j\lambda(\widetilde{J}^i)}{2x^i_j}\right| = \left|\frac{x^N_j\lambda(U^i)}{8x^i_j p^i}\right| \geq \left|\frac{x^N_j L}{x^i_j Q p^i}\right| \geq L.$$

Thus we have (c). Clearly,

$$16\lambda(J^N) \geq 4\lambda(J^N_{v_n}) \geq 2\lambda(\widetilde{J}^N_{v_n}) = \delta_{v_n}|x^N_n|$$

for all $n \in \mathbb{N}$. Thus we have (d).

□

Lemma 4.4.7. *Let $N \in \mathbb{N}, \boldsymbol{a} \in \mathcal{Q}^N(\mathbb{Q}^*)$, $\mathcal{U} = \prod_{i=1}^N U^i \subset [0,1]^N$ be an open interval, $L \in \mathbb{N}$ and $\delta_j = \min\left\{\frac{\lambda(U^i)}{|a_j^i|};\ i = 1,\dots,N\right\}$. Then there exist $\boldsymbol{x} \in \left(\mathbb{Q}^N\right)^{\mathbb{N}}$, a regular matrix $M \in \mathbb{Q}^{N\times N}$, an increasing sequence $\{v_n\}$ of positive integers and an open interval $\mathcal{J} = \prod_{i=1}^N J^i \subset [0,1]^N$ such that*

(a) $\boldsymbol{x} := \{M(a_{v_n})\} \in \mathcal{Q}^N(\mathbb{Q}^*)$,

(b) $H(\boldsymbol{a},\mathcal{U}) \subset H(\boldsymbol{x},\mathcal{J})$,

(c) $\forall n \in \mathbb{N}\ \forall i < N : \left|\frac{x_n^{i+1}\lambda(J^i)}{x_n^i}\right| \geq L$,

(d) $\frac{\lambda(J^N)}{|x_n^N|} \geq \delta_{v_n}/16$.

Proof. We will proceed by induction over N. The case $N = 1$ is trivial. Assume that our statement holds for some $N-1 \in \mathbb{N}$, we show that it also holds for N. By Lemma 4.4.6 there exist a regular matrix $\mathcal{L} \in \mathbb{Q}^{N\times N}$, an increasing sequence $\{p_n\}$ of positive integers and an open interval $\mathcal{V} = \prod_{i=1}^N V^i \subset [0,1]^N$ such that

(i) $\boldsymbol{y} := \{\mathcal{L}(a_{p_n})\} \in \mathcal{Q}^N(\mathbb{Q}^*)$,

(ii) $H(\boldsymbol{a},\mathcal{U}) \subset H(\boldsymbol{y},\mathcal{V})$,

(iii) $\forall n \in \mathbb{N}\ \forall i < N : \left|\frac{y_n^N\lambda(V^i)}{y_n^i}\right| \geq 16L$,

(iv) $\frac{\lambda(V^N)}{|y_n^N|} \geq \delta_{p_n}/16$.

By induction hypothesis there exist $\{x_n\} \in \left(\mathbb{Q}^{N-1}\right)^{\mathbb{N}}$, a regular matrix $\mathcal{T} \in \mathbb{Q}^{(N-1)\times(N-1)}$, an increasing sequence $\{j_n\}$ of positive integers and open intervals $J^i \subset [0,1]$, $0 < i < N$, such that

(1) $\{x_n\} := \{\mathcal{T}(y_{j_n})\} \in \mathcal{Q}^{N-1}(\mathbb{Q}^*)$,

(2) $H(\{y_j\}, \prod_{i=1}^{N-1} V^i) \subset H(\{x_n\}, \prod_{i=1}^{N-1} J^i)$,

(3) $\forall n \in \mathbb{N}\ \forall i < N-1 : \left|\frac{x_n^{i+1}\lambda(J^i)}{x_n^i}\right| \geq L$,

(4) $\frac{\lambda(J^{N-1})}{|x_n^{N-1}|} \geq \frac{1}{16}\min\left\{\frac{\lambda(V^i)}{|y_{j_n}^i|};\ i = 1,\dots,N-1\right\}$.

We set $v_n = p_{j_n}$, $x_n^N = y_{j_n}^N$ and $J^N = V^N$. Using (i) and (1) we easily obtain (a). Using (ii) and (2) we easily obtain (b). Using (3) we obtain (c) for $i < N-1$. From (iii) we have $\min\left\{\frac{\lambda(V^i)}{|y_{j_n}^i|};\ i = 1,\dots,N-1\right\} = \frac{\lambda(V^{N-1})}{|y_{j_n}^{N-1}|}$. Using this, (4) and (iii) again we get the case $i = N-1$. The formula (iv) easily gives (d).

□

Proof of Theorem 4.2.5. The inclusion $H^N(\mathbb{Q}^*) \supset H_L^N(\mathbb{Q}^*)$ is trivial.

Let $A \in H^N(\mathbb{Q}^*)$ be arbitrary. Then there exists $\boldsymbol{a} \in \mathcal{Q}^N(\mathbb{Q}^*)$ and an open interval $\mathcal{U} \subset [0,1]^N$ such that $A \subset H(\boldsymbol{a},\mathcal{U})$. By Lemma 4.4.7 there exists $\boldsymbol{x} \in \left(\mathbb{Q}^N\right)^{\mathbb{N}}$ and an open interval $\mathcal{J} \subset [0,1]^N$ such that $H(\boldsymbol{a},\mathcal{U}) \subset H(\boldsymbol{x},\mathcal{J}) \in H_L^N(\mathbb{Q}^*)$. So, $A \in H_L^N(\mathbb{Q}^*)$. □

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