Royal Holloway University of London

Dr Jens Bolte
Reader in Applied Mathematics
Department of Mathematics
Royal Holloway
University of London
Egham, Surrey TW20 0EX
United Kingdom

Tel +44 1784 276269
Fax +44 1784 430766
Email jens.bolte@rhul.ac.uk

Prof Zdeněk Němeček
Matematicko-Fyzikální Fakulta
Univerzita Karlova v Praze
Ke Karlovu 3
121 16 Praha 2
Czech Republic

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Report on the doctoral thesis "Quantum Graphs and Their Generalizations" by Mgr. Jiří Lipovský

Dear Prof Němeček,

In his thesis the candidate investigates spectral properties of quantum graphs mainly in relation to quantum scattering on graphs. Quantum graphs are mathematical models of quantum systems that are effectively confined to one dimension. They had already been developed in the early days of quantum mechanics, but studies of quantum graphs have surged in the past decade. The original motivation was to set up simplified models of molecules, and to study the effect of the topology of the chemical bonds when details of the binding were believed to be less important. More recently, quantum graphs and related models have been considered as examples of networks of various kinds, e.g., of wires or optical fibres. Common to all such graph models is the combination of a simple, one dimensional quantum motion or wave propagation, and a complicated topology of a graph or network. As a result, one thus obtains convenient models for complex quantum systems, or even for quantum chaos.

Central to most of these studies are spectral properties of quantum graph Hamiltonian operators. In a majority of cases the models were built on compact metric graphs, where these operators possess a purely discrete spectrum. In that context the distribution of eigenvalues and properties of the associated eigenfunctions in terms of topological and metric properties of the graph were the principal objectives of the investigations. In this thesis, however, non-compact graphs are considered. On these, typically, quantum graph Hamiltonians possess spectra with a non-trivial, absolutely continuous part. From a physics perspective, in such situations properties of resonances are of foremost interest as they often are the most easily accessible, measurable quantities. The approach taken in this thesis, to investigate quantum resonances in models of non-compact graphs, hence is natural and of particular interest.

One major topic studied in the thesis is that of resonance counting. This question recently attracted much attention in scattering theory (through, e.g., the work of Zworsky, Nonnenmacher and others) as it was found that in certain cases the leading asymptotics of the resonance count don't follow the naive guess of a Weyl-type law. Rather, when the associated classical dynamics possess a fractal repeller the Weyl asymptotic law is modified in terms of the fractal dimension of the repeller, leading to a so-called fractal Weyl law. On the other hand, as first observed by Kottos and Smilansky, in the compact case quantum graph models share many spectral properties with quantum systems that possess a chaotic classical limit. Based on these observations the question of a potential non-Weyl asymptotics for the resonance count in quantum graphs has been arisen in the quantum chaos community; a search for this provides a strong motivation for the studies performed in this thesis.

The thesis under review consists of a main body of three chapters and an appendix in fours parts. Each part of the appendix contains a paper published in an international journal of high quality. These papers are co-authored by the candidate together with his supervisor and, in one case, a further author. The three chapters of the main body summarise and explain the content of these papers.

More specifically, in Chapter 1 of the main body the construction of quantum graph models is reviewed and some basic properties are established. This chapter summarises well-known results and mainly serves to set up the notation and necessary tools for the later chapters.

Chapter 2 is devoted to the definition and characterisation of resonances in quantum graphs. This chapter summarises results of the paper in Appendix A. A particular first achievement is that the equivalence of resolvent and scattering resonances is proven. Moreover, cases where this equivalence does not hold are characterised and any differences are explained. As in generic quantum systems a proof of such an equivalence is often difficult, if not unknown, this result is a valuable observation in quantum graphs.

Chapter 3 is devoted to a summary and explanations of the papers in Appendices B-D. At first the presence of eigenvalues embedded into the absolutely continuous spectrum is identified in graphs that possess edges with rationally related lengths. These eigenvalues are examples of resolvent resonances that are not scattering resonances, a fact that is exploited in a stability analysis. It is shown that under a small perturbation of the edge lengths the multiplicity of resolvent resonances is preserved. This implies that under the perturbations embedded eigenvalues cannot simply disappear, but can only turn into scattering resonances. This effect is then studied in some examples. The perturbation analysis of embedded eigenvalues confirms the expectation that one has in a general setting and, e.g., plays a crucial role in the context of the Sarnak-Phillips conjecture on embedded eigenvalues for modular (and related arithmetic) groups. Often, however, a detailed analysis as the one performed in this thesis is not feasible. Hence, a principal achievement is that the mechanisms leading to a re-appearance of an embedded eigenvalue as a scattering resonance are uncovered in the greatest possible detail.

The next major topic of Chapter 3 is the analysis of the resonance count in non-compact quantum graphs. In previous work by Davies and Pushnitsky cases were identified where the resonance count follows a non-Weyl asymptotic law, in the sense that the power of the spectral parameter in the main term of the asymptotic law is as in a Weyl case, but the (positive) coefficient turns out to be smaller. In this thesis the previous work, which was confined to Kirchhoff vertex conditions, is extended to general vertex conditions; in fact, all cases with a non-Weyl asymptotic law are characterised. Some examples are given and studied in detail. In a further paper (Appendix C) a presence of magnetic fields is taken into account, the main result being that non-Weyl asymptotics will occur iff this is also the case without magnetic field. The leading coefficient, however, depends on the field.

The last part of Chapter 3 is devoted to the investigations of radial tree graphs presented in the paper in Appendix D. In graphs with an infinite number of edges the spectral type of the Hamiltonian is not immediately clear and absolutely as well as singular continuous components may arise; trees provide convenient examples for such graphs. Previously, cases of (sparse) quantum tree graphs with singular continuous spectrum were identified, and in the paper in Appendix D the class of quantum tree graphs without absolutely continuous spectrum is largely extended by allowing more general than Kirchhoff vertex conditions. Nevertheless, it is shown by way of examples that sparse tree graphs may possess an absolutely continuous spectrum. Hence, the studies presented in the thesis considerably extend our knowledge of spectral types of quantum tree graphs.

Summarising, I consider this thesis a valuable contribution to research in the area of quantum graphs. It contains ample novel results, among which I would particularly high-light the thorough analysis of non-Weyl asymptotic laws for the resonance count in non-compact quantum graphs. These results have important implications for further research in this area as well as for neighbouring areas. The thesis clearly demonstrates that the candidate is able to produce creative scientific work. Therefore, I warmly recommend this thesis to your Faculty.

Your sincerely,

(Jens Bolte)