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DISERTAČNÍ PRÁCE

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Moduly nad Gorensteinovými okruhy

Katedra algebry

Vedoucí disertační práce: Prof. RNDr. Jan Trlifaj, DSc.
Studijní program: Matematika
Studijní obor: Algebra, teorie čísel a matematická logika

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DISSERTATION

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Modules over Gorenstein rings

Department of Algebra

Supervisor: Prof. RNDr. Jan Trlifaj, DSc.
Study programme: Mathematics
Branch of study: Algebra, number theory and mathematical logic

Mé poděkování patří především školiteli prof. Janu Trlifajovi, bez jehož pomoci by tato práce nikdy nevznikla. Dále bych chtěl poděkovat spoluautorovi jednoho z článků, Janu Šťovíčkovi, za jeho inspirativní nápady a přínosné diskuse. V neposlední řadě bych chtěl poděkovat své rodině za vytrvalou podporu během celého mého studia.

Prohlašuji, že jsem svou disertační práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 12.5.2011 David Pospíšil

CONTENTS

Název práce: Moduly nad Gorensteinovými okruhy
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Abstrakt: Disertační práce zahrnuje mé dosavadní příspěvky ke klasifikaci (ko)vychylujících modulů a tříd nad Gorensteinovými okruhy. Oproti původnímu záměru se v ní dokonce podařilo provést klasifikaci (ko)vychylujících tříd pro obecnější případ komutativních noetherovských okruhů (viz. třetí článek této disertace). Disertace se sestává z úvodu a tří článků se spoluautory. První článek (publikovaný v Contemp. Math.) obsahuje klasifikaci všech (ko)vychylujících modulů a tříd nad 1-Gorenstenovými komutativními okruhy. Druhý článek (publikovaný v J. Algebra) obsahuje klasifikaci všech vychylujících tříd nad regulárními okruhy Krullovy dimenze 2 a též klasifikaci všech vychylujících modulů v lokálním případě. Konečne třetí článek (preprint) obsahuje klasifikaci všech (ko)vychylujících tříd a také torzních párů nad obecnými komutativními noetherovskými okruhy. Všechny tyto klasifikace jsou popsány pomocí podmnožin spektra okruhu a asociovaných prvoideálů modulů.

Klíčová slova: (ko)vychylující modul, (ko)vychylující třída, torzní pár, Gorensteinův okruh, regulární okruh, komutativní noetherovský okruh, spektrum okruhu, asociovaný prvoideál.

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Abstract: The dissertation collects my actual contributions to the classification of (co)tilting modules and classes over Gorenstein rings. Compared with the original intent we get a more general result in classification of (co)tilting classes namely for general commutative noetherian rings (see the third paper in this dissertation). The dissertation consists of an introduction and three papers with coauthors. The first paper (published in Contemp. Math.) contains a classification of all (co)tilting modules and classes over 1-Gorenstein commutative rings. The second paper (published in J. Algebra) contains a classification of all tilting classes over regular rings of Krull dimension 2 and also a classification of all tilting modules in the local case. Finally the third paper (preprint) contains a classification of all (co)tilting classes and also torsion pairs over general commutative noetherian rings. All these classifications are in terms of subsets of the spectrum of the ring and by associated prime ideals of modules.

Keywords: (co)tilting module, (co)tilting class, torsion pair, Gorenstein ring, regular ring, commutative noetherian ring, spectrum of a ring, associated prime ideal.

INTRODUCTION

The dissertation consists of this introduction and three papers of which I'm a coauthor. The first paper is already published, the second is in press and the third is a very recent preprint.

(1) J. Trlifaj, D. Pospíšil. Tilting and cotilting classes over Gorenstein rings. In *Rings, modules and representations*, volume 480 of *Contemp. Math.*, pages 319–334. Amer. Math. Soc., Providence, RI, 2009.
(2) D. Pospíšil, J. Trlifaj. Tilting for regular rings of Krull dimension two. *J. Algebra*, doi:10.1016/j.algebra.2011.02.047, 2011.
(3) L. Angeleri-Hügel, D. Pospíšil, J. Trlifaj, J. Šťovíček. Tilting, cotilting, and spectra of commutative noetherian rings. Preprint.

All these three papers focus on tilting modules and tilting classes. The first paper contains a classification of all (co)tilting modules and classes over 1-Gorenstein commutative rings. The second paper contains a classification of all tilting classes over regular rings of Krull dimension two and also a classification of all tilting modules in the local case. The third paper classifies all (co)tilting classes over arbitrary commutative noetherian rings.

0.1. **History.** Tilting modules were introduced by S. Brenner and M. Butler [10] and then generalized by several authors (e.g. [16], [18], [13], [22], [2]). Cotilting modules appeared as vector space duals of tilting modules over finite dimensional (Artin) algebras (e.g. [15, IV.7.8.]) and then generalized in a number of papers (e.g. [12], [2], [23], [4]). The current most general definition of a (co)tilting module and a (co)tilting class are the following.

Definition 0.1. Let R be a ring. A module T is *tilting* provided that

(T1) T has finite projective dimension,
(T2) $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $1 \leq i < \omega$ and all cardinals κ.
(T3) There are $r \geq 0$ and a long exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ where $T_i \in \mathrm{Add}\, T$ for all $i \leq r$.

The class $T^{\perp_\infty} = \{M \in \text{Mod–}R \mid \mathrm{Ext}^i_R(T, M) = 0 \text{ for each } i \geq 1\}$ is the *tilting class* induced by T.

Definition 0.2. Let R be a ring. A right R-module C is *cotilting* provided that

(C1) $\text{inj.dim}_R C \leq n$

(C2) $\text{Ext}^i_R(C^\kappa, C) = 0$ for all $1 \leq i$ and all cardinals κ,

(C3) there is an injective cogenerator W and a long exact sequence $0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to W \to 0$, with $C_i \in \text{Prod}C$.

The class ${}^{\perp_\infty}C = \{M \in \text{Mod-}R \mid \text{Ext}^i_R(M, C) = 0 \text{ for all } i \geq 1\}$ is called *cotilting class* induced by C

Tilting in module categories, viewed as a generalization of the Morita theory, is traditionally restricted to finitely presented tilting modules (see [18], [3, Chap. VI] et al.). Starting with [13] and [2], tilting theory for arbitrary modules over arbitrary rings has been developed over the past two decades, concentrating primarily on connections between tilting and approximation theory of modules.

The recent contributions to the theory, [6] and [8], show that also the derived category aspects of classical tilting extend to the infinitely generated setting. Namely, given a good n-tilting module T, the derived category $D(R)$ is equivalent to a localization of the derived category $D(S)$ where $S = \text{End}\, T$. In particular, there is an infinite dimensional analogue of the main result of [18], providing for an n-tuple of category equivalences between certain subcategories of Mod–R and Mod–S.

This is especially important when R is commutative, because in that case, all finitely generated tilting modules are trivial (i.e., projective), so the classical tilting theory reduces to the Morita theory.

A classification of (co)tilitng modules over special classes of commutative rings and domains was initiated by R. Göbel and J. Trlifaj [14], who classified (co)tilting Abelian groups (under Gödel axiom of constructibility; a condition removed later in [7]). (Co)tilting modules were classified also over Dedekind domains by S.Bazzoni et al. [7] (removing set theoretical assumptions in [21]) and over valuation and Prüfer domains by L. Salce in [19] and [20], and Bazzoni in [5].

Note that all these classification results are up to the tilting equivalence.

Definition 0.3. Let T, T' (C, C') be two tilting (cotilting) modules. We say that T is *equivalent* to T' (C is equivalent to C') if they induces the same tilting (cotilting) class.

The first paper in this dissertation extends this classification to 1-Gorenstein commutative rings. The paper contains a classification of all (co)tilting modules over 1-Gorenstein commutative rings. This result is a positive answer to the second open problem of chapter 6 of [14]. There was a hypothesis by J. Trlifaj that tilting modules over 1-Gorenstein commutative rings are classified by Bass tilting modules.

Definition 0.4. Let R be a 1-Gorenstein commutative ring. Then the minimal injective coresolution of R is of the form

$$0 \to R \to \bigoplus_{\mathfrak{p}\in P_0} E(R/\mathfrak{p}) \to \bigoplus_{\mathfrak{p}\in P_1} E(R/\mathfrak{p}) \to 0 \tag{1}$$

where P_0 (P_1) denotes the set of all prime ideals of height zero (one).

For each set of prime ideals $P \subseteq P_1$, there is a (unique) module R_P such that $R \subseteq R_P \subseteq Q = \bigoplus_{\mathfrak{p}\in P_0} E(R/\mathfrak{p})$ and $R_P/R \cong \bigoplus_{\mathfrak{p}\in P} E(R/\mathfrak{p})$, so there is an exact sequence

$$0 \to R \to R_P \to \bigoplus_{\mathfrak{p}\in P} E(R/\mathfrak{p}) \to 0 \tag{2}$$

The module $T_P = R_P \oplus \bigoplus_{\mathfrak{p}\in P} E(R/\mathfrak{p})$ is tilting a it is called the *Bass tilting module.*

Now the main theorem of the first paper in this dissertation follows.

Theorem 0.5. *Let R be a 1–Gorenstein commutative ring and T a module.*

Then T is tilting if and only if there is a set P consisting of prime ideals of R of height 1 such that T is equivalent to the Bass tilting module T_P.

In the domain case this result was recently extended by Jawad Abuhlail and Mohammed Jarrar in [1], they classified all tilting modules over almost perfect domains.

0.2. **Tilting classes.** Let us stress that all previous classification results for commutative noetherian rings are for rings of Krull dimension at most one. The task to classify all (co)tilting modules over commutative noetherian rings of higher Krull dimensions turned out to be very hard. To briefly illustrate this note that the proofs of previous results for Dedekind domains, Prüfer domains, almost perfect domains and 1-Gorenstein commutative rings are based on that the tilting modules are of finite type (S. Bazzoni, J. Šťovíček in [9]). Roughly speaking this means that tilting modules are parametrized by sets of finitely generated modules, so when you know the structure of finitely generated modules over some ring then using this result you get the structure of tilting modules over this ring. But by [17, Corollary 4.7.] every commutative noetherian ring of Krull dimension at least two is so called finlen-wild which means that there is no hope to classify even modules of finite length, so in this case you can't follow the previous strategy of proof. But there still was a hope to classify at least (co)tilting classes which turned out to be duly justified in a very general setting.

In the second paper in this dissertation we give a classification of all tilting classes over regular rings of Krull dimension two and also a classification of all tilting modules in the local case. The main result of this paper follows. (Note that P_i again denotes the set of all prime ideals of R of height i)

Theorem 0.6. *Let R be a regular ring of Krull dimension 2.*

Then tilting classes in $\text{Mod}–R$ are classified by the pairs (X, Y) where $\text{Ass}_R R \subseteq X \subseteq \text{Spec}(R)$ and $V(X \setminus \text{Ass}_R R) \cap P_2 \subseteq Y \subseteq P_2$.

For each such pair (X, Y), a tilting class $\mathcal{T}_{X,Y}$ is defined by

$$\mathcal{T}_{X,Y} = \bigcap_{\mathfrak{p} \in X} (R/\mathfrak{p})^{\perp_\infty} \cap \bigcap_{\mathfrak{m} \in Y} m^{\perp}.$$

Conversely, each tilting class $\mathcal{T}$ in $\text{Mod}–R$ is of this form, for $X = \text{Ass}_R {}^{\perp}\mathcal{T}$ and $Y = P_2 \cap {}^{\perp}\mathcal{T}$.

And finally the third paper in this dissertation contains a classification of all (co)tilting classes over general commutative noetherian rings. This paper is divided into two sections. The first one classify 1-cotilting classes by using the following result which classify torsion pairs

Theorem 0.7. *Let R be a commutative noetherian ring. Then there are bijections*

$$\begin{aligned} \{\textit{torsion pairs in mod-}R\} &\leftrightarrow \{Y \subseteq \text{Spec}(R) \mid Y \textit{ specialization closed}\} \\ &\leftrightarrow \{\textit{hereditary torsion pairs in Mod-}R\} \end{aligned}$$

and by using the Theorem due Buan and Krause from [11]

Theorem 0.8 (Buan-Krause)**.** *Let R be a right noetherian ring. Then there is a bijection*

$$\begin{aligned} \{\textit{faithful torsion pairs in mod-}R\} &\leftrightarrow \{\textit{1-cotilting classes in Mod-}R\} \\ (\mathcal{T}, \mathcal{F}) &\rightarrow \varinjlim \mathcal{F} \\ (\text{Ker}(\text{Hom}_R(-, \mathcal{C})) \cap \textit{mod-}R, \mathcal{C} \cap \textit{mod-}R) &\leftarrow \mathcal{C} \end{aligned}$$

The classification of 1-tilting classes is obtained by classification of 1-cotilting classes and by the Auslander-Bridger transpose of cyclic modules $R/\mathfrak{p}$, $\mathfrak{p} \in \text{Spec}(R)$.

The second section of this paper extends the first section to general n-(co)tilting classes. The classification is in terms of sequences $(Y_1, \dots, Y_n)$ of subsets of $\operatorname{Spec}(R)$ satisfying the following three conditions

(i) Y_i is closed under specialization for all $1 \leq i \leq n$;

(ii) $Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n$;

(iii) $(\operatorname{Ass}_R \Omega^{-(i-1)}(R)) \cap Y_i = \emptyset$ for all $1 \leq i \leq n$;

Now the main result of this third paper follows.

Theorem 0.9. *Let R be a commutative noetherian ring and $n \geq 1$. Then there are bijections between:*

(i) *Sequence* $(Y_1, \dots, Y_n)$ *of subsets of* $\operatorname{Spec}(R)$ *satisfying the previous three conditions;*

(ii) *n–tilting classes* $\mathcal{T} \subseteq \text{Mod–}R$*;*

(iii) *n–cotilting classes* $\mathcal{C} \subseteq \text{Mod–}R$*.*

The bijections assign to $(Y_1, \dots, Y_n)$ *the n–tilting class*

$$\begin{aligned}\mathcal{T} &= \{M \in \text{Mod–}R \mid \operatorname{Tor}_{i-1}^R(R/\mathfrak{p}, M) = 0 \textit{ for all } i = 1, \dots, n \textit{ and } \mathfrak{p} \in Y_i\} = \\ &= \{M \in \text{Mod–}R \mid \operatorname{Ext}_R^1(\operatorname{Tr}(\Omega^{(n-1)}(R/\mathfrak{p})), M) = 0 \textit{ for all } i = 1, \dots, n \textit{ and } \mathfrak{p} \in Y_i\},\end{aligned}$$

and the n–cotilting class

$$\begin{aligned}\mathcal{C} &= \{M \in \text{Mod–}R \mid \operatorname{Ext}_R^{i-1}(R/\mathfrak{p}, M) = 0 \textit{ for all } i = 1, \dots, n \textit{ and } \mathfrak{p} \in Y_i\} = \\ &= \{M \in \text{Mod–}R \mid \operatorname{Tor}_1^R(\operatorname{Tr}(\Omega^{(n-1)}(R/\mathfrak{p})), M) = 0 \textit{ for all } i = 1, \dots, n \textit{ and } \mathfrak{p} \in Y_i\}.\end{aligned}$$

Note that all classification results of these three papers are in terms of subsets of the spectrum of the ring and by associated prime ideals of modules.

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TILTING AND COTILTING CLASSES OVER GORENSTEIN RINGS

JAN TRLIFAJ AND DAVID POSPÍŠIL

Abstract. Let $n \geq 1$ and R be a Gorenstein ring of Krull dimension n. For each subset P of the set P_n of all prime ideals of height n, we construct a tilting class $\mathcal{T}(P)$. Solving an open problem from [13], we prove that T_P ($P \subseteq P_1$) are the only tilting classes of modules for $n = 1$, that is, all tilting modules are equivalent to the Bass ones. We also prove a dual characterization for cotilting modules, and show that they are hereditary.

However, the analogous result fails for $n = 2$: If $Q = E(R)$ has projective dimension 1 then the pairs (P, J) with $P \subseteq P_2$ and $J \subseteq I$ yield non–equivalent tilting modules (where $\bigoplus_{i \in I} K_i$ is a decomposition of Q/R into countably generated direct summands).

Let R be an n*–Gorenstein ring*, that is, a commutative noetherian ring of injective dimension $\leq n$. By a classical result of Bass [4], n–Gorenstein rings are characterized among commutative noetherian rings by the form of their minimal injective coresolution:

$$0 \to R \to \bigoplus_{p \in P_0} E(R/p) \to \cdots \to \bigoplus_{p \in P_n} E(R/p) \to 0 \tag{1}$$

where P_i denotes the set of all prime ideals of height i for each $i \leq n$.

Assume that R is n–Gorenstein. Then

$$T_{P_n} = \bigoplus_{i \leq n} \bigoplus_{p \in P_i} E(R/p)$$

is easily seen to be an (infinitely generated) n*-tilting module*, that is, to satisfy the following conditions:

(T1) T has projective dimension $\leq n$;

(T2) $\mathrm{Ext}^i_R(T, T^{(I)}) = 0$ for any indexed set I and any $i > 0$;

(T3) There is an exact sequence $0 \to R \to T_0 \to T_1 \to \cdots \to T_r \to 0$ where $0 \leq r < \omega$, $T_i \in \mathrm{Add}(T)$ for all $i \leq r$, and $\mathrm{Add}(T)$ denotes the

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class of all direct summands of a (possibly infinite) direct sum of copies of T.

A module T is *tilting* if it is n–tilting for some $n < \omega$, that is, T satisfies (T2) and (T3), and has finite projective dimension.

The exact sequence witnessing condition (T3) for T_{P_n} is just the minimal injective coresolution (1). Condition (T2) is trivial since T_{P_n} is injective. So the only non–obvious observation is that T_{P_n} has finite projective dimension. But that follows from the fact that $\mathcal{P} = \mathcal{I} = \mathcal{F}$ where $\mathcal{P}$ ($\mathcal{I}$, $\mathcal{F}$) denotes the class of all modules of finite projective (injective, flat) dimension. Moreover

$$\mathcal{P} = \mathcal{P}_n = \mathcal{I}_n = \mathcal{F}_n$$

where $\mathcal{P}_i$ ($\mathcal{I}_i$, $\mathcal{F}_i$) is the class of all modules of projective (injective, flat) dimension $\leq i$ for each $i \leq n$. These equalities hold even in the more general setting of (non-commutative) Iwanaga–Gorenstein rings, see [11, 9.1.10] and [13, 7.1.12].

Given an (n–) tilting module T, we define the (n–) *tilting class* induced by T as $\mathcal{T}_T = \{T\}^{\perp_\infty}$ where for a class of modules $\mathcal{C}$, we define $\mathcal{C}^\perp = \operatorname{KerExt}^1_R(\mathcal{C}, -)$ and $\mathcal{C}^{\perp_\infty} = \bigcap_{i>0} \operatorname{KerExt}^i_R(\mathcal{C}, -)$. Dually, ${}^\perp\mathcal{C}$ and ${}^{\perp_\infty}\mathcal{C}$ are defined using the contravariant Ext functors.

For example, the tilting class induced by T_{P_n} is $\mathcal{GI}$, the class of all *Gorenstein injective modules*, that is, the modules M possessing a (possibly infinite) injective resolution, see [13, 7.1.12].

Two tilting modules T and T' are said to be *equivalent* if their induced tilting classses coincide, or equivalently, if $T' \in \operatorname{Add}(T)$.

Finitely generated tilting modules over any commutative ring are known to be projective (cf. [8]), so all non–trivial tilting modules T considered below will be infinitely generated. However, there is always a set, $\mathcal{S}$, of finitely generated modules of projective dimension bounded by the projective dimension of T such that $\mathcal{S}^{\perp_\infty} = \mathcal{T}_T$. Conversely if R is noetherian and $\mathcal{S}$ is any set of finitely generated modules of projective dimension $\leq m$ then $\mathcal{S}^{\perp_\infty}$ is an m–tilting class (see [7] or [13, §5.2]). In particular if R is noetherian then each tilting module is determined up to equivalence by the indecomposable finitely generated modules in the class ${}^\perp\mathcal{T}_T$. However, indecomposable finitely generated modules are not classified even for 1–Gorenstein local rings, so we suggest a different approach here.

First, in Section 1, we consider the case of 1–Gorenstein rings. There, a simple modification of the minimal injective coresolution (1) is available and yields further 1–tilting modules, not equivalent to T_{P_1}. The idea is that for each set of prime ideals $P \subseteq P_1$, there is a (unique) module R_P such that $R \subseteq R_P \subseteq Q$ and $R_P/R \cong \bigoplus_{p \in P} E(R/p)$, so there is an exact sequence

(2) $$0 \to R \to R_P \to \bigoplus_{p \in P} E(R/p) \to 0$$

This sequence witnesses condition (T3) for the 1–tilting module $T_P = R_P \oplus \bigoplus_{p\in P} E(R/p)$, see [1, §3].

The tilting modules T_P ($P \subseteq P_1$) are called *Bass tilting modules* in [13]. It is easy to see that the tilting class induced by the Bass tilting module T_P equals

$$\mathcal{T}(P) = \bigcap_{p \in P} E(R/p)^{\perp}.$$

We also define

$$Q = \bigoplus_{p \in P_0} E(R/p).$$

Notice that $P_0 = \mathrm{Ass}(R)$ is finite, and Q is the localization of R at the multiplicative set $S = \bigcup_{p\in P_0} p$ of all non–zero divisors of R, so Q is the classical ring of quotients of R. Localizing at the primes $p \in P$, we see that $R_P = \bigcap_{p\in P} R_{(p)}$ is a subring of Q containing R. So R_P is always an intersection of localizations, but it need not be a localization of R at any multiplicative set, cf. [17] and [18] (see also [13, 6.3.13]).

In the particular case when R is a Dedekind domain, $\mathcal{T}(P)$ is just the class of all P–divisible modules, that is

(3) $$\mathcal{T}(P) = \bigcap_{p \in P} (R/p)^{\perp} = \{M \in \text{Mod–}R \mid Mp = M \text{ for all } p \in P\}.$$

In this case, it is known that each tilting module is equivalent to the Bass tilting module T_P for a set $P \subseteq P_1 = \mathrm{mspec}(R)$, see e.g. [13, 6.2.22]. Open Problem 2 in [13, p. 254] asks whether this characterization of tilting modules extends to arbitrary 1–Gorenstein rings. Our first main result shows that this is indeed the case:

Theorem 0.1. *Let R be a 1–Gorenstein ring and T a module.*

Then T is tilting if and only if there is a set P consisting of prime ideals of R of height 1 such that T is equivalent to the Bass tilting module T_P.

The construction of the Bass tilting module T_P for a 1–Gorenstein ring R in [1, §3] gives more: for each $p \in P$ there is a finitely generated module F_p such that $E(R/p)^{\perp} = F_p^{\perp}$, so the 1–tilting class induced by the Bass tilting module T_P equals

$$\mathcal{T}(P) = \bigcap_{p \in P} F_p^{\perp}. \tag{4}$$

The class $F^{\perp}$ is axiomatizable for any finitely generated module F over a noetherian ring R, so (4) yields a first–order description of the tilting class $\mathcal{T}(P)$. The construction of the module F_p goes back to Auslander–Buchweitz [3]; we call it here the Auslander-Buchweitz preenvelope of R/p. (If R is a Dedekind domain, we can take $F_p = R/p$, so (3) is a particular instance of (4).)

Our second main result is presented in Section 2 and shows that Auslander–Buchweitz preenvelopes play the same role for an arbitrary $n \geq 1$:

Theorem 0.2. *Let $n \geq 1$, R be an n–Gorenstein ring, and P a set consisting of prime ideals of R of height n. For each $p \in P$ denote by F_p the Auslander-Buchweitz preenvelope of R/p. Then*

$$\mathcal{N}(P) \overset{def}{=} \bigcap_{p \in P} E(R/p)^{\perp_\infty} = \bigcap_{p \in P} F_p^{\perp_\infty}$$

is an n–tilting class. Moreover, if P' is a set of prime ideals of height n such that $P' \neq P$ then $\mathcal{N}(P') \neq \mathcal{N}(P)$.

In Section 3, we consider various extensions of the construction of the Bass tilting module to the case of 2–Gorenstein rings. If Q has projective dimension 1, we present four different sets of tilting modules parametrized by subsets $P \subseteq P_2$ and $J \subseteq I$ where $Q/R = \bigoplus_{i \in I} K_i$ is a decomposition of Q/R into a direct sum of countably presented modules. In contrast with the case of 1–Gorenstein rings, our sets consist of non–equivalent tilting modules (see Corollary 3.8). In fact, distinct pairs (P, J) with $P \subseteq P_2$ and $J \subseteq I$ yield distinct (non–equivalent) tilting modules by Theorem 3.7.

In Section 4, we combine Theorem 0.1 with the duality between tilting and cotilting modules over 1–Gorenstein rings coming from [13, 8.2.8], and obtain a complete description of all cotilting modules and classes over 1–Gorenstein rings in Theorem 4.2. In particular, we show that in this case all cotilting modules are hereditary in the sense of [14].

We will need the following notation:

A pair of classes of modules $(\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. Each tilting module induces the *tilting cotorsion pair* $({}^{\perp}\mathcal{T}_T, \mathcal{T}_T)$. The projective dimension of modules in the class ${}^{\perp}\mathcal{T}_T$ is bounded by the projective dimension of T. Moreover $\mathcal{T}_T \cap {}^{\perp}\mathcal{T}_T = \mathrm{Add}(T)$, [13, §5.1].

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *hereditary* provided that $\mathrm{Ext}^i_R(A, B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $i \geq 2$. This is equivalent to $\mathcal{A}$ ($\mathcal{B}$) being a resolving (coresolving) class. Here, a class of modules $\mathcal{C}$ is *resolving* (*coresolving*) provided that $\mathcal{C}$ contains all projective (injective) modules, it is closed under extensions, and $A \in \mathcal{C}$ whenever $B, C \in \mathcal{C}$ and there is a short exact sequence $0 \to A \to B \to C \to 0$ ($0 \to C \to B \to A \to 0$).

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *complete* provided that for each module M there is an exact sequence $0 \to M \to B \to A \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ (equivalently, for each module M there is an exact sequence $0 \to B' \to A' \to M \to 0$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$). The module B is called a *special $\mathcal{B}$–preenvelope* of M, and A' a *special $\mathcal{A}$–precover* of M.

Let $\mathcal{X}$ be a class of modules. A module M is *$\mathcal{X}$–filtered* provided there is an increasing chain $(M_\alpha \mid \alpha \leq \sigma)$ of submodules of M (called a *$\mathcal{X}$–filtration* of M) such that $M_0 = 0$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \kappa$, $M_{\alpha+1}/M_\alpha \cong X_\alpha$ for some $X_\alpha \in \mathcal{X}$ for each $\alpha < \kappa$, and $M_\kappa = M$. If $\kappa = \omega$ then M is *countably $\mathcal{X}$–filtered.*

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ over a noetherian ring is said to be *of finite type* provided there is a set of finitely generated modules $\mathcal{S}$ such that $\mathcal{S}^\perp = \mathcal{B}$. Any tilting cotorsion pair is of finite type, and any cotorsion pair of finite type is complete (see [13, §5.2] and [13, §3.2]). The latter fact follows from part (ii) of a result from set–theoretic homological algebra:

Lemma 0.3. *Let R be a ring, and M, N be modules.*

(i) *(Eklof Lemma) Assume that M is ${}^\perp N$–filtered. Then $M \in {}^\perp N$.*
(ii) *The exists a module P containing N such that $P \in M^\perp$ and P/N is $\{M\}$–filtered.*
(iii) *(Bongartz Lemma) Assume $\mathrm{Ext}^1_R(M, M^{(\kappa)}) = 0$ for each cardinal κ. Then there exists a module P containing N such that $P \in M^\perp$ and P/N is isomorphic to a direct sum of copies of M.*

Proof. For (i), we refer e.g. to [13, 3.1.2], for (ii) to [10] (or [13, 3.2.1]), and (iii) is a particular case of (ii). □

If R is n–Gorenstein then there are four distinguished hereditary cotorsion pairs in Mod–R: $(\mathcal{P}_0, \mathrm{Mod}\text{–}R)$, $(\mathcal{P}, \mathcal{GI})$, $(\mathcal{GP}, \mathcal{I})$, and $(\mathrm{Mod}\text{–}R, \mathcal{I}_0)$. The modules in the class $\mathcal{GP}$ are called *Gorenstein projective.* Moreover $\mathcal{GP} \cap \mathcal{P} = \mathcal{P}_0$ and $\mathcal{GI} \cap \mathcal{P} = \mathcal{I}_0$ by [11, 10.2.3 and 10.1.2]. The first two cotorsion pairs are tilting (the first one being induced by the tilting module $T = R$, the second by the tilting module T_{P_n} defined above – see [13, 7.1.12]), and all these cotorsion pairs are complete (see e.g. [13, §4.1]).

In particular, any module M has a special $\mathcal{P}$–preenvelope B. If M is finitely generated and B is also finitely generated, than B is called the *Auslander-Buchweitz preenvelope* of M. The existence of an Auslander–Buchweitz preenvelope for any finitely generated module is a particular feature of our setting (and more generally, of the setting of Iwanaga–Gorenstein rings):

Lemma 0.4. *Let R be an n–Gorenstein ring and M be a finitely generated module. Then M has an Auslander–Buchweitz preenvelope.*

Proof. By [3, 1.8], there is an exact sequence $0 \to M \to F \to G \to 0$ where $F \in \mathcal{P}$ is finitely generated and $G \in {}^{\perp_\infty}R$. Since G is finitely generated, we have even $G \in {}^{\perp_\infty}\mathcal{P}_0$, and hence $G \in {}^{\perp_\infty}\mathcal{P} = {}^{\perp}\mathcal{I} = \mathcal{GP}$. So F is a finitely generated special $\mathcal{P}$–preenvelope of M. □

For further properties of the notions defined above, we refer to [11] and [13].

1. Tilting modules and classes over 1–Gorenstein rings

We start with the case of 1–Gorenstein rings:

Lemma 1.1. *Let R be a 1–Gorenstein ring and $(\mathcal{A}, \mathcal{B})$ be a tilting cotorsion pair. Let $\mathcal{B}' = \mathcal{B} \cap \mathcal{P}$ and $\mathcal{A}' = {}^{\perp}\mathcal{B}'$. Then $(\mathcal{A}', \mathcal{B}')$ is a cotorsion pair of finite type such that $\mathcal{GP} \subseteq \mathcal{A}'$ and $\mathcal{A}'$ is closed under submodules.*

Proof. Since R is commutative and noetherian, we have $\mathcal{I}_0 = \{R/I \mid I \in \mathrm{spec}R\}^{\perp}$ by [11, 2.4.7], so $\mathcal{P} = (\mathrm{spec}R)^{\perp}$. Since $(\mathcal{A}, \mathcal{B})$ is 1–tilting, there is a set $\mathcal{S}$ of finitely generated modules (of projective dimension ≤ 1) such that $\mathcal{S}^{\perp} = \mathcal{B}$. Then $\mathcal{B}' = \mathcal{B} \cap \mathcal{I} = (\mathcal{S} \cup \mathrm{spec}R)^{\perp}$, so $(\mathcal{A}', \mathcal{B}')$ is a cotorsion pair of finite type. Since $\mathcal{B}' \subseteq \mathcal{P}$ is a class of modules of injective dimension ≤ 1, the class $\mathcal{A}'$ is closed under submodules, and $\mathcal{A}' \supseteq {}^{\perp}\mathcal{P} = \mathcal{GP}$. □

Given a 1–Gorenstein ring R, each localization $R_{(p)}$ at a prime ideal p is either 1–Gorenstein (when p has height 1) or quasi-Frobenius (when p has height 0). We consider the local case in more detail:

Lemma 1.2. *Let R be a local 1–Gorenstein ring and T be a tilting module with the induced tilting cotorsion pair $(\mathcal{A}, \mathcal{B})$. Then either T is equivalent to the projective tilting module R, or T is equivalent to the injective tilting module T_{P_1}.*

Proof. Let $(\mathcal{A}', \mathcal{B}')$ be the cotorsion pair from Lemma 1.1. Since $R \in \mathcal{A}'$ and $\mathcal{A}'$ is closed under submodules, we infer that $R/q \in \mathcal{A}'$ for each $q \in \mathrm{Ass}(R)$. However, R is 1–Gorenstein so $\mathrm{Ass}(R) = P_0$, so $\mathcal{A}'$ contains all the cyclic modules R/p ($p \in \mathrm{spec}R$) except possibly for $p = m$, the maximal ideal of R.

We distinguish two cases:

(I) Assume $\mathcal{A}'$ contains all the cyclic modules R/p ($p \in \mathrm{spec}R$). Then $\mathcal{A}'$ contains all finitely generated modules. By Baer Lemma we have $\mathcal{B}' = \mathcal{I}_0$, and $\mathcal{A}' = \mathrm{Mod}\text{–}R$.

We claim that $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ coincides with the cotorsion pair $(\mathcal{P}, \mathcal{GI})$. Clearly $\mathcal{A} \subseteq \mathcal{P}$, because $\mathfrak{C}$ is a tilting cotorsion pair. Conversely, since $\mathrm{Add}(T) = \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{P} \cap \mathcal{B} = \mathcal{B}' = \mathcal{I}_0$, the tilting module T is injective. By [13, 5.1.9], $\mathcal{B}$ consists of the modules possesing a resolution consisting of elements of $\mathrm{Add}(T)$, so $\mathcal{B} \subseteq \mathcal{GI}$ by [13, 7.1.12], and $\mathcal{A} \supseteq {}^{\perp}\mathcal{GI} = \mathcal{P}$. This proves our claim.

Now, $T^{\perp} = \mathcal{GI} = T_{P_1}^{\perp}$, so T is equivalent to the tilting module T_{P_1}.

(II) Assume $R/m \notin \mathcal{A}'$. Consider $0 \neq F \in \mathcal{A}' \cap \mathcal{P}^{<\omega}$. Then F has injective dimension ≤ 1, and since R/m does not embed into F, the socle of F is zero, and F has a minimal injective coresolution of the form

$$0 \to F \to E(F) \to G \to 0$$

where $E(F)$ is a direct sum of copies of $E(R/q)$ ($q \in P_0$), and G is injective. But then $E(F)$ is a flat module by [19, 2.1]. Since G has flat dimension ≤ 1, we infer that F is flat and finitely generated, hence projective. This proves that all modules in $\mathcal{A}'$ that are finitely generated and have finite projective dimension, are projective.

Consider an arbitrary finitely generated module $X \in \mathcal{A}'$ and its Auslander–Buchweitz preenvelope F from Lemma 0.4:

$$0 \to X \to F \to Y \to 0$$

Since $Y \in \mathcal{GP} \subseteq \mathcal{A}'$, we have $F \in \mathcal{P} \cap \mathcal{A}'$. By the argument above, F is projective, so X is a submodule of a projective module. Since the cotorsion pair $(\mathcal{A}', \mathcal{B}')$ is of finite type and ${}^{\perp}\mathcal{P} = {}^{\perp}\mathcal{I}_1$, we infer that $\mathcal{B}' = (\mathrm{spec}R)^{\perp} = \mathcal{P}$, and $\mathcal{A}' = \mathcal{GP}$.

Thus $\mathcal{A} \subseteq \mathcal{GP} \cap \mathcal{P} = \mathcal{P}_0$, $\mathcal{A} = \mathcal{P}_0$, and T is projective, hence equivalent to R. $\square$

We are now in a position to prove our first main result:

Proof of Theorem 0.1. By [13, 5.2.24], a module M belongs to the tilting class of R–modules $T^{\perp}$, if an only if its localization $M_{(m)}$ in each maximal ideal m belongs to the tilting class of $R_{(m)}$–modules $T_{(m)}^{\perp}$. We will use this to prove that T is equivalent to T_P where P denotes the set of all maximal ideals m of R such that m has height 1 and the tilting module $T_{(m)}$ is injective.

Let m be a maximal ideal. If $m \in P$, then $T_{(m)}$ is equivalent to the tilting $R_{(m)}$–module T_{P_1} by Lemma 1.2, so $T_{(m)}^{\perp} = T_{P_1}^{\perp} = E(R_{(m)}/m_{(m)})^{\perp} = ((T_P)_{(m)})^{\perp}$. If $m \notin P$ and m has height 1, then $T_{(m)}$ is equivalent to $R_{(m)}$, so $T_{(m)}^{\perp} = \mathrm{Mod}\text{–}R_{(m)} = ((T_P)_{(m)})^{\perp}$ by Lemma 1.2. If m has height 0, then $T_{(m)}$ is a tilting module over the commutative quasi–Frobenius ring $R_{(m)}$. Let $(\mathcal{A},\mathcal{B})$ be the tilting cotorsion pair induced by $T_{(m)}$ in Mod–$R_{(m)}$, and $\mathcal{B}' = \mathcal{B} \cap \mathcal{I}$, $\mathcal{A}' = {}^{\perp}\mathcal{B}'$. Since $R_{(m)}/m_{(m)} \in \mathcal{A}'$, we have $\mathcal{A}' = $ Mod–$R_{(m)}$, so as above, $\mathcal{A} = \mathcal{P}_0$, and $(T_{(m)})^{\perp} = $ Mod–$R_{(m)} = ((T_P)_{(m)})^{\perp}$.

We have proved that $T_{(m)}^{\perp} = ((T_P)_{(m)})^{\perp}$ for each maximal ideal m of R, q.e.d. □

Remark 1.3. Though infinitely generated tilting R–modules T do not yield classical tilting equivalences between Mod–R and Mod–S for $S = \mathrm{End}(T)$, it is of interest to see what is the 'tilted algebra' $S_P = \mathrm{End}(T_P)$ for $P \subseteq P_1$.

Since $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$, we have the ring isomorphisms $R_P \cong \mathrm{End}(R_P)$, $J_P \cong \prod_{p \in P} J_p$ where $J_p = \mathrm{End}(E(R/p))$ is the 'p–adic ring', and

$$S_P \cong \left(\begin{smallmatrix} R_P & H_P \\ 0 & J_P \end{smallmatrix}\right)$$

Here $H_P = \mathrm{Hom}_R(R_P, \bigoplus_{p \in P} E(R/p)) \cong \mathrm{Hom}_R(Q, \bigoplus_{p \in P} E(R/p))$ is a Q–J_P–bimodule.

Problem 1.4. More generally, let R be a noetherian domain of Krull dimension 1. Then Q has projective dimension ≤ 1 as R–module, so there is a direct sum decomposition $K = Q/R = \bigoplus_{i \in I} K_i$ where K_i ($i \in I$) are countably generated R–modules, cf. [13, 6.3.16]. As in Lemma 3.5 below, one can make I play the role of P_1, that is, use the various direct summands induced by this decomposition to produce non–equivalent 1–tilting modules. Does this procedure yield all 1–tilting R–modules up to equivalence?

2. n–TILTING CLASSES ARISING FROM SETS OF MAXIMAL HEIGHT PRIME IDEALS

We turn to the general case of n–Gorenstein rings. We will need the following version of [6, 5.1]:

Lemma 2.1. *Let R be an n–Gorenstein ring and p be a maximal ideal of R. Let F_p be the Auslander–Buchweitz preenvelope of R/p. Then there exists a countably $\{F_p\}$–filtered module D_p such that $E(R/p)^{\perp_\infty} = D_p^{\perp_\infty}$.*

Proof. Using [15, 18.4 and 18.6], we see that $E(R/p)$ is an artinian countably $\{R/p\}$–filtered module. So $C = E(R/p) \in \mathcal{P}$ is the union of a strictly increasing chain $(C_m \mid m < \omega)$ such that $C_0 = 0$, $C_{m+1}/C_m \cong R/p$ for all $m < \omega$. Starting from the trivial short exact sequence of zeros, and proceeding as in the proof of [6, 5.1] (see also [1, 4.1]), we define by induction a chain of short exact sequence

$$\mathcal{E} : 0 \to C_m \to D_m \to E_m \to 0\,(m < \omega)$$

with $D_{m+1}/D_m \cong F_p$ and $E_{m+1}/E_m \cong G_p$ for all $m < \omega$ where $0 \to R/p \to F_p \to G_p \to 0$ is the exact sequence induced by the Auslander–Buchweitz $\mathcal{P}$–preenvelope F_p of R/p from Lemma 0.4. Let $0 \to C \to D_p \to E \to 0$ be the direct limit of $\mathcal{E}$. Then $D_p \in \mathcal{P}$ by the Eklof Lemma, hence $E \in \mathcal{GP} \cap \mathcal{P}$ is projective. So $D_p \cong C \oplus E$, and in particular, $C^{\perp_\infty} = D_p^{\perp_\infty}$. □

We will take $F_p = R/p$ and $G_p = 0$ in case $R/p \in \mathcal{P}$. Now we can prove our second main result:

Proof of Theorem 0.2. We will prove the result in two steps.

Step I: We will show that $D_p^{\perp_\infty} = F_p^{\perp_\infty}$ for each prime ideal of height n where D_p is the module from Lemma 2.1. Since D_p is countably $\{F_p\}$–filtered, we have the inclusion $F_p^{\perp_\infty} \subseteq D_p^{\perp_\infty}$ by the Eklof Lemma.

For a proof of the reverse inclusion, we first claim that $D_p^{\perp_\infty} \cap \mathcal{P} \subseteq F_p^{\perp_\infty}$. If this is not the case, then there exists $X \in (D_p^{\perp_\infty} \cap \mathcal{P}) \setminus F_p^{\perp_\infty}$ of minimal injective dimension $0 < n < \omega$. There is an exact sequence

$$0 \to X \to G \to P \to 0$$

where G is Gorenstein injective and $P \in \mathcal{P}$. Then $G \in \mathcal{GI} \cap \mathcal{P} = \mathcal{I}_0$. Since $D_p^{\perp_\infty}$ is a coresolving class, we infer that $P \in D_p^{\perp_\infty} \cap \mathcal{P}$ has injective dimension $n-1$. By the minimality of X, also $P \in F_p^{\perp_\infty}$. Since P is the first cosyzygy of X, we have $\mathrm{Ext}^i_R(F_p, X) = 0$ for all $i \geq 2$.

We will prove that also $\mathrm{Ext}^1_R(F_p, X) = 0$; this will contradict our choice of X. Assume $\mathrm{Ext}^1_R(F_p, X) \neq 0$, so there is a homomorphism $f : F_p \to P$ which does not factorize through $\pi : G \to P$. Since $\mathrm{Ext}^1_R(F_p, P) = 0$ and D_p/F_p is countably $\{F_p\}$–filtered by the construction above, the Eklof Lemma yields $\mathrm{Ext}^1_R(D_p/F_p, P) = 0$, so f extends to a homomorphism $f' : D_p \to P$. However, $\mathrm{Ext}^1_R(D_p, X) = 0$ by assumption, so f' has a factorization $f' = \pi g'$. Then $f = \pi(g' \restriction F_p)$ is a factorization of f through π, a contradiction. This proves our claim.

For an arbitrary $M \in D_p^{\perp_\infty}$ there is an exact sequence $0 \to N \to H \to M \to 0$ where $H \in \mathcal{P}$ and N is Gorenstein injective. Since $F_p \in \mathcal{P}$, we have $N \in F_p^{\perp_\infty} \subseteq D_p^{\perp_\infty}$, so $H \in D_p^{\perp_\infty} \cap \mathcal{P} \subseteq F_p^{\perp_\infty}$ by the argument above. Since $F_p^{\perp_\infty}$ is coresolving, also $M \in F_p^{\perp_\infty}$. This proves the reverse inclusion.

By Lemma 2.1, for each set P of prime ideals of R of height n, we have

$$\mathcal{N}(P) = \bigcap_{p \in P} E(R/p)^{\perp_\infty} = \bigcap_{p \in P} F_p^{\perp_\infty}.$$

Since all the modules F_p ($p \in P$) are finitely generated and have projective dimension $\leq n$, we infer that $\mathcal{N}(P)$ is an n–tilting class (see e.g. [13, 5.2.2]).

Step II: We will prove the 'moreover' part of Theorem 0.2. Assume there exists $p \in P \setminus P'$.

First, we construct a module $M \in \mathcal{P}$ such that $\mathrm{Ext}^i_R(R/p', M) = 0$ for all $p' \in P'$ and $i > 0$, but $\mathrm{Ext}^1_R(R/p, M) \neq 0$. For this purpose, we consider a representative set $\mathcal{S}_n$ of all finitely generated nth syzygies of the modules R/q for $q \in \mathrm{spec}(R)$, and let $\mathcal{S} = \{R/p' \mid p' \in P'\} \cup \mathcal{S}'$ where $\mathcal{S}'$ denotes the set of all finitely generated 1st, 2nd,…, and nth syzygies of the modules R/p' ($p' \in P'$). By Lemma 0.3(ii) there is an exact sequence

$$0 \to R/p \to M \to M' \to 0 \tag{5}$$

where (1) $M \in (\mathcal{S}_n \cup \mathcal{S})^{\perp}$, and (2) M' is $(\mathcal{S}_n \cup \mathcal{S})$–filtered. Condition (1) just says that $M \in \mathcal{I}_n = \mathcal{P}$ and $\mathrm{Ext}^i_R(R/p', M) = 0$ for all $p' \in P'$ and $i > 0$. Each $N \in \mathcal{S}_n \cup \mathcal{S}'$ is a submodule of finitely generated free module, hence $\mathrm{Hom}_R(R/p, N) = 0$. Also $\mathrm{Hom}_R(R/p, R/p') = 0$ for all $p' \in P'$, so condition (2) yields that $\mathrm{Hom}_R(R/p, M') = 0$ (otherwise, since $M' = \bigcup_{\alpha \leq \sigma} M_\alpha$ where $(M_\alpha \mid \alpha \leq \sigma)$ is a $(\mathcal{S}_n \cup \mathcal{S})$–filtration of M and R/p is a simple module, there is $\alpha \leq \sigma$ such that $R/p \subseteq M_{\alpha+1}/M_\alpha$ where the latter factor is isomorphic to an element of $(\mathcal{S}_n \cup \mathcal{S})$, a contradiction).

Assume $\mathrm{Ext}^1_R(R/p, M) = 0$. Then $\mathrm{Ext}^1_R(R/p, R/p) = 0$. However, R/p is essential in $E(R/p)$ and $E(R/p)$ is $\{R/p\}$–filtered, so $R/p = E(R/p)$ is a finitely generated injective module, in contradiction with $\mathrm{depth}(R) = n \geq 1$ (see [11, 9.2.17]). This proves that $\mathrm{Ext}^1_R(R/p, M) \neq 0$.

We will show that (1′) $\mathrm{Ext}^i_R(E(R/p'), M) = 0$ for all $p' \in P'$, $i > 0$, and also (2′) $\mathrm{Ext}^1_R(E(R/p), M) \neq 0$. Then M will satisfy $M \in \mathcal{N}(P') \setminus \mathcal{N}(P)$.

(1′) follows from $\mathrm{Ext}^i_R(R/p', M) = 0$ by the Eklof Lemma. If (2′) fails then $\mathrm{Ext}^1_R(F_p, M) = 0$ by Lemma 2.1 and by the first part of the proof. Applying the functor $\mathrm{Hom}_R(-, M)$ to the short exact sequence

$$0 \to R/p \to F_p \to G_p \to 0$$

with G_p Gorenstein projective we get

$$0 = \mathrm{Ext}^1_R(F_p, M) \to \mathrm{Ext}^1_R(R/p, M) \to \mathrm{Ext}^2_R(G_p, M) = 0$$

so $\mathrm{Ext}^1_R(R/p, M) = 0$, a contradiction. This proves that $\mathrm{Ext}^1_R(E(R/p), M) \neq 0$. □

Definition 2.2. For each $P \subseteq P_2$, we will denote by M_P the tilting module inducing the tilting class $\mathcal{N}(P)$. Note that M_P has projective dimension 2 for $P \neq \emptyset$ and $M_\emptyset$ is projective.

3. The 2–Gorenstein case

In this section, we consider several variations of the construction of Bass tilting modules in the setting of 2–Gorenstein rings. The first one uses again the subsets $P \subseteq P_2$ as parameters:

If R is 2–Gorenstein then the minimal injective coresolution of R has the form

$$0 \to R \to Q \to \bigoplus_{p \in P_1} E(R/p) \to \bigoplus_{p \in P_2} E(R/p) \to 0 \tag{6}$$

This coresolution consists of two short exact sequences:

$$0 \to R \to Q \to K \to 0 \tag{7}$$

where $K = Q/R$, and

$$0 \to K \to \bigoplus_{p \in P_1} E(R/p) \to \bigoplus_{p \in P_2} E(R/p) \to 0. \tag{8}$$

In particular, K has injective dimension ≤ 1. For each $P \subseteq P_2$, we define a module

$$T_P = Q \oplus R_P \oplus \bigoplus_{p \in P} E(R/p)$$

where R_P is the unique submodule of $\bigoplus_{p \in P_1} E(R/p)$ containing K and such that $R_P/K \cong \bigoplus_{p \in P} E(R/p)$.

The following lemma gives a necessary and sufficient condition for T_P to be a 2–tilting module:

Lemma 3.1. *Let R be a 2–Gorenstein ring and $P \subseteq P_2$. Then T_P is a 2–tilting module if and only if $Ext^1_R(K, R_P^{(I)}) = 0$ for all sets I. In this case T_P induces the 2–tilting class $K^{\perp_\infty} \cap \mathcal{N}(P)$.*

Proof. We have also the short exact sequences:

$$0 \to K \to R_P \to \bigoplus_{p \in P} E(R/p) \to 0 \tag{9}$$

and

$$0 \to R_P \to \bigoplus_{p \in P_1} E(R/p) \to \bigoplus_{p \in P_2 \setminus P} E(R/p) \to 0. \tag{10}$$

(10) shows that R_P, and hence also T_P, has injective dimension ≤ 1, so T_P always satisfies condition (T1).

Combining (7) and (9), we obtain the exact sequence

$$0 \to R \to Q \to R_P \to \bigoplus_{p \in P} E(R/p) \to 0$$

witnessing condition (T3) for T_P.

If T_P is 2–tilting then clearly $\operatorname{Ext}^1_R(R_P, R_P^{(I)}) = 0$, and hence $\operatorname{Ext}^1_R(K, R_P^{(I)}) = 0$ for all sets I.

To prove the converse, it suffices to verify condition (T2) for T_P. Since T_P has injective dimension ≤ 1, we are left to prove that for each indexed set I, $\operatorname{Ext}^1_R(Q, R_P^{(I)}) = 0$, $\operatorname{Ext}^1_R(R_p, R_P^{(I)}) = 0$, and $\operatorname{Ext}^1_R(\bigoplus_{p \in P} E(R/p), R_P^{(I)}) = 0$.

The first identity follows directly from $\operatorname{Ext}^1_R(K, R_P^{(I)}) = 0$ by (7) while the third follows from the fact that $\operatorname{Hom}_R(\bigoplus_{p \in P} E(R/p), \bigoplus_{p \in P_2 \setminus P} E(R/p)) = 0$, using the direct sum of I copies of the presentation (10). The second identity is then immediate from $\operatorname{Ext}^1_R(K, R_P^{(I)}) = 0$ by (9).

Finally, we prove that T_P induces the 2–tilting class $K^{\perp_\infty} \cap \mathcal{N}(P)$. By definition, $\mathcal{N}(P) = \bigcap_{p \in P} E(R/p)^{\perp_\infty}$, so $T^{\perp_\infty} \subseteq R_P^{\perp_\infty} \cap \mathcal{N}(P)$, and the latter class is contained in $K^{\perp_\infty} \cap \mathcal{N}(P)$ by (9).

Conversely, let $M \in K^{\perp_\infty} \cap \mathcal{N}(P)$. Then $M \in Q^{\perp_\infty}$ by (7) and $M \in R_P^{\perp_\infty}$ by (9), so $M \in T_P^{\perp_\infty}$. □

Of course, T_{P_2} is always an (injective) 2–tilting module inducing the tilting class $\mathcal{GI} = K^{\perp_\infty} \cap \mathcal{N}(P_2)$. However, the assumption of *all* T_P ($P \subseteq P_2$) being tilting is quite a strong one:

Lemma 3.2. *Let R be a Gorenstein ring of Krull dimension 2. Then T_P is a 2–tilting module for each set $P \subseteq P_2$ if and only if Q has projective dimension ≤ 1 as R–module.*

Proof. For all $P \subseteq P' \subseteq P_2$, there is a short exact sequence

$$0 \to R_P \to R_{P'} \to \bigoplus_{p \in P' \setminus P} E(R/p) \to 0. \tag{11}$$

By Lemma 3.1 it follows that T_P is a 2–tilting module for each set $P \subseteq P_2$ if and only if $\operatorname{Ext}^1_R(K, K^{(I)}) = 0$ for all sets I (where the latter just says that $T_\emptyset = Q \oplus K$ is a tilting module). Applying the functors $\operatorname{Hom}_R(K, -)$ and $\operatorname{Hom}_R(-, R^{(I)})$ to (I copies of) (7), we get

$$\operatorname{Ext}^1_R(K, K^{(I)}) \cong \operatorname{Ext}^2_R(K, R^{(I)}) \cong \operatorname{Ext}^2_R(Q, R^{(I)}). \tag{12}$$

Since $Q \in \mathcal{P} = \mathcal{P}_2$, $\operatorname{Ext}^2_R(Q, R^{(I)}) = 0$ for all sets I if and only if $\operatorname{Ext}^2_R(Q, -) = 0$. The latter just says that Q has projective dimension ≤ 1.

Conversely, if Q has projective dimension ≤ 1 then $\operatorname{Ext}^1_R(K, K^{(I)}) = 0$ for all sets I by (12). □

We pause to present examples showing that both alternatives for the projective dimension of the flat module Q are possible in our setting:

Example 3.3. (i) Let R be a countable Gorenstein ring of Krull dimension 2 (e.g., $R = K[x, y]$ where K is a countable field). Since Q is the classical quotient ring of R, also Q is countable, hence Q has projective dimension 1 as R–module because Q is a countably presented flat module, see [19, 2.1] and [12, VI.9].

(ii) Let $R = K[x, y]$ where K is a field of uncountable cardinality. Then R is a regular domain of Krull dimension 2 and Q is its quotient field, so Q has projective dimension 2 by a classical result of Osofsky [16, 2.59].

For the rest of this section we will assume that R is a Gorenstein ring of Krull dimension 2 and Q has projective dimension ≤ 1.

Our assumption on Q is clearly equivalent to $K = Q/R$ having projective dimension ≤ 1. A much deeper fact proved in [2] says that this is further equivalent to K being a direct sum of countably generated submodules $K_i \neq 0$ $(i \in I)$,

$$K = \bigoplus_{i \in I} K_i. \tag{13}$$

In this case $Q \oplus K$ is a 1–tilting module inducing the tilting class of all *divisible* modules (see [2] or [13, 6.3.16]):

$$K^{\perp} = \{M \mid Ms = M \text{ for all non–zero divisors } s \in R\}.$$

In particular, $\mathrm{Ext}^1_R(K, K^{(\kappa)}) = 0$ for any cardinal κ, so if $J \subseteq I$ then any $\{K_i \mid i \in J\}$–filtered module is isomorphic to a direct sum of copies of the modules K_i $(i \in J)$.

For the rest of this section, we fix the decomposition (13).

Under our assumptions, the modules T_P $(P \subseteq P_2)$ form a set of 2–tilting modules which – similarly to the set M_P $(P \subseteq P_2)$ – is parametrized by sets of prime ideals of height 2. The difference from the 1–Gorenstein case is that all these tilting modules are pairwise non–equivalent:

Lemma 3.4. *The tilting module T_P has projective dimension 2 for $P \neq \emptyset$, $T_\emptyset$ has projective dimension 1, and T_P is not equivalent to $T_{P'}$ for $P \neq P' \subseteq P_2$. Moreover, T_P is not equivalent to $M_{P'}$ for all $P, P' \subseteq P_2$.*

Proof. First, if $p \in P$ then $E(R/p)$ has flat, hence projective, dimension 2 by [19, 2.1], and so does T_P. If $P = \emptyset$ then K has projective dimension 1 (since (7) does not split), and so does $T_P = Q \oplus K$.

As in Step II of the proof of Theorem 0.2, if $P' \neq P \subseteq P_2$ and $p \in P \setminus P'$, then there is a module $M \in \mathcal{P}$ such that $M \in K^{\perp} \cap$

$\mathcal{N}(P')$ and $M \notin \mathcal{N}(P)$ (M is an extension of R/p by a $\mathcal{S}_n \cup \mathcal{S} \cup \{K\}$–filtered module M' as in (5); this modification of Step II is possible since $\mathrm{Hom}_R(R/p, K) = 0$). In particular, T_P is not equivalent to $T_{P'}$ and $N_{P'}$ for all $P \neq P' \subseteq P_2$.

Finally we prove that $\mathcal{N}(P_2) \nsubseteq K^{\perp}$ (this will imply that T_P is not equivalent to M_P for all $P \subseteq P_2$). Otherwise $\mathcal{N}(P_2) = T_{P_2}^{\perp_\infty} = \mathcal{GI}$. By Lemma 0.3(ii), there is an exact sequence $0 \to R \subseteq X \to Y \to 0$ such that $X \in \bigcap_{p \in P_2} E(R/p)^{\perp_2}$ and Y is $\mathcal{S}$–filtered where $\mathcal{S}$ is a set of the 1st syzygies of all the modules $E(R/p)$ with $p \in P_2$. Similarly, by the Bongartz Lemma, there is an exact sequence $0 \to X \subseteq Z \to U \to 0$ such that $Z \in \mathcal{N}(P_2)$ and U is a direct sum of copies of the modules $E(R/p)$ ($p \in P_2$).

Then $Z \in \mathcal{GI} \cap \mathcal{P} = \mathcal{I}_0$ is an injective module containing R, hence a copy of Q, and thus $Z/R \cong K \oplus L$ for an injective module L. On one hand, we have $L = L_1 \oplus L_2$ where L_2 is a direct sum of copies of $E(R/p)$ with $p \in P_2$ and with L_1 containing no simple submodule of the form R/p ($p \in P_2$). Notice that L_2 is also the largest $\{R/p \mid p \in P_2\}$–filtered (semiartinian) submodule of Z/R.

On the other hand, we have the exact sequence $0 \to X/R \to Z/R \to U \to 0$, and $Y \cong X/R$, so $X/R \cap L_2 = 0$ because Y, being $\mathcal{S}$–filtered, contains no simple submodule of the form R/p ($p \in P_2$). It follows that $V = K \oplus L_1$ contains a copy, C, of X/R such that V/C is isomorphic to a submodule of $U \cong (Z/R)/(X/R)$, so V/C is $\{R/p \mid p \in P_2\}$–filtered. Since V has no simple submodule of the form R/p ($p \in P_2$), C is essential in V, and $E(K) \oplus L_1 = E(C) \cong E(X/R)$. Each $N \in \mathcal{S}$ is a submodule of a projective module, and hence of $Q^{(\kappa)}$ for a cardinal κ. By induction on the length of an $\mathcal{S}$–filtration, we infer that any $\mathcal{S}$–filtered module embeds into $Q^{(\kappa)}$ for a cardinal κ. In particular, this holds for X/R, so $E(X/R)$ is isomorphic to a direct sum of copies of $E(R/q)$ for $q \in P_0$. However, $E(K) = \bigoplus_{p \in P_1} E(R/p)$, a contradiction. $\square$

Our next goal is to show that there is another source of tilting modules that are analogous to the Bass ones, but this time parametrized by subsets of the set I. We start with the ones of projective dimension 1:

For each subset $J \subseteq I$, denote by R_J be the unique submodule of Q containing R such that $R_J/R = \bigoplus_{j \in J} K_j$, and let $K_J = \bigoplus_{j \in J} K_j$, and $N_J = R_J \oplus K_J$.

Lemma 3.5. *Let $\emptyset \neq J \subseteq I$. Then N_J is a tilting module of projective dimension 1 inducing the tilting class $K_J^{\perp}$. If $J \neq J' \subseteq I$ then N_J is not equivalent to $N_{J'}$.*

Proof. First, we prove that N_J is a 1–tilting module. Since K_J is a direct summand in K, K_J has projective dimension ≤ 1, and so does

R_J and N_J. The exact sequence $0 \to R \to R_J \to K_J \to 0$ witnesses condition (T3) for N_J. Since K_J has projective dimension ≤ 1, in order to prove condition (T2), we are left to show that $\mathrm{Ext}^1_R(K_J, R_J^{(\kappa)}) = 0$ for any cardinal κ. This follows from the existence of the exact sequence $0 \to R_J \to Q \to \bigoplus_{j' \in I \setminus J} K_{j'} \to 0$ and from the fact that $\mathrm{Hom}_R(K_J, \bigoplus_{j' \in I \setminus J} K_{j'}) = 0$ (see [2, 3.2]).

Since $J \neq \emptyset$, K_J has projective dimension 1 (because R is essential in R_J), and so does N_J. The tilting class induced by N_J is $N_J^{\perp} = K_J^{\perp}$.

Assume $J' \neq J \subseteq I$ and $j \in J \setminus J'$. Clearly $R_{J'} \in N_{J'}^{\perp}$. However, the split monomorphism $K_j \hookrightarrow \bigoplus_{m \in I \setminus J'} K_m$ does not factorize through the epimorphism $\pi : Q \to \bigoplus_{m \in I \setminus J'} K_m$ because π has essential kernel (isomorphic to $R_{J'}$). So $\mathrm{Ext}^1_R(K_j, R_{J'}) \neq 0$, and $R_{J'} \notin N_J^{\perp}$. So N_J is not equivalent to $N_{J'}$. □

In order to define the corresponding 2–tilting modules, we need further notation.

For each $i \in I$, there is a countable subset $\emptyset \neq A_i \subseteq P_1$ such that $E(K_i) = \bigoplus_{p \in A_i} E(R/p)$. Since $E(K) = \bigoplus_{i \in I} E(K_i) = \bigoplus_{p \in P_1} E(R/p)$, the Krull–Schmidt–Azumaya Theorem implies that the A_i's actually yield a partition $P_1 = \bigcup_{i \in I} A_i$. Moreover

$$E(K)/K = \bigoplus_{i \in I} (\bigoplus_{p \in A_i} E(R/p))/K_i \cong \bigoplus_{p \in P_2} E(R/p),$$

so there is a partition $P_2 = \bigcup_{i \in I} B_i$ such that $E(K_i)/K_i \cong \bigoplus_{p \in B_i} E(R/p)$ for each $i \in I$ (in particular, $B_i = \emptyset$ iff K_i is injective).

For each subset $J \subseteq I$, we define $X_J = \bigoplus_{j \in J} \bigoplus_{p \in A_j} E(R/p)$, and $Y_J = \bigoplus_{j \in J} \bigoplus_{p \in B_j} E(R/p)$. Let $P_J = R_J \oplus X_J \oplus Y_J$. Note that $P_I = T_{P_2}$ and $P_\emptyset = R$.

Lemma 3.6. *Let $\emptyset \neq J \subseteq I$. Then P_J is a tilting module of projective dimension 2 inducing the tilting class $K_J^{\perp} \cap \mathcal{N}(\bigcup_{j \in J} B_j)$.*

Proof. Since R_J has projective dimension ≤ 1 and X_J, Y_J are injective, P_J is a 2–tilting module. Moreover, Y_J has flat dimension 2 by [19, 2.1], so P_J has projective dimension 2. The exact sequence

$$0 \to R \to R_J \to X_J \to Y_J \to 0 \tag{14}$$

witnesses condition (T3) for P_J. Clearly, $X_J \oplus Y_J$ is injective, R_J, K_J have projective dimension ≤ 11, and $\mathrm{Ext}^1_R(K_J, R_J^{(\kappa)}) = 0$ for any cardinal κ by part (i). So for proving condition (T2), it remains to show that $\mathrm{Ext}^i_R(Y_J, R_J^{(\kappa)}) = 0$ for any cardinal κ and $i = 1, 2$.

However, $0 \to R_J^{(\kappa)} \to Q^{(\kappa)} \to K_{I \setminus J}^{(\kappa)} \to 0$ is exact and $Q^{(\kappa)}$ is the injective envelope of $R_J^{(\kappa)}$, and $\mathrm{Hom}_R(Y_J, K_{I \setminus J}^{(\kappa)}) = 0$ because $K_{I \setminus J}^{(\kappa)} \subseteq \bigoplus_{p \in P_1} E(R/p)$, so $\mathrm{Ext}^1_R(Y_J, R_J^{(\kappa)}) = 0$. And $\mathrm{Ext}^2_R(Y_J, R_J^{(\kappa)}) \cong \mathrm{Ext}^1_R(Y_J, K_{I \setminus J}^{(\kappa)}) \cong$

$\mathrm{Hom}_R(Y_J, Y_{I\setminus J}^{(\kappa)}) = 0$ because $\mathrm{Hom}_R(E(R/p), E(R/p')) = 0$ for all $p \neq p' \in P_2$. This proves that P_J is a 2–tilting module.

Using the fact that the exact sequence (14) consists of the short exact sequences $0 \to R \to R_J \to K_J \to 0$ and $0 \to K_J \to X_J \to Y_J \to 0$, we infer that P_J induces the tilting class $K_J^{\perp} \cap Y_J^{\perp_\infty} = K_J^{\perp} \cap \mathcal{N}(\bigcup_{j\in J} B_j)$. □

Also the 2–tilting modules P_J ($\emptyset \neq J \subsetneq I$) are not equivalent to any of the ones constructed earlier. This is a consequence of a more general result:

Theorem 3.7. *Let $P \subseteq P_2$ and $J \subseteq I$. Then*

$$\mathcal{T}(P, J) \overset{def}{=} M_P^{\perp_\infty} \cap N_J^{\perp_\infty} = \mathcal{N}(P) \cap K_J^{\perp}$$

is a tilting class.

If $P' \subseteq P_2$, $J' \subseteq I$, and $(P', J') \neq (P, J)$ then $\mathcal{T}(P', J') \neq \mathcal{T}(P, J)$.

Proof. First, the intersection of any family of tilting classes is again a tilting class (in our setting, tilting classes coincide with the classes of the form $\mathcal{S}^{\perp_\infty}$ for a set $\mathcal{S}$ of finitely generated modules of projective dimension ≤ 2).

Now, assume there exists $p \in P \setminus P'$. Then we can argue as in the proof of Lemma 3.4: there is a module $M \in \mathcal{P}$ such that $M \in K^{\perp} \cap \mathcal{N}(P')$ but $M \notin \mathcal{N}(P)$, so $M \in \mathcal{T}(P', J') \setminus \mathcal{T}(P, J)$.

Assume $P = P'$ and there is $j \in J \setminus J'$. Denote by $\mathcal{S}$ is a set of the 1st syzygies of all the modules $E(R/p)$ with $p \in P_2$. By Lemma 0.3(ii), there exists an exact sequence $0 \to R_{J'} \subseteq X \to Y \to 0$ such that $X \in K_{J'}^{\perp} \cap \bigcap_{p\in P_2} E(R/p)^{\perp_2}$ and Y is $\mathcal{S} \cup \{K_i \mid i \in J'\}$–filtered. Similarly, by the Bongartz Lemma, there is an exact sequence $0 \to X \subseteq Z \to U \to 0$ such that $Z \in \mathcal{T}(P_2, J')$ and U is a direct sum of copies of the modules $E(R/p)$ ($p \in P_2$). We will prove that $Z \notin \mathcal{T}(P, J)$ by showing that $Z \notin K_j^{\perp}$.

Assume $Z \in K_j^{\perp}$. Let $L = J' \cup \{j\}$. Then $Z \in K_L^{\perp}$. By the Bongartz Lemma, there is an exact sequence $0 \to Z \subseteq V \to W \to 0$ such that $V \in K_{I\setminus L}^{\perp}$ and W is a direct sum of copies of the modules K_i ($i \in I \setminus L$), and hence $V \in K^{\perp} \cap \bigcap_{p\in P_2} E(R/p)^{\perp_2}$ because W has injective dimension ≤ 1. Similarly, we obtain the exact sequence $0 \to V \subseteq E \to F \to 0$ such that $E \in \mathcal{GI} = K^{\perp} \cap \mathcal{N}(P_2)$ and F is a direct sum of copies of the modules $E(R/p)$ ($p \in P_2$).

Notice that $E \in \mathcal{P}$ by construction, so $E \in \mathcal{GI} \cap \mathcal{P}$ is injective and contains $R_{J'}$. Hence it contains a copy of Q, and $G = E/R_{J'} \cong K_{I\setminus J'} \oplus E'$ for an injective module E'.

The injectivity of the modules $E(R/p)$ ($p \in P_2$) makes it possible to reorder the consecutive factors in G: indeed, the injective direct summand Z/X in E/X has a complement C/X with $E/X = Z/X \oplus C/X$. Let $D = C \cap V$. Then $D/X \cong V/Z$ while $C/D \cong E/V$ and

$E/C \cong Z/X$ are injective. So we have the exact sequences $0 \to R_{J'} \to E \to G \to 0$ and $0 \to G' \to G \to E/D \to 0$ where $G' = D/R_{J'}$ is $\mathcal{S} \cup \{K_i \mid j \neq i \in I\}$–filtered, and E/D is a direct sum of copies of the modules $E(R/p)$ $(p \in P_2)$. Now as in the proof of Lemma 3.4, we can assume that E' has no simple submodule of the form R/p for any $p \in P_2$, and hence G' is an essential submodule of G which is $\mathcal{S} \cup \{K_i \mid j \neq i \in I\}$–filtered.

Each $N \in \mathcal{S}$ is a submodule of $Q^{(\kappa)}$ for a cardinal κ, and clearly each $N \in \{K_i \mid j \neq i \in I\}$ is a submodule in $E(K_i)$ for some $j \neq i \in I$. The same holds for the $\mathcal{S}$–filtered module G'. This implies that $E(G') = E(G)$ has no direct summands isomorphic to $E(R/p)$ for any $p \in A_j$. However, $K_{I \setminus J'}$, and hence K_j, is a direct summand in G, a contradiction. □

The idea of parametrizing tilting modules over 1–Gorenstein rings by subsets of the set P_1 of all height 1 prime ideals works well in the sense that it gives all tilting modules up to equivalence (see Section 1).

For 2–Gorestein rings such that Q has projective dimension ≤ 1, we have presented four different variations of the same idea: subsets P of the set P_2 of all height 2 prime ideals have been used to define the tilting modules T_P and M_P while subsets J of the indexing set I from the decomposition (13) parametrize the tilting modules N_J and P_J.

In contrast with the case of $n = 1$, the tilting modules obtained by these variations are pairwise non–equivalent. This follows by Theorem 3.7 using the descriptions of the induced tilting classes given in Definition 2.2, Lemmas 3.4, 3.5, and 3.6, respectively:

Corollary 3.8. *Let R be a Gorenstein rings of Krull dimension 2 such that Q has projective dimension ≤ 1. The set $S_2 = \{T_P \mid \emptyset \neq P \subseteq P_2\} \cup \{M_P \mid \emptyset \neq P \subseteq P_2\} \cup \{P_J \mid \emptyset \neq J \subsetneq I\}$ consists of pairwise non–equivalent tilting modules of projective dimension 2. The set $S_1 = \{N_J \mid \emptyset \neq J \subseteq I\}$ consists of pairwise non–equivalent tilting modules of projective dimension 1.*

Problem 3.9. Let R be a Gorenstein ring of Krull dimension 2.

(1) Assume that Q has projective dimension ≤ 1, so $Q/R = \bigoplus_{i \in I} K_i$ as in (13). Is every tilting class of the form $\mathcal{T}(P, J)$ for $P \subseteq P_2$ and $J \subseteq I$ as in Theorem 3.7?

(This is true of the classes induced by the tilting modules T_P, M_P, N_J, and P_J defined above.)

(2) What is the structure of tilting modules in the case when Q has projective dimension 2?

4. Cotilting modules and classes over 1–Gorenstein rings

We finish by considering the dual case of cotilting modules. The point is that over 1–Gorenstein rings, they are equivalent to duals of the tilting ones, so Theorem 0.1 makes it possible to classify them all up to equivalence. We will use this to prove that in the 1–Gorenstein case, all cotilting modules are hereditary in the sense of [14].

An *n–cotilting module* C over a ring R is defined by the following three conditions:

(C1) C has injective dimension $\leq n$;
(C2) $\mathrm{Ext}^i_R(C^I, C) = 0$ for any indexed set I and any $i > 0$;
(C3) There is an exact sequence $0 \to C_r \to \cdots \to C_0 \to W \to 0$ where $r \geq 0$, W is an injective cogenerator for Mod–R and $C_0, \ldots, C_r$ are direct summands of a (possibly infinite) direct product of copies of the module C.

Each (n–) cotilting module induces an (n–) *cotilting class* $\mathcal{C}_C = {}^{\perp_\infty}C$. Two cotilting modules C and C' are *equivalent* if they induce the same cotilting class.

It is known that given a tilting (right R–) module T, the dual module $C = T^* = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ is a cotilting left R–module. For a class of (right R–) modules $\mathcal{C}$, define $\mathcal{C}^{\intercal} = \mathrm{KerTor}^1_R(\mathcal{C}, -)$ and $\mathcal{C}^{\intercal_\infty} = \bigcap_{i>0} \mathrm{KerTor}^i_R(\mathcal{C}, -)$.

If $C = T^*$ for a tilting module T then the cotilting class induced by C equals $T^{\intercal_\infty}$. Moreover, if $\mathcal{S}$ is a class of finitely presented modules with $\mathcal{S}^{\perp_\infty} = T^{\perp_\infty}$ then $\mathcal{C}_C = \mathcal{S}^{\intercal_\infty}$, see [13, §8].

In general, there exist cotilting modules that are not equivalent to duals of tilting modules. For example, this happens for any valuation domain R which is not strongly discrete, see [5].

However, if R is a 1–Gorenstein ring, then each cotilting module is equivalent to T^* for a tilting module T by [13, 8.2.8]. In particular, if R is a Dedekind domain then the cotilting modules are parametrized by sets of maximal ideals in R.

Moreover, the same holds when dual modules are defined by $T^c = \mathrm{Hom}_R(T, W)$ where $W = \bigoplus_{m \in \mathrm{mspec}\ \mathrm{R}} E(R/m)$ is an injective cogenerator for Mod–R. In this case the cotilting modules $(T_P)^c$ where P runs over all sets of prime ideals of height 1 in a 1–Gorenstein ring R are called the *Bass cotilting modules*, see [13, §8].

Before proceeding we describe the Bass cotilting modules (up to equivalence) more explicitly. For each $p \in P_1$, we denote by $J_p = \mathrm{End}(E(R/p)) = (E(R/p))^c$. Since $E(R/p)$ is injective, J_p is a flat pure–injective R–module.

Lemma 4.1. *Let R be a 1–Gorenstein ring and $P \subseteq P_1$. Then the Bass cotilting module $(T_P)^c$ is equivalent to the cotilting module*

$$C_P = Q \oplus \prod_{p \in P} J_p \oplus \bigoplus_{q \in P_1 \setminus P} E(R/q).$$

Proof. First, we prove that C_P is cotilting. Since $J_P = \prod_{p \in P} J_p$ is a flat module, C_P has injective dimension ≤ 1, so (C1) holds.

Since C is pure–injective and $Q \oplus J_P$ is flat, in order to prove condition (C2), we only have to show that $\mathrm{Ext}^1_R(E_P^I, J_P) = 0$ where $E_P = \bigoplus_{q \in P_1 \setminus P} E(R/q)$ and I is any set. The injective module E_P^I has no direct summands of the form $E(R/p)$ for $p \in P$, so it suffices to prove that $\mathrm{Ext}^1_R(R/q, J_p) = 0$ for each $q \in P_1 \setminus P$ and $p \in P$. But the latter holds because $\mathrm{Tor}_1^R(R/q, R/p)$ is annihilated by $p + q = R$, so $\mathrm{Tor}_1^R(R/q, R/p) = 0$.

For the proof of condition (C3), we first claim that $E(J_p)/J_p \cong E(R/p)^{(X)}$ for a non–empty set X. Indeed, J_p has injective dimension ≤ 1, so $E(J_p)/J_p$ is injective. By the formula for Bass invariants of J_p [11, 9.2.4], the multiplicity of $E(R/q)$ in $E(J_p)/J_p$ for a prime ideal $q \neq p$ is zero, because $\mathrm{Ext}^1_R(R/q, J_p) = 0$ (for $q \in P_1$, this has been proved above; for $q \in P_0$, $E(R/q)$ is flat, hence $\mathrm{Ext}^1_R(E(R/q), J_p) = 0$, and $\mathrm{Ext}^1_R(R/q, J_p) = 0$ because J_p has injective dimension ≤ 1). Since J_p is not injective (as $E(R/p)$ is not flat for $p \in P$), our claim follows.

Our claim implies that the injective module $E(J_P)/J_P$ contains a direct summand isomorphic to $\bigoplus_{p \in P} E(R/p)$. Since $E(J_P)$ is flat (cf. [19, 2.3]), it is isomorphic to a direct summand in a direct product, Π, of copies of Q, and there is an exact sequence

$$0 \to J_P \to \Pi \oplus E_P \to V \to 0 \tag{15}$$

where $V \cong \Pi/J_P \oplus E_P$ is an injective module containing $\bigoplus_{m \in \mathrm{mspec\ R}} E(R/m)$ as a direct summand, hence V is an injective cogenerator. So (15) witnesses condition (C3).

Finally, the cotilitng class induced by C_P is ${}^\perp C_P = {}^\perp J_P = \bigcap_{p \in P} {}^\perp J_p$. But ${}^\perp J_P = \{M \in R\text{–Mod} \mid \mathrm{Tor}_1^R(E(R/p), M) = 0 \text{ for all } p \in P\} = T_P^\intercal$, so C_P is equivalent to $(T_P)^c$. □

The explicit duality $(-)^c$ between tilting and cotilting modules over an arbitrary 1–Gorenstein ring enables us to classify all cotilting modules up to equivalence, solving thus in the positive [13, Open Problem 3, p.292]:

Theorem 4.2. *Let R be a 1–Gorenstein ring and C be a module.*

Then C is cotilting if and only if there is a set P consisting of prime ideals of R of height 1 such that C is equivalent to the cotilting module C_P.

The cotilting class induced by C equals

$$\mathcal{C}(P) = \{M \in R\text{–Mod} \mid Tor_1^R(F_p, M) = 0 \textit{ for all } p \in P\}.$$

where F_p denotes the Auslander-Buchweitz preenvelope of R/p for each $p \in P$.

Proof. By the remarks above, the claim follows from Theorem 0.1 and Lemma 4.1 by applying the duality $(-)^c = \mathrm{Hom}_R(-, W)$ where $W = \bigoplus_{m \in \text{mspec R}} E(R/m)$, to the tilting module T_P. □

If C is any 1–cotilting module and $\mathcal{C} = {}^\perp C$ the corresponding cotilting class, then $\mathcal{C}$ is a torsion–free class of modules, consisting of all modules cogenerated by C (see e.g. [13, §8.2]). So there is a torsion theory $\mathfrak{T} = (\mathcal{T}, \mathcal{C})$. By [14], C is called *hereditary* if $\mathfrak{T}$ is hereditary, that is, $\mathcal{T}$ is closed under submodules or, equivalently, $\mathcal{C}$ is closed under injective envelopes.

In general, 1–cotilting modules need not be hereditary (see [9, Theorem 2.5]), but they are in the 1–Gorenstein case:

Corollary 4.3. *Let R be a 1–Gorenstein ring and C be a cotilting module. Then C is hereditary.*

Proof. By Theorem 4.2, C is equivalent to the cotilting module C_P for a subset $P \subseteq P_1$, so w.l.o.g., $C = C_P$.

It suffices to prove that $\mathrm{Ext}_R^1(E(C_P), C_P) = 0$ because then for each $M \in {}^\perp C_P$, $E(M)$ is a direct summand of a product of copies of $E(C_P)$, hence $E(M) \in {}^\perp C_P$ (cf. [14, Lemma 1.3]).

Since C_P is pure–injective and $Q \oplus E(J_P)$ is flat, it remains to show that $\mathrm{Ext}_R^1(E_P, C_P) = 0$, that is, that $\mathrm{Ext}_R^1(E(R/q), J_p) = 0$ for all $q \in P_1 \setminus P$ and $p \in P$. But this has already been observed above. □

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TILTING FOR REGULAR RINGS OF KRULL DIMENSION TWO

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Abstract. We classify tilting classes over regular rings R of Krull dimension two. They are parametrized by the set of all pairs (X, Y) such that $\mathrm{Ass}_R R \subseteq X \subseteq \mathrm{Spec}(R)$, Y consists of maximal ideals of height 2, and Y contains all the maximal ideals of height 2 that contain some element of $X \setminus \mathrm{Ass}_R R$. For R local, we also classify the corresponding infinitely generated tilting modules.

Introduction

Tilting in module categories, viewed as a generalization of the Morita theory, is traditionally restricted to finitely presented tilting modules (see [23], [4, Chap. VI] et al.). Starting with [12] and [2], tilting theory for arbitrary modules over arbitrary rings has been developed over the past two decades, concentrating primarily on connections between tilting and approximation theory of modules.

The recent contributions to the theory, [5] and [8], show that also the derived category aspects of classical tilting extend to the infinitely generated setting. Namely, given a good n-tilting module T, the derived category $D(R)$ is equivalent to a localization of the derived category $D(S)$ where $S = \mathrm{End}\, T$. In particular, there is an infinite dimensional analogue of the main result of [23], providing for an n-tuple of category equivalences between certain subcategories of Mod–R and Mod–S.

This is especially important when R is commutative, because in that case, all finitely generated tilting modules are trivial (i.e., projective), so the classical tilting theory reduces to the Morita theory.

There is a grain of finiteness even in the infinite setting: each tilting module T is of finite type, that is, there is a set of finitely presented modules $\mathcal{S}$ such that the tilting class induced by T equals $\mathcal{S}^{\perp_\infty}$, [9]. This enables classification of tilting modules and classes over Dedekind domains [6], and is essential in extending this classification in various

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directions: to Prüfer domains [24], almost perfect domains [1], and Gorenstein rings of Krull dimension one [25].

However, commutative noetherian rings of Krull dimension ≥ 2 are known to be finlen-wild [21]. In particular, there is no hope to classify their finitely presented modules, and one needs new methods to approach infinitely generated tilting modules in this setting.

Divisibility and classical localization provide important tools for this purpose, but it is the notion of an associated prime that is essential for the two-dimensional case. Indeed, for regular local rings of Krull dimension 2, we show that non-trivial tilting classes $\mathcal{T}$ are characterized by the sets $\operatorname{Ass}_R {}^{\perp}\mathcal{T}$. We use this fact to classify all tilting modules and classes in that case.

Our main result (Theorem 4.2) then gives a parametrization of all tilting classes over regular rings of Krull dimension 2.

1. Tilting and divisibility

For a ring R, we denote by Mod–R the category of all (unitary right R-) modules. Further, mod–R denotes the class of all modules possessing a projective resolution consisting of finitely generated projective modules. So mod–R is just the class of all finitely generated (finitely presented) modules in case R is right noetherian (right coherent).

For each $n < \omega$, we denote by $\mathcal{P}_n$ ($\mathcal{I}_n$) the class of all modules of projective (injective) dimension at most n. For a module M, $\Omega^n(M)$ is the *nth syzygy* in a projective resolution of M (if $M \in$ mod–R, we will consider only projective resolutions of M consisting of finitely generated modules, hence also $\Omega^n(M) \in$ mod–R).

For a module T, $\operatorname{Sum} T$ denotes the class of all (possibly infinite) direct sums of copies of the module T, and $\operatorname{Add} T$ the class of all direct summands of modules in $\operatorname{Sum} T$. Further, $\operatorname{add} T$ denotes the class of all direct summands of finite direct sums of copies of T.

Let $\mathcal{C}$ be a class of modules. A module M is $\mathcal{C}$*-filtered* provided there exists a chain of submodules of M, $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$, such that $M_\alpha \subseteq M_{\alpha+1}$ and $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for each $\alpha < \sigma$, $M_0 = 0$, $M_\sigma = M$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$. The chain $\mathcal{M}$ is called a $\mathcal{C}$*-filtration* of M.

We also recall some notation for commutative noetherian rings R: $\operatorname{Spec}(R)$ ($\operatorname{mSpec}(R)$) denotes the spectrum (maximal spectrum) of R. The set of all prime ideals of height n is denoted by P_n, and $\operatorname{Kdim} R$ stands for the Krull dimension of R. For $M \in$ Mod–R, $\operatorname{Ass}_R M$ denotes the set of all associated primes of M. If $\mathcal{C} \subseteq$ Mod–R, then $\operatorname{Ass}_R \mathcal{C} = \bigcup_{M \in \mathcal{C}} \operatorname{Ass}_R M$. For R local, we let $\mathfrak{m}$ denote the unique maximal ideal of R, and $k = R/\mathfrak{m}$ the residue field of R.

The ring R is *Gorenstein* provided that each localization $R_{\mathfrak{m}}$ at a maximal ideal $\mathfrak{m}$ satisfies $\text{inj.dim}_{R_{\mathfrak{m}}} R_{\mathfrak{m}} < \infty$; then $\text{inj.dim}_{R_{\mathfrak{m}}} R_{\mathfrak{m}} = \text{Kdim}\, R_{\mathfrak{m}}$. If R has finite Krull dimension then $\text{Kdim}\, R = \text{inj.dim}_R\, R$.

Further, R is *regular* if each localization $R_{\mathfrak{m}}$ at a maximal ideal $\mathfrak{m}$ satisfies $\text{Kdim}\, R_{\mathfrak{m}} = \dim_k \mathfrak{m}/\mathfrak{m}^2$. We will freely use the classic fact that a local ring R has finite global dimension iff it is regular. In this case R is a UFD, its global dimension equals $\text{Kdim}\, R$, and each prime ideal of R of height 1 is principal, see e.g. [13, §19] or [22, §20].

1.1. **Tilting modules and classes.** We recall the notion of an (infinitely generated) tilting module from [19, §5]:

Definition 1.1. Let R be a ring. A module T is *tilting* provided that

- (T1) T has finite projective dimension,
- (T2) $\text{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $1 \leq i < \omega$ and all cardinals κ.
- (T3) There are an integer $r \geq 0$ and a long exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ where $T_i \in \text{Add}\, T$ for all $i \leq r$.

If $n < \omega$ and T is a tilting module of projective dimension at most n, then T is called an *n-tilting* module. The class $T^{\perp_\infty} = \{M \in \text{Mod–}R \mid \text{Ext}^i_R(T, M) = 0 \text{ for each } i \geq 1\}$ is the *tilting class* induced by T.
If T and T' are tilting modules, then T is said to be *equivalent* to T' provided that the induced tilting classes coincide, that is, $T^{\perp_\infty} = (T')^{\perp_\infty}$, or equivalently, $T' \in \text{Add}\, T$.

A tilting module T is *good* provided that all the modules T_i ($i \leq r$) in condition (T3) can be taken in $\text{add}\, T$.

As mentioned above, if R is commutative, then all finitely generated tilting modules are trivial, so the classical tilting theory reduces to the Morita theory. We start with a short proof of this fact (see [11] for the case of $n = 1$):

Lemma 1.2. *Let R be a commutative ring and T be a finitely generated module.*

- (i) *Assume $T \in \text{mod–}R$ and $1 \leq n = \text{proj.dim}_R T < \infty$. Then $\text{Ext}^n_R(T, T) \neq 0$.*
- (ii) *If T is tilting then T is projective.*

Proof. (i) Let $\mathcal{O}$ be a projective resolution of T consisting of finitely generated modules. Consider $M = \Omega^{(n-1)}(T)$, the $(n-1)$th syzygy of T in $\mathcal{O}$. Then M is a finitely presented module of projective dimension 1, so there is a maximal ideal $\mathfrak{m}$ of R such that $\text{proj.dim}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} = 1$. Moreover, $M_{\mathfrak{m}}$ is the $(n-1)$th syzygy of $T_{\mathfrak{m}}$ in $\mathcal{O}_{\mathfrak{m}}$ where $\mathcal{O}_{\mathfrak{m}}$ is the free resolution of the $R_{\mathfrak{m}}$–module $T_{\mathfrak{m}}$ obtained by applying the localization functor $- \otimes_R R_{\mathfrak{m}}$ to $\mathcal{O}$.

Assume that $\text{Ext}^n_R(T, T) = 0$. Then $\text{Ext}^1_R(M, T) \cong \text{Ext}^n_R(T, T) = 0$, so we have $\text{Ext}^1_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, T_{\mathfrak{m}}) = 0$ by [15, 3.2.5]. Since $0 \neq T_{\mathfrak{m}}$ is finitely generated, $T_\mathfrak{m}$ has a maximal $R_\mathfrak{m}$-submodule, and because $M_\mathfrak{m}$ has projective dimension 1, we infer that $\operatorname{Ext}^1_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}) = 0$.

As $R_\mathfrak{m}$ is a local ring, the finitely presented $R_\mathfrak{m}$-module $M_\mathfrak{m}$ has a projective (= free) cover, so there is an exact sequence $0 \to K \subseteq F \to M_\mathfrak{m} \to 0$ where $0 \neq K$ is a finitely generated superfluous $R_\mathfrak{m}$-submodule of a finitely generated free $R_\mathfrak{m}$-module F. In particular, $K \subseteq \operatorname{Rad}(F)$, and there is an $R_\mathfrak{m}$-epimorphism $f : K \to R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$. As $\operatorname{Ext}^1_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}) = 0$, f can be extended to an $R_\mathfrak{m}$–epimorphism $g : F \to R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$. Then $\operatorname{Ker}(g)$ is a maximal submodule of F, so $K \subseteq \operatorname{Rad}(F) \subseteq \operatorname{Ker}(g)$. This implies that $f = g \restriction K = 0$, a contradiction.

(ii) Assume T is tilting. It suffices to prove that $T \in \text{mod–}R$; then part (i), and conditons (T1) and (T2) of Definition 1.1 yield projectivity of T.

By [19, 5.5.20], T is equivalent to a tilting module T' which is $\mathcal{S}$-filtered, where $\mathcal{S} = {}^\perp(T^{\perp_\infty}) \cap \text{mod–}R$. Then $T \in \operatorname{Add}(T')$, that is, T is a direct summand in $(T')^{(\kappa)}$ for a cardinal κ. Since $M = (T')^{(\kappa)}$ is also $\mathcal{S}$–filtered, an application of the Hill Lemma [19, 4.2.6] yields existence of a finitely $\mathcal{S}$-filtered module $N \subseteq M$ such that T is a direct summand in N. Clearly, $N \in \text{mod–}R$, hence $T \in \text{mod–}R$ as well. $\square$

Remark 1.3. The argument in (ii) that a tilting module T satisfies $T \in \text{mod–}R$, if and only if T is finitely generated, clearly works for any (not necessarily commutative) ring R.

However, already in the commutative non-noetherian setting, the stronger assumption of $T \in \text{mod–}R$ rather than T finitely generated is needed in part (i) of Lemma 1.2, as shown by the following example.

Example 1.4. Let R be a commutative von Neumann regular hereditary ring which is not artinian (for example, for a field K, let R be the ring of all eventually constant sequences $(k_i)_{i<\omega}$ from K^ω). Since R is not artinian, there is a simple module T which is not finitely presented (i.e., not projective). Then $\operatorname{proj.dim}_R T = 1$, and $\operatorname{Ext}^1_R(T, T^{(\kappa)}) = 0$ for all cardinals κ; in fact, $T^{(\kappa)}$ is injective, see [18, 6.18].

1.2. **Divisible modules.** We will consider tilting modules over particular kinds of commutative noetherian rings, so in view of Lemma 1.2, our interest will naturally be in infinitely generated modules. For commutative rings, and for domains in particular, infinitely generated 1-tilting modules are closely related to divisible modules:

Definition 1.5. Let R be a ring. A module M is called *divisible* if $Mr = M$ for each non-zero-divisor $r \in R$. The class of all divisible modules is denoted by $\mathcal{D}$. Note that M is divisible iff $\operatorname{Ext}^1_R(R/rR, M) = 0$ for each non-zero-divisor $r \in R$.

Also the more general classes of relatively divisible modules introduced by Fuchs are relevant here:

Definition 1.6. Let R be a domain and S a multiplicative subset of R. Let $\delta_S = F/G$, where F is the free module with the basis given by all sequences $(s_0, \dots, s_n)$ where $n \geq 0$ with $s_i \in S$ for all $i \leq n$ and the empty sequence $w = ()$; the submodule G is generated by the elements of the form $(s_0, \dots, s_n)s_n - (s_0, \dots, s_{n-1})$, where $0 < n$ and $s_i \in S$ for all $i \leq n$, and of the form $(s)s - w$, where $s \in S$.

In fact, δ_S is a 1–tilting module inducing the 1–tilting class $\mathcal{D}_S$ of all S–*divisible* modules, that is, the modules M with $Ms = M$, or equivalently, $\operatorname{Ext}^1_R(R/sR, M) = 0$, for all $s \in S$, see e.g. [19, 2.1.2]. Clearly, $\mathcal{I}_0 \subseteq \mathcal{D} \subseteq \mathcal{D}_S$.

The divisible module $\delta = \delta_{R\setminus\{0\}}$ was discovered by Fuchs, while Facchini [16] proved that δ is a 1-tilting module in the sense of Definition 1.1. The general case of δ_S studied here comes from [17]; hence we will call δ_S the *Fuchs tilting module*.

Clearly, homomorphic images, direct sums, and direct products of S-divisible modules are S-divisible. While the structure of divisible modules over noetherian domains is unknown in general, the injective modules are described by the classic result of Matlis: they are (uniquely) direct sums of copies of the injective envelopes $E(R/\mathfrak{p})$ of the indecomposable cyclic modules $R/\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec}(R)$.

Moreover, for each $\mathfrak{p} \in \operatorname{Spec}(R)$ we have $E(R/\mathfrak{p}) = \bigcup_{n<\omega} L_{\mathfrak{p},n}$ where $L_{\mathfrak{p},0} = 0$, and $L_{\mathfrak{p},n+1} = \{x \in E(R/\mathfrak{p}) \mid x \cdot \mathfrak{p} \in L_{\mathfrak{p},n}\}$ for each $n < \omega$. So each $x \in E(R/\mathfrak{p})$ is annihilated by a power of $\mathfrak{p}$, while multiplication by any element $x \in R \setminus \mathfrak{p}$ is an automorphism of $E(R/\mathfrak{p})$.

If $\mathfrak{m} \in \operatorname{mSpec}(R)$ then $(L_{\mathfrak{m},n} \mid n < \omega)$ is just the socle-sequence of $E(R/\mathfrak{m})$. In this case $L_{\mathfrak{m},1}$ is simple, $L_{\mathfrak{m},n+1}/L_{\mathfrak{m},n}$ is of finite length for all $1 \leq n < \omega$, so $E(R/\mathfrak{m})$ is countably generated.

Lemma 1.7. *Let R be a noetherian UFD, $\mathfrak{p} \in \operatorname{Spec}(R)$ a prime ideal of height* 1*, and D a divisible submodule of $E(R/\mathfrak{p})$ containing $L_{\mathfrak{p},1}$. Then $D = E(R/\mathfrak{p})$.*

Proof. By induction on n, we prove that $L_{\mathfrak{p},n} \subseteq D$. The case of $n = 1$ is our assumption. Let $x \in L_{\mathfrak{p},n+1} \setminus L_{\mathfrak{p},n}$ and $L_{\mathfrak{p},n} \subseteq D$. Then there is $r \in \mathfrak{p}$ such that $rR = \mathfrak{p}$ and $x \cdot r \in L_{\mathfrak{p},n}$. Since D is divisible, $x \cdot r = d \cdot r$ for some $d \in D$, whence $(x-d) \cdot r = 0$, $x - d \in L_{\mathfrak{p},1} \subseteq D$, and $x \in D$. □

Lemma 1.8. *Let R be a noetherian UFD, and M be a module.*

Then M is divisible, if and only if $\operatorname{Ext}^1_R(R/\mathfrak{p}, M) = 0$ *for each prime ideal of height* 1.

A divisible module M is injective, if and only if $\operatorname{Ext}^1_R(R/\mathfrak{p}, M) = 0$ *for each prime ideal of height* > 1.

Proof. Since R is a noetherian UFD, each prime ideal $\mathfrak{p}$ of height one is principal, see [22, Theorem 20.1]; say $\mathfrak{p} = (r_\mathfrak{p})$. Every non-zero $r \in R$ is up to a unit, a product of $r_\mathfrak{p}$'s, so $Mr = M$ for all r is equivalent to

$Mr_{\mathfrak{p}} = M$ for all $r_{\mathfrak{p}}$. Now the exact sequence $0 \to R \stackrel{r_{\mathfrak{p}}}{\to} R \to R/\mathfrak{p} \to 0$ induces $\mathrm{Ext}^1_R(R/\mathfrak{p}, M) \simeq M/Mr_{\mathfrak{p}}$.

The final claim follows from the well-known version of the Baer lemma for commutative noetherian rings which says that a module M is injective, if and only if $\mathrm{Ext}^1_R(R/\mathfrak{p}, M) = 0$ for each $\mathfrak{p} \in \mathrm{Spec}(R)$. $\square$

Example 1.9. Let R be an n-dimensional regular local ring with the quotient field Q. Then $\mathcal{D} = \mathcal{I}_0$ for $n = 1$, but $Q/R \in \mathcal{D} \setminus \mathcal{I}_0$ for $n \geq 2$.

In the latter case, $Q/R \subsetneq E(Q/R) = \bigoplus_{\mathfrak{p} \in P_1} E(R/\mathfrak{p})$. Let $\mathfrak{p} \in P_1$, so $\mathfrak{p} = r_{\mathfrak{p}} R$ for a prime element $r_{\mathfrak{p}} \in R$. Denote by Q_p the localization of R at the multiplicative set $\{r_{\mathfrak{p}}^k \mid k < \omega\}$. Then $0 \neq Q/R \cap E(R/\mathfrak{p}) = Q_p/R$. Since $Q/R \subsetneq E(Q/R)$, there is a $\mathfrak{p} \in P_1$ such that Q_p/R is a proper divisible submodule of $E(R/\mathfrak{p})$. In particular, $L_{\mathfrak{p},1} \nsubseteq Q/R \cap E(R/\mathfrak{p})$ by Lemma 1.7.

1.3. **Cotorsion pairs.** Each tilting module induces a tilting class, and hence a tilting cotorsion pair in the sense of the following definition.

Let R be a ring and $\mathcal{C}$ a class of modules and $1 \leq i < \omega$. Let

$$\mathcal{C}^{\perp_i} = \bigcap_{1 \leq i < \omega} \mathrm{KerExt}^i_R(\mathcal{C}, -) = \{M \in \text{Mod--}R \mid \mathrm{Ext}^i_R(C, M) = 0 \text{ for all } C \in \mathcal{C}\}$$

$$^{\perp_i}\mathcal{C} = \bigcap_{1 \leq i < \omega} \mathrm{KerExt}^i_R(-, \mathcal{C}) = \{M \in \text{Mod--}R \mid \mathrm{Ext}^i_R(M, C) = 0 \text{ for all } C \in \mathcal{C}\}.$$

We also define $\mathcal{C}^{\perp_\infty} = \bigcap_{1 \leq i < \omega} \mathcal{C}^{\perp_i}$ and $^{\perp_\infty}\mathcal{C} = \bigcap_{1 \leq i < \omega} {}^{\perp_i}\mathcal{C}$.

We will use the following shorthand notation: instead of $\mathcal{C}^{\perp_1}$ and $^{\perp_1}\mathcal{C}$, we will write $\mathcal{C}^{\perp}$ and $^{\perp}\mathcal{C}$, respectively; also, if $\mathcal{C} = \{C\}$ then we will write $C^{\perp}$ and $^{\perp}C$ instead of $\{C\}^{\perp}$ and $^{\perp}\{C\}$, respectively, and similarly for $^{\perp_i}$ and $^{\perp_\infty}$.

Definition 1.10. Let R be a ring. A pair of classes of modules $(\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$.

The cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *hereditary* in case $\mathcal{A} = {}^{\perp_\infty}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_\infty}$, that is, $\mathrm{Ext}^i_R(A, B) = 0$ for all $i \geq 2$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Example 1.11. Let R be a ring, and T be a tilting module inducing the tilting class $\mathcal{T} = T^{\perp_\infty}$. Then $(^{\perp}\mathcal{T}, \mathcal{T})$ is a hereditary cotorsion pair, called the *n-tilting cotorsion pair* induced by T.

A more concrete example comes from a recent result of Bazzoni and Herbera [7]: if R is a domain, then the tilting cotorsion pair induced by the Fuchs tilting module δ is $(\mathcal{P}_1, \mathcal{D})$.

Lemma 1.12. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ a cotorsion pair and $\mathcal{S}$ is a set of modules containing R.*

(i) *Let M be an $\mathcal{A}$-filtered module. Then $M \in \mathcal{A}$.*
(ii) *Assume $\mathcal{B} = \mathcal{S}^{\perp}$. Then $\mathcal{A}$ coincides with the class of all direct summands of $\mathcal{S}$-filtered modules.*

(iii) *Assume R is commutative and noetherian, and $\mathcal{B} = \mathcal{S}^{\perp_\infty}$. Then* $\mathrm{Ass}_R\, \mathcal{A} = \mathrm{Ass}_R\, \mathcal{S}$.

Proof. (i) follows by the Eklof Lemma [14, XII.1.5], (ii) by [19, 3.2.4], and (iii) by [25, Lemma 2.1]. □

Tilting classes and tilting cotorsion pairs can also be characterized in abstract terms (see e.g. [19, §5]):

Lemma 1.13. *Let R be a ring.*

(i) *Let $\mathcal{T}$ be a class of modules. Then $\mathcal{T}$ is an n–tilting class, if and only if there is a subset $\mathcal{S} \subseteq \mathcal{P}_n \cap \text{mod–}R$ such that $\mathcal{T} = \mathcal{S}^{\perp_\infty}$.*

(ii) *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then $(\mathcal{A}, \mathcal{B})$ is an n–tilting cotorsion pair, if and only if $\mathcal{A} \subseteq \mathcal{P}_n$, $\mathcal{B}$ is closed under arbitrary direct sums, and $(\mathcal{A}, \mathcal{B})$ is hereditary.*

We recall the following useful facts concerning tilting cotorsion pairs (see e.g. [19, §5.1]):

Lemma 1.14. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ a tilting cotorsion pair induced by a tilting module T. Then*

(i) $\mathrm{Add}\, T = \mathcal{A} \cap \mathcal{B}$.

(ii) *$\mathcal{A} \subseteq \mathcal{P}_n$, and $\mathcal{A}$ coincides with the class of all modules M possessing a short exact sequence of the form $0 \to M \to T_0 \to \dots T_n \to 0$ where $n = \mathrm{proj.dim}_R\, T$ and $T_0, \dots, T_n \in \mathrm{Add}\, T$.*

(iii) *$\mathcal{B}$ coincides with the class of all modules N possessing an $\mathrm{Add}\, T$-resolution (i.e., a long exact sequence of the form $\dots \to T_{n+1} \to T_n \to \dots \to T_0 \to N \to 0$ where $T_n \in \mathrm{Add}\, T$ for each $n < \omega$).*

1.4. **The class $\mathcal{L}_T$.** If T is any tilting module with the induced tilting cotorsion pair $(\mathcal{A}, \mathcal{B})$, then clearly, $(\mathrm{Add}\, T)^{\perp_\infty} = (T)^{\perp_\infty} = \mathcal{B}$. The description of the left orthogonal class $\mathcal{L}_T \overset{def}{=} {}^{\perp_\infty}(\mathrm{Add}\, T) = {}^{\perp_\infty}(\mathrm{Sum}\, T)$ is more complex in general. Of course, $\mathcal{L}_T \supseteq \mathcal{A}$, and the two classes coincide when restricted to modules of finite projective dimension:

Lemma 1.15. *$\mathcal{A} = \mathcal{L}_T \cap \mathcal{P}$ where $\mathcal{P}$ denotes the class of all modules of finite projective dimension. In particular, $\mathcal{A} = \mathcal{L}_T$ in case R has finite global dimension.*

Proof. The inclusion $\mathcal{A} \subseteq \mathcal{L}_T \cap \mathcal{P}$ is clear from Lemma 1.14.

Conversely, let $M \in \mathcal{L}_T$ and assume $\mathrm{proj.dim}_R\, M = d < \infty$. Take $B \in \mathcal{B}$. By Lemma 1.14, there is an $\mathrm{Add}\, T$–resolution of B

$$\dots \to T_{n+1} \to T_n \to \dots \to T_0 \to B \to 0.$$

For each $n \geq 0$, denote by f_n the map $T_n \to T_{n-1}$ (and $T_{-1} = B$). If $d \geq 1$, then

$$\mathrm{Ext}^1_R(M, B) \cong \mathrm{Ext}^2_R(M, \mathrm{Ker}(f_0)) \cong \dots \cong \mathrm{Ext}^{d+1}_R(M, \mathrm{Ker}(f_{d-1})) = 0,$$

hence $M \in {}^{\perp}\mathcal{B} = \mathcal{A}$. □

In particular, $\mathcal{L}_T = \mathcal{A}$ when R is a regular local ring. However, $\mathcal{A} \subsetneq \mathcal{L}_T$ already for Gorenstein local domains that are not regular, as we will see shortly (for $T = \delta$).

Let R be an Gorenstein ring of Krull dimension n. Then $\mathcal{P}_n = \mathcal{I}_n$, and there is a complete hereditary cotorsion pair $(\mathcal{GP}, \mathcal{I}_n)$. If R has infinite global dimension, then $\mathcal{P}_n \neq \text{Mod}-R$, and the class $\mathcal{GP}$ (called the class of all *Gorenstein projective* modules) contains modules of infinite projective dimension (cf. [15, §10.2]).

For a domain R, let $\mathcal{L}_\delta = {}^{\perp_\infty}(\text{Add}\,\delta)$, and recall [7] that the tilting cotorsion pair induced by δ is $(\mathcal{P}_1, \mathcal{D})$. If R is Gorenstein, we have:

Lemma 1.16. *Let R be a Gorenstein domain and M be a module.*

Then $M \in \mathcal{L}_\delta$, if and only if there is a short exact sequence $0 \to M \to P \to G \to 0$ where $P \in \mathcal{P}_1$ and $G \in \mathcal{GP}$.

In particular, if R is a Gorenstein local domain of Krull dimension one which is not regular, then the tilting module δ is injective and $\mathcal{P}_1 \subsetneq \mathcal{L}_\delta = \text{Mod}-R$.

Proof. Let $M \in \text{Mod}-R$. The completeness of the cotorsion pair $(\mathcal{GP}, \mathcal{I}_n)$ yields an exact sequence $0 \to M \to P \to G \to 0$ with $P \in \mathcal{I}_n$ and $G \in \mathcal{GP}$.

Since $\delta \in \mathcal{P}_1 \subseteq \mathcal{I}_n$, we have $\mathcal{L}_\delta \supseteq \mathcal{GP}$.

Now, if $M \in \mathcal{L}_\delta$ then $P \in \mathcal{I}_n \cap \mathcal{L}_\delta = \mathcal{P}_1$ by Lemma 1.15. Conversely, if the middle term $P \in \mathcal{P}_1 (\subseteq \mathcal{L}_\delta)$, then also the left hand term $M \in \mathcal{L}_\delta$ because $\mathcal{L}_\delta$ is resolving.

If R is Gorenstein of Krull dimension one, then $\mathcal{I}_1 = \mathcal{P}_1$, so the final claim folows from the first part and from the completeness of the cotorsion pair $(\mathcal{GP}, \mathcal{I}_1)$. □

Example 1.17. Let R be a Gorenstein local domain of Krull dimension one which is not regular (for instance, $R = k[[x^2, x^3]]$, the ring of all power series over a field k with no x term, that is, the Herzog–Kunz semigroup ring with conductor 2, [13, 21.11, p.553]). Then by [25, Lemma 1.2], up to equivalence, there are only two tilting modules: R and $Q \oplus Q/R$, and hence two tilting classes: Mod–R, and $\mathcal{D}$. So in this case, δ is equivalent to the injective module $Q \oplus Q/R$.

2. Tilting classes in $\mathcal{I}_1$

In this section, we will characterize the tilting classes $\mathcal{T}$ over commutative noetherian rings R with gl.dim$R < \infty$ such that $\mathcal{T}$ consist of modules of injective dimension ≤ 1. We start with a lemma that indicates the role of $\text{Ass}_R\,\mathcal{A}$ in this setting:

Lemma 2.1. *Let R be a commutative noetherian ring and $M \in \text{mod}-R$. Then there exists a chain $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ of submodules of M such that for each $i = 0, \ldots, n-1$, the module M_{i+1}/M_i is isomorphic to a submodule of $R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \text{Ass}_R\,M$.*

Proof. Take $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p}$ is $\subseteq$-maximal in the set $\operatorname{Ass}_R M$. Define $M_1 = \{m \in M \mid m \cdot \mathfrak{p} = 0\}$. Then M_1 is a non-zero submodule of M. Each prime ideal in $\operatorname{Ass}_R M_1$ has to contain $\mathfrak{p}$, so the maximality of $\mathfrak{p}$ yields $\operatorname{Ass}_R M_1 = \{\mathfrak{p}\}$.

Consider M_1 as an $R/\mathfrak{p}$-module. Suppose that there is $0 \neq x \in M_1$ such that $(r + \mathfrak{p})x = 0$ for some $0 \neq (r + \mathfrak{p}) \in R/\mathfrak{p}$. Then $\operatorname{Ann}(x) \supsetneq \mathfrak{p}$, so there is a prime ideal $\mathfrak{q} \supsetneq \mathfrak{p}$ such that $\mathfrak{q} \in \operatorname{Ass}_R M_1$, in contradiction with the maximality of $\mathfrak{p}$. This proves that M_1 is a torsion-free $R/\mathfrak{p}$-module. Since $R/\mathfrak{p}$ is a domain and M_1 is a finitely generated $R/\mathfrak{p}$-module, M_1 is isomorphic to a submodule of the $R/\mathfrak{p}$-module $(R/\mathfrak{p})^{k_0}$ for some integer $k_0 < \omega$ [10, VII.2.5]; this is clearly also an R-isomorphism.

Assume $M_1 \subsetneq M$ (so in particular, $\mathfrak{p} \neq 0$). We will show that $\operatorname{Ass}_R M \supseteq \operatorname{Ass}_R M/M_1$. Take an arbitrary $\mathfrak{q} \in \operatorname{Ass}_R M/M_1$ and distinguish two cases:

Case I: $\mathfrak{p} \not\subseteq \mathfrak{q}$. So there is $x \in \mathfrak{p}$ such that $x \in R \setminus \mathfrak{q}$. It follows that $(M_1)_\mathfrak{q} = 0$. Applying $- \otimes_R R_\mathfrak{q}$ to the short exact sequence $M_1 \to M \to M/M_1 \to 0$ we get $0 \to M_\mathfrak{q} \to (M/M_1)_\mathfrak{q} \to 0$. So $M_\mathfrak{q} \simeq (M/M_1)_\mathfrak{q}$ as $R_\mathfrak{q}$-modules. Since $\mathfrak{q} \in \operatorname{Ass}_R M/M_1$ we have $\mathfrak{q}R_\mathfrak{q} \in \operatorname{Ass}_{R_\mathfrak{q}} (M/M_1)_\mathfrak{q} = \operatorname{Ass}_{R_\mathfrak{q}} M_\mathfrak{q}$, so $\mathfrak{q} \in \operatorname{Ass}_R M$.

Case II: $\mathfrak{p} \subseteq \mathfrak{q}$. If $\mathfrak{p} = \mathfrak{q}$ we are done. So assume that $\mathfrak{p} \subsetneq \mathfrak{q}$. Take a non-zero element $x \in M \setminus M_1$ such that $\mathfrak{q}$ is the annihilator of $x + M_1$ in M/M_1. Since $x \notin M_1$, there is $0 \neq y \in \mathfrak{p}$ such that $x \cdot y \neq 0$. But then $x \cdot y$ is a non-zero element of M such that $\mathfrak{q} \cdot x \cdot y \in (M_1)y = 0$. Thus $\mathfrak{q}$ or a larger prime ideal is in $\operatorname{Ass}_R M$, in contradiction with the maximality of $\mathfrak{p}$.

Now, we can replace M by M/M_1 and repeating the previous procedure to obtain $M_1 \subsetneq M_2 \subseteq M$ such that the R-module M_2/M_1 is isomorphic to a submodule of the R-module $(R/\mathfrak{p})^{k_1}$ where $\mathfrak{p} \in \operatorname{Ass}_R M$ and $k_1 < \omega$. Since M is noetherian, the procedure stops, and yields a chain

$$(*) \quad 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

of submodules of M such that for each $i = 0, \ldots, n-1$, the module M_{i+1}/M_i is isomorphic to a submodule of $(R/\mathfrak{p}_i)^{k_i}$ for some $\mathfrak{p}_i \in \operatorname{Ass}_R M$ and $k_i < \omega$.

Notice that if $N \subseteq (R/\mathfrak{p})^k$ for some $\mathfrak{p} \in \operatorname{Ass}_R M$ and $0 < k < \omega$, then $0 \subseteq N \cap R/\mathfrak{p} \subseteq N \cap (R/\mathfrak{p})^2 \subseteq \cdots \subseteq N \cap (R/\mathfrak{p})^k = N$ is an $\mathcal{S}$-filtration of N where $\mathcal{S}$ is the set of all submodules of $R/\mathfrak{p}$. So the chain $(*)$ can be refined to one having consecutive factors isomorphic to submodules of $R/\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Ass}_R M$. □

The next lemma gives first consequences of the condition $\mathcal{B} \subseteq \mathcal{I}_1$:

Lemma 2.2. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair.*

(i) *$\mathcal{B} \subseteq \mathcal{I}_1$, if and only if the class $\mathcal{A}$ is closed under submodules.*

(ii) *Assume that* $\mathcal{B}$ *is a tilting class and* R *is commutative and noetherian. Let* $X = \operatorname{Ass}_R \mathcal{A}$ *and* $\mathcal{S} = \{N \in \text{mod–}R \mid N \subseteq R/\mathfrak{p}$ *for some* $\mathfrak{p} \in X\}$*. Then the equivalent conditions from (i) imply that* $\mathcal{S}^\perp = \mathcal{B}$ *and* $\mathcal{A}$ *coincides with the class of all* $\mathcal{S}$*-filtered modules.*

Proof. (i) If $\mathcal{A}$ is closed under submodules, $B \in \mathcal{B}$, and $M \in \text{Mod–}R$, then $\operatorname{Ext}^2_R(M, B) \cong \operatorname{Ext}^1_R(\Omega^1(M), B)$. But $\Omega^1(M) \in \mathcal{A}$ by assumption, so $\operatorname{Ext}^2_R(M, B) = 0$. Conversely, if $B \in \mathcal{I}_1$ then ${}^\perp B$ is closed under submodules and the assertion is clear.

(ii) Denote by $\mathcal{F}$ the class of all $\mathcal{S}$-filtered modules. Assume $\mathcal{A}$ is closed under submodules. Then $\mathcal{S} \subseteq \mathcal{A}$, and hence $\mathcal{F} \subseteq \mathcal{A}$ by Lemma 1.12(i). Since $\mathcal{B}$ is tilting, $\mathcal{B} = (\mathcal{A} \cap \text{mod–}R)^\perp$ by Lemma 1.13(i). However, each $M \in \mathcal{A} \cap \text{mod–}R$ is $\mathcal{S}$-filtered by Lemma 2.1, so $\mathcal{S}^\perp \subseteq \mathcal{A}^\perp = \mathcal{B}$ by Lemma 1.12(i). Then $\mathcal{A} = {}^\perp(\mathcal{S}^\perp)$ consists of direct summands of $\mathcal{S}$-filtered modules by Lemma 1.12(ii). Since $\mathcal{S}$ is closed under submodules, so is the class $\mathcal{F}$, and we conclude that $\mathcal{A} = \mathcal{F}$. □

Now, we can prove a structure theorem:

Theorem 2.3. *Let* R *be a regular ring of finite Krull dimension. Then the tilting classes* $\mathcal{T} \subseteq \mathcal{I}_1$ *are classified by the subsets* $P_0 \subseteq X \subseteq \operatorname{Spec}(R)$*.*

For each such subset X*, a tilting class* $\mathcal{T}_X$ *is defined by*

$$\mathcal{T}_X = \bigcap_{\mathfrak{p} \in X} (R/\mathfrak{p})^\perp \cap \mathcal{I}_1.$$

Conversely, each tilting class $\mathcal{T} \subseteq \mathcal{I}_1$ *is of the form* $\mathcal{T} = \mathcal{T}_X$ *for* $X = \operatorname{Ass}_R {}^\perp\mathcal{T}$*.*

Moreover, ${}^\perp\mathcal{T}_X$ *coincides with the class of all* $\mathcal{S}_X$*-filtered modules, where* $\mathcal{S}_X = \{N \in \text{mod–}R \mid N \subseteq R/\mathfrak{p}$ *for some* $\mathfrak{p} \in X\}$*.*

Proof. First, for each subset $X \subseteq \operatorname{Spec}(R)$,

$$\mathcal{T}_X = \bigcap_{\mathfrak{p} \in X} (R/\mathfrak{p})^\perp \cap \mathcal{I}_1 = \bigcap_{\mathfrak{p} \in X} (R/\mathfrak{p})^{\perp_\infty} \cap \bigcap_{I \subseteq R} I^{\perp_\infty}.$$

This is a tilting class contained in $\mathcal{I}_1$ by Lemma 1.13(i).

Conversely, let $\mathcal{T} \subseteq \mathcal{I}_1$ be a tilting class with the induced tilting cotorsion pair $(\mathcal{A}, \mathcal{T})$. Let $X = \operatorname{Ass}_R \mathcal{A}$. Then $P_0 \subseteq X$ because $P_0 = \operatorname{Ass}_R R$, and $\mathcal{T} = \mathcal{S}_X^\perp$ by Lemma 2.2(ii). Since $\mathcal{T}_X \subseteq \mathcal{I}_1$, Lemma 2.2(i) gives $\mathcal{S}_X \subseteq {}^\perp\mathcal{T}_X$, and $\mathcal{T} = \mathcal{T}_X$ because $\mathcal{S}_X^\perp \subseteq \mathcal{I}_1$. As $X = \operatorname{Ass}_R {}^\perp\mathcal{T}_X$, we have $\mathcal{T}_X \neq \mathcal{T}_{X'}$ for $X \neq X'$.

The final claim follows again from Lemma 2.2(ii). □

The next corollary gives a useful test for equality of tilting classes of the kind studied in this section:

Corollary 2.4. *Let R be a regular ring of finite Krull dimension, $\mathcal{T}$, $\mathcal{T}'$ be tilting classes contained in $\mathcal{I}_1$, and let $\mathcal{S}, \mathcal{S}' \subseteq$ mod–R satisfy $R \in \mathcal{S} \cap \mathcal{S}'$, $\mathcal{T} = \mathcal{S}^{\perp_\infty}$ and $\mathcal{T}' = (\mathcal{S}')^{\perp_\infty}$.*

Then $\mathcal{T} = \mathcal{T}'$, if and only if $\operatorname{Ass}_R \mathcal{S} = \operatorname{Ass}_R \mathcal{S}'$.

Proof. By Theorem 2.3, the class $\mathcal{T}$ is determined by the set $\operatorname{Ass}_R {}^{\perp}\mathcal{T}$, where the latter equals $\operatorname{Ass}_R \mathcal{S}$ by Lemma 1.12(iii); similarly for $\mathcal{T}'$. □

3. Tilting classes in the local case

In this section, we concentrate on the case when R is a regular local ring of Krull dimension n where $1 \leq n < \omega$.

Recall that $\operatorname{depth} R = n$, and R is a UFD. In particular, each $\mathfrak{p} \in P_1$ is principal, $\mathfrak{p} = r_\mathfrak{p} R$ for a prime element $r_\mathfrak{p} \in R$. Denote by $\{r_\mathfrak{p} \mid \mathfrak{p} \in P_1\}$ a representative set of all prime elements of R. Then each non-zero non-invertible element $r \in R$ is uniquely of the form $r = u \prod_{\mathfrak{p} \in P_1} r_\mathfrak{p}^{n_\mathfrak{p}}$ where $n_\mathfrak{p}$ ($\mathfrak{p} \in P_1$) are natural numbers, almost all of them zero, and $u \in R$ is invertible.

In [25], it was observed that if R is a Gorenstein local ring of Krull dimension one, then all non-projective tilting modules T are injective, and the induced tilting class $\mathcal{B} = T^{\perp_\infty}$ is the class of all Gorenstein injective modules. In particular, if R is regular, $\mathcal{B} = \mathcal{I}_0$. We will now prove a similar result for n-dimensional regular local rings when $n \geq 2$.

First, we recall several classic facts in our particular setting (see e.g. [15, §9]):

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a regular local ring of Krull dimension $n \geq 1$ and $M, N \in$ mod–R. Then*

(i) $\operatorname{depth} M = \inf \{i \mid \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\}$.

(ii) $\operatorname{proj.dim}_R M = n - \operatorname{depth} M$. *In particular,* $\operatorname{proj.dim}_R M = n$, *if and only if* $\mathfrak{m} \in \operatorname{Ass}_R M$.

(iii) $\operatorname{inj.dim}_R N = n = \operatorname{depth} M + \sup \{i \mid \operatorname{Ext}^i_R(M, N) \neq 0\}$.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a regular local ring of Krull dimension $n \geq 2$. Let T be a tilting module of projective dimension $\geq n - 1$. Let $\mathcal{T} = T^{\perp_\infty}$ be the induced tilting class. Then $\mathcal{T} \subseteq \mathcal{I}_{n-1}$.*

Proof. We have to prove that $\operatorname{Ext}^n_R(R/\mathfrak{p}, B) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and $B \in \mathcal{T}$. However, if $\mathfrak{p} \neq \mathfrak{m}$, then $\mathfrak{m} \notin \operatorname{Ass}_R R/\mathfrak{p}$, so $\operatorname{proj.dim}_R R/\mathfrak{p} \leq n - 1$ by Lemma 3.1(ii).

It remains to show that $\operatorname{Ext}^n_R(R/\mathfrak{m}, B) = 0$ for all $B \in \mathcal{T}$. By Lemma 1.13(i), the tilting class $\mathcal{T}$ is of the form $\mathcal{T} = \mathcal{S}^{\perp_\infty}$ for some $\mathcal{S} \subseteq$ mod–R. Let $X = \operatorname{Ass}_R \mathcal{S}$. We distinguish two cases:

Case I: $\mathfrak{m} \in X$. Then there is an exact sequence $0 \to R/\mathfrak{m} \to S \to N \to 0$ for some $S \in \mathcal{S}$. For each $B \in \mathcal{B}$, an application of $\operatorname{Hom}_R(-, B)$ to this sequence yields exactness of $0 = \operatorname{Ext}^n_R(S, B) \to \operatorname{Ext}^n_R(R/\mathfrak{m}, B) \to \operatorname{Ext}^{n+1}_R(N, B) = 0$, so we infer that $\operatorname{Ext}^n_R(R/\mathfrak{m}, B) = 0$ as desired.

Case II: $\mathfrak{m} \notin X$. By the assumption on T we have $\mathcal{S} \nsubseteq \mathcal{P}_{n-2}$. Consider $M \in \mathcal{S} \setminus \mathcal{P}_{n-2}$. Since $\mathfrak{m} \notin \mathrm{Ass}_R M$, $\mathrm{proj.dim}_R M = n-1$ by Lemma 3.1(ii). So $\mathrm{Ext}^1_R(R/\mathfrak{m}, M) \neq 0$ by Lemma 3.1(i) and (ii), and there is a non-split short exact sequence $0 \to M \to N \to R/\mathfrak{m} \to 0$. Since the sequence does not split, $\mathfrak{m} \notin \mathrm{Ass}_R N$, whence $\mathrm{proj.dim}_R N \leq n-1$.

Let K be a module. Applying the functor $\mathrm{Hom}_R(-, K)$ to the short exact sequence above, we obtain exactness of

$$\mathrm{Ext}^{n-1}_R(M, K) \to \mathrm{Ext}^n_R(R/\mathfrak{m}, K) \to \mathrm{Ext}^n_R(N, K).$$

But the latter Ext is zero because $N \in \mathcal{P}_{n-1}$. It follows that $M^{\perp_{n-1}} \subseteq (R/\mathfrak{m})^{\perp_n}$. So $\mathcal{T} \subseteq \mathcal{S}^{\perp_{n-1}} \subseteq M^{\perp_{n-1}} \subseteq (R/\mathfrak{m})^{\perp_n}$. □

There is another case where the injective dimension of a tilting class is always bounded by $n-1$, namely the case of divisible modules:

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a regular local ring of Krull dimension $n \geq 2$. Then $\mathcal{D} \subseteq \mathcal{I}_{n-1}$, but $\mathcal{D} \nsubseteq \mathcal{I}_{n-2}$.*

Proof. As above, for the first claim, we only have to check that $\mathrm{Ext}^n_R(R/\mathfrak{m}, D) = 0$ for each $D \in \mathcal{D}$. Let $N = \Omega^{n-1}(R/\mathfrak{m})$. Then $\mathrm{proj.dim}_R N = 1$, so $\mathrm{Ext}^n_R(R/\mathfrak{m}, D) \cong \mathrm{Ext}^1_R(N, D) = 0$ for each divisible module D, because the 1-tilting cotorsion pair induced by δ is $(\mathcal{P}_1, \mathcal{D})$.

For the second claim, consider the divisible module $D = Q/R$. Since R is Gorenstein of Krull dimension n, the minimal injective coresolution of R has the form

$$0 \to R \to Q \to \bigoplus_{\mathfrak{p} \in P_1} E(R/\mathfrak{p}) \to \cdots \to E(R/\mathfrak{m}) \to 0,$$

and D is the first cosyzygy of R. So $\mathrm{inj.dim}_R D = n-1$. □

In the two-dimensional case, we infer that all non-trivial tilting classes fit in the setting of Section 2:

Corollary 3.4. *Let R be a regular local ring of Krull dimension 2, T a non-projective tilting module, and $\mathcal{T} = T^{\perp_\infty}$ the induced tilting class.*

Then $\mathcal{T} \subseteq \mathcal{I}_1$, and $\mathcal{T}$ contains no non-zero finitely generated modules.

Proof. Theorem 3.2 for $n = 2$ yields $\mathcal{T} \subseteq \mathcal{I}_1$. The final claim follows by Lemma 3.1(iii), as all non-zero finitely generated modules have injective dimension 2. □

In particular, Lemma 1.2(ii) holds in a stronger form here: if T is a non-projective tilting module, then T has no non-zero finitely generated direct summands.

Theorem 2.3 now applies directly and gives

Theorem 3.5. *Let R be a regular local ring of Krull dimension 2. Then all tilting classes $\mathcal{T} \subsetneq \mathrm{Mod}\text{–}R$ are contained in $\mathcal{I}_1$, and they are classified by the subsets $X \subseteq \mathrm{Spec}(R)$ containing the zero ideal.*

For each such subset X, a tilting class $\mathcal{T}_X$ is defined by

$$\mathcal{T}_X = \bigcap_{\mathfrak{p} \in X} (R/\mathfrak{p})^\perp \cap \mathcal{I}_1.$$

Conversely, each tilting class $\mathcal{T} \subsetneq$ Mod–R is of the form $\mathcal{T} = \mathcal{T}_X$ for $X = \mathrm{Ass}_R\, {}^\perp\mathcal{T}$.

Moreover, ${}^\perp\mathcal{T}_X$ coincides with the class of all $\mathcal{S}_X$-filtered modules, where $\mathcal{S}_X = \{N \in \text{Mod–}R \mid N \subseteq R/\mathfrak{p} \text{ for some } \mathfrak{p} \in X\}$.

4. The global case

The tools needed for a transfer from the local to the global case are collected in the following lemma. For a class $\mathcal{T} \subseteq$ Mod–R and a multiplicative subset S of a commutative ring R, we denote by $S^{-1}\mathcal{T}$ the class $\{N \in \text{Mod–}S^{-1}R \mid N \cong S^{-1}M \text{ for some } M \in \mathcal{T}\}$; for $\mathfrak{p} \in \mathrm{Spec}(R)$ and $S = R \setminus \mathfrak{p}$, we will also use the notation $\mathcal{T}_\mathfrak{p} = S^{-1}\mathcal{T}$.

Lemma 4.1. *Let R be a commutative noetherian ring, $n > 0$, and T be an n-tilting module inducing the tilting class $\mathcal{T} = T^{\perp_\infty}$ and the cotorsion pair $(\mathcal{A}, \mathcal{T})$. Let $\mathcal{S} \subseteq \text{mod–}R \cap \mathcal{P}_n$ be such that $\mathcal{S}^{\perp_\infty} = \mathcal{T}$.*

(i) *Let S be a multiplicative subset of R. Then the localization $S^{-1}T$ is an n-tilting module inducing the tilting class*

$$(S^{-1}\mathcal{S})^{\perp_\infty} = S^{-1}\mathcal{T} = \mathcal{T} \cap \text{Mod–}S^{-1}R$$

(where ${}^{\perp_\infty}$ is considered in Mod–$S^{-1}R$*).*

(ii) *Let $M \in$ Mod–R. Then $M \in \mathcal{T}$, if and only if $M_\mathfrak{m} \in \mathcal{T}_\mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{mSpec}(R)$.*

(iii) *Let $\mathfrak{m} \in \mathrm{mSpec}(R)$. Let $\mathcal{C}_\mathfrak{m}$ denote the class of all direct summands of the elements of $(\mathcal{A} \cap \text{mod–}R)_\mathfrak{m}$. Then $\mathcal{C}_\mathfrak{m} = {}^\perp\mathcal{T}_\mathfrak{m} \cap$ mod–$R_\mathfrak{m}$ (where ${}^\perp$ is considered in* Mod–$R_\mathfrak{m}$*).*

Proof. (i) and (ii) follow by [3, Proposition 4.3] (see also [19, 5.2.24]).

(iii) First, recall that for each $M \in$ Mod–R, $F \in$ mod–R, and $i > 0$, we have

$$\mathrm{Ext}^i_{R_\mathfrak{m}}(F_\mathfrak{m}, M_\mathfrak{m}) \cong (\mathrm{Ext}^i_R(F, M))_\mathfrak{m} \cong \mathrm{Ext}^i_R(F, M_\mathfrak{m})$$

by [15, 3.2.6 and 3.2.15]. So $\mathcal{A}_\mathfrak{m} \subseteq {}^\perp\mathcal{T}_\mathfrak{m}$, and $\mathcal{C}_\mathfrak{m} \subseteq ({}^\perp\mathcal{T}_\mathfrak{m}) \cap \text{mod–}R_\mathfrak{m}$.

Conversely, $(\mathcal{A} \cap \text{mod–}R)_\mathfrak{m}^\perp = \mathcal{T}_\mathfrak{m}$ by part (i), so ${}^\perp\mathcal{T}_\mathfrak{m}$ is the class of all direct summands of $(\mathcal{A} \cap \text{mod–}R)_\mathfrak{m}$-filtered modules by Lemma 1.12(ii). However, $(\mathcal{A} \cap \text{mod–}R)_\mathfrak{m}$ is closed under extensions, so $({}^\perp\mathcal{T}_\mathfrak{m}) \cap \text{mod–}R_\mathfrak{m} \subseteq \mathcal{C}_\mathfrak{m}$. □

Since the 1-dimensional case has already been treated in [25], we will consider here the case of Krull dimension 2. Recalling that for a set $X \subseteq \mathrm{Spec}(R)$, $V(X)$ denotes the set of all prime ideals of R containing at least one element of X, we can now formulate our main result:

Theorem 4.2. *Let R be a regular ring of Krull dimension 2.*

Then tilting classes in $\mathrm{Mod}\text{–}R$ *are classified by the pairs* (X, Y) *where* $\mathrm{Ass}_R R \subseteq X \subseteq \mathrm{Spec}(R)$ *and* $V(X \setminus \mathrm{Ass}_R R) \cap P_2 \subseteq Y \subseteq P_2$.

For each such pair (X, Y)*, a tilting class* $\mathcal{T}_{X,Y}$ *is defined by*

$$\mathcal{T}_{X,Y} = \bigcap_{\mathfrak{p} \in X} (R/\mathfrak{p})^{\perp_\infty} \cap \bigcap_{\mathfrak{m} \in Y} m^{\perp}.$$

Conversely, each tilting class $\mathcal{T}$ *in* $\mathrm{Mod}\text{–}R$ *is of this form, for* $X = \mathrm{Ass}_R {}^{\perp}\mathcal{T}$ *and* $Y = P_2 \cap {}^{\perp}\mathcal{T}$.

Proof. By assumption, $\mathrm{proj.dim}_R \, \mathfrak{m} = \mathrm{proj.dim}_{R_\mathfrak{m}} \mathfrak{m}_\mathfrak{m} = 1$ for each $\mathfrak{m} \in P_2$, hence $\mathfrak{m}^\perp = \mathfrak{m}^{\perp_\infty}$. So the finitely generated modules $R/\mathfrak{p}$ ($\mathfrak{p} \in X$) and $\mathfrak{m}$ ($\mathfrak{m} \in Y$) have projective dimension at most 2, and $\mathcal{T}_{X,Y}$ is a 2-tilting class by Lemma 1.13(i) (Note that this includes the trivial case of $X = \mathrm{Ass}_R R$ and $Y = \emptyset$, when $\mathcal{T}_{X,Y} = \mathrm{Mod}\text{–}R$ since $R/\mathfrak{p}$ is a projective R-module for $\mathfrak{p} \in \mathrm{Ass}_R R = P_0$).

Conversely, let $\mathcal{T}$ be a tilting class and $\mathcal{A} = {}^{\perp}\mathcal{T}$. Let $X = \mathrm{Ass}_R \mathcal{A}$ and $Y = P_2 \cap \mathcal{A}$.

If $\mathfrak{m} \in P_2$ and $\mathfrak{p} \in X \setminus \mathrm{Ass}_R R$ are such that $\mathfrak{p} \subseteq \mathfrak{m}$, then $\mathfrak{p}_\mathfrak{m} \in \mathrm{Ass}_{R_\mathfrak{m}} {}^{\perp}\mathcal{T}_\mathfrak{m}$, so $R_\mathfrak{m}/\mathfrak{p}_\mathfrak{m} \in {}^{\perp}\mathcal{T}_\mathfrak{m}$, $\mathcal{T}_\mathfrak{m} \neq \mathrm{Mod}\text{–}R_\mathfrak{m}$, and $\mathfrak{m}_\mathfrak{m} \in {}^{\perp}\mathcal{T}_\mathfrak{m}$ by Theorem 3.5, hence $\mathfrak{m} \in Y$ by Lemma 4.1(iii).

We will show that $\mathcal{T} = \mathcal{T}_{X,Y}$. By Lemma 4.1(ii), it suffices to show that for each $\mathfrak{m} \in \mathrm{mSpec}(R)$, $\mathcal{T}_\mathfrak{m} = (\mathcal{T}_{X,Y})_\mathfrak{m}$.

First, assume that $\mathfrak{m} \in P_2 \setminus Y$. Then $\mathcal{T}_\mathfrak{m} = \mathrm{Mod}\text{–}R_\mathfrak{m} = (\mathcal{T}_{\mathcal{X},\mathcal{Y}})_\mathfrak{m}$ by Lemma 4.1(i), because $\mathfrak{m} \not\supseteq \mathfrak{p}$ for all $\mathfrak{p} \in (X \cup Y) \setminus \mathrm{Ass}_R R$.

Now assume that either $\mathfrak{m} \in P_1 \cap \mathrm{mSpec}(R)$ or $\mathfrak{m} \in Y$. In the former case, $R_\mathfrak{m}$ is hereditary, and in the latter $\mathfrak{m} \in \mathcal{A}$, hence in both cases $\mathcal{T}_\mathfrak{m}$ consists of $R_\mathfrak{m}$-modules of injective dimension ≤ 1. By Lemma 4.1(iii) and Corollary 2.4, it suffices to show that $\mathrm{Ass}_{R_\mathfrak{m}} (\mathcal{A} \cap \mathrm{mod}\text{–}R)_\mathfrak{m} = \mathrm{Ass}_{R_\mathfrak{m}} ({}^{\perp}\mathcal{T}_{(X,Y)} \cap \mathrm{mod}\text{–}R)_\mathfrak{m}$. Since there is a bijective correspondence between the sets $\mathrm{Ass}_R M \cap \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \subseteq \mathfrak{m}\}$ and $\mathrm{Ass}_{R_\mathfrak{m}} M_\mathfrak{m}$ for each module $M \in \mathrm{Mod}\text{–}R$ because R is noetherian, it is enough to prove that $\mathrm{Ass}_R (\mathcal{A} \cap \mathrm{mod}\text{–}R) = \mathrm{Ass}_R ({}^{\perp}\mathcal{T}_{(X,Y)} \cap \mathrm{mod}\text{–}R)$. However, $\mathrm{Ass}_R (\mathcal{A} \cap \mathrm{mod}\text{–}R) = \mathrm{Ass}_R \mathcal{A} = X$ by definition, while $\mathrm{Ass}_R ({}^{\perp}\mathcal{T}_{(X,Y)} \cap \mathrm{mod}\text{–}R) = X$ by Lemma 1.12(iii).

If $\mathfrak{m} \in \mathrm{Ass}_R R = P_0$, then $\mathcal{T}_\mathfrak{m} = \mathrm{Mod}\text{–}R_\mathfrak{m} = (\mathcal{T}_{\mathcal{X},\mathcal{Y}})_\mathfrak{m}$ because $R_\mathfrak{m}$ is a field.

Finally, note that the pair (X, Y) is completely determined by the class $\mathcal{A}$, and hence by $\mathcal{T}$. This finishes the classification. □

5. Tilting modules in the local case

In this section, we will return to the setting of regular local rings $(R, \mathfrak{m})$ of Krull dimension 2 and compute the representing tilting modules.

Then $P_1 \subseteq \mathcal{P}_0$ implies $\mathfrak{m}^\perp = (R/\mathfrak{m})^{\perp_2} = \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(R)} (R/\mathfrak{p})^{\perp_2} = \mathcal{I}_1$.

Moreover, since R is Gorenstein of Krull dimension two, the minimal injective coresolution of R has the form

$$0 \to R \to Q \to \bigoplus_{\mathfrak{p}\in P_1} E(R/\mathfrak{p}) \to E(R/\mathfrak{m}) \to 0$$

and it is obtained by glueing together the following two short exact sequences: $0 \to R \to Q \to Q/R \to 0$, and $0 \to Q/R \to \bigoplus_{\mathfrak{p}\in P_1} E(R/\mathfrak{p}) \to E(R/\mathfrak{m}) \to 0$. Notice that Q/R is a proper divisible, but non-injective, submodule of $\bigoplus_{\mathfrak{p}\in P_1} E(R/\mathfrak{p})$ (see Example 1.9).

We will present the structure of the tilting modules T_X inducing the classes $\mathcal{T}_X$ from Theorem 3.5, that is, such that $T_X^{\perp_\infty} = \mathcal{T}_X$.

Our first result concerns the form of the minimal injective coresolution of T_X:

Lemma 5.1. *Let $(R, \mathfrak{m})$ be a regular local ring of Krull dimension 2. Let X be a subset of* $\operatorname{Spec}(R)$ *containing the zero ideal. Then the minimal injective coresolution of T_X has the form*

$$0 \to T_X \to \bigoplus_{\mathfrak{p}\in X} E(R/\mathfrak{p})^{(\alpha_\mathfrak{p})} \to \bigoplus_{\mathfrak{p}\in\operatorname{Spec}(R)\setminus X} E(R/\mathfrak{p})^{(\alpha_\mathfrak{p})} \to 0$$

where $\alpha_\mathfrak{p}$ ($\mathfrak{p} \in \operatorname{Spec}(R)$) are non-zero cardinals.

Proof. By Theorem 3.5, $\operatorname{Ass}_R T_X = \operatorname{Ass}_R {}^\perp\mathcal{T}_X = X$, so $E(T_X) \cong \bigoplus_{\mathfrak{p}\in X} E(R/\mathfrak{p})^{(\alpha_\mathfrak{p})}$ for some non-zero cardinals $\alpha_\mathfrak{p}$ ($\mathfrak{p} \in X$).

Since $T_X \in \mathcal{I}_1$, it remains to determine the first Bass invariants of T_X. For $\mathfrak{p} \in \operatorname{Spec}(R)$, they are computed as $\mu_1(\mathfrak{p}, T_X) = \dim_{k(\mathfrak{p})}(\operatorname{Ext}^1_R(R/\mathfrak{p}, T_X))_\mathfrak{p}$ (cf. [15, 9.2.4]). If $\mathfrak{p} \in X$ then $T_X \in (R/\mathfrak{p})^\perp$, hence $\mu_1(\mathfrak{p}, T_X) = 0$. If $\mathfrak{q} \in \operatorname{Spec}(R) \setminus X$, then $\operatorname{Ext}^1_R(R/\mathfrak{q}, T_X) \neq 0$, hence $\alpha_\mathfrak{q} = \mu_1(\mathfrak{q}, T_X) \neq 0$ in case $\mathfrak{q} = \mathfrak{m}$ (because R is local). If ht $\mathfrak{q} = 1$, then $R_\mathfrak{q}$ is a DVR, but not a field. By Theorem 3.5, T_X is $\mathcal{S}_X$-filtered, hence $(T_X)_\mathfrak{q}$ is $(\mathcal{S}_X)_\mathfrak{q}$-filtered. However, $(R/\mathfrak{p})_\mathfrak{q} = 0$ for all $\mathfrak{p} \in X \setminus \{0\}$, and since all ideals of $R_\mathfrak{q}$ are principal, $(T_X)_\mathfrak{q}$ is a free $R_\mathfrak{q}$-module. By condition (T3) of Definition 1.1, R embeds into a finite direct sum of copies of T_X, hence $(T_X)_\mathfrak{q} \neq 0$. So $\alpha_\mathfrak{q} = \mu_1(\mathfrak{q}, T_X) \neq 0$ follows from $\operatorname{Ext}^1_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}, R_\mathfrak{p}) \neq 0$ since $\mathfrak{p}_\mathfrak{p} = rR_\mathfrak{p}$ for some $r \in R_\mathfrak{p}$, and $R_\mathfrak{p}$ is not r-divisible. □

Next, we will distinguish four cases:

(I) $\mathfrak{m} \in X$ and $X \neq \{0, \mathfrak{m}\}$, (II) $\mathfrak{m} \notin X$ and $X \neq \{0\}$, (III) $X = \{0\}$, and (IV) $X = \{0, \mathfrak{m}\}$.

Each of these cases requires a different approach for construction of the tilting module T_X. In the case (I), T_X can be obtained via classical localization at a multiplicative subset of R:

Theorem 5.2. *Let $(R, \mathfrak{m})$ be a regular local ring of Krull dimension* 2. *Let $X \subseteq \operatorname{Spec}(R)$ be such that $\{0, \mathfrak{m}\} \subsetneq X$.*

Denote by S_X the submonoid of $(R, \cdot, 1)$ generated by all invertible elements of R and by the set $\{r_\mathfrak{p} \mid \mathfrak{p} \in X \cap P_1\}$. Let Q_X be the localization of R at S_X, and $T_X = \bigoplus_{\mathfrak{p} \in X \setminus \{0\}} E(R/\mathfrak{p}) \oplus Q_X$.

Then T_X is a good tilting module of projective dimension 2, and $(T_X)^{\perp_\infty} = \mathcal{T}_X$.

Proof. By [26, 5.1.2], $\mathrm{flat.dim}_R\, E(R/\mathfrak{m}) = \mathrm{ht}\ \mathfrak{m} = 2$, so $\mathrm{proj.dim}_R\, T_X = 2$.

In order to verify condition (T2) for the module T_X, we consider its minimal injective coresolution.

First, we claim that

$$0 \to Q_X \to Q \to \bigoplus_{\mathfrak{q} \in P_1 \setminus X} E(R/\mathfrak{q}) \to 0$$

is the minimal injective coresolution of Q_X.

Clearly, $Q = E(R) = E(Q_X)$. Next, we prove that the module Q/Q_X is injective. Since Q/Q_X is divisible, we only have to verify that $\mathrm{Ext}^1_R(R/\mathfrak{m}, Q/Q_X) = 0$ (see Lemma 1.8). In other words, we have to show that each $f \in \mathrm{Hom}_R(\mathfrak{m}, Q/Q_X)$ extends to some $g \in \mathrm{Hom}_R(R, Q/Q_X)$. Since $X \neq \{0, \mathfrak{m}\}$, we have $r_\mathfrak{p} \in \mathfrak{m}$ for some $\mathfrak{p} \in X \cap P_1$, and $\mathfrak{m} = \mathfrak{p} + \sum_{i<k} r_i R$ for some $r_i \in \mathfrak{m} \setminus \mathfrak{p}$ $(i < k)$. Let $f(r_\mathfrak{p}) = q_\mathfrak{p} + Q_X$ and $f(r_i) = q_i + Q_X$ $(i < k)$. Evaluating $f(r_i \cdot r_\mathfrak{p})$ in two ways, we see that $q_\mathfrak{p} \cdot r_i - q_i \cdot r_\mathfrak{p} \in Q_X$. Since Q_X is a subring of Q containing $1/r_\mathfrak{p}$, also $(q_\mathfrak{p}/r_\mathfrak{p}) \cdot r_i - q_i \in Q_X$, for each $i < k$. This implies that the map $g \in \mathrm{Hom}_R(R, Q/Q_X)$ defined by $g(1) = q_\mathfrak{p}/r_\mathfrak{p} + Q_X$ extends f.

In order to prove our claim, it remains to show that $\mathrm{Ass}_R\, Q/Q_X = P_1 \setminus X$, and that $E(R/\mathfrak{q})$ occurs with multiplicity 1 in the decomposition of Q/Q_X for each $\mathfrak{q} \in P_1 \setminus X$. Since R is a UFD and Q_X is the localization of R at S_X, we have $\mathrm{Ann}(1/r_\mathfrak{q} + Q_X) = r_\mathfrak{q} R = \mathfrak{q}$ for each $\mathfrak{q} \in P_1 \setminus X$. Similarly, $\mathrm{Ass}_R\,(Q/Q_X) \cap X = \emptyset$.

So $Q/Q_X \cong \bigoplus_{\mathfrak{q} \in P_1 \setminus X} E(R/\mathfrak{q})^{(\alpha_\mathfrak{q})}$ where $\alpha_\mathfrak{q} > 0$ for all $\mathfrak{q} \in P_1 \setminus X$. We will prove that $\alpha_\mathfrak{q} = 1$ for all $\mathfrak{q} \in P_1 \setminus X$, that is, the $\mathfrak{q}$-component $C_\mathfrak{q}$ of Q/Q_X is uniform. If $u_i/v_i + Q_X \in C_\mathfrak{q}$ $(i = 1, 2)$ are such that $\mathrm{Ann}(u_i/v_i + Q_X) = \mathfrak{q}$, then $(u_i/v_i) \cdot r_\mathfrak{q} = r_i/s_i$ where $r_i \in R$ is not divisible by $r_\mathfrak{q}$ and $s_i \in S_X$ $(i = 1, 2)$. Then $r_2 \cdot s_1 \cdot (u_1/v_1 + Q_X) = r_1 \cdot r_2/r_\mathfrak{q} + Q_X = r_1 \cdot s_2 \cdot (u_2/v_2 + Q_X) \neq 0$. This shows that $C_\mathfrak{q}$ is a uniform module, hence $C_\mathfrak{q} \cong E(R/\mathfrak{q})$.

It follows that the minimal injective coresolution of T_X has the form

$$0 \to T_X \to Q \oplus \bigoplus_{\mathfrak{p} \in X \setminus \{0\}} E(R/\mathfrak{p}) \to \bigoplus_{\mathfrak{q} \in P_1 \setminus X} E(R/\mathfrak{q}) \to 0.$$

Since $\mathrm{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{q})) = 0$ for all $\mathfrak{p} \neq \mathfrak{q}$ such that $\mathfrak{q} \in P_1$ and $\mathfrak{p} \in P_1 \cup \{\mathfrak{m}\}$, we have $\mathrm{Ext}^1_R(\bigoplus_{\mathfrak{p} \in X \setminus \{0\}} E(R/\mathfrak{p}), T_X^{(\kappa)}) = 0$ for any cardinal κ.

The proof of condition (T2) is now completed by recalling that that Q_X is a localization of R at S_X, hence it satisfies $\operatorname{Ext}^1_R(Q_X, Q_X^{(\kappa)}) = \operatorname{Ext}^1_{Q_X}(Q_X, Q_X^{(\kappa)}) = 0$ for all cardinals κ.

In order to verify condition (T3), we show that there is an exact sequence

$$0 \to R \to Q_X \to \bigoplus_{\mathfrak{p} \in X \cap P_1} E(R/\mathfrak{p}) \to E(R/\mathfrak{m}) \to 0$$

obtained by glueing together the short exact sequences $0 \to R \to Q_X \to Q_X/R \to 0$ and $0 \to Q_X/R \to \bigoplus_{\mathfrak{p} \in X \cap P_1} E(R/\mathfrak{p}) \to E(R/\mathfrak{m}) \to 0$. We only have to prove the isomorphisms $E(Q_X/R) \cong \bigoplus_{\mathfrak{p} \in X \cap P_1} E(R/\mathfrak{p})$, and $E(Q_X/R)/(Q_X/R) \cong E(R/\mathfrak{m})$.

The first isomorphism follows from $Q_X/R \subseteq Q/R \subseteq E(Q/R) \cong \bigoplus_{\mathfrak{p} \in P_1} E(R/\mathfrak{p})$ and $\operatorname{Ass}_R Q_X/R = X \cap P_1$ (where the latter holds because Q_X/R is essential in $\bigoplus_{\mathfrak{p} \in X \cap P_1} E(R/\mathfrak{p})$. For the second, consider the pushout of the embeddings $Q_X/R \hookrightarrow \bigoplus_{\mathfrak{p} \in P_1} E(R/\mathfrak{p})$ and $Q_X/R \hookrightarrow Q/R$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Q_X/R & \longrightarrow & E(Q_X/R) & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Q/R & \xrightarrow{\nu} & X & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & M & = & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

By the first part of the proof, $M \cong \bigoplus_{\mathfrak{q} \in P_1 \setminus X} E(R/\mathfrak{q})$. Also, we get $E(Q_X/R) \cong \bigoplus_{\mathfrak{p} \in X \cap P_1} E(R/\mathfrak{p})$ by the above. Since the middle column splits, we have $X \cong \bigoplus_{\mathfrak{p} \in P_1} E(R/\mathfrak{p})$. As N is torsion, $\nu(Q/R)$ is essential in X, so $X = E(\nu(Q/R))$, and $N \cong X/\nu(Q/R) \cong E(Q/R)/(Q/R) = E(R/\mathfrak{m})$. This proves the second isomorphism.

Let $(\mathcal{A}_X, \mathcal{B}_X)$ be the tilting cotorsion pair induced by T_X. We will show that $\mathcal{B}_X = \mathcal{T}_X$.

Since $\mathcal{T}_X \subseteq \mathcal{I}_1$, we have $\mathcal{A}_X = {}^{\perp}\operatorname{Sum} T$ by Lemmma 1.15. In particular, if M is a finitely generated module, then $M \in \mathcal{A}_X$, if and only if $\operatorname{Ext}^1_R(M, T_X) = 0$. However, $\operatorname{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{q})) = 0$ for all $0 \neq \mathfrak{p} \in X$ and $\mathfrak{q} \in P_1 \setminus X$, so the minimal injective coresolution of T_X constructed in Lemma 5.1 yields $\operatorname{Ext}^1_R(R/\mathfrak{p}, T_X) = 0$ for all

$\mathfrak{p} \in X$. Finally, $\mathrm{Hom}_R(R/\mathfrak{q}, Q) = 0$ and $\mathrm{Hom}_R(R/\mathfrak{q}, Q/Q_X) \neq 0$ for all $\mathfrak{q} \in P_1 \setminus X$, gives $\mathrm{Ext}^1_R(R/\mathfrak{q}, Q_X) \neq 0$.

This proves that $\mathrm{Ass}_R \mathcal{A}_X = X$, so $\mathcal{B}_X = \mathcal{T}_X$ by Theorem 3.5. □

In the case (II), we employ the Fuchs tilting modules introduced in Section 1:

Theorem 5.3. *Let $(R, \mathfrak{m})$ be a regular local ring of Krull dimension 2. Let $0 \in X \subseteq \mathrm{Spec}(R)$ be such that $\mathfrak{m} \notin X$ and $X \neq \{0\}$.*

Denote by S_X the submonoid of $(R, \cdot, 1)$ generated by all invertible elements of R and by the set $\{r_\mathfrak{p} \mid \mathfrak{p} \in X\}$. Let δ_{S_X} be the Fuchs tilting module corresponding to S_X.

Then δ_{S_X} is a good tilting module of projective dimension 1 inducing the 1-tilting class

$$\mathcal{T}_X = \{M \in \text{Mod–}R \mid M\mathfrak{p} = M \text{ for all } \mathfrak{p} \in X \setminus \{0\}\}.$$

Proof. First, since $X \setminus \{0\}$ contains only prime ideals of height 1, $\mathcal{T}_X = \bigcap_{\mathfrak{p} \in X} (R/\mathfrak{p})^\perp$ equals the class of all S_X-divisible modules. However, by [17] and [19, 5.1.2], δ_{S_X} is a 1-tilting module inducing the same class, that is, $(\delta_{S_X})^\perp = \mathcal{T}_X$. □

There remain the two 'special' cases, of $X = \{0\}$ and $X = \{0, \mathfrak{m}\}$. Here, the description of the tilting module T_X is less transparent. In both cases, however, T_X can be taken countably generated.

Theorem 5.4. *Let $(R, \mathfrak{m})$ be a regular local ring of Krull dimension 2. Let $X = \{0\}$.*

Then there is a short exact sequence $0 \to R \to T_1 \to T_2 \to 0$ consisting of countably generated modules such that $T_1 \in \mathfrak{m}^\perp$ and T_2 is $\{\mathfrak{m}\}$-filtered.

Let $T_X = T_1 \oplus T_2$. Then T_X is a countably generated good tilting module of projective dimension 1 such that $T_X^\perp = \mathcal{T}_X = \mathcal{I}_1$.

Proof. Fix a presentation $0 \to K \to F \to \mathfrak{m} \to 0$ with F finitely generated free.

The construction of the required short exact sequence witnessing condition (T3) for T_X proceeds as in the universal construction of [19, 3.2.1], but with the modifications from [20, Theorem 2.2] (for $\kappa = \omega$ and $\mathcal{S} = \{\mathfrak{m}\}$) that make it possible to consider only countable chains. So $T_1 = \bigcup_{n<\omega} U_n$ and $T_2 = T_1/U_0$, where $(U_n \mid n < \omega)$ is an increasing chain of finitely generated submodules of T_1 such that $U_0 = R$, $U_{n+1}/U_n \cong \mathfrak{m}^{(G_n)}$ for each $n < \omega$, and G_n is a finite generating set of the R-module $\mathrm{Hom}_R(K, U_n)$.

Since $\mathrm{proj.dim}_R\, \mathfrak{m} = 1$, also $\mathrm{proj.dim}_R\, T \leq 1$. By the construction, $T_1 \in \mathfrak{m}^\perp$, hence also $T_2 \in \mathfrak{m}^\perp (= \mathcal{I}_1)$.

Also, by the above and by Lemma 1.12(i), $T_X^\perp = T_2^\perp \supseteq \mathfrak{m}^\perp$, and condition (T2) follows.

Finally, let $M \in T_2^{\perp}$. Let $f \in \mathrm{Hom}_R(R^{(\lambda)}, M)$ be an epimorphism and consider the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^{(\lambda)} & \longrightarrow & T_1^{(\lambda)} & \longrightarrow & T_2^{(\lambda)} & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & T_2^{(\lambda)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0. & & & & \end{array}$$

Since $M \in T_2^{\perp}$, the bottom row splits, showing that M is an epimorphic image of $T_1^{(\lambda)}$, hence $M \in \mathfrak{m}^{\perp}$. This proves that $T_X^{\perp} = \mathcal{T}_X = \mathcal{I}_1$. □

Remark 5.5. The construction of the countably generated tilting module $T_{\{0\}}$ reveals some new phenomena. First, it does not involve any localization or divisibility. Moreover, by Theorem 3.5, the 1-tilting cotorsion pair induced by $T_{\{0\}}$ is $(\mathcal{A}, \mathcal{I}_1)$, where $\mathcal{A}$ is the class of all torsion-free modules that are $\mathcal{S}$-filtered, and $\mathcal{S}$ is the set of all ideals of R. The class $\mathcal{A}$ does not contain all torsion-free modules, otherwise $Q \in \mathcal{A}$, so Q is $\mathcal{S}$-filtered, and there is a torsion-free module $R \subseteq N \subseteq Q$ such that $0 \neq Q/N$ is torsion-free, but there is also an epimorphism of the torsion module Q/R onto Q/N.

Though the tilting modules $T_{\{0\}}$ and R are not equivalent, we have $\mathrm{Ass}_R \mathcal{A} = \mathrm{Ass}_R \mathcal{P}_0 = \{0\}$. This contrasts with the case of Gorenstein rings of Krull dimension one (see [25]). Also, $T_{\{0\}}$ is a torsion-free tilting module which is not projective. Such tilting modules do not exist over Prüfer domains and almost perfect domains by [24, Corollary 2.6] and [1, Proposition 4.9(2)], respectively.

Theorem 5.6. *Let $(R, \mathfrak{m})$ be a regular local ring of Krull dimension 2. Let $X = \{0, \mathfrak{m}\}$.*

Then there exist short exact sequences consisting of countably generated modules $0 \to R \to T_1 \to T_2 \to 0$ and $0 \to T_2 \to T_3 \to E(R/\mathfrak{m})^{(\omega)} \to 0$, such that $T_1 \in \mathcal{T}_X$ is $\{R/\mathfrak{m}, \mathfrak{m}\}$-filtered, and $T_3 \in (R/\mathfrak{m})^{\perp}$.

Let $T_X = T_1 \oplus T_3 \oplus E(R/\mathfrak{m})^{(\omega)}$. Then T_X is a countably generated good 2-tilting module such that $T_X^{\perp} = \mathcal{T}_X = (R/\mathfrak{m})^{\perp_\infty} = (E(R/\mathfrak{m}))^{\perp_\infty}$.

Proof. The first short exact sequence is obtained as in the proof of Theorem 5.4, except that the set $\mathcal{S} = \{\mathfrak{m}\}$ is replaced by $\mathcal{S} = \{\mathfrak{m}, R/\mathfrak{m}\}$. So T_1, T_2 are countably generated and $\{\mathfrak{m}, R/\mathfrak{m}\}$–filtered, $T_1 \in \{\mathfrak{m}, R/\mathfrak{m}\}^{\perp} = \mathcal{T}_X$, and also hence $T_2 \in \mathfrak{m}^{\perp}$.

The second sequence is obtained similarly, but using the set $\mathcal{S} = \{R/\mathfrak{m}\}$ as follows. First, we consider the projective resolution of $R/\mathfrak{m}$, $0 \to K \to F \to R/\mathfrak{m} \to 0$ with F a finitely generated free module.

We will construct T_3 as $T_3 = \bigcup_{n<\omega} U'_n$ where $(U'_n \mid n < \omega)$ is an increasing chain of submodules of T_3 such that $U'_0 = T_2$, and $U'_{n+1}/U'_n \cong E(R/\mathfrak{m})^{(\kappa_n)}$ where $0 < \kappa_n \leq \omega$ for each $n < \omega$.

If U'_n is defined for $n > 1$, let C_n be a countable generating subset of the R-module $\mathrm{Hom}_R(K, U'_n)$ and $\kappa_n = \mathrm{card}\, C_n$. By the universal construction in [19, 3.2.1], there is a module $U \supseteq U'_n$ such that $U/U'_n \cong (R/\mathfrak{m})^{(\kappa_n)}$ and each $f \in \mathrm{Hom}_R(K, U'_n)$ extends to some $g \in \mathrm{Hom}_R(F, U)$.

Consider the exact sequence $0 \to R/\mathfrak{m} \to E(R/\mathfrak{m}) \to D \to 0$. Then the module D is $\{R/\mathfrak{m}\}$-filtered, so $\mathrm{Ext}^2_R(D, U'_n) = 0$, because $\mathrm{Ext}^2_R(R/\mathfrak{m}, U'_n) \cong \mathrm{Ext}^1_R(\mathfrak{m}, U'_n)$, and the exact sequence $0 \to T_2 \to U'_n \to E(R/\mathfrak{m})^{(\sum_{i<n} \kappa_i)} \to 0$ with $T_2 \in \mathfrak{m}^\perp$ yields $\mathrm{Ext}^1_R(\mathfrak{m}, U'_n) = 0$.

The exactness of the sequence $\mathrm{Ext}^1_R(E(R/\mathfrak{m})^{(\kappa_n)}, U'_n) \to \mathrm{Ext}^1_R((R/\mathfrak{m})^{(\kappa_n)}, U'_n) \to \mathrm{Ext}^2_R(D^{(\kappa_n)}, U'_n) = 0$ implies existence of a module $U'_{n+1} \supseteq U$ such that $U'_{n+1}/U'_n \cong E(R/\mathfrak{m})^{(\kappa_n)}$. This finishes the construction of T_3.

Now, $T_2 \in \mathfrak{m}^\perp$ implies $T_3 \in \mathfrak{m}^\perp$. Moreover, $T_3 \in (R/\mathfrak{m})^\perp$ because if $f \in \mathrm{Hom}_R(K, T_3)$, then $\mathrm{Im}\, f \subseteq U'_n$ for some n, hence f extends to some $g \in \mathrm{Hom}_R(F, T_3)$ by construction.

Since $\mathrm{flat.dim}_R\, E(R/\mathfrak{m}) = 2$ by [26, 5.1.2], T_X has projective dimension 2. By construction, T_X is $\{\mathfrak{m}, R/\mathfrak{m}\}$-filtered and $T_X \in \mathcal{T}_X$. Thus $T_X \in \mathcal{T}_X \cap {}^\perp\mathcal{T}_X$, so condition (T2) holds for T_X, and $T_X^\perp = \mathcal{T}_X = (R/\mathfrak{m})^{\perp_\infty} = (E(R/\mathfrak{m}))^{\perp_\infty}$.

The two short exact sequences constructed above connect into a long exact sequence witnessing condition (T3), $0 \to R \to T_1 \to T_3 \to (E(R/\mathfrak{m}))^{(\omega)} \to 0$. □

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TILTING, COTILTING, AND SPECTRA OF COMMUTATIVE NOETHERIAN RINGS

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Abstract. We classify tilting and cotilting classes over commutative noetherian rings in terms of descending sequences of specialization closed subsets of the Zariski spectrum.

Introduction

We give a complete classification of tilting and cotilting classes in Mod–R, where R is a commutative noetherian ring, extending previous results such as [18]. The classification is given in Theorem 3.19 in terms of finite sequences of subsets of the Zariski spectrum of R. Along the way, we prove other non–trivial results about tilting and cotilting modules for commutative noetherian rings, which fail for general rings. Namely:

(i) The elementary duality (cf. Remark 1.12) gives a bijection between tilting and cotilting classes. For general rings, there are more cotilting classes than duals of tilting classes. Bazzoni constructed such examples for certain commutative non–noetherian rings in [5].
(ii) Cotilting classes are closed under taking injective envelopes by Proposition 3.10(ii).
(iii) In particular, 1–cotilting classes are precisely the torsion–free classes of faithful hereditary torsion pairs (Theorem 2.6). Note that 1–cotilting classes over general rings need not be hereditary; see [12, Theorem 2.5].
(iv) Up to adding an injective summand, a minimal cosyzygy of an n–cotilting module is $(n-1)$–cotilting (Corollary 3.16). This typically fails for non–commutative rings, ever for finite dimensional algebras over a field, since the cosyzygy often has self–extensions.

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1. Preliminaries

1.1. **Basic notations.** For a ring R, we denote by Mod–R the category of all (unitary right R-) modules, and by mod–R its subcategory consisting of all finitely generated modules. Similarly, we define R–Mod and R–mod using left R–modules.

For a module M, $\operatorname{Add} M$ denotes the class of all direct summands of (possibly infinite) direct sums of copies of the module M. Similarly, $\operatorname{Prod} M$ denotes the class of all summands of direct products of copies of M. Further, we denote by $\Omega(M)$ a syzygy of M and by $\mho(M)$ a minimal cosyzygy of M. That is, $\mho(M) = E(M)/M$, where $E(M)$ is an injective envelope of M. As usual, we define also higher cosyzygies: Given a module M,

$$0 \longrightarrow M \longrightarrow E_0(M) \longrightarrow E_1(M) \longrightarrow E_2(M) \longrightarrow \cdots$$

will stand for the minimal injective coresolution and the image of $E_{i-1}(M) \to E_i(M)$ for $i \geq 1$ will be denoted by $\mho_i(M)$. That is, $\mho(M) = \mho_1(M)$. We refrain from the usual notation $\Omega^{-i}(M)$ for the i-th cosyzygy for we adopt the following convention:

$$\mho_0(M) = M \qquad \text{and} \qquad \mho_i(M) = 0 \text{ for all } i < 0.$$

Thus, we need to clearly distinguish between syzygies and negative cosyzygies.

Given a class $\mathcal{S}$ of right modules, we denote:

$$\mathcal{S}^\perp = \{M \in \text{Mod–}R \mid \operatorname{Ext}^i_R(S, M) = 0 \text{ for all } S \in \mathcal{S} \text{ and } i \geq 1\},$$
$${}^\perp\mathcal{S} = \{M \in \text{Mod–}R \mid \operatorname{Ext}^i_R(M, S) = 0 \text{ for all } S \in \mathcal{S} \text{ and } i \geq 1\}.$$

If $\mathcal{S} = \{S\}$ is a singleton, we shorten the notation to $S^\perp$ and ${}^\perp S$. A similar notation is used for the classes of modules orthogonal with respect to the Tor functor:

$$\mathcal{S}^\intercal = \{M \in R\text{–Mod} \mid \operatorname{Tor}_i^R(S, M) = 0 \text{ for all } S \in \mathcal{S} \text{ and } i \geq 1\}.$$

Given a class $\mathcal{S} \subseteq$ Mod–R and a module M, a well–ordered chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\alpha \subseteq M_{\alpha+1} \subseteq \cdots M_\kappa = M,$$

is called an $\mathcal{S}$*–filtration* of M if $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ for every limit ordinal $\beta \leq \kappa$ and up to isomorphism $M_{\alpha+1}/M_\alpha \in \mathcal{S}$ for each $\alpha < \kappa$. A module is called $\mathcal{S}$*–filtered* if it has at least one $\mathcal{S}$–filtration.

Further, given an abelian category $\mathcal{A}$ (in our case typically $\mathcal{A} =$ Mod–R, or $\mathcal{A} =$ mod–R if R is right noetherian), a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ is called a *torsion pair* if

(i) $\operatorname{Hom}_\mathcal{A}(T, F) = 0$ for each $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
(ii) For each $M \in \mathcal{A}$ there is an exact sequence $0 \to T \to M \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

In such a case, $\mathcal{T}$ is called a *torsion class* and $\mathcal{F}$ a *torsion–free class.* A standard and easy but useful observation is the following:

Lemma 1.1. *Let $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}', \mathcal{F}')$ be torsion pairs in an abelian category. If $\mathcal{T}' \subseteq \mathcal{T}$ and $\mathcal{F}' \subseteq \mathcal{F}$, then $\mathcal{T} = \mathcal{T}'$ and $\mathcal{F} = \mathcal{F}'$.*

If $\mathcal{A} = \text{Mod–}R$, it is well–known that $\mathcal{F}$ is a torsion–free class of a torsion pair if and only if $\mathcal{F}$ is closed under submodules, extensions and direct products. Similarly, torsion classes are precisely those closed under factor modules, extensions and direct sums. For $\mathcal{A} = \text{mod–}R$ and R right noetherian, any torsion–free class $\mathcal{F}$ is closed under submodules and extensions (so also under finite products), but some caution is due here as these closure properties do not characterize torsion–free classes. Consider for instance $R = \mathbb{Z}$ and the class $\mathcal{F}$ of all finite abelian groups.

Let us conclude this discussion with two more properties which torsion pairs in Mod–R can possess.

Definition 1.2. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in Mod–R. Then $(\mathcal{T}, \mathcal{F})$ is *hereditary* if $\mathcal{T}$ is closed under submodules, or equivalently by [20, Proposition 3.2], if $\mathcal{F}$ is closed under taking injective envelopes. The torsion pair is called *faithful* if $R \in \mathcal{F}$.

1.2. **Commutative algebra essentials.** For a commutative noetherian ring R, we denote by Spec(R) the spectrum of R. The spectrum is well-known to carry the Zariski topology, where the closed sets are those of the form

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \supseteq I\},$$

for some subset $I \subseteq R$. If $I = \{f\}$ is a singleton, we again write just $V(f)$.

Given $M \in \text{Mod–}R$, $\operatorname{Ass} M$ denotes the set of all associated primes of M, and $\operatorname{Supp} M$ the support of M. For $\mathcal{C} \subseteq \text{Mod–}R$, we let

$$\operatorname{Ass} \mathcal{C} = \bigcup_{M \in \mathcal{C}} \operatorname{Ass} M \quad \text{ and } \quad \operatorname{Supp} \mathcal{C} = \bigcup_{M \in \mathcal{C}} \operatorname{Supp} M.$$

For $\mathfrak{p} \in \operatorname{Spec}(R)$, we denote by $R_\mathfrak{p}$ the localization of R at $\mathfrak{p}$, and by $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ the residue field.

If $M \in \text{Mod–}R$, $\mathfrak{p} \in \operatorname{Spec}(R)$ and $i \geq 0$, the *Bass invariant* $\mu_i(\mathfrak{p}, M)$ is defined as the number of summands isomorphic to $E(R/\mathfrak{p})$ in the injective module $E_i(M)$ in a minimal injective coresolution of M (see e.g. [13, §9.2] or [8, §3.2]). That is,

$$E_i(M) = \bigoplus_{\mathfrak{p} \in \operatorname{Spec}(R)} E(R/\mathfrak{p})^{(\mu_i(\mathfrak{p}, M))}.$$

The relation of associated primes to Bass invariants is captured by the following lemma:

Lemma 1.3. *Let M be an R-module, $\mathfrak{p} \in \operatorname{Spec}(R)$ and $i \geq 0$. Then*

$$\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}),$$

and we have the following equivalences:

$$\mathfrak{p} \in \operatorname{Ass} \mho_i(M) \quad \Longleftrightarrow \quad \mathfrak{p} \in \operatorname{Ass} E_i(M) \quad \Longleftrightarrow \quad \mu_i(\mathfrak{p}, M) \neq 0.$$

Proof. For the above equality we refer for instance to [8, Proposition 3.2.9] or [13, Theorem 9.2.4]. The first equivalence below is proved in [8, Lemma 3.2.7]. For the second, we use the equality $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E_i(M_\mathfrak{p}))$ from the proof of [8, Proposition 3.2.9] or [13, Theorem 9.2.4]. □

As a consequence, we can relate associated prime ideals of the terms of a short exact sequence and their cosyzygies.

Lemma 1.4. *Let $0 \to K \to L \to M \to 0$ be a short exact sequence of R-modules and $i \in \mathbb{Z}$. Then the following hold:*

(i) $\operatorname{Ass} \mho_i(K) \subseteq \operatorname{Ass} \mho_{i-1}(M) \cup \operatorname{Ass} \mho_i(L)$.
(ii) $\operatorname{Ass} \mho_i(L) \subseteq \operatorname{Ass} \mho_i(K) \cup \operatorname{Ass} \mho_i(M)$.
(iii) $\operatorname{Ass} \mho_i(M) \subseteq \operatorname{Ass} \mho_i(L) \cup \operatorname{Ass} \mho_{i+1}(K)$.

Proof. Given any $\mathfrak{p} \in \operatorname{Spec}(R)$, we consider the long exact sequence of Hom and Ext groups, which we obtain by applying the functor $\operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), -)$ on the localized short exact sequence

$$0 \longrightarrow K_\mathfrak{p} \longrightarrow L_\mathfrak{p} \longrightarrow M_\mathfrak{p} \longrightarrow 0.$$

The lemma is then an easy consequence of Lemma 1.3. □

In particular, we obtain information on associated primes of syzygy modules.

Corollary 1.5. *Let M be an R-module, $\ell \geq 1$ and K be an ℓ-th syzygy of M. Then for any $i \in \mathbb{Z}$ we have:*

$$\operatorname{Ass} \mho_i(K) \subseteq \operatorname{Ass} \mho_{i-\ell}(M) \cup \bigcup_{j=0}^{\ell-1} \operatorname{Ass} \mho_{i-j}(R).$$

Remark 1.6. We stress that according to our convention, $\mho_{i-\ell}(M) = 0$ for $i - \ell < 0$. Thus, the right hand term does not depend on M for $i < \ell$.

Proof. This is easily obtained from Lemma 1.4 by induction on ℓ. We also use that $\operatorname{Ass} \mho_j(P) \subseteq \operatorname{Ass} \mho_j(R)$ for any $j \in \mathbb{Z}$ and any projective module P. □

1.3. **Tilting and cotilting modules and classes.** Next, we recall the notion of an (infinitely generated) tilting module from [11, 1]:

Definition 1.7. Let R be a ring. A module T is *tilting* provided that

(T1) T has finite projective dimension.
(T2) $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $i \geq 1$ and all cardinals κ.
(T3) There is a short exact sequence $0 \to R \to T_0 \to T_1 \to \cdots \to T_r \to 0$ where $T_0, T_1, \ldots, T_r \in \operatorname{Add} T$.

The class $T^\perp = \{M \in \text{Mod–}R \mid \mathrm{Ext}^i_R(T, M) = 0 \text{ for each } i \geq 1\}$ is called the *tilting class* induced by T. Given an integer $n \geq 0$, a tilting module as well as its associated class are called *n–tilting* provided the projective dimension of T is at most n. We recall that in such a case we can chose the sequence in (T3) so that $r \leq n$ (see [4, Proposition 3.5]).
If T and T' are tilting modules, then T is said to be *equivalent* to T' provided that $T^\perp = (T')^\perp$, or equivalently by [14, Lemma 5.1.12], $T' \in \operatorname{Add} T$.

The structure of tilting modules over commutative noetherian rings is rather different from the classic case of artin algebras. The key point is absence of non–trivial finitely generated tilting modules:

Lemma 1.8. [10, 18] *Let R be a commutative noetherian ring and T be a finitely generated module. Then T is tilting, if and only if T is projective.*

Even though the tilting module T is infinitely generated, the tilting class $T^\perp$ is always determined by a set $\mathcal{S}$ of finitely generated modules of bounded projective dimension. This was proved in [7], based on the corresponding result [6] for 1–tilting modules. We will call a subclass $\mathcal{S}$ of mod–R *resolving* in case $\mathcal{S}$ is closed under extensions, direct summands, kernels of epimorphisms, and $R \in \mathcal{S}$. If $\mathcal{S}$ consists of modules of projective dimension ≤ 1, the requirement of $\mathcal{S}$ being closed under kernels of epimorphisms is redundant by [14, Lemma 5.2.22]. Using results from [2, 6, 7], we learn that resolving subclasses of mod–R parametrize tilting classes (and hence also the tilting modules up to equivalence):

Lemma 1.9. [14, 5.2.23] *Let R be a right noetherian ring and $n \geq 0$. Then there is a bijective correspondence between*

(i) *n–tilting classes $\mathcal{T}$ in* Mod–R*, and*
(ii) *resolving subclasses $\mathcal{S}$ of* mod–R *consisting of modules of projective dimension $\leq n$.*

The correspondence is given by the assignments $\mathcal{T} \mapsto {}^\perp\mathcal{T} \cap \text{mod–}R$ and $\mathcal{S} \mapsto \mathcal{S}^\perp$.

The dual notions of a cotilting module and a cotilting class are defined as follows:

Definition 1.10. Let R be a ring. A module C is *cotilting* provided that

(C1) C has finite injective dimension.
(C2) $\operatorname{Ext}^i_R(C^\kappa, C) = 0$ for all $i \geq 1$ and all cardinals κ.
(C3) There is a short exact sequence $0 \to C_r \to \cdots \to C_1 \to C_0 \to W \to 0$ where W is an injective cogenerator of Mod–R and $C_0, C_1, \dots C_r \in \operatorname{Prod} C$.

The class ${}^\perp C = \{M \in \text{Mod–}R \mid \operatorname{Ext}^i_R(M, C) = 0 \text{ for all } i \geq 1\}$ is the *cotilting class* induced by C. Again, if the injective dimension of C is at most n, we call C and ${}^\perp C$ an n*–cotilting* module and class, respectively.
If C and C' are cotilting modules, then C is said to be *equivalent* to C' provided that ${}^\perp C = {}^\perp C'$, or equivalently by [14, Remark 8.1.6], $C' \in \operatorname{Prod} C$.

If T is an n–tilting right R–module, then the character module

$$C = T^+ = \operatorname{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$$

is an n–cotilting left R–module; see [2, Proposition 2.3]. By Lemma 1.9, the induced tilting class $\mathcal{T} = T^\perp$ equals $\mathcal{S}^\perp$ where $\mathcal{S} = {}^\perp\mathcal{T} \cap \text{mod–}R$ is a resolving subclass of mod–R. The cotilting class $\mathcal{C}$ induced by C in R–Mod is then easily seen to be

$$\mathcal{C} = {}^\perp C = T^\intercal = \mathcal{S}^\intercal = \{M \in R\text{–Mod} \mid \operatorname{Tor}^R_1(S, M) \text{ for all } S \in \mathcal{S}\}.$$

We will call $\mathcal{C}$ the cotilting class *associated* to the tilting class $\mathcal{T}$.

It follows that that tilting modules T and T' are equivalent, if and only if the character modules T^+ and $(T')^+$ are equivalent as cotilting left R–modules; see [14, Theorem 8.1.13]. Therefore, the assignment $T \mapsto T^+$ induces an injective map from equivalence classes of tilting to equivalence classes of cotilting modules. For R commutative noetherian, this map, as we will show, turns out to be a bijection, but for non–noetherian commutative rings the surjectivity may fail; see [5]. Let us summarize the properties we need.

Lemma 1.11. *Let R be right noetherian ring and $n \geq 0$. Then the following holds:*

(i) *If $\mathcal{S} \subseteq$ mod–R is a class of finitely generated modules of projective dimension bounded by n, then $\mathcal{S}^\perp$ is an n–tilting class in* Mod–R *and $\mathcal{S}^\intercal$ is the associated n–cotilting class in R*–Mod.
(ii) *An n–cotilting class $\mathcal{C}$ in R*–Mod *is associated to a tilting class if and only if there exists a class $\mathcal{S}$ of finitely generated modules of projective dimension $\leq n$ such that $\mathcal{C} = \mathcal{S}^\intercal$.*

Proof. For (i), $\mathcal{S}^\perp$ is an n–tilting class by [14, Theorem 5.2.2] and $\mathcal{S}^\intercal$ is n–cotilting by [14, Theorem 8.1.12]. The cotilting class $\mathcal{S}^\intercal$ is associated to the tilting class $\mathcal{S}^\perp$ by [14, Theorem 8.1.2]. Part (ii) is proved in [14, Theorem 8.1.13(a)]. □

Remark 1.12. The relation between a tilting class $\mathcal{T}$ and the associated cotilting class $\mathcal{C}$ can be interpreted using model–theoretic means in terms of the so–called elementary duality. Namely, $\mathcal{T}$ and $\mathcal{C}$ can be axiomatized in the first order language of the right (left, resp.) R–modules (cf. [14, 5.2.2 and 8.1.7]) and the corresponding theories are given by mutually dual primitive positive formulas. We refer to [19, Section 1.3] for more details and references on the model–theoretic background.

2. The one–dimensional case

We will treat separately the case of 1–tilting and 1–cotilting modules. We have chosen such presentation for two reasons. First, the arguments for this special situation are simpler and more transparent. Second, the one–dimensional case is tightly connected to the classical notion of Gabriel topology and the abelian quotients of the category Mod–R. We refer to [20] for details on the latter concepts.

To start with, we recall [14, Lemma 6.1.2]: $T \in$ Mod–R is 1–tilting if and only if $T^\perp = \operatorname{Gen}(T)$ where the latter denotes the class of all modules generated by T. In particular, $T^\perp$ is a torsion class in Mod–R. Dually by [14, Lemma 8.2.2], a module C is 1–cotilting if and only if ${}^\perp C = \operatorname{Cog}(C)$ where the latter denotes the class of all modules cogenerated by C. Thus, ${}^\perp C$ is a torsion free class.

From this point on, we will assume that our base ring R is commutative noetherian if not specified otherwise. Our aim is to show that a torsion pair in Mod–R is of the form $(\mathcal{T}, \operatorname{Cog}(C))$ for a 1–cotilting module C if and only if it is faithful and hereditary. Moreover, we are going to classify such torsion pairs in terms of certain subsets of $\operatorname{Spec}(R)$. To this end, we introduce the following terminology:

Definition 2.1. For any subset $X \subseteq \operatorname{Spec}(R)$ we say that X is *closed under generalization* (under *specialization*, resp.) if for any $\mathfrak{p} \in X$ and any $\mathfrak{q} \in \operatorname{Spec}(R)$ we have $\mathfrak{q} \in X$ whenever $\mathfrak{q} \subseteq \mathfrak{p}$ ($\mathfrak{q} \supseteq \mathfrak{p}$, resp.). In other words, P is a lower (upper, resp.) set in the poset $(\operatorname{Spec}(R), \subseteq)$.

Further, we recall that Gabriel established a one–to–one correspondence between the subsets of $\operatorname{Spec}(R)$ closed under specialization and certain linear topologies on R. On the other hand, there is a bijective correspondence between these Gabriel topologies and hereditary torsion pairs in Mod–R. Let us look closer at this relationship.

Proposition 2.2. *Every subset $Y \subseteq \operatorname{Spec}(R)$ closed under specialization gives rise to a Gabriel topology on R (in the sense of* [20, §VI.5]*), given by the following set of open neighbourhoods of $0 \in R$, where all the I are ideals:*

$$\mathcal{G}_Y = \{I \subseteq R \mid V(I) \subseteq Y\}.$$

Then $\mathcal{G}_Y \cap \operatorname{Spec}(R) = Y$ *and the set* Y *also determines a hereditary torsion pair* $(\mathcal{T}(Y), \mathcal{F}(Y))$*, where:*

$$\mathcal{T}(Y) = \{M \in \text{Mod–}R \mid \operatorname{Supp} M \subseteq Y\},$$
$$\mathcal{F}(Y) = \{M \in \text{Mod–}R \mid \operatorname{Ass} M \cap Y = \emptyset\}.$$

We further have the following:

(i) *The assignments* $Y \mapsto \mathcal{G}_Y$ *and* $Y \mapsto (\mathcal{T}(Y), \mathcal{F}(Y))$ *define bijective correspondences between the subsets of* $\operatorname{Spec}(R)$ *closed under specialization, the Gabriel topologies on* R*, and the hereditary torsion pairs in* Mod–R.

(ii) $\mathcal{T}(Y) = \{M \in \text{Mod–}R \mid \operatorname{Hom}_R(M, E(R/\mathfrak{q})) = 0 \textit{ for all } \mathfrak{q} \notin Y\}$ *and* $\mathcal{T}(Y)$ *contains all* $E(R/\mathfrak{p})$ *with* $\mathfrak{p} \in Y$.

(iii) $\mathcal{F}(Y) = \{M \in \text{Mod–}R \mid \operatorname{Hom}_R(R/\mathfrak{p}, M) = 0 \textit{ for all } \mathfrak{p} \in Y\}$ *and* $\mathcal{F}(Y)$ *contains all* $E(R/\mathfrak{q})$ *with* $\mathfrak{q} \notin Y$.

(iv) $(\mathcal{T}(Y), \mathcal{F}(Y))$ *is a torsion theory of finite type, that is,*

$$\mathcal{T}(Y) = \varinjlim(\mathcal{T}(Y) \cap \text{mod–}R) \textit{ and } \mathcal{F}(Y) = \varinjlim(\mathcal{F}(Y) \cap \text{mod–}R).$$

Proof. First of all, observe that $\mathcal{G}_Y \cap \operatorname{Spec}(R) = Y$ as Y is closed under specialization. For the fact that $\mathcal{G}_Y$ is a Gabriel topology we refer to [20, Theorem VI.5.1 and §VI.6.6]. Next, $\mathcal{T}(Y)$ defined as above is clearly closed under submodules, factor modules, extensions and direct sums, so it is a torsion class in a hereditary torsion pair. We claim that $\mathcal{F}(Y)$ is the corresponding torsion–free class. Indeed, given $M \in \mathcal{F}(Y)$, denote by $t(M)$ the $\mathcal{T}(Y)$–torsion part of M. Then

$$\operatorname{Ass} t(M) \subseteq \operatorname{Ass} M \cap \operatorname{Ass} \mathcal{T} \subseteq \operatorname{Ass} M \cap Y = \emptyset.$$

Hence $t(M) = 0$ by [13, 2.4.3] and M is torsion–free. Conversely, if M is torsion–free, we must have $\operatorname{Ass} M \cap Y = \emptyset$. This is since for any $\mathfrak{p} \in \operatorname{Ass} M$ we have an embedding $R/\mathfrak{p} \hookrightarrow M$, but if $\mathfrak{p} \in Y$, we have $R/\mathfrak{p} \in \mathcal{T}(Y)$ owing to the fact that Y is closed under specialization and $\operatorname{Supp} R/\mathfrak{p} = V(\mathfrak{p}) \subseteq Y$. This proves the claim, showing that the latter correspondence is well–defined.

For statement (i), note that the inverse of $Y \mapsto \mathcal{G}_Y$ is given by the assignment $\mathcal{G} \mapsto \mathcal{G} \cap \operatorname{Spec}(R)$, where $\mathcal{G}$ is a Gabriel topology. This follows from the equality $\mathcal{G}_Y \cap \operatorname{Spec}(R) = Y$ and [20, VI.6.13 and VI.6.15]. It is well–known that Gabriel topologies are in bijection with hereditary torsion pairs; the hereditary torsion pair $(\mathcal{T}'(Y), \mathcal{F}'(Y))$ corresponding to $\mathcal{G}_Y$ is given by

$$\mathcal{T}'(Y) = \{M \in \text{Mod–}R \mid \operatorname{Ann}(x) \in \mathcal{G}_Y \text{ for all } x \in M\},$$

see [20, Theorem VI.5.1]. Equivalently,

$$\mathcal{T}'(Y) = \{M \in \text{Mod–}R \mid M_{\mathfrak{p}} = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R) \setminus Y\},$$

see [20, Example, p. 168]. Since $Y \subseteq \operatorname{Spec}(R)$ is closed under specialization, we infer that $\mathcal{T}(Y) = \mathcal{T}'(Y)$, hence the bijection between specialization closed subsets Y and hereditary torsion pairs in Mod–R.

For statements (ii) and (iii), we refer to [20, Proposition VI.3.6 and Exercise VI.24] and [13, Theorem 3.3.8].

Finally for (iv), we know from [14, Lemma 4.5.2] that $(\mathcal{T}(Y) \cap \text{mod–}R, \mathcal{F}(Y) \cap \text{mod–}R)$ is a torsion pair in mod–R and that

$$(\varinjlim(\mathcal{T}(Y) \cap \text{mod–}R), \varinjlim(\mathcal{F}(Y) \cap \text{mod–}R))$$

is a torsion pair in Mod–R. Note that both $\mathcal{T}(Y)$ and $\mathcal{F}(Y)$ are closed under taking direct limits. In the case of $\mathcal{F}(Y)$ this follows from (iii). Hence

$$\varinjlim(\mathcal{T}(Y) \cap \text{mod–}R) \subseteq \mathcal{T}(Y) \quad \text{and} \quad \varinjlim(\mathcal{F}(Y) \cap \text{mod–}R) \subseteq \mathcal{F}(Y),$$

and by Lemma 1.1 we have equalities. □

Remark 2.3. The bijections from Proposition 2.2 can be reinterpreted in terms of the one-to-one-correspondence

$$Y \mapsto \{M \in \text{mod–}R \mid \operatorname{Ass} M \subseteq Y\},$$

established by Takahashi in [21, Theorem 4.1], between all subsets of Spec(R) and the subcategories of mod–R which are closed under submodules and extensions. Indeed, this correspondence restricts to a bijection $Y \mapsto \{M \in \text{mod–}R \mid \operatorname{Supp} M \subseteq Y\}$ between the subsets of Spec(R) closed under specialization and the Serre subcategories (i.e. subcategories closed under submodules, factor modules and extensions) of mod–R, which in turn correspond bijectively to the hereditary torsion pairs in Mod–R via the assignment $\mathcal{S} \mapsto \varinjlim \mathcal{S}$, see [15, Lemma 2.3].

We also give an alternative description of the class $\{M \in \text{mod–}R \mid \operatorname{Ass} M \subseteq Y\}$. Given a subset $Y \subseteq \operatorname{Spec}(R)$, we say that a module $M \in \text{mod–}R$ is *Y–subfiltered* provided there exists a chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$$

of submodules of M such that for each $i = 0, \ldots, \ell - 1$, the module M_{i+1}/M_i is isomorphic to a submodule of $R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in Y$.

It was shown by Hochster (cf. [18, Lemma 2.1]) that any module $M \in \text{mod–}R$ is ($\operatorname{Ass} M$)-subfiltered. Thus $\{M \in \text{mod–}R \mid \operatorname{Ass} M \subseteq Y\}$ is the subcategory of mod–R given by all Y-subfiltered modules. Indeed, If $0 \to N \to M \to M/N \to 0$ is a short exact sequence in mod–R, then $\operatorname{Ass} N \subseteq \operatorname{Ass} M$ and $\operatorname{Ass} M \subseteq \operatorname{Ass} N \cup \operatorname{Ass} M/N$, so the claim follows directly by Hochster's result.

For our classification, we need to decide, which of the classes in mod–R closed under submodules and extensions are torsion–free classes in mod–R. As it turns out, these again correspond bijectively to subsets of Spec(R) closed under specialization.

Proposition 2.4. *The assignment*

$$Y \mapsto (\mathcal{T}(Y) \cap \text{mod–}R, \mathcal{F}(Y) \cap \text{mod–}R),$$

using the notation from Proposition 2.2, gives a bijective correspondence between subsets $Y \subseteq \mathrm{Spec}(R)$ *closed under specialization and torsion pairs in* mod–R.

Proof. By Proposition 2.2, $(\mathcal{T}(Y) \cap \text{mod–}R, \mathcal{F}(Y) \cap \text{mod–}R)$ is clearly a torsion pair in mod–R for every specialization closed set Y, and the assignment is injective since $\mathfrak{p} \in Y$ if and only if $R/\mathfrak{p} \in \mathcal{T}(Y)$. We must prove the surjectivity.

To this end, suppose that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in mod–R. By [21, Theorem 4.1] (cf. Remark 2.3) there is a subset $X \subseteq \mathrm{Spec}(R)$ such that $\mathcal{F} = \{M \in \text{mod–}R \mid \mathrm{Ass}\, M \subseteq X\}$. Denoting $Y = \mathrm{Spec}(R) \setminus X$, we claim that

$$\mathcal{T} \subseteq \{M \in \text{mod–}R \mid \mathrm{Supp}\, M \subseteq Y\}.$$

Indeed, given $\mathfrak{p} \in X$, we have $R/\mathfrak{p} \in \mathcal{F}$. Then for any $N \in \mathcal{T}$, $\mathrm{Hom}_R(N, R/\mathfrak{p}) = 0$ implies $\mathrm{Hom}_{R_\mathfrak{p}}(N_\mathfrak{p}, k(\mathfrak{p})) = 0$, so the finitely generated $R_\mathfrak{p}$–module $N_\mathfrak{p}$ has no maximal submodules. That is, $N_\mathfrak{p} = 0$ by the Nakayama Lemma (see e.g. [13, 1.2.28]). This proves the claim. We have shown that

$$\mathcal{T} \subseteq \mathcal{T}(Y) \cap \text{mod–}R \quad \text{and} \quad \mathcal{F} \subseteq \mathcal{F}(Y) \cap \text{mod–}R.$$

Thus $\mathcal{T} = \mathcal{T}(Y) \cap \text{mod–}R$ and $\mathcal{F} = \mathcal{F}(Y) \cap \text{mod–}R$ by Lemma 1.1. □

Let us now give a relation to 1–cotilting modules, using a result of Buan and Krause.

Proposition 2.5. *Let* R *be a not necessarily commutative right noetherian ring. Then the cotilting classes* $\mathcal{C}$ *in* Mod–R *correspond bijectively to the torsion–free classes* $\mathcal{F}$ *in* mod–R *containing* R. *The correspondence is given by the assignments*

$$\mathcal{C} \mapsto \mathcal{F} = \mathcal{C} \cap \text{mod–}R \quad \textit{and} \quad \mathcal{F} \mapsto \varinjlim \mathcal{F}.$$

Proof. This has been proved in [9, Theorem 1.5]. See also [14, Theorem 8.2.5]. □

As a direct consequence, we get a characterization and a classification of 1–cotilting classes in mod–R. Note that the fact that they induce hereditary torsion pair may fail if R is non–commutative, see [12, Theorem 2.5].

Theorem 2.6. *Let* R *be a commutative noetherian ring and* $\mathcal{C} \subseteq$ Mod–R. *Then* $\mathcal{C}$ *is* 1*–cotilting if and only if* $\mathcal{C}$ *is the torsion–free class in a faithful hereditary torsion pair* $(\mathcal{T}, \mathcal{C})$. *In particular, the* 1*–cotilting classes* $\mathcal{C}$ *in* Mod–R *are parametrized by the subsets* Y *of* $\mathrm{Spec}(R)$ *closed under specialization with* $\mathrm{Ass}\, R \cap Y = \emptyset$. *The parametrization is given by*

$$\mathcal{C} \mapsto \mathrm{Spec}(R) \setminus \mathrm{Ass}\,(\mathcal{C} \cap \text{mod–}R) \quad \textit{and} \quad Y \mapsto \{M \in \text{Mod–}R \mid \mathrm{Ass}\, M \cap Y = \emptyset\}.$$

Proof. By Proposition 2.5, 1–cotilting classes in Mod–R correspond bijectively to torsion–free classes in mod–R containing R, which by Propositions 2.2 and 2.4 and [14, Lemma 4.5.2] correspond bijectively to faithful hereditary torsion pairs in Mod–R. Composing the two assignments amounts to identifying a cotilting class $\mathcal{C}$ with the torsion–free part of the hereditary torsion pair. This shows the first part.

For the parametrization, we can use Proposition 2.2, as soon as we prove that

$$\operatorname{Ass}(\mathcal{C} \cap \text{mod–}R) = \operatorname{Ass}\mathcal{C}$$

for any 1–cotilting class $\mathcal{C}$. Clearly, $\operatorname{Ass}(\mathcal{C} \cap \text{mod–}R) \subseteq \operatorname{Ass}\mathcal{C}$. For the other implication, we can express by Proposition 2.5 every $M \in \mathcal{C}$ as $M = \varinjlim_{i \in I} M_i$, where $M_i \in \mathcal{C} \cap \text{mod–}R$ for all $i \in I$. If $\mathfrak{p} \notin \operatorname{Ass}(\mathcal{C} \cap \text{mod–}R)$, then $\operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), (M_i)_\mathfrak{p}) = 0$ for all $i \in I$ by Lemma 1.3, and so

$$\operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \cong \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), \varinjlim_{i \in I}(M_i)_\mathfrak{p}) \cong \varinjlim_{i \in I} \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), (M_i)_\mathfrak{p}) = 0.$$

Thus, $\mathfrak{p} \notin \operatorname{Ass} M$ by Lemma 1.3, as required. □

Now, we will give a connection to tilting classes. For this purpose, we recall a concept from [3].

Definition 2.7. Let $C \in \text{Mod–}R$ and $P_1 \xrightarrow{f} P_0 \to C \to 0$ be a projective presentation in Mod–R. Then an *Auslander-Bridger transpose* of C, denoted by $\operatorname{Tr}(C)$, is the cokernel of f^*, where $(-)^* = \operatorname{Hom}_R(-, R)$. That is, we have an exact sequence

$$P_0^* \xrightarrow{f^*} P_1^* \longrightarrow \operatorname{Tr}(C) \longrightarrow 0.$$

Note that by [3, Corollary 2.3], $\operatorname{Tr}(C)$ is uniquely determined up to adding or splitting off a projective summand. An easy lemma shows that for indecomposable cyclic modules, the transpose has in some cases projective dimension ≤ 1.

Lemma 2.8. *Let* $\mathfrak{p} \in \operatorname{Spec}(R)$ *such that* $\operatorname{Ass} R \cap V(\mathfrak{p}) = \emptyset$. *Then we have:*

(i) $\operatorname{proj.dim}_R \operatorname{Tr}(R/\mathfrak{p}) \leq 1$;
(ii) $\operatorname{Hom}_R(R/\mathfrak{p}, -)$ *and* $\operatorname{Tor}_1^R(\operatorname{Tr}(R/\mathfrak{p}), -)$ *are isomorphic functors.*

Proof. (i) Let $x_1, \dots, x_n$ be a set of generators of $\mathfrak{p}$ and form them into a row vector (x_n). Then we have an exact sequence

$$R^n \xrightarrow{(x_n)} R \longrightarrow R/\mathfrak{p} \longrightarrow 0,$$

and the transpose fits into an exact sequence

$$0 \longrightarrow K \longrightarrow R \xrightarrow{(x_n)^T} R^n \longrightarrow \operatorname{Tr}(R/\mathfrak{p}) \longrightarrow 0.$$

In particular, $K = \{x \in R \mid \mathfrak{p} \cdot x = 0\}$, so $K \in \text{Mod–}R/\mathfrak{p}$ and $\operatorname{Ass} K \subseteq V(\mathfrak{p})$. Since we assume $\operatorname{Ass} R \cap V(\mathfrak{p}) = \emptyset$, we have $K = 0$. Thus, $\text{proj.dim}_R \text{Tr}(R/\mathfrak{p}) \leq 1$.

(ii) With the notation above, one immediately deduces that both $\operatorname{Hom}_R(R/\mathfrak{p}, M)$ and $\operatorname{Tor}_1^R(\text{Tr}(R/\mathfrak{p}), M)$, where $M \in \text{Mod–}R$, are computed as the kernel of the map

$$M \xrightarrow{(x_n)^T} M^n. \qquad \square$$

Now we summarize our findings in the main theorem of the section.

Theorem 2.9. *Let R be a commutative noetherian ring. Then there are bijections between the following sets:*

(i) *1–tilting classes $\mathcal{T}$ in* Mod–R*.*
(ii) *1–cotilting classes $\mathcal{C}$ in* Mod–R*.*
(iii) *Subsets $Y \subseteq \operatorname{Spec}(R)$ closed under specialization such that $\operatorname{Ass} R \cap Y = \emptyset$.*
(iv) *Faithful hereditary torsion pairs $(\mathcal{T}', \mathcal{F}')$ in* Mod–R*.*
(v) *Torsion pairs $(\mathcal{T}'', \mathcal{F}'')$ in* mod–R *such that $R \in \mathcal{F}''$.*

Proof. Let us first explicitly state the bijections:

Bijection	Assignment
(i) → (ii)	$\mathcal{T} \mapsto ({}^\perp\mathcal{T} \cap \text{mod–}R)^\intercal$
(ii) → (iii)	$\mathcal{C} \mapsto \operatorname{Spec}(R) \setminus \operatorname{Ass}(\mathcal{C} \cap \text{mod–}R)$
(iii) → (ii)	$Y \mapsto \{M \in \text{Mod–}R \mid \operatorname{Ass} M \cap Y = \emptyset\}$
(ii) → (iv)	$\mathcal{C} \mapsto \mathcal{F}'$
(iv) → (v)	$\mathcal{F}' \mapsto \mathcal{F}' \cap \text{mod–}R$
(v) → (ii)	$\mathcal{F}'' \mapsto \varinjlim \mathcal{F}''$

For the first line of the table, the assignment is injective by the discussion at the end of Section 1.3. On the other hand, given a 1–cotilting class $\mathcal{C}$, $X = \operatorname{Ass}(\mathcal{C} \cap \text{mod–}R)$ and $Y = \operatorname{Spec}(R) \setminus X$, then Theorem 2.6, Proposition 2.2 and Lemma 2.8 tell us that

$$\mathcal{C} = \{M \mid \operatorname{Hom}_R(R/\mathfrak{p}, M) = 0 \text{ for all } \mathfrak{p} \in Y\} = \{M \mid \operatorname{Tor}_1^R(\text{Tr}(R/\mathfrak{p}), M) = 0 \text{ for all } \mathfrak{p} \in Y\}.$$

The preimage of $\mathcal{C}$ under the assignment is by Lemma 1.11 the 1–tilting class

$$\mathcal{T} = \{M \mid \operatorname{Ext}_R^1(\text{Tr}(R/\mathfrak{p}), M) = 0 \text{ for all } \mathfrak{p} \in Y\},$$

Hence we have a bijection.

The second, third and fourth line in the table are covered by Theorem 2.6. The fifth line follows from Propositions 2.2 and 2.4, while the sixth line is implied by Proposition 2.5. $\square$

3. General tilting and cotilting classes

In this section we classify all n–tilting and n–cotilting classes in Mod–R. Unfortunately, our methods do not seem to provide much information on the corresponding n–(co)tilting modules. Except for special classes of examples in [14, Chapters 5, 6 and 8], the only known way to construct, say, a cotilting module for a cotilting class $\mathcal{C}$, seems to be as in the proof of [14, Theorem 8.1.9], using so–called special $\mathcal{C}$–precovers.

Let us first introduce the sequences of subsets of $\mathrm{Spec}(R)$ (where R is commutative noetherian under the standing assumption) which will parametrize both n–tilting and n–cotilting classes for given $n \geq 1$.

Definition 3.1. In the following $(Y_1, \ldots, Y_n)$ will always denote the sequence of subsets of $\mathrm{Spec}(R)$ such that

(i) Y_i is closed under specialization for all $1 \leq i \leq n$;
(ii) $Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n$;
(iii) $(\mathrm{Ass}\, \mho_{i-1}(R)) \cap Y_i = \emptyset$ for all $1 \leq i \leq n$;

and X_i will always denote $\mathrm{Spec}(R) \setminus Y_i$. For any such $(Y_1, \ldots, Y_n)$ we define the class of modules

$$\mathcal{C}_{(Y_1,\ldots,Y_n)} = \{M \in \text{Mod–}R \mid (\mathrm{Ass}\, \mho_{i-1}(M)) \cap Y_i = \emptyset \text{ for all } 1 \leq i \leq n\}$$

Remark 3.2. Equivalently by Lemma 1.3, we can write

$$\mathcal{C}_{(Y_1,\ldots,Y_n)} = \{M \in \text{Mod–}R \mid \mu_{i-1}(\mathfrak{p}, M) = 0 \text{ for all } 1 \leq i \leq n \text{ and } \mathfrak{p} \in Y_i\}.$$

Using a well–known result on Bass invariants of finitely generated modules, see e.g. [13, Proposition 9.2.13], it follows that

(iii*) $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{ht}\, \mathfrak{p} = i - 1\} \subseteq X_i$ for all $1 \leq i \leq n$.

For Gorenstein rings we have (iii) $\Leftrightarrow$ (iii*) by [17, Theorem 18.8], but for general commutative noetherian rings condition (iii) may be more restrictive. In an extreme case, it may prevent existence of any non-trivial sequences $(Y_1, \ldots, Y_n)$ as in the following example.

Example 3.3. Let k be a field, $S = k[x, y]/(x^2, xy)$, and let $(R, \mathfrak{m}, k)$ be the localization of S at the maximal ideal (x, y). It is easy to check that the ideal $(x) \subseteq R$ is simple, so $\mathfrak{m} \in \mathrm{Ass}\, R$. Hence given any $(Y_1, \ldots, Y_n)$ as in Definition 3.1, we necessarily have $Y_i = \emptyset$ for all $1 \leq i \leq n$ and $\mathcal{C}_{(Y_1,\ldots,Y_n)} = \text{Mod–}R$. In view of the main theorem below, this implies that there are no non-trivial tilting or cotilting classes over this ring R.

Our next task is to prove that $\mathcal{C}_{(Y_1,\ldots,Y_n)}$ are precisely the n–cotilting classes in Mod–R. The following definition and lemma will allow us to use induction on n.

Definition 3.4. For any cotilting module $C \in \text{Mod–}R$, the corresponding cotilting class $\mathcal{C} = {}^{\perp}C$ and $j \geq 1$, we define the class

$$\mathcal{C}_{(j)} = {}^{\perp}\mho_j(C) = \{M \in \text{Mod–}R \mid \mathrm{Ext}^i_R(M, C) = 0 \text{ for all } i \geq j\}.$$

Lemma 3.5. *Let $\mathcal{C} = {}^{\perp}C$ be an n–cotilting class. Then $\mathcal{C}_{(j)}$ is an $(n-j+1)$–cotilting class for any $j \leq n+1$.*

Proof. The class $\mathcal{C}_{(j)}$ is closed under direct products by [4, Lemma 3.4] (see also [14, Proposition 8.1.5(a)]). The rest follows from the characterization of cotilting classes in [14, Corollary 8.1.10]. There, one uses the notion of cotorsion pairs introduced below in Definition 3.12 □

Now we can state one of our main classification results.

Theorem 3.6. *Let R be a commutative noetherian ring and $n \geq 1$. Then the assignments*

$$\Phi\colon \qquad \mathcal{C} \longmapsto (\operatorname{Spec}(R) \setminus \operatorname{Ass}\mathcal{C}_{(1)}, \dots, \operatorname{Spec}(R) \setminus \operatorname{Ass}\mathcal{C}_{(n)}),$$

$$\Psi\colon \quad (Y_1, \dots, Y_n) \longmapsto \mathcal{C}_{(Y_1,\dots,Y_n)}$$

give mutually inverse bijections between the sequences of subsets $(Y_1, \dots, Y_n)$ of $\operatorname{Spec}(R)$ *satisfying the three conditions of Definition 3.1, and the n–cotilting classes $\mathcal{C}$ in* Mod–R*.*

We will prove the theorem in several steps. Let us start with preparatory results. We first prove that Ψ is injective. However, we postpone for the moment a proof of the important fact that Ψ is well-defined in the sense that each class of the form $\mathcal{C}_{(X_1,\dots,X_n)}$ is actually cotilting.

Lemma 3.7. *Let $(Y_1, \dots, Y_n)$ and $(Y_1', \dots, Y_n')$ be two sequences as in Definition 3.1. Then $\mathcal{C}_{(Y_1,\dots,Y_n)} = \mathcal{C}_{(Y_1',\dots,Y_n')}$ if and only if $(Y_1, \dots, Y_n) = (Y_1', \dots, Y_n')$.*

Proof. We only have to prove that $\mathcal{C}_{(Y_1,\dots,Y_n)} \neq \mathcal{C}_{(Y_1',\dots,Y_n')}$ whenever $(Y_1, \dots, Y_n) \neq (Y_1', \dots, Y_n')$. Thus suppose that there is $1 \leq i \leq n$ and $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} \in Y_i' \setminus Y_i$. Note that by examining condition (iii) of Definition 3.1 for Y_i', this only can happen if

$$\mu_j(\mathfrak{p}, R) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^j_{R_\mathfrak{p}}(k(\mathfrak{p}), R_\mathfrak{p}) = 0 \qquad \text{for all } 0 \leq j \leq i-1.$$

Denoting by M an $(i-1)$-th syzygy module of $k(\mathfrak{p})$, we claim that $M \in \mathcal{C}_{(Y_1,\dots,Y_n)} \setminus \mathcal{C}_{(Y_1',\dots,Y_n')}$. Indeed, by [13, Theorem 3.3.8] it is clear that $\operatorname{Ass} k(\mathfrak{p}) = \{\mathfrak{p}\}$ and Corollary 1.5 gives us for each $0 \leq j \leq i-1$:

$$\operatorname{Ass} \mho_j(M) \subseteq \begin{cases} \bigcup_{k=0}^{i-1} \operatorname{Ass} \mho_{j-k}(R) \cup \{\mathfrak{p}\} & \text{for } j = i-1 \\ \bigcup_{k=0}^{i-1} \operatorname{Ass} \mho_{j-k}(R) & \text{for } j \neq i-1. \end{cases}$$

Using Definition 3.1 for $(Y_1, \dots, Y_n)$, one easily checks that $M \in \mathcal{C}_{(Y_1,\dots,Y_n)}$.

On the other hand, a straightforward dimension shifting argument based on the fact that $\operatorname{Ext}^j_{R_\mathfrak{p}}(k(\mathfrak{p}), R_\mathfrak{p}) = 0$ for all $0 \leq j \leq i-1$ gives us that

$$\operatorname{Ext}^{i-1}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \cong \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), k(\mathfrak{p})) \neq 0,$$

so $\mu_{i-1}(\mathfrak{p}, M) \neq 0$ by Lemma 1.3 and $M \notin \mathcal{C}_{(Y_1',\dots,Y_n')}$. □

Next, observe a consequence of the fact that every cotilting class is closed under taking direct limits (see [14, Theorem 8.1.7]).

Lemma 3.8. *Let $\mathcal{C}$ be a cotilting class in* Mod–R*, and let $M \in \mathcal{C}$ and F be a flat R-module. Then $M \otimes_R F \in \mathcal{C}$. In particular, $M_\mathfrak{p} \in \mathcal{C}$ for any $M \in \mathcal{C}$ and $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Proof. Using Lazard's theorem (see e.g. [14, Corollary 1.2.16]), we can express F as a direct limit $F = \varinjlim_{i \in I} F_i$ of finitely generated free modules F_i. In particular, $M \otimes_R F_i \cong M^{n_i} \in \mathcal{C}$ for each $i \in I$. Since $\mathcal{C}$ is closed under taking direct limits by [14, Theorem 8.1.7], we have $M \otimes_R F \cong \varinjlim_{i \in I} M \otimes_R F_i \in \mathcal{C}$. The last assertion follows since $M_\mathfrak{p} \cong M \otimes_R R_\mathfrak{p}$ and $R_\mathfrak{p}$ is flat as an R-module. □

Another observation gives us relation between $\mathcal{C}$ and $\mathcal{C}_{(2)}$ (see Definition 3.4 and Lemma 3.5).

Lemma 3.9. *Let $\mathcal{C}$ be a cotilting class and*

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$$

be a short exact sequence such that $L \in \mathcal{C}$. Then $K \in \mathcal{C}$ if and only if $M \in \mathcal{C}_{(2)}$.

Proof. Let C be a cotilting module for $\mathcal{C}$. Then $\operatorname{Ext}^i_R(K, C) \cong \operatorname{Ext}^{i+1}_R(M, C)$ for each $i \geq 1$. The conclusion follows directly from the definition. □

Now we prove another part of Theorem 3.6, namely that $\Psi \circ \Phi = id$. In fact, we postpone at the moment a proof that Φ is well defined in the sense that the sequence $(\operatorname{Spec}(R) \setminus \operatorname{Ass} \mathcal{C}_{(1)}, \dots, \operatorname{Spec}(R) \setminus \operatorname{Ass} \mathcal{C}_{(n)})$ of subsets of $\operatorname{Spec}(R)$ satisfies for each cotilting class $\mathcal{C}$ the conditions in Definition 3.1.

Proposition 3.10. *Let $n \geq 1$ and $\mathcal{C}$ be an n–cotilting class. Then the following hold:*

(i) *If $\mathfrak{p} \in \operatorname{Ass} \mathcal{C}$, then $k(\mathfrak{p}) \in \mathcal{C}$.*
(ii) *$\mathcal{C}$ is closed under taking injective envelopes.*
(iii) *If we put $X_i = \operatorname{Ass} \mathcal{C}_{(i)}$ and $Y_i = \operatorname{Spec}(R) \setminus X_i$ for $1 \leq i \leq n$, then*

$$\mathcal{C} = \{M \in \text{Mod–}R \mid \operatorname{Ass} E_{i-1}(M) \cap Y_i = \emptyset \text{ for all } 1 \leq i \leq n\}.$$

Proof. We will prove the statement by induction. More precisely, we will show that (i) holds for $n = 1$, that (i) ⇒ (ii) ⇒ (iii) for each fixed $n \geq 1$, and finally that (iii) for $n - 1$ implies (i) for n if $n > 1$.

(i) for $n = 1$: Suppose that $\mathfrak{p} \in \operatorname{Ass} \mathcal{C}$. That is, $R/\mathfrak{p} \subseteq M$ for some $M \in \mathcal{C}$. By possibly using Lemma 3.8 and passing from M to $M_\mathfrak{p}$, we may assume that $k(\mathfrak{p}) \subseteq M$. Since by Theorem 2.6 any 1–cotilting class $\mathcal{C}$ is torsion–free and hence closed under taking submodules, then also $k(\mathfrak{p}) \in \mathcal{C}$.

(ii) By [16, Theorem 3.4], $E(R/\mathfrak{p})$ is $k(\mathfrak{p})$-filtered and by Lemma 1.3 for $i = 0$, $\operatorname{Ass} M = \operatorname{Ass} E(M)$. Thus $\mathcal{C}$ is closed under injective envelopes.

(iii) Using (ii) as the inductive hypothesis and Lemma 3.9, it is easy to see that a module M belongs to $\mathcal{C}$ if and only if $E(M) \in \mathcal{C}$ and $\mho(M) \in \mathcal{C}_{(2)}$. Moreover, we know that the indecomposable summands of $E(M)$ are precisely $E(R/\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Ass} M$, and (i) and (ii) imply that

$$E(R/\mathfrak{p}) \in \mathcal{C} \quad \Rightarrow \quad \mathfrak{p} \in \operatorname{Ass} \mathcal{C} \quad \Rightarrow \quad k(\mathfrak{p}) \in \mathcal{C} \quad \Rightarrow \quad E(R/\mathfrak{p}) \in \mathcal{C}.$$

Thus, we have shown that $E(M) \in \mathcal{C}$ if and only if $\operatorname{Ass} M \subseteq X_1 = \operatorname{Ass} \mathcal{C}$.

If $n = 1$, then $\mathcal{C}_{(2)} = \text{Mod–}R$ and we are done. If $n > 1$, it follows from (iii) for $n - 1$ and Lemma 1.3 for the sets X_i that

$$\mathcal{C}_{(2)} = \{L \in \text{Mod–}R \mid \operatorname{Ass} E_{i-2}(L) \subseteq X_i \text{ for all } 2 \leq i \leq n\}.$$

In particular, $\mho(M) \in \mathcal{C}_{(2)}$ if and only if $\operatorname{Ass} E_{i-1}(M) \subseteq X_i$ for all $2 \leq i \leq n$. The conclusion follows.

(i) for $n > 1$: Suppose that $\mathfrak{p} \in \operatorname{Ass} \mathcal{C}$. As above, we find $M \in \mathcal{C}$ such that $k(\mathfrak{p}) \subseteq M$. To show that $k(\mathfrak{p}) \in \mathcal{C}$, it suffices in view of Lemma 3.9 to show that $M/k(\mathfrak{p}) \in \mathcal{C}_{(2)}$. Using [13, Theorem 3.3.8], it follows by induction that each cosyzygy of $k(\mathfrak{p})$ is $k(\mathfrak{p})$–filtered and hence $\operatorname{Ass} \mho_i(k(\mathfrak{p})) \subseteq \{\mathfrak{p}\}$ for each $i \geq 0$. In particular, Lemma 1.4(iii) then implies that

$$\operatorname{Ass} \mho_i(M/k(\mathfrak{p})) \subseteq \operatorname{Ass} \mho_i(M) \cup \{\mathfrak{p}\} \qquad \text{for each } i \geq 0.$$

Now clearly, $M \in \mathcal{C} \subseteq \mathcal{C}_{(2)}$, so $\operatorname{Ass} E_{i-1}(M) \subseteq \operatorname{Ass} \mathcal{C}_{(i+1)}$ for all $1 \leq i \leq n - 1$. Further, $\mathfrak{p} \in \operatorname{Ass} \mathcal{C}_{(i+1)}$ for all $1 \leq i \leq n-1$ since $\mathfrak{p} \in \operatorname{Ass} \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{C}_{(i+1)}$. Using the inductive hypothesis, that is, assertion (iii) for $n - 1$, we infer that $M/k(\mathfrak{p}) \in \mathcal{C}_{(2)}$ and conclude the proof. □

Let us summarize what has been done so far. We have proved that the assignment Ψ in Theorem 3.6 is injective, and that $\Psi \circ \Phi = id$. What is left to show is that any sequence of subsets in the image of Φ meets the requirements of Definition 3.1, and that any class obtained by an application of Ψ is actually cotilting. We start with the former statement, which is easier.

Lemma 3.11. *Let $n \geq 1$ and $\mathcal{C}$ be an n–cotilting class. If we put $X_i = \operatorname{Ass} \mathcal{C}_{(i)}$ and $Y_i = \operatorname{Spec}(R) \setminus X_i$ for $1 \leq i \leq n$, then the sequence $(Y_1, \dots, Y_n)$ of subsets of* Spec(R) *satisfies conditions* (i)–(iii) *in Definition 3.1.*

Proof. Clearly, (ii) and (iii) are satisfied since $\mathcal{C} = \mathcal{C}_{(1)} \subseteq \mathcal{C}_{(2)} \subseteq \cdots \subseteq \mathcal{C}_{(n)}$ and $R \in \mathcal{C}$. We have to show (i), that is, show that each Y_i is closed under specialization or equivalently that each X_i is closed under generalization. To this end, suppose that $\mathfrak{p} \in X_i$. Then $k(\mathfrak{p}) \in \mathcal{C}_{(i)}$ by Proposition 3.10(i), which implies that $E(k(\mathfrak{p})) \in \mathcal{C}_{(i)}$. Since

$E_R(k(\mathfrak{p})) \cong E_{R_\mathfrak{p}}(k(\mathfrak{p})) = E_R(R/\mathfrak{p})$ by [13, Theorem 3.3.3], $\mathcal{C}_{(i)}$ contains an injective cogenerator for Mod–$R_\mathfrak{p}$. Given any $\mathfrak{q} \subseteq \mathfrak{p}$ in $\operatorname{Spec}(R)$, $E(R/\mathfrak{q})$ is an injective $R_\mathfrak{p}$-module by [13, Theorem 3.3.8(1)], so it is a summand in $E_R(R/\mathfrak{p})^I$ for some set I. It follows, using that $\mathcal{C}_{(i)}$ is closed under arbitrary direct products and direct summands, that $E(R/\mathfrak{q}) \in \mathcal{C}_{(i)}$ and $\mathfrak{q} \in X_i = \operatorname{Ass} \mathcal{C}_{(i)}$. □

Finally, we are going to prove that each class $\mathcal{C} = \mathcal{C}_{(Y_1,\dots,Y_n)}$ as in Definition 3.1 is n–cotilting. However, for that we need a few definitions first.

Definition 3.12. A class $\mathcal{C}$ of modules is called *definable* if it is closed under products, direct limits and pure submodules. A pair $(\mathcal{C}, \mathcal{D})$ of classes of modules is called a *cotorsion pair* if

$$\mathcal{D} = \{D \in \text{Mod–}R \mid \operatorname{Ext}^1_R(C, D) = 0 \text{ for all } C \in \mathcal{C}\} \quad \text{and}$$
$$\mathcal{C} = \{C \in \text{Mod–}R \mid \operatorname{Ext}^1_R(C, D) = 0 \text{ for all } D \in \mathcal{D}\}.$$

A cotorsion pair $(\mathcal{C}, \mathcal{D})$ is called hereditary if $\mathcal{C}$ is closed under taking syzygies.

The following characterization of n–cotilting classes will be useful for completing our feat:

Proposition 3.13. *Let $n \geq 0$ and $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is n–cotilting if and only if all of the following conditions are satisfied:*

(i) *$\mathcal{C}$ is definable,*
(ii) *$R \in \mathcal{C}$,*
(iii) *$\mathcal{C}$ is closed under taking syzygies, and*
(iv) *each n-th syzygy module belongs to $\mathcal{C}$.*

Proof. If $\mathcal{C}$ is cotilting, then $\mathcal{C}$ is definable by [14, Theorem 8.1.7]. Clearly $R \in \mathcal{C}$ and conditions (iii) and (iv) are obtained by simple dimension shifting.

Assume on the other hand that (i)–(iv) hold. Using [14, Lemma 1.2.17], we can construct for each $M \in \mathcal{C}$ a well–ordered chain of pure submodules

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\alpha \subseteq M_{\alpha+1} \subseteq \cdots M_\kappa = M,$$

of M such that $|M_{\alpha+1}/M_\alpha| \leq |R| + \aleph_0$ for each $\alpha < \kappa$ and $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ for every limit ordinal $\beta \leq \kappa$. Since definable classes are closed under taking pure epimorphic images by [19, Theorem 3.4.8], it follows that each subfactor $M_{\alpha+1}/M_\alpha$ belongs to $\mathcal{C}$. In particular, it follows easily that $M \in \mathcal{C}$ if and only if M is $\mathcal{S}$–filtered, where $\mathcal{S}$ is a representative set for modules of cardinality $\leq |R| + \aleph_0$ in $\mathcal{C}$. Since clearly $R \in \mathcal{S}$, we can use [14, Corollary 3.2.4 and Lemma 4.2.10] to infer that $\mathcal{C}$ fits into some hereditary cotorsion pair $(\mathcal{C}, \mathcal{D})$. A simple dimension shifting using condition (iv) tells us that all modules in $\mathcal{D}$

have injective dimension at most n. Thus, $\mathcal{C}$ is an n–cotilting class by [14, Corollary 8.1.10]. □

Now we are ready to give the last piece of the proof of Theorem 3.6.

Proposition 3.14. *Let* $(Y_1, \dots, Y_n)$ *be a sequence of subsets of* $\mathrm{Spec}(R)$ *meeting the requirements of Definition 3.1. Then the class* $\mathcal{C} = \mathcal{C}_{(Y_1,\dots,Y_n)}$ *is* n*–cotilting.*

Proof. We use the characterization of n–cotilting classes from Proposition 3.13. Clearly, $R \in \mathcal{C}$ by the assumptions on $(Y_1, \dots, Y_n)$, and conditions (iii) and (iv) of Proposition 3.13 follow easily from Lemma 1.4 and Corollary 1.5 (see also Remark 1.6). Thus, it only remains to prove that $\mathcal{C}$ is definable.

To this end, note first that for a family of modules, the product of injective coresolutions of the modules is a (possibly non–minimal) injective coresolution of the product of the modules. Using the fact that all $\mathrm{Spec}(R) \setminus Y_i$ are closed under generalization and an argument similar to the one in the proof of Lemma 3.11, one checks that $\mathcal{C}$ is closed under products.

Assume next that $M \in \mathcal{C}$ and $K \subseteq M$ is a pure submodule. To prove that $K \in \mathcal{C}$, we must show that for each $1 \le i \le n$ and $\mathfrak{p} \in Y_i$, we have

$$\mu_i(\mathfrak{p}, K) = \dim_{k(\mathfrak{p})} \mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), K_\mathfrak{p}) = 0.$$

Since the embedding $K \subseteq M$ is a direct limit of split monomorphisms and localizing at $\mathfrak{p}$ preserves direct limits, also the embedding $K_\mathfrak{p} \subseteq M_\mathfrak{p}$ is pure. The conclusion that $\mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), K_\mathfrak{p}) = 0$ then follows from the fact that $k(\mathfrak{p})$ is a finitely generated $R_\mathfrak{p}$-module and thus the class

$$\{N \in \mathrm{Mod}\text{–}R_\mathfrak{p} \mid \mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), N) = 0\}$$

is definable in $\mathrm{Mod}\text{–}R_\mathfrak{p}$, see [14, Example 3.1.11].

The proof that $\mathcal{C}$ is closed under direct limits is similar. Namely for each $1 \le i \le n$ and $\mathfrak{p} \in Y_i$, the class

$$\{M \in \mathrm{Mod}\text{–}R \mid \mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = 0\}$$

is the kernel of the composition of two direct limit preserving functors: the localization at $\mathfrak{p}$ and the functor $\mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), -)$; and $\mathcal{C}$ is the intersection of all these classes. □

Proof of Theorem 3.6. Lemma 3.11 and Proposition 3.14 show that Φ assigns to each n–cotilting class a sequence satisfying the conditions of Definition 3.1, and conversely that Ψ assigns to each such sequence an n–cotilting class. Further, we have proved in Lemma 3.7 and Proposition 3.10 that Ψ is injective and $\Psi \circ \Phi = id$. Thus, Φ and Ψ are mutually inverse bijections. □

We conclude the discussion by two consequences. We clarify the relation of passing from $\mathcal{C}$ to $\mathcal{C}_{(j)}$ in the sense of Definition 3.4, to the corresponding filtrations of subsets of the spectrum.

Corollary 3.15. *Let $(Y_1, \dots, Y_n)$ be as in Definition 3.1. Then for any natural number $j \leq n$ we have $(\mathcal{C}_{(Y_1,\dots,Y_n)})_{(j)} = \mathcal{C}_{(Y_j,\dots,Y_n)}$.*

Proof. Since we know that $\mathcal{C}_{(Y_1,\dots,Y_n)}$ is an n–cotilting class, the statement follows directly from Proposition 3.10(iii). $\square$

Further, we show that the dimension shifting in the sense of Definition 3.4 works nicely also at the level of cotilting modules.

Corollary 3.16. *Let C be an n–cotilting module ($n \geq 2$) with the corresponding cotilting class given by $(Y_1, \dots, Y_n)$. Then $D = \mho(C) \oplus \bigoplus_{\mathfrak{p} \in X_2} E(R/\mathfrak{p})$ is an $(n-1)$–cotilting module with the corresponding cotilting class given by $(Y_2, \dots, Y_n)$.*

Proof. Denote $\mathcal{C} = {}^{\perp}C$ the cotilting class. Clearly ${}^{\perp}D = {}^{\perp}\mho(C) = \mathcal{C}_{(2)}$ which is the $(n-1)$–cotilting class given by $(Y_2, \dots, Y_n)$.

The module D obviously satisfies (C1) of Definition 1.10. Condition (C2) also holds for D since for any $i \geq 1$ and any cardinal κ we have $D \in \mathcal{C}_{(2)}$ and $D^\kappa \in \mathcal{C}_{(2)} = {}^{\perp}D$, so $\operatorname{Ext}^i_R(D^\kappa, D) = 0$. To prove (C3), it is by [4, Lemma 3.12] enough to show that $\mathcal{C}_{(2)} \subseteq \operatorname{Cog} D$, that is, each $M \in \mathcal{C}_{(2)}$ is cogenerated by D. We will show more, namely that

$$\{M \in \text{Mod–}R \mid \operatorname{Ass} M \subseteq X_2\} \subseteq \operatorname{Cog} D.$$

Indeed, taking any M with $\operatorname{Ass} M \subseteq X_2$, we have

$$M \subseteq E(M) = \bigoplus_{\mathfrak{p} \in \operatorname{Ass} M} E(R/\mathfrak{p})^{(\mu_0(\mathfrak{p},M))} \subseteq \prod_{\mathfrak{p} \in \operatorname{Ass} M} E(R/\mathfrak{p})^{\mu_0(\mathfrak{p},M)} \in \operatorname{Cog} D.$$

$\square$

The final aim is to prove that the correspondence $T \mapsto T^+$ induces a bijection between n–tilting and n–cotilting classes. We start with an extending of Lemma 2.8 to higher projective dimensions.

Lemma 3.17. *Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and $n \geq 0$ such that $\operatorname{Ass} \mho_i(R) \cap V(\mathfrak{p}) = \emptyset$ for each $i = 0, 1, \dots, n$. Then we have:*

(i) $\operatorname{proj.dim}_R \operatorname{Tr}(R/\mathfrak{p}) \leq n$;
(ii) $\operatorname{Ext}^n_R(R/\mathfrak{p}, -)$ *and* $\operatorname{Tor}^R_1(\operatorname{Tr}(\Omega^{n-1}(R/\mathfrak{p})), -)$ *are isomorphic functors.*
(iii) $\operatorname{Ext}^1_R(\operatorname{Tr}(\Omega^{n-1}(R/\mathfrak{p})), -)$ *and* $\operatorname{Tor}^R_n(R/\mathfrak{p}, -)$ *are isomorphic functors.*

Proof. (i) Consider the beginning of a projective resolution of $R/\mathfrak{p}$:

$$P_{n+1} \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} \dots \xrightarrow{f_1} P_0 \longrightarrow R/\mathfrak{p} \longrightarrow 0.$$

Denoting as in Definition 2.7 by $(-)^*$ the functor $\mathrm{Hom}_R(-, R)$, we get a sequence

$$0 \longleftarrow \mathrm{Tr}(\Omega^{n-1}(R/\mathfrak{p})) \longleftarrow P^*_{n+1} \xleftarrow{f^*_{n+1}} P^*_n \xleftarrow{f^*_n} \dots \xleftarrow{f^*_1} P^*_0 \longleftarrow 0.$$

We claim that the latter sequence is exact. Equivalently, we must prove that

$$\mathrm{Ext}^i_R(R/\mathfrak{p}, R) = 0 \qquad \text{for all } i = 0, 1, \dots, n.$$

Using the isomorphisms $(\mathrm{Ext}^i_R(R/\mathfrak{p}, R))_\mathfrak{m} \cong \mathrm{Ext}^i_{R_\mathfrak{m}}((R/\mathfrak{p})_\mathfrak{m}, R_\mathfrak{m})$ (see [13, Corollary 3.2.6]), we must equivalently prove that

$$\mathrm{Ext}^i_{R_\mathfrak{m}}((R/\mathfrak{p})_\mathfrak{m}, R_\mathfrak{m}) \qquad \text{for all } i = 0, 1, \dots, n \text{ and } \mathfrak{m} \subseteq R \text{ maximal.}$$

However, the latter follows from the assumption and Lemma 1.3 if $\mathfrak{m} \supseteq \mathfrak{p}$, and from the fact that $(R/\mathfrak{p})_\mathfrak{m} = 0$ if $\mathfrak{m} \not\supseteq \mathfrak{p}$. This proves the claim.

(ii), (iii) These parts follow immediately using the well–known natural isomorphisms $\mathrm{Hom}_R(P, M) \cong P^* \otimes_R M$ and $\mathrm{Hom}_R(P^*, M) \cong P \otimes_R M$ for any $P, M \in \text{Mod–}R$ with P finitely generated projective. □

Next we need a translation of the definition of $\mathcal{C}_{(Y_1,\dots,Y_n)}$ to a homological condition.

Lemma 3.18. *Let* $(Y_1, \dots, Y_n)$ *be as in Definition 3.1. Then*

$$\mathcal{C}_{(Y_1,\dots,Y_n)} = \{M \in \text{Mod–}R \mid \mathrm{Ext}^{i-1}_R(R/\mathfrak{p}, M) = 0 \textit{ for all } i = 1, \dots, n \textit{ and } \mathfrak{p} \in Y_i\}.$$

Proof. If M is in the class on the right hand side, then the isomorphisms

$$0 = (\mathrm{Ext}^{i-1}_R(R/\mathfrak{p}, M))_\mathfrak{p} \cong \mathrm{Ext}^{i-1}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}),$$

together with Lemma 1.3 yield $\mu_{i-1}(\mathfrak{p}, M) = 0$ for all $i = 1, \dots, n$ and $\mathfrak{p} \in Y_i$. Thus, $M \in \mathcal{C}_{(Y_1,\dots,Y_n)}$.

Conversely, suppose that $M \in \mathcal{C}_{(Y_1,\dots,Y_n)}$, $1 \leq i \leq n$ and $\mathfrak{p} \in Y_i$. We must prove that $\mathrm{Ext}^{i-1}_R(R/\mathfrak{p}, M) = 0$. Consider the beginning of an injective coresolution of M

$$0 \longrightarrow M \longrightarrow E_0(M) \longrightarrow E_1(M) \longrightarrow \cdots \longrightarrow E_{n-1}(M) \longrightarrow E_n(M).$$

Then any element of $\mathrm{Ext}^{i-1}_R(R/\mathfrak{p}, M)$ is represented by a homomorphism $f \in \mathrm{Hom}_R(R/\mathfrak{p}, E_{i-1}(M))$. Now on one hand, $\mathrm{Im}\, f$ is an $R/\mathfrak{p}$–module, so $\mathrm{Ass}\,(\mathrm{Im}\, f) \subseteq V(\mathfrak{p}) \subseteq Y_i$. On the other hand, $\mathrm{Ass}\,(\mathrm{Im}\, f) \subseteq \mathrm{Ass}\, E_{i-1}(M) \subseteq \mathrm{Spec}(R) \setminus Y_i$ since $M \in \mathcal{C}_{(Y_1,\dots,Y_n)}$; see Definition 3.1. Thus, $f = 0$ and so is $\mathrm{Ext}^{i-1}_R(R/\mathfrak{p}, M)$. □

Now we are in a position to state and prove the main classification result.

Theorem 3.19. *Let* R *be a commutative noetherian ring and* $n \geq 1$. *Then there are bijections between:*

(i) *Sequence* $(Y_1, \dots, Y_n)$ *of subsets of* $\mathrm{Spec}(R)$ *as in Definition 3.1;*

(ii) *n–tilting classes* $\mathcal{T} \subseteq \text{Mod–}R$*;*
(iii) *n–cotilting classes* $\mathcal{C} \subseteq \text{Mod–}R$.

The bijections assign to $(Y_1, \ldots, Y_n)$ *the n–tilting class*

$$\mathcal{T} = \{M \in \text{Mod–}R \mid \text{Tor}_{i-1}^R(R/\mathfrak{p}, M) = 0 \text{ for all } i = 1, \ldots, n \text{ and } \mathfrak{p} \in Y_i\} =$$
$$\{M \in \text{Mod–}R \mid \text{Ext}_R^1(\text{Tr}(\Omega^{n-1}(R/\mathfrak{p})), M) = 0 \text{ for all } i = 1, \ldots, n \text{ and } \mathfrak{p} \in Y_i\},$$

and the n–cotilting class

$$\mathcal{C} = \{M \in \text{Mod–}R \mid \text{Ext}_R^{i-1}(R/\mathfrak{p}, M) = 0 \text{ for all } i = 1, \ldots, n \text{ and } \mathfrak{p} \in Y_i\} =$$
$$\{M \in \text{Mod–}R \mid \text{Tor}_1^R(\text{Tr}(\Omega^{n-1}(R/\mathfrak{p})), M) = 0 \text{ for all } i = 1, \ldots, n \text{ and } \mathfrak{p} \in Y_i\}.$$

Proof. The bijection between (i) and (iii) follows directly from Theorem 3.6, taking into account Lemmas 3.18 and 3.17(ii). The bijection between (ii) and (iii) is an immediate consequence of Lemmas 3.17 and 1.11. □

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