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VARIATIONAL SEQUENCES IN MECHANICS ON GRASSMANN FIBRATIONS

PHD THESIS

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Preface

The aim of this work is to extend that part of modern variational theory considering the basic variational objects as a Lagrangian, Euler-Lagrange equations and Helmholtz equations as elements of a differential sequence, introduced by D. Krupka as the *variational sequence*, from its basic structures of fibred manifolds and their finite order jet prolongations to Grassmann prolongations of manifolds. The classes of differential forms, entering the second order variational sequence, are determined by means of charts. We find that the meaning of classes is different from the fibred situation; this also implies important consequences for the global considerations which is a part of author's present studies.

I wish to thank my advisor Professor Demeter Krupka for all the scientific and material support when he was going with me through my studies from its early beginnings. I also acknowledge Professors Willy Sarlet, Frans Cantrijn, Raffaele Vitolo and their collaborators for their kind hospitality during my stays abroad.

Not at least, my warmest thanks are to my wife and parents.

Olomouc, March 2011 Zbyněk Urban

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V Ostravě dne 25. 3. 2011

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Abstract

Extension of the variational sequence theory in mechanics to the Grassmann fibrations (prolongations) of 1-dimensional submanifolds is presented. The coordinate expressions of classes of differential forms, entering the variational sequence, are determined for arbitrary second order forms. In particular, the meaning of classes as the well-known variational objects (Lagrangian, Euler-Lagrange form, Helmholtz-Sonin form) is pointed out. The correspondence with the variational theory of parameter-invariant problems on manifolds is discussed in terms of the theory of jets (slit tangent bundles) and contact elements.

Key Words: jet, contact element, Grassmann fibration, contact form, variational sequence, Euler-Lagrange equations, Helmholtz conditions, parameter-invariant variational theory.

Abstrakt

Tato práce se zabývá zobecněním teorie variační posloupnosti na Grassmannova prodloužení 1-rozměrných podvariet. Třídy diferenciálních forem, jakožto elementy variační posloupnosti, jsou určeny lokálně pro libovolné formy druhého řádu. Zabýváme se variačním významem tříd (Lagrangián, Eulerova-Lagrangeova forma, Helmholtzova-Soninova forma). Pomocí teorie jetů a kontaktních elementů ukazujeme vztah s variační teorií parametricky invariatních problemů.

Klíčová slova: jet, kontaktní element, Grassmannova fibrace, kontaktní forma, variační posloupnost, Eulerovy-Lagrangeovy rovnice, Helmholtzovy podmínky, invariantní variační teorie.

CONTENTS

1. Introduction

This work is devoted to the problem of extension of the variational sequence theory, whose basic geometric structures are fibred manifolds, to Grassmann fibrations, the underlying structures for global variational functionals, describing submanifolds; we refer to Urban and Krupka [57]. The approach is based on Ehresmann's theory of jets and contact elements [8]; we refer to Saunders [51] and, for the most appropriate setting, to Grigore and Krupka [15], D. Krupka and M. Krupka [31, 39], Krupka and Urban [38].

The theory of variational sequences on *finite order* jet bundles, as introduced by Krupka [25], was a consequence of previous results from 70's on the Lepage forms [20, 23] and was supported by ideas of the variational bicomplexes [61, 54, 56, 1, 2]. The basic purpose of these theories was to introduce adequate variational constructions which allow us to understand global characteristics of the Euler–Lagrange mapping, the problem of characterizing the kernel and the image of the Euler-Lagrange mapping. It is not our aim in this work to compare these two approaches; for differences between these theories we refer to e.g. Krupka [29] and Vitolo [60].

Generally speaking, the underlying structure for both variational sequences and bicomplexes is a fibred manifold and its jet prolongations. However, there is an important class of variational principles with a different underlying structure, namely integral variational functionals for parameter-independent curves in a manifold. Our objective is to study the variational sequences, associated with variational functionals for 1-dimensional submanifolds or, in other words, functionals for non-parametric curves in a manifold. The basic concept, defining the variational integral, we introduce by means of a differential form; the notion of the Lagrange function appears as a secondary one. In this sense we follow basic ideas on variational theory on fibred manifolds (Krupka [20, 21, 22, 23, 25], Krupková [40, 41], and references therein). The underlying structures, the manifolds of higher order velocities (higher order tangent bundles) are comparatively simple, and allow us to avoid technicalities, present in multidimensional problems; in particular, we wish to make basic formulas explicit, to give clear ideas for proofs. We also discuss differences between the variational sequence theory for submanifolds and the fibred mechanics (cf. Krupka [27], Krupková and Prince [43, 42] and Musilová [48]). For the classical and modern geometric analysis of parameter-invariant integrals and geometric structures, related with them, we refer to Gelfand and Fomin [9], Grigore [13, 14], Grigore and Krupka [15].

The variational principles, corresponding with the theory explained in this work, belong to the foundations of the Riemann-Finsler geometry (the theory of geodesics, see e.g. Gromoll et al. [16], Chern et al. [5]); it turns out that if a Lagrangian satisfies a positive homogeneity conditions, the corresponding variational functional can be defined by means of a 1-form, the well-known Hilbert form (Crampin and Saunders [6], Chern et al. [5], Urban and Krupka [58]). It should also be pointed out that the variational principles of the same type appear in relativistic particle mechanics (see e.g. Landau and Lifshitz [44]) and in physical applications of Finsler geometry (see e.g. Ingarden [17]).

To characterize differences between "parameter-invariant" and "fibred" variational integrals, consider for example the two variational principles in Finsler geometry. Let X be a manifold and $TX \setminus \{0\}$ its slit tangent bundle. Any regular curve $t \to \gamma(t)$ in X induces a curve $t \to \tilde{\gamma}(t)$ in $TX \setminus \{0\}$. If a Finsler function F is given, we have two integrals of the form

$$L(\gamma) = \int_a^b F(\tilde{\gamma}(t))\,dt$$

and

$$E(\gamma) = \frac{1}{2}\int_a^b F^2(\tilde{\gamma}(t))\,dt,$$

the *length* and the *energy* of the segment $\gamma\colon [a,b] \to X$. While the energy depends on parametrization, the length is parameter-invariant. The homogeneity of F implies that the integrand of $L(\gamma)$ can be written by means of a differential form (the Hilbert form). The number $L(\gamma)$ depends on the "directions" (i.e. contact elements) of γ rather than on its tangent vectors. We consider in this work the variational sequence for the integrals of the type $L(\gamma)$.

In standard classical sources, usually only basic properties of parameter-invariant variational integrals are considered (see e.g. Gelfand and Fomin [9]). Our aim is not only to extend the classical theory to global theory whose underlying spaces are manifolds, but also state some new results on the local and global structure of those integrals. One of our principal results is an analogue of the Helmholtz variationality conditions, which are very-well known for the integrals of the type $E(\gamma)$.

Basic ideas and the method of constructing the variational sequence are formulated for higher order Grassmann fibrations. Nevertheless, the classes of differential forms are completely determined for *second order* variational sequence. The theory can be extended to variational sequences for n-dimensional submanifolds; however, this is not the objective of this work. In a different geometric setting, for particle mechanics with constrains, represented by smooth manifolds, and "higher order mechanics", we refer to Grácia et al. [11] and Krupková [41]. For different approach to the subject we refer to Manno and Vitolo [46]; however, we do not compare the obtained results because [46] is more or less oriented to applications, and we consider a detail comparison non-adequate.

In Section 2 we present the concepts and properties of the jet theory of manifolds of higher order velocities, the parameter groups, acting on these manifolds, and the higher order Grassmann fibrations, whose points are the orbits. In particular, the structure of first order grassmannians is considered separately. Our treatment in next sections is based on introducing of specific invariant coordinate systems, needed, among others, for the proofs. For further use, we also discuss the concepts of formal derivative morphism, horizontal and vertical vectors and horizontalization in associated charts on manifolds of regular velocities.

Section 3 is devoted to contact forms both on manifolds of regular velocities and Grassmann fibrations. In particular, on the basis of theory of contact forms on fibred manifolds (Krupka [24, 28]), we study the canonical decomposition of forms into its contact components.

Main results of this work are contained in Section 4. The observation that the exterior algebra $\tilde{\Omega}^r_k Y$ of smooth differential k-forms on the r-th Grassmann prolongation $G^r Y$ of a manifold Y contains an ideal of “contact” forms, allows us to introduce the “contact subsequence” and the corresponding quotient sequence of the De Rham sequence on $G^r Y$. The “horizontalization” mapping that can be used for this construction differs from an analogous concept applied in the fibred case. The sequence can serve as an abstract framework for introducing basic variational concepts like Lagrangian, Euler-Lagrange expressions and Helmholtz expressions, known from the local theory. Utilizing existence of charts, adapted to immersions, we transform the study of the sequence morphisms to the form that has been applied in the fibred case (Krupka [27]). The classes of forms then represent the Lagrangians (1-forms), Euler-Lagrange expressions (2-forms), Helmholtz expressions (3-forms), etc. We derive explicit formulas for classes entering first and second order sequence by the method used by Krupka [27, 33], and we find that the local expressions of classes coincides with formulas described in the fibred situation (Krupka [25, 27], Krbek and Musilová [19]). The mapping, assigning to the Lagrange class its Euler-Lagrange expressions, represents the *Euler-Lagrange mapping*; assigning to a system of source Euler-Lagrange expressions its Helmholtz expressions, we get the *Helmholtz mapping*. The meaning of the classes, however, differs essentially from the variational sequence in fibred mechanics. This is explicitly apparent from the transformation laws of the components of the classes.

In Section 5 we introduce, in the context considered, the integral variational functionals for parametrized problems on velocity spaces, and study independence of the integral on parametrization. In our geometric setting this is equivalent to projectability of the Lagrangian onto Grassmann fibration. We also give a direct interpretation of classes and their morphisms in terms of the variational theory of differential forms on first order Grassmann fibrations.

As a principal consequence of the variational sequence we find that the global properties of the Euler-Lagrange mapping for submanifolds can be derived in the same way as in the fibred case. The presented theory shows that the variational sequence admits a sheaf representation; this implies, in particular, that we can always characterize differences between local and global properties of the Euler-Lagrange and Helmholtz morphisms in terms of the cohomology groups, measuring local and global variationality, of the underlying manifolds $G^r Y$; for fibred mechanics, compare with Krupka [27]. However, we do not include in this work the discussion of global aspects of the theory of parameter-invariant problems, which lie outside the classical variational theory (e.g. the global inverse problem).

CONVENTIONS.

Throughout this work, we denote by Y a smooth manifold of dimension $m+1$, with $m \geq 1$ be an integer. For the local coordinates on Y we reserve the letters $y^1, y^2, \ldots, y^{m+1}$, or briefly (y^K), and their jet prolongations $(y^K, y^K_1, \ldots, y^K_r)$. In lower order considerations, we use the dot symbol to denote $\dot{y}^K, \ddot{y}^K, \dddot{y}^K$ instead of y^K_1, y^K_2, y^K_3, respectively. All mappings and curves are supposed to be smooth on their domain of definition. As usual, the k-th derivative of a function $f : \mathbf{R} \to \mathbf{R}$ is denoted by $D^k f = d^k f/dt^k$, and the i-th partial derivative of a function $F : \mathbf{R}^n \to \mathbf{R}$ is denoted by $D_i F = \partial F/\partial t^i$, with t and $(t^1, t^2, \ldots, t^n)$ the canonical coordinates on $\mathbf{R}$ and $\mathbf{R}^n$, respectively.

We use the Einstein summation convention. Also, we often distinguish antisymmetric indices in this way: $A_{\sigma_1\sigma_2\ldots\sigma_k, \nu_1\nu_2\ldots\nu_l}$ is antisymmetric in two mutually distinct sets of indices, $\{\sigma_1\sigma_2\ldots\sigma_k\}$ and $\{\nu_1\nu_2\ldots\nu_l\}$. The symbol

$$alt(\sigma_1\sigma_2\ldots\sigma_k\nu_1\nu_2\ldots\nu_l)$$

then means that we antisymmetrize in all the indices.

Let $l \geq 1$ and $1 \leq p \leq l$ be fixed integers. We need a convention regarding summation through all partitions of the set $\{i_1, i_2, \ldots, i_l\}$ of integers. A p-tuple $(I_1, I_2, \ldots, I_p)$ is called a *p-partition* of the set $\{i_1, i_2, \ldots, i_l\}$, if all I_j, $1 \leq j \leq p$, are mutually disjoint subsets of $\{i_1, i_2, \ldots, i_l\}$, and $\cup_j I_j = \{i_1, i_2, \ldots, i_l\}$. In particular, we consider in this work the case when $i_1 = i_2 = \ldots = i_l = 1$. By the *length* $|I_j|$ we mean simply the number of its elements. Then the symbol

$$\sum_{(I_1, I_2, \ldots, I_p)} \tag{1.1}$$

denotes the summation through all p-partitions of the set $\{i_1, i_2, \ldots, i_l\}$, where $i_1 = i_2 = \ldots = i_l = 1$.

The composite of two differentiable mappings is again differentiable, and the formula for its derivative is known as the chain rule. We often need the higher order generalization of this formula in the following form.

Lemma 1.1. *Let $U \subset \mathbf{R}$ and $V \subset \mathbf{R}^m$ be open sets. Let $g : U \to V$, $g = (g^\sigma)$, $1 \leq \sigma \leq m$, and $f : V \to \mathbf{R}$ be smooth mappings. Then*

(1.2)

$$D^l(f \circ g)(t) = \sum_{p=1}^{l} \sum_{(I_1, I_2, \ldots, I_p)} D_{\sigma_p} \ldots D_{\sigma_2} D_{\sigma_1} f(g(t)) D^{|I_p|} g^{\sigma_p}(t) \ldots D^{|I_2|} g^{\sigma_2}(t) D^{|I_1|} g^{\sigma_1}(t).$$

Proof. The proof can be found in D. Krupka and M. Krupka [31]; see also Munkres [47]. □

Example 1.1. Let us give the formula 1.2 without the summation convention; for $l = 3$ we have

$$D^3(f \circ g)(t) = D_{\sigma_3} D_{\sigma_2} D_{\sigma_1} f(g(t)) D g^{\sigma_3}(t) D g^{\sigma_2}(t) D g^{\sigma_1}(t) + 3 D_{\sigma_2} D_{\sigma_1} f(g(t)) D^2 g^{\sigma_2}(t) D g^{\sigma_1}(t) + D_{\sigma_1} f(g(t)) D^3 g^{\sigma_1}(t).$$

Because of frequent use, we state here the standard result.

Theorem 1.1 (Rank Theorem)**.** *Let X and Y be manifolds, $n = \dim X$, $m = \dim Y$, and let q be a positive integer such that $q \leq \min(n,m)$. Let $W \subset X$ be an open set, and let $f : W \to Y$ be a C^r mapping. The following two conditions are equivalent:*

(a) f has constant rank on W equal to q.

(b) To every point $x_0 \in W$ there exists a chart (U,φ), $\varphi = (x^i)$, at x_0, an open rectangle $P \subset \mathbf{R}^n$ with centre $0 \in \mathbf{R}^n$ such that $\varphi(U) = P$, $\varphi(x_0) = 0$, a chart (V,ψ), $\psi = (y^\sigma)$, at $y_0 = f(x_0)$, and an open rectangle $Q \subset \mathbf{R}^m$ with centre $0 \in \mathbf{R}^m$ such that $\psi(V) = Q$, $\psi(y_0) = 0$, and

$$\begin{aligned} &y^\sigma \circ f = x^\sigma, \ \sigma = 1,2,\ldots,q, \\ &y^\sigma \circ f = 0, \ \sigma = q+1,q+2,\ldots,m. \end{aligned} \tag{1.3}$$

Proof. This theorem is proved in standard books on analysis on manifolds, see e.g. Krupka [30], Narasimhan [49]. □

2. MANIFOLDS OF JETS AND CONTACT ELEMENTS OF CURVES

In this section we present some basic aspects of the jet theory of higher order velocities and regular velocities, the associated parameter group actions, and the higher order Grassmann fibrations of 1-dimensional submanifolds. The latter form natural underlying structures of variational functionals for non-parametric curves in a manifold. The basic references for this material are Ehresmann [8], Grigore and Krupka [15], Krupka and Krupka [31], Krupka and Urban [38] and Saunders [51]. Another reference for differential invariants on velocity spaces is Olver [50].

2.1. **Velocities.**

Basic notions and statements. Let $r \geq 0$ be an integer. By a *velocity of order r* at a point $y \in Y$ we mean an r-jet $P \in J^r_{(0,y)}(\mathbf{R},Y)$, $P = J^r_0\zeta$, such that $y = \zeta(0)$, whose representative is a curve ζ in manifold Y, defined on a neighbourhood of the origin $0 \in \mathbf{R}$. Velocities of order r are also called *tangent vectors* of order r. We denote

$$T^rY = \bigcup_{y\in Y} J^r_{(0,y)}(\mathbf{R},Y),$$

and define surjective mappings, the *canonical r-jet projections* $\tau^{r,s} : T^rY \to T^sY$, where $0 \leq s \leq r$, by $\tau^{r,s}(J^r_0\zeta) = J^s_0\zeta$.

We consider the set T^rY with standard geometric structures. Recall that every chart (V,ψ) on Y, with coordinate functions $\psi = (y^K)$, where $1 \leq K \leq m+1$, induces on the set $V^r = (\tau^{r,0})^{-1}(V)$ a collection of functions $\psi^r = (y^K, y^K_1, y^K_2, \dots, y^K_r)$, defined by

$$y^K_l(J^r_0\zeta) = D^l(y^K\zeta)(0).$$

Denoting tr^K_ξ the K-component of the translation $x \to x - \xi$ of $\mathbf{R}^{m+1}$, note that we can also use an equivalent formula $y^K_l(J^r_0\zeta) = D^l(\mathrm{tr}^K_{\psi\zeta(0)}\,\psi\zeta)(0)$. Then it is straightforward to verify that the pairs (V^r,ψ^r), the *associated charts*, define a smooth structure of T^rY. In particular, the higher order chain formula (1.2) shows that the coordinate transformation is polynomial in coordinate functions. Together with this structure, we call T^rY the *manifold of velocities of order r* over Y, and its dimension is equal $(m+1)\cdot(r+1)$. The associated local trivialization

$$V^r \ni J^r_0\zeta \to (\zeta(0), J^r_0(\mathrm{tr}_{\psi\zeta(0)}\,\psi\zeta)) \in V \times J^r_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1}) \tag{2.1}$$

shows that T^rY is a fibration with base Y, projection $\tau^{r,0}$, and type fibre the manifold of r-jets with source $0 \in \mathbf{R}$ and target $0 \in \mathbf{R}^{m+1}$, $J^r_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$.

Group actions. Recall here the standard definition of a group action on a differentiable manifold. We say that a Lie group G *acts differentiably to the right* on a manifold X if there is a differentiable map $\phi : X \times G \to X$ such that for any element $a \in G$ the map $p \to \phi(p,a)$ is a diffeomorphism of P onto itself, and $\phi(\phi(p,a),b) = \phi(p,ab)$ for any $a,b \in G$ and $p \in X$. If for any element $a \in G$ that is *not* the identity element of G and for any $p \in X$ we have $\phi(p,a) \neq p$, then we say that the group action of G on X is *free*. For details on the geometry of G-structures, see e.g. Sternberg [52].

The r-th *differential group* of $\mathbf{R}$, denoted by L^r, is the set of r-jets $J_0^r\alpha$ of diffeomorphisms $\alpha : I \to J$, where I and J are open intervals in $\mathbf{R}$ containing the origin 0, such that $\alpha(0) = 0$. The group operation is given by the composition of jets then. The *canonical coordinates* $a_1, a_2, \ldots, a_r$ on L^r are defined by $a_l(J_0^r\alpha) = D^l\alpha(0)$. In these coordinates $L^r = \{J_0^r\alpha \in J_{(0,0)}^r(\mathbf{R},\mathbf{R}) \mid a_1(J_0^r\alpha) \neq 0\}$.

The differential group L^r acts on itself and on T^rY to the right by composition of jets; we have

$$L^r \times L^r \ni (J_0^r\alpha, J_0^r\beta) \to J_0^r\alpha \circ J_0^r\beta = J_0^r(\alpha \circ \beta) \in L^r, \tag{2.2}$$

and

$$T^rY \times L^r \ni (J_0^r\zeta, J_0^r\alpha) \to J_0^r\zeta \circ J_0^r\alpha = J_0^r(\zeta \circ \alpha) \in T^rY, \tag{2.3}$$

respectively. To formulate the coordinate equations of these actions, denote $a_l = a_l(J_0^r\alpha)$, $b_l = a_l(J_0^r\beta)$, $\bar{a}_l = a_l(J_0^r\alpha \circ J_0^r\beta)$, and $\bar{y}_l^K = y_l^K(J_0^r\zeta \circ J_0^r\alpha)$ in a chart (V,ψ), $\psi = (y^K)$.

Lemma 2.1. *The group actions (2.2) and (2.3) are expressed by the equations*

$$\bar{a}_l = \sum_{p=1}^{l} \sum_{(I_1,I_2,\ldots,I_p)} a_p b_{|I_1|} b_{|I_2|} \ldots b_{|I_p|}, \tag{2.4}$$

$$\bar{y}^K = y^K, \quad \bar{y}_l^K = \sum_{p=1}^{l} \sum_{(I_1,I_2,\ldots,I_p)} y_p^K a_{|I_1|} a_{|I_2|} \ldots a_{|I_p|}, \tag{2.5}$$

respectively.

Proof. In the chart (V,ψ), $\psi = (y^K)$ on Y, $y_l^K(J_0^r\zeta \circ J_0^r\alpha) = D^l(y^K\zeta \circ \alpha)(0)$. Hence, using the higher order chain rule (Lemma 1.1),

$$y_l^K(J_0^r\zeta \circ J_0^r\alpha) = \sum_{p=1}^{l} \sum_{(I_1,I_2,\ldots,I_p)} D^p(y^K\zeta)(0) D^{|I_1|}\alpha(0) D^{|I_2|}\alpha(0) \ldots D^{|I_p|}\alpha(0),$$

proving (2.5). By the same argument we obtain equations (2.4) of the group operation in L^r. □

Jet prolongation of curve. We introduce the concept of the *prolongation* of a curve in Y to a curve in the manifold of velocities T^rY.

Let γ be a smooth curve in Y, defined on an open interval $I \subset \mathbf{R}$. Then for any $t \in I$ we have the mapping $s \to \gamma \circ \mathrm{tr}_{-t}(s)$, defined on a neighbourhood of the origin $0 \in \mathbf{R}$ so that the r-jet $J_0^r(\gamma \circ \mathrm{tr}_{-t})$ is defined; we get the curve

$$I \ni t \to T^r\gamma(t) = J_0^r(\gamma \circ \mathrm{tr}_{-t}) \in T^rY. \tag{2.6}$$

We call $T^r\gamma$ the *r-jet prolongation* of the curve γ.

Note that for every $\mu : J \to I$, an isomorphism of open intervals, and every $s \in J$, the r-jet $J_0^r(\mathrm{tr}_{\mu(s)} \circ \mu \circ \mathrm{tr}_{-s})$ belongs to the differential group L^r; denote

$$\mu_s = \mathrm{tr}_{\mu(s)} \circ \mu \circ \mathrm{tr}_{-s}, \quad \mu^r(s) = J_0^r\mu_s.$$

Lemma 2.2. (a) *The mapping* (2.6) *has the chart expression*

$$y_l^K \circ T^r\gamma(t) = D(y_{l-1}^K \circ T^{r-1}\gamma)(t). \tag{2.7}$$

(b) *The mapping* (2.6) *satisfies*

$$T^r(\gamma \circ \mu)(s) = T^r\gamma(\mu(s)) \circ \mu^r(s). \tag{2.8}$$

Proof. (a) Differentiating the curve $s \to y_{l-1}^K \circ T^{r-1}\gamma(s) = D^{l-1}(y^K\gamma)(s)$ at a point $t \in I$, we get

$$\begin{aligned} D(y_{l-1}^K \circ T^{r-1}\gamma)(t) &= D(D^{l-1}(y^K\gamma))(t) = D^l(y^K\gamma)(t) \\ &= D^l(y^K\gamma \circ \mathrm{tr}_{-t})(0) = y_l^K \circ T^r\gamma(t). \end{aligned}$$

(b) By the definition of $T^r\gamma$ and μ^r, we have on J,

$$\begin{aligned} T^r(\gamma \circ \mu)(s) &= J_0^r(\gamma \circ \mu \circ \mathrm{tr}_{-s}) = J_0^r(\gamma \circ \mathrm{tr}_{-\mu(s)} \circ \mathrm{tr}_{\mu(s)} \circ \mu \circ \mathrm{tr}_{-s}) \\ &= J_0^r(\gamma \circ \mathrm{tr}_{-\mu(s)}) \circ J_0^r(\mathrm{tr}_{\mu(s)} \circ \mu \circ \mathrm{tr}_{-s}) = T^r\gamma(\mu(s)) \circ \mu^r(s), \end{aligned}$$

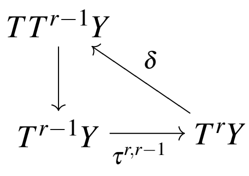
which proves the property (2.8). □

Formal differentiation. We now introduce the concepts of formal derivative morphism and of formal derivative of a function, both on r-th order velocities bundle. Formal derivatives play a basic role in higher order computations. In the context of this work the reader can find its definition and basic properties in D. Krupka and M. Krupka [31], Tulczyjew [55]; for analogues in jet prolongations of fibred manifolds we refeer to Krupka [28].

We can canonically identify the tangent space $T_0\mathbf{R}$ and the vector space $\mathbf{R}$. Denote by t the canonical coordinate on $\mathbf{R}$, and $(d/dt)_0$ the vector of canonical basis of the 1-dimensional vector space $T_0\mathbf{R}$. We define a mapping $\delta : T^rY \to TT^{r-1}Y$ by

$$\delta(J_0^r\zeta) = T_0T^{r-1}\zeta \cdot \left(\frac{d}{dt}\right)_0, \tag{2.9}$$

with $T^{r-1}\zeta$ the $(r-1)$-jet prolongation of the representative ζ, defined by (2.6). $\delta(J_0^r\zeta)$ is a tangent vector of $T^{r-1}Y$ at the point $T^{r-1}\zeta(0) = J_0^{r-1}\zeta = \tau^{r,r-1}(J_0^r\zeta)$. From the definition, note that δ does not depend on the choice of a chart. The morphism of fibrations δ is a vector field *along the projection* $\tau^{r,r-1}$, meaning that the following diagram is commutative



We express δ in terms of the chart (V^r,ψ^r) on T^rY. Using Lemma 2.2, (2.7), for arbitrary $J^r_0\zeta \in T^rY$ we obtain

$$\delta(J^r_0\zeta) = T_0T^{r-1}\zeta \cdot \left(\frac{d}{dt}\right)_0 = \sum_{l=0}^{r-1} D(y^K_l \circ T^{r-1}\zeta)(0)\left(\frac{\partial}{\partial y^K_l}\right)_{J^{r-1}_0\zeta}$$
$$= \sum_{l=0}^{r-1} y^K_{l+1} \circ T^r\zeta(0)\left(\frac{\partial}{\partial y^K_l}\right)_{J^{r-1}_0\zeta} = \sum_{l=0}^{r-1} y^K_{l+1}(J^r_0\zeta)\left(\frac{\partial}{\partial y^K_l}\right)_{J^{r-1}_0\zeta}.$$

Let $W \subset Y$ be an opet set, $W^{r-1} = (\tau^{r-1,0})^{-1}(W)$, and let $f : W^{r-1} \to \mathbf{R}$ be a function. In analogy with the formal derivative morphism δ, (2.9), (2.12), for every chart (V,ψ), $\psi = (y^K)$, on Y such that $V \subset W$, formula

$$\delta f(J^r_0\zeta) = \sum_{l=0}^{r-1} y^K_{l+1}(J^r_0\zeta)\left(\frac{\partial f(\psi^{r-1})^{-1}}{\partial y^K_l}\right)_{\psi^{r-1}\tau^{r,r-1}(J^r_0\zeta)} \tag{2.10}$$

defines a function $\delta f : W^r = (\tau^{r,0})^{-1}(W) \to \mathbf{R}$.

Note, in particular, that if $f = y^K_l$ is a coordinate function, it holds on V^{l+1}

$$\delta y^K_l = y^K_{l+1}. \tag{2.11}$$

Then by 2.2, (2.7), for any smooth curve γ in Y we have

$$D(y^K_l \circ T^{r-1}\gamma)(t) = D^{l+1}(y^K \circ \gamma)(t) = D^{l+1}(y^K \circ \gamma \circ \mathrm{tr}_{-t})(0)$$
$$= y^K_{l+1}(J^r_0(\gamma \circ \mathrm{tr}_{-t})) = y^K_{l+1}(T^r\gamma(t)),$$

hence

$$D(f \circ T^{r-1}\gamma)(t) = D(f \circ (\psi^{r-1})^{-1} \circ \psi^{r-1} \circ T^{r-1}\gamma)(t)$$
$$= \sum_{l=0}^{r-1}\left(\frac{\partial(f \circ (\psi^{r-1})^{-1})}{\partial y^K_l}\right)_{\psi^{r-1}(T^{r-1}\gamma(t))} D(y^K_l \circ T^{r-1}\gamma)(t)$$
$$= \sum_{l=0}^{r-1} y^K_{l+1}(T^r\gamma(t))\left(\frac{\partial(f \circ (\psi^{r-1})^{-1})}{\partial y^K_l}\right)_{\psi^{r-1}(T^{r-1}\gamma(t))}$$
$$= \delta f(T^r\gamma(t)) = (\delta f \circ T^r\gamma)(t).$$

We call δ the *formal derivative morphism* of order r, and δf the *formal derivative* of a function f. The following lemma summarize their properties.

Lemma 2.3. (a) δ *has a chart expression*

$$\delta = \sum_{l=0}^{r-1} y^K_{l+1}\frac{\partial}{\partial y^K_l}. \tag{2.12}$$

(b) *For any functions* $f, g : W^{r-1} \to \mathbf{R}$,

$$\delta(f+g) = \delta(f) + \delta(g), \quad \delta(f \cdot g) = \delta(f)\cdot g + f \cdot \delta(g).$$

(c) *For every function* $f : W^{r-1} \to \mathbf{R}$ *and every curve* $I \ni t \to \gamma(t) \in Y$,

$$\delta(f)(T^r\gamma(t)) = D(f \circ T^{r-1}\gamma)(t).$$

In the following lemma we give explicit transformation formulas between charts, needed for proofs. Suppose we have the transformation equations from a chart (V,ψ), $\psi=(y^K)$, to a chart (U,φ), $\varphi=(\bar{y}^M)$, of the form $\bar{y}^M=F^M(y^K)$. We want to determine the functions F_l^M, defining the induced transformations $\bar{y}_l^M = F_l^M(y^K,y_1^K,y_2^K,\dots,y_r^K)$.

Lemma 2.4. *Let (V,ψ), $\psi=(y^K)$, and (U,φ), $\varphi=(\bar{y}^M)$, be two charts on Y such that $V\cap U\neq\emptyset$. Suppose that the transformation equations from (V,ψ), $\psi=(y^K)$, to (U,φ), $\varphi=(\bar{y}^M)$, are known in the form*

$$\bar{y}^M = F^M(y^K). \tag{2.13}$$

Then, by means of associated charts on T^rY, the transformation equations from (V^r,ψ^r), to (U^r,φ^r), are given by

$$F_l^M=\sum_{p=1}^{l}\sum_{(I_1,I_2,\dots,I_p)}\frac{\partial^p F^M}{\partial y^{K_1}\partial y^{K_2}\dots\partial y^{K_p}}y_{|I_1|}^{K_1}y_{|I_2|}^{K_2}\dots y_{|I_p|}^{K_p},\quad 1\le l\le r. \tag{2.14}$$

Proof. We proceed by induction. First note that for $l=1$, (2.14) is of the form

$$F_1^M=\frac{\partial F^M}{\partial y^K}y_1^K=\delta F^M.$$

Since the functions $F^M: V\cap U\to\mathbf{R}$ satisfy equations (2.13), we have on $V^1\cap U^1$

$$\begin{aligned}&\bar{y}_1^M(J_0^1\zeta)=D(\bar{y}^M\zeta)(0)=D(F^M\psi^{-1}\circ\psi\zeta)(0)\\&=\left(\frac{\partial F^M\psi^{-1}}{\partial y^K}\right)_{\psi\zeta(0)}D(y^K\zeta)(0)=y_1^K(J_0^1\zeta)\left(\frac{\partial F^M\psi^{-1}}{\partial y^K}\right)_{\psi\tau^{1,0}(J_0^1\zeta)}\\&=\delta F^M(J_0^1\zeta),\end{aligned}$$

thus the assertion holds for $r=1$.

Suppose now that $r>1$ and (2.14) holds for some fixed l, $1<l<r-1$. We shall prove the assertion for $l+1$, provided the functions F^M, (2.13), are defined on V^{l+1}. By assumption, we have

$$\bar{y}_l^M=F_l^M(y^K,y_1^K,y_2^K,\dots,y_l^K)$$

on $V^l \cap U^l$. Then by (2.10), and (2.11), $\delta \bar{y}^M_l = \bar{y}^M_{l+1}$,

$$\begin{aligned}
F^M_{l+1} &= \delta \bar{y}^M_l = \delta F^M_l \\
&= \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} \delta \left(\frac{\partial^p F^M}{\partial y^{K_1} \partial y^{K_2} \dots \partial y^{K_p}} \right) y^{K_1}_{|I_1|} y^{K_2}_{|I_2|} \dots y^{K_p}_{|I_p|} \\
&+ \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} \frac{\partial^p F^M}{\partial y^{K_1} \partial y^{K_2} \dots \partial y^{K_p}} \\
&\cdot \left(\delta y^{K_1}_{|I_1|} y^{K_2}_{|I_2|} \dots y^{K_p}_{|I_p|} + y^{K_1}_{|I_1|} \delta y^{K_2}_{|I_2|} \dots y^{K_p}_{|I_p|} + \dots + y^{K_1}_{|I_1|} y^{K_2}_{|I_2|} \dots \delta y^{K_p}_{|I_p|} \right) \\
&= \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} \frac{\partial^{p+1} F^M}{\partial y^K \partial y^{K_1} \partial y^{K_2} \dots \partial y^{K_p}} y^K_1 y^{K_1}_{|I_1|} y^{K_2}_{|I_2|} \dots y^{K_p}_{|I_p|} \\
&+ \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} \frac{\partial^p F^M}{\partial y^{K_1} \partial y^{K_2} \dots \partial y^{K_p}} \\
&\cdot \left(y^{K_1}_{|I_1|+1} y^{K_2}_{|I_2|} \dots y^{K_p}_{|I_p|} + y^{K_1}_{|I_1|} y^{K_2}_{|I_2|+1} \dots y^{K_p}_{|I_p|} + \dots + y^{K_1}_{|I_1|} y^{K_2}_{|I_2|} \dots y^{K_p}_{|I_p|+1} \right) \\
&= \sum_{p=1}^{l+1} \sum_{(J_1,J_2,\dots,J_p)} \frac{\partial^p F^M}{\partial y^{K_1} \partial y^{K_2} \dots \partial y^{K_p}} y^{K_1}_{|J_1|} y^{K_2}_{|J_2|} \dots y^{K_p}_{|J_p|},
\end{aligned}$$

where $(J_1, J_2, \dots, J_p)$ denotes p-partitions of the set $\{j_1, j_2, \dots, j_{l+1}\}$, with $j_1 = j_2 = \dots = j_{l+1} = 1$. □

2.2. **Regular velocities.**

Basic notions and statements. From now on we are concerned with velocities which are regular by means of the jet composition. A velocity $P \in T^rY$, $P = J^r_0\zeta$, is said to be *regular*, if there exists an r-jet $Q \in J^r_{(y,0)}(Y, \mathbf{R})$, $y = \zeta(0)$, such that $Q \circ P = J^r_0 \mathrm{id}_{\mathbf{R}}$. The following two equivalent conditions, characterizing regular velocities, are direct consequences of the rank theorem on manifolds (cf. Theorem 1.1).

Lemma 2.5. *A velocity $P \in T^rY$, $P = J^r_0\zeta$, is regular if and if only one of the equivalent conditions is satisfied:*

(a) *Every representative of P is an immersion at the origin* $0 \in \mathbf{R}$.

(b) *There exists a chart (V, ψ), $\psi = (y^K)$, at $y = \zeta(0)$ and an index L,* $1 \le L \le m+1$, *such that*

$$D(y^L\zeta)(0) \neq 0. \tag{2.15}$$

Denote by $\mathrm{Imm}\, T^rY$ the subset of regular velocities in T^rY, and analyze its basic topological and differentiable structure. Let $P \in \mathrm{Imm}\, T^rY$, $P = J^r_0\zeta$, be a point. Fix a chart (V, ψ), $\psi = (y^K)$, at $y = \zeta(0)$, and consider the associated chart (V^r, ψ^r), $\psi^r = (y^K, y^K_1, y^K_2, \dots, y^K_r)$, on T^rY; restricting the coordinate functions to the set $V^r \cap \mathrm{Imm}\, T^rY$, we get a chart on $\mathrm{Imm}\, T^rY$. Then by Lemma 2.5, we can suppose that $y^L_1(P) \neq 0$ for some L, $1 \le L \le m+1$. From the continuity of coordinate functions, it follows that the point P has a neighbourhood on which y^L_1 is not vanishing.

Also, since for every local diffeomorphism α at $0 \in \mathbf{R}$ and every immersion ζ the composition $\zeta \circ \alpha$ is again an immersion, we see that $P \circ A$ belongs to $\operatorname{Imm} T^r Y$ for every $A = J^r_0 \alpha$. Thus, the canonical right action on $T^r Y$, defined by (2.3), induces a right action on $\operatorname{Imm} T^r Y$, and we have

Lemma 2.6. *The set of regular velocities* $\operatorname{Imm} T^r Y$ *forms an open, dense, and L^r-invariant subset of $T^r Y$. The manifold structure of* $\operatorname{Imm} T^r Y$ *is induced by the canonical structure of $T^r Y$.*

Note that local trivialization (2.1) of $T^r Y$ induces local trivialization of $\operatorname{Imm} T^r Y$ over Y with projection $\tau^{r,0}$ and type fibre $\operatorname{Imm} J^r_{(0,0)}(\mathbf{R}, \mathbf{R}^{m+1})$.

For a regular velocity $P = J^r_0 \zeta$, the structure of manifold $\operatorname{Imm} T^r Y$ allows us to assign to every chart (V, ψ), $\psi = (y^K)$, at $y = \zeta(0)$, the collection of $(m+1)$ charts $(V^{r,L}, \psi^{r,L})$, $1 \leq L \leq m+1$, at $P \in \operatorname{Imm} T^r Y$, by shrinking the coordinate functions $\psi^r = (y^K, y^K_1, y^K_2, \ldots, y^K_r)$ to the domains of the form

$$V^{r,L} = \{P \in V^r \,|\, y^L_1(P) \neq 0\}.$$

We set $\psi^{r,L} = (y^L, y^L_1, y^L_2, \ldots, y^L_r, y^\sigma, y^\sigma_1, y^\sigma_2, \ldots, y^\sigma_r)$ for every index L, with, according to this new coordinates on $\operatorname{Imm} T^r Y$, the index σ is supposed to run through the sequence $(1, 2, \ldots, L-1, L+1, \ldots, m, m+1)$, complementary to the index L. Obviously, the domains $V^{r,L}$, $1 \leq L \leq m+1$, cover the set $V^r = (\tau^{r,0})^{-1}(V)$, and the charts $(V^{r,L}, \psi^{r,L})$ form an atlas on $\operatorname{Imm} T^r Y$.

Consider the group action $\operatorname{Imm} T^r Y \times L^r \ni (P, A) \to P \circ A \in \operatorname{Imm} T^r Y$, induced by the canonical right action (2.3) on $T^r Y$, and the equivalence relation $\mathscr{R}$ on $\operatorname{Imm} T^r Y$ "*there exists $A \in L^r$ such that $Q = P \circ A$*". The following lemma characterizes this equivalence and is used to prove invariance of certain new coordinates on $\operatorname{Imm} T^r Y$, adapted to the canonical group action of L^r.

Lemma 2.7. *Let $P, Q \in \operatorname{Imm} T^r Y$. The following conditions are equivalent:*

(a) $(P, Q) \in \mathscr{R}$.

(b) *There exist a chart (V, ψ), $\psi = (y^K)$, on Y, an index L, $1 \leq L \leq m+1$, and an element $A \in L^r$ such that $P, Q \in V^{r,L}$, and the coordinates $y^K_l = y^K_l(P)$, $\bar{y}^K_l = \bar{y}^K_l(Q)$, $a_l = a_l(A)$ satisfy*

$$\bar{y}^K = y^K, \quad \bar{y}^\sigma_l = \sum_{p=1}^{l} \sum_{(I_1, I_2, \ldots, I_p)} y^\sigma_p a_{|I_1|} a_{|I_2|} \ldots a_{|I_p|}, \tag{2.16}$$

and the recurrence formula

$$a_1 = \frac{\bar{y}^L_1}{y^L_1}, \quad a_l = \frac{1}{y^L_1} \left(\bar{y}^L_l - \sum_{p=2}^{l} \sum_{(I_1, I_2, \ldots, I_p)} y^L_p a_{|I_1|} a_{|I_2|} \ldots a_{|I_p|} \right). \tag{2.17}$$

Proof. First suppose that (a) is satisfied. There exists a chart (V, ψ), $\psi = (y^K)$, on Y such that $P, Q \in V^r$, and the group action $(P, A) \to Q = P \circ A$ is expressed by (2.5) of Lemma 2.1. Then in terms of L-adapted chart, for some L, $1 \leq L \leq m+1$,

$P, Q \in V^{r,L}$ and (2.5) becomes of the desired form

$$\bar{y}^L = y^L, \quad \bar{y}^\sigma = y^\sigma, \quad \bar{y}^\sigma_l = \sum_{p=1}^{l} \sum_{(I_1, I_2, \ldots, I_p)} y^\sigma_p a_{|I_1|} a_{|I_2|} \ldots a_{|I_p|},$$

and

$$y^L_1 a_l = \bar{y}^L_l - \sum_{p=2}^{l} \sum_{(I_1, I_2, \ldots, I_p)} y^L_p a_{|I_1|} a_{|I_2|} \ldots a_{|I_p|}.$$

Conversely, assume that condition (b) is satisfied. Then (2.16) and (2.17) are equivalent with the chart expression of the L^r-action meaning that P and Q belong to the same L^r-orbit which is the condition (a). □

Vertical vectors, horizontal forms. Let (V, ψ), $\psi = (y^K)$, be a chart on Y, and consider a tangent vector ξ to $\operatorname{Imm} T^r Y$ at a point $P \in V^{r,L}$, $P = J^r_0 \zeta$. Note that an r-jet $Q \in V^r$, $Q = J^r_0 \gamma$, belongs to $V^{r,L}$ if and only if the mapping $t \to y^L \circ \gamma(t)$ is a diffeomorphism at $0 \in \mathbf{R}$. The mapping $(y^L \circ \zeta)^{-1} \circ y^L \circ \tau^{r,0}$ is defined on a neighbourhood of P in $V^{r,L}$, and it stands here for a projection.

Then, we call ξ *L-vertical*, if

$$T_P((y^L \circ \zeta)^{-1} \circ y^L \circ \tau^{r,0}) \cdot \xi = 0. \tag{2.18}$$

One can now define the induced concept of a horizontal form as follows. A differential k-form η on $V^{r,L}$ is said to be *L-horizontal*, if the contraction $i_\xi \eta(P)$ vanishes for every point $P \in V^{r,L}$ and whenever ξ is an L-vertical vector tangent to $\operatorname{Imm} T^r Y$ at P. However, in this sense every k-form, $k \geq 2$, is horizontal if and only if it is everywhere zero form. Horizontal 1-forms are then of the form $A dy^L$.

We can easily find the chart expression of an L-vertical vector. Suppose that ξ is expressed in the chart $(V^{r,L}, \psi^{r,L})$ by

$$\xi = \sum_{l=0}^{r} \xi^K_l \left(\frac{\partial}{\partial y^K_l} \right)_P. \tag{2.19}$$

Denoting t the canonical coordinate on $\mathbf{R}$, we get

$$T_P((y^L \circ \zeta)^{-1} \circ y^L \circ \tau^{r,0}) \cdot \xi = \frac{1}{y^L_1(P)} \xi^L \left(\frac{d}{dt} \right)_0, \tag{2.20}$$

(no summation through L) hence (2.18) gives us $\xi^L = 0$. Thus, the vertical vector ξ, (2.19), is expressed by

$$\xi = \xi^\sigma \left(\frac{\partial}{\partial y^\sigma} \right)_P + \sum_{l=1}^{r} \xi^K_l \left(\frac{\partial}{\partial y^K_l} \right)_P, \tag{2.21}$$

with σ running through $1, 2, \ldots, m+1$, $\sigma \neq L$.

Remark 1. Note that the definition of a vertical vector to $\operatorname{Imm} T^r Y$ is *not* chart independent. However, if we restrict ourselves to the choice of charts with a fixed non-vanishing coordinate w^L on $\operatorname{Imm} T^r Y$, we get a good geometric meaning of this vertical vector. For that purpose, consider two charts on Y, (V, ψ), $\psi = (y^K)$, and (U,φ), $\varphi=(\bar{y}^K)$, such that $V\cap U\neq\emptyset$. Then also $V^{r,L}\cap U^{r,M}\neq\emptyset$ for some $1\leq L,M\leq m+1$. Let ξ be a tangent vector to $\mathrm{Imm}\,T^rY$ at a point $P=J^r_0\zeta$,

$$\xi=\sum_{l=0}^{r}\xi^N_l\left(\frac{\partial}{\partial y^N_l}\right)_P=\sum_{l=0}^{r}\bar{\xi}^K_l\left(\frac{\partial}{\partial \bar{y}^K_l}\right)_P,$$

and let the coordinate transformation $\varphi\psi^{-1}$ be expressed by the equations

$$\bar{y}^K=\bar{y}^K(y^N).$$

Hence, on $V^{1,L}\cap U^{1,M}$, we get

$$\bar{y}^K_1(P)=D(\bar{y}^K\zeta)(0)=D_N(\bar{y}^K\psi^{-1})(\psi\zeta(0))D(y^N\zeta)(0),$$

and

$$\xi^L=D(y^L\zeta)(0)=D_K(y^L\varphi^{-1})(\varphi\zeta(0))D(\bar{y}^K\zeta)(0)=D_K(y^L\varphi^{-1})(\varphi\zeta(0))\bar{\xi}^K.$$

Suppose now that ξ is L-vertical in the chart $(V^{r,L},\psi^{r,L})$, i.e. (2.18) holds hence $\xi^L=0$. Then

$$\frac{1}{\bar{y}^M_1(P)}\bar{\xi}^M\left(\frac{d}{dt}\right)_0=\frac{D_\sigma(\bar{y}^M\psi^{-1})(\psi\gamma(0))\xi^\sigma}{D_N(\bar{y}^M\psi^{-1})(\psi\zeta(0))y^N_1(P)}$$

does not vanish for non-zero tangent vector and ξ is not M-vertical in the chart $(U^{r,M},\varphi^{r,M})$.

Horizontalization of vectors. In the next paragraph, the concept of horizontalization of tangent vectors to the manifold of regular velocities $\mathrm{Imm}\,T^rY$ is considered. Similarly to fibred manifolds, the horizontalization mapping is introduced as a morphism of fibrations over the canonical projection. But since here we are not equipped with the base manifold projection, the construction is proceeded with the help of non-vanishing coordinate functions again.

A new mapping on $V^{r,L}\subset\mathrm{Imm}\,T^rY$ with values in $T\mathrm{Imm}\,T^{r-1}Y$, arising from the formal derivative morphism δ and associated with the non-vanishing L-coordinate, we define by

$$\Delta_L=\frac{1}{y^L_1}\,\delta. \tag{2.22}$$

Choose now a point $P\in V^{r,L}$, $P=J^r_0\zeta$. Its representative ζ then defines the $(r-1)$-prolongation of ζ, the curve $t\to T^{r-1}\zeta(t)$ in $\mathrm{Imm}\,T^{r-1}Y$, defined on some neighbourhood of the origin $0\in\mathbf{R}$ by (2.6). Then the composite

$$T^{r-1}\zeta\circ(y^L\circ\zeta)^{-1}\circ y^L\circ\tau^{r,0}$$

is a mapping from a neighbourhood of P in $\mathrm{Imm}\,T^rY$ into $\tau^{r,r-1}(V^{r,L})\subset\mathrm{Imm}\,T^{r-1}Y$. Further, let ξ be a tangent vector to $\mathrm{Imm}\,T^rY$ at the point P, locally expressed by (2.19), and consider the vector $h^L\xi$,

$$h^L\xi=T_P(T^{r-1}\zeta\circ(y^L\circ\zeta)^{-1}\circ y^L\circ\tau^{r,0})\cdot\xi. \tag{2.23}$$

Then $h^L\xi$ is tangent to $\mathrm{Imm}\,T^{r-1}Y$ at the point $T^{r-1}\zeta\circ(y^L\circ\zeta)^{-1}\circ y^L\circ\tau^{r,0}(P) = \tau^{r,r-1}(P)$. Applying the chain rule, we get

$$\begin{aligned} h^L\xi &= T_{(y^L\circ\zeta)^{-1}y^L\tau^{r,0}(P)}T^{r-1}\zeta\circ T_P((y^L\circ\zeta)^{-1}\circ y^L\circ\tau^{r,0})\cdot\xi \\ &= T_0T^{r-1}\zeta\circ T_P((y^L\circ\zeta)^{-1}\circ y^L\circ\tau^{r,0})\cdot\xi. \end{aligned}$$

Then

$$t\circ(y^L\circ\zeta)^{-1}\circ y^L\circ\tau^{r,0}\circ(\psi^r)^{-1}(y^K,y_1^K,y_2^K,\dots,y_r^K) = t\circ(y^L\circ\zeta)^{-1}(y^L),$$

hence

$$T_P((y^L\circ\zeta)^{-1}\circ y^L\circ\tau^{r,0})\cdot\xi = \left(\frac{d(y^L\circ\zeta)^{-1}}{dt}\right)_{y^L(y)}\xi^L\left(\frac{d}{dt}\right)_0 = \frac{1}{y_1^L(P)}\xi^L\left(\frac{d}{dt}\right)_0,$$

and consequently by Lemma 2.2, (a),

$$\begin{aligned} h^L\xi &= \frac{1}{y_1^L(P)}\xi^L T_0T^{r-1}\zeta\cdot\left(\frac{d}{dt}\right)_0 = \frac{1}{y_1^L(P)}\xi^L\sum_{l=0}^{r-1}y_{l+1}^K(P)\left(\frac{\partial}{\partial y_l^K}\right)_{\tau^{r,r-1}(P)} \\ &= \frac{1}{y_1^L(P)}\xi^L\delta(P) = \Delta_L(P)\xi^L \end{aligned} \tag{2.24}$$

(with no summation through L). The assignment $\xi\to h^L\xi$ from $T\mathrm{Imm}\,T^rY$ to $T\mathrm{Imm}\,T^{r-1}Y$ is called *L-horizontalization*, and vector $h^L\xi$ the *L-horizontal component* of ξ. It is easy to verify that h^L is a morphism of fibrations over $\tau^{r,r-1}$, in other words, the following diagram is commutative

$$\begin{array}{ccc} T\mathrm{Imm}\,T^rY\supset TV^{r,L} & \xrightarrow{h} & T\mathrm{Imm}\,T^{r-1}Y \\ \tau_{T^rY}\downarrow & & \downarrow\tau_{T^{r-1}Y} \\ \mathrm{Imm}\,T^rY\supset V^{r,L} & \xrightarrow[\tau^{r,r-1}]{} & \mathrm{Imm}\,T^{r-1}Y \end{array}$$

The transformation properties of the morphism Δ_L are described as follows.

Lemma 2.8. *Let (V,ψ), $\psi=(y^K)$, and (U,φ), $\varphi=(\bar{y}^K)$, are two overlapping charts such that $V^{r,L}\cap U^{r,M}\neq\emptyset$ for some $1\le L,M\le m+1$. Then*

$$\bar{\Delta}_M = \frac{y_1^L}{\bar{y}_1^M}\Delta_L.$$

Proof. The proof is straightforward. □

Remark 2. In comparison with the fibred manifolds, we have now a set of $m+1$ horizontalization morphisms. However, it is worth to note that the structure of chart expression of horizontal component of tangent vector to regular velocity bundles is different from that one tangent to prolongations of fibred manifolds. This concept of L-horizontalization of vectors can be generalized to regular n-velocities whose representatives are immersions from $\mathbf{R}^n$. The morphism Δ could be then defined by the inverse of regular matrix of appropriate coordinate functions.

In an analogous way to the fibred case, we can now use the complementary construction to introduce the contact components of a tangent vector $\xi \in T_P \mathrm{Imm}\, T^r Y$, $P = J^r_0 \zeta \in V^{r,L}$. Keeping the notation from preceding paragraphs, we observe that both vectors $h^L \xi$ and $T_P \tau^{r,r-1} \cdot \xi$ belong to the tangent space of $\mathrm{Imm}\, T^{r-1} Y$ at the same point $\tau^{r,r-1}(P)$. We define the *L-contact component* $p^L \xi$ of ξ by

$$p^L \xi = T_P \tau^{r,r-1} \cdot \xi - h^L \xi. \tag{2.25}$$

The chart expression of $p^L \xi$ follows immediately from the definition and (2.24). Since

$$T_P \tau^{r,r-1} \cdot \xi = \sum_{l=0}^{r-1} \xi^K_l \left(\frac{\partial}{\partial y^K_l} \right)_{\tau^{r,r-1}(P)},$$

we obtain

$$\begin{aligned} p^L \xi &= \sum_{l=0}^{r-1} \left(\xi^K_l - \xi^L \frac{y^K_{l+1}}{y^L_1}(P) \right) \left(\frac{\partial}{\partial y^K_l} \right)_{\tau^{r,r-1}(P)} \\ &= \sum_{l=1}^{r-1} \eta^L_l(P)\xi \left(\frac{\partial}{\partial y^L_l} \right)_{\tau^{r,r-1}(P)} + \sum_{l=0}^{r-1} \eta^\sigma_l(P)\xi \left(\frac{\partial}{\partial y^\sigma_l} \right)_{\tau^{r,r-1}(P)}, \end{aligned} \tag{2.26}$$

where

$$\eta^L_l = dy^L_l - \frac{y^L_{l+1}}{y^L_1} dy^L, \quad \eta^\sigma_l = dy^\sigma_l - \frac{y^\sigma_{l+1}}{y^L_1} dy^L. \tag{2.27}$$

Note that the L-contact component $p^L \xi$ of ξ is L-vertical in the sense of (2.18).

Invariant charts. In what follows, we prove the structure theorems on invariant coordinates and orbit manifolds of regular velocities with respect to the differential group L^r (for details, see Section 2.1). By means of a quotient projection, the L^r-invariant functions on $\mathrm{Imm}\, T^r Y$ constitute new coordinates on an orbit manifold.

Theorem 2.1. *Let* (V, ψ), $\psi = (y^K)$, *be a chart on* Y, *let* L *be an index,* $1 \le L \le m+1$, *and let* σ *be an index, belonging to the complementary sequence.*

(a) *There exist unique functions* $w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r$, *defined on* $V^{r,L}$, *such that*

$$y^\sigma = w^\sigma, \quad y^\sigma_l = \sum_{p=1}^{l} \sum_{(I_1, I_2, \dots, I_p)} y^L_{|I_1|} y^L_{|I_2|} \cdots y^L_{|I_p|} w^\sigma_p. \tag{2.28}$$

These functions are L^r*-invariant and satisfy the recurrence formula*

$$w^\sigma_{l+1} = \Delta_L w^\sigma_l. \tag{2.29}$$

(b) *The pair* $(V^{r,L}, \chi^{r,L})$, $\chi^{r,L} = (w^L, w^L_1, w^L_2, \dots, w^L_r, w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r)$, *in which the functions* $w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r$ *are defined by* (2.28), *and*

$$w^L = y^L, \; w^L_1 = y^L_1, \; w^L_2 = y^L_2, \; \dots, \; w^L_r = y^L_r,$$

is a chart on $\mathrm{Imm}\, T^r Y$.

(c) *The canonical group action of the differential group L^r on* $\operatorname{Imm} T^r Y$ *is described by the equations*

$$\bar{w}^L = w^L, \quad \bar{w}^\sigma = w^\sigma, \quad \bar{w}^\sigma_l = w^\sigma_l,$$

$$\bar{w}^L_l = \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} w^L_p a_{|I_1|} a_{|I_2|} \dots a_{|I_p|}.$$

Equations of the orbits are

$$w^L = c^L, \quad w^\sigma = c^\sigma, \quad w^\sigma_l = c^\sigma_l,$$

where $c^L, c^\sigma, c^\sigma_l \in \mathbf{R}$.

Proof. We proceed by induction to prove existence of $w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r$.

For $r = 1$, let $\psi^{1,L} = (y^K, y^K_1)$ be associated coordinates on $V^{1,L}$, and consider functions w^σ, w^σ_1, defined by relations $w^\sigma = y^\sigma, y^\sigma_1 = y^L_1 w^\sigma_1$ (2.28). Then, obviously, $w^\sigma_1 = y^\sigma_1 / y^L_1$, and w^σ, w^σ_1 are unique and well-defined functions on $V^{1,L}$. The functions w^σ, w^σ_1 are L^1-invariant. To show this, we need equations of L^1 action on $\operatorname{Imm} T^1 Y$; from Lemma 2.1, we have $\bar{y}^K = y^K$, $\bar{y}^K_1 = y^K_1 a_1$. Hence $\bar{w}^\sigma = w^\sigma$, and $\bar{w}^\sigma_1 = \bar{y}^\sigma_1 / \bar{y}^L_1 = (y^\sigma_1 a_1)/(y^L_1 a_1) = y^\sigma_1 / y^L_1 = w^\sigma_1$. It remains to show the recurrence formula. From the expression of formal derivative δ of coordinate functions (2.11) it follows that $w^\sigma_1 = (1/y^L_1) y^\sigma_1 = (1/y^L_1)\delta y^\sigma = (1/y^L_1)\delta w^\sigma = \Delta w^\sigma$ proving (2.29).

Now suppose that we are given the functions $w^\sigma, w^\sigma_1, \dots, w^\sigma_l$, $1 \le l \le r-1$, satisfying properties of (a). We apply induction and prove (a) for the functions $w^\sigma, w^\sigma_1, \dots, w^\sigma_l, w^\sigma_{l+1}$, where $w^\sigma_{l+1} = \Delta_L w^\sigma_l$. Using the formal derivative of a function (2.11) and morphism Δ_L we get

$$\begin{aligned}
y^\sigma_{l+1} = \delta y^\sigma_l &= \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} \delta\left(y^L_{|I_1|} y^L_{|I_2|} \dots y^L_{|I_p|} w^\sigma_p\right) \\
&= \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} \delta\left(y^L_{|I_1|} y^L_{|I_2|} \dots y^L_{|I_p|}\right) w^\sigma_p \\
&+ \sum_{p=1}^{l-1} \sum_{(I_1,I_2,\dots,I_p)} y^L_{|I_1|} y^L_{|I_2|} \dots y^L_{|I_p|} y^L_1 \Delta_L w^\sigma_p + (y^L_1)^{l+1} \Delta_L w^\sigma_l \\
&= \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} \Big(y^L_{|I_1|+1} y^L_{|I_2|} \dots y^L_{|I_p|} w^\sigma_p + y^L_{|I_1|} y^L_{|I_2|+1} \dots y^L_{|I_p|} w^\sigma_p \\
&+ \dots + y^L_{|I_1|} y^L_{|I_2|} \dots y^L_{|I_p|+1} w^\sigma_p\Big) \\
&+ \sum_{p=1}^{l-1} \sum_{(I_1,I_2,\dots,I_p)} y^L_{|I_1|} y^L_{|I_2|} \dots y^L_{|I_p|} y^L_1 w^\sigma_{p+1} + (y^L_1)^{l+1} w^\sigma_{l+1}.
\end{aligned} \tag{2.30}$$

But we need to sum through partitions $(J_1, J_2, \dots, J_p)$ of the set $\{1, 1, \dots, 1\}$ of $l+1$ elements in order to obtain y^σ_{l+1} in the form (2.28). However, such partitions $(J_1, J_2, \dots, J_p)$ arise from partitions $(I_1, I_2, \dots, I_p)$ either by adding the element 1 to some I_s, or by adding the element 1 to get a new partition of the form

$(I_1, I_2, \dots, I_p, \{1\})$. One can see that these partitions are exactly applied in (2.30) hence

$$y^{\sigma}_{l+1} = \sum_{p=1}^{l} \sum_{(J_1,J_2,\dots,J_p)} y^L_{|J_1|} y^L_{|J_2|} \dots y^L_{|J_p|} w^{\sigma}_p + (y^L_1)^{l+1} w^{\sigma}_{l+1}$$

(2.31)

$$= \sum_{p=1}^{l+1} \sum_{(J_1,J_2,\dots,J_p)} y^L_{|J_1|} y^L_{|J_2|} \dots y^L_{|J_p|} w^{\sigma}_p .$$

This proves the existence of functions $w^{\sigma}, w^{\sigma}_1, \dots, w^{\sigma}_r$ satisfying (2.28) and (2.29).

Uniqueness of these functions follows immediately from the fact that y^{σ}_l, $1 \le l \le r$, are polynomial in functions $w^{\sigma}, w^{\sigma}_1, \dots, w^{\sigma}_l$, and w^L_1 is not equal zero on $V^{l,L}$. The L^r-invariance condition is a consequence of Lemma 2.7.

The assertions (b) and (c) are immediate. □

Each of the charts $(V^{r,L}, \psi^{r,L})$ and $(V^{r,L}, \chi^{r,L})$ is referred to as *subordinate* to the chart (V, ψ).

Remark 3. We can introduce the coordinates w^K, w^L_l, w^{σ}_l, $l = 1, 2, \dots, r$, on the set $V^{r,L}$ less formally as follows. First consider the first order $r = 1$. Let $J^1_0 \zeta \in V^{1,L}$. Then by definition, $D(y^L \zeta)(0) \neq 0$, so the function $t \to (y^L \zeta)(t)$ is a diffeomorphism on a neighbourhood of the origin $0 \in \mathbf{R}$. We assign to this diffeomorphism another one, with the same domain of definition, by the formula

$$\alpha_{\zeta}(t) = (y^L \zeta)(t) - (y^L \zeta)(0) = (\mathrm{tr}_{y^L \zeta(0)} \circ y^L \zeta)(t). \tag{2.32}$$

Then

$$(y^L \zeta)(t) = \alpha_{\zeta}(t) + (y^L \zeta)(0) = (\mathrm{tr}_{-y^L \zeta(0)} \circ \alpha_{\zeta})(t).$$

α_{ζ} may be viewed as a reparametrisation of $\mathbf{R}$, satisfying $\alpha_{\zeta}(0) = 0$. Clearly,

$$J^1_0 \zeta = J^1_0 (\zeta \alpha^{-1}_{\zeta} \circ \alpha_{\zeta}) = J^1_0 \zeta \alpha^{-1}_{\zeta} \circ J^1_0 \alpha_{\zeta},$$
$$\dot{y}^K (J^1_0 \zeta) = D(y^K \zeta \alpha^{-1}_{\zeta})(0) D\alpha_{\zeta}(0) = \dot{y}^K (J^1_0 (\zeta \alpha^{-1}_{\zeta})) D\alpha_{\zeta}(0),$$

and all derivatives of $y^L \zeta$ and α_{ζ} coincide. We set

$$\begin{aligned} w^K (J^1_0 \zeta) &= y^K \zeta \alpha^{-1}_{\zeta}(0) = y^K \zeta(0) = y^K (J^1_0 \zeta), \\ \dot{w}^L (J^1_0 \zeta) &= D(y^L \zeta)(0) = \dot{y}^L (J^1_0 \zeta), \\ \dot{w}^{\sigma} (J^1_0 \zeta) &= D(y^{\sigma} \zeta \alpha^{-1}_{\zeta})(0), \end{aligned} \tag{2.33}$$

$\sigma = 1, 2, \dots, L-1, L+1, \dots, m+1$. These formulas define real functions $w^K, \dot{w}^L, \dot{w}^{\sigma}$ on the set $V^{1,L}$. Note that from the identity $D(\alpha^{-1}_{\zeta} \alpha_{\zeta})(0) = D\alpha^{-1}_{\zeta}(0) D\alpha_{\zeta}(0) = 1$ and from (2.32) and (2.33) it follows that

$$D\alpha^{-1}_{\zeta}(0) = \frac{1}{D\alpha_{\zeta}(0)} = \frac{1}{\dot{y}^L (J^1_0 \zeta)}.$$

Then

$$\begin{aligned} w^K(J_0^1\zeta) &= y^K(J_0^1\zeta), \\ \dot{w}^L(J_0^1\zeta) &= \dot{y}^L(J_0^1\zeta), \\ \dot{w}^\sigma(J_0^1\zeta) &= D(y^\sigma\zeta)(0)D\alpha_\zeta^{-1}(0) = \frac{\dot{y}^\sigma(J_0^1\zeta)}{\dot{y}^L(J_0^1\zeta)}, \end{aligned}$$

or, which is the same,

$$\begin{aligned} w^K &= y^K, \\ \dot{w}^L &= \dot{y}^L, \\ \dot{w}^\sigma &= \frac{\dot{y}^\sigma}{\dot{y}^L}, \end{aligned}$$

$\sigma = 1, 2, \dots, L-1, L+1, \dots, m+1$.

More generally, for higher order derivatives, we set

$$\begin{aligned} w^K(J_0^r\zeta) &= y^K\zeta\alpha_\zeta^{-1}(0) = y^K\zeta(0) = y^K(J_0^r\zeta), \\ w_p^L(J_0^r\zeta) &= D^p(y^L\zeta)(0) = y_{(p)}^L(J_0^r\zeta), \\ w_p^\sigma(J_0^r\zeta) &= D^p(y^\sigma\zeta\alpha_\zeta^{-1})(0), \end{aligned} \tag{2.34}$$

$\sigma = 1, 2, \dots, L-1, L+1, \dots, m+1$. Let for example, $r = 2$. We have

$$D\alpha_\zeta^{-1}(\alpha_\zeta(t))D\alpha_\zeta(t) = 1;$$

differentiating at $t = 0$ we obtain

$$D^2\alpha_\zeta^{-1}(0)D\alpha_\zeta(0)D\alpha_\zeta(0) + D\alpha_\zeta^{-1}(0)D^2\alpha_\zeta(0) = 0,$$

and

$$D^2\alpha_\zeta^{-1}(0) = -\frac{D\alpha_\zeta^{-1}(0)D^2\alpha_\zeta(0)}{D\alpha_\zeta(0)D\alpha_\zeta(0)} = -\frac{\ddot{y}^L(J_0^1\zeta)}{(\dot{y}^L(J_0^1\zeta))^3}.$$

Thus, we have on $V^{2,L}$

$$\begin{aligned} w^K(J_0^2\zeta) &= y^K(J_0^2\zeta), \\ \dot{w}^L(J_0^2\zeta) &= \dot{y}^L(J_0^2\zeta), \\ \dot{w}^\sigma(J_0^2\zeta) &= \frac{\dot{y}^\sigma(J_0^2\zeta)}{\dot{y}^L(J_0^2\zeta)}, \\ \ddot{w}^L(J_0^2\zeta) &= \ddot{y}^L(J_0^2\zeta), \\ \ddot{w}^\sigma(J_0^2\zeta) &= D^2(y^\sigma\zeta\alpha_\zeta^{-1})(0) \\ &= D^2(y^\sigma\zeta)(\alpha_\zeta^{-1}(0))D\alpha_\zeta^{-1}(0)D\alpha_\zeta^{-1}(0) + D(y^\sigma\zeta)(\alpha_\zeta^{-1}(0))D^2\alpha_\zeta^{-1}(0) \\ &= D^2(y^\sigma\zeta)(0)D\alpha_\zeta^{-1}(0)D\alpha_\zeta^{-1}(0) + D(y^\sigma\zeta)(0)D^2\alpha_\zeta^{-1}(0) \\ &= \ddot{y}^\sigma(J_0^2\zeta)\frac{1}{(\dot{y}^L(J_0^2\zeta))^2} - \dot{y}^\sigma(J_0^2\zeta)\frac{\ddot{y}^L(J_0^2\zeta)}{(\dot{y}^L(J_0^2\zeta))^3}. \end{aligned}$$

Omitting the argument $J_0^2\zeta$, we have

$$\begin{aligned} w^K &= y^K, \\ \dot{w}^L &= \dot{y}^L, \\ \dot{w}^\sigma &= \frac{\dot{y}^\sigma}{\dot{y}^L}, \\ \ddot{w}^L &= \ddot{y}^L, \\ \ddot{w}^\sigma &= \frac{\ddot{y}^\sigma}{(\dot{y}^L)^2} - \frac{\ddot{y}^L \dot{y}^\sigma}{(\dot{y}^L)^3}, \end{aligned}$$

$\sigma = 1,2,\dots,L-1,L+1,\dots,m+1$. These equations can be solved with respect to $y^K, \dot{y}^K, \ddot{y}^K$; we get

$$\begin{aligned} y^K &= w^K, \\ \dot{y}^L &= \dot{w}^L, \\ \dot{y}^\sigma &= \dot{y}^L \dot{w}^\sigma = \dot{w}^L \dot{w}^\sigma, \\ \ddot{y}^L &= \ddot{w}^L, \\ \ddot{y}^\sigma &= (\dot{y}^L)^2 \ddot{w}^\sigma + \frac{\ddot{y}^L \dot{y}^\sigma}{\dot{y}^L} = (\dot{w}^L)^2 \ddot{w}^\sigma + \ddot{w}^L \dot{w}^\sigma, \end{aligned} \tag{2.35}$$

$\sigma = 1,2,\dots,L-1,L+1,\dots,m+1$.

Remark 4. Let us find the coordinate transformation equations from $(V^{1,L}, \chi^{1,L})$ to $(\bar{V}^{1,M}, \bar{\chi}^{1,M})$. Suppose we have two charts on Y, (V,ψ), $\psi = (y^K)$, and $(\bar{V},\bar{\psi})$, $\bar{\psi} = (\bar{y}^K)$, such that $V \cap \bar{V} \neq \emptyset$. Let the coordinate transformation $\bar{\psi}\psi^{-1}$ be expressed by the equations

$$\bar{y}^K = \bar{y}^K(y^N).$$

Then on $V^1 \cap \bar{V}^1$

$$\dot{\bar{y}}^K = \frac{\partial \bar{y}^K}{\partial y^N} \dot{y}^N.$$

We restrict both sides of this transformation equation to the set $V^{1,L} \cap \bar{V}^{1,M}$, and write on this set

$$\begin{aligned} \dot{\bar{y}}^\nu &= \sum_{\sigma \neq L} \frac{\partial \bar{y}^\nu}{\partial y^\sigma} \dot{y}^\sigma + \frac{\partial \bar{y}^\nu}{\partial y^L} \dot{y}^L, \quad 1 \leq \nu \leq m+1,\ \nu \neq M, \\ \dot{\bar{y}}^M &= \sum_{\sigma \neq L} \frac{\partial \bar{y}^M}{\partial y^\sigma} \dot{y}^\sigma + \frac{\partial \bar{y}^M}{\partial y^L} \dot{y}^L. \end{aligned}$$

But from (2.35), for $1 \leq \nu, \sigma \leq m+1$, $\nu \neq M$, $\sigma \neq L$,

$$\begin{aligned} \dot{\bar{y}}^\nu &= \dot{\bar{w}}^M \dot{\bar{w}}^\nu, \\ \dot{\bar{y}}^M &= \dot{\bar{w}}^M, \\ \dot{y}^\sigma &= \dot{w}^L \dot{w}^\sigma, \\ \dot{y}^L &= \dot{w}^L, \end{aligned}$$

so we have

$$\dot{\bar{w}}^M \dot{\bar{w}}^\nu = \dot{w}^L \left(\sum_{\sigma \neq L} \frac{\partial \bar{y}^\nu}{\partial y^\sigma} \dot{w}^\sigma + \frac{\partial \bar{y}^\nu}{\partial y^L} \right),$$

$$\dot{\bar{w}}^M = \dot{w}^L \left(\sum_{\sigma \neq L} \frac{\partial \bar{y}^M}{\partial y^\sigma} \dot{w}^\sigma + \frac{\partial \bar{y}^M}{\partial y^L} \right),$$

hence

$$\dot{\bar{w}}^\nu = \frac{\sum\limits_{\sigma \neq L} (\partial \bar{y}^\nu / \partial y^\sigma) \dot{w}^\sigma + (\partial \bar{y}^\nu / \partial y^L)}{\sum\limits_{\sigma \neq L} (\partial \bar{y}^M / \partial y^\sigma) \dot{w}^\sigma + (\partial \bar{y}^M / \partial y^L)},$$

$$\dot{\bar{w}}^M = \dot{w}^L \left(\sum_{\sigma \neq L} \frac{\partial \bar{y}^M}{\partial y^\sigma} \dot{w}^\sigma + \frac{\partial \bar{y}^M}{\partial y^L} \right).$$

Note that the functions $\dot{\bar{w}}^\nu$ are *independent* of $\dot{w}^L$.

Remark 5. We wish to express Δ_L in terms of the chart $(V^{r,L}, \chi^{r,L})$ (Theorem 2.1, (b)). The local expressions

$$\frac{\partial}{\partial y^K}, \frac{\partial}{\partial y^K_1}, \dots, \frac{\partial}{\partial y^K_{r-1}}$$

are tangent vectors to $\operatorname{Imm} T^{r-1}Y$, which determines the transformation properties of δ and Δ_L. We have

$$\begin{aligned}
\delta &= \sum_{l=0}^{r-1} y^K_{l+1} \frac{\partial}{\partial y^K_l} = \sum_{l=0}^{r-1} y^K_{l+1} \sum_{p=0}^{r-1} \frac{\partial w^M_p}{\partial y^K_l} \frac{\partial}{\partial w^M_p} \\
&= \sum_{p=0}^{r-1} \left(\sum_{l=0}^{r-1} y^K_{l+1} \frac{\partial w^L_p}{\partial y^K_l} \right) \frac{\partial}{\partial w^L_p} + \sum_{p=0}^{r-1} \left(\sum_{l=0}^{r-1} y^K_{l+1} \frac{\partial w^\sigma_p}{\partial y^K_l} \right) \frac{\partial}{\partial w^\sigma_p} \\
&= y^L_1 \frac{\partial}{\partial w^L} + \sum_{l=1}^{r-1} w^L_{l+1} \frac{\partial}{\partial w^L_l} + \sum_{p=0}^{r-1} \sum_{l=0}^{r-1} \left(y^\nu_{l+1} \frac{\partial w^\sigma_p}{\partial y^\nu_l} + y^L_{l+1} \frac{\partial w^\sigma_p}{\partial y^L_l} \right) \frac{\partial}{\partial w^\sigma_p}.
\end{aligned}$$

By the definition of the formal derivative (2.10),

$$w^\sigma_{p+1} = \Delta_L w^\sigma_p = \frac{1}{y^L_1} \delta w^\sigma_p = \frac{1}{y^L_1} \left(\sum_{l=0}^{r-1} y^L_{l+1} \frac{\partial w^\sigma_p}{\partial y^L_l} + \sum_{l=0}^{r-1} y^\nu_{l+1} \frac{\partial w^\sigma_p}{\partial y^\nu_l} \right),$$

hence we get

$$\Delta_L = \frac{\partial}{\partial w^L} + \sum_{p=0}^{r-1} w^\sigma_{p+1} \frac{\partial}{\partial w^\sigma_p} + \sum_{l=1}^{r-1} \frac{w^L_{l+1}}{w^L_1} \frac{\partial}{\partial w^L_l}. \tag{2.36}$$

2.3. **Contact elements, Grassmannians and Grassmann fibrations.**

Contact elements. We denote by

$$G^rY = \operatorname{Imm} T^rY/L^r \tag{2.37}$$

the quotient set, endowed with the quotient topology. The points of G^rY are called *contact elements of order* r and *type* 1. Clearly, the type refers to immersions of open intervals in $\mathbf{R} = \mathbf{R}^1$ into Y; in this work, we do not consider contact elements of type different from 1. The contact element, containing a regular velocity $P \in \operatorname{Imm} T^rY$, $P = J^r_0\zeta$, is denoted by $[P]$; the r-jet $J^r_0\zeta$ is called a *representative* of $[P]$. Let us consider two C^r immersions ζ_1, ζ_2 in Y, defined on a neighbourhood of $0 \in \mathbf{R}$. We say that ζ_1 and ζ_2 have *contact up to order* r at 0, if there exist charts (V, ψ) and (U, φ) at 0 such that $J^r_0(\zeta_1\psi^{-1}\operatorname{tr}_{-\psi(0)}) = J^r_0(\zeta_2\varphi^{-1}\operatorname{tr}_{-\varphi(0)})$. "$\zeta_1$ and ζ_2 have contact up to order r" is equivalence on the set of immersions in Y, defined on a neighbourhood of 0, and this relation induces an equivalence on the manifold of regular velocities $\operatorname{Imm} T^rY$. The next lemma is a simple observation.

Lemma 2.9. *For any two immersions ζ_1 and ζ_2 in Y, defined on a neighbourhood of* $0 \in \mathbf{R}$*, the following two conditions are equivalent:*

(a) *ζ_1 and ζ_2 have contact of order r at* 0.

(b) *There exists an element $J^r_0\alpha \in L^r$ such that $J^r_0\zeta_1 = J^r_0\zeta_2 \circ J^r_0\alpha$, i.e.* $[J^r_0\zeta_1] = [J^r_0\zeta_2]$.

We denote by $\pi^r : \operatorname{Imm} T^rY \to G^rY$ the quotient projection $P \to [P]$. By the definition of the quotient topology, the quotient projection π^r is continuous, and by Lemma 2.1, we can make the diagram

$$\begin{array}{ccc} \operatorname{Imm} T^rY & \xrightarrow{\pi^r} & G^rY \\ \Big\downarrow \tau^{r,0} & & \\ Y & & \end{array} \tag{2.38}$$

commutative by putting $\rho^{r,0}([J^r_0\zeta]) = \tau^{r,0}(J^r_0\zeta)$. Generally, we have the canonical projections $\rho^{r,s} : G^rY \to G^sY$, where $0 \le s \le r$ and $G^0Y = Y$, that satisfy

$$\rho^{r,s} \circ \pi^r = \pi^s \circ \tau^{r,s}, \quad \rho^{r,0} \circ \pi^r = \tau^{r,0}. \tag{2.39}$$

The mapping $\rho^{r,0}$ is also continuous: this observation follows from the continuity of projections $\tau^{r,0}$ and π^r, and from the properties of final topology on G^rY.

The next theorem shows the quotient set of $\operatorname{Imm} T^rY$ with respect to the differential group L^r to be the base manifold of a principal L^r-bundle. Roughly speaking, a manifold M is called the principle G-bundle, if (a) the quotient set of M modulo G has a manifold structure, (b) M is locally trivial over M/G and (c) G acts freely on M (for details see Bourbaki [4] and Dieudonné [7]).

As a consequence of Theorem 2.1, we have the following assertion.

Theorem 2.2. *If Y is Hausdorff, then the canonical action of L^r defines on* $\operatorname{Imm} T^rY$ *the structure of a right principal L^r-bundle with base G^rY. The projection π^r of this principal L^r-bundle is an open mapping.*

Proof. We can trivialize the manifold $\operatorname{Imm} T^rY$ over G^rY, (2.37), in a natural way by the mapping $J^r_0\zeta \to ([J^r_0\zeta], J^r_0(\operatorname{tr}_{w^L_1\zeta(0)} w^L_1\zeta))$ (cf. local trivialization of $\operatorname{Imm} T^rY$ over Y (2.1)).

It remains to show that (1) the equivalence relation on $\operatorname{Imm} T^rY$ from Lemma 2.7, $(P,Q) \in \mathscr{R}$ if and only if there exists an element $A \in L^r$ satisfying $Q = P \circ A$, is a closed submanifold of the manifold product $\operatorname{Imm} T^rY \times \operatorname{Imm} T^rY$, and (2) that the right group action of L^r on $\operatorname{Imm} T^rY$, $(P,A) \to P \circ A$ (cf. 2.3), is free.

From Theorem 2.1, (c), it is immediate that the relation $\mathscr{R}$ is a submanifold of $\operatorname{Imm} T^rY \times \operatorname{Imm} T^rY$. We shall verify that the complement set to $\mathscr{R}$ in this manifold product is open. Let (P,Q) be a point of $\operatorname{Imm} T^rY \times \operatorname{Imm} T^rY$ where P and Q are *not* equivalent velocities, i.e. $(P,Q) \notin \mathscr{R}$. Obviously then $P \neq Q$, and since Y is Hausdorff, there exist open neighbourhoods of P and Q which do not intersect, and the product of which does not intersect the equivalence $\mathscr{R}$. It doesn't matter whether there is a chart V^r with $P,Q \in V^r$, or not. Hence we see that $\mathscr{R}$ is closed.

Let us prove that the action (2.3) on $\operatorname{Imm} T^rY$ is free. To this purpose, suppose that there exist $P \in \operatorname{Imm} T^rY$, $P = J^r_0\zeta$, and $A \in L^r$ such that $P = P \circ A$. Since P is a regular velocity, there is an r-jet $Q \in J^r_{(\zeta(0),0)}(Y,\mathbf{R})$ such that $Q \circ P = J^r_0 \operatorname{id}_{\mathbf{R}}$, hence $A = J^r_0 \operatorname{id}_{\mathbf{R}}$, the identity element of L^r.

Finally, we show that the principle bundle projection, π^r, is an open mapping. For every open set W in $\operatorname{Imm} T^rY$, the set $\tau^{r,0}(W)$ is open in Y since the projection $\tau^{r,0} : \operatorname{Imm} T^rY \to Y$ is an open mapping. From the continuity of $\rho^{r,0} : G^rY \to Y$, it follows that $\pi^r(W) = (\rho^{r,0})^{-1}(\tau^{r,0}(W))$ is an open set in G^rY. □

In this section we suppose that the manifold Y is Hausdorff. We study the structure of the base G^rY of the principal L^r-bundle $\operatorname{Imm} T^rY$.

For simplicity, first let us consider *grassmannians* of the first order.

Grassmannians of type $\binom{1}{m+1}$ *as jet manifolds.* The set $J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$ of 1-jets with source $0 \in \mathbf{R}$ and target $0 \in \mathbf{R}^{m+1}$ is canonically identified with the tangent space $T_0\mathbf{R}^{m+1}$, and also with $\mathbf{R}^{m+1}$, and is considered with its canonical vector space, topological and smooth structures. The open set $\operatorname{Imm} J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1}) = J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1}) \setminus \{0\}$ consists of *regular* 1-jets $J^1_0\mu$, whose representatives are curves $\mu : I \to \mathbf{R}^{m+1}$ that are *immersions* at the origin $0 \in \mathbf{R}$. Denote by $z^1, z^2, \dots, z^{m+1}$ the *canonical* global coordinates on $J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$ and $\operatorname{Imm} J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$. By definition, an element $J^1_0\mu \in J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$ belongs to the set $\operatorname{Imm} J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$ if and only if there exists an integer L such that $1 \le L \le m+1$, and $z^L(J^1_0\mu) = D(z^L\mu)(0) \neq 0$. The manifold $\operatorname{Imm} J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$ is endowed with the right group action of the differential group of $\mathbf{R}$ of order 1, $L^1 = \operatorname{Imm} J^1_{(0,0)}(\mathbf{R},\mathbf{R})$,

$$\begin{aligned} &\operatorname{Imm} J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1}) \times L^1 \ni (J^1_0\mu, J^1_0\alpha) \\ &\to J^1_0\mu \cdot J^1_0\alpha = J^1_0(\mu\alpha) \in \operatorname{Imm} J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1}) \end{aligned} \tag{2.40}$$

(see also (2.3)). We denote the quotient set $\operatorname{Imm} J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})/L^1$ by G^1_{m+1}. The class of an element $J^1_0\mu \in \operatorname{Imm} J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$, i.e., the L^1-*orbit* of $J^1_0\mu$, is denoted by $[J^1_0\mu]$, and the quotient projection $\mathrm{Imm}\, J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1}) \ni J^1_0\mu \to [J^1_0\mu] \in G^1_{m+1}$ is denoted by π^1. We consider G^1_{m+1} with the quotient topology. Our aim now will be to study topological and smooth properties of G^1_{m+1}.

Since $D(\mu\alpha)(0) = D\mu(\alpha(0))D\alpha(0)$, the group action (2.40) is expressed in the canonical coordinates z^K on $\mathrm{Imm}\, J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$ and the canonical coordinate a on L^1 by

(2.41) $$z^K(J^1_0\mu \cdot J^1_0\alpha) = a(J^1_0\alpha)\cdot z^K(J^1_0\mu).$$

We introduce some charts on $\mathrm{Imm}\, J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$, adapted to the group action (2.40). Denote for every integer L such that $1 \le L \le m+1$

(2.42) $$U^L = \{J^1_0\mu \in \mathrm{Imm}\, J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1}) \,|\, z^L(J^1_0\mu) \neq 0\}.$$

U^L is an open subset of $\mathrm{Imm}\, J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$, and we can write

(2.43) $$\mathrm{Imm}\, J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1}) = \bigcup_{K=1}^{m+1} U^K.$$

Note that U^L is canonically diffeomorphic with the product $\mathbf{R}^m \times (\mathbf{R}\setminus\{0\})$.

Two points $J^1_0\mu_1$ and $J^1_0\mu_2$ belong to the same orbit of the group action (2.40) if and only if there exists $J^1_0\alpha \in L^1$ such that $z^K(J^1_0\mu_2) = a(J^1_0\alpha)\cdot z^K(J^1_0\mu_1)$ for all K. Suppose we have $J^1_0\mu_1$ and $J^1_0\mu_2$. Then by hypothesis, $z^L(J^1_0\mu_1) \neq 0$ for some L, and since $a(J^1_0\alpha)$ is always different from 0, also $z^L(J^1_0\mu_2) \neq 0$. For this index L condition $z^L(J^1_0\mu_2) = a(J^1_0\alpha)\cdot z^L(J^1_0\mu_1)$ implies

(2.44) $$a(J^1_0\alpha) = \frac{z^L(J^1_0\mu_2)}{z^L(J^1_0\mu_1)},$$

so according to (2.41), for all $\sigma = 1,2,\ldots,L-1,L+1,\ldots,m+1$

(2.45) $$\frac{z^\sigma(J^1_0\mu_1)}{z^L(J^1_0\mu_1)} = \frac{z^\sigma(J^1_0\mu_2)}{z^L(J^1_0\mu_2)}.$$

In particular, the functions $u^\sigma : U^L \to \mathbf{R}$, defined by

(2.46) $$u^\sigma(J^1_0\mu) = \frac{z^\sigma(J^1_0\mu)}{z^L(J^1_0\mu)},$$

are *constant* along the L^1-orbits. We set $u^L(J^1_0\mu) = z^L(J^1_0\mu)$. We claim that the pair (U^L,χ^L), $\chi^L = (u^L,u^\sigma)$, is a chart on $\mathrm{Imm}\, J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$. We have on U^L

(2.47) $$u^L = z^L, \quad u^\sigma = \frac{z^\sigma}{z^L}.$$

Clearly, the *image* of the mapping χ^L consists of the solutions of equations (2.47)

(2.48) $$z^L = u^L, \quad z^\sigma = u^L u^\sigma,$$

and can be canonically identified with the open set U^L. The charts (U^L,χ^L) are said to be *subordinate* to the canonical chart on $\mathrm{Imm}\, J^1_{(0,0)}(\mathbf{R},\mathbf{R}^{m+1})$ (cf. Theorem 2.1). Transformation equations (2.47) and (2.48) are obviously of class C^∞.

The group action (2.40) is expressed in the subordinate chart (U^L, χ^L) by

$$
(2.49)\quad
\begin{aligned}
u^\sigma(J_0^1\mu \cdot J_0^1\alpha) &= \frac{z^\sigma(J_0^1\mu \cdot J_0^1\alpha)}{z^L(J_0^1\mu \cdot J_0^1\alpha)} = \frac{z^\sigma(J_0^1\mu)}{z^L(J_0^1\mu)} = u^\sigma(J_0^1\mu),\\
u^L(J_0^1\mu \cdot J_0^1\alpha) &= z^L(J_0^1\mu \cdot J_0^1\alpha) = a(J_0^1\alpha)z^L(J_0^1\mu) = a(J_0^1\alpha)u^L(J_0^1\mu).
\end{aligned}
$$

Suppose we have two subordinate charts (U^K, χ^K), $\chi^K = (u^K, u^\nu)$, and (U^L, χ^L), $\chi^L = (v^L, v^\sigma)$, such that $K \neq L$, and a point $J_0^1\mu \in U^K \cap U^L$. Then

$$
(2.50)\quad
\begin{aligned}
u^K(J_0^1\mu) &= z^K(J_0^1\mu),\\
u^L(J_0^1\mu) &= \frac{z^L(J_0^1\mu)}{z^K(J_0^1\mu)},\\
u^\nu(J_0^1\mu) &= \frac{z^\nu(J_0^1\mu)}{z^K(J_0^1\mu)}, \quad \nu \in \{1,2,\dots,m+1\},\ \nu \neq K, L,
\end{aligned}
$$

and

$$
(2.51)\quad
\begin{aligned}
z^L(J_0^1\mu) &= v^L(J_0^1\mu),\\
z^K(J_0^1\mu) &= v^L(J_0^1\mu)v^K(J_0^1\mu),\\
z^\sigma(J_0^1\mu) &= v^L(J_0^1\mu)v^\sigma(J_0^1\mu), \quad \sigma \in \{1,2,\dots,m+1\},\ \nu \neq K, L.
\end{aligned}
$$

Substituting from (2.51) in (2.50), we have

$$
(2.52)\quad
\begin{aligned}
u^K(J_0^1\mu) &= z^K(J_0^1\mu) = v^L(J_0^1\mu)v^K(J_0^1\mu),\\
u^L(J_0^1\mu) &= \frac{z^L(J_0^1\mu)}{z^K(J_0^1\mu)} = \frac{v^L(J_0^1\mu)}{v^L(J_0^1\mu)v^K(J_0^1\mu)} = \frac{1}{v^K(J_0^1\mu)},\\
u^\nu(J_0^1\mu) &= \frac{z^\nu(J_0^1\mu)}{z^K(J_0^1\mu)} = \frac{v^L(J_0^1\mu)v^\nu(J_0^1\mu)}{v^L(J_0^1\mu)v^K(J_0^1\mu)} = \frac{v^\nu(J_0^1\mu)}{v^K(J_0^1\mu)},
\end{aligned}
$$

thus, the transformation equations are

$$
(2.53)\quad u^K = v^L v^K, \quad u^L = \frac{1}{v^K}, \quad u^\nu = \frac{v^\nu}{v^K}.
$$

The inverse transformation is indeed of the same form,

$$
(2.54)\quad v^K = \frac{1}{u^L}, \quad v^L = u^K u^L, \quad v^\sigma = \frac{u^\sigma}{u^L}.
$$

We are now in a position to define a smooth structure on the topological space G^1_{m+1}. First we show that every subordinate chart (U^L, χ^L), $\chi^L = (u^L, u^\sigma)$, on $\operatorname{Imm} J^1_{(0,0)}(\mathbf{R}, \mathbf{R}^{m+1})$, induces a chart on G^1_{m+1}. By equation (2.49), the functions $u^\sigma : U^L \to \mathbf{R}$ are constant along the L^1_1-orbits; we set $\tilde{u}^\sigma([J_0^1\mu]) = u^\sigma(J_0^1\mu)$ for every point $J_0^1\mu \in U^L$. This formula can equivalently be written as

$$
(2.55)\quad \tilde{u}^\sigma \circ \pi^1 = u^\sigma,
$$

and defines some functions $\tilde{u}^\sigma : \tilde{U}^L \to \mathbf{R}$, where $\tilde{U}^L = \theta(U^L)$. Denote by $\tilde{\chi}^L$ the mapping of $\tilde{U}^L$ into $\mathbf{R}^m$ whose components are $\tilde{u}^\sigma$; we have the commutative diagram

(2.56)
$$\begin{array}{ccc} U^L & \xrightarrow{\chi^L} & \mathbf{R}^m \times (\mathbf{R} \setminus \{0\}) \\ \downarrow \theta & & \downarrow \mathrm{pr}_1 \\ \tilde{U}^L & \xrightarrow{\tilde{\chi}^L} & \mathbf{R}^m \end{array}$$

The pair $(\tilde{U}^L, \tilde{\chi}^L)$, is obviously a chart on G^1_{m+1}: by the definition of the quotient topology, the projection $\tilde{U}^L$ of the set U^L is an open set in G^1_{m+1}, the set $\tilde{\chi}^L(\tilde{U}^L) \subset \mathbf{R}^m$ is also open, and $\tilde{\chi}^L$ is a homeomorphism.

Let (U^K, χ^K), $\chi^K = (u^K, u^\nu)$, and (U^L, χ^L), $\chi^L = (v^L, v^\sigma)$, be subordinate charts such that $K \neq L$, and let $(\tilde{U}^K, \tilde{\chi}^K)$, $\tilde{\chi}^K = (\tilde{u}^\nu)$, and $(\tilde{U}^L, \tilde{\chi}^L)$, $\tilde{\chi}^L = (\tilde{v}^\sigma)$, be the associated charts on G^1_{m+1}. Suppose we have an element $[J^1_0\mu]$ of the intersection $\tilde{U}^K \cap \tilde{U}^L$. Since the orbit $[J^1_0\mu]$ is contained in $U^K \cap U^L$, any representative $J^1_0\mu$ belongs to $U^K \cap U^L$, and its coordinates satisfy the transformation equations (2.53). Consequently, the classes $[J^1_0\mu] \in \tilde{U}^K \cap \tilde{U}^L$ satisfy the transformation equations

(2.57)
$$\begin{aligned} \tilde{u}^L &= \frac{1}{\tilde{v}^K}, \\ \tilde{u}^\sigma &= \frac{\tilde{v}^\sigma}{\tilde{v}^K}, \quad \sigma = 1, 2, \dots, K-1, K+1, \dots, m+1. \end{aligned}$$

The inverse transformation has equations of the same form.

In the next lemma, we summarize properties of the topological and smooth structures on the sets $\mathrm{Imm}\, J^1_{(0,0)}(\mathbf{R}, \mathbf{R}^{m+1})$ and G^1_{m+1}.

Lemma 2.10. *The canonical group action (2.40) defines the structure of a right principal L^1-bundle on the manifold* $\mathrm{Imm}\, J^1_{(0,0)}(\mathbf{R}, \mathbf{R}^{m+1})$. *The base of the bundle G^1_{m+1} is compact and Hausdorff.*

Proof. Lemma 2.10 is a special case of Theorem 2.2 for $r = 1$ and $Y = \mathbf{R}^{m+1}$.

Let us show the local trivialization in this case. For this purpose, consider a subordinate chart (U^L, χ^L), $\chi^L = (u^L, u^\sigma)$, and the associated chart $(\tilde{U}^L, \tilde{\chi}^L)$, $\tilde{\chi}^L = (\tilde{u}^\sigma)$; obviously, $U^L = \theta^{-1}(\tilde{U}^L)$. We have the commutative diagram

(2.58)
$$\begin{array}{ccccc} U^L & \xrightarrow{\chi^L} & \mathbf{R}^m \times (\mathbf{R} \setminus \{0\}) & \xrightarrow{(\tilde{\chi}^L)^{-1} \times \mathrm{id}} & \tilde{U}^L \times L^1 \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{U}^L & \xrightarrow{\tilde{\chi}^L} & \mathbf{R}^m & \xrightarrow{(\tilde{\chi}^L)^{-1}} & \tilde{U}^L \end{array}$$

defining a local trivialization of $\mathrm{Imm}\, J^1_{(0,0)}(\mathbf{R}, \mathbf{R}^{m+1})$ over $\tilde{U}^L \subset G^1_{m+1}$ with projection π^1 and type fibre $L^1 = \mathbf{R} \setminus \{0\}$,

(2.59)
$$U^L \ni J^1_0\mu \to ([J^1_0\mu], u^L(J^1_0\mu)) \in \tilde{U}^L \times L^1.$$

This local trivialization is L^1-equivariant by formula (2.49).

It remains to prove compactness of G^1_{m+1}. We have a continuous mapping π^1 of $\mathbf{R}^{m+1} \setminus \{0\}$ onto G^1_{m+1}, sending a (nonzero) vector to the vector subspace of $\mathbf{R}^{m+1}$, generated by this vector. Then G^1_{m+1} is compact as the continuous image of the unit sphere. $\square$

Grassmann fibrations. If $Y = \mathbf{R}^{m+1}$, we have the canonical projections

$\tau^{r,0} : \operatorname{Imm} T^r\mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$, $\pi^r : \operatorname{Imm} T^r\mathbf{R}^{m+1} \to G^r\mathbf{R}^{m+1}$, $\rho^{r,0} : G^r\mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$.

The fibre over $0 \in \mathbf{R}^{m+1}$ in $\operatorname{Imm} T^r\mathbf{R}^{m+1}$, $\operatorname{Imm} J^r_{(0,0)}(\mathbf{R}, \mathbf{R}^{m+1})$, is a closed, L^r-invariant submanifold. We define

$$G^r_{m+1} = \operatorname{Imm} J^r_{(0,0)}(\mathbf{R}, \mathbf{R}^{m+1})/L^r.$$

Since $G^r_{m+1} = (\rho^{r,0})^{-1}(0)$, G^r_{m+1} is a closed submanifold of $G^r\mathbf{R}^{m+1}$. We call G^r_{m+1} the *Grassmannian of order r* and *type* 1 over $\mathbf{R}^{m+1}$.

Lemma 2.11. *The canonical group action of L^r defines the structure of a right principal L^r-bundle on* $\operatorname{Imm} J^r_{(0,0)}(\mathbf{R}, \mathbf{R}^{m+1})$. *If $r = 1$, then the base G^r_{m+1} is compact. If $r \neq 1$, the base is not compact.*

Proof. This is a direct consequence of Theorem 2.2 and Lemma 2.10. $\square$

We now give an explicit description of the smooth structure on G^rY. Let $(V^{r,L}, \chi^{r,L})$, $\chi^{r,L} = (w^L, w^L_1, w^L_2, \dots, w^L_r, w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r)$, be a subordinate chart on $\operatorname{Imm} T^rY$ to the chart (V, ψ) on Y. Denote $\tilde{V}^{r,L} = \pi^r(V^{r,L})$, and $\tilde{\chi}^{r,L} = (\tilde{w}^L, \tilde{w}^\sigma, \tilde{w}^\sigma_1, \tilde{w}^\sigma_2, \dots, \tilde{w}^\sigma_r)$, where

$$\tilde{w}^L([P]) = w^L(P),\ \tilde{w}^\sigma([P]) = w^\sigma(P),$$
$$\tilde{w}^\sigma_1([P]) = w^\sigma_1(P),\ \tilde{w}^\sigma_2([P]) = w^\sigma_2(P),\ \dots,\ \tilde{w}^\sigma_r([P]) = w^\sigma_r(P).$$

Then the pair $(\tilde{V}^{r,L}, \tilde{\chi}^{r,L})$ is a chart on G^rY, which is said to be *associated* with the subordinate chart $(V^{r,L}, \chi^{r,L})$.

We describe transformation equations between associated charts.

Lemma 2.12. *Let (V, ψ), $\psi = (y^K)$, and (U, φ), $\varphi = (\bar{y}^K)$, be two charts on Y such that $V \cap U \neq \emptyset$, and let $(\tilde{V}^{r,L}, \tilde{\chi}^{r,L})$ and $(\tilde{U}^{r,M}, \tilde{\phi}^{r,M})$ be charts on G^rY, associated with the subordinate charts $(V^{r,L}, \chi^{r,L})$ and $(U^{r,M}, \phi^{r,M})$, respectively. Let the transformation formulas from (V, ψ) to (U, φ) be given by the equations*

$$\bar{y}^M = F^M(y^L, y^\sigma), \quad \bar{y}^\nu = F^\nu(y^L, y^\sigma).$$

Then

$$\tilde{\bar{w}}^\nu_1 = \frac{\tilde{w}^L_1}{\tilde{\bar{w}}^M_1} \left(\frac{\partial F^\nu}{\partial y^L} + \tilde{w}^\sigma_1 \frac{\partial F^\nu}{\partial y^\sigma} \right), \tag{2.60}$$

and the functions $\tilde{\bar{w}}^\nu_{l+1}$ satisfy the recurrence formula

$$\tilde{\bar{w}}^\nu_{l+1} = \frac{\tilde{w}^L_1}{\tilde{\bar{w}}^M_1} \Delta_L \tilde{\bar{w}}^\nu_l. \tag{2.61}$$

Proof. From Theorem 2.1 (a), (2.29), and Lemma 2.8, we obtain

$$\bar{\tilde{w}}^{\nu}_{l+1} = \bar{\Delta}_M \bar{\tilde{w}}^{\nu}_l = \frac{\tilde{w}^L_1}{\bar{\tilde{w}}^M_1} \Delta_L \bar{\tilde{w}}^{\nu}_l ,$$

which proves (2.61). The expression (2.60) is now a consequence of the chart expression of Δ_L (cf. (2.36)). □

Note that from the structure of the morphism Δ_L (2.22), we can write $\Delta_L = d/d\tilde{w}^L$.

It is easily seen that every chart (V, ψ) on Y induces a local trivialization

$$(\rho^{r,0})^{-1}(V) \ni [J^r_0 \zeta] \to (\zeta(0), [J^r_0(\mathrm{tr}_{\psi\zeta(0)}\, \psi\zeta)]) \in V \times G^r_{m+1}.$$

Summarizing, we have this direct consequence of Theorem 2.1. For details on *fibration structures*, we refer the reader e.g. Krupka [30].

Theorem 2.3. *The orbit manifold G^rY has the structure of a fibration with base Y, projection $\rho^{r,0}$, and type fibre G^r_{m+1}. The dimension of G^rY is*

$$\dim G^rY = m(r+1) + 1.$$

The manifold G^rY together with the fibration structure described by Theorem 2.3, is called the *Grassmann fibration of order r* over the manifold Y.

If $\gamma : I \to Y$ is an immersion, such that $T^r\gamma(I) \subset V^{r,L}$ for some L, then the curve

$$I \ni t \to [T^r\gamma](t) = [T^r\gamma(t)] \in G^rY \tag{2.62}$$

is called the *Grassmann prolongation* of γ of order r. We have

Lemma 2.13. *The mapping* (2.62) *has the chart expression*

$$\begin{aligned} &\tilde{w}^L \circ [T^r\gamma](t) = w^L \circ \gamma(t) = \mu(t), \\ &\tilde{w}^{\sigma}_l \circ [T^r\gamma](t) = D(\tilde{w}^{\sigma}_{l-1} \circ [T^{r-1}\gamma] \circ \mu^{-1})(\mu(t)),\ 1 \le l \le r. \end{aligned} \tag{2.63}$$

Proof. The chart expression of $[T^r\gamma]$ (2.62) easily follows from Lemma 2.2 and Remark 3, (2.34), in Sect. 2.1. □

3. CONTACT FORMS

In this section we study differential forms, defined on manifolds of regular velocities and Grassmann fibrations, that vanish identically with respect to the canonical prolongations of curves in these manifolds. In Section 2.1, we have introduced the canonical projections $\tau^{r,0}$ and $\rho^{r,0}$ of $\operatorname{Imm} T^rY$ and G^rY, respectively. Let Y be a smooth manifold of dimension $m+1$, $m \geq 1$. For an open subset W of Y we put $W^r = (\tau^{r,0})^{-1}(W) \subset \operatorname{Imm} T^rY$, and $\tilde{W}^r = (\rho^{r,0})^{-1}(W) \subset G^rY$.

3.1. **Contact forms.**

Contact forms on $\operatorname{Imm} T^rY$. We denote by $\Omega_0^r W$ the ring of smooth functions on W^r, and by $\Omega_k^r W$ the $\Omega_0^r W$-module of smooth differential k-forms on W^r. Let $\eta \in \Omega_1^r W$ be a 1-form. We say that η is *contact*, if

$$T^r\zeta^*\eta = 0 \tag{3.1}$$

for all immersions ζ, defined on an open interval in $\mathbf{R}$ with values in W, where $T^r\zeta$ is the canonical jet prolongation (see Sect. 2.1, (2.6)). We note that in the context of this definition of contactness, every function f on W^r is contact if and only if f vanishes identically, and every k-form, $k \geq 2$, is contact.

In the next lemma we give a description of the contact ideal in terms of charts.

Lemma 3.1. *Let W be an open set in Y, let η be a 1-form on $(\tau^{r,0})^{-1}(W)$, and let (V,ψ), $\psi = (y^K)$, be an arbitrary chart on Y such that $V \subset W$.*

(a) *η is contact if and only if for every subordinate chart $(V^{r,L},\psi^{r,L})$,*

$$\eta = \sum_{s=1}^{r-1} A^s\eta_s^L + \sum_{l=0}^{r-1} A_\sigma^l \eta_l^\sigma,$$

where

$$\eta_s^L = dy_s^L - \frac{y_{s+1}^L}{y_1^L}dy^L, \quad \eta_l^\sigma = dy_l^\sigma - \frac{y_{l+1}^\sigma}{y_1^L}dy^L. \tag{3.2}$$

The forms (3.2) *are linearly independent.*

(b) *η is contact if and only if for every subordinate chart $(V^{r,L},\chi^{r,L})$,*

$$\eta = \sum_{s=1}^{r-1} B^s\omega_s^L + \sum_{l=0}^{r-1} B_\sigma^l \omega_l^\sigma,$$

where

$$\omega_s^L = dw_s^L - \frac{w_{s+1}^L}{w_1^L}dw^L, \quad \omega_l^\sigma = dw_l^\sigma - w_{l+1}^\sigma dw^L. \tag{3.3}$$

The forms (3.3) *are linearly independent.*

Proof. (a) Let $I \ni t \to \gamma(t) \in Y$ be a curve (immersion), defined on an open interval in $\mathbf{R}$ with values in $W \subset Y$. We derive a formula for the pull-back $T^r\gamma^*\eta$. Let η be expressed in associated chart $(V^{r,L},\psi^{r,L})$ by

$$\eta = \sum_{l=0}^{r} A_L^l dy_l^L + \sum_{l=0}^{r} A_\sigma^l dy_l^\sigma.$$

Since by Lemma 2.2, (a), $y^K_l \circ T^r\gamma(t) = D^l(y^K\gamma)(t)$, we have for every $t_0 \in I$, and for every $\xi \in T_{t_0}\mathbf{R}$, $\xi = \xi_0(d/dt)_{t_0}$,

$$\begin{aligned}(T^r\gamma^*\eta)(t_0)\cdot\xi &= \sum_{l=0}^{r}(A^l_K \circ T^r\gamma)(t_0)d(y^K_l \circ T^r\gamma)(t_0)\cdot\xi \\ &= \sum_{l=0}^{r}(A^l_K \circ T^r\gamma)(t_0)D(y^K_l \circ T^r\gamma)(t_0)dt(t_0)\cdot\xi \qquad (3.4)\\ &= \sum_{l=0}^{r}A^l_K(J^r_0(\gamma\mathrm{tr}_{-t_0}))y^K_{l+1}(J^{r+1}_0(\gamma\mathrm{tr}_{-t_0}))dt(t_0)\cdot\xi.\end{aligned}$$

If the expression (3.4) vanishes for all γ, we have

$$\begin{aligned}&A^r_L = 0, \quad A^r_\sigma = 0,\\ &A^0_L y^L_1 + A^1_L y^L_2 + \ldots + A^{r-1}_L y^L_r + A^0_\sigma y^\sigma_1 + A^1_\sigma y^\sigma_2 + \ldots + A^{r-1}_\sigma y^\sigma_r = 0,\end{aligned}$$

hence on $V^{r,L}$

$$A^0_L + A^1_L\frac{y^L_2}{y^L_1} + \ldots + A^{r-1}_L\frac{y^L_r}{y^L_1} + A^0_\sigma\frac{y^\sigma_1}{y^L_1} + A^1_\sigma\frac{y^\sigma_2}{y^L_1} + \ldots + A^{r-1}_\sigma\frac{y^\sigma_r}{y^L_1} = 0.$$

Now we get

$$\begin{aligned}\eta &= \sum_{l=1}^{r-1}A^l_L dy^L_l + \sum_{l=0}^{r-1}A^l_\sigma dy^\sigma_l\\ &- \left(A^1_L\frac{y^L_2}{y^L_1} + \ldots + A^{r-1}_L\frac{y^L_r}{y^L_1} + A^0_\sigma\frac{y^\sigma_1}{y^L_1} + A^1_\sigma\frac{y^\sigma_2}{y^L_1} + \ldots + A^{r-1}_\sigma\frac{y^\sigma_r}{y^L_1}\right)dy^L\\ &= \sum_{l=1}^{r-1}A^l_L\left(dy^L_l - \frac{y^L_{l+1}}{y^L_1}dy^L\right) + \sum_{l=0}^{r-1}A^l_\sigma\left(dy^\sigma_l - \frac{y^\sigma_{l+1}}{y^L_1}dy^L\right).\end{aligned}$$

(b) Using the following chart properties of the canonical prolongation $I \ni t \to T^r\gamma(t) \in V^{r,L}$ of an immersion $\gamma : I \to Y$,

$$\begin{aligned}&(w^\sigma \circ T^r\gamma)(t) = w^\sigma \circ \gamma \circ \mu^{-1}(\mu(t)),\\ &(w^\sigma_k \circ T^r\gamma)(t) = D(w^\sigma_{k-1} \circ T^{r-1}\gamma \circ \mu^{-1})(\mu(t)),\\ &(w^L \circ T^r\gamma)(t) = \mu(t) = (w^L \circ \gamma)(t),\\ &(w^L_k \circ T^r\gamma)(t) = D^k(\mu)(t),\end{aligned}$$

the proof of assertion (b) is proceeded analogously to (a). □

The ideal of the exterior algebra $\Omega^r_c W$ of differential forms on W^r, locally generated by contact 1-forms, is called the *contact ideal*. By a *contact k-form* we mean any k-form, belonging to the contact ideal. We note that the sets of forms $\{dy^L, \eta^L_s, \eta^\sigma_l, dy^L_r, dy^\sigma_r\}$, $\{dw^L, \omega^L_s, \omega^\sigma_l, dw^L_r, dw^\sigma_r\}$, where $l = 0,1,\ldots,r-1$, $s = 1,2,\ldots,r-1$, both define a basis of linear forms on $V^{r,L} \subset \mathrm{Imm}\,T^rY$, called the *contact basis*.

Contact forms on G^rY. Our aim now is to analyze contact forms on Grassmann fibration G^rY. We denote by $\tilde{\Omega}^r_0 W$ the ring of smooth functions on $\tilde{W}^r \subset G^rY$, and by $\tilde{\Omega}^r_k W$ the $\tilde{\Omega}^r_0 W$-module of smooth differential k-forms on $\tilde{W}^r$. Let η be a 1-form, $\eta \in \tilde{\Omega}^r_1 W$. We say that η is *contact*, if

$$[T^r\zeta]^*\eta = 0$$

for all immersions ζ, defined on an open interval in $\mathbf{R}$ with values in W, where $[T^r\zeta]$ denotes the Grassmann prolongation of a curve ζ (see Sect. 2.3, (2.62)).

In the following lemma we give a description of the ideal $\tilde{\Omega}^r_c W$ in terms of charts.

Lemma 3.2. *Let W be an open set in Y, let η be a 1-form on $\tilde{W}^r$, and let (V,ψ), $\psi = (y^K)$, be an arbitrary chart on Y such that $V \subset W$. Then η is contact if and only if for every chart $(\tilde{V}^{r,L}, \tilde{\chi}^{r,L})$, $\tilde{\chi}^{r,L} = (\tilde{w}^L, \tilde{w}^\sigma, \tilde{w}^\sigma_1, \tilde{w}^\sigma_2, \dots, \tilde{w}^\sigma_r)$, associated with the subordinate chart $(V^{r,L}, \chi^{r,L})$,*

$$\eta = \sum_{i=0}^{r-1} B^i_\sigma \tilde{\omega}^\sigma_i,$$

where

$$\tilde{\omega}^\sigma_i = d\tilde{w}^\sigma_i - \tilde{w}^\sigma_{i+1} d\tilde{w}^L. \tag{3.5}$$

Proof. We can express η in chart $(\tilde{V}^{r,L}, \tilde{\chi}^{r,L})$ in the form

$$\eta = B_L d\tilde{w}^L + \sum_{l=0}^{r} B^l_\sigma d\tilde{w}^\sigma_l.$$

Let $I \ni t \to T^r\gamma(t) \in W$ be an immersion such that $T^r\gamma(I) \subset V^{r,L}$. The Grassmann prolongation of γ, $t \to [T^r\gamma](t)$ (2.62), has the chart expression

$$\begin{aligned}
&(\tilde{w}^L \circ [T^r\gamma])(t) = \mu(t) = w^L \circ \gamma(t),\\
&(\tilde{w}^\sigma \circ [T^r\gamma])(t) = \tilde{w}^\sigma \circ \gamma \circ \mu^{-1}(\mu(t)),\\
&(\tilde{w}^\sigma_l \circ [T^r\gamma])(t) = D(\tilde{w}^\sigma_{l-1} \circ [T^{r-1}\gamma] \circ \mu^{-1})(\mu(t)) = D^l(\tilde{w}^\sigma \circ \gamma \circ \mu^{-1})(\mu(t)),
\end{aligned} \tag{3.6}$$

$l = 1, 2, \dots, r$. We have for every $t_0 \in I$,

$$\begin{aligned}
([T^r\gamma]^*\eta)(t_0) &= (B_L \circ [T^r\gamma])(t_0) d(\tilde{w}^L \circ [T^r\gamma])(t_0)\\
&\quad + \sum_{l=0}^{r} (B^l_\sigma \circ [T^r\gamma])(t_0) d(\tilde{w}^\sigma_l \circ [T^r\gamma])(t_0).
\end{aligned}$$

Using (3.6), we obtain

$$\begin{aligned}
&d(\tilde{w}^\sigma_l \circ [T^r\gamma])(t_0) = d(D^l(\tilde{w}^\sigma\gamma\mu^{-1}) \circ \mu)(t_0) = D(D^l(\tilde{w}^\sigma\gamma\mu^{-1}) \circ \mu)(t_0) dt(t_0)\\
&= D^{l+1}(\tilde{w}^\sigma\gamma\mu^{-1})(\mu(t_0)) D\mu(t_0) dt(t_0) = D^{l+1}(\tilde{w}^\sigma\gamma\mu^{-1})(\mu(t_0)) d\mu(t_0).
\end{aligned}$$

Thus we have

$$\begin{aligned}
([T^r\gamma]^*\eta)(t_0) &= (B_L \circ [T^r\gamma])(t_0) d\mu(t_0)\\
&\quad + \sum_{l=0}^{r} (B^l_\sigma \circ [T^r\gamma])(t_0) D^{l+1}(\tilde{w}^\sigma\gamma\mu^{-1})(\mu(t_0)) d\mu(t_0).
\end{aligned}$$

Now, if $[T^r\gamma]^*\eta$ vanishes for all γ, we have

$$(B_L \circ [T^r\gamma])(t_0) + \sum_{l=0}^{r-1} (B^l_\sigma \circ [T^r\gamma])(t_0) D^{l+1}(\tilde{w}^\sigma \gamma\mu^{-1})(\mu(t_0)) + (B^r_\sigma \circ [T^r\gamma])(t_0) D^{r+1}(\tilde{w}^\sigma \gamma\mu^{-1})(\mu(t_0)) = 0.$$

Then, however, because γ is arbitrary,

$$B^r_\sigma = 0, \quad B_L + \sum_{l=0}^{r-1} B^l_\sigma \tilde{w}^\sigma_{l+1} = 0,$$

and turning back to 1-form η, we get the desired result. □

In particular, the 1-forms $\tilde{\omega}^\sigma_i$, $0 \le i \le r-1$, are contact, and every contact 1-form is expressible as a linear combination of $\tilde{\omega}^\sigma_i$. By definition, we see that $\tilde{\omega}^\sigma_i$ are linearly independent.

The ideal of the exterior algebra of differential forms on $\tilde{W}^r$, locally generated by contact 1-forms, is called the *contact ideal*, and is denoted by $\tilde{\Omega}^r_c W$. By a *contact k-form* we mean any k-form, belonging to the contact ideal.

From now on, we adopt this notational *convention*: we will *not* distinguish between coordinates on $\operatorname{Imm} T^rY$ and G^rY, and so we will omit the sign ˜ upon the w-coordinates and the 1-forms ω as well; it is always clear from the context where the coordinates or forms are defined.

A smooth differential k-form $\eta \in \tilde{\Omega}^r_k W$ is said to be *locally generated* by l-forms $\omega^{\sigma_1}_{i_1} \wedge \omega^{\sigma_2}_{i_2} \wedge \ldots \wedge \omega^{\sigma_l}_{i_l}$ if in some associated chart on $\tilde{W}^r = (\rho^{r,0})^{-1}(W) \subset G^rY$, η is expressible as

$$\eta = \omega^{\sigma_1}_{i_1} \wedge \omega^{\sigma_2}_{i_2} \wedge \ldots \wedge \omega^{\sigma_l}_{i_l} \wedge \eta^{i_1 i_2 \ldots i_l}_{\sigma_1\sigma_2\ldots\sigma_l}, \tag{3.7}$$

for some $(k-l)$-form $\eta^{i_1 i_2 \ldots i_l}_{\sigma_1\sigma_2\ldots\sigma_l} \in \tilde{\Omega}^r_{k-l} W$. We call k-form η *locally l-contact* if in the decomposition (3.7) at least one $(k-l)$-form $\eta^{i_1 i_2 \ldots i_l}_{\sigma_1\sigma_2\ldots\sigma_l}$ does not contain any contact 1-form ω^σ_i.

Lemma 3.3. (a) *Let W be an open set in Y, and let (V,ψ), $\psi = (y^K)$, be an arbitrary chart on Y such that $V \subset W$. If $(\tilde{V}^{r,L}, \tilde{\chi}^{r,L})$ is a chart on $\tilde{W}^r \subset G^rY$, associated with the subordinate chart $(V^{r,L}, \chi^{r,L})$, then the forms*

$$dw^L,\ \omega^\sigma_i,\ dw^\sigma_r, \tag{3.8}$$

where $0 \le i \le r-1$, define a basis of linear forms on $\tilde{V}^{r,L}$.

(b) *Let W be an open set in Y. If (V,ψ), $\psi = (y^K)$, and (U,φ), $\varphi = (\bar{y}^K)$, are two charts on Y such that $V, U \subset W$, $V \cap U \neq \emptyset$, and if $(\tilde{V}^{r,L}, \tilde{\chi}^{r,L})$ and $(\tilde{U}^{r,M}, \tilde{\phi}^{r,M})$ are charts on $\tilde{W}^r \subset G^rY$, associated with the subordinate charts $(V^{r,L}, \chi^{r,L})$ and $(U^{r,M}, \phi^{r,M})$ respectively, then*

$$\bar{\omega}^\nu_l = \left(\frac{\partial \bar{w}^\nu_l}{\partial w^\sigma} - \bar{w}^\nu_{l+1} \frac{\partial \bar{w}^M}{\partial w^\sigma} \right) \omega^\sigma + \sum_{p=1}^{l} \frac{\partial \bar{w}^\nu_l}{\partial w^\sigma_p} \omega^\sigma_p, \quad 0 \le l \le r-1. \tag{3.9}$$

Proof. The proof of assertion (a) is straightforward.

Let us prove (b). To derive the transformation formula (3.9) for 1-forms ω_l^σ, we consider two associated charts $(\tilde{V}^{r,L},\tilde{\chi}^{r,L})$, $\tilde{\chi}^{r,L}=(w^L,w^\sigma,w_1^\sigma,w_2^\sigma,\dots,w_r^\sigma)$, and $(\tilde{U}^{r,M},\tilde{\phi}^{r,M})$, $\tilde{\phi}^{r,M}=(\bar{w}^M,\bar{w}^\nu,\bar{w}_1^\nu,\bar{w}_2^\nu,\dots,\bar{w}_r^\nu)$, on W^r. On $V\cap U$ we have $\bar{w}^M=\bar{w}^M(w^L,w^\sigma)$, $\bar{w}^\nu=\bar{w}^\nu(w^L,w^\sigma)$, where $1\le\sigma,\nu\le m+1$, $\sigma\ne L$, $\nu\ne M$, and let $\omega_l^\sigma=dw_l^\sigma-w_{l+1}^\sigma dw^L$, $\bar{\omega}_l^\nu=d\bar{w}_l^\nu-\bar{w}_{l+1}^\nu d\bar{w}^M$, $l=0,1,\dots,r-1$, be 1-forms on $\tilde{V}^{r,L}$ and $\tilde{U}^{r,M}$, respectively. Since $\bar{w}_l^\nu=\bar{w}_l^\nu(w^L,w^\sigma,w_1^\sigma,\dots,w_l^\sigma)$, we have

$$\begin{aligned}d\bar{w}_l^\nu&=\frac{\partial\bar{w}_l^\nu}{\partial w^L}dw^L+\sum_{p=0}^{l}\frac{\partial\bar{w}_l^\nu}{\partial w_p^\sigma}dw_p^\sigma\\&=\left(\frac{\partial\bar{w}_l^\nu}{\partial w^L}+\sum_{p=0}^{l}\frac{\partial\bar{w}_l^\nu}{\partial w_p^\sigma}w_{p+1}^\sigma\right)dw^L+\sum_{p=0}^{l}\frac{\partial\bar{w}_l^\nu}{\partial w_p^\sigma}\omega_p^\sigma,\end{aligned}$$

$$d\bar{w}^M=\frac{\partial\bar{w}^M}{\partial w^L}dw^L+\frac{\partial\bar{w}^M}{\partial w^\sigma}dw^\sigma=\left(\frac{\partial\bar{w}^M}{\partial w^L}+\frac{\partial\bar{w}^M}{\partial w^\sigma}w_1^\sigma\right)dw^L+\frac{\partial\bar{w}^M}{\partial w^\sigma}\omega^\sigma.$$

and applying the transformation properties of morphism Δ, Lemma 2.8, we get

$$\bar{w}_{l+1}^\nu=\bar{\Delta}_M\bar{w}_l^\nu=\frac{w_1^L}{\bar{w}_1^M}\Delta_L\bar{w}_l^\nu=\frac{w_1^L}{\bar{w}_1^M}\left(\frac{\partial\bar{w}_l^\nu}{\partial w^L}+\sum_{p=0}^{l}\frac{\partial\bar{w}_l^\nu}{\partial w_p^\sigma}w_{p+1}^\sigma\right).$$

Hence

(3.10)

$$\begin{aligned}\bar{\omega}_l^\nu&=d\bar{w}_l^\nu-\bar{w}_{l+1}^\nu d\bar{w}^M\\&=\left(\frac{\partial\bar{w}_l^\nu}{\partial w^L}+\sum_{p=0}^{l}\frac{\partial\bar{w}_l^\nu}{\partial w_p^\sigma}w_{p+1}^\sigma\right)dw^L+\sum_{p=0}^{l}\frac{\partial\bar{w}_l^\nu}{\partial w_p^\sigma}\omega_p^\sigma\\&-\frac{w_1^L}{\bar{w}_1^M}\left(\frac{\partial\bar{w}_l^\nu}{\partial w^L}+\sum_{p=0}^{l}\frac{\partial\bar{w}_l^\nu}{\partial w_p^\sigma}w_{p+1}^\sigma\right)\left(\left(\frac{\partial\bar{w}^M}{\partial w^L}+\frac{\partial\bar{w}^M}{\partial w^\sigma}w_1^\sigma\right)dw^L+\frac{\partial\bar{w}^M}{\partial w^\sigma}\omega^\sigma\right).\end{aligned}$$

Since

$$\frac{w_1^L}{\bar{w}_1^M}\left(\frac{\partial\bar{w}^M}{\partial w^L}+\frac{\partial\bar{w}^M}{\partial w^\sigma}w_1^\sigma\right)=\frac{w_1^L}{\bar{w}_1^M}\Delta_L\bar{w}^M=\bar{\Delta}_M\bar{w}^M=1,$$

the terms in (3.10) containing dw^L vanish, and we obtain

$$\begin{aligned}\bar{\omega}_l^\nu&=\sum_{p=0}^{l}\frac{\partial\bar{w}_l^\nu}{\partial w_p^\sigma}\omega_p^\sigma-\left(\frac{\partial\bar{w}_l^\nu}{\partial w^L}+\sum_{p=0}^{l}\frac{\partial\bar{w}_l^\nu}{\partial w_p^\sigma}w_{p+1}^\sigma\right)\frac{w_1^L}{\bar{w}_1^M}\frac{\partial\bar{w}^M}{\partial w^\sigma}\omega^\sigma\\&=\sum_{p=0}^{l}\frac{\partial\bar{w}_l^\nu}{\partial w_p^\sigma}\omega_p^\sigma-\frac{w_1^L}{\bar{w}_1^M}\Delta_L\bar{w}_l^\nu\frac{\partial\bar{w}^M}{\partial w^\sigma}\omega^\sigma\\&=\left(\frac{\partial\bar{w}_l^\nu}{\partial w^\sigma}-\bar{w}_{l+1}^\nu\frac{\partial\bar{w}^M}{\partial w^\sigma}\right)\omega^\sigma+\sum_{p=1}^{l}\frac{\partial\bar{w}_l^\nu}{\partial w_p^\sigma}\omega_p^\sigma.\end{aligned}$$

□

Corollary. *Locally k-contact k-forms on $\tilde{W}^r$ form a submodule of the module of differential k-forms $\tilde{\Omega}_k^rW$, which we denote by $\tilde{\Omega}_{k,c}^rW$.*

The basis of 1-forms, constituted by the forms (3.8), is said to be the *contact* basis on $\tilde{V}^{r,L}$.

We conclude this section with a version of Volterra-Poincaré lemma for contact forms, with domain of definition to be the Cartesian product of open sets in Euclidean spaces. The standard Volterra-Poincaré lemma for forms on star-shaped open subsets in Euclidean spaces can be found in e.g. Narasimhan [49], Warner [63]. Let U be an open interval in $\mathbf{R}$, V an open ball in $\mathbf{R}^m$ with centre 0. We define the homotopy mapping $\chi : [0,1]\times(U\times V)\to U\times V$ by $\chi(s,(t,y^\sigma)) = (t,sy^\sigma)$, where (t,y^σ) are the canonical coordinates on $U\times V$. Consider a k-form η, $k\geq 1$, on $U\times V$. The pull-back $\chi^*\eta$ is a k-form on the set $[0,1]\times U\times V$. Clearly, we can uniquely decompose $\chi^*\eta = ds\wedge\eta_0(s)+\eta'(s)$, where $\eta_0(s)$ and $\eta'(s)$ are k-forms that do *not* contain ds; the dependence on s is in coefficients only. Define $I\eta = \int_0^1 \eta_0(s)$, where the right-hand side means that we integrate the coefficients of the form $\eta_0(s)$ over s from 0 to 1. I is called the homotopy operator. Denoting π the first Cartesian projection of $U\times V$, and $\zeta : U\to U\times V$ the zero section, $\zeta(t) = (t,0)$, then it is standard to prove that

$$\eta = Id\eta + dI\eta + \pi^*\zeta^*\eta; \tag{3.11}$$

for the proof we refer to e.g. Krupka [30]. Regarding the structure of contact forms, from (3.11) we now obtain the following result.

Lemma 3.4 (Volterra-Poincaré lemma)**.** *Let U be an open interval in $\mathbf{R}$, V be an open ball in $\mathbf{R}^m$ with centre 0. Let π be the first Cartesian projection of $U\times V$ onto U. If $\eta\in\tilde{\Omega}^r_{k,c}(U\times V)$ is a k-contact k-form such that $d\eta = 0$, then there exists a $(k-1)$-contact $(k-1)$-form $\tau\in\tilde{\Omega}^r_{k-1,c}(U\times V)$ satisfying $d\tau=\eta$.*

3.2. **Canonical decomposition of forms.** For our purpose of calculus of differential forms, we establish the basic considerations of decomposition of forms on $\operatorname{Imm}T^rY$ and G^rY into *contact components*. It is well-known that the decompositions play a fundamental role in the geometric variational theory on fibred spaces; the general theory of modules of contact forms on *fibred spaces* can be found in Krupka [25], [28]. In particular, it is shown that a certain pull-back of every k-form defined on J^rY (the manifold of r-jets of smooth sections $\gamma : X\to Y$) can be decomposed into its *horizontal* and *contact* components. We note that this decomposition is, however, different in the case of manifolds of regular n-velocities for $n\geq 2$. In this work we consider $n=1$ only, and it shows up that the local decompositions by means of associated charts on $\operatorname{Imm}T^rY$ are formally of the same formula as in the fibred case. However, by means of *subordinate* charts (cf. Sec. 2.2, Theorem 2.1) on $\operatorname{Imm}T^rY$ we get different formulas for the contact components. We note that, with respect to the structure of our manifolds, contact decompositions on G^rY are naturally induced by decompositions on $\operatorname{Imm}T^rY$. In Section 4, the crucial computational task is to determine certain classes of differential forms. We start with a particular but in calculations important case of the contact decomposition of 1-form df, where f is a function defined on $V^{r,L}$ in $\operatorname{Imm}T^rY$.

Decomposition of forms on $\operatorname{Imm}T^rY$. Let $W\subset Y$ is open and consider charts (V,ψ), $\psi=(y^K)$, on Y with $V\subset W$, and associated chart $(V^{r,L},\psi^{r,L})$, $\psi^{r,L} = (y^K, y^K_1,\dots,y^K_r)$

on $\mathrm{Imm}\, T^rY$. For a C^{r-1} function $f : W^{r-1} \to \mathbf{R}$, where $W^{r-1} = (\tau^{r-1,0})^{-1}(W)$, df is 1-form on W^{r-1}. Let $J^r_0\zeta \in V^{r,L} \subset W^r$ be an arbitrary point, and let ξ be a tangent vector to $\mathrm{Imm}\, T^rY$ at this point, expressed by $\xi = \xi^K_l (\partial/\partial y^K_l)_{J^r_0\zeta}$. Using the notation of horizontal and contact components of a tangent vector (cf. (2.24) and (2.26)), we have

$$\begin{aligned}(\tau^{r,r-1})^* df(J^r_0\zeta)\cdot\xi &= df(J^{r-1}_0\zeta)\cdot(T_{J^r_0\zeta}\tau^{r,r-1}\cdot\xi)\\ &= df(J^{r-1}_0\zeta)\cdot(h^L\xi + p^L\xi),\\ &= df(J^{r-1}_0\zeta)\cdot h^L\xi + df(J^{r-1}_0\zeta)\cdot p^L\xi,\end{aligned}$$

where $h^L\xi$ and $p^L\xi$ are the L-horizontal and L-contact components of ξ, respectively. We put

$$\begin{aligned}h^L df(J^r_0\zeta)\cdot\xi &= df(J^{r-1}_0\zeta)\cdot h^L\xi = \partial_{h^L\xi} f,\\ p^L df(J^r_0\zeta)\cdot\xi &= df(J^{r-1}_0\zeta)\cdot p^L\xi = \partial_{p^L\xi} f.\end{aligned} \tag{3.12}$$

Then

$$\begin{aligned}&h^L df(J^r_0\zeta)\cdot\xi = df(J^{r-1}_0\zeta)\cdot h^L\xi = \partial_{h^L\xi} f\\ &= \sum_{l=0}^{r-1}\left(\frac{\partial f}{\partial y^K_l}\right)_{J^{r-1}_0\zeta}\frac{y^K_{l+1}}{y^L_1}(J^r_0\zeta)\,dy^L(J^{r-1}_0\zeta)\cdot\xi\\ &= \left(\sum_{l=0}^{r-1}\left(\frac{\partial f}{\partial w^\sigma_l}\right)_{J^{r-1}_0\zeta} w^\sigma_{l+1}(J^r_0\zeta) + \sum_{l=0}^{r-1}\left(\frac{\partial f}{\partial w^L_l}\right)_{J^{r-1}_0\zeta}\frac{w^L_{l+1}}{w^L_1}(J^r_0\zeta)\right)dw^L(J^{r-1}_0\zeta)\cdot\xi\\ &= \Delta_L f(J^r_0\zeta)\,dw^L(J^{r-1}_0\zeta)\cdot\xi,\end{aligned}$$

and

$$\begin{aligned}p^L df(J^r_0\zeta)\cdot\xi &= df(J^{r-1}_0\zeta)\cdot p^L\xi = \partial_{p^L\xi} f\\ &= \sum_{s=1}^{r-1}\left(\frac{\partial f}{\partial y^L_s}\right)_{J^{r-1}_0\zeta}\eta^L_s(J^r_0\zeta)\cdot\xi + \sum_{l=0}^{r-1}\left(\frac{\partial f}{\partial y^\sigma_l}\right)_{J^{r-1}_0\zeta}\eta^\sigma_l(J^r_0\zeta)\cdot\xi\\ &= \sum_{s=1}^{r-1}\left(\frac{\partial f}{\partial w^L_s}\right)_{J^{r-1}_0\zeta}\omega^L_s(J^r_0\zeta)\cdot\xi + \sum_{l=0}^{r-1}\left(\frac{\partial f}{\partial w^\sigma_l}\right)_{J^{r-1}_0\zeta}\omega^\sigma_l(J^r_0\zeta)\cdot\xi,\end{aligned}$$

where η^L_s and η^σ_l are contact forms defined by (3.2). Hence we have the following

Lemma 3.5. *Let W be an open set in Y, and let (V,ψ), $\psi = (y^K)$, be a chart on Y such that $V \subset W$. If $f \in \Omega^r_0 W$, then*

$$(\tau^{r,r-1})^* df = h^L df + p^L df, \tag{3.13}$$

where

$$\begin{aligned}h^L df &= \left(\sum_{l=0}^{r-1}\frac{\partial f}{\partial w^\sigma_l} w^\sigma_{l+1} + \sum_{l=0}^{r-1}\frac{\partial f}{\partial w^L_l}\frac{w^L_{l+1}}{w^L_1}\right)dw^L = (\Delta_L f)dw^L,\\ p^L df &= \sum_{l=0}^{r-1}\frac{\partial f}{\partial w^\sigma_l}\omega^\sigma_l + \sum_{s=1}^{r-1}\frac{\partial f}{\partial w^L_s}\omega^L_s.\end{aligned} \tag{3.14}$$

In (3.14), 1*-forms* $\omega_l^\sigma, \omega_s^L$ *and morphism* Δ_L *are given by Lemma 3.1, (3.3), and by (2.22), respectively.*

Note that the decomposition of df, given by (3.13), concerns the pull-back of df rather than df itself. The 1-forms $h^L df$ and $p^L df$, defined by (3.12) and satisfying (3.14), are called the *L-horizontal* and *L-contact* components of df.

Consider now a general k-form $\eta \in \Omega_k^{r-1} W$, and let $\xi_1, \xi_2, \dots, \xi_k$ be tangent vectors to $\operatorname{Imm} T^r Y$ at a point $J_0^r \zeta$. From (2.25), we have

$$T_{J_0^r \zeta} \tau^{r,r-1} \cdot \xi_j = h^L \xi_j + p^L \xi_j \tag{3.15}$$

for every $j = 1, 2, \dots, k$. By the definition of pull-back of a differential form, we get

$$\begin{aligned}&(\tau^{r,r-1})^* \eta(J_0^r \zeta)(\xi_1, \xi_2, \dots, \xi_k)\\ &= \eta(J_0^{r-1} \zeta)(T_{J_0^r \zeta} \tau^{r,r-1} \cdot \xi_1, T_{J_0^r \zeta} \tau^{r,r-1} \cdot \xi_2, \dots, T_{J_0^r \zeta} \tau^{r,r-1} \cdot \xi_k)\\ &= \eta(J_0^{r-1} \zeta)(h^L \xi_1 + p^L \xi_1, h^L \xi_2 + p^L \xi_2, \dots, h^L \xi_k + p^L \xi_k),\end{aligned}$$

and decomposing this k-form into terms homogeneous of order $k-l$, $l = 0, 1, \dots, k$, in horizontal components $h^L \xi_j$, $j = 1, 2, \dots, k$, we can write

$$(\tau^{r,r-1})^* \eta = \sum_{l=0}^{k} p_l^L \eta,$$

where

$$\begin{aligned}&p_l^L \eta(J_0^r \zeta)(\xi_1, \xi_2, \dots, \xi_k)\\ &= \frac{1}{l!(k-l)!} \varepsilon^{i_1 i_2 \dots i_l i_{l+1} \dots i_k} \eta(J_0^{r-1} \zeta)(p^L \xi_{i_1}, p^L \xi_{i_2}, \dots, p^L \xi_{i_l}, h^L \xi_{i_{l+1}}, \dots, h^L \xi_{i_k})\end{aligned}$$

with summation through all values of the indices $i_1, i_2, \dots, i_k$. But it follows from the definition of L-horizontal components that $h^L \xi_j$, $j = 1, 2, \dots, k$, all belong to a 1-dimensional subspace of the tangent space to $\operatorname{Imm} T^r Y$ at $J_0^r \zeta$ (cf. (2.24)). Thus

$$(\tau^{r,r-1})^* \eta = p_{k-1}^L \eta + p_k^L \eta, \tag{3.16}$$

where

$$\begin{aligned}&p_{k-1}^L \eta(J_0^r \zeta)(\xi_1, \xi_2, \dots, \xi_k)\\ &\qquad = \frac{1}{(k-1)!} \varepsilon^{i_1 i_2 \dots i_{k-1} i_k} \eta(J_0^{r-1} \zeta)(p^L \xi_{i_1}, p^L \xi_{i_2}, \dots, p^L \xi_{i_{k-1}}, h^L \xi_{i_k})\\ &\qquad = \eta(J_0^{r-1} \zeta)(h^L \xi_1, p^L \xi_2, \dots, p^L \xi_k)\\ &\qquad + \eta(J_0^{r-1} \zeta)(p^L \xi_1, h^L \xi_2, p^L \xi_3 \dots, p^L \xi_k)\\ &\qquad + \dots\\ &\qquad + \eta(J_0^{r-1} \zeta)(p^L \xi_1, p^L \xi_2, \dots, p^L \xi_{k-1}, h^L \xi_k),\end{aligned}$$

and

$$p_k^L \eta(J_0^r \zeta)(\xi_1, \xi_2, \dots, \xi_k) = \eta(J_0^{r-1} \zeta)(p^L \xi_1, p^L \xi_2, \dots, p^L \xi_k).$$

We call the forms $p_{k-1}^L \eta$ (resp. $p_k^L \eta$) the $(k-1)$*-contact* (resp. k*-contact*) *component* of η, *associated* to the chart $(V^{r,L}, \psi^{r,L})$. The k-form η is then called

$(k-1)$-*contact* (resp. k-*contact*) *with respect to* $(V^{r,L},\psi^{r,L})$, whenever $p^L_k\eta=0$ (resp. $p^L_{k-1}\eta=0$).

If $k=1$, it is convenient to denote $h^L\eta=p^L_0\eta$ and $p^L\eta=p^L_1\eta$, and extend the definition of h^L to functions. For a function $f\in\Omega^r_0W$, we put $h^Lf=(\tau^{r,r-1})^*f$. Now, the decomposition (3.16) is of the form

$$(\tau^{r,r-1})^*\eta=h^L\eta+p^L\eta. \tag{3.17}$$

Formulas (3.16), (3.17) are referred to as the *canonical decomposition* of the form η, *associated* to the chart $(V^{r,L},\psi^{r,L})$. However, this decomposition concerns rather $(\tau^{r,r-1})^*\eta$ than η itself.

The next lemma describes the $(k-1)$-contact and k-contact components of a form in any associated chart on $\operatorname{Imm}T^rY$.

Lemma 3.6. *Let W be an open set in Y,* (V,ψ), $\psi=(y^K)$, *a chart on Y such that* $V\subset W$, *and* $(V^{r,L},\psi^{r,L})$, $\psi^{r,L}=(y^L,y^\sigma_l,y^L_s)$ *an associated chart on* $\operatorname{Imm}T^rY$ *where* $l=0,1,\dots,r$, $s=1,2,\dots,r$.

(a) *If a* 1*-form* $\eta\in\Omega^{r-1}_1W$ *has a chart expression*

$$\eta=Ady^L+\sum_{l=0}^{r-1}B^l_\sigma dy^\sigma_l+\sum_{s=1}^{r-1}C^s_Ldy^L_s \tag{3.18}$$

(no summation through index L), then

$$\begin{aligned} h^L\eta&=\left(A+\sum_{l=0}^{r-1}B^l_\sigma\frac{y^\sigma_{l+1}}{y^L_1}+\sum_{s=1}^{r-1}C^s_L\frac{y^L_{s+1}}{y^L_1}\right)dy^L,\\ p^L\eta&=\sum_{l=0}^{r-1}B^l_\sigma\eta^\sigma_l+\sum_{s=1}^{r-1}C^s_L\eta^L_s,\end{aligned} \tag{3.19}$$

where η^L_s *and* η^σ_l *are contact forms defined by (3.2).*

(b) *Let a k-form* $\eta\in\Omega^{r-1}_kW$, $k\geq 2$, *has a chart expression of the form*

$$\eta=\phi\wedge dy^L+\chi,$$

where the forms $\phi\in\Omega^{r-1}_{k-1}W$, $\chi\in\Omega^{r-1}_kW$ *are expressed by*

$$\phi=\sum_{j=0}^{k-1}\frac{1}{j!(k-1-j)!}A^{l_1\,\dots\,l_j\;s_{j+1}\dots s_{k-1}}_{\sigma_1\dots\sigma_j}dy^{\sigma_1}_{l_1}\wedge\dots\wedge dy^{\sigma_j}_{l_j}\wedge dy^L_{s_{j+1}}\wedge\dots\wedge dy^L_{s_{k-1}},$$

$$\chi=\sum_{j=0}^{k}\frac{1}{j!(k-j)!}B^{l_1\,\dots\,l_j\;s_{j+1}\dots s_k}_{\sigma_1\dots\sigma_j}dy^{\sigma_1}_{l_1}\wedge\dots\wedge dy^{\sigma_j}_{l_j}\wedge dy^L_{s_{j+1}}\wedge\dots\wedge dy^L_{s_k},$$

with coefficients $A^{l_1\ \dots\ l_j\ s_{j+1}\dots s_{k-1}}_{\sigma_1\dots\sigma_j}$ *and* $B^{l_1\ \dots\ l_j\ s_{j+1}\dots s_k}_{\sigma_1\dots\sigma_j}$ *antisymmetric in the double indices* $\binom{l_1}{\sigma_1},\dots,\binom{l_j}{\sigma_j}$*, and in the indices* $s_{j+1},\dots,s_{k-1},s_k$*. Then*

(3.20)
$$p^L_{k-1}\eta = \frac{1}{(k-1)!}\left(A^{l_1\ l_2\ \dots\ l_{k-1}}_{\sigma_1\sigma_2\dots\sigma_{k-1}} + B^{l_1\ l_2\ \dots\ l_k}_{\sigma_1\sigma_2\dots\sigma_k}\frac{y^{\sigma_k}_{l_k+1}}{y^L_1}\right)\eta^{\sigma_1}_{l_1}\wedge\eta^{\sigma_2}_{l_2}\wedge\dots\wedge\eta^{\sigma_{k-1}}_{l_{k-1}}\wedge dy^L$$
$$+\frac{1}{(k-1)!}B^{l_1\ l_2\ \dots\ l_{k-1}\ s_k}_{\sigma_1\sigma_2\dots\sigma_{k-1}}\eta^{\sigma_1}_{l_1}\wedge\eta^{\sigma_2}_{l_2}\wedge\dots\wedge\eta^{\sigma_{k-1}}_{l_{k-1}}\wedge\eta^L_{s_k},$$

and

$$p^L_k\eta = \frac{1}{k!}B^{l_1\ l_2\ \dots\ l_k}_{\sigma_1\sigma_2\dots\sigma_k}\eta^{\sigma_1}_{l_1}\wedge\eta^{\sigma_2}_{l_2}\wedge\dots\wedge\eta^{\sigma_k}_{l_k}. \tag{3.21}$$

Proof. The assertion (a) is a particular case of (b) for $\phi = A$ and $\chi = B^l_\sigma dy^\sigma_l + C^s_L dy^L_s$. Note that using the coordinate expressions (2.24) and (2.26) of horizontal and contact components of a tangent vector we immediately obtain

$$\begin{aligned}
&dy^L(J^{r-1}_0\zeta)(p^L\xi) = 0,\\
&dy^L_s(J^{r-1}_0\zeta)(p^L\xi) = \eta^L_s(J^r_0\zeta)(\xi),\\
&dy^\sigma_l(J^{r-1}_0\zeta)(p^L\xi) = \eta^\sigma_l(J^r_0\zeta)(\xi),\\
&dy^K_l(J^{r-1}_0\zeta)(h^L\xi) = \frac{y^K_{l+1}}{y^L_1}dy^L(J^r_0\zeta)(\xi),
\end{aligned}\tag{3.22}$$

$s = 1,2,\dots,r-1$, $l = 0,1,\dots,r-1$, for every tangent vector ξ to $\operatorname{Imm} T^rY$ at the point $J^r_0\zeta$. Consider now, for instance, the $(k-1)$-contact component of η. From linearity of p^L_{k-1} we obtain $p^L_{k-1}\eta = p^L_{k-1}(\phi\wedge dy^L) + p^L_{k-1}\chi$. Applying the coordinate expression of χ, the non-vanishing terms in the $(k-1)$-contact component $p^L_{k-1}\chi$ are such where at most one exterior derivative of y^L_s coordinate appears. Thus, it is sufficient to consider χ of the form

$$\chi = \frac{1}{(k-1)!}B^{l_1\ l_2\ \dots\ l_{k-1}\ s_k}_{\sigma_1\sigma_2\dots\sigma_{k-1}}dy^{\sigma_1}_{l_1}\wedge dy^{\sigma_2}_{l_2}\wedge\dots\wedge dy^{\sigma_{k-1}}_{l_{k-1}}\wedge dy^L_{s_k}$$
$$+\frac{1}{k!}B^{l_1\ l_2\ \dots\ l_k}_{\sigma_1\sigma_2\dots\sigma_k}dy^{\sigma_1}_{l_1}\wedge dy^{\sigma_2}_{l_2}\wedge\dots\wedge dy^{\sigma_k}_{l_k}.$$

Using (3.22), it is straightforward to compute

$$\chi(J^{r-1}_0\zeta)(h^L\xi_1,p^L\xi_2,p^L\xi_3,\dots,p^L\xi_k)$$
$$=\frac{1}{k!}B^{l_1\ l_2\ \dots\ l_k}_{\sigma_1\sigma_2\dots\sigma_k}\frac{y^{\sigma_k}_{l_k+1}}{y^L_1}\eta^{\sigma_1}_{l_1}\wedge\eta^{\sigma_2}_{l_2}\wedge\dots\wedge\eta^{\sigma_{k-1}}_{l_{k-1}}\wedge dy^L(J^r_0\zeta)(\xi_1,\xi_2,\dots,\xi_k)$$
$$+\frac{1}{(k-1)!}B^{l_1\ l_2\ \dots\ l_{k-1}\ s_k}_{\sigma_1\sigma_2\dots\sigma_{k-1}}\eta^{\sigma_1}_{l_1}\wedge\eta^{\sigma_2}_{l_2}\wedge\dots\wedge\eta^{\sigma_{k-1}}_{l_{k-1}}(J^r_0\zeta)(\xi_2,\xi_3,\dots,\xi_k)\,\eta^L_{s_k}(J^r_0\zeta)(\xi_1),$$

hence we get

$$p^L_{k-1}\chi = \frac{1}{(k-1)!} B^{l_1\ l_2\ \dots\ l_k}_{\sigma_1\sigma_2\dots\sigma_k} \frac{y^{\sigma_k}_{l_k+1}}{y^L_1} \eta^{\sigma_1}_{l_1} \wedge \eta^{\sigma_2}_{l_2} \wedge \dots \wedge \eta^{\sigma_{k-1}}_{l_{k-1}} \wedge dy^L$$
$$+ \frac{1}{(k-1)!} B^{l_1\ l_2\ \dots\ l_{k-1}\ s_k}_{\sigma_1\sigma_2\dots\sigma_{k-1}} \eta^{\sigma_1}_{l_1} \wedge \eta^{\sigma_2}_{l_2} \wedge \dots \wedge \eta^{\sigma_{k-1}}_{l_{k-1}} \wedge \eta^L_{s_k}.$$

Analogously, we obtain

$$p^L_{k-1}(\phi \wedge dy^L) = \frac{1}{(k-1)!} A^{l_1\ l_2\ \dots\ l_{k-1}}_{\sigma_1\sigma_2\dots\sigma_{k-1}} \eta^{\sigma_1}_{l_1} \wedge \eta^{\sigma_2}_{l_2} \wedge \dots \wedge \eta^{\sigma_{k-1}}_{l_{k-1}} \wedge dy^L.$$

The coordinate expression of the k-contact component $p^L_k\eta$ can be derived by the same way. □

Remark 6. It is not difficult to express the $(k-1)$-contact and k-contact components of a form in a subordinate chart $(V^{r,L}, \chi^{r,L})$, $\chi^{r,L} = (w^L, w^\sigma_l, w^L_s)$, $l = 0, 1, \dots, r$, $s = 1, 2, \dots, r$, on $\mathrm{Imm}\, T^rY$. For simplicity, consider a 1-form $\eta \in \Omega^{r-1}_1 W$, expressed by (3.18). Obviously, for the horizontal component $h^L\eta$ in (3.19) we get

$$h^L\eta = \left(A + \sum_{l=0}^{r-1} B^l_\sigma \frac{1}{w^L_1} \sum_{p=1}^{l+1} \sum_{(I_1,I_2,\dots,I_p)} w^L_{|I_1|} w^L_{|I_2|} \dots w^L_{|I_p|} w^\sigma_p + \sum_{s=1}^{r-1} C^s_L \frac{w^L_{s+1}}{w^L_1} \right) dw^L,$$

where we sum through all p-partitions of the set $\{i_1, i_2, \dots, i_l, i_{l+1}\}$ with $i_1 = i_2 = \dots = i_{l+1} = 1$. To express the contact component $p^L\eta$ of η (3.19) by means of a subordinate chart, it is sufficient to find the expression of the contact 1-forms η^L_s and η^σ_l. We get $\eta^L_s = \omega^L_s$, $s = 1, 2, \dots, r-1$, and after straightforward calculation

$$\begin{aligned}
\eta^\sigma &= \omega^\sigma, \\
\eta^\sigma_1 &= w^L_1 \omega^\sigma_1 + w^\sigma_1 \omega^L_1, \\
\eta^\sigma_2 &= (w^L_1)^2 \omega^\sigma_2 + w^L_2 \omega^\sigma_1 + 2 w^L_1 w^\sigma_2 \omega^L_1 + w^\sigma_1 \omega^L_2, \\
\eta^\sigma_3 &= (w^L_1)^3 \omega^\sigma_3 + 3 w^L_1 w^L_2 \omega^\sigma_2 + w^L_3 \omega^\sigma_1 \\
&\quad + 3(w^\sigma_3 (w^L_1)^2 + w^\sigma_2 w^L_2)\omega^L_1 + 3 w^L_1 w^\sigma_2 \omega^L_2 + w^\sigma_1 \omega^L_3, \\
&\ \vdots \\
\eta^\sigma_l &= \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} w^L_{|I_1|} w^L_{|I_2|} \dots w^L_{|I_p|} \omega^\sigma_p \\
&\quad + \sum_{s=1}^{l} \binom{l}{s} \omega^L_s \sum_{p=1}^{l-s} \sum_{(I_1,I_2,\dots,I_p)} w^L_{|I_1|} w^L_{|I_2|} \dots w^L_{|I_p|} w^\sigma_{p+1},
\end{aligned} \tag{3.23}$$

where $l = 0, 1, \dots, r-1$, and in the latter term of η^σ_l we sum through all p-partitions of the set $\{i_1, i_2, \dots, i_{l-s}\}$ with $i_1 = i_2 = \dots = i_{l-s} = 1$.

Note that from (3.23) it follows that for a general k-form $\eta \in \Omega^{r-1}_k W$ the k-contact component $p^L_k\eta$ (3.21) is a linear combination of k-forms $\omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_s}_{l_s} \wedge \omega^L_{j_{s+1}} \wedge \omega^L_{j_{s+2}} \wedge \dots \wedge \omega^L_{j_k}$.

Decomposition of forms on G^rY. Let π^r be the quotient projection from $\mathrm{Imm}\,T^rY$ to G^rY (cf. (2.38)), and $\rho^{r,r-1}$ the canonical projection from G^rY to $G^{r-1}Y$ (cf. (2.39)). Consider a general k-form $\eta \in \tilde{\Omega}_k^{r-1}W$, defined on $\tilde{W}^{r-1} \subset G^{r-1}Y$.

Analogously to the preceding paragraphs, we can construct the contact decomposition of forms, defined on the Grassmann fibration G^rY. To this end, consider first the concept of *horizontalization* of a tangent vector. Choose a point $[J_0^r\zeta] \in G^rY$, and let $(\tilde{V}^{r,L}, \tilde{\chi}^{r,L})$ be an associated chart at $[J_0^r\zeta]$. Let ξ be a tangent vector to G^rY at a point $[J_0^r\zeta]$, locally expressed by

$$\xi = \xi^L \left(\frac{\partial}{\partial w^L}\right) + \sum_{l=0}^{r} \xi_l^\sigma \left(\frac{\partial}{\partial w_l^\sigma}\right).$$

We assign to ξ a tangent vector $\tilde{h}^L\xi$ as follows (cf. (2.23)). Define

$$\tilde{h}^L\xi = T_{[J_0^r\zeta]}([T^{r-1}\zeta] \circ (w^L\zeta)^{-1} \circ w^L \circ \rho^{r,0}) \cdot \xi, \tag{3.24}$$

and we can easily observe that (3.24) is a tangent vector to $G^{r-1}Y$ at a point $[J_0^{r-1}\zeta]$, with a coordinate expression given by

$$\tilde{h}^L\xi = dw^L([J_0^{r-1}\zeta]) \cdot \xi \left(\left(\frac{\partial}{\partial w^L}\right)_{[J_0^{r-1}\zeta]} + \sum_{l=0}^{r-1} w_{l+1}^\sigma(J_0^r\zeta) \left(\frac{\partial}{\partial w_l^\sigma}\right)_{[J_0^{r-1}\zeta]}\right).$$

Using complementary construction, we define a tangent vector $\tilde{p}^L\xi$ by the formula (cf. 2.25)

$$T_{[J_0^r\zeta]}\rho^{r,r-1} \cdot \xi = \tilde{h}^L\xi + \tilde{p}^L\xi, \tag{3.25}$$

where

$$\tilde{p}^L\xi = \omega_l^\sigma([J_0^r\zeta]) \cdot \xi \left(\frac{\partial}{\partial w_l^\sigma}\right)_{[J_0^{r-1}\zeta]}.$$

The tangent vectors $\tilde{h}^L\xi$ and $\tilde{p}^L\xi$ are again called the *L-horizontal* and *L-contact components* of ξ, respectively.

Let $\xi_1, \xi_2, \dots, \xi_k$ be tangent vectors to G^rY at a point $[J_0^r\zeta]$, and compute the pull-back of η by the canonical projection $\rho^{r,r-1}$. By the definition of pull-back and applying (3.25) we get

$$\begin{aligned}
&(\rho^{r,r-1})^*\eta([J_0^r\zeta])(\xi_1, \xi_2, \dots, \xi_k) \\
&= \eta([J_0^{r-1}\zeta])(T_{[J_0^r\zeta]}\rho^{r,r-1} \cdot \xi_1, T_{[J_0^r\zeta]}\rho^{r,r-1} \cdot \xi_2, \dots, T_{[J_0^r\zeta]}\rho^{r,r-1} \cdot \xi_k) \\
&= \eta([J_0^{r-1}\zeta])(\tilde{h}^L\xi_1 + \tilde{p}^L\xi_1, \tilde{h}^L\xi_2 + \tilde{p}^L\xi_2, \dots, \tilde{h}^L\xi_k + \tilde{p}^L\xi_k).
\end{aligned}$$

Now, using similar arguments as in (3.16), we get

$$(\rho^{r,r-1})^*\eta = \tilde{p}_{k-1}^L\eta + \tilde{p}_k^L\eta, \tag{3.26}$$

where

$$\begin{aligned}
&\tilde{p}^L_{k-1}\eta([J^r_0\zeta])(\xi_1,\xi_2,\dots,\xi_k)\\
&=\frac{1}{(k-1)!}\varepsilon^{i_1 i_2\dots i_{k-1} i_k}\eta([J^{r-1}_0\zeta])\cdot(\tilde{p}^L\xi_{i_1},\tilde{p}^L\xi_{i_2},\dots,\tilde{p}^L\xi_{i_{k-1}},\tilde{h}^L\xi_{i_k})\\
&=\eta([J^{r-1}_0\zeta])(\tilde{h}^L\xi_1,\tilde{p}^L\xi_2,\dots,\tilde{p}^L\xi_k)\\
&+\eta([J^{r-1}_0\zeta])(\tilde{p}^L\xi_1,\tilde{h}^L\xi_2,\tilde{p}^L\xi_3\dots,\tilde{p}^L\xi_k)\\
&+\dots\\
&+\eta([J^{r-1}_0\zeta])(\tilde{p}^L\xi_1,\tilde{p}^L\xi_2,\dots,\tilde{p}^L\xi_{k-1},\tilde{h}^L\xi_k),
\end{aligned}$$

and

$$\tilde{p}^L_k\eta([J^r_0\zeta])(\xi_1,\xi_2,\dots,\xi_k)=\eta([J^{r-1}_0\zeta])(\tilde{p}^L\xi_1,\tilde{p}^L\xi_2,\dots,\tilde{p}^L\xi_k).$$

are the $(k-1)$-*contact* and k-*contact components* of η, respectively.

Now, we can describe the contact components of a form by means of associated charts on G^rY.

Lemma 3.7. *Let W be an open set in Y, (V,ψ), $\psi=(y^K)$, a chart on Y such that $V\subset W$, $(V^{r,L},\chi^{r,L})$, $\chi^{r,L}=(w^L,w^\sigma_l,w^L_s)$, be subordinate chart on* $\mathrm{Imm}\,T^rY$*, and let $(\tilde{V}^{r,L},\tilde{\chi}^{r,L})$, $\tilde{\chi}^{r,L}=(w^L,w^\sigma_l)$, be associated chart on G^rY, where $l=0,1,\dots,r$, $s=1,2,\dots,r$.*

(a) *If a 1-form $\eta\in\tilde{\Omega}^{r-1}_1W$ has a chart expression*

$$\eta=B_L dw^L+\sum_{l=0}^{r-1}B^l_\sigma dw^\sigma_l \tag{3.27}$$

(no summation through index L), then

$$\begin{aligned}
&\tilde{h}^L\eta=\tilde{p}^L_0\eta=\left(B_L+\sum_{l=0}^{r-1}B^l_\sigma w^\sigma_{l+1}\right)dw^L,\\
&\tilde{p}^L\eta=\tilde{p}^L_1\eta=\sum_{l=0}^{r-1}B^l_\sigma\omega^\sigma_l,
\end{aligned} \tag{3.28}$$

where ω^σ_l are contact forms defined by (3.5).

(b) *Let a k-form $\eta\in\tilde{\Omega}^{r-1}_kW$, $k\geq 2$, has a chart expression of the form*

$$\eta=\phi\wedge dw^L+\chi,$$

where the forms $\phi\in\tilde{\Omega}^{r-1}_{k-1}W$, $\chi\in\tilde{\Omega}^{r-1}_kW$ are expressed by

$$\begin{aligned}
\phi&=\frac{1}{(k-1)!}A^{l_1\ l_2\ \dots\ l_{k-1}}_{\sigma_1\sigma_2\dots\sigma_{k-1}}dw^{\sigma_1}_{l_1}\wedge dw^{\sigma_2}_{l_2}\wedge\dots\wedge dw^{\sigma_{k-1}}_{l_{k-1}},\\
\chi&=\frac{1}{k!}B^{l_1\ l_2\ \dots\ l_k}_{\sigma_1\sigma_2\dots\sigma_k}dw^{\sigma_1}_{l_1}\wedge dw^{\sigma_2}_{l_2}\wedge\dots\wedge dw^{\sigma_k}_{l_k},
\end{aligned}$$

with coefficients antisymmetric in the double indices $\binom{l_1}{\sigma_1},\binom{l_2}{\sigma_2},\dots,\binom{l_{k-1}}{\sigma_{k-1}},\binom{l_k}{\sigma_k}$.

Then

(3.29)
$$\tilde{p}^L_{k-1}\eta = \frac{1}{(k-1)!}\left(A^{l_1\, l_2\, \dots\, l_{k-1}}_{\sigma_1\sigma_2\dots\sigma_{k-1}} + B^{l_1\, l_2\, \dots\, l_k}_{\sigma_1\sigma_2\dots\sigma_k}\, w^{\sigma_k}_{l_k+1}\right)\omega^{\sigma_1}_{l_1}\wedge\omega^{\sigma_2}_{l_2}\wedge\dots\wedge\omega^{\sigma_{k-1}}_{l_{k-1}}\wedge dw^L$$

and

$$\tilde{p}^L_k\eta = \frac{1}{k!}B^{l_1\, l_2\, \dots\, l_k}_{\sigma_1\sigma_2\dots\sigma_k}\,\omega^{\sigma_1}_{l_1}\wedge\omega^{\sigma_2}_{l_2}\wedge\dots\wedge\omega^{\sigma_k}_{l_k}. \tag{3.30}$$

Proof. Analogously to the proof of Lemma 3.6, the result is now straightforward. □

Corollary. *For function $f\in\tilde{\Omega}^{r-1}_0 W$, $(\rho^{r,r-1})^* df = \tilde{h}^L df + \tilde{p}^L df$, where*

$$\begin{aligned}\tilde{h}^L df &= \left(\frac{\partial f}{\partial w^L} + \sum_{l=0}^{r-1}\frac{\partial f}{\partial w^\sigma_l}w^\sigma_{l+1}\right)dw^L = \Delta_L f\, dw^L,\\ \tilde{p}^L df &= \frac{\partial f}{\partial w^\sigma_l}\,\omega^\sigma_l.\end{aligned} \tag{3.31}$$

Corollary. *A k-form $\eta\in\tilde{\Omega}^{r-1}_k W$ is k-contact if and only if $\tilde{p}^L_{k-1}\eta$ vanishes or, which is the same,*

$$\eta = \frac{1}{k!}B^{l_1\, l_2\, \dots\, l_k}_{\sigma_1\sigma_2\dots\sigma_k}\,\omega^{\sigma_1}_{l_1}\wedge\omega^{\sigma_2}_{l_2}\wedge\dots\wedge\omega^{\sigma_k}_{l_k}, \tag{3.32}$$

where $0\le l_1, l_2, \dots, l_k \le r-2$.

We note that for $n=1$, a k-contact k-form corresponds to the concept of a *strongly contact* form, introduced by Krupka [25] in the context of fibred manifolds with an n-dimensional base manifold.

Note that, of course, the formula (3.28) for contact component of a 1-form coincides with the expression for a contact 1-form, Lemma 3.2.

Remark 7. We can obtain the results of Lemma 3.7 in a slightly different way. Note that for an arbitrary tangent vector ξ to $\mathrm{Imm}\, T^rY$ at $J^r_0\zeta$, $T_{J^r_0\zeta}\pi^r\cdot\xi$ is a tangent vector to G^rY at $\pi^r(J^r_0\zeta) = [J^r_0\zeta]$, and one can consider tangent vectors to G^rY of this form. Indeed, an arbitrary tangent vector $\tilde{\xi}$ to G^rY at $[J^r_0\zeta]$, represented by a curve $t\to[T^r\zeta](t)$ (see 2.62), has its pre-image in the tangent mapping $T\pi^r$, represented by $t\to T^r\zeta(t)$. From (2.39), the quotient projection π^r and canonical projections $\rho^{r,r-1}: G^rY\to G^{r-1}Y$ and $\tau^{r,r-1}:\mathrm{Imm}\, T^rY\to\mathrm{Imm}\, T^{r-1}Y$ satisfy the identity $\rho^{r,r-1}\circ\pi^r = \pi^{r-1}\circ\tau^{r,r-1}$. Hence and from (3.15) we obtain

$$\begin{aligned}&T_{[J^r_0\zeta]}\rho^{r,r-1}\circ T_{J^r_0\zeta}\pi^r\cdot\xi = T_{J^r_0\zeta}(\rho^{r,r-1}\circ\pi^r)\cdot\xi\\ &= T_{J^r_0\zeta}(\pi^{r-1}\circ\tau^{r,r-1})\cdot\xi = T_{J^{r-1}_0\zeta}\pi^{r-1}\circ T_{J^r_0\zeta}\tau^{r,r-1}\cdot\xi\\ &= T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot(h^L\xi + p^L\xi).\end{aligned} \tag{3.33}$$

The tangent vectors $T_{J^r_0\zeta}\pi^{r-1}\cdot h^L\xi$ and $T_{J^r_0\zeta}\pi^{r-1}\cdot p^L\xi$ have then the following coordinate expressions,

$$T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot h^L\xi = \xi^L\left(\frac{\partial}{\partial w^L}\right)_{[J^{r-1}_0\zeta]} + \frac{1}{w^L_1}\xi^L\left(\frac{\partial w^\nu_p}{\partial y^K_l}\right)_{J^{r-1}_0\zeta} y^K_{l+1}\left(\frac{\partial}{\partial w^\nu_p}\right)_{[J^{r-1}_0\zeta]},$$

$$T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot p^L\xi = \left(\frac{\partial w^\nu_p}{\partial y^K_l}\right)_{J^{r-1}_0\zeta}\eta^K_l(J^r_0\zeta)\cdot\xi\left(\frac{\partial}{\partial w^\nu_p}\right)_{[J^{r-1}_0\zeta]},$$

and for arbitrary tangent vectors $\xi_1,\xi_2,\dots,\xi_k$ to $\operatorname{Imm} T^rY$ at a point $J^r_0\zeta$, we compute the pull-back of η by the canonical projection $\rho^{r,r-1}$. By the definition of pull-back and using (3.33) we get

$$\begin{aligned}&(\rho^{r,r-1})^*\eta([J^r_0\zeta])(T_{J^r_0\zeta}\pi^r\cdot\xi_1, T_{J^r_0\zeta}\pi^r\cdot\xi_2,\dots,T_{J^r_0\zeta}\pi^r\cdot\xi_k)\\ &= \eta([J^{r-1}_0\zeta])(T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot(h^L\xi_1+p^L\xi_1), T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot(h^L\xi_2+p^L\xi_2),\\ &\dots,T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot(h^L\xi_k+p^L\xi_k)).\end{aligned}$$

Hence

$$(\rho^{r,r-1})^*\eta = \tilde{p}^L_{k-1}\eta + \tilde{p}^L_k\eta,$$

where

$$\begin{aligned}&\tilde{p}^L_{k-1}\eta([J^r_0\zeta])(T_{J^r_0\zeta}\pi^r\cdot\xi_1, T_{J^r_0\zeta}\pi^r\cdot\xi_2,\dots,T_{J^r_0\zeta}\pi^r\cdot\xi_k)\\ &= \frac{1}{(k-1)!}\varepsilon^{i_1i_2\dots i_{k-1}i_k}\eta([J^{r-1}_0\zeta])\\ &\cdot(T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot p^L\xi_{i_1}, T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot p^L\xi_{i_2},\dots,T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot p^L\xi_{i_{k-1}}, T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot h^L\xi_{i_k})\end{aligned}$$

and

$$\begin{aligned}&\tilde{p}^L_k\eta([J^r_0\zeta])(T_{J^r_0\zeta}\pi^r\cdot\xi_1, T_{J^r_0\zeta}\pi^r\cdot\xi_2,\dots,T_{J^r_0\zeta}\pi^r\cdot\xi_k)\\ &= \eta([J^{r-1}_0\zeta])(T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot p^L\xi_1, T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot p^L\xi_2,\dots,T_{J^{r-1}_0\zeta}\pi^{r-1}\cdot p^L\xi_k).\end{aligned}$$

Now, if $\eta\in\tilde{\Omega}^{r-1}_1W$ is a 1-form on $\tilde{W}^{r-1}$, expressed by

$$\eta = B_L dw^L + \sum_{l=0}^{r-1} B^l_\sigma dw^\sigma_l,$$

then

$$\tilde{h}^L\eta([J^r_0\zeta])(T_{J^r_0\zeta}\pi^r\cdot\xi) = \left(B_L + \frac{1}{w^L_1}\sum_{l=0}^{r-1} B^l_\sigma\frac{\partial w^\sigma_l}{\partial y^K_p}y^K_{p+1}\right)dw^L(J^r_0\zeta)\cdot\xi,$$

$$\tilde{p}^L\eta([J^r_0\zeta])(T_{J^r_0\zeta}\pi^r\cdot\xi) = \sum_{l=0}^{r-1} B^l_\sigma\omega^\sigma_l(J^r_0\zeta)\cdot\xi,$$

where ω^σ_l are contact forms defined by (3.3). The contact components of a k-form, $k\geq 2$, can be derived analogously.

Remark 8. Finally, let us mention here the basic formulas for exterior derivative of contact forms, useful in many coordinate calculations. We restrict to first order case since the various higher order analogues are straightforward. We have the following contact k-forms, antisymmetric in all indices,

$$
\begin{aligned}
& d\omega^{\sigma} = -\omega_1^{\sigma} \wedge dw^L, \\
& d(\omega^{\sigma_1} \wedge \omega^{\sigma_2}) = (\omega_1^{\sigma_1} \wedge \omega^{\sigma_2} - \omega_1^{\sigma_2} \wedge \omega^{\sigma_1}) \wedge dw^L, \\
& d(\omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \omega^{\sigma_3}) = -(\omega_1^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \omega^{\sigma_3} - \omega_1^{\sigma_2} \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_3} \\
& + \omega_1^{\sigma_3} \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2}) \wedge dw^L, \\
& \vdots \\
& d(\omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \ldots \wedge \omega^{\sigma_k}) \\
& = (-1)^k \sum_{j=1}^{k} (-1)^{j-1} \omega_1^{\sigma_j} \wedge \omega^{\sigma_2} \wedge \ldots \wedge \omega^{\sigma_{j-1}} \wedge \omega^{\sigma_{j+1}} \wedge \ldots \wedge \omega^{\sigma_k} \wedge dw^L.
\end{aligned}
$$

4. Variational sequences: Mechanics

In this section we define the variational sequence on r-th order Grassmann fibration G^rY and we prove the basic structure results. The classes of differential forms, entering the variational sequence, are determined for an arbitrary first and second order k-form. The ideas and methods are motivated by the existing theory on fibred manifolds; we refer to D. Krupka [25, 27], M. Krbek and J. Musilová [18], D. Krupka and J. Šeděnková [34] and references therein. We consider the structure of manifolds of velocities which arise in the study of variational principles for curves in the manifold Y. It should be pointed out that in this context higher order prolongations of manifolds of regular velocities are considered.

4.1. **The structure of T^4Y.** To illustrate the basic structures, we collect here coordinate formulas, needed in the theory of variational sequences on *first* order Grassmann fibration G^1Y where the prolongations of *fourth* order arise. The higher order case is proceed analogously.

We use a simplified notation for the charts on manifolds of regular velocities $\operatorname{Imm} T^1Y$, $\operatorname{Imm} T^2Y$, $\operatorname{Imm} T^3Y$ and $\operatorname{Imm} T^4Y$, and the corresponding charts on the Grassmann fibrations G^1Y, G^2Y, G^3Y and G^4Y. For a given chart (V,ψ), $\psi = (y^K)$, on Y, we denote by (V^1,ψ^1), $\psi^1 = (y^K,\dot{y}^K)$, (V^2,ψ^2), $\psi^2 = (y^K,\dot{y}^K,\ddot{y}^K)$, (V^3,ψ^3), $\psi^3 = (y^K,\dot{y}^K,\ddot{y}^K,\dddot{y}^K)$ and (V^4,ψ^4), $\psi^4 = (y^K,\dot{y}^K,\ddot{y}^K,\dddot{y}^K,\ddddot{y}^K)$ the associated charts. The corresponding subordinate charts on the manifold of regular velocities $\operatorname{Imm} T^1Y$ are denoted by $(V^{1,L},\psi^{1,L})$, $\psi^{1,L} = (y^L,\dot{y}^L,y^\sigma,\dot{y}^\sigma)$, etc.; the second subordinate charts are denoted by $(V^{1,L},\chi^{1,L})$, $\chi^{1,L} = (w^L,\dot{w}^L,w^\sigma,w_1^\sigma)$, etc. For further use we need transformation formulas between these subordinate charts. The transformation equations between $(V^{4,L},\psi^{4,L})$ and $(V^{4,L},\chi^{4,L})$ are of the form

$$
\begin{aligned}
& w^L = y^L, \quad \dot{w}^L = \dot{y}^L, \quad \ddot{w}^L = \ddot{y}^L, \quad \dddot{w}^L = \dddot{y}^L,\\
& w^\sigma = y^\sigma, \quad w_1^\sigma = \frac{1}{\dot{y}^L}\dot{y}^\sigma, \quad w_2^\sigma = \frac{1}{(\dot{y}^L)^2}\left(\ddot{y}^\sigma - \frac{\ddot{y}^L}{\dot{y}^L}\dot{y}^\sigma\right),\\
& w_3^\sigma = \frac{1}{(\dot{y}^L)^3}\left(\dddot{y}^\sigma - 3\frac{\ddot{y}^L}{\dot{y}^L}\ddot{y}^\sigma - \frac{1}{\dot{y}^L}\left(\dddot{y}^L - 3\frac{(\ddot{y}^L)^2}{\dot{y}^L}\right)\dot{y}^\sigma\right),\\
& w_4^\sigma = \frac{1}{(\dot{y}^L)^4}\left(\ddddot{y}^\sigma - 6\frac{\ddot{y}^L}{\dot{y}^L}\dddot{y}^\sigma - \frac{1}{(\dot{y}^L)^2}\left(4\dddot{y}^L\dot{y}^L - 15(\ddot{y}^L)^2\right)\ddot{y}^\sigma\right.\\
& \qquad \left. - \frac{1}{(\dot{y}^L)^3}\left((\dot{y}^L)^2\ddddot{y}^L - 10\dot{y}^L\ddot{y}^L\dddot{y}^L + 15(\ddot{y}^L)^3\right)\dot{y}^\sigma\right)
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
& y^L = w^L, \quad \dot{y}^L = \dot{w}^L, \quad \ddot{y}^L = \ddot{w}^L, \quad \dddot{y}^L = \dddot{w}^L,\\
& y^\sigma = w^\sigma, \quad \dot{y}^\sigma = w_1^\sigma\dot{w}^L, \quad \ddot{y}^\sigma = w_2^\sigma(\dot{w}^L)^2 + w_1^\sigma\ddot{w}^L,\\
& \dddot{y}^\sigma = w_3^\sigma(\dot{w}^L)^3 + 3w_2^\sigma\dot{w}^L\ddot{w}^L + w_1^\sigma\dddot{w}^L,\\
& \ddddot{y}^\sigma = w_4^\sigma(\dot{w}^L)^4 + 6w_3^\sigma(\dot{w}^L)^2\ddot{w}^L + w_2^\sigma(4\dot{w}^L\dddot{w}^L + 3(\ddot{w}^L)^2) + w_1^\sigma\ddddot{w}^L.
\end{aligned}
\tag{4.2}
$$

Consider the differential group L^4; in the context of this subsection, L^4 describes the change of parameters in variational functionals for parametric variational problems, whose Lagrangians are of order 4. L^4 acts canonically on T^4Y and $\operatorname{Imm} T^4Y$ to the right by composition of jets,

$$T^4Y \times L^4 \ni (J_0^4\zeta, J_0^4\alpha) \to J_0^4\zeta \circ J_0^4\alpha = J_0^4(\zeta\circ\alpha) \in T^4Y. \tag{4.3}$$

Recall that the canonical coordinates a_1, a_2, a_3 and a_4 on L^4 are defined by $\dot{a}(J_0^4\alpha) = D\alpha(0)$, $\ddot{a}(J_0^4\alpha) = D^2\alpha(0)$, $\dddot{a}(J_0^4\alpha) = D^3\alpha(0)$, and $\ddddot{a}(J_0^4\alpha) = D^4\alpha(0)$. To derive the equations of this action we consider the coordinate expression $t \to (y^K\zeta\circ\alpha)(t)$ of a representative of $J_0^4(\zeta\circ\alpha)$. Denote the coordinates of regular velocities $J_0^4\alpha$, $J_0^4\zeta$ and $J_0^4(\zeta\circ\alpha)$ by $(\dot{a}, \ddot{a}, \dddot{a}, \ddddot{a})$, $(y^K, \dot{y}^K, \ddot{y}^K, \dddot{y}^K, \ddddot{y}^K)$ and $(z^K, \dot{z}^K, \ddot{z}^K, \dddot{z}^K, \ddddot{z}^K)$, respectively. Differentiating at $t = 0$, we obtain the equations of the group action (4.3), expressed in the subordinate chart $(V^{4,L}, \psi^{4,L})$, $\psi^{4,L} = (y^K, \dot{y}^K, \ddot{y}^K, \dddot{y}^K, \ddddot{y}^K)$,

$$\begin{aligned}
z^K &= y^K, \quad \dot{z}^K = \dot{y}^K\dot{a},\\
\ddot{z}^K &= \ddot{y}^K(\dot{a})^2 + \dot{y}^K\ddot{a}, \quad \dddot{z}^K = \dddot{y}^K(\dot{a})^3 + 3\ddot{y}^K\dot{a}\ddot{a} + \dot{y}^K\dddot{a},\\
\ddddot{z}^K &= \ddddot{y}^K(\dot{a})^4 + 6\dddot{y}^K(\dot{a})^2\ddot{a} + \ddot{y}^K(4\dot{a}\dddot{a} + 3(\ddot{a})^2) + \dot{y}^K\ddddot{a}.
\end{aligned}$$

In the subordinate chart $(V^{4,L}, \chi^{4,L})$, denote the coordinates of $J_0^4\zeta$ and $J_0^4(\zeta\circ\alpha)$ by $(w^L, \dot{w}^L, \ddot{w}^L, \dddot{w}^L, \ddddot{w}^L, w^\sigma, w_1^\sigma, w_2^\sigma, w_3^\sigma, w_4^\sigma)$ and $(z^L, \dot{z}^L, \ddot{z}^L, \dddot{z}^L, \ddddot{z}^L, z^\sigma, z_1^\sigma, z_2^\sigma, z_3^\sigma, z_4^\sigma)$, respectively. The group action (4.3), restricted to $\operatorname{Imm} T^4Y$, is then expressed in the chart $(V^{4,L}, \chi^{4,L})$ by the equations

$$\begin{aligned}
z^L &= w^L, \quad \dot{z}^L = \dot{w}^L\dot{a},\\
\ddot{z}^L &= \ddot{w}^L(\dot{a})^2 + \dot{w}^L\ddot{a}, \quad \dddot{z}^L = \dddot{w}^L(\dot{a})^3 + 3\ddot{w}^L\dot{a}\ddot{a} + \dot{w}^L\dddot{a},\\
\ddddot{z}^L &= \ddddot{w}^L(\dot{a})^4 + 6\dddot{w}^L(\dot{a})^2\ddot{a} + \ddot{w}^L(4\dot{a}\dddot{a} + 3(\ddot{a})^2) + \dot{w}^L\ddddot{a},\\
z^\sigma &= w^\sigma, \quad z_1^\sigma = w_1^\sigma, \quad z_2^\sigma = w_2^\sigma, \quad z_3^\sigma = w_3^\sigma, \quad z_4^\sigma = w_4^\sigma.
\end{aligned}$$

Hence, equations of the L^4-orbits of (4.3) are

$$w^L = c^L, \quad w^\sigma = c^\sigma, \quad w_1^\sigma = c_1^\sigma, \quad w_2^\sigma = c_2^\sigma, \quad w_3^\sigma = c_3^\sigma, \quad w_4^\sigma = c_4^\sigma, \tag{4.4}$$

where $c^L, c^\sigma, c_1^\sigma, c_2^\sigma, c_3^\sigma, c_4^\sigma \in \mathbf{R}$.

Let G^4Y be the Grassmann fibration of order 4 over Y. Recall that the elements of G^4Y, called the contact elements of order 4, are such classes of regular velocities of $\operatorname{Imm} T^4Y$, whose representatives belong to the same L^4-orbit. We denote by $[J_0^4\zeta]$ the contact element represented by $J_0^4\zeta$. For a chart (V, ψ), $\psi = (y^K)$, on Y, we have a subordinate chart $(V^{4,L}, \chi^{4,L})$, $\chi^{4,L} = (w^L, \dot{w}^L, \ddot{w}^L, \dddot{w}^L, \ddddot{w}^L, w^\sigma, w_1^\sigma, w_2^\sigma, w_3^\sigma, w_4^\sigma)$, on $\operatorname{Imm} T^4Y$. By (4.4), we get real valued functions on $\tilde{V}^{4,L} = \pi^4(V^{4,L})$, defined by

$$\tilde{w}^L([J_0^4\zeta]) = w^L(J_0^4\zeta), \quad \tilde{w}^\sigma([J_0^4\zeta]) = w^\sigma(J_0^4\zeta), \quad \tilde{w}_k^\sigma([J_0^4\zeta]) = w_k^\sigma(J_0^4\zeta),$$

for all $J_0^4\zeta \in V^{4,L}$, where $1 \le k \le 4$. For every index L, $1 \le L \le m+1$, the pair $(\tilde{V}^{4,L}, \tilde{\chi}^{4,L})$, $\tilde{\chi}^{4,L} = (\tilde{w}^L, \tilde{w}^\sigma, \tilde{w}_1^\sigma, \tilde{w}_2^\sigma, \tilde{w}_3^\sigma, \tilde{w}_4^\sigma)$, is a chart on G^4Y.

Let γ be a smooth curve in Y, defined on an open interval $I \subset \mathbf{R}$. The canonical 4-jet prolongation of γ is a smooth curve in T^4Y, defined by

$$I \ni t \to T^4\gamma(t) = J_0^4(\gamma \circ \mathrm{tr}_{-t}) \in T^4Y.$$

Let $I \ni t \to \gamma(t) \in Y$ be an immersion such that $T^4\gamma(I) \subset V^{4,L}$. Then the canonical prolongation $T^4\gamma$ of γ has the chart expressions

$$\begin{aligned} &y^K \circ T^4\gamma(t) = y^K \circ \gamma(t), \quad \dot{y}^K \circ T^4\gamma(t) = D(y^K \circ \gamma)(t),\\ &\ddot{y}^K \circ T^4\gamma(t) = D^2(y^K \circ \gamma)(t), \quad \dddot{y}^K \circ T^4\gamma(t) = D^3(y^K \circ \gamma)(t),\\ &\ddddot{y}^K \circ T^4\gamma(t) = D^4(y^K \circ \gamma)(t), \end{aligned}$$

and

$$\begin{aligned} &w^L \circ T^4\gamma(t) = \mu(t) = w^L \circ \gamma(t),\\ &\dot{w}^L \circ T^4\gamma(t) = D\mu(t) = D(w^L \circ T^3\gamma)(t),\\ &\ddot{w}^L \circ T^4\gamma(t) = D^2\mu(t) = D^2(w^L \circ T^2\gamma)(t),\\ &\dddot{w}^L \circ T^4\gamma(t) = D^3\mu(t) = D^3(w^L \circ T^1\gamma)(t),\\ &\ddddot{w}^L \circ T^4\gamma(t) = D^4\mu(t) = D^4(w^L \circ \gamma)(t),\\ &w^\sigma \circ T^4\gamma(t) = w^\sigma \circ \gamma \circ \mu^{-1}(\mu(t)),\\ &w_k^\sigma \circ T^4\gamma(t) = D(w_{k-1}^\sigma \circ T^3\gamma \circ \mu^{-1})(\mu(t)), \quad 1 \leq k \leq 4, \end{aligned}$$

compare with Lemma 2.2, Sect. 2.1, and Lemma 2.13, Sect. 2.3.

4.2. **The contact subsequence.** Let $W \subset Y$ be an open set, $\rho^{r,0} : G^rY \to Y$ the canonical projection (2.39), $\tilde{W}^r = (\rho^{r,0})^{-1}(W)$, and $\tilde{\Omega}_k^r W$ the module of smooth differential k-forms on $\tilde{W}^r$. Let $\tilde{\Omega}_{k,c}^r W$ be the submodule of $\tilde{\Omega}_k^r W$ of k-contact k-forms on $\tilde{W}^r$, defined by (3.7). Recall that by Lemma 3.7, k-contact k-forms are expressed as a linear combinations of $\omega \wedge \omega \wedge \ldots \wedge \omega$ (k factors). We extend the definition of $\tilde{\Omega}_k^r W = \ker \tilde{p}_{k-1}^L$, $k \geq 1$, in the following sense. We put $\tilde{\Omega}_0^r W = \{0\}$, $\tilde{\Theta}_1^r W = \tilde{\Omega}_{1,c}^r W$, and

$$\tilde{\Theta}_k^r W = \tilde{\Omega}_{k,c}^r W + d\tilde{\Omega}_{k-1,c}^r W, \tag{4.5}$$

meaning that a k-form $\eta \in \tilde{\Omega}_k^r W$ belongs to $\tilde{\Theta}_k^r W$ if and only if every point of $\tilde{W}^r$ has a neighbourhood where η is decomposable as $\eta = \mu + d\mu'$ for some k-contact k-form $\mu \in \tilde{\Omega}_{k,c}^r W$ and some $(k-1)$-contact $(k-1)$-form $\mu' \in \tilde{\Omega}_{k-1,c}^r W$. $\tilde{\Theta}_k^r W$ is a subgroup of the Abelian group $\tilde{\Omega}_k^r W$, and we get a subsequence of Abelian groups

$$0 \longrightarrow \tilde{\Theta}_1^r W \longrightarrow \tilde{\Theta}_2^r W \longrightarrow \tilde{\Theta}_3^r W \longrightarrow \ldots \longrightarrow \tilde{\Theta}_M^r W \longrightarrow 0 \tag{4.6}$$

of the De Rham sequence $0 \to \mathbf{R} \to \tilde{\Omega}_0^r W \to \tilde{\Omega}_1^r W \to \tilde{\Omega}_2^r W \to \ldots \to \tilde{\Omega}_N^r W \to 0$, where $M = mr + 1$, $N = \dim G^rY = m(r+1) + 1$. In both preceding sequences, all arrows denote the *exterior derivative operator* d.

If any misunderstanding may not arise with previous definitions of contactness of differential forms, the elements of $\tilde{\Theta}_k^r W$ are simply said to be the *contact* forms. In all following diagrams we omit the underlying set W.

Remark 9. The subset $\tilde{\Theta}^r_k W$ of $\tilde{\Omega}^r_k W$ has the structure of a real vector space but *not* of a submodule of the module $\tilde{\Omega}^r_k W$. For if $f \in \tilde{\Omega}^r_0 W$ is a function and η belongs to $\tilde{\Theta}^r_k W$, $\eta = \mu + d\mu'$ on some neighbourhood in $\tilde{W}^r$, then we have $f\eta = f\mu + fd\mu' = f\mu + d(f\mu') - df \wedge \mu'$, which belongs to $\tilde{\Theta}^r_k W$ if and only if f is constant.

Theorem 4.1. *The subsequence* (4.6) *of the De Rham sequence is exact.*

Proof. We prove this theorem directly, employing the structure of spaces $\tilde{\Theta}^r_k W$ of contact forms. Let $W \subset Y$ be open, (V, ψ), $\psi = (y^K)$, a chart on Y such that $V \subset W$, and $(\tilde{V}^{r,L}, \tilde{\chi}^{r,L})$, $\tilde{\chi}^{r,L} = (w^L, w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r)$, the associated chart on $\tilde{W}^r$.

Let $k = 1$. For 1-form $\eta \in \tilde{\Theta}^r_1 W$, $\eta = A^l_\sigma \omega^\sigma_l$ (the sum through $0 \le l \le r-1$), it is sufficient to show that if $d\eta = 0$, then $\eta = 0$. We have $0 = d\eta = dA^l_\sigma \wedge \omega^\sigma_l + A^l_\sigma d\omega^\sigma_l = dA^l_\sigma \wedge \omega^\sigma_l - A^l_\sigma dw^\sigma_{l+1} \wedge dw^L$. Since the term $dA^l_\sigma \wedge \omega^\sigma_l$ does not contain $dw^\sigma_{l+1} \wedge dw^L$ of the contact basis, it follows that $A^l_\sigma = 0$, and $\eta = 0$. This is the exactness of (4.6) in the first term $\tilde{\Theta}^r_1 W$.

Let $2 \le k \le m+1$, and let $\eta \in \tilde{\Theta}^r_k W$. By the definition of $\tilde{\Theta}^r_k W$ (4.5), $\eta = \mu + d\mu'$ on some neighbourhood in $\tilde{W}^r$, where $\mu \in \tilde{\Omega}^r_{k,c} W$ and $\mu' \in \tilde{\Omega}^r_{k-1,c} W$. We wish to show that if $d\eta = 0$, then there exists a contact $(k-1)$-form $\eta_0 \in \tilde{\Theta}^r_{k-1} W$ such that $\eta = d\eta_0$. The condition $d\eta = 0$ implies $d\mu = 0$. Suppose that μ is of the form $\mu = A^{l_1\, l_2\, \dots\, l_k}_{\sigma_1 \sigma_2 \dots \sigma_k} \omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_k}_{l_k}$ (the sum through $0 \le l_1, l_2, \dots, l_k \le r-1$; cf. (3.32)). Differentiating μ we get $d\mu = dA^{l_1\, l_2\, \dots\, l_k}_{\sigma_1 \sigma_2 \dots \sigma_k} \wedge \omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_k}_{l_k} + A^{l_1\, l_2\, \dots\, l_k}_{\sigma_1 \sigma_2 \dots \sigma_k} d(\omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_k}_{l_k})$, where the first term does not contain $dw^{\sigma_s}_{l_s} \wedge dw^L$, $l_s = \max\{l_1, l_2, \dots, l_k\} + 1$, whereas in the latter term $d(\omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_k}_{l_k})$ is a linear combination of terms of contact basis, all containing dw^L and $dw^{\sigma_s}_{l_s}$; for details see Remark 8 of Sect. 3.2. Hence we get $A^{l_1\, l_2\, \dots\, l_k}_{\sigma_1 \sigma_2 \dots \sigma_k} = 0$, and $\mu = 0$. This means $\eta = d\mu'$, and we put $\eta_0 = \mu'$, an element of $\tilde{\Theta}^r_{k-1} W$. This however verifies the exactness of (4.6) in the term $\Theta^r_k W$. □

The subsequence (4.6) is called the *contact subsequence* of the De Rham sequence of smooth differential forms on an open subset $\tilde{W}^r$ in the Grassmann fibration $G^r Y$.

Remark 10. The space of contact k-forms $\tilde{\Theta}^r_k W$ is a *direct sum* of the module $\tilde{\Omega}^r_{k,c} W$ and of the image of the module $\tilde{\Omega}^r_{k-1,c} W$ in the mapping d. In other words: *For every contact form $\eta \in \tilde{\Theta}^r_k W$ there exist a unique k-contact k-form $\mu \in \tilde{\Omega}^r_{k,c} W$ and a unique $(k-1)$-contact $(k-1)$-form $\mu' \in \tilde{\Omega}^r_{k-1,c} W$ such that $\eta = \mu + d\mu'$.* For if $\eta = 0$, then it is sufficient to prove that $\mu = 0$ and $\mu' = 0$. Differentiating η, we get $d\mu = 0$. Applying the same steps used in the proof of Theorem 4.1 we get the desired result.

Remark 11. We note that Theorem 4.1 is indeed a direct consequence of Volterra-Poincaré lemma for contact forms on $G^r Y$, Lemma 3.4. Suppose $\eta \in \tilde{\Theta}^r_k W$ is a contact form, uniquely decomposed as $\eta = \mu + d\mu'$ for some $\mu \in \tilde{\Omega}^r_{k,c} W$ and $\mu' \in \tilde{\Omega}^r_{k-1,c} W$. From the assumption $d\eta = 0$ it follows that $d\mu = 0$, and by Lemma

3.4 we get $d\mu_0 = \mu$, where μ_0 belongs to $\tilde{\Omega}^r_{k-1,c}W$. Now, we have $\eta = d\eta_0$ for $\eta_0 = \mu_0 + \mu'$, the element of $\tilde{\Theta}^r_{k-1}W$.

4.3. **The variational sequence.** Now, we are in a position to define the Krupka's *variational sequence* as the quotient sequence of De Rham sequence by its subsequence of contact forms (4.6).

It is well known that the quotient sequence by a subsequence is exact if and only if the subsequence is exact (see e.g. Warner [63]). Thus, for the quotient sequence

$$0 \to \mathbf{R} \to \tilde{\Omega}^r_0 \to \tilde{\Omega}^r_1/\tilde{\Theta}^r_1 \to \ldots \to \tilde{\Omega}^r_{m+1}/\tilde{\Theta}^r_{mr+1} \to \ldots \to \tilde{\Omega}^r_N \to 0 \tag{4.7}$$

we have the following result.

Theorem 4.2. *The quotient sequence* (4.7) *is exact.*

We call (4.7) the *r-th order variational sequence* on Grassmann fibration G^rY. Let $[\eta]$ denotes the class of a differential k-form $\eta \in \tilde{\Omega}^r_k W$, and we define quotient mappings $E : \tilde{\Omega}^r_k W/\tilde{\Theta}^r_k W \to \tilde{\Omega}^r_{k+1}W/\tilde{\Theta}^r_{k+1}W$ in the variational sequence (4.7) by

$$E([\rho]) = [d\rho]. \tag{4.8}$$

We have the diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \tilde{\Omega}^r_{k-1}/\tilde{\Theta}^r_{k-1} & \longrightarrow & \tilde{\Omega}^r_k/\tilde{\Theta}^r_k & \longrightarrow & \tilde{\Omega}^r_{k+1}/\tilde{\Theta}^r_{k+1} & \longrightarrow & \ldots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\ldots & \xrightarrow{d} & \tilde{\Omega}^r_{k-1} & \xrightarrow{d} & \tilde{\Omega}^r_k & \xrightarrow{d} & \tilde{\Omega}^r_{k+1} & \xrightarrow{d} & \ldots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\ldots & \xrightarrow{d} & \tilde{\Theta}^r_{k-1} & \xrightarrow{d} & \tilde{\Theta}^r_k & \xrightarrow{d} & \tilde{\Theta}^r_{k+1} & \xrightarrow{d} & \ldots
\end{array}$$

where the upper arrows denote the quotient mappings E, making all upper squares commutative.

Our goal now is to study the inclusion of the r-th order variational sequence to variational sequence of order $(r+1)$. The situation is represented by the following "section" diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \dashrightarrow & \tilde{\Theta}^{r+1}_k/\tilde{\Theta}^r_k & \longrightarrow & \tilde{\Omega}^{r+1}_k/\tilde{\Omega}^r_k & \longrightarrow & \Psi & \dashrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \tilde{\Theta}^{r+1}_k & \xrightarrow{\iota} & \tilde{\Omega}^{r+1}_k & \longrightarrow & \tilde{\Omega}^{r+1}_k/\tilde{\Theta}^{r+1}_k & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle (\rho^{r+1,r})^*} & & \uparrow{\scriptstyle (\rho^{r+1,r})^*} & & \uparrow & & \\
0 & \longrightarrow & \tilde{\Theta}^r_k & \xrightarrow{\iota} & \tilde{\Omega}^r_k & \longrightarrow & \tilde{\Omega}^r_k/\tilde{\Theta}^r_k & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4.9}$$

where ι and $\rho^{r+1,r}$ denote the canonical injection $\tilde{\Theta}^r_k W \to \tilde{\Omega}^r_k W$ and the canonical projection $G^{r+1}Y \to G^r Y$, respectively, and Ψ is well defined by

$$\Psi = (\tilde{\Omega}^{r+1}_k W/\tilde{\Omega}^r_k W)/(\tilde{\Theta}^{r+1}_k W/\tilde{\Theta}^r_k W).$$

The quotient mappings E,

$$\tilde{\Omega}^r_k W/\tilde{\Theta}^r_k W \to \tilde{\Omega}^{r+1}_k W/\tilde{\Theta}^{r+1}_k W, \quad \tilde{\Theta}^{r+1}_k W/\tilde{\Theta}^r_k W \to \tilde{\Omega}^{r+1}_k W/\tilde{\Omega}^r_k W$$

are defined independently of the choise of a representant, making all correspondent squares commutative. The exactness of diagram (4.9) insure the correctness of using the increasing degree method for differential forms, the technique used to calculating explicit expressions of the quotient mappings in the variational sequence. What remains to show is the exactness of the upper row and the right column of (4.9).

Theorem 4.3. *The quotient mapping $\tilde{\Omega}^r_k W/\tilde{\Theta}^r_k W \to \tilde{\Omega}^{r+1}_k W/\tilde{\Theta}^{r+1}_k W$ is injective.*

Proof. Let $\eta \in \tilde{\Omega}^r_k W$ be a k-form on $\tilde{W}^r$. It is necessary and sufficient to show that if η satisfies $(\rho^{r+1,r})^*\eta \in \tilde{\Theta}^{r+1}_k W$, then $\eta \in \tilde{\Theta}^r_k W$. By assumption, $(\rho^{r+1,r})^*\eta$ is uniquely decomposable as

$$(\rho^{r+1,r})^*\eta = \mu + d\mu',$$

where $\mu \in \tilde{\Omega}^{r+1}_{k,c} W$ and $\mu' \in \tilde{\Omega}^{r+1}_{k-1,c} W$. The forms μ' and μ have the following expressions in the contact basis $dw^L, \omega^\sigma, \omega^\sigma_1, \dots, \omega^\sigma_r, dw^\sigma_{r+1}$ on $\tilde{\Omega}^{r+1}_1 W$,

$$\begin{aligned}
\mu' &= \frac{1}{(k-1)!} \sum_{0\le l_1,l_2,\dots,l_{k-1}\le r} A^{l_1\,l_2\,\dots\,l_{k-1}}_{\sigma_1\sigma_2\dots\sigma_{k-1}} \omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_{k-1}}_{l_{k-1}},\\
\mu &= \frac{1}{k!} \sum_{0\le l_1,l_2,\dots,l_k\le r} B^{l_1\,l_2\,\dots\,l_k}_{\sigma_1\sigma_2\dots\sigma_k} \omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_k}_{l_k},
\end{aligned}$$

where $A^{l_1\,l_2\,\dots\,l_{k-1}}_{\sigma_1\sigma_2\dots\sigma_{k-1}}$, $B^{l_1\,l_2\,\dots\,l_k}_{\sigma_1\sigma_2\dots\sigma_k}$ are smooth functions on $\tilde{W}^{r+1} \subset G^{r+1}Y$, antisymmetric in double indices (l_s, σ_s). Because of the $\rho^{r+1,r}$-projectability of the form $(\rho^{r+1,r})^*\eta$, the terms in $(\rho^{r+1,r})^*\eta$ containing the base form dw^ν_{r+1} should vanish. Obviously, these terms appear only in the form $d\mu'$. We obtain

$$\sum_{0\le l_1,l_2,\dots,l_{k-1}\le r} \frac{\partial A^{l_1\,l_2\,\dots\,l_{k-1}}_{\sigma_1\sigma_2\dots\sigma_{k-1}}}{\partial w^\nu_{r+1}} \omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_{k-1}}_{l_{k-1}} \wedge dw^\nu_{r+1} = 0, \tag{4.10}$$

hence $\partial A^{l_1\,l_2\,\dots\,l_{k-1}}_{\sigma_1\sigma_2\dots\sigma_{k-1}}/\partial w^\nu_{r+1} = 0$. From analysis of the exterior derivative of contact forms in

$$\sum_{\substack{0\le l_1,l_2,\dots,l_{k-1}\le r\\ l_1+l_2+\dots+l_k>0}} A^{l_1\,l_2\,\dots\,l_{k-1}}_{\sigma_1\sigma_2\dots\sigma_{k-1}} d(\omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_{k-1}}_{l_{k-1}}),$$

we immediately obtain that dw^ν_{r+1} is contained in such terms $d(\omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_{k-1}}_{l_{k-1}})$ where some ω^ν_r appears. Hence $A^{l_1\,l_2\,\dots\,l_{k-1}}_{\sigma_1\sigma_2\dots\sigma_{k-1}} = 0$ whenever $l_{k-1} = r$. Thus, the form μ' must be of the form

$$\mu' = \frac{1}{(k-1)!} \sum_{0 \le l_1, l_2, \dots, l_{k-1} \le r-1} A^{l_1\ l_2\ \dots\ l_{k-1}}_{\sigma_1 \sigma_2 \dots \sigma_{k-1}} \omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_{k-1}}_{l_{k-1}},$$

and is defined on G^rY. Then, however, the form μ is also defined on G^rY, and so is the form $d\mu$. The terms in $d\mu$ containing $dw^{\nu}_{r+1} \wedge dw^L$ should vanish separately. From this requirement, we derive similar conditions for coefficients $B^{l_1\ l_2\ \dots\ l_k}_{\sigma_1 \sigma_2 \dots \sigma_k}$ of μ, defined on G^rY, and we get

$$\mu = \frac{1}{k!} \sum_{0 \le l_1, l_2, \dots, l_k \le r-1} B^{l_1\ l_2\ \dots\ l_k}_{\sigma_1 \sigma_2 \dots \sigma_k} \omega^{\sigma_1}_{l_1} \wedge \omega^{\sigma_2}_{l_2} \wedge \dots \wedge \omega^{\sigma_k}_{l_k},$$

and the form μ belongs to $\tilde{\Omega}^r_{k,c}W$.

Then, however, the form $\eta = \mu + d\mu'$, defined on $\tilde{W}^r$, belongs to $\tilde{\Theta}^r_k W$ as required. □

The following theorem now completes the exactness of diagram (4.9) in all terms, and it is immediate consequence of the 3×3 lemma (see e.g. Greub, Halperin and Vanstone [12]).

Theorem 4.4. *The quotient mapping* $\tilde{\Theta}^{r+1}_k W / \tilde{\Theta}^r_k W \to \tilde{\Omega}^{r+1}_k W / \tilde{\Omega}^r_k W$ *is injective.*

4.4. **Classes as elements of variational sequence.** In this part we give a variational meaning to the classes of differential 1, 2, and 3-forms, the elements of the variational sequence. The classes of *first* and *second* order k-forms are determined by means on certain prolongations of a Grassmann fibration. Naturally, we may ask whether there exists an appropriate representative of a class of differential forms belonging to this class. This is the *representation problem* of the variational sequence by means of forms. For its general solution on *fibred* manifolds, we refer to Krbek and Musilová [18, 19]. In this context, one of the main requirements on representatives was to be a *globally* defined differential form satisfying the transformation rules. However, considering the Grassmann prolongations as the underlying structures we shall see that important variational object such as the Euler-Lagrange class or Helmholtz class are not longer represented by differential forms.

As usual, we denote

$$\omega^\sigma_l = dw^\sigma_l - w^\sigma_{l+1} dw^L, \quad l \ge 0,$$

the contact linear forms defined on prolongation of a Grassmann fibration. The contact basis on G^rY is formed by dw^L, ω^σ_l, dw^σ_r, where $0 \le l \le r-1$.

Classes entering 1st order variational sequence. The next theorem describes the classes of first order variational sequence by means of associated charts.

Theorem 4.5. *Let* (V, ψ), $\psi = (y^K)$, *be a chart on* Y, *and let* $(\tilde{V}^{1,L}, \tilde{\chi}^{1,L})$, $\tilde{\chi}^{1,L} = (w^L, w^\sigma, w^\sigma_1)$, *be an associated chart on* $\tilde{W}^1 \subset G^1Y$.

(a) *Let* $\eta \in \tilde{\Omega}^1_1 W$ *be expressed in the contact basis by* $\eta = A dw^L + B_\sigma \omega^\sigma + C_\sigma dw^\sigma_1$. *Then the class* $[\eta]$ *is an element of* $\tilde{\Omega}^2_1 W / \tilde{\Theta}^2_1 W$ *defined by*

$$[\eta] = (A + C_\sigma w^\sigma_2) dw^L. \tag{4.11}$$

(b) *Let* $\eta \in \tilde{\Omega}^1_2 W$ *be expressed in the contact basis by*

$$\eta = A_\sigma \omega^\sigma \wedge dw^L + B_\nu dw_1^\nu \wedge dw^L + \frac{1}{2} C_{\sigma_1\sigma_2} \omega^{\sigma_1} \wedge \omega^{\sigma_2} + D_{\nu,\sigma} dw_1^\nu \wedge \omega^\sigma + \frac{1}{2} D_{\nu_1\nu_2} dw_1^{\nu_1} \wedge dw_1^{\nu_2}.$$

Then the class $[\eta]$ *is an element of* $\tilde{\Omega}^3_2 W / \tilde{\Theta}^3_2 W$ *defined by*

$$[\eta] = E_\sigma([\eta])\, \omega^\sigma \wedge dw^L,$$

where

$$E_\sigma([\eta]) = A_\sigma - D_{\nu,\sigma} w_2^\nu - \frac{d}{dw^L}\left(B_\sigma + D_{\sigma\nu} w_2^\nu\right). \tag{4.12}$$

(c) *Let* $\eta \in \tilde{\Omega}^1_3 W$ *be expressed in the contact basis by*

$$\begin{aligned} \eta \;=\; & \frac{1}{2} A_{\sigma_1\sigma_2} \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge dw^L \\ & + B_{\nu,\sigma} dw_1^\nu \wedge \omega^\sigma \wedge dw^L + \frac{1}{2} B_{\nu_1\nu_2} dw_1^{\nu_1} \wedge dw_1^{\nu_2} \wedge dw^L \\ & + \frac{1}{6} C_{\sigma_1\sigma_2\sigma_3} \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \omega^{\sigma_3} + \frac{1}{2} D_{\nu,\sigma_1\sigma_2} dw_1^\nu \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2} \\ & + \frac{1}{2} D_{\nu_1\nu_2,\sigma} dw_1^{\nu_1} \wedge dw_1^{\nu_2} \wedge \omega^\sigma + \frac{1}{6} D_{\nu_1\nu_2\nu_3} dw_1^{\nu_1} \wedge dw_1^{\nu_2} \wedge dw_1^{\nu_3}. \end{aligned}$$

Then the class $[\eta]$ *is an element of* $\tilde{\Omega}^4_3 W / \tilde{\Theta}^4_3 W$ *defined by*

$$[\eta] = \frac{1}{2} E_{\sigma_1\sigma_2}([\eta])\, \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge dw^L + F_{\nu,\sigma}([\eta])\, \omega_1^\nu \wedge \omega^\sigma \wedge dw^L + \frac{1}{2} F_{\nu_1\nu_2}([\eta])\, \omega_2^{\nu_1} \wedge \omega^{\nu_2} \wedge dw^L,$$

where

$$\begin{aligned} E_{\sigma_1\sigma_2}([\eta]) &= A_{\sigma_1\sigma_2} + D_{\mu,\sigma_1\sigma_2} w_2^\mu + \frac{1}{2}\frac{d}{dw^L}\bigg(B_{\sigma_2,\sigma_1} - B_{\sigma_1,\sigma_2} + (D_{\mu\sigma_2,\sigma_1} - D_{\mu\sigma_1,\sigma_2}) w_2^\mu \\ &\quad + \frac{d}{dw^L}(B_{\sigma_1\sigma_2} + D_{\sigma_1\sigma_2\mu} w_2^\mu)\bigg), \\ F_{\nu,\sigma}([\eta]) &= \frac{1}{2}\left(B_{\nu,\sigma} + B_{\sigma,\nu} + (D_{\mu\nu,\sigma} + D_{\mu\sigma,\nu}) w_2^\mu\right), \\ F_{\nu_1\nu_2}([\eta]) &= B_{\nu_2\nu_1} + D_{\nu_2\nu_1\mu} w_2^\mu. \end{aligned}$$

(c′) *The class* $[\eta]$ *of* $\eta \in \tilde{\Omega}^1_3 W$*, expressed as in* (c)*, is an element of* $\tilde{\Omega}^3_3 W / \tilde{\Theta}^3_3 W$ *defined by*

$$[\eta] = \frac{1}{2} E'_{\sigma_1\sigma_2}([\eta])\, \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge dw^L + F'_{\nu,\sigma}([\eta])\, \omega_1^\nu \wedge \omega^\sigma \wedge dw^L + \frac{1}{2} F'_{\nu_1\nu_2}([\eta])\, \omega_1^{\nu_1} \wedge \omega_1^{\nu_2} \wedge dw^L,$$

where

$$E'_{\sigma_1\sigma_2}([\eta]) = A_{\sigma_1\sigma_2} + D_{\mu,\sigma_1\sigma_2}w_2^\mu - \frac{d}{dw^L}\left(B_{\sigma_1,\sigma_2} + D_{\mu\sigma_1,\sigma_2}w_2^\mu\right) \quad alt(\sigma_1\sigma_2),$$
$$F'_{\nu,\sigma}([\eta]) = \frac{1}{2}\left(B_{\nu,\sigma} + B_{\sigma,\nu} + (D_{\mu\nu,\sigma} + D_{\mu\sigma,\nu})w_2^\mu\right),$$
$$F'_{\nu_1\nu_2}([\eta]) = B_{\nu_1\nu_2} + D_{\mu\nu_1\nu_2}w_2^\mu.$$

(d) *Let* $k \geq 3$. *Let* $\eta \in \tilde{\Omega}^1_k W$ *be expressed in the contact basis by*

$$\begin{aligned}\eta &= \frac{1}{(k-1)!}A_{\sigma_1\sigma_2\dots\sigma_{k-1}}\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\dots\wedge\omega^{\sigma_{k-1}}\wedge dw^L \\ &+ \sum_{1\leq j\leq k-1}\frac{1}{j!(k-1-j)!}B_{\nu_1\nu_2\dots\nu_j,\sigma_1\sigma_2\dots\sigma_{k-1-j}}dw_1^{\nu_1}\wedge dw_1^{\nu_2}\wedge\dots\wedge dw_1^{\nu_j} \\ &\quad \wedge\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\dots\wedge\omega^{\sigma_{k-1-j}}\wedge dw^L \\ &+ \frac{1}{k!}C_{\sigma_1\sigma_2\dots\sigma_k}\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\dots\wedge\omega^{\sigma_k} \\ &+ \sum_{1\leq j\leq k}\frac{1}{j!(k-j)!}D_{\nu_1\nu_2\dots\nu_j,\sigma_1\sigma_2\dots\sigma_{k-j}}dw_1^{\nu_1}\wedge dw_1^{\nu_2}\wedge\dots\wedge dw_1^{\nu_j} \\ &\quad \wedge\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\dots\wedge\omega^{\sigma_{k-j}}.\end{aligned}$$

Then the class $[\eta]$ *is an element of* $\tilde{\Omega}^3_k W/\tilde{\Theta}^3_k W$ *defined by*

$$\begin{aligned}[\eta] &= \frac{1}{(k-1)!}E_{\sigma_1\sigma_2\dots\sigma_{k-1}}([\eta])\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\dots\wedge\omega^{\sigma_{k-1}}\wedge dw^L \\ &+ \sum_{1\leq j\leq k-1}\frac{1}{j!(k-1-j)!}F_{\nu_1\nu_2\dots\nu_j,\sigma_1\sigma_2\dots\sigma_{k-1-j}}([\eta])\omega_1^{\nu_1}\wedge\omega_1^{\nu_2}\wedge\dots\wedge\omega_1^{\nu_j} \\ &\wedge\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\dots\wedge\omega^{\sigma_{k-1-j}}\wedge dw^L,\end{aligned}$$

where

$$E_{\sigma_1\sigma_2\dots\sigma_{k-1}}([\eta]) = A_{\sigma_1\sigma_2\dots\sigma_{k-1}} + (-1)^{k-1}D_{\mu,\sigma_1\sigma_2\dots\sigma_{k-1}}w_2^\mu$$
$$-\frac{d}{dw^L}\left(B_{\sigma_1,\sigma_2\sigma_3\dots\sigma_{k-1}} + (-1)^{k-1}D_{\mu\sigma_1,\sigma_2\sigma_3\dots\sigma_{k-1}}w_2^\mu\right) \quad alt(\sigma_1\sigma_2\dots\sigma_{k-1}),$$
$$F_{\nu_1,\sigma_1\sigma_2\dots\sigma_{k-2}}([\eta]) = \frac{k-2}{k-1}\left(B_{\nu_1,\sigma_1\sigma_2\sigma_3\dots\sigma_{k-2}} + B_{\sigma_1,\nu_1\sigma_2\sigma_3\dots\sigma_{k-2}}\right.$$
$$\left.+(-1)^{k-1}\left(D_{\mu\nu_1,\sigma_1\sigma_2\sigma_3\dots\sigma_{k-2}} + D_{\mu\sigma_1,\nu_1\sigma_2\sigma_3\dots\sigma_{k-2}}\right)w_2^\mu\right) \ alt(\sigma_1\sigma_2\dots\sigma_{k-2}),$$
$$F_{\nu_1\nu_2\dots\nu_j,\sigma_1\sigma_2\dots\sigma_{k-1-j}}([\eta]) =$$
$$B_{\nu_1\nu_2\dots\nu_j,\sigma_1\sigma_2\dots\sigma_{k-1-j}} + (-1)^{k-1}D_{\mu\nu_1\nu_2\dots\nu_j,\sigma_1\sigma_2\dots\sigma_{k-1-j}}w_2^\mu, \ 2\leq j\leq k-1.$$

Proof. We compute pull-backs of forms η in projections $\rho^{r,s}: G^rY \to G^sY$, and factorize by contact forms in sense of (4.5).

(a) The result is immediate from $(\rho^{2,1})^*\eta = (A + C_\sigma w_2^\sigma)dw^L + B_\sigma\omega^\sigma + C_\sigma\omega_1^\sigma$.

(b) For the 2-form η we obtain

$$(\rho^{2,1})^*\eta = (A_\sigma - D_{\nu,\sigma}w_2^\nu)\omega^\sigma \wedge dw^L + (B_{\nu_1} + D_{\nu_1,\nu_2}w_2^{\nu_2})\omega_1^{\nu_1} \wedge dw^L + \frac{1}{2}C_{\sigma_1\sigma_2}\omega^{\sigma_1} \wedge \omega^{\sigma_2} + D_{\nu,\sigma}\omega_1^\nu \wedge \omega^\sigma + \frac{1}{2}D_{\nu_1\nu_2}\omega_1^{\nu_1} \wedge \omega_1^{\nu_2}.$$

But from $d\omega^\nu = -\omega_1^\nu \wedge dw^L$, we have

$$(B_{\nu_1} + D_{\nu_1,\nu_2}w_2^{\nu_2})\omega_1^{\nu_1} \wedge dw^L = d(B_{\nu_1} + D_{\nu_1,\nu_2}w_2^{\nu_2}) \wedge \omega^{\nu_1} - d((B_{\nu_1} + D_{\nu_1,\nu_2}w_2^{\nu_2})\omega^{\nu_1})$$
$$= -\frac{d}{dw^L}\left(B_{\nu_1} + D_{\nu_1,\nu_2}w_2^{\nu_2}\right)\omega^{\nu_1} \wedge dw^L - d((B_{\nu_1} + D_{\nu_1,\nu_2}w_2^{\nu_2})\omega^{\nu_1}) + \tilde{p}^L d(B_{\nu_1} + D_{\nu_1,\nu_2}w_2^{\nu_2}) \wedge \omega^{\nu_1},$$

where a function is decomposed by Lemma 3.7, (3.31). Hence we get the class of $(\rho^{3,1})^*\eta$ (4.12).

(c) From $dw_1^\sigma \wedge dw^L = \omega_1^\sigma \wedge dw^L$, $dw_1^\sigma = \omega_1^\sigma + w_2^\sigma dw^L$ we have

$$(\rho^{2,1})^*\eta = \frac{1}{2}(A_{\sigma_1\sigma_2} + D_{\nu,\sigma_1\sigma_2}w_2^\nu)\omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge dw^L + (B_{\nu,\sigma} - D_{\nu\mu,\sigma}w_2^\mu)\omega_1^\nu \wedge \omega^\sigma \wedge dw^L + \frac{1}{2}(B_{\nu_1\nu_2} + D_{\nu_1\nu_2\nu_3}w_2^{\nu_3})\omega_1^{\nu_1} \wedge \omega_1^{\nu_2} \wedge dw^L + \text{ contact forms.} \tag{4.13}$$

First, we apply the formula

$$d(\omega_1^\nu \wedge \omega^\sigma) = (\omega_2^\nu \wedge \omega^\sigma + \omega_1^\nu \wedge \omega_1^\sigma) \wedge dw^L,$$

and obtain η pull-backed on $\tilde{W}^3$,

(4.14)

$$(\rho^{3,1})^*\eta = \frac{1}{2}(A_{\sigma_1\sigma_2} + D_{\mu,\sigma_1\sigma_2}w_2^\mu)\omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge dw^L + \left((B_{\nu,\sigma} - D_{\nu\mu,\sigma}w_2^\mu) + \frac{1}{2}\frac{d}{dw^L}\left((B_{\sigma\nu} + D_{\sigma\nu\mu}w_2^\mu)\right)\right)\omega_1^\nu \wedge \omega^\sigma \wedge dw^L + \frac{1}{2}(B_{\sigma\nu} + D_{\sigma\nu\mu}w_2^\mu)\omega_2^\nu \wedge \omega^\sigma \wedge dw^L + \text{ contact forms.}$$

Decomposing the middle term in (4.14) into forms with antisymmetric and symmetric coefficients, and utilizing the formula

$$d(\omega^\nu \wedge \omega^\sigma) = (\omega_1^\nu \wedge \omega^\sigma - \omega_1^\sigma \wedge \omega^\nu) \wedge dw^L, \tag{4.15}$$

we easily obtain 3-form on $\tilde{W}^4$,

$$\left(B_{\nu,\sigma} - D_{\nu\mu,\sigma}w_2^\mu + \frac{1}{2}\frac{d}{dw^L}\left((B_{\sigma\nu} + D_{\sigma\nu\mu}w_2^\mu)\right)\right)\omega_1^\nu \wedge \omega^\sigma \wedge dw^L$$
$$= \frac{1}{2}\left((B_{\nu,\sigma} + B_{\sigma,\nu}) + (D_{\mu\nu,\sigma} + D_{\mu\sigma,\nu})w_2^\mu\right)\omega_1^\nu \wedge \omega^\sigma \wedge dw^L - \frac{1}{4}\frac{d}{dw^L}\left((B_{\nu,\sigma} - B_{\sigma,\nu}) + (D_{\mu\nu,\sigma} - D_{\mu\sigma,\nu})w_2^\mu + \frac{d}{dw^L}(B_{\sigma\nu} + D_{\sigma\nu\mu}w_2^\mu)\right) \cdot \omega^\nu \wedge \omega^\sigma \wedge dw^L + \text{ contact forms.}$$

Hence and from (4.14) we get $(\rho^{4,1})^*\eta$ of the required form.

(c$'$) Since the second term in $(\rho^{2,1})^*\eta$ (4.13) is not antisymmetric in indices ν_1, σ, we utilize formula (4.15) to obtain

$$\begin{aligned}
&(B_{\nu_1,\sigma} - D_{\nu_1\nu_2,\sigma} w_2^{\nu_2})\omega_1^{\nu_1} \wedge \omega^\sigma \wedge dw^L \\
&\quad = -\frac{1}{2}\frac{d}{dw^L}(B_{\nu_1,\sigma} - D_{\nu_1\nu_2,\sigma} w_2^{\nu_2})\omega^{\nu_1} \wedge \omega^\sigma \wedge dw^L \quad \mathrm{alt}(\nu_1\sigma) \\
&\quad + \frac{1}{2}(B_{\nu_1,\sigma} + B_{\sigma,\nu_1} - (D_{\nu_1\nu_2,\sigma} + D_{\sigma\nu_2,\nu_1}) w_2^{\nu_2})\omega_1^{\nu_1} \wedge \omega^\sigma \wedge dw^L \\
&\quad + \text{ contact forms.}
\end{aligned}$$

This implies $(\rho^{3,1})^*\eta$ of the required form.

(d) The class of a k-form, $k \geq 3$, on G^1Y, can be derived analogously to the method used in (c$'$) (compare also with Krupka [27, 33]). We use the elementary theory of Young diagrams, decomposing tensors into its symmetric and antisymmetric components. To the author's knowledge, there is no adequate reference for this material. However, since we discuss only the four "variational" terms in the sequence, we omit here a tedious calculation. □

Now we determine the quotient mappings (4.8) in first order variational sequence.

Theorem 4.6. *Let* (V,ψ), $\psi = (y^K)$, *be a chart on* Y, *and let* $(\tilde{V}^{1,L}, \tilde{\chi}^{1,L})$, $\tilde{\chi}^{1,L} = (w^L, w^\sigma, w_1^\sigma)$, *be an associated chart on* $\tilde{W}^1 \subset G^1Y$.

(a) *If* $f \in \tilde{\Omega}_0^1 W$, *then* $E(f) = (df/dw^L)dw^L$.

(b) *Let* $\eta \in \tilde{\Omega}_1^1 W$ *be expressed in the contact basis by* $\eta = A dw^L + B_\sigma \omega^\sigma + C_\sigma dw_1^\sigma$. *Then*

$$E([\eta]) = E_\sigma([d\eta])\, \omega^\sigma \wedge dw^L, \tag{4.16}$$

where

$$E_\sigma([d\eta]) = \frac{\partial A}{\partial w^\sigma} + \frac{\partial C_\nu}{\partial w^\sigma} w_2^\nu - \frac{d}{dw^L}\left(\frac{\partial A}{\partial w_1^\sigma} + \frac{\partial C_\nu}{\partial w_1^\sigma} w_2^\nu\right) + \frac{d^2 C_\sigma}{d(w^L)^2}.$$

(c) *Let* $\eta \in \tilde{\Omega}_2^1 W$ *be expressed in the contact basis by*

$$\begin{aligned}
\eta &= A_\sigma \omega^\sigma \wedge dw^L + B_\nu dw_1^\nu \wedge dw^L \\
&\quad + \frac{1}{2} C_{\sigma_1\sigma_2}\omega^{\sigma_1} \wedge \omega^{\sigma_2} + D_{\nu,\sigma} dw_1^\nu \wedge \omega^\sigma + \frac{1}{2} D_{\nu_1\nu_2} dw_1^{\nu_1} \wedge dw_1^{\nu_2}.
\end{aligned}$$

Then

$$\begin{aligned}
E([\eta]) &= \frac{1}{2} E_{\sigma_1\sigma_2}([d\eta])\, \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge dw^L + F_{\nu,\sigma}([d\eta])\, \omega_1^\nu \wedge \omega^\sigma \wedge dw^L \\
&\quad + \frac{1}{2} F_{\nu\sigma}([d\eta])\, \omega_2^\nu \wedge \omega^\sigma \wedge dw^L,
\end{aligned} \tag{4.17}$$

where

$$E_{\sigma\nu}([d\eta]) = \frac{\partial A_{\sigma_2}}{\partial w^{\sigma_1}} - \frac{\partial A_{\sigma_1}}{\partial w^{\sigma_2}} + \left(\frac{\partial D_{\mu,\sigma_1}}{\partial w^{\sigma_2}} - \frac{\partial D_{\mu,\sigma_2}}{\partial w^{\sigma_1}}\right) w_2^{\mu} - \frac{1}{2}\frac{d}{dw^L}\left(\frac{\partial A_{\sigma_2}}{\partial w_1^{\sigma_1}} - \frac{\partial A_{\sigma_1}}{\partial w_1^{\sigma_2}}\right.$$
$$+ \frac{\partial B_{\sigma_2}}{\partial w^{\sigma_1}} - \frac{\partial B_{\sigma_1}}{\partial w^{\sigma_2}} + \left.\left(\frac{\partial D_{\mu,\sigma_1}}{\partial w_1^{\sigma_2}} - \frac{\partial D_{\mu,\sigma_2}}{\partial w_1^{\sigma_1}} + \frac{\partial D_{\sigma_2\mu}}{\partial w^{\sigma_1}} - \frac{\partial D_{\sigma_1\mu}}{\partial w^{\sigma_2}}\right) w_2^{\mu}\right)$$
$$+ \frac{1}{2}\frac{d^2}{d(w^L)^2}\left(\frac{\partial B_{\sigma_2}}{\partial w_1^{\sigma_1}} - \frac{\partial B_{\sigma_1}}{\partial w_1^{\sigma_2}} + \left(\frac{\partial D_{\sigma_2\mu}}{\partial w_1^{\sigma_1}} - \frac{\partial D_{\sigma_1\mu}}{\partial w_1^{\sigma_2}}\right) w_2^{\mu}\right) + \frac{1}{2}\frac{d^3 D_{\sigma_1\sigma_2}}{d(w^L)^3},$$

$$F_{\nu,\sigma}([d\eta]) = \frac{1}{2}\left(\frac{\partial A_{\sigma}}{\partial w_1^{\nu}} + \frac{\partial A_{\nu}}{\partial w_1^{\sigma}} - \frac{\partial B_{\sigma}}{\partial w^{\nu}} - \frac{\partial B_{\nu}}{\partial w^{\sigma}} - \left(\frac{\partial D_{\mu,\sigma}}{\partial w_1^{\nu}} + \frac{\partial D_{\mu,\nu}}{\partial w_1^{\sigma}} + \frac{\partial D_{\sigma\mu}}{\partial w^{\nu}}\right.\right.$$
$$\left.\left.+ \frac{\partial D_{\nu\mu}}{\partial w^{\sigma}}\right) w_2^{\mu} + \frac{d}{dw^L}(D_{\nu,\sigma} + D_{\sigma,\nu})\right),$$

$$F_{\nu\sigma}([d\eta]) = \frac{\partial B_{\nu}}{\partial w_1^{\sigma}} - \frac{\partial B_{\sigma}}{\partial w_1^{\nu}} + D_{\sigma,\nu} - D_{\nu,\sigma} + \left(\frac{\partial D_{\nu\mu}}{\partial w_1^{\sigma}} - \frac{\partial D_{\sigma\mu}}{\partial w_1^{\nu}}\right) w_2^{\mu} - \frac{dD_{\nu\sigma}}{dw^L}.$$

Proof. (a) By definition of quotient mapping $E : \tilde{\Omega}_0^1 W \to \tilde{\Omega}_1^1 W / \tilde{\Theta}_1^1 W$ (4.8), we have $E(f) = E([f]) = [df]$. We obtain the result immediately from

$$(\rho^{2,1})^*(df) = \frac{df}{dw^L} dw^L + \frac{\partial f}{\partial w^{\sigma}} \omega^{\sigma} + \frac{\partial f}{\partial w_1^{\sigma}} \omega_1^{\sigma}.$$

(b) Differentiating $\eta \in \tilde{\Omega}_1^1 W$ we have

$$d\eta = P_{\sigma}\omega^{\sigma} \wedge dw^L + Q_{\nu} dw_1^{\nu} \wedge dw^L$$
$$+ \frac{1}{2} R_{\sigma_1\sigma_2} \omega^{\sigma_1} \wedge \omega^{\sigma_2} + S_{\nu,\sigma} dw_1^{\nu} \wedge \omega^{\sigma} + \frac{1}{2} S_{\nu_1\nu_2} dw_1^{\nu_1} \wedge dw_1^{\nu_2},$$

where

$$\begin{aligned} &P_{\sigma} = \frac{\partial A}{\partial w^{\sigma}} - \frac{\partial B_{\sigma}}{\partial w^L} - \frac{\partial B_{\sigma}}{\partial w^{\mu}} w_1^{\mu}, \quad Q_{\nu} = \frac{\partial A}{\partial w_1^{\nu}} - B_{\nu} - \frac{\partial C_{\nu}}{\partial w^L} - \frac{\partial C_{\nu}}{\partial w^{\mu}} w_1^{\mu}, \\ &R_{\sigma_1\sigma_2} = \frac{\partial B_{\sigma_2}}{\partial w^{\sigma_1}} - \frac{\partial B_{\sigma_1}}{\partial w^{\sigma_2}}, \quad S_{\nu,\sigma} = \frac{\partial B_{\sigma}}{\partial w_1^{\nu}} - \frac{\partial C_{\nu}}{\partial w^{\sigma}}, \quad S_{\nu_1\nu_2} = \frac{\partial C_{\nu_2}}{\partial w_1^{\nu_1}} - \frac{\partial C_{\nu_1}}{\partial w_1^{\nu_2}}. \end{aligned} \tag{4.18}$$

Now we find the class of 2-form $d\eta$, as determined by Theorem 4.5, (b), (4.12). We get

$$[(\rho^{3,1})^* d\eta] = E_{\sigma}([d\eta])\omega^{\sigma} \wedge dw^L,$$

where

$$E_{\sigma}([d\eta]) = \left(P_{\sigma} - S_{\nu,\sigma} w_2^{\nu} - \frac{d}{dw^L}(Q_{\sigma} + S_{\sigma\nu} w_2^{\nu})\right) \omega^{\sigma} \wedge dw^L.$$

Substituting the coefficients from (4.18) into $E_{\sigma}([d\eta])$, we obtain (4.16).

(c) Differentiating $\eta \in \tilde{\Omega}_2^1 W$ we have

$$\begin{aligned} d\eta = {} & \frac{1}{2} P_{\sigma_1\sigma_2} \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge dw^L \\ & + Q_{\nu,\sigma} dw_1^\nu \wedge \omega^\sigma \wedge dw^L + \frac{1}{2} Q_{\nu_1\nu_2} dw_1^{\nu_1} \wedge dw_1^{\nu_2} \wedge dw^L \\ & + \frac{1}{6} R_{\sigma_1\sigma_2\sigma_3} \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \omega^{\sigma_3} + \frac{1}{2} S_{\nu,\sigma_1\sigma_2} dw_1^\nu \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2} \\ & + \frac{1}{2} S_{\nu_1\nu_2,\sigma} dw_1^{\nu_1} \wedge dw_1^{\nu_2} \wedge \omega^\sigma + \frac{1}{6} S_{\nu_1\nu_2\nu_3} dw_1^{\nu_1} \wedge dw_1^{\nu_2} \wedge dw_1^{\nu_3}, \end{aligned}$$

where

$$\begin{aligned} P_{\sigma_1\sigma_2} &= \frac{\partial A_{\sigma_2}}{\partial w^{\sigma_1}} - \frac{\partial A_{\sigma_1}}{\partial w^{\sigma_2}} + \frac{\partial C_{\sigma_1\sigma_2}}{\partial w^L} + \frac{\partial C_{\sigma_1\sigma_2}}{\partial w^\mu} w_1^\mu, \\ Q_{\nu,\sigma} &= \frac{\partial A_\sigma}{\partial w_1^\nu} - \frac{\partial B_\nu}{\partial w^\sigma} + C_{\nu\sigma} + \frac{\partial D_{\nu,\sigma}}{\partial w^L} + \frac{\partial D_{\nu,\sigma}}{\partial w^\mu} w_1^\mu, \\ Q_{\nu_1\nu_2} &= \frac{\partial B_{\nu_2}}{\partial w_1^{\nu_1}} - \frac{\partial B_{\nu_1}}{\partial w_1^{\nu_2}} + D_{\nu_1,\nu_2} - D_{\nu_2,\nu_1} + \frac{\partial D_{\nu_1\nu_2}}{\partial w^L} + \frac{\partial D_{\nu_1\nu_2}}{\partial w^\mu} w_1^\mu, \qquad (4.19) \\ R_{\sigma_1\sigma_2\sigma_3} &= \frac{C_{\sigma_1\sigma_2}}{\partial w^{\sigma_3}} + \frac{C_{\sigma_2\sigma_3}}{\partial w^{\sigma_1}} + \frac{C_{\sigma_3\sigma_1}}{\partial w^{\sigma_2}}, \quad S_{\nu,\sigma_1\sigma_2} = \frac{D_{\nu,\sigma_1}}{\partial w^{\sigma_2}} - \frac{D_{\nu,\sigma_2}}{\partial w^{\sigma_1}} + \frac{C_{\sigma_1\sigma_2}}{\partial w_1^\nu}, \\ S_{\nu_1\nu_2,\sigma} &= \frac{D_{\nu_2,\sigma}}{\partial w_1^{\nu_1}} - \frac{D_{\nu_1,\sigma}}{\partial w_1^{\nu_2}} + \frac{D_{\nu_1\nu_2}}{\partial w^\sigma}, \quad S_{\nu_1\nu_2\nu_3} = \frac{D_{\nu_1\nu_2}}{\partial w_1^{\nu_3}} + \frac{D_{\nu_2\nu_3}}{\partial w_1^{\nu_1}} + \frac{D_{\nu_3\nu_1}}{\partial w_1^{\nu_2}}. \end{aligned}$$

We apply Theorem 4.5, (c), and find the class of 3-form $d\eta$. We get

$$\begin{aligned} [(\rho^{4,1})^* d\eta] = {} & \frac{1}{2} E_{\sigma_1\sigma_2}([d\eta])\, \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge dw^L + F_{\nu,\sigma}([d\eta])\, \omega_1^\nu \wedge \omega^\sigma \wedge dw^L \\ & + \frac{1}{2} F_{\nu_1\nu_2}([d\eta])\, \omega_2^{\nu_1} \wedge \omega^{\nu_2} \wedge dw^L, \end{aligned}$$

where

$$\begin{aligned} E_{\sigma_1\sigma_2}([d\eta]) = {} & P_{\sigma_1\sigma_2} + S_{\nu,\sigma_1\sigma_2} w_2^\nu + \frac{1}{2} \frac{d}{dw^L} \bigg(Q_{\sigma_2,\sigma_1} - Q_{\sigma_1,\sigma_2} + (S_{\nu\sigma_2,\sigma_1} - S_{\nu\sigma_1,\sigma_2}) w_2^\nu \\ & + \frac{d}{dw^L} (Q_{\sigma_1\sigma_2} + S_{\sigma_1\sigma_2\nu} w_2^\nu) \bigg), \\ F_{\nu,\sigma}([d\eta]) = {} & \frac{1}{2} \left(Q_{\nu,\sigma} + Q_{\sigma,\nu} + (S_{\mu\nu,\sigma} + S_{\mu\sigma,\nu}) w_2^\mu \right), \\ F_{\nu_1\nu_2}([d\eta]) = {} & Q_{\nu_2\nu_1} + S_{\nu_2\nu_1\mu} w_2^\mu. \end{aligned}$$

Substituting back the coefficients from (4.19) into $E_{\sigma_1\sigma_2}([d\eta])$, $F_{\nu,\sigma}([d\eta])$ and $F_{\nu_1\nu_2}([d\eta])$, we obtain (4.17). □

Classes entering higher order order variational sequence. The next theorem describes the classes of second order variational sequence by means of associated charts. Classes of 1 and 2-forms are described also for arbitrary finite order.

Theorem 4.7. *Let (V,ψ), $\psi=(y^K)$, be a chart on Y, and let $(\tilde{V}^{r,L},\tilde{\chi}^{r,L})$, $\tilde{\chi}^{r,L}=(w^L,w^\sigma,w^\sigma_1,w^\sigma_2,\dots,w^\sigma_r)$, be an associated chart on $\tilde{W}^r\subset G^rY$.*

(a) *Let $\eta\in\tilde{\Omega}^r_1W$ be expressed in the contact basis by*

$$\eta=Adw^L+\sum_{l=0}^{r-1}B^l_\sigma\omega^\sigma_l+C_\sigma dw^\sigma_r.$$

Then the class $[\eta]$ is an element of $\tilde{\Omega}^{r+1}_1W/\tilde{\Theta}^{r+1}_1W$ defined by

$$[\eta]=(A+C_\sigma w^\sigma_{r+1})dw^L. \tag{4.20}$$

(b) *Let $\eta\in\tilde{\Omega}^r_2W$ be expressed in the contact basis by*

$$\begin{aligned}\eta=&\sum_{l=0}^{r-1}A^l_\sigma\omega^\sigma_l\wedge dw^L+B_\nu dw^\nu_r\wedge dw^L\\&+\frac{1}{2}\sum_{l=0}^{r-1}C^l_{\sigma_1\sigma_2}\omega^{\sigma_1}_l\wedge\omega^{\sigma_2}_l+\sum_{l=1}^{r-1}\sum_{s=0}^{l-1}C^{l,s}_{\nu,\sigma}\omega^\nu_l\wedge\omega^\sigma_s\\&+\sum_{l=0}^{r-1}D^l_{\nu,\sigma}dw^\nu_r\wedge\omega^\sigma_l+\frac{1}{2}D_{\nu_1\nu_2}dw^{\nu_1}_r\wedge dw^{\nu_2}_r.\end{aligned}\tag{4.21}$$

Then the class $[\eta]$ is an element of $\tilde{\Omega}^{2r+1}_2W/\tilde{\Theta}^{2r+1}_2W$ defined by

$$[\eta]=E_\sigma([\eta])\,\omega^\sigma\wedge dw^L,$$

where

(4.22)

$$E_\sigma([\eta])=\sum_{l=0}^{r-1}(-1)^l\frac{d^l}{d(w^L)^l}\left(A^l_\sigma-D^l_{\nu,\sigma}w^\nu_{r+1}\right)+(-1)^r\frac{d^r}{d(w^L)^r}\left(B_\sigma-D_{\nu\sigma}w^\nu_{r+1}\right).$$

(c) *Let $\eta\in\tilde{\Omega}^2_3W$ be expressed in the contact basis by*

$$\begin{aligned}\eta\;=\;&\frac{1}{2}A_{\sigma_1\sigma_2}\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge dw^L+A^1_{\nu,\sigma}\omega^\nu_1\wedge\omega^\sigma\wedge dw^L+\frac{1}{2}A^1_{\nu_1\nu_2}\omega^{\nu_1}_1\wedge\omega^{\nu_2}_1\wedge dw^L\\+\;&B_{\nu,\sigma}dw^\nu_2\wedge\omega^\sigma\wedge dw^L+B^1_{\nu,\sigma}dw^\nu_2\wedge\omega^\sigma_1\wedge dw^L+\frac{1}{2}B_{\nu_1\nu_2}dw^{\nu_1}_2\wedge dw^{\nu_2}_2\wedge dw^L\\+\;&\frac{1}{6}C_{\sigma_1\sigma_2\sigma_3}\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\omega^{\sigma_3}+\frac{1}{2}C^1_{\nu,\sigma_1\sigma_2}\omega^\nu_1\wedge\omega^{\sigma_1}\wedge\omega^{\sigma_2}\\+\;&\frac{1}{2}C^1_{\nu_1\nu_2,\sigma}\omega^{\nu_1}_1\wedge\omega^{\nu_2}_1\wedge\omega^\sigma+\frac{1}{6}C^1_{\nu_1\nu_2\nu_3}\omega^{\nu_1}_1\wedge\omega^{\nu_2}_1\wedge\omega^{\nu_3}_1\end{aligned}$$

$$\begin{aligned}
&+ \quad \frac{1}{2}D_{\nu,\sigma_1\sigma_2}dw_2^{\nu}\wedge\omega^{\sigma_1}\wedge\omega^{\sigma_2}+\frac{1}{2}D_{\nu_1\nu_2,\sigma}dw_2^{\nu_1}\wedge dw_2^{\nu_2}\wedge\omega^{\sigma}\\
&+ \quad \frac{1}{6}D_{\nu_1\nu_2\nu_3}dw_2^{\nu_1}\wedge dw_2^{\nu_2}\wedge dw_2^{\nu_3}+D^1_{\nu,\mu,\sigma}dw_2^{\nu}\wedge\omega_1^{\mu}\wedge\omega^{\sigma}\\
&+ \quad \frac{1}{2}D^1_{\nu,\sigma_1\sigma_2}dw_2^{\nu}\wedge\omega_1^{\sigma_1}\wedge\omega_1^{\sigma_2}+\frac{1}{2}D^1_{\nu_1\nu_2,\sigma}dw_2^{\nu_1}\wedge dw_2^{\nu_2}\wedge\omega_1^{\sigma}.
\end{aligned}$$

Then the class $[\eta]$ is an element of $\tilde{\Omega}^7_3W/\tilde{\Theta}^7_3W$ defined by

$$\begin{aligned}
[\eta] &= \frac{1}{2}E_{\nu\sigma}([\eta])\,\omega^{\nu}\wedge\omega^{\sigma}\wedge dw^L\\
&+F_{\nu,\sigma}([\eta])\,\omega_1^{\nu}\wedge\omega^{\sigma}\wedge dw^L+\frac{1}{2}F_{\nu\sigma}([\eta])\,\omega_2^{\nu}\wedge\omega^{\sigma}\wedge dw^L\\
&+G_{\nu,\sigma}([\eta])\,\omega_3^{\nu}\wedge\omega^{\sigma}\wedge dw^L+\frac{1}{2}G_{\nu\sigma}([\eta])\,\omega_4^{\nu}\wedge\omega^{\sigma}\wedge dw^L,
\end{aligned}$$

where

$$\begin{aligned}
&E_{\nu\sigma}([\eta]) = A_{\nu\sigma}+D_{\mu,\nu\sigma}w_3^{\mu}-\frac{1}{2}\frac{d}{dw^L}\left(A^1_{\nu,\sigma}-A^1_{\sigma,\nu}+(D^1_{\mu,\nu,\sigma}-D^1_{\mu,\sigma,\nu})w_3^{\mu}\right)\\
&+\frac{1}{2}\frac{d^2}{d(w^L)^2}(A^1_{\nu\sigma}+D^1_{\mu,\nu\sigma}w_3^{\mu})-\frac{1}{4}\frac{d^3}{d(w^L)^3}\left(B^1_{\nu,\sigma}-B^1_{\sigma,\nu}+(D^1_{\mu\nu,\sigma}-D^1_{\mu\sigma,\nu})w_3^{\mu}\right)\\
&+\frac{1}{2}\frac{d^4}{d(w^L)^4}(B_{\nu\sigma}+D_{\mu\nu\sigma}w_3^{\mu}),
\end{aligned}$$

$$\begin{aligned}
F_{\nu,\sigma}([\eta]) &= \frac{1}{2}\Big(A^1_{\nu,\sigma}+A^1_{\sigma,\nu}+(D^1_{\mu,\nu,\sigma}+D^1_{\mu,\sigma,\nu})w_3^{\mu}\\
&\quad -\frac{d}{dw^L}\left(B_{\nu,\sigma}+B_{\sigma,\nu}+(D_{\mu\nu,\sigma}+D_{\mu\sigma,\nu})w_3^{\mu}\right)\\
&\quad +\frac{d^2}{d(w^L)^2}\left(B^1_{\nu,\sigma}+B^1_{\sigma,\nu}+(D^1_{\mu\nu,\sigma}+D^1_{\mu\sigma,\nu})w_3^{\mu}\right)\Big),\\
F_{\nu\sigma}([\eta]) &= A^1_{\sigma\nu}+D^1_{\mu,\sigma\nu}w_3^{\mu}+(B_{\nu,\sigma}-B_{\sigma,\nu})+(D_{\mu\nu,\sigma}-D_{\mu\sigma,\nu})w_3^{\mu}\\
&\quad +\frac{1}{2}\frac{d}{dw^L}\left(B^1_{\nu,\sigma}-B^1_{\sigma,\nu}+(D^1_{\mu\nu,\sigma}-D^1_{\mu\sigma,\nu})w_3^{\mu}\right)\\
&\quad -2\frac{d^2}{d(w^L)^2}(B_{\nu\sigma}+D_{\mu\nu\sigma}w_3^{\mu}),\\
G_{\nu,\sigma}([\eta]) &= -\frac{1}{2}\left(B^1_{\nu,\sigma}+B^1_{\sigma,\nu}+(D^1_{\mu\nu,\sigma}+D^1_{\mu\sigma,\nu})w_3^{\mu}\right),\\
G_{\nu\sigma}([\eta]) &= B_{\nu,\sigma}+D_{\mu\nu\sigma}w_3^{\mu}.
\end{aligned}$$

(d) *Let $k \geq 3$. Let $\eta \in \tilde{\Omega}^2_k W$ be expressed in the contact basis by*

$$\begin{aligned}
\eta \;=\; & \frac{1}{(k-1)!} A_{\sigma_1\sigma_2\ldots\sigma_{k-1}} \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \ldots \wedge \omega^{\sigma_{k-1}} \wedge dw^L \\
& + \sum_{1\leq j\leq k-1} \frac{1}{j!(k-1-j)!} A^1_{\mu_1\mu_2\ldots\mu_j,\sigma_1\sigma_2\ldots\sigma_{k-1-j}} \omega_1^{\mu_1} \wedge \omega_1^{\mu_2} \wedge \ldots \wedge \omega_1^{\nu_j} \\
& \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \ldots \wedge \omega^{\sigma_{k-1-j}} \wedge dw^L \\
& + \sum_{1\leq j\leq k-1} \frac{1}{j!(k-1-j)!} B_{\nu_1\nu_2\ldots\nu_j,\sigma_1\sigma_2\ldots\sigma_{k-1-j}} dw_2^{\nu_1} \wedge dw_2^{\nu_2} \wedge \ldots \wedge dw_2^{\nu_j} \\
& \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \ldots \wedge \omega^{\sigma_{k-1-j}} \wedge dw^L \\
& + \sum_{1\leq j\leq k-2} \sum_{1\leq l\leq k-1-j} \frac{1}{j!l!(k-1-j-l)!} B^1_{\nu_1\nu_2\ldots\nu_j,\mu_1,\mu_2,\ldots,\mu_l,\sigma_1\sigma_2\ldots\sigma_{k-1-j-l}} \\
& dw_2^{\nu_1} \wedge \ldots \wedge dw_2^{\nu_j} \wedge \omega_1^{\mu_1} \wedge \ldots \wedge \omega_1^{\nu_l} \wedge \omega^{\sigma_1} \wedge \ldots \wedge \omega^{\sigma_{k-1-j-l}} \wedge dw^L \\
& + \sum_{1\leq j\leq k} \frac{1}{j!(k-j)!} D_{\nu_1\nu_2\ldots\nu_j,\sigma_1\sigma_2\ldots\sigma_{k-j}} dw_2^{\nu_1} \wedge dw_2^{\nu_2} \wedge \ldots \wedge dw_2^{\nu_j} \\
& \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \ldots \wedge \omega^{\sigma_{k-j}} \\
& + \sum_{1\leq j\leq k-1} \sum_{1\leq l\leq k-j} \frac{1}{j!l!(k-j-l)!} D^1_{\nu_1\nu_2\ldots\nu_j,\mu_1,\mu_2,\ldots,\mu_l,\sigma_1\sigma_2\ldots\sigma_{k-j-l}} \\
& dw_2^{\nu_1} \wedge \ldots \wedge dw_2^{\nu_j} \wedge \omega_1^{\mu_1} \wedge \ldots \wedge \omega_1^{\nu_l} \wedge \omega^{\sigma_1} \wedge \ldots \wedge \omega^{\sigma_{k-j-l}} + \text{ contact forms.}
\end{aligned}$$

Then the class $[\eta]$ is an element of $\tilde{\Omega}^5_k W / \tilde{\Theta}^5_k W$ defined by

$$\begin{aligned}
[\eta] = & \frac{1}{(k-1)!} E_{\sigma_1\sigma_2\ldots\sigma_{k-1}}([\eta]) \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \ldots \wedge \omega^{\sigma_{k-1}} \wedge dw^L \\
& + \sum_{1\leq j\leq k-1} \frac{1}{j!(k-1-j)!} E^1_{\mu_1\mu_2\ldots\mu_j,\sigma_1\sigma_2\ldots\sigma_{k-1-j}}([\eta]) \omega_1^{\mu_1} \wedge \omega_1^{\mu_2} \wedge \ldots \wedge \omega_1^{\mu_j} \\
& \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \ldots \wedge \omega^{\sigma_{k-1-j}} \wedge dw^L \\
& + \sum_{2\leq j\leq k-1} \frac{1}{j!(k-1-j)!} F_{\nu_1\nu_2\ldots\nu_j,\sigma_1\sigma_2\ldots\sigma_{k-1-j}}([\eta]) \omega_2^{\nu_1} \wedge \omega_2^{\nu_2} \wedge \ldots \wedge \omega_2^{\nu_j} \\
& \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \ldots \wedge \omega^{\sigma_{k-1-j}} \wedge dw^L \\
& + \sum_{1\leq j\leq k-2} \sum_{1\leq l\leq k-1-j} \frac{1}{j!l!(k-1-j-l)!} F^1_{\nu_1\nu_2\ldots\nu_j,\mu_1\mu_2\ldots\mu_l,\sigma_1\sigma_2\ldots\sigma_{k-1-j-l}}([\eta]) \\
& \omega_2^{\nu_1} \wedge \ldots \wedge \omega_2^{\nu_j} \wedge \omega_1^{\mu_1} \wedge \ldots \wedge \omega_1^{\mu_l} \wedge \omega^{\sigma_1} \wedge \ldots \wedge \omega^{\sigma_{k-1-j-l}} \wedge dw^L,
\end{aligned}$$

where

$$\begin{aligned}
& E_{\sigma_1\sigma_2\ldots\sigma_{k-1}}([\eta]) = A_{\sigma_1\sigma_2\ldots\sigma_{k-1}} + (-1)^{k-1} D_{\mu,\sigma_1\sigma_2\ldots\sigma_{k-1}} w_3^{\mu} \\
& - \frac{d}{dw^L} \left(A^1_{\sigma_1,\sigma_2\sigma_3\ldots\sigma_{k-1}} + (-1)^{k-1} D^1_{\nu,\sigma_1,\sigma_2\sigma_3\ldots\sigma_{k-1}} w_3^{\nu} \right) \\
& + \frac{d^2}{d(w^L)^2} \left(B_{\sigma_1,\sigma_2\sigma_3\ldots\sigma_{k-1}} + (-1)^{k-1} D_{\mu\sigma_1,\sigma_2\sigma_3\ldots\sigma_{k-1}} w_3^{\mu} \right) \quad alt(\sigma_1\sigma_2\ldots\sigma_{k-1}),
\end{aligned}$$

$$E^1_{\mu,\sigma_1\sigma_2\ldots\sigma_{k-2}}([\eta]) = \frac{k-2}{k-1}\Big(A^1_{\mu,\sigma_1\sigma_2\ldots\sigma_{k-2}} + A^1_{\sigma_1,\mu\sigma_2\ldots\sigma_{k-2}}$$
$$+(-1)^{k-1}(D^1_{\nu,\mu,\sigma_1\sigma_2\ldots\sigma_{k-2}} + D^1_{\nu,\sigma_1,\mu\sigma_2\ldots\sigma_{k-2}})w^\nu_3$$
$$-\frac{d}{dw^L}\big(B_{\mu,\sigma_1,\sigma_2\ldots\sigma_{k-2}} + B_{\sigma_1,\mu\sigma_2\ldots\sigma_{k-2}}$$
$$+(-1)^{k-1}(D_{\nu\mu,\sigma_1\sigma_2\ldots\sigma_{k-2}} + D_{\nu\sigma_1,\mu\sigma_2\ldots\sigma_{k-2}})w^\nu_3\big)\Big) \quad alt(\sigma_1\sigma_2\ldots\sigma_{k-2}),$$
$$E^1_{\mu_1\mu_2,\sigma_1\sigma_2\ldots\sigma_{k-3}}([\eta]) = A^1_{\mu_1\mu_2,\sigma_1\sigma_2\ldots\sigma_{k-3}} + (-1)^{k-1}D^1_{\nu,\mu_1\mu_2,\sigma_1\sigma_2\ldots\sigma_{k-3}}w^\nu_3$$
$$-(B_{\mu_1,\mu_2\sigma_1\ldots\sigma_{k-3}} - B_{\mu_2,\mu_1\sigma_1\ldots\sigma_{k-3}}) + (-1)^k(D_{\nu\mu_1,\mu_2\sigma_1\ldots\sigma_{k-3}} - D_{\nu\mu_2,\mu_1\sigma_1\ldots\sigma_{k-3}})w^\nu_3,$$
$$E^1_{\mu_1\mu_2\ldots\mu_j,\sigma_1\sigma_2\ldots\sigma_{k-1-j}}([\eta])$$
$$= A^1_{\mu_1\mu_2\ldots\mu_j,\sigma_1\sigma_2\ldots\sigma_{k-1-j}} + (-1)^{k-1}D^1_{\nu,\mu_1\mu_2\ldots\mu_j,\sigma_1\sigma_2\ldots\sigma_{k-1-j}}w^\nu_3,\ 3 \le j \le k-2,$$
$$E^1_{\mu_1\mu_2\ldots\mu_{k-1}}([\eta]) = A^1_{\mu_1\mu_2\ldots\mu_{k-1}} + (-1)^{k-1}D^1_{\nu,\mu_1\mu_2\ldots\mu_{k-1}}w^\nu_3$$
$$-\frac{d}{dw^L}\Big(B^1_{\mu_1,\mu_2\ldots\nu_{k-1}} + (-1)^{k-1}D^1_{\nu\mu_1,\mu_2\ldots\mu_{k-1}}w^\nu_3\Big) \quad alt(\mu_1\mu_2\ldots\mu_{k-1}),$$
$$F_{\mu_1\mu_2\ldots\mu_j,\sigma_1\sigma_2\ldots\sigma_{k-1-j}}([\eta])$$
$$= B_{\nu_1\nu_2\ldots\nu_j,\sigma_1\sigma_2\ldots\sigma_{k-1-j}} + (-1)^{k-1}D_{\mu\nu_1\nu_2\ldots\nu_j,\sigma_1\sigma_2\ldots\sigma_{k-1-j}}w^\mu_3,\ 2 \le j \le k-1,$$
$$F^1_{\nu,\mu_1\mu_2\ldots\mu_{k-2}}([\eta]) = \frac{k-2}{k-1}\Big(B^1_{\nu,\mu_1\mu_2\ldots\mu_{k-2}} + B_{\mu_1,\nu\mu_2\ldots\mu_{k-2}}$$
$$+(-1)^{k-1}\Big(D^1_{\sigma\nu,\mu_1\mu_2\ldots\mu_{k-2}} + D^1_{\sigma\mu_1,\nu\mu_2\ldots\mu_{k-2}}\Big)w^\sigma_3\Big) \quad alt(\mu_1\mu_2\ldots\mu_{k-2}),$$
$$F^1_{\nu_1\nu_2\ldots\nu_j,\mu_1\mu_2\ldots\mu_l,\sigma_1\sigma_2\ldots\sigma_{k-1-j-l}}([\eta])$$
$$= B^1_{\nu_1\nu_2\ldots\nu_j,\mu_1\mu_2\ldots\mu_l,\sigma_1\sigma_2\ldots\sigma_{k-1-j-l}} + (-1)^{k-1}D^1_{\tau\nu_1\nu_2\ldots\nu_j,\mu_1\mu_2\ldots\mu_l,\sigma_1\sigma_2\ldots\sigma_{k-1-j-l}}w^\tau_3,$$
$$1 \le j \le k-2,\ 1 \le l \le k-3.$$

Proof. Factorizing by contact forms in sense of (4.5) we obtain the classes in a routine way.

(a) The result follows from $(\rho^{r+1,r})^*\eta = (A + C_\sigma w^\sigma_{r+1})dw^L + \sum_{l=0}^{r-1} B^l_\sigma \omega^\sigma_l + C_\sigma \omega^\sigma_r$.

(b) Using $dw^\sigma_r = \omega^\sigma_r + w^\sigma_{r+1}dw^L$ and $d\omega^\sigma_l = -\omega^\sigma_{l+1} \wedge dw^L$, $l = 0, 1, \ldots, r-1$, in chart expression (4.21) of 2-form η we get

$$(\rho^{r+1,r})^*\eta = (A_\sigma - D_{\nu,\sigma}w^\nu_{r+1})\omega^\sigma \wedge dw^L - \sum_{l=1}^{r-1}(A^l_\sigma - D^l_{\nu,\sigma}w^\nu_{r+1})d\omega^\sigma_{l-1}$$
$$-(B_\sigma + D_{\sigma\nu}w^\nu_{r+1})d\omega^\sigma_{r-1}$$

up to contact forms. But from $-fd\omega = df \wedge \omega - d(f\omega)$ for arbitrary function f, applied to $A^l_\sigma - D^l_{\nu,\sigma}w^\nu_{r+1}$ and $B_\sigma + D_{\sigma\nu}w^\nu_{r+1}$, we obtain

$$(\rho^{r+2,r})^*\eta = \left(A_\sigma - D_{\nu,\sigma}w^\nu_{r+1} - \frac{d}{dw^L}(A^1_\sigma - D^1_{\nu,\sigma}w^\nu_{r+1})\right)\omega^\sigma \wedge dw^L$$
$$+\sum_{l=2}^{r-1}\frac{d}{dw^L}(A^l_\sigma - D^l_{\nu,\sigma}w^\nu_{r+1})d\omega^\sigma_{l-2} + \frac{d}{dw^L}(B_\sigma - D_{\nu\sigma}w^\nu_{r+1})d\omega^\sigma_{r-2}$$

up to contact forms. Taking now for f the derivatives of functions $A^l_\sigma - D^l_{\nu,\sigma} w^\nu_{r+1}$ and $B_\sigma + D_{\sigma\nu} w^\nu_{r+1}$, we obtain after $r-1$ steps the pull-back $(\rho^{2r+1,r})^* \eta$ of η, as required.

The proofs of assertions (c) and (d) are routine and follows from the previous analysis of contact forms. □

Now we analyze the quotient mappings (4.8) in second order variational sequence which are related to basic variational concepts as we shall see later; namely the mappings $E : \tilde{\Omega}^r_0 W \to \tilde{\Omega}^r_1 W/\tilde{\Theta}^r_1 W$, $E : \tilde{\Omega}^r_1 W/\tilde{\Theta}^r_1 W \to \tilde{\Omega}^r_2 W/\tilde{\Theta}^r_2 W$, and $E : \tilde{\Omega}^r_2 W/\tilde{\Theta}^r_2 W \to \tilde{\Omega}^r_3 W/\tilde{\Theta}^r_3 W$; the first two mappings are determined also for arbitrary finite order.

Theorem 4.8. *Let (V,ψ), $\psi = (y^K)$, be a chart on Y, and let $(\tilde{V}^{r,L}, \tilde{\chi}^{r,L})$, $\tilde{\chi}^{r,L} = (w^L, w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r)$, be an associated chart on $\tilde{W}^r \subset G^r Y$.*

(a) *If $f \in \tilde{\Omega}^r_0 W$, then $E(f) = (df/dw^L) dw^L$.*

(b) *Let $\eta \in \tilde{\Omega}^r_1 W$ be expressed in the contact basis by*

$$\eta = A dw^L + \sum_{l=0}^{r-1} B^l_\sigma \omega^\sigma_l + C_\sigma dw^\sigma_r.$$

Then

$$E([\eta]) = E_\sigma([d\eta])\, \omega^\sigma \wedge dw^L, \tag{4.23}$$

where

$$\begin{aligned} E_\sigma([d\eta]) = {} & \frac{\partial A}{\partial w^\sigma} + \frac{\partial C_\nu}{\partial w^\sigma} w^\nu_{r+1} \\ & + \sum_{l=1}^{r} (-1)^l \frac{d^l}{d(w^L)^l} \left(\frac{\partial A}{\partial w^\sigma_l} + \frac{\partial C_\nu}{\partial w^\sigma_l} w^\nu_{r+1} \right) + (-1)^{r+1} \frac{d^{r+1} C_\sigma}{d(w^L)^{r+1}}. \end{aligned}$$

(c) *Let $\eta \in \tilde{\Omega}^2_2 W$ be expressed in the contact basis by*

$$\begin{aligned} \eta = {} & A_\sigma \omega^\sigma \wedge dw^L + A^1_\nu \omega^\nu_1 \wedge dw^L + B_\nu dw^\nu_2 \wedge dw^L \\ & + \frac{1}{2} C_{\sigma_1\sigma_2} \omega^{\sigma_1} \wedge \omega^{\sigma_2} + \frac{1}{2} C^1_{\nu_1\nu_2} \omega^{\nu_1}_1 \wedge \omega^{\nu_2}_1 + C^1_{\nu,\sigma} \omega^\nu_1 \wedge \omega^\sigma \\ & + D_{\nu,\sigma} dw^\nu_2 \wedge \omega^\sigma + D^1_{\nu,\sigma} dw^\nu_2 \wedge \omega^\sigma_1 + \frac{1}{2} D_{\nu_1\nu_2} dw^{\nu_1}_2 \wedge dw^{\nu_2}_2. \end{aligned}$$

Then

$$\begin{aligned} E([\eta]) = {} & \frac{1}{2} E_{\nu\sigma}([d\eta])\, \omega^\nu \wedge \omega^\sigma \wedge dw^L \\ & + F_{\nu,\sigma}([d\eta])\, \omega^\nu_1 \wedge \omega^\sigma \wedge dw^L + \frac{1}{2} F_{\nu\sigma}([d\eta])\, \omega^\nu_2 \wedge \omega^\sigma \wedge dw^L \\ & + G_{\nu,\sigma}([d\eta])\, \omega^\nu_3 \wedge \omega^\sigma \wedge dw^L + \frac{1}{2} G_{\nu\sigma}([d\eta])\, \omega^\nu_4 \wedge \omega^\sigma \wedge dw^L, \end{aligned} \tag{4.24}$$

where

$$E_{\nu\sigma}([d\eta]) = \frac{\partial A_\sigma}{\partial w^\nu} - \frac{\partial A_\nu}{\partial w^\sigma} + \left(\frac{\partial D_{\mu,\nu}}{\partial w^\sigma} - \frac{\partial D_{\mu,\sigma}}{\partial w^\nu}\right) w_3^\mu - \frac{1}{2}\frac{d}{dw^L}\left(\frac{\partial A_\sigma}{\partial w_1^\nu} - \frac{\partial A_\nu}{\partial w_1^\sigma}\right.$$
$$+ \frac{\partial A_\sigma^1}{\partial w^\nu} - \frac{\partial A_\nu^1}{\partial w^\sigma} + \left.\left(\frac{\partial D_{\mu,\nu}}{\partial w_1^\sigma} - \frac{\partial D_{\mu,\sigma}}{\partial w_1^\nu} + \frac{\partial D_{\mu,\nu}^1}{\partial w^\sigma} - \frac{\partial D_{\mu,\sigma}^1}{\partial w^\nu}\right) w_3^\mu\right)$$
$$+ \frac{1}{2}\frac{d^2}{d(w^L)^2}\left(\frac{\partial A_\sigma^1}{\partial w_1^\nu} - \frac{\partial A_\nu^1}{\partial w_1^\sigma} + \left(\frac{\partial D_{\mu,\nu}^1}{\partial w_1^\sigma} - \frac{\partial D_{\mu,\sigma}^1}{\partial w_1^\nu}\right) w_3^\mu\right)$$
$$- \frac{1}{4}\frac{d^3}{d(w^L)^3}\left(\frac{\partial A_\sigma^1}{\partial w_2^\nu} - \frac{\partial A_\nu^1}{\partial w_2^\sigma} + \frac{\partial B_\sigma}{\partial w_1^\nu} - \frac{\partial B_\nu}{\partial w_1^\sigma} + \left(\frac{\partial D_{\mu\nu}}{\partial w_1^\sigma} - \frac{\partial D_{\mu\sigma}}{\partial w_1^\nu}\right) w_3^\mu\right.$$
$$\left. + (D_{\nu,\sigma} - D_{\sigma,\nu}) + \left(\frac{\partial D_{\mu,\nu}^1}{\partial w_2^\sigma} - \frac{\partial D_{\mu,\sigma}^1}{\partial w_2^\nu}\right) w_3^\mu\right)$$
$$+ \frac{1}{2}\frac{d^4}{d(w^L)^4}\left(\frac{\partial B_\sigma}{\partial w_2^\nu} - \frac{\partial B_\nu}{\partial w_2^\sigma} + \frac{1}{2}(D_{\nu,\sigma}^1 - D_{\sigma,\nu}^1) + \left(\frac{\partial D_{\mu\nu}}{\partial w_2^\sigma} - \frac{\partial D_{\mu,\sigma}}{\partial w_2^\nu}\right) w_3^\mu\right)$$
$$+ \frac{1}{2}\frac{d^5}{d(w^L)^5}(D_{\nu\sigma}),$$

$$F_{\nu,\sigma}([d\eta]) = \frac{1}{2}\left(\frac{\partial A_\sigma}{\partial w_1^\nu} + \frac{\partial A_\nu}{\partial w_1^\sigma} - \frac{\partial A_\sigma^1}{\partial w^\nu} - \frac{\partial A_\nu^1}{\partial w^\sigma} - \left(\frac{\partial D_{\mu,\sigma}}{\partial w_1^\nu} + \frac{\partial D_{\mu,\nu}}{\partial w_1^\sigma}\right.\right.$$
$$\left.\left. + \frac{\partial D_{\mu,\nu}^1}{\partial w^\sigma} + \frac{\partial D_{\mu,\sigma}^1}{\partial w^\nu}\right) w_3^\mu\right) - \frac{d}{dw^L}\left(\frac{\partial A_\sigma}{\partial w_2^\nu} + \frac{\partial A_\nu}{\partial w_2^\sigma} - \left(\frac{\partial B_\sigma}{\partial w^\nu} + \frac{\partial B_\nu}{\partial w^\sigma}\right)\right.$$
$$\left. - \left(\frac{\partial D_{\mu,\nu}}{\partial w_2^\sigma} + \frac{\partial D_{\mu,\sigma}}{\partial w_2^\nu} - \frac{\partial D_{\mu\sigma}}{\partial w^\nu} - \frac{\partial D_{\mu\nu}}{\partial w^\sigma}\right) w_3^\mu\right)$$
$$+ \frac{d^2}{d(w^L)^2}\left(\frac{\partial A_\sigma^1}{\partial w_2^\nu} + \frac{\partial A_\nu^1}{\partial w_2^\sigma} - \left(\frac{\partial B_\sigma}{\partial w_1^\nu} + \frac{\partial B_\nu}{\partial w_1^\sigma}\right) + \left(\frac{\partial D_{\mu\nu}}{\partial w_1^\sigma} + \frac{\partial D_{\mu\sigma}}{\partial w_1^\nu}\right.\right.$$
$$\left.\left. - \frac{\partial D_{\mu,\nu}^1}{\partial w_2^\sigma} - \frac{\partial D_{\mu,\sigma}^1}{\partial w_2^\nu}\right) w_3^\mu + \frac{d}{d(w^L)^3}(D_{\nu,\sigma}^1 + D_{\sigma,\nu}^1)\right),$$

$$F_{\nu\sigma}([d\eta]) = \left(\frac{\partial A_\sigma}{\partial w_2^\nu} - \frac{\partial A_\nu}{\partial w_2^\sigma}\right) + \left(\frac{\partial A_\nu^1}{\partial w_1^\sigma} - \frac{\partial A_\sigma^1}{\partial w_1^\nu}\right) + \left(\frac{\partial B_\sigma}{\partial w^\nu} - \frac{\partial B_\nu}{\partial w^\sigma}\right)$$
$$+ \left(\frac{\partial D_{\mu\nu}}{\partial w^\sigma} - \frac{\partial D_{\mu\sigma}}{\partial w^\nu} + \frac{\partial D_{\mu,\nu}}{\partial w_2^\sigma} - \frac{\partial D_{\mu,\sigma}}{\partial w_2^\nu} + \frac{\partial D_{\mu,\sigma}^1}{\partial w_1^\nu} - \frac{\partial D_{\mu\nu}^1}{\partial w_1^\sigma}\right) w_3^\mu$$
$$+ \frac{1}{2}\frac{d}{dw^L}\left(\frac{\partial A_\sigma^1}{\partial w_2^\nu} - \frac{\partial A_\nu^1}{\partial w_2^\sigma} + \frac{\partial B_\sigma}{\partial w_1^\nu} - \frac{\partial B_\nu}{\partial w_1^\sigma} + \left(\frac{\partial D_{\mu\nu}}{\partial w_1^\sigma} - \frac{\partial D_{\mu\sigma}}{\partial w_1^\nu}\right.\right.$$
$$\left.\left. + \frac{\partial D_{\mu,\nu}^1}{\partial w_2^\sigma} - \frac{\partial D_{\mu\sigma}^1}{\partial w_2^\nu}\right) w_3^\mu\right) + \frac{3}{2}\frac{d}{dw^L}(D_{\nu,\sigma} - D_{\sigma,\nu} - \frac{d}{d(w^L)}(D_{\nu,\sigma}^1 - D_{\sigma,\nu}^1))$$
$$- 2\frac{d^2}{d(w^L)^2}\left(\frac{\partial B_\sigma}{\partial w_2^\nu} - \frac{\partial B_\nu}{\partial w_2^\sigma} + \left(\frac{\partial D_{\mu\nu}}{\partial w_2^\sigma} - \frac{\partial D_{\mu\sigma}}{\partial w_2^\nu}\right) w_3^\mu\right) - 2\frac{d^3}{d(w^L)^3}(D_{\nu\sigma}),$$

$$G_{\nu,\sigma}([d\eta]) = -\frac{1}{2}\left(\frac{\partial A^1_\nu}{\partial w^\sigma_2} + \frac{\partial A^1_\sigma}{\partial w^\nu_2} - \left(\frac{\partial B_\sigma}{\partial w^\nu_1} + \frac{\partial B_\nu}{\partial w^\sigma_1}\right) + (D_{\nu,\sigma} + D_{\sigma,\nu})\right.$$
$$\left. + \left(\frac{\partial D_{\mu\nu}}{\partial w^\sigma_1} + \frac{\partial D_{\mu\sigma}}{\partial w^\nu_1} - \frac{\partial D^1_{\mu,\sigma}}{\partial w^\nu_2} - \frac{\partial D^1_{\mu\nu}}{\partial w^\sigma_2}\right) w^\mu_3 + \frac{d}{dw^L}(D^1_{\nu,\sigma} + D^1_{\sigma,\nu})\right),$$
$$G_{\nu\sigma}([d\eta]) = \frac{\partial B_\sigma}{\partial w^\nu_2} - \frac{\partial B_\nu}{\partial w^\sigma_2} + D^1_{\nu,\sigma} - D^1_{\sigma,\nu} + \left(\frac{\partial D_{\mu\nu}}{\partial w^\sigma_2} - \frac{\partial D_{\mu\sigma}}{\partial w^\nu_2}\right) w^\mu_3 + \frac{d}{dw^L}(D_{\nu\sigma}).$$

Proof. Analogously to the proof of Theorem 4.6, all the classes can be determined by direct computation. □

Remark 12. We remark that in the previous theorem, for example the class $[d\eta] = E([\eta])$ of $\eta \in \tilde{\Omega}^2_2 W$ can be described in a different basis, and on a different prolongation of a manifold. Namely,

$$E([\eta]) = \frac{1}{2}E'_{\nu\sigma}([d\eta])\,\omega^\nu \wedge \omega^\sigma \wedge dw^L$$
$$+ F'_{\nu,\sigma}([d\eta])\,\omega^\nu_1 \wedge \omega^\sigma \wedge dw^L + \frac{1}{2}F'_{\nu\sigma}([d\eta])\,\omega^\nu_1 \wedge \omega^\sigma_1 \wedge dw^L \tag{4.25}$$
$$+ G'_{\nu,\sigma}([d\eta])\,\omega^\nu_2 \wedge \omega^\sigma_1 \wedge dw^L + \frac{1}{2}G'_{\nu\sigma}([d\eta])\,\omega^\nu_2 \wedge \omega^\sigma_2 \wedge dw^L;$$

please compare with (4.24). The reason why to prefer some class of a form and why another not lies in the fact that classes of forms need not to be well-defined forms. This circumstance is even more apparent in the case of fibred manifolds; in Grassmann fibrations, in general, the classes do *not* define forms. Roughly speaking, we prefer such classes witch correspond to the variational objects, known from the local theory. This fact suggests to study invariance of classes with respect to isomorphisms of underlying manifolds.

In Theorem 4.6 we deal with the quotient mappings E, defined in Sect. 4.3 by (4.8). Now we wish to characterize the mappings $E : \tilde{\Omega}^2_1 W/\tilde{\Theta}^2_1 W \to \tilde{\Omega}^2_2 W/\tilde{\Theta}^2_2 W$ and $E : \tilde{\Omega}^2_2 W/\tilde{\Theta}^2_2 W \to \tilde{\Omega}^2_3 W/\tilde{\Theta}^2_3 W$ in a different way.

Although the classes (4.11) and (4.12) are defined in an abstract way in the variational sequence theory, they are closely related to the variational theory on Grassmann fibrations (cf. Sect. 5). This is the motivation for the terminology, we now introduce.

Let (V,ψ), $\psi = (y^K)$, be a fixed chart on Y. Consider a 1-form $\rho \in \tilde{\Omega}^1_1 W$, expressed in the contact basis by $\eta = A dw^L + B_\sigma \omega^\sigma + C_\sigma dw^\sigma_1$, and define a *Lagrange function* $\mathscr{L}_L : \tilde{V}^{2,L} \to \mathbf{R}$ by

$$\mathscr{L}_L = A + C_\sigma w^\sigma_2. \tag{4.26}$$

The corresponding *Euler-Lagrange* expressions $\mathscr{E}_\sigma(\mathscr{L}_L) : \tilde{V}^{3,L} \to \mathbf{R}$ are of the form

$$\mathscr{E}_\sigma(\mathscr{L}_L) = \frac{\partial \mathscr{L}_L}{\partial w^\sigma} - \frac{d}{dw^L}\frac{\partial \mathscr{L}_L}{\partial w^\sigma_1} + \frac{d^2}{d(w^L)^2}\frac{\partial \mathscr{L}_L}{\partial w^\sigma_2}. \tag{4.27}$$

Then by Theorem 4.5 we have $[\eta] = \mathscr{L}_L dw^L$.

Let a 1-form $\eta \in \tilde{\Omega}_1^r W$ be expressed in the contact basis by $Adw^L + \sum_{l=0}^{r-1} B_\sigma^l \omega_l^\sigma + C_\sigma dw_r^\sigma$. We define a Lagrange function for η, $\mathscr{L}_L : \tilde{V}^{r+1,L} \to \mathbf{R}$, by

$$\mathscr{L}_L = A + C_\sigma w_{r+1}^\sigma, \tag{4.28}$$

and the corresponding *Euler-Lagrange* expressions $\mathscr{E}_\sigma(\mathscr{L}_L) : \tilde{V}^{2r+1,L} \to \mathbf{R}$ are of the form

$$\mathscr{E}_\sigma(\mathscr{L}_L) = \frac{\partial \mathscr{L}_L}{\partial w^\sigma} + \sum_{l=1}^{r} (-1)^l \frac{d^l}{d(w^L)^l} \frac{\partial \mathscr{L}_L}{\partial w_l^\sigma}. \tag{4.29}$$

Let $\eta \in \tilde{\Omega}_2^1 W$ be a 2-form, expressed in the contact basis by

$$\begin{aligned}\eta &= A_\sigma \omega^\sigma \wedge dw^L + B_\sigma dw_1^\sigma \wedge dw^L \\ &+ \frac{1}{2} C_{\sigma\nu} \omega^\sigma \wedge \omega^\nu + D_{\nu,\sigma} dw_1^\nu \wedge \omega^\sigma + \frac{1}{2} D_{\sigma\nu} dw_1^\sigma \wedge dw_1^\nu,\end{aligned}$$

and we set

$$\varepsilon_\sigma = A_\sigma - D_{\nu,\sigma} w_2^\nu - \frac{d}{dw^L}\left(B_\sigma + D_{\sigma\nu} w_2^\nu\right), \tag{4.30}$$

the class of η (cf. Theorem 4.5, (4.12)). Then the *Helmholtz* expressions, defined by ε_σ, are given by

(4.31)

$$\begin{aligned}
\mathscr{H}_{\sigma\nu}^3(\varepsilon_\kappa) &= \frac{\partial \varepsilon_\sigma}{\partial w_3^\nu} + \frac{\partial \varepsilon_\nu}{\partial w_3^\sigma}, \\
\mathscr{H}_{\sigma\nu}^2(\varepsilon_\kappa) &= \frac{\partial \varepsilon_\sigma}{\partial w_2^\nu} - \frac{\partial \varepsilon_\nu}{\partial w_2^\sigma} - \frac{3}{2}\frac{d}{dw^L}\left(\frac{\partial \varepsilon_\sigma}{\partial w_3^\nu} - \frac{\partial \varepsilon_\nu}{\partial w_3^\sigma}\right), \\
\mathscr{H}_{\sigma\nu}^1(\varepsilon_\kappa) &= \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial w_1^\nu} + \frac{\partial \varepsilon_\nu}{\partial w_1^\sigma} - \frac{d}{dw^L}\left(\frac{\partial \varepsilon_\sigma}{\partial w_2^\nu} + \frac{\partial \varepsilon_\nu}{\partial w_2^\sigma}\right)\right), \\
\mathscr{H}_{\sigma\nu}^0(\varepsilon_\kappa) &= \frac{\partial \varepsilon_\sigma}{\partial w^\nu} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma} - \frac{1}{2}\frac{d}{dw^L}\left(\frac{\partial \varepsilon_\sigma}{\partial w_1^\nu} - \frac{\partial \varepsilon_\nu}{\partial w_1^\sigma} - \frac{1}{2}\frac{d^2}{d(w^L)^2}\left(\frac{\partial \varepsilon_\sigma}{\partial w_3^\nu} - \frac{\partial \varepsilon_\nu}{\partial w_3^\sigma}\right)\right).
\end{aligned}$$

Let $\eta \in \tilde{\Omega}_2^r W$ be a 2-form, expressed in the contact basis by

$$\begin{aligned}\eta &= \sum_{l=0}^{r-1} A_\sigma^l \omega_l^\sigma \wedge dw^L + B_\nu dw_r^\nu \wedge dw^L + \frac{1}{2}\sum_{l=0}^{r-1} C_{\sigma_1\sigma_2}^l \omega_l^{\sigma_1} \wedge \omega_l^{\sigma_2} \\ &+ \sum_{l=1}^{r-1}\sum_{s=0}^{l-1} C_{\nu,\sigma}^{l,s} \omega_l^\nu \wedge \omega_s^\sigma + \sum_{l=0}^{r-1} D_{\nu,\sigma}^l dw_r^\nu \wedge \omega_l^\sigma + \frac{1}{2} D_{\nu_1\nu_2} dw_r^{\nu_1} \wedge dw_r^{\nu_2}.\end{aligned}$$

and we set

(4.32)

$$\varepsilon_\sigma = \sum_{l=0}^{r-1} (-1)^l \frac{d^l}{d(w^L)^l}\left(A_\sigma^l - D_{\nu,\sigma}^l w_{r+1}^\nu\right) + (-1)^r \frac{d^r}{d(w^L)^r}\left(B_\sigma - D_{\nu\sigma} w_{r+1}^\nu\right),$$

the class of η (cf. Theorem 4.7, (4.22)). Then for $r = 2$ the *Helmholtz* expressions, defined by ε_σ, are given by

(4.33)

$$\mathscr{H}^5_{\sigma\nu}(\varepsilon_\kappa) = \frac{\partial \varepsilon_\sigma}{\partial w^\nu_5} + \frac{\partial \varepsilon_\nu}{\partial w^\sigma_5},$$

$$\mathscr{H}^4_{\sigma\nu}(\varepsilon_\kappa) = \frac{\partial \varepsilon_\sigma}{\partial w^\nu_4} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma_4} - \frac{5}{2}\frac{d}{dw^L}\left(\frac{\partial \varepsilon_\sigma}{\partial w^\nu_5} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma_5}\right),$$

$$\mathscr{H}^3_{\sigma\nu}(\varepsilon_\kappa) = \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial w^\nu_3} + \frac{\partial \varepsilon_\nu}{\partial w^\sigma_3} - 2\frac{d}{dw^L}\left(\frac{\partial \varepsilon_\sigma}{\partial w^\nu_4} + \frac{\partial \varepsilon_\nu}{\partial w^\sigma_4}\right)\right),$$

$$\mathscr{H}^2_{\sigma\nu}(\varepsilon_\kappa) = \frac{\partial \varepsilon_\sigma}{\partial w^\nu_2} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma_2} - \frac{3}{2}\frac{d}{dw^L}\left(\frac{\partial \varepsilon_\sigma}{\partial w^\nu_3} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma_3}\right) + \frac{5}{2}\frac{d^3}{d(w^L)^3}\left(\frac{\partial \varepsilon_\sigma}{\partial w^\nu_5} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma_5}\right),$$

$$\mathscr{H}^1_{\sigma\nu}(\varepsilon_\kappa) = \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial w^\nu_1} + \frac{\partial \varepsilon_\nu}{\partial w^\sigma_1} - \frac{d}{dw^L}\left(\frac{\partial \varepsilon_\sigma}{\partial w^\nu_2} + \frac{\partial \varepsilon_\nu}{\partial w^\sigma_2}\right) + \frac{d^3}{d(w^L)^3}\left(\frac{\partial \varepsilon_\sigma}{\partial w^\nu_4} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma_4}\right)\right),$$

$$\mathscr{H}^0_{\sigma\nu}(\varepsilon_\kappa) = \frac{\partial \varepsilon_\sigma}{\partial w^\nu} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma} - \frac{1}{2}\frac{d}{dw^L}\left(\frac{\partial \varepsilon_\sigma}{\partial w^\nu_1} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma_1}\right) + \frac{1}{4}\frac{d^3}{d(w^L)^3}\left(\frac{\partial \varepsilon_\sigma}{\partial w^\nu_3} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma_3}\right)$$
$$- \frac{1}{2}\frac{d^5}{d(w^L)^5}\left(\frac{\partial \varepsilon_\sigma}{\partial w^\nu_5} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma_5}\right)$$

One can see that Euler-Lagrange expressions (4.27) and (4.29) and Helmholtz expressions (4.31) and (4.33) coincide with the coefficients of classes (4.16), (4.23), (4.17) and (4.24), respectively. The following result is important for applications.

Theorem 4.9. *Let* (V, ψ)*,* $\psi = (y^K)$*, be a chart on* Y*, let* $(\tilde{V}^{1,L}, \tilde{\chi}^{1,L})$*,* $\tilde{\chi}^{1,L} = (w^L, w^\sigma, w^\sigma_1, w^\sigma_2, \ldots, w^\sigma_r)$*, be an associated chart on* $\tilde{W}^r \subset G^rY$*.*

(a) *Let* $\eta \in \tilde{\Omega}^r_1 W$ *be expressed in the contact basis by* $A dw^L + \sum_{l=0}^{r-1} B^l_\sigma \omega^\sigma_l + C_\sigma dw^\sigma_r$*. Then*

$$E([\eta]) = E_\sigma([d\eta])\,\omega^\sigma \wedge dw^L,$$

where

$$E_\sigma([d\eta]) = \mathscr{E}_\sigma(\mathscr{L}_L),$$

and $\mathscr{L}_L$ *is defined by* (4.28)*, resp.* (4.26) *for* $r = 1$*.*

(b) *Let* $\eta \in \tilde{\Omega}^1_2 W$ *be expressed in the contact basis by*

$$\eta = A_\sigma \omega^\sigma \wedge dw^L + B_\sigma dw^\sigma_1 \wedge dw^L$$
$$+ \frac{1}{2}C_{\sigma\nu}\omega^\sigma \wedge \omega^\nu + D_{\nu,\sigma} dw^\nu_1 \wedge \omega^\sigma + \frac{1}{2}D_{\sigma\nu} dw^\sigma_1 \wedge dw^\nu_1.$$

Then

$$E([\eta]) = \frac{1}{2}E_{\sigma\nu}([d\eta])\,\omega^\sigma \wedge \omega^\nu \wedge dw^L + F_{\nu,\sigma}([d\eta])\,\omega^\nu_1 \wedge \omega^\sigma \wedge dw^L$$
$$+ \frac{1}{2}F_{\nu\sigma}([d\eta])\,\omega^\nu_2 \wedge \omega^\sigma \wedge dw^L,$$

where

$$E_{\nu\sigma}([d\eta]) = \mathscr{H}^0_{\sigma\nu}(\varepsilon_\kappa), \quad F_{\nu,\sigma}([d\eta]) = \mathscr{H}^1_{\sigma\nu}(\varepsilon_\kappa),$$
$$F_{\nu\sigma}([d\eta]) = \mathscr{H}^2_{\sigma\nu}(\varepsilon_\kappa), \quad \mathscr{H}^3_{\sigma\nu}(\varepsilon_\kappa) = 0,$$

are the Helmholtz expressions (4.31) and ε_κ *is defined by* (4.30).

(c) *Let* $\eta \in \tilde{\Omega}^2_2 W$ *be expressed in the contact basis by*

$$\begin{aligned}\eta = {} & A_\sigma \omega^\sigma \wedge dw^L + A^1_\nu \omega^\nu_1 \wedge dw^L + B_\nu dw^\nu_2 \wedge dw^L \\ & + \frac{1}{2} C_{\sigma_1\sigma_2} \omega^{\sigma_1} \wedge \omega^{\sigma_2} + \frac{1}{2} C^1_{\nu_1\nu_2} \omega^{\nu_1}_1 \wedge \omega^{\nu_2}_1 + C^1_{\nu,\sigma} \omega^\nu_1 \wedge \omega^\sigma \\ & + D_{\nu,\sigma} dw^\nu_2 \wedge \omega^\sigma + D^1_{\nu,\sigma} dw^\nu_2 \wedge \omega^\sigma_1 + \frac{1}{2} D_{\nu_1\nu_2} dw^{\nu_1}_2 \wedge dw^{\nu_2}_2.\end{aligned}$$

Then

$$\begin{aligned}E([\eta]) = {} & \frac{1}{2} E_{\nu\sigma}([d\eta])\, \omega^\nu \wedge \omega^\sigma \wedge dw^L \\ & + F_{\nu,\sigma}([d\eta])\, \omega^\nu_1 \wedge \omega^\sigma \wedge dw^L + \frac{1}{2} F_{\nu\sigma}([d\eta])\, \omega^\nu_2 \wedge \omega^\sigma \wedge dw^L \\ & + G_{\nu,\sigma}([d\eta])\, \omega^\nu_3 \wedge \omega^\sigma \wedge dw^L + \frac{1}{2} G_{\nu\sigma}([d\eta])\, \omega^\nu_4 \wedge \omega^\sigma \wedge dw^L,\end{aligned}$$

where

$$E_{\nu\sigma}([d\eta]) = \mathscr{H}^0_{\sigma\nu}(\varepsilon_\kappa), \quad F_{\nu,\sigma}([d\eta]) = \mathscr{H}^1_{\sigma\nu}(\varepsilon_\kappa), \quad F_{\nu\sigma}([d\eta]) = \mathscr{H}^2_{\sigma\nu}(\varepsilon_\kappa),$$
$$G_{\nu,\sigma}([d\eta]) = \mathscr{H}^3_{\sigma\nu}(\varepsilon_\kappa), \quad G_{\nu\sigma}([d\eta]) = \mathscr{H}^4_{\sigma\nu}(\varepsilon_\kappa), \quad \mathscr{H}^5_{\sigma\nu}(\varepsilon_\kappa) = 0,$$

are the Helmholtz expressions (4.33) and ε_κ *is defined by* (4.32).

Proof. The proof follows from Theorems 4.6, 4.8 and is routine. □

The following theorem describes the transformation properties of coefficients of quotient mappings in first order variational sequence.

Theorem 4.10. *Let* $(\tilde{V}^{1,L}, \tilde{\chi}^{1,L})$ *and* $(\tilde{U}^{1,M}, \tilde{\phi}^{1,M})$ *be associated charts on* $\tilde{W}^1$ *such that* $\tilde{V}^{1,L} \cap \tilde{U}^{1,M} \neq \emptyset$*, where* $\tilde{\chi}^{1,L} = (w^L, w^\sigma, w^\sigma_1)$*,* $\tilde{\phi}^{1,M} = (\bar{w}^M, \bar{w}^\nu, \bar{w}^\nu_1)$.

(a) *If* $\eta \in \tilde{\Omega}^1_1 W$*, and its class* $[\eta] \in \tilde{\Omega}^2_1 W / \tilde{\Theta}^2_1 W$ *is expressed by* $[\eta] = \mathscr{L}\, dw^L = \bar{\mathscr{L}}\, d\bar{w}^M$*, then*

$$\mathscr{L} = \bar{\mathscr{L}}\, \frac{d\bar{w}^M}{dw^L}.$$

(b) *If* $\eta \in \tilde{\Omega}^1_2 W$*, and if* $[\eta] \in \tilde{\Omega}^3_2 W / \tilde{\Theta}^3_2 W$ *is expressed by*

$$[\eta] = E_\sigma([\eta])\, \omega^\sigma \wedge dw^L = \bar{E}_\nu([\rho])\, \bar{\omega}^\nu \wedge d\bar{w}^M,$$

then

$$E_\sigma([\eta]) = \sum_{\substack{\nu=1 \\ \nu\neq M}}^{m+1} \bar{E}_\nu([\eta]) \left(\frac{\partial \bar{w}^\nu}{\partial w^\sigma} - \bar{w}^\nu_1 \frac{\partial \bar{w}^M}{\partial w^\sigma} \right) \frac{d\bar{w}^M}{dw^L}.$$

Proof. Both results (a), (b) are immediate consequences of coordinate transformations on $G^1 Y$. □

Remark 13. Theorem 4.10 shows that in the Grassmann fibrations the transformation properties of local expressions for classes are completely different as in the fibred case. In particular, the Euler-Lagrange class and the Helmholtz class cannot be described by a differential form on some suitable higher order Grassmann fibration.

5. Applications: The calculus of variations

5.1. **Variational functionals.** Suppose we have a 1-form η on $\operatorname{Imm} T^1 Y$. Let I be an open interval, and let $\gamma : I \to Y$ be an immersion. Any *compact* subinterval K of I defines the *variational integral*, associated with η,

$$\eta_K(\gamma) = \int_K (T^1\gamma)^* \eta. \tag{5.1}$$

Lemma 5.1. *Let $\gamma : I \to Y$ be an immersion, J an open interval, and $\mu : J \to I$ an isomorphism. The following conditions are equivalent:*

(a) *For any two compact intervals $L \subset J$ and $K \subset I$ such that $\mu(L) = K$,*

$$\eta_K(\gamma) = \eta_L(\gamma \circ \mu).$$

(b) η *satisfies*

$$(T^1\gamma)^*\eta = (\mu^{-1})^* T^1(\gamma \circ \mu)^* \eta. \tag{5.2}$$

Proof. We show that (a) implies (b). We transform the variational integral $\eta_L(\gamma \circ \mu)$ by means of the change of variables formula. We have

$$\eta_L(\gamma \circ \mu) = \int_K (\mu^{-1})^* (T^1(\gamma \circ \mu))^* \eta.$$

If $\eta_K(\gamma) = \eta_L(\gamma \circ \mu)$, then

$$\int_K (T^1\gamma)^*\eta = \int_K (\mu^{-1})^*(T^1(\gamma \circ \mu))^*\eta.$$

Since this equality is satisfied for all K, we have the condition (5.2).

The converse is obvious. □

Condition (5.2), called the *invariance condition*, expresses independence of the variational integral (5.1) on parametrization. We say that the 1-form η and the immersion γ *satisfy the invariance condition*, if condition (5.2) holds for all diffeomorphisms μ.

Let $\Psi_{J_0^1\alpha}$ denote the diffeomorphism $J_0^1\zeta \to J_0^1\zeta \circ J_0^1\alpha$ of T^1Y.

Lemma 5.2. *Let $\gamma : I \to Y$ be an immersion. The following conditions are equivalent:*

(a) η *satisfies the invariance condition for all diffeomorphisms $\mu : J \to I$.*

(b) *For all $J_0^1\alpha \in L^1$*

$$T^1\gamma^*\eta = T^1\gamma^*\Psi_{J_0^1\alpha}{}^*\eta.$$

Proof. This is just a restatement of Lemma 5.1. □

If the invariance condition is satisfied, we denote

$$\Omega = \gamma(K). \tag{5.3}$$

Then the number $\eta_K(\gamma)$ depends only on the *segment* Ω, and we sometimes denote

$$\eta_K(\Omega) = \int_K (T^1\gamma)^*\eta.$$

For every compact interval $K \subset I$ the variational integral (5.1) defines the *variational functional*, associated with η,

$$C_K^2 Y \ni \gamma \to \eta_K(\gamma) = \int_K (T^1\gamma)^* \eta \in \mathbf{R}, \tag{5.4}$$

where $C_K^2 Y$ is the set of immersions $\gamma : K \to Y$ of class C^2.

As before, let W be an open set in Y, and let $\gamma : I \to Y$ be an immersion. By the *Grassmann prolongation* of γ of order r we mean the curve $[T^r\gamma]$ in the Grassmann fibration G^rY, defined by

$$[T^r\gamma](t) = T^r\gamma(t).$$

Let η be expressed in a chart (V, ψ), $\psi = (y^K)$, by

$$\eta = A_L dw^L + A_\sigma \omega^\sigma + B_L d\dot{w}^L + B_\sigma dw_1^\sigma,$$

and let $\gamma : I \to Y$ be an immersion of an open interval $I \subset \mathbf{R}$ into Y such that $\gamma(I) \subset V$ and $T^1\gamma(I) \subset V^{1,L}$ for some index L. One can easily determine the chart expression for the 1-form $T^1\gamma^*\eta$. We get

$$T^1\gamma^*\eta = \Bigg((A_L \circ T^1\gamma) + (B_L \circ T^1\gamma) \frac{D^2(w^L\gamma)}{D(w^L\gamma)} + (B_\sigma \circ T^1\gamma) \frac{D^2(w^\sigma\gamma) D(w^L\gamma) - D(w^\sigma\gamma) D^2(w^L\gamma)}{D(w^L\gamma)^3} \Bigg) dw^L.$$

Introducing the *Lagrange function* $\mathscr{L}_L$ by

$$\mathscr{L}_L = A_L + B_\sigma w_2^\sigma + B_L \frac{\ddot{w}^L}{\dot{w}^L},$$

we can also write

$$T^1\gamma^*\eta = (\mathscr{L}_L \circ T^2\gamma) dw^L. \tag{5.5}$$

Lemma 5.3. η *satisfies the invariance condition if and only if*

$$B_L = 0, \quad A_L = A_L(w^L, w^\sigma, w_1^\sigma), \quad B_\sigma = B_\sigma(w^L, w^\sigma, w_1^\sigma).$$

If η satisfies the invariance condition, then η is projectable onto G^1Y; the variational integral (5.4) depends only on the Grassmann prolongation $[T^1\gamma]$. Denoting the projection of η by the same letter, we can write

$$\eta_K(\Omega) = \int_K [T^1\gamma]^* \eta. \tag{5.6}$$

From now on we suppose that the 1-form η satisfies the invariance condition. In this case

$$\mathscr{L}_L = A_L + B_\sigma w_2^\sigma, \tag{5.7}$$

where $A_L = A_L(w^L, w^\sigma, w_1^\sigma)$, $B_\sigma = B_\sigma(w^L, w^\sigma, w_1^\sigma)$. The integral (5.6) can be written as

$$\eta_K(\Omega) = \int_K (\mathscr{L}_L \circ T^2\gamma) dw^L. \tag{5.8}$$

Formula (5.7) shows that $\mathscr{L}_L$ coincides with the class of η in the variational sequence (Theorem 4.5, (a)).

5.2. **The Euler-Lagrange equations.** Let η be a 1-form, satisfying the invariance condition (5.2), and let $\mathscr{L}_L$ (5.7) be the corresponding Lagrange function. In this case we can regard the functions $w^L, w^\sigma, w_1^\sigma, w_2^\sigma$ as coordinates on G^2Y (previously denoted by $\tilde{w}^L, \tilde{w}^\sigma, \tilde{w}_1^\sigma, \tilde{w}_2^\sigma$). From (5.8) it now follows that the *extremals* of the variational functional η_K are determined by the *Euler-Lagrange expressions*

$$\mathscr{E}_\sigma(\mathscr{L}_L) = \frac{\partial \mathscr{L}_L}{\partial w^\sigma} - \frac{d}{dw^L}\frac{\partial \mathscr{L}_L}{\partial w_1^\sigma} + \frac{d^2}{d(w^L)^2}\frac{\partial \mathscr{L}_L}{w_2^\sigma}. \tag{5.9}$$

Note that the 1st order form η defines a 2nd *order* Lagrangian and, from (5.7), 3rd order Euler-Lagrange expressions (5.9). Thus, the extremals satisfy the system of the 3rd order differential equations $\mathscr{E}_\sigma(\mathscr{L}_L) \circ T^3\gamma = 0$ for a curve γ.

The Euler-Lagrange expressions define a 2-form on $V^{3,L}$,

$$\mathscr{E} = \mathscr{E}_\sigma(\mathscr{L}_L)\omega^\sigma \wedge dw^L. \tag{5.10}$$

In the variational problems on fibred manifolds, (5.10) is a global 2-form on the corresponding jet space. On the Grassmann fibration G^2Y, the expressions $\mathscr{E}_\sigma(\mathscr{L}_L)$ do not define a global 2-form; instead, we have a *class* as expressed in the variational sequence theory in Sect. 4.4.

In particular, this observation illustrates the differences in the geometric structures of variational principles on fibred manifolds and Grassmann fibrations.

5.3. **The Helmholtz equations.** We are now interested in the variationality of the expressions

$$\varepsilon_\sigma = \varepsilon_\sigma(w^L, w^\nu, w_1^\nu, w_2^\nu, w_3^\nu). \tag{5.11}$$

The basic local theory is well known. It was discovered by direct calculation that the system of functions ε_σ is variational if and only if the Helmholtz expressions (4.31) vanish identically,

$$\begin{aligned}
&\frac{\partial \varepsilon_\sigma}{\partial w_3^\nu} + \frac{\partial \varepsilon_\nu}{\partial w_3^\sigma} = 0,\\
&\frac{\partial \varepsilon_\sigma}{\partial w_2^\nu} - \frac{\partial \varepsilon_\nu}{\partial w_2^\sigma} - \frac{3}{2}\frac{d}{dw^L}\left(\frac{\partial \varepsilon_\sigma}{\partial w_3^\nu} - \frac{\partial \varepsilon_\nu}{\partial w_3^\sigma}\right) = 0,\\
&\frac{\partial \varepsilon_\sigma}{\partial w_1^\nu} + \frac{\partial \varepsilon_\nu}{\partial w_1^\sigma} - \frac{d}{dw^L}\left(\frac{\partial \varepsilon_\sigma}{\partial w_2^\nu} + \frac{\partial \varepsilon_\nu}{\partial w_2^\sigma}\right) = 0,\\
&\frac{\partial \varepsilon_\sigma}{\partial w^\nu} - \frac{\partial \varepsilon_\nu}{\partial w^\sigma} - \frac{1}{2}\frac{d}{dw^L}\left(\frac{\partial \varepsilon_\sigma}{\partial w_1^\nu} - \frac{\partial \varepsilon_\nu}{\partial w_1^\sigma} - \frac{1}{2}\frac{d^2}{d(w^L)^2}\left(\frac{\partial \varepsilon_\sigma}{\partial w_3^\nu} - \frac{\partial \varepsilon_\nu}{\partial w_3^\sigma}\right)\right) = 0.
\end{aligned} \tag{5.12}$$

Originally, these expressions were discovered for functions ε_σ of the form $\varepsilon_\sigma = \varepsilon_\sigma(t, q^\nu, \dot{q}^\nu, \ddot{q}^\nu)$ by Helmholtz.

In this work we derived (5.12) from the variational sequence theory, and we generalized the results to second order variational sequence. In particular, the observation that the mapping assigning to the expressions ε_σ (5.11) the Helmholtz expressions (5.12) is a part of the variational sequence (see Sect. 4.3, (4.7), and Sect. 4.4, Theorem 4.9, (b), and higher order analogues), allows us to discuss the concept of global variationality by the same way as in the fibred case (Krupka [27]).

The variational sequence theory allows us to assign to an arbitrary 2-form ρ defined on $\mathrm{Imm}\, T^1Y$ its class $[\rho]$ in the quotient Ω_2^1/Θ_2^1. The factorization defines an expression for $[\rho]$ explicitly. If ρ is of the form $d\eta$ for some 1-form η on $\mathrm{Imm}\, T^1Y$, whose class is $[\eta] = \mathscr{L}_L dw^L$, then the class $[d\rho]$ is uniquely determined by the functions (5.12). For the second order, compare with (4.33).

Thus, the variationality problem for differential equations on Grassmann fibrations is locally described in adapted formulas similarly as in the fibred case. Differences arise in the structure of classes, see Remark 13.

The existence of the interpretation of the Helmholtz mapping as one arrow of an exact sequence allows us to understand differences between local and global aspects of the inverse variational problem for submanifolds; the differences are characterized by the cohomology of the manifold G^rY.

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