Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

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Samodlážditelné simplexy

Katedra aplikované matematiky

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Studijní program: Matematika, Matematické struktury

2010

I would like to express my sincere thanks to the supervisor of my thesis Prof. Jiří Matoušek for inspirational consultations, valuable advice, comments and motivation as well as for his effort to improve my English. Naturally, I thank my parents for their many sided support during the years of my studies.

Prohlašuji, že jsem svou diplomovou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 12.4.2010 Zuzana Safernová

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Název práce: Samodlážditelné simplexy
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Abstrakt: V předložené práci se zabýváme problémem k-samodlážditelnosti čtyřstěnů. Simplex S je *k-samodlážditelný*, pokud se dá rozdělit na k navzájem shodných simplexů (s disjunktními vnitřky), jež jsou navíc podobné původnímu simplexu S. V rovině jsou všechny k-samodlážditelné trojúhelníky charakterizovány, na druhou stranu jediné k-samodlážditelné simplexy v dimenzi $d \geq 3$ jsou známy pro hodnotu $k = m^d$, kde $m \geq 2$, tzv. *Hillovy simplexy*.

V práci dokážeme, že v dimenzi 3 existují k-samodlážditelné čtyřstěny *pouze* pro $k = m^3$, což částečně potvrzuje Hertelovu domněnku, že jediné k-samodlážditelné čtyřstěny jsou Hillovy. Domníváme se, že $k = m^d$ je nutná podmínka pro existenci k-samodlážditelných simplexů ($d > 3$).

Klíčová slova: simplex, k-samodlážditelnost, čtyřstěn

Title: Reptile simplices
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Abstract: In the present work we study tetrahedral k-reptiles. A d-dimensional simplex is called a *k-reptile* if it can be tiled in k simplices with disjoint interiors that are all congruent and similar to S. For $d = 2$, triangular k-reptiles exist for many values of k and they have been completely characterized. On the other hand, the only simplicial k-reptiles that are known for $d \geq 3$ have $k = m^d$, where $m \geq 2$ (*Hill simplices*).

We prove that for $d = 3$, tetrahedral k-reptiles exist *only* for $k = m^3$. This partially confirms the Hertel's conjecture, asserting that the only tetrahedral k-reptiles are the Hill tetrahedra. We conjecture that $k = m^d$ is necessary condition for existence of d-dimensional simplicial k-reptiles, $d > 3$.

Keywords: simplex, k-reptile, tetrahedron

Chapter 1

Introduction

In this thesis we continue with the research of tetrahedral reptiles, which was initiated in my bachelor thesis.

In this chapter we introduce the problem of *simplicial reptiles*, and we describe its history and known results. At the end we present the results obtained in this thesis.

1.1 History and motivation

We start with brief history of reptiles and related areas.

A reptile is a geometric figure such that copies (replicas) of the figure fit together to form a larger similar figure. For example, it is easy to see that four congruent squares fit together to form another square. It may look as a funny play for children, on the other hand, by repeating this replicating process infinitely many times with still larger squares, we can tile the plane, and tiling the plane has many interesting applications.

What happens if we want to tile the space? This is a fascinating question and one of the oldest geometric problems. It has a rich history. We restrict ourselves to tiling the space by congruent polyhedra. *Which polyhedra can tile the space?* This question arose first in ancient times in relation to Plato. Aristotle incorrectly thought that not only the cube but also the regular tetrahedron could fill space. Aristotle had big influence and authority, so when many of his later followers realized that something was wrong, they assumed that somehow they must be mistaken. Later they asked themselves which tetrahedra do actually tile space. Many of methods developed by these scholars are still used today in the study of space-tiling polyhedra. The early history of the space-tiling problem was discussed in detail by Struik in 1925 [1].

Figure 1.1: The Menger sponge

The tiling problem arose from nature, because natural structures in their ideal forms provide physical models of mathematical concepts. The space-filling model also provides useful interpretations of crystal structures, thus the problem is a subject of research by crystallographers as well as by mathematicians. We could mention M. J. M. Hill and E. S. Fedorov. Hill worked on determining which tetrahedra are congruent by dissection to cubes. He is also known for his achievement in the field of crystallography. From his work we know that a space-tiling tetrahedron is congruent by dissection to a cube [8]. Fedorov, a great Russian crystallographer and geometer, proved that the convex *parallelohedra* (convex polyhedra that could tile space with only translations) can be classified into five types: the cube, the hexagonal prism, a dodecahedron with eight rhombic and four hexagonal faces, the rhombic dodecahedron, and the truncated octahedron. A simplified proof of this theorem can be found in [2].

The crystallography and the reptiles are also closely associated with fractals. In fact, many fractals are constructed to be reptiles. Probably the simplest example in three dimension is the Menger sponge, a generalization of the Sierpinski carpet. See Figure 1.1 – it is taken from [21].

The space-filling is connected with Hilbert's problems, namely with the first part of 18th problem, which asks, whether there exists a polyhedron that tiles the 3-dimensional Euclidean space but does not admit an isohedral (tile-transitive) tiling. The first such tile in three dimensions was found by Karl Reinhardt in 1928. The second part of Hilbert's 18th problem is about densest sphere packing, also known as Kepler's conjecture.

Many people were fascinated by space-tiling polyhedra, among others Goldberg, Grünbaum, Shephard, Wells and Conway.

The first systematic study of space-tiling tetrahedra was made by D.M.Y. Sommerville. The immediate inspiration for his study of tetrahedra was an error made by a student. Sommerville wrote, "In the answer to the book-work question, set in a recent examination to investigate the volume of a pyramid, one candidate state that the three tetrahedra into which a triangular prism can be divided are *congruent*, instead of only equal in volume. It was an interesting question to determine the conditions in order that the three tetrahedra should be congruent, and this led to the wider problem – to determine what tetrahedra can fill up space by repetitions." [3].

Sommerville [4] discovered a list of exactly four tilings (up to isometry and rescaling), but he assumed that all tiles are properly congruent (i.e., congruent by an orientation-preserving isometry) and meet face-to-face. The gap at the end of his proof was patched by Edmonds [5] in 2007. In the non-proper and non face-to-face situations there are infinite families of non-similar tetrahedral tilers [9].

Many other famous people were interested in tetrahedra-tiling, for example H.S.M. Coxeter, H.L. Davies, L. Baumgartner, M.Goldberg and E. Koch.

More about the history of space-filling questions can be found in the paper of Senechal [10].

For additional information about reptiles and their connection to tiling, see Golomb [6], Gardner [7], Gelbrich [15] and Bandt [14].

In recent years the subject of tilings has received a certain impulse from computer graphics and other computer applications. In fact, our main motivation for studying simplices that are k-reptiles comes from a paper by Adler [11] on probabilistic marking of Internet packets. Matoušek says [13] that from this point of view, it would be interesting to find a d-dimensional simplex that is a k-reptile with k as small as possible.

In this thesis we will focus our attention on tetrahedra which can be reptiles. We will prove that tetrahedron can be a k-reptile only for k of the form m^3.

The thesis is organised as follows: In the rest of this chapter we present known results about simplices that are reptiles, and at the end we introduce our contribution to this problem and state our conjecture. Chapter 2 is devoted to theorems which will be useful for our proof. Namely, the first part will be about scissor congruence and connection with Hilbert 3rd problem, the second part will be about the geometry of a tetrahedron, and in the third part we focus on theory of angles that are rational multiples of π. In Chapter 3 we will present the proof of the theorem mentioned above, i.e., a necessary condition for k such that there exists a tetrahedron that is a k-reptile.

1.2 Formulation of the problem

Recall that a *d-dimensional simplex* is the convex hull of $(d+1)$ affinely independent points in $\mathbb{R}^d$.

Definition 1. *A d-dimensional simplex S is called a* k-reptile *if there exist d-simplices $S_1, S_2, \ldots, S_k$ with disjoint interiors and with $S = S_1 \cup S_2 \cup \ldots \cup S_k$ that are all mutually congruent and similar to S.*

This definition prompts a question:

Question 2. *For what k and d there exist d-dimensional simplicial k-reptiles?*

1.3 Known results

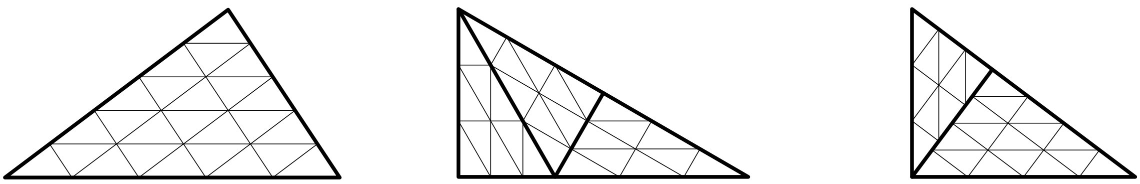
For $d = 2$ the question was completely solved by Snover, Waiveris and Williams [16]:

Theorem 3. *The triangle S is a k-reptile if and only if:*

(a) $k = m^2$ ($m \geq 2$ and S is an arbitrary triangle) or

(b) $k = 3m^2$ ($m \geq 1$ and S is a right triangle with angles $\frac{\pi}{3}$ and $\frac{\pi}{6}$) or

(c) $k = m^2 + l^2$ ($m, l \geq 2$ and S is a right triangle whose two shorter sides have ratio $k : l$).



(a) $k = m^2$ (b) $k = 3m^2$ (c) $k = m^2 + l^2$

Figure 1.2: Triangular k-reptiles in plane

This statement says (in case (a)) that each triangle is a k^2-reptile. The corresponding dissection is called *standard* [12]: We divide each side of the triangle by $k - 1$ points into k parts of equal length. Then dissect S by straight lines through these points parallel to the sides of S.

One can think that the standard dissection of a triangle into four smaller triangles can be generalized to a dissection of tetrahedron, but the situation is more complicated, because when we cut off the corners of tetrahedron we obtain an octahedron, which can almost never be tiled by congruent tetrahedra. It is known that the standard tiling exists only if the original tetrahedron is a Hill tetrahedron.

Definition 4. *A d-dimensional* Hill simplex *is the convex hull of the vectors* $0, b_1, b_1+b_2, \ldots, b_1 + \ldots + b_d$, *where* $b_1, \ldots, b_d$ *are vectors of equal length such that the angle between any two of them is the same and lies in the interval* $(0, \frac{2\pi}{3})$ *(see [13]).*

In Figure 1.3 there is the decomposition of a Hill tetrahedron with $(b_1, b_2, b_3) = (e_1, e_2, e_3)$ the standard orthonormal basis, into 8 congruent pieces.

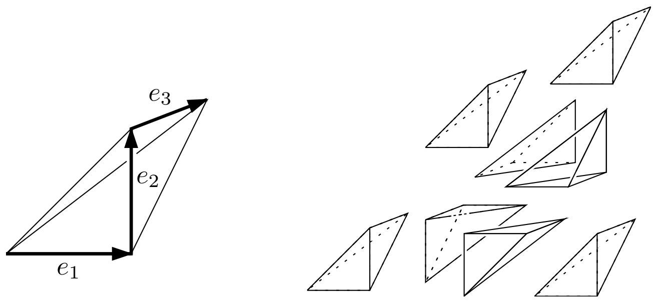


Figure 1.3: A Hill tetrahedron is an 8-reptile

Hertel proved in [12] that a 3-dimensional simplex S is a k^3-reptile using the standard dissection if and only if S is a Hill simplex. He conjectures that Hill simplices are the only 3-dimensional simplicial reptiles.

Another partial result was obtained by Matoušek [13]: For $d \geq 3$ no d-dimensional simplex is a 2-reptile. He also mentioned the possibility of trying to prove nonexistence of simplicial m-reptiles for $m < 2^d$.

Finally, in my bachelor thesis I showed that no 3-dimensional tetrahedron can be a 3-reptile.

1.4 Goals of this thesis

In this thesis we continue with studying 3-dimensional simplicial k-reptiles. The main result of this thesis is the following theorem:

Theorem 5. *A 3-dimensional simplex (tetrahedron) can be a k-reptile only for k of the form m^3 for some positive integer m.*

This partially confirms the Hertel's conjecture (the only tetrahedral k-reptiles are the Hill tetrahedra).

Consequence 6. *There is no 3-dimensional tetrahedral k-reptile for $k < 8$. Thus the smallest k for which there can exist a tetrahedral k-reptile is 8. And such a tetrahedron really exists – at least the Hill one.*

Furthermore, we wonder whether $k = m^d$ is a necessary condition for existence of simplicial d-dimensional k-reptiles.

Conjecture 7. *A d-dimensional simplex can be a k-reptile only for k of the form m^d for some positive integer m ($d > 3$).*

If there is an affirmative answer, we will know that the least k for which there exist simplices that are k-reptiles is 2^d.

Chapter 2

Preliminaries

In this chapter we present statements, which will be useful later for proving our main theorem. First we focus on polytopes and the scissor congruence—this gives a connection between rectifiable polytopes and tetrahedral reptiles. Then we focus on the geometry of a tetrahedron, and the last part will be devoted to properties of rational angles.

2.1 The scissor congruence

Here we introduce the notions and definitions related to Hilbert's third problem. See, Pak's book [17], chapters 15–16 for more background, or Sah's book [18] entirely devoted to this topic.

Hilbert's 3rd problem was the first in Hilbert's famous problem list to be solved. It is related to the following question: *Given any two polyhedra of equal volume, can the first one be cut into finitely many polyhedral pieces so that they can be reassembled to form the second one?* It was known that for planar polygons it is true – this is the well-known Bolyai-Gerwien theorem. Hilbert expected the negative answer for the three dimensional version of the problem, which was confirmed within a year by his student Max Dehn giving a counterexample [19], [20].
Recall that a *convex polytope* is the convex hull of a finite set of points. A *convex polyhedron* is a 3-dimensional convex polytope. General *polytopes* are defined as finite unions of convex polytopes. Two polytopes $P, Q \in \mathbb{R}^3$ are *scissor congruent* if P can be cut into finitely many smaller polytopes which can be rearranged and assembled into Q. Thus the Hilbert third problem can be reformulated: Are two polytopes of the same volume always scissor congruent? Dehn showed that a regular tetrahedron is not scissor congruent to a cube of the same volume.

It leads to the following definitions:

We say that a polytope in $\mathbb{R}^3$ is *rectifiable* if it is scissor congruent to a cube of equal volume, and a polytope $P \in \mathbb{R}^3$ is called *self-similar* if it is scissor congruent to a disjoint union of two or more polytopes similar to P.

By a *dihedral angle* of a polytope (also called the face angle) we mean the internal angle at which two adjacent faces meet.

A polytope P in $\mathbb{R}^3$ is *fortunate* if π can be written as a positive rational combination of its dihedral angles β_i:

$$c_1\beta_1 + \cdots + c_m\beta_m = \pi, \qquad c_i > 0, c_i \in \mathbb{Q} \text{ and } i = 1, \ldots, m,$$

where m is number of edges in P. Otherwise, P is *unfortunate*.

Note that each rational coefficient in previous definition is strictly greater than zero.

The Bricard condition says that an unfortunate polytope in $\mathbb{R}^3$ is not scissor congruent to a cube.

Dehn's counterexample immediately follows from this condition. It is not hard to show that the regular tetrahedron is an unfortunate polytope. Indeed, the Bricard condition is a very special consequence of the Dehn's invariants, which are global invariants involving dihedral angles and edge lengths.

Sydler's criterion states that a polytope $P \subset \mathbb{R}^3$ is rectifiable if and only if P is self-similar.

If we turn back to tetrahedral reptiles, we can conclude:

Lemma 8. *Let tetrahedron S be a k-reptile. Then the positive rational combination of dihedral angles of S equals π.*

Proof. Since the k-reptile tetrahedron S is a self-similar polytope in $\mathbb{R}^3$, it is rctifiable due to Sydler's criterion S. According to the Bricard condition S is fortunate and the lemma follows. □

2.2 The geometry of a tetrahedron

In this section we show some necessary conditions for the existence of a tetrahedron of given properties.

The next observation is well-known; it can be found for example in [5].

Observation 9. *The three dihedral angles at any vertex of a tetrahedron have sum greater than π.*

Proof. The intersection of a tetrahedron with a small sphere (unit for convenience) centered at a vertex yields a spherical triangle whose angles are the dihedral angles centered at that vertex. The surface area of the spherical triangle is the sum of its angles minus π and thus dihedral angles add up to more than π. ☐

There are two useful statements about simplices by Fiedler, which can be found in [22]. Before stating them we define some notation. Given a simplex S with vertices v_i $(i = 1, \dots, d+1)$, we will use the notation F_i for the facet opposite to v_i and φ_{ij} for the dihedral angle formed by the facets F_i, F_j.

Theorem 10. *(Fiedler) Let φ_{ij} $(i, j = 1, \dots, d+1)$ be the dihedral angles of a d-dimensional simplex. Then the matrix*

$$A = \begin{pmatrix} -1 & \cos\varphi_{1,2} & \dots & \cos\varphi_{1,d+1} \\ \cos\varphi_{2,1} & -1 & \dots & \cos\varphi_{2,d+1} \\ \vdots & & \ddots & \vdots \\ \cos\varphi_{d+1,1} & \dots & \dots & -1 \end{pmatrix}$$

satisfies the following conditions:

- *A is negative semidefinite of rank d.*
- *The kernel of A is one-dimensional and it is generated by a vector with strictly positive coordinates, i.e. there exists a vector w, $w^T = (w_1, \dots, w_{d+1})$, such that $w_i > 0$ for $i = 1, 2, \dots, d+1$ and $Aw = 0$.*

The proof can be found in Fiedler's book [22], chapter 4.

Now we prove a straightforward but useful lemma:

Lemma 11. *Let A be a matrix as above. Let vector v be a linear combination of rows of A such that only two coordinates of v are nonzero. Then the values at these coordinates must have the opposite signs.*

Proof. Let $v = a_1 v_1 + \dots + a_n v_n$, where v_i are rows of A and $a_i \in \mathbb{Z}$, $i = 1, 2, \dots, n$, $n \leq d+1$. Suppose that w is an arbitrary nonzero vector from the kernel K of the matrix A. Thus $v_i \cdot w^T = 0$, $i = 1, 2, \dots, n$, so $v \cdot w^T = 0$. Since the vector w has strictly positive coordinates (by the Fiedler theorem) and the vector v has only two nonzero coordinates (by the assumption), we get that these coordinates must have the opposite signs (otherwise, w has coordinates with opposite signs or zeros). ☐

The second statement by Fiedler is:

Theorem 12. *If two simplices in $\mathbb{R}^d$ coincide with length of one edge and in the values of $\binom{d+1}{2} - 1$ corresponding dihedral angles, then they are congruent.*

The proof can be found in Fiedler's book [22].

2.3 Rational angles

By a *rational angle* we mean an angle that is a rational multiple of π, i.e., it can be written in the form $q\pi$, where $q \in \mathbb{Q}$. Here we investigate which values can be obtained from the cosine function at rational angles.

A *quadratic irrationality* is an irrational number that is a solution to some quadratic equation with rational coefficients. Since we can get rid of fractions from a quadratic equation by multiplying both sides by their common denominator, this is the same as saying that quadratic irrationality is an irrational root of some quadratic polynomial with integer coefficients. Therefore, quadratic irrationalities are all those numbers that can be expressed in the form $\frac{a+b\sqrt{c}}{d}$ for integers a, b, c, d with b and d nonzero and with $c > 1$ and square-free. Analogously we can define *cubic* and *quartic irrationality* as an irrational solution of some cubic and quartic equation with integer coefficients, respectively.

The following statement by Jahnel is from [23]:

Theorem 13. *(Jahnel) Let $\alpha = \frac{m}{n} \cdot 2\pi$ be a rational angle. Assume that $m, n \in \mathbb{Z}$, $n \neq 0$ have no common factors. Then*

1. *$\cos\alpha$ is a rational number if and only if $\varphi(n) \leq 2$, i.e. for $n = 2, 3, 4,$ and 6;*
2. *$\cos\alpha$ is an algebraic number of degree $d > 1$ if and only if $\varphi(n) = 2d$.*

Here φ means Euler's totient function.

Recall that Euler's totient function $\varphi(n)$ is defined to be the number of integers $\{1, 2, \ldots, n\}$ that are coprime to n. The integers m, n are *coprime* if they have no prime factors in common, i.e., $\gcd(m, n) = 1$, where gcd denotes the greatest common divisor of m and n. It is easy to see that $\varphi(1) = 1$ and $\varphi(p) = p - 1$ for p prime.

We will present a proof of the Jahnel theorem, because it is a nice illustration of cyclotomic fields. Furthermore, we explain it in more detail than it is in the paper [23].

Proof. A *cyclotomic field* $\mathbb{Q}(\zeta)$ is a field obtained by adjoining a complex primitive root of unity ζ to the rational numbers $\mathbb{Q}$. An n-th root of unity z is *primitive* if $z^m \neq 1$ for all $m \in \{1, 2, \ldots, n-1\}$. It is clear that one of the primitive roots of unity is $e^{\frac{2\pi i}{n}}$, denoted by ζ_n. Observe that the others primitive roots are $\zeta_n^m := e^{\frac{m}{n}2\pi i}$ for m, n coprime. Indeed, if $\gcd(m, n) = 1$ then there exist integers a, b such that $am + bn = 1$ (Bézoute's identity) and ζ_n^m is the m-th power of ζ_n. We can conclude that $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_n^m)$ for m, n coprime.

The degree of the minimal polynomial Φ_n of the cyclotomic field $\mathbb{Q}(\zeta_n)$ is $\varphi(n)$, because

$$\Phi_n(x) = \prod_{\substack{1 \leq j \leq n, \\ \gcd(j,n)=1}} (x - \zeta_n^j).$$

The well-known formula $\cos\alpha = \cos\frac{2\pi m}{n} = \frac{\zeta_n^m + \zeta_n^{-m}}{2}$ implies that ζ_n^m solves the equation

$$x^2 - 2(\cos\alpha)x + 1 = 0 \qquad \text{over } \mathbb{Q}(\cos\alpha).$$

By the assumption m, n are coprime, thus $[\mathbb{Q}(\zeta_n) : \mathbb{Q}(\cos\alpha)] = 1$ or 2.

Since $\mathbb{Q}(\cos\alpha) \subseteq \mathbb{R}$, we get that $[\mathbb{Q}(\zeta_n) : \mathbb{Q}(\cos\alpha)] = 1$ only if $\zeta_n \in \mathbb{R}$, i.e., only for $n = 1, 2$. Otherwise,

$$d = [\mathbb{Q}(\cos\alpha) : \mathbb{Q}] = \frac{[\mathbb{Q}(\zeta_n) : \mathbb{Q}]}{[\mathbb{Q}(\zeta_n) : \mathbb{Q}(\cos\alpha)]} = \frac{\varphi(n)}{2},$$

and the proof is finished. □

The Euler's function satisfies the following inequality [24]:

Theorem 14. *(Kendall, Osborn)* $\varphi(n) \geq \sqrt{n}$, *for* $n \neq 2$ *and* $n \neq 6$.

Note that $\varphi(2) = 1$ and $\varphi(6) = 2$.

With regard to this theorem it is now easily possible to list all quadratic, cubic and quartic irrationalities that occur as special values of the cosine function. In fact, we will need only quadratic and quartic irrationalities, so we will investigate only these two cases.

For the quadratic irrationalities $\varphi(n) = 4$, thus according to Theorem 14, $n \leq 16$. It can be easily checked that appropriate values of n are $5, 8, 10$ and 12.

Similarly for the quartic irrationalities $\varphi(n) = 8$ we can restrict to $n \leq 64$. The appropriate values of n in this case are $15, 16, 20, 24$ and 30.

The possible rational angles corresponding to quadratic and quartic irrationalities are summarized in Table 2.1.

Table 2.1: Possible rational angles α

(a) quadratic irrationalities

$0^\circ < \alpha < 180^\circ$			
$\varphi(n)$	n	angles α	values of $\cos\alpha$
4	5	$72^\circ, 144^\circ$	$\frac{-1+\sqrt{5}}{4}, \frac{-1-\sqrt{5}}{4}$
	8	$45^\circ, 135^\circ$	$\pm\frac{\sqrt{2}}{2}$
	10	$36^\circ, 108^\circ$	$\frac{1+\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4}$
	12	$30^\circ, 150^\circ$	$\pm\frac{\sqrt{3}}{2}$

(b) quartic irrationalities

$60^\circ < \alpha < 120^\circ$		
$\varphi(n)$	n	angles α
8	15	96°
	16	$67\frac{1}{2}^\circ, 112\frac{1}{2}^\circ$
	20	–
	24	$75^\circ, 105^\circ$
	30	84°

Chapter 3

Proof of Theorem 5

The proof presented in this chapter is based on the irrationality of $k^{1/3}$ and characterization of dihedral angles.

3.1 Outline of the proof

Fix k not of the form m^3 and assume for contradiction that S is a tetrahedral k-reptile. Let f_i be the similarity map sending S to S_i, $i = 1, 2, \ldots, k$. Thus f_i is an isometry followed by scaling in the ratio $k^{-1/3}$ (since the volume of S_i is k-times smaller than volume of S). Set $\alpha = k^{1/3}$; the map f_i reduces the length of edge by α^{-1}. That is, the length of image of edge e is α-times smaller than the length of e. The proof of Theorem 5 goes as follows:

1. First we show that α is irrational. This will play a key role in the whole proof.

2. Then we show that there must be at least three edges corresponding to the same dihedral angle in S.

3. Since a positive rational combination of dihedral angles of S must be equal to π (according to Lemma 8), we show that it suffices to distinguish two cases:

   (a) All dihedral angles in S are multiples of the minimal dihedral angle, which is of the form $\frac{\pi}{n}$. To exclude this case we use Lemma 11 and the Fiedler theorem.

   (b) There are only two different dihedral angles in S. We exclude this by the help of the Fiedler theorem and the knowledge of values of the cosine function at rational angles (Table 2.1).

3.2 Step 1 – the irrationality of α

Observation 15. *For k not of the form m^3, $\alpha = k^{1/3}$ is irrational.*

Proof. The proof is simple. Assume for contradiction that $k^{1/3} = \frac{p}{q}$ for $p, q \in \mathbb{N}$ coprime ($k \neq m^3$ thus q cannot be 1). Cubing both sides we get: $k = \frac{p^3}{q^3}$, but k is an integer, so $q^3 | p^3$ and thus also $q | p$, which is a contradiction. □

3.3 Step 2 – indecomposable angles

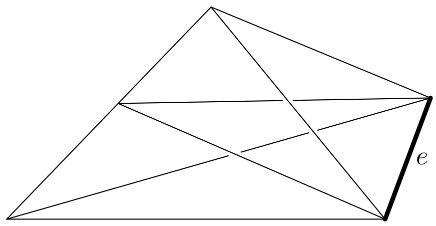


Figure 3.1: The dihedral angle corresponding to the bold edge is not indecomposable

Choose one of the dihedral angles that cannot be subdivided [1] and denote it by γ. Sometimes we will call such an angle *indecomposable* in S. Therefore, situation from Figure 3.1 cannot happen for an indecomposable dihedral angle.

Lemma 16. *There are at least three edges with dihedral angle γ in S. Moreover, no two these edges have the same length.*

Proof. First observe that there are at least two edges with dihedral angle γ in S because no edge can be subdivided by itself while α^{-1} is irrational (by Observation 15). In more detail, assume for contradiction there is only one edge e with dihedral angle γ in S. From the definition of a k-reptile (see Definition 1) there exists a subdivision of S into smaller congruent tetrahedra. Consider such a subdivision. Because γ is indecomposable in S and e is the only edge corresponding to γ, we can conclude that in this subdivision e can consist only of its smaller copies (images). Recall that the length of image of edge e is α-times smaller than the length of e. Therefore, there exists k natural such that $|e| = k\alpha^{-1}|e|$, where $|e|$ denotes the length of e. This is a contradiction, because α is irrational.

[1] it means there are no dihedral angles $\delta_1, \delta_2, \ldots, \delta_m$ different from γ in S such that $\gamma = \sum_{i=1}^{m} l_i \delta_i$, where $l_i \in \mathbb{N}$

Now suppose there are only two edges e_1, e_2 with dihedral angle γ; denote their lengths by x_1, x_2. We proceed analogously as above. Consider a subdivision of S into smaller congruent tetrahedra from the definition of a k-reptile. The edge e_1 is composed only of smaller copies of the edges e_1, e_2 and so is the edge e_2. Thus there exist $k_1, k_2, l_1, l_2 \in \mathbb{N}_0$ such that

$$\begin{aligned} k_1\alpha^{-1} \cdot x_1 + k_2\alpha^{-1}x_2 &= x_1 \\ l_1\alpha^{-1} \cdot x_1 + l_2\alpha^{-1}x_2 &= x_2. \end{aligned}$$

This means that the edge e_1 is constructed from k_1 parts of the image of e_1 and k_2 parts of the image of e_2, and similarly, e_2 is constructed from l_1 parts of the image of e_1 and l_2 parts of the image of e_2.

We want to describe a subdivision of S so we try to find some restrictive conditions for k_1, k_2, l_1, l_2. In this setting x_1, x_2 are unknowns. One condition for solving this system of linear equations is that the determinant of left side must be equal to zero, which yields the following equation:

$$\alpha^2 - (k_1 + l_2)\alpha + k_1 l_2 - l_1 k_2 = 0.$$

After substitution $p := -(k_1 + l_2), q := k_1 l_2 - l_1 k_2$ we get:

$$\alpha^2 + p\alpha + q = 0, \tag{3.1}$$

where $p, q \in \mathbb{Q}$. We check that $p \neq 0 \neq q$. Indeed, if $p = 0$ then $\alpha^2 + q = 0$ which is impossible (α^2 is irrational). If $q = 0$ then $\alpha^2 + p\alpha = 0$. It means that $\alpha + p = 0$ which is also a contradiction. Multiplying the equation 3.1 by the factor $q\alpha$ and $p\alpha^2$, respectively, and using fact that $\alpha^3 = k$ we get:

$$qp\alpha^2 + q^2\alpha + qk = 0 \tag{3.2}$$

$$qp\alpha^2 + kp\alpha + p^2k = 0. \tag{3.3}$$

Subtracting (3.2) - (3.3):

$$(q^2 - kp)\alpha + (qk - p^2k) = 0$$

Since α is irrational we get $q^2 = kp$ and $qk = p^2k$. We know that $p \neq 0$, thus $k = p^3$, but $k = \alpha^3$ which is a contradiction (p is rational, α isn't). □

Consequence 17. *There are at most* 4 *different dihedral angles in* S.

Proof. It suffices to show that there is at least one indecomposable angle in S. Indeed, consider the minimal dihedral angle in S. □

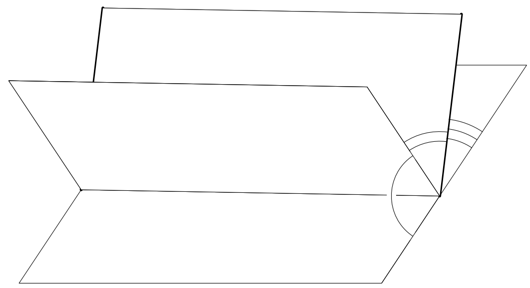


Figure 3.2: The sum of dihedral angles inside a face of S is equal to π

3.4 Step 3 – the elimination of cases

Observation 18. *Consider a subdivision of S as in Definition 1. The dihedral angles around any edge of this subdivision inside a face of S sum to π (see Fig. 3.2). This sum is a non-negative integral combination of dihedral angles of S.*

Note that there is always a face of a k-reptile with an edge inside. We can consider a face with maximal area.

Lemma 19. *There are exactly two different dihedral angles in S (each of them corresponds to 3 edges of S), or every dihedral angle is a multiple of the minimal dihedral angle β, which is of the form $\frac{\pi}{n}$ for some natural n.*

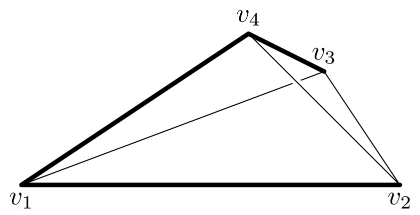
Proof. Let φ be the smallest dihedral angle that is not a multiple of the minimal dihedral angle β. Thus φ cannot be subdivided in S. According to Lemma 16, φ corresponds to at least three edges in S, and so does β, and therefore, there are exactly two different dihedral angles β and φ in S.

It remains to prove that if every dihedral angle is a multiple of β, then there is a natural number n such that $\beta = \frac{\pi}{n}$. Consider dihedral angles around an edge inside a face with maximal area. These angles sum to π (Observation 18), and so there exist natural numbers $m_1, \ldots, m_i$ for a suitable i such that $m_1\beta + \ldots + m_i\beta = \pi$. For $n := m_1 + \ldots + m_i$ we get that $\beta = \frac{\pi}{n}$. □

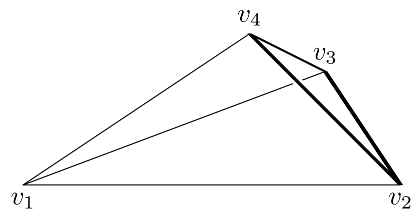
From now on, let us denote the minimal dihedral angle in S by β.

Lemma 8 says that there exists a positive rational combination of the dihedral angles of S equal to π. In particular, it means that there cannot be only one irrational multiple of π. So we can conclude:

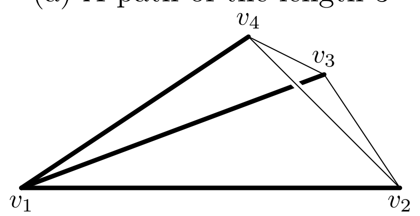
Observation 20. *If there are two different dihedral angles in S, then either both of them are rational multiples of π or both of them are irrational multiples of π.*



(a) A path of the length 3



(b) A triangle



(c) A tripod

Figure 3.3: The bold edges correspond to the minimal dihedral angle of S

3.4.1 Multiples of the minimal dihedral angle β

We know that $\beta = \frac{\pi}{n}$ for some n. The angle β is minimal, thus $\beta < \frac{\pi}{2}$, and so $n \geq 3$. Otherwise, all dihedral angles in S would be obtuse (greater than or equal to $\frac{\pi}{2}$) which is not possible.

Observation 21. *There is no vertex in S whose three edges all have the minimal dihedral angle β.*

Proof. Assume for contradiction that there exists such a vertex v. According to Observation 9 the sum of dihedral angles at the vertex v must be greater than π, so $\beta > \frac{\pi}{3}$. On the other hand, $\beta < \frac{\pi}{2}$, thus there is no β of the form $\frac{\pi}{m}$. □

Thus there are only two possibilities for placing β into S (see, Fig. 3.3a, Fig. 3.3b). The bold edges in the figures correspond to the minimal dihedral angle β. The other dihedral angles are multiples of β (some of them could also be β).

Let φ be a minimal dihedral angle in S such that the following condition holds:

$$2\beta + \varphi > \pi.$$

Note that such a φ always exists. Indeed, the inequality corresponds to the sum at the vertex v_1 or v_4 in the case in the Fig. 3.3a, and to the sum at the vertices v_2, v_3, v_4 in the case in the Fig. 3.3b. We express φ as a multiple of β: $\varphi = m\beta$ for a suitable natural number m. Since $\varphi < \pi$, we get $\beta < \frac{\pi}{m}$. On the other hand, $\pi < 2\beta + \varphi = (m+2)\beta$, and so $\beta > \frac{\pi}{m+2}$. Thus we can conclude that $\beta = \frac{\pi}{m+1}$ and

(a) A four-cycle (b) A path

Figure 3.4: The tetrahedron S

$\varphi = \frac{m}{m+1}\pi$. With regard to Observation 21 it is sufficient to distinguish the following tree cases:

1. There are only 2 different dihedral angles in S: Since $\beta + \varphi = \pi$, the case with 3 and 3 edges will be solved separately later. Therefore, the only remaining possibility is when β corresponds to a four-cycle (Fig. 3.4a).

2. There exists ψ such that $\beta < \varphi < \psi$. Then $\pi > \psi \geq (m+1)\beta$ and thus $\beta < \frac{\pi}{m+1}$ which is a contradiction with $\beta = \frac{\pi}{m+1}$.

3. Neither case 1. nor case 2. happened. Then there exists ψ such that $\beta < \psi < \varphi$ and $2\beta + \psi \leq \pi$. The only possibility of such an S is in Figure 3.4b.

It remains to show that such tetrahedra (Fig. 3.4a, 3.4b) cannot exist. We exclude these cases at once, with the aid of the Fiedler theorem and Lemma 11.

Denote $\cos\beta$ by t $(0 < t < 1)$; then $\cos\varphi = -t$ since $\beta + \varphi = \pi$. According to the Fiedler theorem we construct a matrix corresponding to S.

$$A = \begin{pmatrix} -1 & t & -t & s \\ t & -1 & t & -t \\ -t & t & -1 & t \\ s & -t & t & -1 \end{pmatrix}.$$

The matrix A corresponds to the tetrahedron with a four-cycle for $s = t$ (Fig. 3.4a) and to the path-tetrahedron for $s = \cos\psi$ (Fig. 3.4b).

Adding second and third rows of A we get the vector $u = (0, -1+t, t-1, 0)$. The second and third coordinate are the same, which is a contradiction to Lemma 11.

3.4.2 Only two dihedral angles in S

There are only three possible placings of dihedral angles in S – Fig. 3.3. The bold edges correspond to the minimal dihedral angle in S.

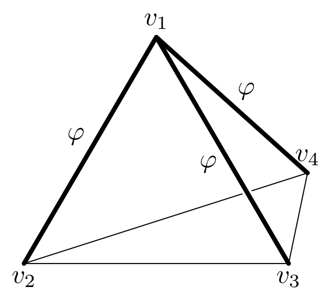


Figure 3.5: The regular pyramid T

First we consider a tetrahedron S with a triangle or with a tripod (if we do not distinguish angles, these two cases are the same) – Fig. 3.3b, 3.3c. Let φ be a dihedral angle corresponding to the tripod and ψ angle corresponding to the triangle. Let $t := \cos\varphi, s := \cos\psi$. Observe that $s \in (0,1)$ because ψ has to be smaller than $\frac{\pi}{2}$ (otherwise the planes F_2, F_3, F_4 would not intersect). Let s be fixed. We want to find some restrictive conditions for t. The matrix A of the tetrahedron S has the form

$$\begin{pmatrix} -1 & s & s & s \\ s & -1 & t & t \\ s & t & -1 & t \\ s & t & t & -1 \end{pmatrix}.$$

The A is singular (semidefinite), thus the determinant of A must be equal to zero and so t is a root of the polynomial:

$$\det(A) = -(t+1)^2(2t + 3s^2 - 1).$$

There are only two roots:

$$t_1 = -1, \quad t_2 = \frac{1-3s^2}{2},$$

but only one of them is in interval $(-1,1)$. The dihedral angle φ is uniquely determined by the angle ψ. Consider a regular pyramid with a tripod and a triangle (see Fig. 3.5). This pyramid T has two different dihedral angles and corresponding edges have the same length. According to Theorem 12, also S is a regular pyramid, but this contradicts Lemma 16 because edge-lengths corresponding to same dihedral angle must be different in a tetrahedral k-reptile.

It remains to exclude a path-tetrahedron (Fig. 3.3a). Let φ, ψ be the dihedral angles in S. Observe that

$$\begin{aligned} 2\varphi + \psi &> \pi & (3.4) \\ 2\psi + \varphi &> \pi, & (3.5) \end{aligned}$$

where (3.4) and (3.5) are the sums of dihedral angles at a vertex v_1 and v_2, respectively.

By Observation 18 there exist $m, n \in \mathbb{N}_0$ such that $m\varphi + n\psi = \pi$. If φ, ψ are both irrational multiples of π, we know that $m, n > 0$ (by Observation 20), so $m = n = 1$ and $\varphi + \psi = \pi$. If not, there are 3 possibilities: $m = n = 1$, $m = 0$ or $n = 0$. Therefore, we distinguish two cases:

(a) $\varphi + \psi = \pi$, and

(b) $\varphi = \frac{\pi}{m}$, ψ is a rational multiple of π

Case (a) $\varphi + \psi = \pi$

Without loss of generality $\varphi < \psi$ (note that the equality is not possible). Denote $\cos\varphi$ by t $(0 < t < 1)$; then $\cos\psi = -t$. The matrix A of a path-tetrahedron S has the form:

$$\begin{pmatrix} -1 & t & -t & -t \\ t & -1 & t & -t \\ -t & t & -1 & t \\ -t & -t & t & -1 \end{pmatrix}.$$

Adding the second and third rows of the matrix A we get the vector $(0, -1+t, -1+t, 0)$ which is a contradiction to Lemma 11.

Finally, the last case we solve with regard to knowledge of values of the cosine function at rational angles.

Case (b) $\varphi = \frac{\pi}{m}$, ψ is rational

It remains to exclude a path-tetrahedron (Fig. 3.3a). From (3.4), (3.5) is obvious that $\max(\varphi, \psi) > \frac{\pi}{3}$ and $\min(\varphi, \psi) < \frac{\pi}{2}$, because there must be at least one acute angle in S.

Again we distinguish 2 cases:

i) $\max(\varphi, \psi) = \varphi$, thus $\varphi = \frac{\pi}{2}$,

ii) $\max(\varphi, \psi) = \psi$.

Ad i) Let $t = \cos\psi$ $(0 < t < 1)$. The matrix A is:

$$\begin{pmatrix} -1 & 0 & t & t \\ 0 & -1 & 0 & t \\ t & 0 & -1 & 0 \\ t & t & 0 & -1 \end{pmatrix}.$$

This matrix is singular (semidefinite), thus the determinant must be equal to zero.

$$\det(A) = t^4 - 3t^2 + 1 = 0$$

The only solution from the interval $(0, 1)$ is $t = \frac{\sqrt{5}-1}{2}$, but this is not a value of the cosine at rational angle (see Tab. 2.1a), thus ψ cannot be a rational angle.

Ad ii) Let $t = \cos\varphi, s = \cos\psi$. We know that $\varphi \leq \frac{\pi}{3} < \psi$.
The matrix A corresponding to a path-tetrahedron (Fig 3.3a) has the form:

$$\begin{pmatrix} -1 & t & s & s \\ t & -1 & t & s \\ s & t & -1 & t \\ s & s & t & -1 \end{pmatrix}.$$

The eigenvalues of A are:

$$\lambda_1 = -\frac{(\sqrt{5}+1)t + (1-\sqrt{5})s + 2}{2}, \lambda_2 = \frac{(\sqrt{5}-1)t + (-\sqrt{5}-1)s - 2}{2},$$

$$\lambda_3 = -\frac{\sqrt{5t^2 + 6st + 5s^2} - t - s + 2}{2}, \lambda_4 = \frac{\sqrt{5t^2 + 6st + 5s^2} + t + s - 2}{2}.$$

The matrix A is negative semidefinite, thus all eigenvalues must be nonpositive. In addition, $t \geq \frac{1}{2}$ because $\varphi \leq \frac{\pi}{3}$. Now we can estimate the value of s:

$$\begin{aligned} 0 &\geq \lambda_2 = \frac{(\sqrt{5}-1)t + (-\sqrt{5}-1)s - 2}{2} \\ 0 &\geq (\sqrt{5}-1)t + (-\sqrt{5}-1)s - 2 \geq \frac{(\sqrt{5}-1)}{2} + (-\sqrt{5}-1)s - 2 \\ s &\geq \frac{\sqrt{5}-5}{2(\sqrt{5}+1)} = \frac{5-3\sqrt{5}}{4}, \end{aligned}$$

We used that $t \geq \frac{1}{2}$ and got $s \geq \frac{5-3\sqrt{5}}{4}$. Since the arccosine function is decreasing in the interval $(-1, 1)$, we get $\arccos s \leq \arccos \frac{5-3\sqrt{5}}{4} < 115{,}3°$, and thus $\psi < 116°$. On the other hand, $180° < 2\varphi + \psi < 2\varphi + 116°$, so $\varphi > 32°$. Because $\varphi = \frac{\pi}{n}$ we can conclude that $n = 3, 4, 5$ and so $t = \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1+\sqrt{5}}{4}$.

We can easily compute that:

- Since $t > s$, then $\lambda_1 < \lambda_2$.
- Since $s \in \left(-1, \frac{1}{2}\right)$ and $t \in \left[\frac{1}{2}, 1\right)$, then $\lambda_3 < 0$

The matrix A must be singular (semidefinite), so either $\lambda_2 = 0$ or $\lambda_4 = 0$.

1. $\lambda_2 = 0$ for $s = \frac{5-3\sqrt{5}}{4}$ and $t = \frac{1}{2}$. Since s is not a value of the cosine at rational angle (see Tab. 2.1a), ψ cannot be a rational angle.

2. $\lambda_4 = 0$ leads to the equation

$$s^2 + s(t+1) + t^2 + t - 1 = 0. \tag{3.6}$$

   - For $t_1 = \frac{1}{2}$ it has a solution $s_1 = \frac{-3+\sqrt{13}}{4}$. Again, s_1 is not among admissible values of quadratic irrationalities, see Tab. 2.1a, so ψ cannot be a rational angle.
   - For $t_2 = \frac{\sqrt{2}}{2}$ we get $97° < \psi = \arccos s < 98°$. This value is not among angles corresponding to quartic irrationalities, see Tab. 2.1b. How do we know, that s is a quartic irrationality? We postpone the answer to the end of this section.
   - For $t_3 = \frac{1+\sqrt{5}}{2}$ we get $s_3 = \frac{1-\sqrt{5}}{4}$, thus $\psi = 108°$. In this case both $\varphi = 36°$ and ψ are rational angles, but $2\varphi + \psi = 180°$ which is a contradiction with $2\varphi + \psi > 180°$ (the inequality 3.4).

It remains to show that s from the case (2) is really a quartic irrationality. Substituting $t = \frac{\sqrt{2}}{2}$ into the equation 3.6 we get the equation

$$s^2 + s\left(\frac{\sqrt{2}}{2} + 1\right) + \frac{\sqrt{2}}{2} - \frac{1}{2} = 0.$$

Multiplying by 2 and rearranging we get:

$$\sqrt{2}(s+1) = 1 - 2s - 2s^2 \tag{3.7}$$
$$4s^4 + 8s^3 - 2s^2 - 8s - 1 = 0 \tag{3.8}$$

In general, the squaring of an equation is not an equivalent operation, but the non-equivalent operations may only put on a spurious solution but cannot cancel the original one, thus if s satisfies the equation 3.7, then also satisfies the equation 3.8.

Now we claim that the polynomial on the left side of the equation 3.8 is irreducible over $\mathbb{Q}$, i.e., it has no rational roots. Showing this we will be done, because it will mean that s cannot be a root of a polynomial of degree lower than four.

We will use the *rational root theorem* that states: If a polynomial in x with integral coefficients: $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ has rational roots $\frac{p}{q}$, then $p | a_0$ and $q | a_n$.

By this theorem every rational solution of the equation 3.8 must be among the numbers $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}$. But we can easily check that none of these candidates satisfy the equation 3.8.

We excluded all cases, therefore the proof of Theorem 5 is finished.

Chapter 4

Conclusion

We proved that for $d = 3$, tetrahedral k-reptiles exist *only* for k of the form m^3. This partially confirms the Hertel's conjecture, asserting that the only tetrahedral k-reptiles are the Hill tetrahedra.

We conjecture, that $k = m^d$ is a necessary condition also for existence simplicial k-reptiles in dimensions $d > 3$. The dihedral angles have proven themselves as a useful invariant. It is now a question which invariants will be helpful in higher dimensions. Yet there remains still another open problem whether the standard division is the only one for simplicial m^d-reptiles.

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