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Velký dík patří také mým rodičům za podporu při studiu.

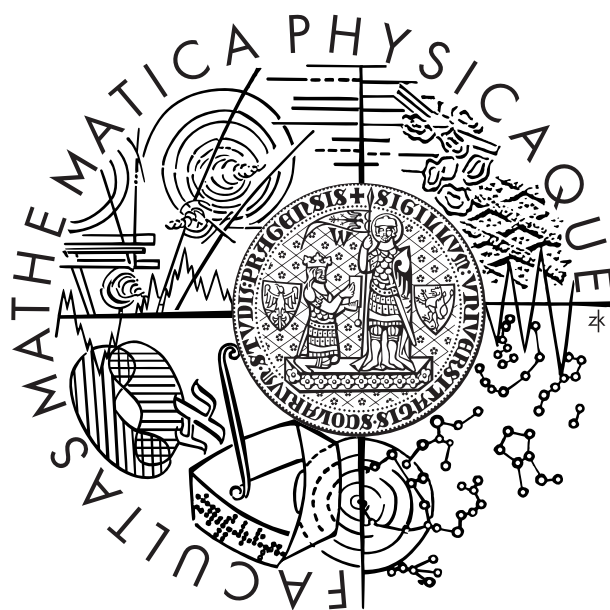
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# DIPLOMA THESIS



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## *Risk measures - dynamics, sensitivity*

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## Abstrakt/Abstract

**Název práce:** Míry rizika - dynamika, citlivost

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**Abstrakt:** Míry rizika jsou tématem mnoha prací, které se věnují optimalizaci portfolia v rámci stochastického programování. V současnosti je navrženo mnoho ukazatelů, které měří riziko dle požadavků správce portfolia. Méně je již prozkoumána jejich citlivost, a to zejména vzhledem ke změně charakteru vstupních dat, popř. vzhledem k alokaci portfolia. Tato práce je věnována studiu citlivosti dvou často diskutovaných měř - Value at Risk (VaR) a Conditional Value at Risk (CVaR). Po uvedení formálních definic a základních vlastností se zabývá aplikacemi kontaminačních technik ve stress testingu těchto měř. S použitím obecných výsledků parametrické optimalizace platných pro kvadratické programování jsou vyjádřeny explicitní kontaminační meze pro úlohu s relativním VaR v účelové funkci, je provedena numerická studie. Je rozšířen heuristický postup pro stresování korelačních matic. Dále je studována citlivost měř pomocí derivací vzhledem k alokaci portfolia. Jsou zformulovány předpoklady, odvozeny Hessiány měř a diskutována konvexita. Poslední část se věnuje dynamickým mírám vhodným pro měření rizika investic na více časových obdobích.

**Klíčová slova:** Míry rizika, Value at Risk, Conditional Value at Risk, citlivost, dynamika.

**Title:** Risk Measures - Dynamics, Sensitivity

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**Abstract:** Risk measures are subject to many scientific papers and monographs published on financial portfolio optimization problem within stochastic programming. Currently there are many functionals which measure risk of random future losses according to risk managers preferences. However, their sensitivity is studied less commonly, especially according to possible changes of input data or with respect to the portfolio allocation. This thesis deals with sensitivity of two frequently discussed measures - Value at Risk (VaR) and Conditional Value at Risk (CVaR). Explicit contamination bounds for relative VaR optimization problem are expressed using general results of parametric optimization valid for quadratic programming. A numerical study and a heuristic algorithm for correlation matrices stressing are involved. Sensitivity of VaR and CVaR is studied through their derivatives with respect to the portfolio allocation. Assumptions for the derivatives are formulated, Hessians introduced and convexity is discussed. At last, some dynamic risk measures for multi-period investory models are proposed.

**Keywords:** Risk measures, Value at Risk, Conditional Value at Risk, sensitivity, dynamics.

# Chapter 1

## Introduction

The problem of portfolio optimization is a classical problem in theoretical and computational finance, where by portfolio we mean a group of financial assets. Risk measures were introduced in order to quantify the riskiness of a financial position and to provide a criterion to determine whether the risk of a portfolio was acceptable or not. A variety of different criteria for optimal portfolio selection has been proposed.

Since the seminar work of Markowitz (1952), the portfolio performance is measured in two distinct dimensions - the mean describing the expected return and the risk which measures the uncertainty of the return. Another approach is based on utility maximization, where the risk and the expected return are incorporated into one function, called utility function. Such approach is not involved in this thesis. The classical Markowitz model measures the risk by portfolio variance. Markowitz developed a theory of portfolio selection on basis of the minimization of a quadratic function (portfolio variance) subject to linear constraints (minimal acceptable portfolio mean return and budget constraint) under some simplifying assumptions, for details see [10].

Value at Risk (VaR) has become the standard measure that financial institutions use or even have to use to quantify market and credit risk. Conditional Value at Risk (CVaR) is often proposed as an alternative for VaR, that has many problematic properties, for discussion see [13] and [34]. Above all, the lack of subadditivity makes VaR a problematic criterion for portfolio optimization. VaR associated with a combination of two assets can be greater than the sum of the risks of the individual assets, which the portfolio diversification makes very difficult. On the other hand, CVaR has superior properties in many respects, it is coherent, convex and it can be expressed by a minimization formula, for details see [35], [36]. The marginal behaviour of VaR and CVaR, if a new position is added to the portfolio, is studied in [34] through their derivatives with respect to the portfolio allocation. First and second order derivatives of VaR were published earlier in [15], where local behaviour of VaR was studied. However, some assumptions for the derivatives are missing in both articles. Therefore, our goal is to formulate these assumptions and to prove the derivative formulas properly. We will try to extend so far introduced results and to find Hessians of VaR and CVaR. If we know the Hessians, we will be able to discuss convexity of these measures with respect to the portfolio allocation.

Portfolio risk can be measured by many different performance functionals, depending on investor's risk preferences and can be reflected by the choice of a suitable

utility function or incorporated into constraints of a stochastic programming model. In [13] risk measures commonly used in asset allocation problems are classified into dispersion risk measures (variance, mean absolute deviation, Gini's risk measure) and safety risk measures (safety-first, Value at Risk, Conditional Value at Risk, MiniMax) and a comparison of their properties on theoretical and empirical basis is proposed. All of these risk criteria are theoretically valid. However, all of them are based on one-period model, that does not take into account the sequence of the asset's rates of return, cashflows, sales and purchases of assets (portfolio revision) within an investment horizon. So many multiperiod stochastic programming models for risk management have been proposed and the problem of dynamic (multi-period) risk measures has gained much attention recently. A risk measure that is defined over a process or time series rather than for an one-period future value (loss) is called the multi-period or dynamic risk measure. One possible way how to define new dynamic risk measure is an extension of one-period one. Conditional Drawdown at Risk (CDaR) represents such extension of CVaR, for details about drawdown measures see [5], [6], [21]. Another approach can be found in [31], [32], where a completely new dynamic risk measure for income streams is introduced. The measure is based on the nonanticipativity princip and is closely related to the expected value of perfect information (EVPI) of stochastic programming problem, for details see [10], [37].

Stochastic programming is becoming more popular in finance, because every investment is uncertain with respect to the gain or loss that occur in the future. The main sources of error in practical applications of stochastic programming come from simulation, sampling, estimation and also from incomplete input information about underlying probability distribution, which we does not usually know exactly. The distribution plays a role of an abstract parameter which is estimated or approximated by another probability distribution, by parametric or nonparametric methods or simulation techniques. Moreover, the goal is to get a sensible approximation of the optimal solution and of the optimal value, not an approximation of the probability distribution. For this purpose, methods of output analysis for stochastic programs were introduced to investigate how the optimal value and the solution behave when some changes in input of the stochastic programming model are made. We refer to [9] for complex information about the methods of output analysis. Contamination techniques represent one of such methods which can be also used in stress testing. The output of all stress tests in finance is an estimate of loss that would be suffered by the portfolio (or institution) if a particular, mostly extremal scenario is realized. The contamination techniques provide a way to construct stress test estimates and contamination bounds, which quantify the effect of considered input data change. Practical use of the contamination techniques in stress testing for VaR and CVaR is discussed in [11]. We will concentrate on application of contamination techniques to the optimization problem with relative VaR objective function.

These are the main general assumptions of this thesis: all assets are infinitely divisible, there are no transaction costs and taxes, all the assets in question are marketable, short sales are allowed, no investor can affect the returns of the respective assets substantially. For a more realistic approach see [19]. This article shows how a long-short portfolio optimization problem under concave transaction costs and non-convex commision fee can be solved by a branch and bound algorithm, using the



mean absolute deviation as the risk criterion.

This thesis is organized as follows: In Chapter 2 general definitions of the loss function and the risk measure are proposed. Then we focus on two frequently discussed risk measures - Value at Risk (VaR) and Conditional Value at Risk (CVaR). Chapter 3 deals with the contamination techniques in stress testing for VaR and CVaR with a special attention to relative VaR. Explicit contamination bounds for the relative VaR optimization problem are expressed and the numerical study is involved. In Chapter 4 sensitivity of VaR and CVaR is studied through their derivatives with respect to the portfolio allocation. Assumptions for the derivatives are formulated, Hessians introduced and convexity is discussed. Chapter 5 concentrates on dynamic risk measures for multi-period investory models. A dynamic risk measure based on nonanticipativity princip and drawdown risk measures are proposed. Chapter 6 concludes this thesis. Chapter 7 - Appendix - contains models for financial returns modelling, a methodology for correlation matrices stressing, a brief review of RiskMetrics framework and some numerical results.

# Chapter 2

## Risk Measures for Random Losses

In this chapter we concentrate on one-period risk measures for random losses. First, we propose a general definition of the loss random variable as a function of a random future value and a decision. The risk measure is then defined as a probability functional on a set of loss random variables (functions). Because the distribution of the loss random variable depends on our (risk manager's) decision, we can affect the risk by the decision. We also define some required and relevant properties of risk measures, such as convexity, consistency with stochastic dominance order and coherence. Next, we focus on two frequently discussed risk measures - Value at Risk (VaR) and Conditional Value at Risk (CVaR). The definitions and main properties of these measures for general loss distributions are borrowed from [36]. Close relations between VaR, CVaR and standard deviation can be found for normally distributed loss random variables. In [35] is even shown that these risk measures are equivalent in certain portfolio optimization problems.

This chapter is organized as follows: In Section 2.1 we propose basic definitions of the loss random function, the risk measure and its general properties. In Section 2.2 Value at Risk for general loss distributions is defined and its basic properties and disadvantages are summarized. In Section 2.3 we define Conditional Value at Risk and introduce many of its relevant properties. In Section 2.4 Value at Risk and Conditional Value at Risk are compared under the assumption that the losses are normally distributed.

### 2.1 Basic Definitions and Assumptions

We define the **loss function** (random variable)  $Z = g(x, Y)$  dependent on a decision  $x \in X \subset \mathbb{R}^n$  (weights of assets in our portfolio, portfolio allocation) and a random future value  $Y$  (rate of interest, returns, yields), where  $Y^T = (Y_1, \dots, Y_m)$  is a real random vector with components defined on a probability space  $(\Omega, \mathcal{A}, P)$  with values in  $(E, \mathcal{B}(E))$ , where  $\mathcal{B}(E)$  denotes the Borel  $\sigma$ -algebra generated by a metric space  $E$ , in this case  $E \subseteq \mathbb{R}$ .  $I = -Z$  can be seen as a random income variable (function).

For example

$$\begin{aligned} X &= \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, 0 \leq x_i \leq 0.5, i = 1, \dots, n\}, \\ g_1(x, Y) &= x_1 f_1(Y) + \dots + x_n f_n(Y), \\ g_2(x, Y) &= x_1 Y_1 + \dots + x_n Y_n = x'Y, n = m. \end{aligned}$$

Let  $\omega \in \Omega$  represent a realization of  $Y$ . If  $Y$  is a random vector with known distribution  $P$  (independent on decision  $x$ ), then  $Z = g(x, Y)$  is a random variable with distribution dependent on the decision  $x \in X$ . In later sections we will assume that:

1.  $g(x, \omega)$  is ( $\forall \omega \in \Omega$  continuous in  $x$ ) and ( $\forall x \in X$  measurable in  $\omega$ ),
2.  $\mathbb{E}_P[|g(x, Y)|] < \infty, \forall x \in X$ .

Then we define the cumulative distribution function of  $Z$  as

$$\psi(x, P; \xi) = P(g(x, Y) \leq \xi), \forall x \in X$$

and its left limit in point  $\xi$

$$\psi(x, P; \xi^-) = P(g(x, Y) < \xi), \forall x \in X.$$

In this chapter we simplify the notion  $\psi(x, P; \xi)$  to  $\psi(x, \xi)$ , because we suppose that the distribution  $P$  of losses is known and it does not change. If the difference

$$\psi(x, \xi) - \psi(x, \xi^-) = P(g(x, Y) = \xi)$$

is positive, then there is a probability atom in  $\xi$  and the distribution function  $\psi(x, \cdot)$  has a "jump" in the point  $\xi$ .

Formal definition of risk measure can be given as follows.

**Definition 2.1.1.** (*Risk measure*)

Let  $\mathbb{Z}$  be a set of loss random variables, then the one-period risk measure is a probability functional  $\rho : \mathbb{Z} \rightarrow \mathbb{R}$  which "returns values according to investor's (risk manager's) preferences".

Multi-period risk measure can be defined for loss random processes in similar manner. Thanks to our assumptions, the set of loss random variables is some subset of the space  $L_1(\Omega, \mathcal{A}, P)$ . By analogy to [13] we define some additional properties of risk measures of random losses.

**Definition 2.1.2.** (*Additional properties of risk measures*)

We assume that  $\mathbb{Z}$  is a set of loss random variables,  $Z, Z_1, Z_2 \in \mathbb{Z}$  and  $\rho : \mathbb{Z} \rightarrow \mathbb{R}$  is a risk measure. Let  $I_1 = -Z_1, I_2 = -Z_2$  be income random variables. We will say that

1.  $\rho(Z)$  is consistent with order relation  $\succ$  if  $I_1 \succ I_2$  implies that  $\rho(-I_1) \leq \rho(-I_2)$ ; we consider the following order relations:
  - First stochastic dominance order (FSD)  $\stackrel{def}{\Leftrightarrow} \mathbb{E}[u(I_1)] \geq \mathbb{E}[u(I_2)]$  for every non-decreasing function  $u$  for which these expectations are finite.

- *Second stochastic dominance order (SSD)  $\stackrel{def}{\Leftrightarrow} \mathbb{E}[u(I_1)] \geq \mathbb{E}[u(I_2)]$  for every non-decreasing concave function  $u$  for which these expectations are finite.*
- 2.  $\rho(Z)$  is convex if for all loss random variables  $Z_1, Z_2 \in \mathbb{Z}$  and for every  $\lambda \in [0, 1]$  such that  $(\lambda Z_1 + (1 - \lambda)Z_2) \in \mathbb{Z}$

$$\rho(\lambda Z_1 + (1 - \lambda)Z_2) \leq \lambda \rho(Z_1) + (1 - \lambda)\rho(Z_2).$$

- 3.  $\rho(Z)$  is translation invariant if for every loss random variable  $Z \in \mathbb{Z}$  and for every real  $c$  such that  $Z + c \in \mathbb{Z}$ ,  $\rho(Z + c) = \rho(Z)$ .
- 4.  $\rho(Z)$  is translation equivariant if for every loss random variable  $Z \in \mathbb{Z}$  and for every real  $c$  such that  $Z + c \in \mathbb{Z}$ ,  $\rho(Z + c) = \rho(Z) + c$ .

Consistency with second stochastic dominance order and convexity are the most required and relevant properties of risk measures, which allow us to get efficient portfolios by solving a mean-risk model.

**Definition 2.1.3.** (*Axiomatic definition of coherent risk measures*)

Let  $\mathbb{Z}$  be a set of loss random variables and  $\rho : \mathbb{Z} \rightarrow \mathbb{R}$  be a risk measure. We will say that  $\rho(Z)$  is the coherent risk measure if it satisfies

1. *sublinearity:*
  - *subadditivity:*  
 $\rho(Z_1 + Z_2) \leq \rho(Z_1) + \rho(Z_2), \forall Z_1, Z_2 \in \mathbb{Z} : Z_1 + Z_2 \in \mathbb{Z},$
  - *homogeneity:*  
 $\rho(\lambda Z_1) = \lambda \rho(Z_1)$  for  $\lambda \geq 0, \forall Z_1 \in \mathbb{Z} : \lambda Z_1 \in \mathbb{Z},$
2.  $(Z_1 \equiv c \text{ (constant)}, Z_1 \in \mathbb{Z}) \Rightarrow (\rho(Z_1) = c),$
3.  $(\forall Z_1, Z_2 \in \mathbb{Z} : Z_1 \prec Z_2) \Rightarrow (\rho(Z_1) \leq \rho(Z_2)),$  where  $\prec$  denotes first stochastic dominance order.

## 2.2 Value at Risk

Value at Risk (VaR) has become the standard measure that financial institutions use to quantify market and credit risk. VaR is defined as: Probability of loss greater than VaR is at most  $1 - \alpha$  for some  $\alpha \in (0, 1)$ .

**Definition 2.2.1.** (*Value at Risk (VaR)*) [36]

The Value at Risk  $VaR_\alpha(x)$  of the loss  $Z = g(x, Y)$  associated with a decision  $x$  is the value

$$VaR_\alpha(x) = \min\{\xi : \psi(x, \xi) \geq \alpha\}, \quad (2.1)$$

$\alpha \in (0, 1)$ , usually  $\alpha = 0.95$  or  $\alpha = 0.99$ .

We again simplify notion  $VaR_\alpha(x, P) = \min\{\xi : \psi(x, P; \xi) \geq \alpha\}$ .

**Definition 2.2.2.** (*Quantile function, quantiles*) [1]

Let  $F(x) = P(X < x)$  be a distribution function (left-continuous). Then we define the quantile (fractile) function as

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}, \quad 0 < u < 1.$$

The values  $F^{-1}(u)$  are called quantiles (fractiles).

Minimum in definition (2.1) is always attained, because  $\psi(x, \xi)$  is nondecreasing and right continuous. We can compare this definition with the quantile function, that returns the same values, but it is not exactly the same.

**Definition 2.2.3.** (*"upper" VaR*) [36]

The upper Value at Risk  $VaR_{\alpha}^{+}(x)$  of the loss  $Z = g(x, Y)$  associated with a decision  $x$  is the value

$$VaR_{\alpha}^{+}(x) = \inf\{\xi : \psi(x, \xi) > \alpha\}. \quad (2.2)$$

**Remark 2.2.4.**

We can discuss the number of solutions  $\xi$  of the equation  $\psi(x, \xi) = \alpha$ :

1. unique  $\Leftrightarrow \psi(x, \xi)$  is continuous and strictly increasing in  $\xi$ ,
2. no solution  $\Leftrightarrow$  "vertical gap"  $\psi(x, \xi)$  in point  $\xi = VaR_{\alpha}(x)$ , we denote

$$\begin{aligned} \alpha^{-}(x) &= \psi(x, VaR_{\alpha}(x)^{-}), \\ \alpha^{+}(x) &= \psi(x, VaR_{\alpha}(x)) \end{aligned}$$

and it holds  $\alpha^{-} < \alpha \leq \alpha^{+}$ .

3. many solutions  $\Leftrightarrow \psi(x, \xi)$  is constant over some interval of  $\xi$  at level  $\alpha$ , the solution is then the interval  $[VaR_{\alpha}(x), VaR_{\alpha}^{+}(x))$  or  $[VaR_{\alpha}(x), VaR_{\alpha}^{+}(x)]$  depending on whether there is or is not vertical gap of  $\psi(x, \xi)$  in  $VaR_{\alpha}^{+}(x)$ .

The cases 2, 3 occur for discrete distributions and scenario models, where  $\psi(x, \xi)$  is constant between jumps.

**Remark 2.2.5.** (*Main disadvantages of VaR*) [13], [34]

Main disadvantages of Value at Risk:

1. Unstability in behaviour of VaR, i.e. VaR is not continuous in  $\alpha$  - previous cases 2 and 3.
2. VaR is not coherent risk measure - it is not always subadditive, which disables effective portfolio diversification; an example can be found in [39].
3. VaR can violate second stochastic dominance order.
4. There are many computational difficulties in portfolio optimization with VaR, VaR is not convex risk measure.

5. VaR does not take into account extreme events.

**Remark 2.2.6.** (Computational methods for VaR)

There are many methods for VaR computation. These methods can be divided into three main families as it is done in [26]:

1. parametric methods - RiskMetrics<sup>TM</sup> (see Section 7.3) and GARCH,
2. semi-parametric methods - extreme value theory, historical simulation and the hybrid model,
3. nonparametric methods - conditional autoregressive value-at-risk(CAViaR), see [26], quasi-maximum likelihood GARCH, nonparametric optimization models, see [33].

## 2.3 Conditional Value at Risk

### 2.3.1 Basic Definitions and Properties

Conditional Value at Risk (CVaR) is often proposed as an alternative for Value at Risk, that is widely used in practise even though it is not adequate risk measure, see Remark 2.2.5. CVaR is defined as the conditional mean of losses on condition that we are beyond VaR. The formal Definition 2.3.1 solves successfully the problem if there is a probability atom at  $VaR_\alpha(x)$  and so the interval  $[VaR_\alpha(x), \infty)$  has probability greater or equal to  $1 - \alpha$ . The "  $\alpha$ -tail" distribution is then "  $1 - \alpha$ " part of the distribution function  $\psi(x, \xi)$  rescaled from  $[\alpha, 1]$  onto  $[0, 1]$ .

**Definition 2.3.1.** (Conditional Value at Risk (CVaR)) [36]

The Conditional Value at Risk  $CVaR_\alpha(x)$  associated with a decision  $x$  is the mean of the loss function  $Z = g(x, Y)$  in "  $\alpha$ -tail" distribution defined by

$$\psi_\alpha(x, \xi) = \begin{cases} 0 & \text{for } \xi < VaR_\alpha(x), \\ [\psi(x, \xi) - \alpha]/[1 - \alpha] & \text{for } \xi \geq VaR_\alpha(x). \end{cases} \quad (2.3)$$

**Definition 2.3.2.** (Upper and lower CVaR) [36]

The upper Conditional Value at Risk  $CVaR_\alpha^+(x)$  associated with a decision  $x$  we define as

$$CVaR_\alpha^+(x) = \mathbb{E}[g(x, Y) | g(x, Y) > VaR_\alpha(x)]. \quad (2.4)$$

The lower Conditional Value at Risk  $CVaR_\alpha^-(x)$  associated with a decision  $x$  we define as

$$CVaR_\alpha^-(x) = \mathbb{E}[g(x, Y) | g(x, Y) \geq VaR_\alpha(x)]. \quad (2.5)$$

**Remark 2.3.3.**

1.  $CVaR_\alpha^-(x)$  is well defined because it always holds  $P\{g(x, Y) \geq VaR_\alpha(x)\} \geq 1 - \alpha > 0$ .

2. With  $CVaR_\alpha^+(x)$  there is no problem as far as  $P\{g(x, Y) > VaR_\alpha(x)\} > 0$ . If  $\psi(x, VaR_\alpha(x)) = 1$ , i.e.  $VaR_\alpha(x)$  is the greatest loss, which can occur, then  $CVaR_\alpha^+(x)$  is ill-defined.
3. We can express distributions for "upper" and "lower" CVaR by analogy to the definition (2.3) of the  $\alpha$ -tail distribution as

$$\psi_\alpha^+(x, \xi) = \begin{cases} 0 & \text{for } \xi < VaR_\alpha(x), \\ [\psi(x, \xi) - \alpha^+]/[1 - \alpha^+] & \text{for } \xi \geq VaR_\alpha(x), \end{cases}$$

$$\psi_\alpha^-(x, \xi) = \begin{cases} 0 & \text{for } \xi < VaR_\alpha(x), \\ [\psi(x, \xi) - \alpha^-]/[1 - \alpha^-] & \text{for } \xi \geq VaR_\alpha(x). \end{cases}$$

$CVaR_\alpha^+(x)$  is sometimes called mean shortfall,  $CVaR_\alpha(x)$  expected shortfall and  $\mathbb{E}[g(x, Y) - VaR_\alpha(x) | g(x, Y) > VaR_\alpha(x)] = CVaR_\alpha^+(x) - VaR_\alpha(x)$  mean excess loss.

The following theorem shows how important is to take care of the correct definition of CVaR, especially in the case, when the distribution function  $\psi(x, \xi)$  is not continuous.

**Theorem 2.3.4.** (Basic relations between CVaR's) [36]

1. There is not probability atom in  $VaR_\alpha(x)$ , then

$$CVaR_\alpha^-(x) = CVaR_\alpha(x) = CVaR_\alpha^+(x). \quad (2.6)$$

2. There is a probability atom in  $VaR_\alpha(x)$ , then we must differ 3 cases:

- $\alpha = \psi(x, VaR_\alpha(x))$ , then

$$CVaR_\alpha^-(x) < CVaR_\alpha(x) = CVaR_\alpha^+(x). \quad (2.7)$$

- $\psi(x, VaR_\alpha(x)) = 1$ , then

$$CVaR_\alpha^-(x) < CVaR_\alpha(x) \quad (2.8)$$

and  $CVaR_\alpha^+(x)$  is ill defined.

- If  $\alpha^-(x) < \alpha < \alpha^+(x) < 1$ , then

$$CVaR_\alpha^-(x) < CVaR_\alpha(x) < CVaR_\alpha^+(x). \quad (2.9)$$

### 2.3.2 CVaR as a Weighted Average

As a consequence of previous statements we obtain the following expression for CVaR, which we apply to discrete - scenario model. These expressions can be used as alternative definitions of CVaR.

**Theorem 2.3.5.** (*CVaR as a weighted average*) [36]

Let  $\lambda_\alpha(x)$  be the probability of loss  $Z = VaR_\alpha(x)$  in " $\alpha$ -tail" distribution, thus

$$\lambda_\alpha(x) = [\psi(x, VaR_\alpha(x)) - \alpha] / [1 - \alpha].$$

If  $\psi(x, VaR_\alpha(x)) < 1$ , then

$$\begin{aligned} CVaR_\alpha(x) &= \\ &= \lambda_\alpha(x) VaR_\alpha(x) + [1 - \lambda_\alpha(x)] CVaR_\alpha^+(x). \end{aligned} \quad (2.10)$$

If  $\psi(x, VaR_\alpha(x)) = 1$ , then

$$CVaR_\alpha(x) = VaR_\alpha(x). \quad (2.11)$$

CVaR can be viewed as a weighted average of VaR and the mean of the worst losses strictly exceeding VaR.

**Consequence 2.3.6.** (*CVaR vs. VaR*) [36]

$CVaR_\alpha(x) \geq VaR_\alpha(x)$ .

**Consequence 2.3.7.** (*CVaR for discrete-scenario models*) [36]

Let the random vector  $Y$  be discrete distributed with the probability measure  $P$  concentrated in finitely many points, then distribution of the loss random variable  $Z = g(x, Y)$  for fixed  $x \in X$  is likewise concentrated in finitely many points  $z^{[1]} < z^{[2]} < \dots < z^{[N]}$  with probabilities  $P(Z = z^{[k]}) = p^{[k]}$ ,  $\sum_{k=1}^N p^{[k]} = 1$ .

For  $\alpha \in (0, 1)$  we can find such unique index  $k_\alpha$ , that it holds

$$\sum_{k=1}^{k_\alpha-1} p^{[k]} < \alpha \leq \sum_{k=1}^{k_\alpha} p^{[k]}.$$

Then we have

$$VaR_\alpha(x) = z^{[k_\alpha]} \quad (2.12)$$

and if  $\alpha > 1 - p^{[N]}$ , then

$$VaR_\alpha(x) = CVaR_\alpha(x) = z^{[N]}, \quad (2.13)$$

else

$$CVaR_\alpha(x) = \frac{1}{1 - \alpha} \left[ \left( \sum_{k=1}^{k_\alpha} p^{[k]} - \alpha \right) z^{[k_\alpha]} + \sum_{k=k_\alpha+1}^N p^{[k]} z^{[k]} \right]. \quad (2.14)$$



**Proof.**

Explanation of formula for  $CVaR_\alpha^+(x)$ :

$$\psi_\alpha^+(x, \xi) = \begin{cases} 0 & \text{for } \xi < VaR_\alpha(x), \\ [\psi(x, \xi) - \alpha^+]/[1 - \alpha^+] & \text{for } \xi \geq VaR_\alpha(x), \end{cases}$$

where

$$\alpha^+ = \sum_{k=1}^{k_\alpha} p^{[k]}, \quad 1 - \alpha^+ = \sum_{k=k_\alpha+1}^N p^{[k]},$$

for probabilities  $p^{[k]^+}$  we obtain

$$\begin{cases} 0 & \text{for } z^{[k]} \leq VaR_\alpha(x), \\ p^{[k]}/\sum_{k=k_\alpha+1}^N p^{[k]} & \text{for } z^{[k]} > VaR_\alpha(x). \end{cases}$$

Then

$$\begin{aligned} \lambda_\alpha(x) &= \frac{1}{1 - \alpha} \left( \sum_{k=1}^{k_\alpha} p^{[k]} - \alpha \right), \\ 1 - \lambda_\alpha(x) &= \frac{1}{1 - \alpha} \left( \sum_{k=k_\alpha+1}^N p^{[k]} \right), \\ CVaR_\alpha^+(x) &= \frac{\sum_{k=k_\alpha+1}^N p^{[k]} z^{[k]}}{\sum_{k=k_\alpha+1}^N p^{[k]}}. \end{aligned}$$

And finally

$$CVaR_\alpha(x) = \frac{1}{1 - \alpha} \left[ \left( \sum_{k=1}^{k_\alpha} p^{[k]} - \alpha \right) z^{[k_\alpha]} + \sum_{k=k_\alpha+1}^N p^{[k]} z^{[k]} \right].$$

□

### 2.3.3 Minimization Formula for CVaR

Conditional Value at Risk can be expressed by a minimization formula which represents the best computational advantage of CVaR over VaR and many other measures. Under some additional assumptions it enables us to use linear programming techniques to minimize CVaR and to prove many its useful properties such as convexity and coherence.

First we define the auxiliary function

$$F_\alpha(x, \xi) = \xi + \frac{1}{1 - \alpha} \mathbb{E} \left[ [g(x, Y) - \xi]^+ \right], \quad (2.15)$$

where  $[t]^+ = \max\{0, t\}$ .

**Theorem 2.3.8.** (Minimization formula for CVaR) [36]

As a function of  $\xi \in \mathbb{R}$ ,  $F_\alpha(x, \xi)$  is finite and convex, hence continuous and has finite one-sided derivatives in all  $\xi$ . It holds

$$CVaR_\alpha(x) = \min_{\xi \in \mathbb{R}} F_\alpha(x, \xi) \quad (2.16)$$

and moreover  $\arg \min_{\xi \in \mathbb{R}} F_\alpha(x, \xi) = [VaR_\alpha(x), VaR_\alpha^+(x)]$  which has to be a nonempty closed bounded interval (perhaps reducing to a single point). In particular, one always has

$$CVaR_\alpha(x) = F_\alpha(x, VaR_\alpha(x)).$$

**Proof.** The basic properties of the auxiliary function  $F_\alpha(x, \xi)$  follow from general assumptions and from convexity of the function  $[g(x, Y) - \xi]^+$  in  $\xi$ . Next, we find explicit formula for right derivative of the auxiliary function  $F_\alpha(x, \xi)$  in  $\xi$ .

$$\begin{aligned} & \frac{F(x, \xi') - F(x, \xi)}{\xi' - \xi} = \\ & 1 + \frac{1}{1 - \alpha} \mathbb{E} \left[ \underbrace{\frac{[g(x, Y) - \xi']^+ - [g(x, Y) - \xi]^+}{\xi' - \xi}}_{\clubsuit} \right] \end{aligned}$$

If  $\xi' > \xi$ , then

$$\clubsuit \begin{cases} = -1 & \text{for } g(x, Y) > \xi', \\ = 0 & \text{for } g(x, Y) \leq \xi, \\ \in (-1, 0) & \text{for } \xi < g(x, Y) \leq \xi'. \end{cases}$$

Furthermore

$$\begin{aligned} P(g(x, Y) > \xi') &= 1 - \psi(x, \xi'), \\ P(\xi < g(x, Y) \leq \xi') &= \psi(x, \xi') - \psi(x, \xi), \end{aligned}$$

whereas

$$\psi(x, \xi') \searrow \psi(x, \xi), \text{ if } \xi' \searrow \xi.$$

Interchanging limit and the mean  $\mathbb{E}$  we obtain

$$\lim_{\xi' \searrow \xi} \mathbb{E}\{\clubsuit\} = \mathbb{E}\{\lim_{\xi' \searrow \xi} \clubsuit\} = -[1 - \psi(x, \xi)]$$

and finally, for the right derivative we have

$$\frac{\partial^+ F_\alpha}{\partial \xi}(x, \xi) = \frac{\psi(x, \xi) - \alpha}{1 - \alpha}. \quad (2.17)$$

By analogy for left derivative of the auxiliary function it holds

$$\frac{\partial^- F_\alpha}{\partial \xi}(x, \xi) = \frac{\psi(x, \xi^-) - \alpha}{1 - \alpha}. \quad (2.18)$$

Convexity of  $F_\alpha(x, \cdot)$  implies that one-sided derivatives are nondecreasing in  $\xi$ . For the limits it holds

$$\begin{aligned} \lim_{\xi \rightarrow \infty} \frac{\partial^+ F_\alpha}{\partial \xi}(x, \xi) &= \lim_{\xi \rightarrow \infty} \frac{\partial^- F_\alpha}{\partial \xi}(x, \xi) = 1, \\ \lim_{\xi \rightarrow -\infty} \frac{\partial^+ F_\alpha}{\partial \xi}(x, \xi) &= \lim_{\xi \rightarrow -\infty} \frac{\partial^- F_\alpha}{\partial \xi}(x, \xi) = \\ &= -\frac{\alpha}{1-\alpha}. \end{aligned}$$

So the derivatives are finite, bounded and nondecreasing. Hence  $\operatorname{argmin}_\xi F(x, \xi)$  is a closed bounded interval, and the points, where the minimum is attained, are characterized by

$$\begin{aligned} \frac{\partial^- F_\alpha}{\partial \xi}(x, \xi) \leq 0 \leq \frac{\partial^+ F_\alpha}{\partial \xi}(x, \xi), \\ \Leftrightarrow \\ \psi(x, \xi^-) \leq \alpha \leq \psi(x, \xi). \end{aligned}$$

This condition is satisfied for  $VaR_\alpha(x)$  like lower limit and  $VaR_\alpha^+(x)$  like upper limit of the interval of  $\operatorname{argmin}_\xi F(x, \xi)$ . □

**Consequence 2.3.9.** [13]

*CVaR is consistent with second stochastic dominance order (SSD) and translation equivariant.*

**Consequence 2.3.10.** (Stability of CVaR) [36]

*CVaR $_\alpha(x)$  behaves continuously with respect to the choice of  $\alpha \in (0, 1)$ .*

**Consequence 2.3.11.** (Convexity of CVaR) [36]

*If the loss function  $g(x, Y)$  is convex in  $x$ , then  $CVaR_\alpha(x)$  is convex with respect to  $x$  and  $F_\alpha(x, \xi)$  is jointly convex in  $(x, \xi)$ .*

**Consequence 2.3.12.** (Coherence of CVaR) [36]

*CVaR is a coherent risk measure. When the loss function  $g(x, Y)$  is linear with respect to  $x$ , not only  $CVaR_\alpha(x)$  is sublinear with respect to  $x$ , but furthermore it satisfies*

$$CVaR_\alpha(x) = c \text{ when } g(x, Y) \equiv c,$$

*for some constant  $c \in \mathbb{R}$ , and it obeys the monotocity rule that*

$$CVaR_\alpha(x_1) \leq CVaR_\alpha(x_2) \text{ when } g(x_1, Y) \leq g(x_2, Y),$$

*for arbitrary  $x_1, x_2 \in X$ .*

In [40] **expected regret** is defined for a treshold  $\zeta \in \mathbb{R}$  and a continuously distributed loss function  $g(x, Y)$  as

$$ER_\zeta(x) \stackrel{def.}{=} \mathbb{E}[g(x, Y) - \zeta]^+. \quad (2.19)$$

There is also shown that  $CVaR_\alpha(x)$  and  $ER_\zeta(x)$  efficient portfolios are equivalent, i.e. a portfolio minimizes the expected regret for a threshold  $\zeta$  just when a confidence level  $\alpha$  exists so that the portfolio minimizes the  $CVaR_\alpha$ . This result follows from the relation

$$F_\alpha(x, \xi) = \xi + \frac{1}{1-\alpha} ER_\xi(x).$$

The main computational advantages of CVaR are introduced in the following theorems, where we assume that  $F_\alpha(x, \xi)$  is convex with respect to  $(x, \xi)$ .

**Theorem 2.3.13.** (*Optimization shortcut*) [36]

$$\min_{x \in X} CVaR_\alpha(x) = \min_{(x, \xi) \in X \times \mathbb{R}} F_\alpha(x, \xi), \quad (2.20)$$

where moreover

$$\begin{aligned} x^* &\in \operatorname{argmin}_{x \in X} CVaR_\alpha(x), \\ \xi^* &\in \operatorname{argmin}_{\xi \in \mathbb{R}} F_\alpha(x^*, \xi) \\ &\Leftrightarrow \\ (x^*, \xi^*) &\in \operatorname{argmin}_{(x, \xi) \in X \times \mathbb{R}} F_\alpha(x, \xi). \end{aligned}$$

**Theorem 2.3.14.** (*Risk-shaping with CVaR*) [36]

The problem

$$\begin{aligned} &\min_{x \in X} h(x) \\ &\text{s.t.} \\ &CVaR_{\alpha_l}(x) \leq \vartheta_l, \quad l = 1, \dots, L, \end{aligned} \quad (2.21)$$

is equivalent to

$$\begin{aligned} &\min_{(x, \xi_1, \dots, \xi_L) \in X \times \mathbb{R}^L} h(x) \\ &\text{s.t.} \\ &F_{\alpha_l}(x, \xi_l) \leq \vartheta_l, \quad l = 1, \dots, L \end{aligned} \quad (2.22)$$

for any selection of probability thresholds  $\alpha_l$  and loss tolerances  $\vartheta_l$ ,  $l = 1, \dots, L$ .

Indeed,  $(x^*, \xi_1^*, \dots, \xi_L^*)$  solves the second problem (2.22)  $\Leftrightarrow x^*$  solves the first problem (2.21).

Moreover, one then has  $CVaR_{\alpha_l}(x) \leq \vartheta_l$  for every  $l$ , and actually  $CVaR_{\alpha_l}(x) = \vartheta_l$  for each  $l$  such that  $F_{\alpha_l}(x, \xi_l) = \vartheta_l$ .

**Example 2.3.15.** We denote  $I_t$  the value of an index and  $P_{tj}$  the prices of instruments (assets)  $j = 1, \dots, J$  at time  $t = 1, \dots, T$ .  $\theta$  denotes the imaginary number of units of the financial index at time  $T$ ,  $\omega$  is a given threshold. Then we get an example of linear CVaR constraint in tracking of given financial index:

$$\xi + \frac{1}{(1-\alpha)T} \sum_{t=1}^T \left[ \left[ \theta I_t - \sum_{j=1}^n P_{tj} x_j / \theta I_t \right] - \xi \right]^+ \leq \omega,$$

where the loss function is defined as

$$g(x, Y) = \theta I_t - \sum_{j=1}^J Y_j x_j / \theta I_t.$$

Numerical results are demonstrated in [36].

**Remark 2.3.16.** Let  $I = -Z = -g(x, Y)$  be a random profit (income) variable, then

$$\psi_Z(z) = P(Z \leq z) = P(-I \leq z) = P(I \geq -z) = 1 - \psi_I(-z) + P(I = -z)$$

and for the discussed risk measures it holds

$$\begin{aligned} -VaR_\alpha(x) &= \inf\{i : P(I \leq i) > 1 - \alpha\}, \\ -VaR_\alpha^+(x) &= \inf\{i : P(I \leq i) \geq 1 - \alpha\}, \\ CVaR_\alpha(x) &= \max_{a \in \mathbb{R}} \left\{ a - \frac{1}{1 - \alpha} \mathbb{E}[|I - a|^-] \right\}. \end{aligned} \quad (2.23)$$

## 2.4 Parametric VaR and CVaR

Under the assumption that the loss function is normally distributed, there are close relations between standard deviation, VaR and CVaR. These relations are introduced in [35] and bring very interesting result in portfolio optimization problems.

**Definition 2.4.1.** (Absolute and Relative VaR) [10]

If we suppose that  $Y \sim \mathcal{N}_n(\mu, \Sigma)$  and  $g(x, \omega) = x'\omega$ , then  $Z = g(x, Y) \sim \mathcal{N}(\mu(x), \sigma^2(x))$ , where  $\mu(x) = \mu'x$  and  $\sigma^2(x) = x'\Sigma x$ . We can define the absolute Value at Risk associated with a decision  $x$  as

$$VaR_\alpha^{abs}(x) = \mu(x) + u_\alpha \sigma(x) \quad (2.24)$$

and the relative Value at Risk associated with a decision  $x$  as

$$VaR_\alpha^{rel}(x) = u_\alpha \sigma(x), \quad (2.25)$$

where  $u_\alpha = \phi^{-1}(\alpha)$ ,  $\phi$  is the distribution function of the standard normal distribution.

The expression (2.24) represents the decomposition of VaR into two components: expected loss and risk. We can also compute first and second order derivatives of absolute VaR with respect to the portfolio allocation.

$$\begin{aligned} \frac{\partial VaR_\alpha^{abs}(x)}{\partial x} &= \mu + \frac{\Sigma x}{(x'\Sigma x)^{1/2}} u_\alpha = \\ &= \mu + \frac{\Sigma x}{x'\Sigma x} (VaR_\alpha^{abs}(x) - x'\mu) = \\ &\stackrel{[15]}{=} \mathbb{E}[Y | x'Y = VaR_\alpha^{abs}(x)], \end{aligned} \quad (2.26)$$

$$\begin{aligned} \frac{\partial^2 VaR_\alpha^{abs}(x)}{\partial x^2} &= \frac{u_\alpha}{(x'\Sigma x)^{1/2}} \left[ \Sigma - \frac{\Sigma x x' \Sigma}{x'\Sigma x} \right] = \\ &\stackrel{[15]}{=} \frac{u_\alpha}{(x'\Sigma x)^{1/2}} \mathbb{V}\mathbb{A}\mathbb{R}[Y | x'Y = VaR_\alpha^{abs}(x)]. \end{aligned} \quad (2.27)$$

We express the derivatives of VaR under more general assumptions in Chapter 3.

**Theorem 2.4.2.** [35]

Let  $\alpha \geq 0.5$  and  $X = \{x \in \mathbb{R}^n : \sum_{j=1}^n x_j = 1, x_j \geq 0 \forall_j\}$ ,  $X_r = X \cap \{x : -\mu(x) \geq r\}$ . Under the assumption of Definition 2.4.1, if we solve any two of the following problems

$$\begin{aligned} \min_{x \in X_r} \quad & VaR_\alpha^{abs}(x), \\ \min_{x \in X_r} \quad & CVaR_\alpha(x), \\ \min_{x \in X_r} \quad & \sigma^2(x) \end{aligned}$$

and the constraint  $-\mu(x) \geq r$  is active at both, then the solutions to those two problems are the same; a common portfolio  $x^*$  is optimal by both criteria.

**Example 2.4.3.** [28]

Under the assumption  $Z \sim \mathcal{N}(0, \sigma^2)$  we express the relation between VaR and CVaR explicitly. Let

$$\begin{aligned} 1 - \alpha &= \int_{VaR_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}} dz, \\ VaR_\alpha(x) &= u_\alpha \cdot \sigma, \\ CVaR_\alpha &= \frac{1}{1 - \alpha} \int_{VaR_\alpha}^{\infty} z \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}} dz = \frac{1}{1 - \alpha} \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{VaR_\alpha^2}{2\sigma^2}}. \end{aligned} \tag{2.28}$$

Using substitution  $z = \sigma a$  we obtain

$$CVaR_\alpha = K_\alpha \sigma, \tag{2.29}$$

where

$$K_\alpha = \frac{1}{1 - \alpha} \int_{u_\alpha}^{\infty} a \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}} da = \frac{1}{1 - \alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{u_\alpha^2}{2}}. \tag{2.30}$$

From (2.28) and (2.29) we finally have

$$CVaR_\alpha = \frac{K_\alpha}{u_\alpha} VaR_\alpha. \tag{2.31}$$

The relation (2.31) enables us to extend RiskMetrics framework (see Section 7.3) for CVaR measuring.

Table 2.1: Quantiles, constants.

$\alpha$	0.90	0.95	0.99
$u_\alpha$	1.2815	1.6448	2.3263
$K_\alpha$	1.7550	2.0627	2.6652
$K_\alpha/u_\alpha$	1.3694	1.2540	1.1457

# Chapter 3

## Stress Testing for VaR and CVaR

Stress testing is one of general methods used in finance to estimate the loss that would be suffered by portfolio (or institution) if a particular, mostly extremal scenario is realized. Contamination techniques represent one of possible methods for stress test estimates construction. We focus on contamination bounds which quantify the effect of considered input data change on solution of the stochastic programming problem. We refer to [9] for complex information about the methods of output analysis for stochastic programs.

In this chapter we summarize and extend the application of the contamination techniques for Value at Risk (VaR) and Conditional Value at Risk (CVaR) from [11]. We use the family of contaminated probability distributions of the random vector  $Y$

$$P_\lambda = (1 - \lambda)P + \lambda Q, \quad \lambda \in [0, 1],$$

where  $Q$  is a fixed stress distribution and  $\lambda$  is a contamination parameter. We say that the probability distribution  $P$  is contaminated by the probability distribution  $Q$ . The contamination method does not require any specific properties of the probability distributions  $P$  and  $Q$ . Such approach enables us to exploit the existing results of parametric optimization.

This Chapter is organized as follows: In Section 3.1 the contamination techniques for CVaR are described. In Section 3.2 we focus on the contamination techniques for VaR with a special attention to relative VaR. Explicit contamination bounds for optimization problem with the relative VaR objective function are introduced and the numerical study is involved. Section 3.3 summarizes some useful results of parametric optimization.

### 3.1 Stress Testing for CVaR

We extend our marking for the auxiliary objective function  $F_\alpha(x, \xi, \lambda) := F_\alpha(x, \xi, P_\lambda)$ , which is linear in  $\lambda$  and convex in  $\xi$ , for definition see (2.15). For its optimal value for fixed  $x$  it holds

$$CVaR_\alpha(x, \lambda) := CVaR_\alpha(x, P_\lambda) = \min_{\xi \in \mathbb{R}} F_\alpha(x, \xi, P_\lambda). \quad (3.1)$$

For  $\lambda = 0$  we will speak about initial (unperturbed) problem with optimal solution  $\xi^*(x, P) \in [VaR_\alpha(x, P), VaR_\alpha^+(x, P)]$ , see Theorem 2.3.8.

### 3.1.1 Contamination Bounds for CVaR

Applying the result (3.20) to (3.1) for a fixed vector  $x \in X$  we get

$$\frac{d}{d\lambda} CVaR_\alpha(x, 0^+) = \min_{\xi \in [VaR_\alpha(x, P), VaR_\alpha^+(x, P)]} F_\alpha(x, \xi, Q) - CVaR_\alpha(x, P).$$

The contamination bounds for  $CVaR_\alpha(x, \lambda)$  follow from concavity of  $CVaR_\alpha(x, \lambda)$  with respect to  $\lambda$  which is a consequence of linearity of  $F_\alpha(x, \xi, P_\lambda)$  in  $\lambda$ , thus

$$\begin{aligned} (1 - \lambda)CVaR_\alpha(x, 0) + \lambda CVaR_\alpha(x, 1) &\leq \\ &\leq CVaR_\alpha(x, \lambda) \leq \\ &\leq CVaR_\alpha(x, 0) + \lambda \frac{d}{d\lambda} CVaR_\alpha(x, 0^+), \end{aligned} \quad (3.2)$$

$$0 \leq \lambda \leq 1,$$

which can be rewritten as

$$\begin{aligned} (1 - \lambda)CVaR_\alpha(x, P) + \lambda CVaR_\alpha(x, Q) &\leq \\ &\leq CVaR_\alpha(x, P_\lambda) \leq \\ &\leq (1 - \lambda)CVaR_\alpha(x, P) + \lambda F_\alpha(x, \xi^*(x, P), Q), \end{aligned} \quad (3.3)$$

$$0 \leq \lambda \leq 1,$$

where  $\xi^*(x, P)$  denotes one of the optimal solution of the initial problem, i.e. (3.1) for  $\lambda = 0$ .

### 3.1.2 Discrete-scenario CVaR

We use the results of Consequence 2.3.7. Let  $Q$  be a discrete probability distribution of  $Y$  carried by  $M$  stress out-of-sample scenarios  $\omega^s$ ,  $s = N + 1, \dots, M$  with probabilities  $p_s$ ,  $s = N + 1, \dots, M$  ( $g(x, \omega^s) \neq g(x, \omega^t)$ ,  $\forall 1 \leq s < t \leq M$ ). Then the upper contamination bound has the form

$$F_\alpha(x, \xi^*(x, P), P_\lambda), \quad \forall \lambda \in [0, 1].$$

Specially, if  $Q$  is a degenerate probability distribution with only one scenario  $\omega^*$ , we can get easy the difference between the upper and the lower bound

$$\begin{aligned} &\lambda \left[ F_\alpha(x, \xi^*(x, P), Q) - CVaR_\alpha(x, Q) \right] = \\ &= \lambda \left[ \xi^*(x, P) - g(x, \omega^*) + \frac{1}{1 - \alpha} [g(x, \omega^*) - \xi^*(x, P)]^+ \right]. \end{aligned} \quad (3.4)$$

### 3.1.3 Optimization with the CVaR Objective Function

From Theorem 2.3.13

$$\varphi_C(P) := \min_{x \in X} CVaR_\alpha(x, P) = \min_{(x, \xi) \in \mathbb{R} \times X} F_\alpha(x, \xi, P), \quad (3.5)$$



where  $X \subset \mathbb{R}^n$  is a compact, convex, nonempty set independent on  $P$  and  $Q$ , e.g.  $X = \{x \in \mathbb{R}^n : \sum_i x_i = 1, 0 \leq x_i \leq 0.25 \forall_i\}$ . If  $g(\cdot, \omega)$  is convex for all  $\omega \in \Omega$ , then from Theorem 2.3.11  $F_\alpha(x, \xi, P)$  is convex in  $(x, \xi)$ . Let  $(x_C^*(P), \xi_C^*(P))$  be an optimal solution of the problem (3.5) and  $(x_C^*(Q), \xi_C^*(Q))$  its optimal solution for  $Q$ , then

$$\begin{aligned} (1 - \lambda)\varphi_C(P) + \lambda\varphi_C(Q) &\leq \\ &\leq \varphi_C(P_\lambda) \leq \\ &\leq \min \left\{ (1 - \lambda)\varphi_C(P) + \lambda F_\alpha(\xi_C^*(P), x_C^*(P), Q), \right. \\ &\quad \left. \lambda\varphi_C(Q) + (1 - \lambda)F_\alpha(\xi_C^*(Q), x_C^*(Q), P) \right\}. \end{aligned} \quad (3.6)$$

In [11] can be found a numerical experiment.

### 3.1.4 Mean-CVaR Optimization Problem

We solve a bi-criteria problem in which one minimizes  $CVaR_\alpha(x, P)$  and simultaneously maximizes the expected return  $\mathbb{E}_P r(x, Y)$  for  $r(x, \omega) = -x'\omega$  on  $X$ . Thus, using knowledge from [10], either we can solve the parametrized objective function problem

$$\min_{x \in X} CVaR_\alpha(x, P) - \rho x' \mathbb{E}_P Y$$

for some value of parameter  $\rho > 0$  or "ε-constrained" problem with a threshold  $r$  for the expected return objective function, i.e.

$$\min CVaR_\alpha(x, P) \text{ on the set } X(P, r) = \{x \in X : -x' \mathbb{E}_P Y \geq r\}. \quad (3.7)$$

Let  $\varphi_r(P)$  denote the optimal value and  $X_r^*(P) \neq \emptyset$  be the bounded set of optimal solutions of (3.7).

1. Under the assumption  $\mathbb{E}_P Y = \mathbb{E}_Q Y = \bar{Y}$  the set of feasible solutions  $X_r(P) = \{x \in X : -x'\bar{Y} \geq r\}$  does not depend on  $P$ ,  $Q$  and  $P_\lambda$ . If we replace  $X$  by  $X_r(P)$  in the optimization problem (3.5), the contamination bounds have a similar form as (3.6).
2. If  $\mathbb{E}_P Y \neq \mathbb{E}_Q Y$  it is possible to get from (3.6) a raw estimate for the lower bound

$$\varphi_r(P_\lambda) \geq \varphi_C(P_\lambda) \geq (1 - \lambda)\varphi_C(P) + \lambda\varphi_C(Q). \quad (3.8)$$

To get an upper bound we define the set  $X(P, Q, r) := X(P, r) \cap X(Q, r) \subset X(P_\lambda, r)$ , which does not depend on  $\lambda$ . If  $X(P, Q, r) \neq \emptyset$ , we set

$$U_r(\lambda) := \min_{x \in X(P, Q, r)} CVaR_\alpha(x, P_\lambda) \geq \varphi_r(P_\lambda).$$

$U_r(\lambda)$  is concave in  $\lambda$ , thus

$$\varphi_r(P_\lambda) \leq U_r(\lambda) \leq (1 - \lambda)U_r(0) + \lambda F_\alpha(\hat{x}_r(P), \hat{\xi}_r(P), Q), \quad (3.9)$$

where  $(\hat{x}_r(P), \hat{\xi}_r(P))$  is one of optimal solutions of

$$\min_{(x, \xi) \in \mathbb{R} \times X(P, Q, r)} F_\alpha(x, \xi, P).$$

It is necessary to remark, that the bounds (3.8) and (3.9) can be quite loose.

## 3.2 Stress Testing for VaR

Value at Risk can be seen as a solution of the stochastic program with one probabilistic constraint, i.e.

$$\begin{aligned} VaR_\alpha(x, P) &:= \min \xi \\ &\quad s.t. \\ &P(g(x, Y) \leq \xi) \geq \alpha. \end{aligned}$$

This expression enables us to use the stability results for problems of such form, but we will not apply it here. Some general results are in [37]. In Sections 3.2.1 and 3.2.2 results from [11] are summarized, in Section 3.2.3 new results are introduced.

### 3.2.1 Optimal Solution of the Minimization Formula

Let  $x \in X$  is fixed. We assume that the distribution function  $\psi(x, P; \xi)$  is continuous with a positive, continuous density  $p(x, P; \xi)$  on a neighborhood of the unique solution of the initial problem (3.1). The unique solution equals  $VaR_\alpha(x, P)$ , see Theorem 2.3.8. For sensitivity study we can use the following general theorem.

**Theorem 3.2.1.** (About implicit function)[42]

Let  $p, n \in \mathbb{N}$ ,  $a \in \mathbb{R}^n$ ,  $b \in \mathbb{R}$ ,  $h(x_1, \dots, x_n, y) = h(x, y)$  is a function of  $(n + 1)$  variables and

1.  $h(a, b) = 0$ ,
2.  $h \in C^p(V)$ , where  $V := U_r(a, b)$  for some  $r > 0$ ,
3.  $\frac{\partial h}{\partial y}(a, b) \neq 0$ .

Then there exist  $\delta > 0$  and  $\Delta > 0$  such that it holds

1.  $U_\delta(a) \times U_\Delta(b) \subset V$ .
2.  $\forall_{x \in U_\delta(a)} \exists!_{y \in U_\Delta(b)} y := f(x)$  and  $h(x, y) = 0$ .
3.  $f \in C^p(U_\delta(a))$ .

The optimal solution of the contaminated problem (3.1) solves the equation

$$h(\lambda, \xi) := \frac{d}{d\xi} F_\alpha(x, \xi, \lambda) = \frac{\psi(x, P_\lambda; \xi) - \alpha}{1 - \alpha} = 0, \quad (3.10)$$

see proof of Theorem 2.3.8. We apply Theorem 3.2.1 to previous implicit function. Filling the assumptions for "point"  $(0, VaR_\alpha(x, P))$

1.  $h(0, VaR_\alpha(x, P)) = 0$ ,
2.  $h \in C^1(V)$ , where  $V := U_r(0, VaR_\alpha(x, P))$  for some  $r > 0$ ,
- 3.

$$\frac{\partial h}{\partial \xi}(0, VaR_\alpha(x, P)) = \frac{p(x, P; VaR_\alpha(x, P))}{1 - \alpha} > 0$$

we get  $\delta > 0$  and  $\Delta > 0$  so that for any  $0 \leq \lambda < \delta$  just one solution  $U_\Delta(VaR_\alpha(x, P)) \ni VaR_\alpha(x, P_\lambda) \in C^1([0, \delta])$  of the contaminated problem (3.1) exists and it fulfils (3.10). Thanks to the assumptions it holds

$$VaR_\alpha(x, P_\lambda) = \psi^{-1}(x, P_\lambda; \alpha),$$

and for the derivative we have

$$\left. \frac{\partial}{\partial \lambda} VaR_\alpha(x, P_\lambda) \right|_{\lambda=0+} = - \frac{\psi(x, Q; VaR_\alpha(x, P)) - \alpha}{p(x, P; VaR_\alpha(x, P))}.$$

For small  $\lambda > 0$  we can use the following approximation of the VaR optimal value

$$\min_{x \in X} VaR_\alpha(x, P_\lambda) \cong VaR_\alpha(x^*(0), P) + \lambda \cdot \partial VaR_\alpha(x, P_\lambda) / \partial \lambda|_{\lambda=0+}.$$

### 3.2.2 Discrete-scenario VaR

Using Theorem 2.3.7 for a fixed  $x \in X$  and one additional stress scenario  $\omega^*$  we can construct under some additional conditions a finite number of non-overlapping intervals  $[0, \lambda_1], (\lambda_1, \lambda_2], \dots, (\lambda_I, 1]$  and study stability of  $VaR_\alpha(x, P_\lambda)$  separately on each of them. (We consider degenerate intervals.)

Setting

$$z^{[1]} < \dots < z^{[k_{\omega^*}-1]} \leq g(x, \omega^*) < z^{[k_{\omega^*}]} < \dots < z^{[N]}$$

with probabilities

$$(1 - \lambda)p^{[1]}, \dots, (1 - \lambda)p^{[k_{\omega^*}-1]}, \lambda, (1 - \lambda)p^{[k_{\omega^*}]}, \dots, (1 - \lambda)p^{[N]}$$

and with index  $k_{\alpha, P_\lambda} \in \{1, \dots, N + 1\}$  such that

$$\sum_{k=1}^{k_{\alpha, P_\lambda}-1} (1 - \lambda)p^{[k]} < \alpha \leq \sum_{k=1}^{k_{\alpha, P_\lambda}} (1 - \lambda)p^{[k]},$$

we get  $VaR_\alpha(x, P_\lambda) = z^{[k_{\alpha, P_\lambda}]}$ . Let **the stress scenario satisfy**  $z^{[k_{\alpha, P}]} < z^{[k_{\omega^*}-1]}$ . If  $\lambda$  is sufficiently small, i.e.

$$0 \leq \lambda \leq 1 - \frac{\alpha}{\sum_{k=1}^{k_{\alpha, P}} p^{[k]}} =: \lambda_1$$

then  $VaR_\alpha(x, P_\lambda) = VaR_\alpha(x, P) = z^{[k_{\alpha, P}]}$ . Thus we have got the first interval  $[0, \lambda_1]$ . Similarly we can construct the intervals  $(\lambda_i, \lambda_{i+1}]$ , where

$$\lambda_i = 1 - \frac{\alpha}{\sum_{k=1}^{k_{\alpha, P+i-1}} p^{[k]}}, \quad i = 1, \dots, k_{\omega^*} - k_{\alpha, P} - 1,$$

on which  $VaR_\alpha(x, P_\lambda) = z^{[k_{\alpha, P+i}]} \leq g(x, \omega^*) = VaR_\alpha(x, Q)$ . Finally, for  $(\lambda_I, 1]$ ,  $I := k_{\omega^*} - k_{\alpha, P}$ ,  $VaR_\alpha(x, P_\lambda)$  equals  $VaR_\alpha(x, Q)$ .

### 3.2.3 Optimization with the Relative VaR Objective Function

In this section we provide a way to construct contamination bounds for portfolio optimization problem with the relative VaR objective function, where we consider volatility and correlations shocks. A crucial question in using relative VaR is estimation of asset returns correlations and volatilities. We can suppose that the volatilities and the correlations are stressed to reflect a forecast that differs from their historical estimates.

Under the assumptions  $\alpha > 0.5$ ,  $g(x, \omega) = x'\omega$ ,  $X \subset \mathbb{R}^n$  nonempty, convex polyhedral set,  $0 \notin X$ ,  $Y \sim \mathcal{N}_n(\mu, \Sigma)$  we can get "relative VaR optimal" portfolio  $x_{rel}^*(\Sigma)$  (dependent on the covariance matrix  $\Sigma$ ) by solving the following optimization problem

$$VaR_\alpha^{rel}(x_{rel}^*(\Sigma), \mathcal{N}_n(\mu, \Sigma)) = \min_{x \in X} u_\alpha \sqrt{x' \Sigma x} \quad (3.11)$$

or the convex quadratic program

$$\min_{x \in X} x' \Sigma x. \quad (3.12)$$

We can rewrite the variance matrix as

$$\Sigma = DCD$$

(as it is done in [22]) with the diagonal matrix  $D$  of standard deviations of marginal distributions and the correlation matrix  $C$ .

#### Correlations stressing

Using a positive semidefinite correlation matrix  $\hat{C}$ , we get the contaminated correlation matrix as  $C(\lambda) = (1 - \lambda)C + \lambda\hat{C}$ ,  $\lambda \in [0, 1]$ . We may exploit stability results valid for quadratic programming, see [3], to the perturbed problem

$$\varphi_V(\lambda) = \min_{x \in X} x' DC(\lambda) Dx, \quad \lambda \in [0, 1]. \quad (3.13)$$

The optimal value  $\varphi_V(\lambda)$  is concave and continuous in  $\lambda$ , the optimal solution of (3.13)  $x^*(\lambda)$  is continuous in the range of  $\lambda$  where  $C(\lambda)$  is positive definite and using the result (3.20) we get

$$\begin{aligned} \varphi_V'(0^+) &= \min_{x \in X^*(0)} x' D\hat{C} Dx - \varphi_V(0), \\ \varphi_V'(1^-) &= \min_{x \in X^*(1)} x' DC Dx - \varphi_V(1). \end{aligned} \quad (3.14)$$

where  $X^*(\lambda) = \arg \min_{x \in X} x' DC(\lambda) Dx$ ,  $\lambda = 0, 1$ . Now we can construct contamination bounds for (3.13) as

$$\begin{aligned} &(1 - \lambda)\varphi_V(0) + \lambda\varphi_V(1) \leq \\ &\leq \varphi_V(\lambda) \leq \\ &\leq \min \left\{ \varphi_V(0) + \lambda\varphi_V'(0^+), \varphi_V(1) + (1 - \lambda)\varphi_V'(1^-) \right\}, \\ &\lambda \in [0, 1] \end{aligned}$$

which can be rewritten using (3.14) as

$$\begin{aligned}
& (1 - \lambda)\varphi_V(0) + \lambda\varphi_V(1) \leq \\
& \leq \varphi_V(\lambda) \leq \\
& \leq \min \left\{ (1 - \lambda)\varphi_V(0) + \lambda x^*(0)' D \hat{C} D x^*(0), \right. \\
& \quad \left. \lambda\varphi_V(1) + (1 - \lambda)x^*(1)' D C D x^*(1) \right\}, \\
& \quad \lambda \in [0, 1].
\end{aligned} \tag{3.15}$$

where  $x^*(\lambda) \in X^*(\lambda)$ ,  $\lambda = 0, 1$ . When we apply the increasing function  $u_\alpha \sqrt{\cdot}$  to the bounds (3.15), we can get "rough" contamination bounds for (3.11):

$$\begin{aligned}
& u_\alpha \sqrt{(1 - \lambda)\varphi_V(0) + \lambda\varphi_V(1)} \leq \\
& \leq VaR_\alpha^{rel} \left( x_{rel}^*(\lambda), \mathcal{N}_n(\mu, DC(\lambda)D) \right) \leq \\
& \leq \min \left\{ u_\alpha \sqrt{(1 - \lambda)\varphi_V(0) + \lambda x^*(0)' D \hat{C} D x^*(0)}, \right. \\
& \quad \left. u_\alpha \sqrt{\lambda\varphi_V(1) + (1 - \lambda)x^*(1)' D C D x^*(1)} \right\}, \\
& \quad \lambda \in [0, 1].
\end{aligned} \tag{3.16}$$

### Volatility stressing

We consider a stress test scenario of volatility shocks. For this purpose we construct a diagonal matrix  $\Delta$  in which the elements are equal to the increments by which historical estimates of standard deviations differ from those desired in the stress test scenario. The covariance matrix  $\hat{\Sigma}$  that should be used for stressing is given by

$$\hat{\Sigma} = (D + \Delta)C(D + \Delta).$$

Using this modified covariance matrix, we can get the contaminated covariance matrix as  $\Sigma(\lambda) = (1 - \lambda)\Sigma + \lambda\hat{\Sigma}$ ,  $\lambda \in [0, 1]$ . We can express contamination bounds for the contaminated problem

$$\phi_V(\lambda) = \min_{x \in X} x' \Sigma(\lambda) x, \quad \lambda \in [0, 1] \tag{3.17}$$

with concave optimal value function, analogously to (3.13), as

$$\begin{aligned}
& (1 - \lambda)\phi_V(0) + \lambda\phi_V(1) \leq \\
& \leq \phi_V(\lambda) \leq \\
& \leq \min \left\{ \phi_V(0) + \lambda\phi_V'(0^+), \phi_V(1) + (1 - \lambda)\phi_V'(1^-) \right\}, \\
& \quad \lambda \in [0, 1].
\end{aligned}$$

For the optimal value of the relative VaR optimization problem (3.11) with the contaminated covariance matrix  $\Sigma(\lambda)$  we get

$$\begin{aligned}
& u_\alpha \sqrt{(1-\lambda)\phi_V(0) + \lambda\phi_V(1)} \leq \\
& \leq \text{VaR}_\alpha^{\text{rel}}\left(x_{\text{rel}}^*(\lambda), \mathcal{N}_n(\mu, \Sigma(\lambda))\right) \leq \\
& \leq \min \left\{ u_\alpha \sqrt{(1-\lambda)\phi_V(0) + \lambda x^*(0)' \hat{\Sigma} x^*(0)}, \right. \\
& \quad \left. u_\alpha \sqrt{\lambda\phi_V(1) + (1-\lambda)x^*(1)' \Sigma x^*(1)} \right\}, \\
& \quad \lambda \in [0, 1],
\end{aligned} \tag{3.18}$$

where  $x^*(\lambda)$  are some solutions of (3.17) for  $\lambda = 0, 1$ .

Of course, it is possible to use another method to construct the stressing covariance matrix  $\hat{\Sigma}$  than above. Contamination bounds has then the identical form as (3.18).

### 3.2.4 Numerical Example

In this subsection the practical implementation of stress testing techniques is demonstrated. We compute the contamination bounds (3.15) for the quadratic program (3.13) and then we use them to estimate the "rough" contamination bounds (3.16) for the relative VaR optimization problem with correlation shocks.

We use monthly prices and dividends from 6.2.1996 to 6.2.2006 to estimate the correlation matrix  $C$ , the matrix  $D$  of marginal standard deviations and means (see Table 3.6) of log-returns of 22 american corporate shares. We have got 120 observations of the monthly log-returns for each of the following companies: Advanced Micro Devices Inc. (AMD), Alcoa Inc. (AA), American Electric Power Co. Inc. (AEP), Advanced Micro Devices Inc. (AMD), Avon Products Inc. (AVP), American Express Co. (AXP), Boeing Co. (BA), Bank of America Corp. (DE) (BAC), Caterpillar Inc. (CAT), Colgate-Palmolive Co. (CL), Cisco Systems Inc. (CSCO), Dell Inc. (DELL), General Dynamics Corp. (GD), Harrah's Entertainment Inc. (HET), Hewlett-Packard Co. (HPQ), International Business Machines Corp. (IBM), Intel Corp. (INTC), Lockheed Martin Corp. (LMT), Microsoft Corp. (MSFT), Northrop Grumman Corp. (NOC), Oracle Corp. (ORCL), Texas Instruments Inc. (TXN), Unisys Corp. (UIS).

Statistical tests (Kolmogorov-Smirnov) and histograms do not reject normality of the log-returns (Kendall hypothesis) on level 0.95.  $X = \{x \in \mathbb{R}^{22} : \sum_{i=1}^{22} x_i = 1, 0 \leq x_i \leq 0.25, \forall_{i=1, \dots, 22}\}$  is taken as the set of feasible solution for our optimization problems, we do not involve the constraint on return. The relative VaR level equals 0.95, i.e.  $\alpha = 0.95$ .

A methodology for correlation matrices stressing is introduced in Section 7.2. Using it we create the stressing correlation matrix  $\hat{C}$ , where we suppose that the correlations between assets of companies from the same sector and industry will increase, while the others will not change significantly. We have got the following "groups" of companies:

- Technology (sector) - Application Software (industry), companies: MSFT, ORCL

[coefficient of increase  $\theta = 0.4$ , for details see Section 7.2], correlation before and after stressing are entered in Table 3.1,

- Technology - Diversified Computer Systems: HPQ, IBM [0.2], Table 3.1,
- Consumer Goods - Personal Products: AVP, CL [0.2], Table 3.1,
- Technology - Semiconductor/Broad Line: AMD, INTC, TXN [0.1], Table 3.2,
- Industrial Goods - Aerospace/Defense-Major Diversified: BA, GD, LMT, NOC [0.15], Table 3.3.

The whole correlation matrices  $C, \hat{C}$  and statistical tests can be found in Section 7.5.

As you can see from Table 3.4 and Figure 3.1., although the gaps between bounds are not too wide, behaviour of the relative VaR optimal value is very unstable. Even, if a little change in correlations between the log-returns occurs, change in the optimal value will be relatively large.

Figure 3.1: Contamination bounds for the relative VaR optimization problem

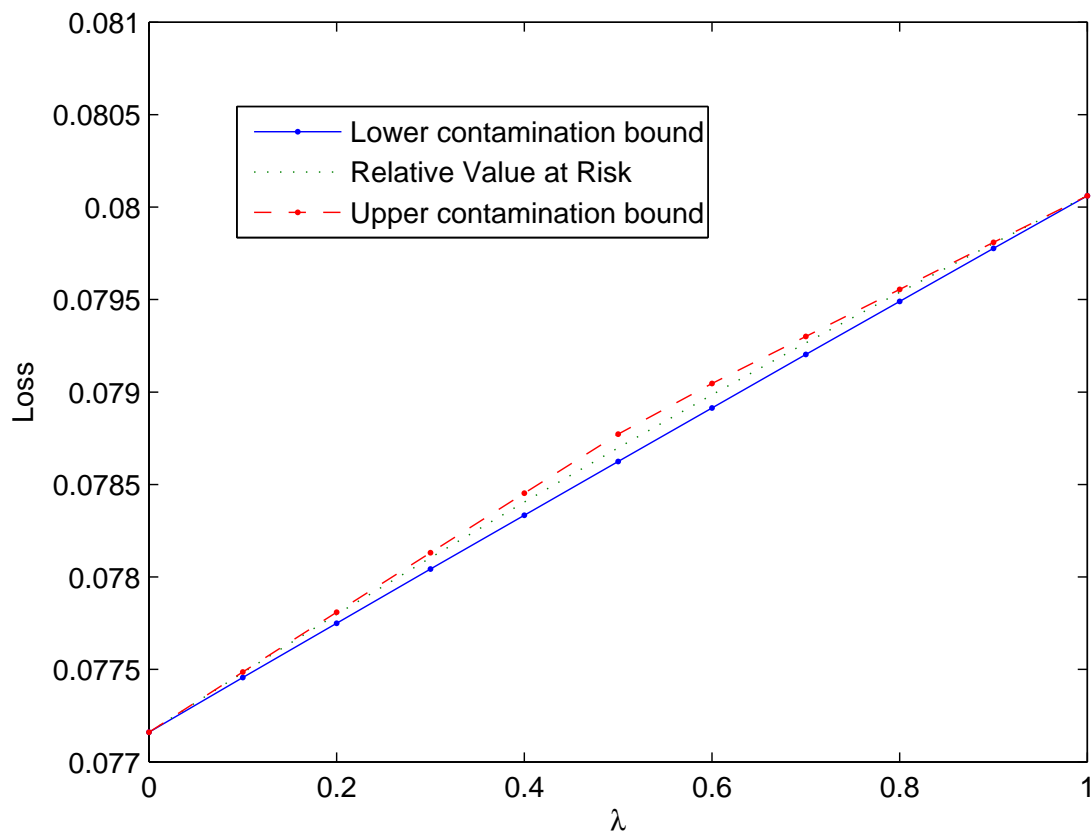


Table 3.1: Correlation estimates and stressing correlations.

	Correlation estimates	Stressing correlations
	ORCL	ORCL
MSFT	0.3335	0.6950
	IBM	IBM
HPQ	0.5137	0.6589
	CL	CL
AVP	0.5768	0.7069

Table 3.2: Submatrix of the correlation matrix and the stressing correlation matrix.

	Correlation estimates			Stressing correlations		
	AMD	INTC	TXN	AMD	INTC	TXN
AMD	1.0000	0.5937	0.5354	1.0000	0.6537	0.6033
INTC	0.5937	1.0000	0.7104	0.6537	1.0000	0.7541
TXN	0.5354	0.7104	1.0000	0.6033	0.7541	1.0000

Table 3.3: Submatrix of the correlation matrix and the stressing correlation matrix.

	Correlation estimates				Stressing correlations			
	BA	GD	LMT	NOC	BA	GD	LMT	NOC
BA	1.0000	0.3234	0.2499	0.3003	1.0000	0.4358	0.3767	0.4194
GD	0.3234	1.0000	0.4703	0.4537	0.4358	1.0000	0.5658	0.5528
LMT	0.2499	0.4703	1.0000	0.6272	0.3767	0.5658	1.0000	0.6959
NOC	0.3003	0.4537	0.6272	1.0000	0.4194	0.5528	0.6959	1.0000

Table 3.4: Contamination bounds and relative VaR.

$\lambda$	0	0.1	0.2	0.3	0.4	0.5
Lower contamination bound	0.0772	0.0775	0.0777	0.0780	0.0783	0.0786
Relative VaR	0.0772	0.0775	0.0778	0.0780	0.0784	0.0787
Upper contamination bound	0.0772	0.0775	0.0778	0.0781	0.0785	0.0788
$\lambda$	0.6	0.7	0.8	0.9	1.0	-
Lower contamination bound	0.0789	0.0792	0.0795	0.0798	0.0801	-
Relative VaR	0.0790	0.0793	0.0795	0.0798	0.0801	-
Upper contamination bound	0.0790	0.0793	0.0796	0.0798	0.0801	-



Table 3.5: Portfolio weights.

$\lambda$	0	1	$\lambda$	0	1
AA	0	0	GD	0.0368	0.0405
AEP	0.2500	0.2500	HET	0.0536	0.0659
AMD	0	0	HPQ	0.0313	0.0295
AVP	0	0	IBM	0.0138	0.0033
AXP	0	0	INTC	0.0097	0.0118
BA	0	0	LMT	0.1416	0.1437
BAC	0.1647	0.1749	MSFT	0.0627	0.0626
CAT	0	0	NOC	0.0358	0.0192
CL	0.1519	0.1583	ORCL	0.0620	0.0403
CSCO	0	0.	TXN	0	0
DELL	0	0	UIS	0	0

Table 3.6: Monthly mean logarithmic returns and standard deviations.

	Mean log-return	Standard deviation
AA	0.0083	0.1015
AEP	0.0031	0.0666
AMD	0.0117	0.2082
AVP	0.0104	0.1104
AXP	0.0126	0.0774
BA	0.0058	0.0902
BAC	0.0099	0.0777
CAT	0.0138	0.0873
CL	0.0103	0.0790
CSCO	0.0103	0.1324
DELL	0.0276	0.1402
GD	0.0135	0.0710
HET	0.0087	0.0968
HPQ	0.0046	0.1227
IBM	0.0085	0.0945
INTC	0.0090	0.1392
LMT	0.0062	0.0864
MSFT	0.0135	0.1124
NOC	0.0083	0.0875
ORCL	0.0096	0.1582
TXN	0.0137	0.1430
UIS	0.0004	0.1682

## 3.3 Appendix

### 3.3.1 Bounds and Worst Case Analysis

The worst case analysis is an additional approach of output analysis for stochastic programming with respect to the input probability distribution which we does not know exactly, see [9]. Therefore we use some empirical approximations of it.

If we assume that  $P$  belongs to a family  $\mathcal{P}$  of probability distributions, we can construct *minmin* and *minmax bounds* for the optimal value  $\varphi(P) = \min_{x \in X} G(x, P)$  as

$$\min_{x \in X} \inf_{P \in \mathcal{P}} G(x, P) \leq \varphi(P) \leq \min_{x \in X} \sup_{P \in \mathcal{P}} G(x, P).$$

### 3.3.2 Parametric Programming

The aim of this section is to summarize some results of parametric programming from [3] and [9] which are useful for the contamination techniques. We assume that

1.  $\Lambda$  is a non-empty convex subset of  $\mathbb{R}^s$ ,
2.  $\mathcal{X}$  is a non-empty convex subset of  $\mathbb{R}^n$ ,
3.  $G(x, \lambda)$  is a real-valued function defined on  $\mathbb{R}^s \times \mathbb{R}^n$  and  $G(x, \cdot)$  is for each  $x \in \mathbb{R}^n$  *affine-linear* function on  $\mathbb{R}^s$ , i.e. for all  $x \in \mathcal{X}$  there exists an  $a \in \mathbb{R}^s$  and  $b \in \mathbb{R}$  such that  $G(x, \lambda) = a\lambda + b$ ,  $\lambda \in \Lambda$ .

The basic parametric problem considered in this section has the form

$$\varphi(\lambda) := \min_{x \in \mathcal{X}} G(x, \lambda), \quad \lambda \in \Lambda,$$

where  $\varphi(\lambda) : \Lambda \rightarrow \mathbb{R} \cup \{-\infty\}$  is the *extreme value function*. We also define the *optimal set mapping*  $\psi$ , the *convexity set*  $\mathcal{G}_{G, \mathcal{X}}$  of the function  $G$  with respect to  $\mathcal{X}$ , and  $\text{fin } \varphi$  by

$$\begin{aligned} \psi(\lambda) &:= \{x \in \mathcal{X} : G(x, \lambda) = \varphi(\lambda)\}, \\ \mathcal{G}_{G, \mathcal{X}} &:= \{\lambda \in \mathbb{R}^s : G(\cdot, \lambda) \text{ is convex on } \mathcal{X}\}, \\ \text{fin } \varphi &:= \{\lambda \in \Lambda : \varphi(\lambda) > -\infty\}. \end{aligned}$$

#### Theorem 3.3.1. [3]

*Under previous assumptions*

1.  $\varphi(\lambda)$  is concave and  $\text{fin } \varphi$  is a convex subset of  $\Lambda$ .
2.  $\mathcal{G}_{G, \mathcal{X}}$  is a closed convex subset of  $\mathbb{R}^s$ .
3. If  $\mathcal{G}_{G, \mathcal{X}}$  is non-empty then the functions  $G(\cdot, \lambda)$  are strictly convex on  $\mathcal{X}$  either for all  $\lambda \in \text{ri } \mathcal{G}_{G, \mathcal{X}}$  or for no  $\lambda \in \mathcal{G}_{G, \mathcal{X}}$ , where  $\text{ri } C$  denotes the relative interior<sup>1</sup> of a convex set  $C$ .

---

<sup>1</sup>A point  $x \in \mathbb{R}^s$  belongs to the relative interior of the convex set  $C$  if  $x$  is an interior point of  $C$  relative to the affine space generated by  $C$ , i.e. there exists a neighborhood of  $x$  such that its intersection with the affine space generated by  $C$  is included in  $C$ . The affine space generated by  $C$  is the space of points in  $\mathbb{R}^s$  of the form  $tx + (1-t)y$ , where  $x, y \in C$  and  $t \in \mathbb{R}$ .

4. If  $G(x, \lambda)$  is in addition continuous on  $\mathcal{X} \times \Lambda$ , then the mapping  $\psi$  is closed on  $\Lambda$ .

Let  $\Lambda \subset \mathbb{R}$ . We denote  $G(u, \lambda) = G(u, P_\lambda)$  the objective function for the contaminated distribution and by

$$\varphi(\lambda) := \min_{u \in \mathcal{U}} G(u, \lambda), \quad \lambda \in [0, 1] \quad (3.19)$$

the optimal value function. Suppose that we have nonempty, compact sets of optimal solutions  $\mathcal{U}^*(P), \mathcal{U}^*(Q)$  of the initial problem ((3.19) for  $\lambda = 0$ ) and the completely contaminated problem ((3.19) for  $\lambda = 1$ ). If  $G(u, \lambda)$  is a linear function in  $\lambda$  and convex in  $u$ , then the optimal value  $\varphi(\lambda)$  is a finite concave function on  $[0, 1]$ , right continuous at 0 and left continuous at 1. For the directional derivatives of the optimal value function it holds

$$\begin{aligned} \varphi'(0^+) &= \left. \frac{d}{d\lambda} \varphi(\lambda) \right|_{\lambda=0^+} = \min_{u \in \mathcal{U}^*(P)} G(u, Q) - \varphi(0), \\ \varphi'(1^-) &= \left. \frac{d}{d\lambda} \varphi(\lambda) \right|_{\lambda=1^-} = \min_{u \in \mathcal{U}^*(Q)} G(u, P) - \varphi(1). \end{aligned} \quad (3.20)$$

# Chapter 4

## Sensitivity Analysis of VaR and CVaR

The main objective of this chapter is to study sensitivity of the measures Value at Risk and Conditional Value at Risk through their derivatives with respect to the portfolio allocation. These derivatives are introduced in articles [15], [34], however their proofs are not always obvious or some assumptions are missing.

This Chapter is organized as follows: In Section 4.1 basic assumptions and notion are introduced. In Sections 4.2, 4.3, the goal is to specify the conditions under which it is possible to get explicit expressions for the first and second order derivatives and we prove these expressions properly. In Section 4.4 we extend so far introduced results and find Hessians of CVaR and VaR, which allow us to discuss convexity of these measures. Section 4.5 contains needful statements from mathematical analysis.

### 4.1 Assumptions and Marking

In this chapter we denote Value at Risk of the loss random variable  $Z$  as  $VaR_\alpha(Z)$  and Conditional Value at Risk as  $CVaR_\alpha(Z)$ . We assume that  $(Z_1, Y_2)$  is a continuously distributed random loss vector on  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), P_1 \otimes P_2)$  with density  $f(z_1, y_2)$ , where  $Z_1$  denotes random loss of a portfolio without an asset with random loss  $Y_2$ , and  $L$  is an open interval,  $0 \in L$ . For the density of the loss random variable  $Z(x_2) := Z_1 + x_2 Y_2$ ,  $x_2 \in L$ , it holds

$$f_{Z(x_2)}(z) = \begin{cases} \int_{-\infty}^{\infty} f(z - x_2 y_2, y_2) dy_2 & \text{if } \int_{-\infty}^{\infty} f(z - x_2 y_2, y_2) dy_2 < \infty, \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

Then the conditional density is defined by

$$f_{Y_2|Z(x_2)}(y_2|z) = \begin{cases} \frac{f(z - x_2 y_2, y_2)}{f_{Z(x_2)}(z)} & \text{if } f_{Z(x_2)}(z) > 0, \\ 0 & \text{if } f_{Z(x_2)}(z) = 0, \end{cases} \quad (4.2)$$

Specially,  $f_{Z_1}(z_1)$  denotes the marginal density of  $Z_1$ , i.e.

$$f_{Z_1}(z_1) = \begin{cases} \int_{-\infty}^{\infty} f(z_1, y_2) dy_2 & \text{if } \int_{-\infty}^{\infty} f(z_1, y_2) dy_2 < \infty, \\ 0 & \text{otherwise} \end{cases} \quad (4.3)$$

and the conditional density is defined by

$$f_{Y_2|Z_1}(y_2|z_1) = \begin{cases} \frac{f(z_1, y_2)}{f_{Z_1}(z_1)} & \text{if } f_{Z_1}(z_1) > 0 \\ 0 & \text{if } f_{Z_1}(z_1) = 0, \end{cases} \quad (4.4)$$

We denote (A1) the assumptions of Theorem 4.5.1 about interchanging of derivative and integral, i.e. if  $h(t, x)$  satisfies

1.  $h(t, \cdot)$  is measurable on  $(X, \mathcal{A})$  for all  $t \in I$ ,
2. it exists a set  $N \subseteq X$  with  $\mu(N) = 0$  such that  $\forall t \in I \forall x \in X \setminus N$  the derivative  $\frac{d}{dt}h(t, x)$  exists and is finite,
3. it exists  $g(x) \in L^1(\mu)$  such that  $\forall t \in I \forall x \in X \setminus N \left| \frac{d}{dt}h(t, x) \right| \leq g(x)$ ,
4.  $\exists t_0 \in I h(t_0, \cdot) \in L^1(\mu)$ .

The symbol (F) means "according to Fubini's theorem 4.5.3".

## 4.2 First Order Derivatives of VaR and CVaR

In this section we assume that  $\mathbb{E}[Z_1] < \infty$  and  $\mathbb{E}[Y_2] < \infty$ .

**Theorem 4.2.1.** (First order derivative of VaR)

If  $h_1(x, y) = \int_{VaR_\alpha(Z(x)) - xy}^{\infty} f(z, y) dz$  fills the assumptions (A1), then

$$\frac{d VaR_\alpha(Z(x_2))}{dx_2} = \mathbb{E}\left[Y_2 \mid Z(x_2) = VaR_\alpha(Z(x_2))\right]. \quad (4.5)$$

Specially

$$\left. \frac{d VaR_\alpha(Z(x_2))}{dx_2} \right|_{x_2=0} = \mathbb{E}\left[Y_2 \mid Z_1 = VaR_\alpha(Z_1)\right]. \quad (4.6)$$

**Proof.**

$$\mathbb{P}\left[Z(x_2) > VaR_\alpha(Z(x_2))\right] = 1 - \alpha, \quad (4.7)$$

$$\begin{aligned} \mathbb{P}\left[Z_1 + x_2 Y_2 > VaR_\alpha(Z(x_2))\right] &\stackrel{(F)}{=} \\ &\stackrel{(F)}{=} \int_{-\infty}^{\infty} \left[ \int_{VaR_\alpha(Z(x_2)) - x_2 y_2}^{\infty} f(z_1, y_2) dz_1 \right] dy_2, \end{aligned} \quad (4.8)$$

From (4.7) and (4.8)

$$\begin{aligned}
0 &= \frac{d}{dx_2} \int_{-\infty}^{\infty} \left[ \overbrace{\int_{VaR_\alpha(Z(x_2)) - x_2 y_2}^{\infty} f(z_1, y_2) dz_1}^{=: h_1(x_2, y_2)} \right] dy_2 = \\
&= \int_{-\infty}^{\infty} \left[ \frac{d}{dx_2} \int_{VaR_\alpha(Z(x_2)) - x_2 y_2}^{\infty} f(z_1, y_2) dz_1 \right] dy_2 = \\
&= - \int_{-\infty}^{\infty} \left[ \frac{d VaR_\alpha(Z(x_2))}{dx_2} - y_2 \right] \cdot f(VaR_\alpha(Z(x_2)) - x_2 y_2, y_2) dy_2.
\end{aligned} \tag{4.9}$$

Combining (4.1) and (4.2), if for  $x_2 \in \mathbb{R}$  it holds

$$\int_{-\infty}^{\infty} f(VaR_\alpha(Z(x_2)) - x_2 y_2, y_2) dy_2 = f_{Z(x_2)}(VaR_\alpha(Z(x_2))) > 0,$$

then

$$\begin{aligned}
\frac{d VaR_\alpha(Z(x_2))}{dx_2} &= \frac{\int_{-\infty}^{\infty} y_2 f(VaR_\alpha(Z(x_2)) - x_2 y_2, y_2) dy_2}{\int_{-\infty}^{\infty} f(VaR_\alpha(Z(x_2)) - x_2 y_2, y_2) dy_2} = \\
&= \int_{-\infty}^{\infty} y_2 \cdot \frac{f(VaR_\alpha(Z(x_2)) - x_2 y_2, y_2)}{f_{Z(x_2)}(VaR_\alpha(Z(x_2)))} dy_2 =
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} y_2 f_{Y_2|Z(x_2)}(y_2 | VaR_\alpha(Z(x_2))) dy_2 = \\
&= \mathbb{E}[Y_2 | Z(x_2) = VaR_\alpha(Z(x_2))].
\end{aligned} \tag{4.11}$$

If

$$\int_{-\infty}^{\infty} f(VaR_\alpha(Z(x_2)) - x_2 y_2, y_2) dy_2 = f_{Z(x_2)}(VaR_\alpha(Z(x_2))) = 0,$$

then from (4.2)

$$\begin{aligned}
f_{Y_2|Z(x_2)}(y_2 | VaR_\alpha(Z(x_2))) &= 0 \Rightarrow \\
\Rightarrow \frac{d VaR_\alpha(Z(x_2))}{dx_2} &= \mathbb{E}[Y_2 | Z(x_2) = VaR_\alpha(Z(x_2))] = 0.
\end{aligned}$$

Specially, if  $\int_{-\infty}^{\infty} f(VaR_{\alpha}(Z_1), y_2) dy_2 > 0$ , then

$$\begin{aligned}
\left. \frac{d VaR_{\alpha}(Z(x_2))}{dx_2} \right|_{x_2=0} &= \frac{\int_{-\infty}^{\infty} y_2 f(VaR_{\alpha}(Z_1), y_2) dy_2}{\int_{-\infty}^{\infty} f(VaR_{\alpha}(Z_1), y_2) dy_2}, \\
&= \int_{-\infty}^{\infty} y_2 \cdot \frac{f(VaR_{\alpha}(Z_1), y_2)}{f_{Z_1}(VaR_{\alpha}(Z_1))} dy_2 = \\
&= \int_{-\infty}^{\infty} y_2 f_{Y_2|Z_1}(y_2|VaR_{\alpha}(Z_1)) dy_2, \\
&= \mathbb{E}[Y_2 | Z_1 = VaR_{\alpha}(Z_1)]
\end{aligned} \tag{4.12}$$

and, if  $\int_{-\infty}^{\infty} f(VaR_{\alpha}(Z_1), y_2) dy_2 = 0$ , then

$$\left. \frac{d VaR_{\alpha}(Z(x_2))}{dx_2} \right|_{x_2=0} = 0.$$

**Theorem 4.2.2.** (First order derivative of CVaR)

If  $h_1(x, y) = \int_{VaR_{\alpha}(Z(x)) - xy}^{\infty} f(z, y) dz$

and  $h_2(x, y) = \int_{VaR_{\alpha}(Z(x)) - xy}^{\infty} (z + xy) f(z, y) dz$  satisfy (A1), then

$$\frac{d CVaR_{\alpha}(Z(x_2))}{dx_2} = \mathbb{E}[Y_2 | Z(x_2) > VaR_{\alpha}(Z(x_2))]. \tag{4.13}$$

Specially

$$\left. \frac{d CVaR_{\alpha}(Z(x_2))}{dx_2} \right|_{x_2=0} = \mathbb{E}[Y_2 | Z_1 > VaR_{\alpha}(Z_1)]. \tag{4.14}$$

**Proof.**

$$\begin{aligned}
CVaR_{\alpha}(Z(x_2)) &= \mathbb{E}[Z_1 + x_2 Y_2 | Z_1 + x_2 Y_2 > VaR_{\alpha}(Z(x_2))] = \\
&= \frac{\mathbb{E}[(Z_1 + x_2 Y_2) I_{\{Z_1 + x_2 Y_2 > VaR_{\alpha}(Z(x_2))\}}]}{\mathbb{P}[Z_1 + x_2 Y_2 > VaR_{\alpha}(Z(x_2))]} \stackrel{(F)}{=} \\
&\stackrel{(F)}{=} \frac{1}{1 - \alpha} \int_{-\infty}^{\infty} \left[ \int_{VaR_{\alpha}(Z(x_2)) - x_2 y_2}^{\infty} (z_1 + x_2 y_2) f(z_1, y_2) dz_1 \right] dy_2,
\end{aligned} \tag{4.15}$$

Hence

$$(1 - \alpha) \cdot \frac{d CVaR_{\alpha}(Z(x_2))}{dx_2} =$$

$$\begin{aligned}
 &= \frac{d}{dx_2} \int_{-\infty}^{\infty} \left[ \overbrace{\int_{VaR_\alpha(Z(x_2)) - x_2 y_2}^{\infty} (z_1 + x_2 y_2) f(z_1, y_2) dz_1}^{=: h_2(x_2, y_2)} \right] dy_2 = \\
 &= \int_{-\infty}^{\infty} \left[ \frac{d}{dx_2} \int_{VaR_\alpha(Z(x_2)) - x_2 y_2}^{\infty} (z_1 + x_2 y_2) f(z_1, y_2) dz_1 \right] dy_2 = \\
 &= \int_{-\infty}^{\infty} \left[ \overbrace{\frac{d}{dx_2} \left( \int_{VaR_\alpha(Z(x_2)) - x_2 y_2}^{\infty} z_1 f(z_1, y_2) dz_1 \right)}^{\spadesuit} + \right. \\
 &\quad \left. + \underbrace{\frac{d}{dx_2} \left( x_2 y_2 \int_{VaR_\alpha(Z(x_2)) - x_2 y_2}^{\infty} f(z_1, y_2) dz_1 \right)}_{\clubsuit} \right] dy_2 \tag{4.16}
 \end{aligned}$$

$$\begin{aligned}
 \spadesuit &= - \left[ \frac{d VaR_\alpha(Z(x_2))}{dx_2} - y_2 \right] \left( VaR_\alpha(Z(x_2)) - x_2 y_2 \right) \times \\
 &\quad \times f \left( VaR_\alpha(Z(x_2)) - x_2 y_2, y_2 \right), \\
 \clubsuit &= y_2 \int_{VaR_\alpha(Z(x_2)) - x_2 y_2}^{\infty} f(z_1, y_2) dz_1 - \\
 &\quad - x_2 y_2 \left[ \frac{d VaR_\alpha(Z(x_2))}{dx_2} - y_2 \right] f \left( VaR_\alpha(Z(x_2)) - x_2 y_2, y_2 \right).
 \end{aligned}$$

Using  $\spadesuit$ ,  $\clubsuit$  and (4.9) we get

$$\frac{d CVaR_\alpha(Z(x_2))}{dx_2} = \frac{1}{1 - \alpha} \int_{-\infty}^{\infty} \left[ \int_{VaR_\alpha(Z(x_2)) - x_2 y_2}^{\infty} y_2 f(z_1, y_2) dz_1 \right] dy_2 \tag{4.17}$$

and finally

$$\frac{d CVaR_\alpha(Z(x_2))}{dx_2} = \mathbb{E} \left[ Y_2 \mid Z(x_2) > VaR_\alpha(Z(x_2)) \right].$$

Specially

$$\begin{aligned}
 \left. \frac{d CVaR_\alpha(Z(x_2))}{dx_2} \right|_{x_2=0} &= \frac{1}{1 - \alpha} \int_{-\infty}^{\infty} \left[ \int_{VaR_\alpha(Z_1)}^{\infty} y_2 f(z_1, y_2) dz_1 \right] dy_2, \\
 &= \mathbb{E} \left[ Y_2 \mid Z_1 > VaR_\alpha(Z_1) \right].
 \end{aligned}$$

□



### 4.3 Second Order Derivatives of VaR and CVaR

In this section we assume that  $\mathbb{E}[Z_1^2] < \infty$  and  $\mathbb{E}[Y_2^2] < \infty$ .

**Theorem 4.3.1.** (Second order derivative of CVaR)

If  $h_2(x, y) = \int_{\text{VaR}_\alpha(Z(x)) - xy}^{\infty} (z + xy) f(z, y) dz$  and  $h_3(x, y) = \int_{\text{VaR}_\alpha(Z(x)) - xy}^{\infty} y f(z, y) dz$  satisfy (A1), then

$$\begin{aligned} \frac{d^2 \text{CVaR}_\alpha(Z(x_2))}{dx_2^2} &= \frac{f_{Z(x_2)}(\text{VaR}_\alpha(Z(x_2)))}{1 - \alpha} \times \\ &\times \text{VAR}[Y_2 | Z(x_2) = \text{VaR}_\alpha(Z(x_2))]. \end{aligned} \quad (4.18)$$

Specially

$$\left. \frac{d^2 \text{CVaR}_\alpha(Z(x_2))}{dx_2^2} \right|_{x_2=0} = \frac{f_{Z_1}(\text{VaR}_\alpha(Z_1))}{1 - \alpha} \text{VAR}[Y_2 | Z_1 = \text{VaR}_\alpha(Z_1)]. \quad (4.19)$$

**Proof.** From (4.17)

$$\begin{aligned} &\frac{d^2 \text{CVaR}_\alpha(Z(x_2))}{dx_2^2} = \\ &= \frac{1}{1 - \alpha} \int_{-\infty}^{\infty} \underbrace{\frac{d}{dx_2} \left[ \int_{\text{VaR}_\alpha(Z(x_2)) - x_2 y_2}^{\infty} \overbrace{y_2 f(z_1, y_2)}^{=: h_3(x_2, y_2)} dz_1 \right]}_{\spadesuit} dy_2 = \\ &\spadesuit = - \left( \frac{d \text{VaR}_\alpha(Z(x_2))}{dx_2} - y_2 \right) f(\text{VaR}_\alpha(Z(x_2)) - x_2 y_2, y_2). \end{aligned} \quad (4.20)$$

Using  $\spadesuit$  and (4.2) for (4.20) we get

$$\begin{aligned} &= \frac{1}{1 - \alpha} \left[ - \frac{d \text{VaR}_\alpha(Z(x_2))}{dx_2} \int_{-\infty}^{\infty} y_2 f(\text{VaR}_\alpha(Z(x_2)) - x_2 y_2, y_2) dy_2 + \right. \\ &\left. + \int_{-\infty}^{\infty} y_2^2 f(\text{VaR}_\alpha(Z(x_2)) - x_2 y_2, y_2) dy_2 \right] = \end{aligned} \quad (4.21)$$

$$\begin{aligned} &= \frac{1}{1 - \alpha} \int_{-\infty}^{\infty} f(\text{VaR}_\alpha(Z(x_2)) - x_2 y_2, y_2) dy_2 \times \\ &\times \left[ - \frac{d \text{VaR}_\alpha(Z(x_2))}{dx_2} \int_{-\infty}^{\infty} y_2 \frac{f(\text{VaR}_\alpha(Z(x_2)) - x_2 y_2, y_2)}{\int_{-\infty}^{\infty} f(\text{VaR}_\alpha(Z(x_2)) - x_2 y_2, y_2) dy_2} dy_2 + \right. \\ &\left. + \int_{-\infty}^{\infty} y_2^2 \frac{f(\text{VaR}_\alpha(Z(x_2)) - x_2 y_2, y_2)}{\int_{-\infty}^{\infty} f(\text{VaR}_\alpha(Z(x_2)) - x_2 y_2, y_2) dy_2} dy_2 \right]. \end{aligned} \quad (4.22)$$

From (4.10) and (4.22)

$$\begin{aligned}
 & \frac{d^2 CVaR_\alpha(Z(x_2))}{dx_2^2} = \\
 &= \frac{\int_{-\infty}^{\infty} f(VaR_\alpha(Z(x_2)) - x_2 y_2, y_2) dy_2}{1 - \alpha} \times \\
 & \times \left[ \mathbb{E}[Y_2^2 | Z(x_2) = VaR_\alpha(Z(x_2))] - \left( \mathbb{E}[Y_2 | Z(x_2) = VaR_\alpha(Z(x_2))] \right)^2 \right] = \\
 &= \frac{f_{Z(x_2)}(VaR_\alpha(Z(x_2)))}{1 - \alpha} \text{VAR}[Y_2 | Z(x_2) = VaR_\alpha(Z(x_2))].
 \end{aligned}$$

Specially

$$\begin{aligned}
 & \left. \frac{d^2 CVaR_\alpha(Z(x_2))}{dx_2^2} \right|_{x_2=0} = \\
 &= \frac{\int_{-\infty}^{\infty} f(VaR_\alpha(Z_1), y_2) dy_2}{1 - \alpha} \times \\
 & \times \left[ \mathbb{E}[Y_2^2 | Z_1 = VaR_\alpha(Z_1)] - \left( \mathbb{E}[Y_2 | Z_1 = VaR_\alpha(Z_1)] \right)^2 \right] = \\
 &= \frac{f_{Z_1}(VaR_\alpha(Z_1))}{1 - \alpha} \text{VAR}[Y_2 | Z_1 = VaR_\alpha(Z_1)].
 \end{aligned}$$

□

**Theorem 4.3.2.** (Second order derivative of VaR)

If  $h_2(x, y) = \int_{VaR_\alpha(Z(x)) - xy}^{\infty} (z + xy) f(z, y) dz$

and  $h_3(x, y) = \int_{VaR_\alpha(Z(x)) - xy}^{\infty} y f(z, y) dz$  satisfy (A1) and if  $CVaR_\alpha(Z_1 + x_2 Y_2) = h_4(\alpha, x_2)$  as a function of  $\alpha$  and  $x_2$  is three times continuously differentiable, i.e.  $h_4(\alpha, x_2) \in C^3((0, 1) \times I)$ , then

$$\begin{aligned}
 & \frac{d^2 VaR_\alpha(Z(x_2))}{dx_2^2} = \\
 &= - \left[ \frac{d \text{VAR}[Y_2 | Z(x_2) = z]}{dz} + \text{VAR}[Y_2 | Z(x_2) = z] \frac{d \ln f_{Z(x_2)}(z)}{dz} \right] \Bigg|_{z=VaR_\alpha(Z(x_2))}.
 \end{aligned} \tag{4.23}$$

Specially

$$\begin{aligned} & \left. \frac{d^2 \text{VaR}_\alpha(Z(x_2))}{dx_2^2} \right|_{x_2=0} = \\ & = - \left[ \frac{d \text{VAR}[Y_2|Z_1 = z_1]}{dz_1} + \text{VAR}[Y_2|Z_1 = z_1] \frac{d \ln f_{Z_1}(z_1)}{dz_1} \right] \Bigg|_{z_1 = \text{VaR}_\alpha(Z_1)}. \end{aligned} \quad (4.24)$$

**Lemma 4.3.3.**

If  $Z$  is a continuously distributed loss random variable, then

1.

$$\begin{aligned} \text{CVaR}_\alpha(Z) &= \frac{1}{1-\alpha} \int_\alpha^1 z dF_Z = \\ &\stackrel{(\text{subst.})}{=} \left[ \begin{array}{l} z = \text{VaR}_s(Z) \\ s = \mathbb{P}(Z \leq z) = F(z) \end{array} \right] = \\ &= \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_s(Z) ds. \end{aligned} \quad (4.25)$$

2.

$$\begin{aligned} \text{VaR}_\alpha(Z) &= F_Z^{-1}(\alpha) \\ \frac{d F_Z^{-1}(\alpha)}{d\alpha} &= \frac{1}{f_Z(F_Z^{-1}(\alpha))}, \text{ i.e.} \\ \frac{d \text{VaR}_\alpha(Z)}{d\alpha} &= \frac{1}{f_Z(\text{VaR}_\alpha(Z))} \end{aligned} \quad (4.26)$$

**Proof of Theorem 4.3.2**

From (4.25)

$$\begin{aligned} \text{VaR}_\alpha(Z(x_2)) &= \frac{d \left[ -(1-\alpha) \text{CVaR}_\alpha(Z(x_2)) \right]}{d\alpha}, \\ \frac{-d^2 \text{VaR}_\alpha(Z)}{dx_2^2} &= \frac{d^2}{dx_2^2} \left[ \frac{d \left[ (1-\alpha) \text{CVaR}_\alpha(Z(x_2)) \right]}{d\alpha} \right], \end{aligned} \quad (4.27)$$

If it is possible to interchange the order of partial derivatives in (4.27), i.e. if  $\text{CVaR}_\alpha(Z_1 + x_2 Y_2) = f(\alpha, x_2) \in C^3((0, 1) \times I)$ , see Theorem 4.5.2, then

$$\frac{d}{d\alpha} \left[ \frac{d^2 \left[ (1-\alpha) \text{CVaR}_\alpha(Z(x_2)) \right]}{dx_2^2} \right] = \frac{d^2}{dx_2^2} \left[ \frac{d \left[ (1-\alpha) \text{CVaR}_\alpha(Z(x_2)) \right]}{d\alpha} \right] \quad (4.28)$$

and from (4.18) and (4.28)

$$\begin{aligned}
 & \frac{-d^2 \text{VaR}_\alpha(Z(x_2))}{dx_2^2} = \\
 & = \frac{d}{d\alpha} \left[ f_{Z(x_2)} \left( \text{VaR}_\alpha(Z(x_2)) \right) \mathbb{V}\text{AR} \left[ Y_2 \mid Z(x_2) = \text{VaR}_\alpha(Z(x_2)) \right] \right] \\
 & = \underbrace{\frac{d f_{Z(x_2)} \left( \text{VaR}_\alpha(Z(x_2)) \right)}{d\alpha}}_{\spadesuit} \cdot \mathbb{V}\text{AR} \left[ Y_2 \mid Z(x_2) = \text{VaR}_\alpha(Z(x_2)) \right] + \\
 & + f_{Z(x_2)} \left( \text{VaR}_\alpha(Z(x_2)) \right) \cdot \underbrace{\frac{d \mathbb{V}\text{AR} \left[ Y_2 \mid Z(x_2) = \text{VaR}_\alpha(Z(x_2)) \right]}{d\alpha}}_{\clubsuit}.
 \end{aligned} \tag{4.29}$$

From (4.26)

$$\begin{aligned}
 \spadesuit & = \left. \frac{d f_{Z(x_2)}(z)}{dz} \right|_{z=\text{VaR}_\alpha(Z(x_2))} \cdot \frac{1}{f_{Z(x_2)} \left( \text{VaR}_\alpha(Z(x_2)) \right)} = \\
 & = \left. \frac{d \ln f_{Z(x_2)}(z)}{dz} \right|_{z=\text{VaR}_\alpha(Z(x_2))}. \\
 \clubsuit & = \left. \frac{d \mathbb{V}\text{AR} [Y_2 \mid Z(x_2) = z]}{dz} \right|_{z=\text{VaR}_\alpha(Z(x_2))} \cdot \frac{d \text{VaR}_\alpha(Z(x_2))}{d\alpha} = \\
 & = \left. \frac{d \mathbb{V}\text{AR} [Y_2 \mid Z(x_2) = z]}{dz} \right|_{z=\text{VaR}_\alpha(Z(x_2))} \cdot \frac{1}{f_{Z(x_2)} \left( \text{VaR}_\alpha(Z(x_2)) \right)}.
 \end{aligned}$$

Now we can express from (4.29) the second order derivative of VaR (4.23). Specially

$$\begin{aligned}
 & \left. \frac{-d^2 \text{VaR}_\alpha(Z(x_2))}{dx_2^2} \right|_{x_2=0} = \\
 & = \frac{d}{d\alpha} \left[ f_{Z_1} \left( \text{VaR}_\alpha(Z_1) \right) \mathbb{V}\text{AR} \left[ Y_2 \mid Z_1 = \text{VaR}_\alpha(Z_1) \right] \right] \\
 & = \underbrace{\frac{d f_{Z_1} \left( \text{VaR}_\alpha(Z_1) \right)}{d\alpha}}_{\spadesuit^0} \cdot \mathbb{V}\text{AR} \left[ Y_2 \mid Z_1 = \text{VaR}_\alpha(Z_1) \right] + \\
 & + f_{Z_1} \left( \text{VaR}_\alpha(Z_1) \right) \cdot \underbrace{\frac{d \mathbb{V}\text{AR} \left[ Y_2 \mid Z_1 = \text{VaR}_\alpha(Z_1) \right]}{d\alpha}}_{\clubsuit^0}.
 \end{aligned} \tag{4.30}$$

From (4.26)

$$\begin{aligned}
\spadesuit^0 &= \left. \frac{d f_{Z_1}(z_1)}{dz_1} \right|_{z_1=VaR_\alpha(Z_1)} \cdot \frac{1}{f_{Z_1}(VaR_\alpha(Z_1))} = \\
&= \left. \frac{d \ln f_{Z_1}(z_1)}{dz_1} \right|_{z_1=VaR_\alpha(Z_1)} \\
\clubsuit^0 &= \left. \frac{d \text{VAR}[Y_2|Z_1 = z_1]}{dz_1} \right|_{z_1=VaR_\alpha(Z_1)} \cdot \frac{d VaR_\alpha(Z_1)}{d\alpha} = \\
&= \left. \frac{d \text{VAR}[Y_2|Z_1 = z_1]}{dz_1} \right|_{z_1=VaR_\alpha(Z_1)} \cdot \frac{1}{f_{Z_1}(VaR_\alpha(Z_1))}.
\end{aligned}$$

□

Some conditions for partial derivatives interchanging are also introduced in [17] (Theorem 195).

Interpretation of (4.5), (4.13) as marginal risk contributions in credit risk and proofs (not so exact as previous and without many assumptions) of these formulas and of the second order derivatives (4.18), (4.23) for incomes (profits)  $I = -Z$  can be found in [34].

## 4.4 Hessian of CVaR and VaR

Below we show that under certain assumptions it is possible to find Hessians of VaR and CVaR with respect to the portfolio allocation vector.

Let  $I = \{1, \dots, n\}$  and  $I(i, j) = I \setminus \{i, j\}$ ,  $\forall i, j \in I$ ,  $i \neq j$  (the case  $i = j$  is introduced in previous section) be index sets, the random future values have finite second order moments, i.e.  $\mathbb{E}[Y_k^2] < \infty$ ,  $\forall k \in I$ , the portfolio allocations belong to open intervals, i.e.  $x_k \in L_k \subset \mathbb{R}$ ,  $0 \in L_k$ ,  $\forall k \in I$ ,  $L_k$  be an open interval. We denote  $\mathbf{x}^T = (x_1, \dots, x_n)$  the decision vector and

$$\begin{aligned}
Z(\mathbf{x}) &= \sum_{i \in I} x_i Y_i, \\
Z_{I(i,j)} &= \sum_{k \in I(i,j)} x_k Y_k \text{ for fixed } x_k \in L_k, k \in I(i, j), \\
Z(x_i, x_j) &= Z_{I(i,j)} + x_i Y_i + x_j Y_j
\end{aligned}$$

the bilinear loss random functions. Let  $(Z_{I(i,j)}, Y_i, Y_j)$  be continuously distributed random vector on  $(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3), P_{I(i,j)} \otimes P_i \otimes P_j)$  with the density  $f(z, y_i, y_j)$ . Then

$$f_{Z(x_i, x_j)}(z) = \begin{cases} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z - x_i y_i - x_j y_j, y_i, y_j) dy_i dy_j \\ \text{if } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z - x_i y_i - x_j y_j, y_i, y_j) dy_i dy_j < \infty, \\ 0 \\ \text{otherwise,} \end{cases} \quad (4.31)$$

$$f_{Y_i|Z(x_i, x_j)}(y_i|z) = \begin{cases} \frac{\int_{-\infty}^{\infty} f(z-x_i y_i - x_j y_j, y_i, y_j) dy_j}{f_{Z(x_i, x_j)}(z)} & \text{if } f_{Z(x_i, x_j)}(z) > 0, \\ 0 & \text{if } f_{Z(x_i, x_j)}(z) = 0, \end{cases} \quad (4.32)$$

Because the prepositions and their proofs are analogous to the previous ones, we will proceed more quickly, denote steps and assumptions.

(A)

If  $h_4(x_i, y_i, y_j) = \int_{VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j}^{\infty} f(VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j, y_i, y_j) dz$  satisfies the assumptions (A1) for fixed  $x_j$ , then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{\partial VaR_\alpha(Z(x_i, x_j))}{\partial x_i} - y_i \right] \cdot f(VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j, y_i, y_j) dy_i dy_j = 0. \quad (4.33)$$

(B)

Under previous assumptions and if  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j, y_i, y_j) dy_i dy_j > 0$ , then

$$\begin{aligned} & \frac{\partial VaR_\alpha(Z(x_i, x_j))}{\partial x_i} = \\ &= \int_{-\infty}^{\infty} y_i \frac{\int_{-\infty}^{\infty} f(VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j, y_i, y_j) dy_j}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j, y_i, y_j) dy_i dy_j} dy_i = \\ &= \int_{-\infty}^{\infty} y_i f_{Y_i|Z(x_i, x_j)}(y_i|VaR_\alpha(Z(x_i, x_j))) dy_i = \end{aligned} \quad (4.34)$$

$$= \mathbb{E}[Y_i | Z(x_i, x_j) = VaR_\alpha(Z(x_i, x_j))]. \quad (4.35)$$

(C)

Under previous assumptions and if  $h_5(x_i, y_i, y_j) = \int_{VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j}^{\infty} (z + x_i y_i + x_j y_j) \cdot f(z, y_i, y_j) dz$  satisfies (A1) for fixed  $x_j$ , then

$$\begin{aligned} \frac{\partial CVaR_\alpha(Z(x_i, x_j))}{\partial x_i} &= \frac{1}{1-\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial}{\partial x_i} \overbrace{\left[ \int_{VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j}^{\infty} \right]}^{h(x_i, y_i, y_j)} \\ & \quad (z + x_i y_i + x_j y_j) \cdot f(z, y_i, y_j) dz dy_i dy_j = \\ &= \frac{1}{1-\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j}^{\infty} y_i \cdot f(z, y_i, y_j) dz dy_i dy_j = \\ &= \mathbb{E}[Y_i | Z(x_i, x_j) > VaR_\alpha(Z(x_i, x_j))]. \end{aligned} \quad (4.36)$$

(D)

Under previous assumptions and if  $h(x_j, y_i, y_j) = \int_{VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j}^{\infty} y_i \cdot f(z, y_i, y_j) dz$  satisfies (A1) for fixed  $x_i$ , then from (4.36)

$$\begin{aligned}
& (1 - \alpha) \cdot \frac{\partial^2 CVaR_\alpha(Z(x_i, x_j))}{\partial x_j \partial x_i} = \\
& = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial}{\partial x_j} \overbrace{\left[ \int_{VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j}^{\infty} y_i \cdot f(z, y_i, y_j) dz \right]}^{h(x_j, y_i, y_j)} dy_i dy_j \\
& = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_i \left[ \frac{-\partial VaR_\alpha(Z(x_i, x_j))}{\partial x_j} + y_j \right] \cdot \\
& \quad \cdot f(VaR_\alpha(Z(x_i, x_j)) - x_i y_i - x_j y_j, y_i, y_j) dy_i dy_j = \\
& = f_{Z(x_i, x_j)}(VaR_\alpha(Z(x_i, x_j))) \cdot \left[ \mathbb{E}[Y_i Y_j | Z(x_i, x_j) = VaR_\alpha(Z(x_i, x_j))] - \right. \\
& \quad \left. - \mathbb{E}[Y_i | Z(x_i, x_j) = VaR_\alpha(Z(x_i, x_j))] \cdot \mathbb{E}[Y_j | Z(x_i, x_j) = VaR_\alpha(Z(x_i, x_j))] \right] = \\
& = f_{Z(x_i, x_j)}(VaR_\alpha(Z(x_i, x_j))) \cdot \text{COV}[Y_i, Y_j | Z(x_i, x_j) = VaR_\alpha(Z(x_i, x_j))]. \tag{4.37}
\end{aligned}$$

(E)

If the previous assumptions hold for every  $i, j \in I$ , then from (4.37) we can obtain the **Hessian of CVaR**

$$\begin{aligned}
\mathfrak{H}_{CVaR}(\mathbf{x}) & = \left\{ \frac{\partial^2 CVaR_\alpha(Z(\mathbf{x}))}{\partial x_j \partial x_i} \right\}_{i, j \in \{1 \dots n\}} \\
& = \frac{f_{Z(\mathbf{x})}(VaR_\alpha(Z(\mathbf{x})))}{1 - \alpha} \cdot \left\{ \text{COV}[Y_i, Y_j | Z(\mathbf{x}) = VaR_\alpha(Z(\mathbf{x}))] \right\}_{i, j \in \{1 \dots n\}}. \tag{4.38}
\end{aligned}$$

$\mathfrak{H}_{CVaR}(\mathbf{x})$  is positive semidefinite that implies convexity of  $CVaR_\alpha(Z(\mathbf{x}))$  in  $\mathbf{x}$ .

(F)

Under previous assumptions and if  $CVaR_\alpha(Z(x_2, x_3)) \in C^3((0, 1) \times I_2 \times I_3)$ , then from Lemma 4.3.3 and (4.37) it holds

$$\begin{aligned}
& \frac{\partial^2 VaR_\alpha(Z(x_i, x_j))}{\partial x_j \partial x_i} = - \left[ \frac{d \text{COV}[Y_i, Y_j | Z(x_i, x_j) = z]}{dz} + \right. \\
& \quad \left. + \text{COV}[Y_i, Y_j | Z(x_i, x_j) = z] \frac{d \ln f_{Z(x_i, x_j)}(z)}{dz} \right] \Bigg|_{z=VaR_\alpha(Z(x_i, x_j))}. \tag{4.39}
\end{aligned}$$

(G)

If the previous assumptions hold for every  $i, j \in I$ , using result (4.39), the **Hessian of VaR** equals

$$\mathfrak{H}_{VaR}(\mathbf{x}) = \left\{ - \left[ \frac{d \text{COV}[Y_i, Y_j | Z(\mathbf{x}) = z]}{dz} + \text{COV}[Y_i, Y_j | Z(\mathbf{x}) = z] \frac{d \ln f_{Z(\mathbf{x})}(z)}{dz} \right] \Big|_{z=VaR_\alpha(Z(\mathbf{x}))} \right\}_{i,j \in \{1 \dots n\}}. \quad (4.40)$$

In [15] some conditions which ensure convexity of VaR in each decision variable are introduced. However, this conditions does not imply joint convexity in the decision vector  $\mathbf{x}$  and we are not able to find any general conditions which would ensure convexity of VaR with respect to the whole decision vector.

## 4.5 Appendix

**Theorem 4.5.1.** (Interchanging of derivative and integral) [25]

Let  $(X, \mathcal{A}, \mu)$  be a space with a measure,  $I \subseteq \mathbb{R}$  an open interval and  $f : I \times X \rightarrow \mathbb{R}$  a function with following properties:

1.  $f(t, \cdot)$  is measurable on  $(X, \mathcal{A})$  for all  $t \in I$ ,
2. it exists a set  $N \subseteq X$  with  $\mu(N) = 0$  such that  $\forall t \in I \ \forall x \in X \setminus N$  the derivative  $\frac{d}{dt} f(t, x)$  exists and is finite,
3. it exists  $g(x) \in L^1(\mu)$  such that  $\forall t \in I \ \forall x \in X \setminus N \ \left| \frac{d}{dt} f(t, x) \right| \leq g(x)$ ,
4.  $\exists t_0 \in I \ f(t_0, \cdot) \in L^1(\mu)$ .

Then  $f(t, \cdot) \in L^1(\mu)$ ,  $\forall t \in I$ , the function  $F : t \rightarrow \int_X f(t, x) \, d\mu(x)$  is differentiable on  $I$  and it holds

$$F'(t) = \int_X \frac{d}{dt} f(t, x) \, d\mu(x), \quad t \in I.$$

**Theorem 4.5.2.** (Interchanging of partial derivatives) [42]

Let the function  $f$  be in class  $C^k(G)$  ( $k \geq 2$ ), where  $G \subset \mathbb{R}^n$  is an open set and  $x \in G$ . Then the value

$$\frac{\partial^k f}{\partial x_{i_k} \dots \partial x_{i_1}}(x) = f_{i_k, \dots, i_1}(x)$$

does not depend on order of the indices  $i_1, \dots, i_k \in \{1, \dots, n\}$ .



**Theorem 4.5.3.** (Fubini) [25]

Let  $(\Omega_1, \mathcal{A}_1, \mu_1)$  and  $(\Omega_2, \mathcal{A}_2, \mu_2)$  be spaces with  $\sigma$ -finite measures,  $f$  be  $\mathcal{A}_1 \times \mathcal{A}_2$ -measurable function on  $\Omega_1 \times \Omega_2$ . If  $\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) d(\mu_1 \times \mu_2)(\omega_1, \omega_2) < \infty$ , then

$$\begin{aligned} & \int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) d(\mu_1 \times \mu_2)(\omega_1, \omega_2) = \\ &= \int_{\Omega_2} \left( \int_{\Omega_1} f(\omega_1, \omega_2) d\mu_1(\omega_1) \right) d\mu_2(\omega_2) = \\ &= \int_{\Omega_1} \left( \int_{\Omega_2} f(\omega_1, \omega_2) d\mu_2(\omega_2) \right) d\mu_1(\omega_1). \end{aligned}$$

# Chapter 5

## Dynamic Risk Measures

A risk measure that is defined over a process or a time series is called the multi-period or dynamic risk measure. The problem of dynamic risk measures has gained much attention recently and is discussed in many monographs and scientific papers, i.a. in [2], [6], [38]. We focus on risk measures for discrete time decision model with a given finite time horizon.

This chapter is organized as follows: In Section 5.1 we define the dynamic risk measure for income streams which is introduced in [31], summarize its basic properties and incorporate it into multistage stochastic programming problem for a scenario tree of income streams. In Section 5.2 we concentrate on applications of drawdown functional in portfolio optimization, we describe some drawdown measures and its properties.

### 5.1 A Risk Measure for Income Processes

In this section we introduce the dynamic risk measure from [31] and [32] which is based on nonanticipativity princip and is closely related to expected value of perfect information (EVPI) of stochastic programming problems, for details see [10] or [37]. The basic idea follows.

[31]: "The risk is not only a function of the random distribution of the random income, but also a function of the available information about it. Risk is viewed as the cost for uncertainty and lack of information and is measured in money units, in the same units as the value." The amount of information is discribed by appropriate  $\sigma$ -algebra (filtration of a stochastic process).

#### 5.1.1 Nonanticipativity

Basic principles of anticipativity and non-anticipativity can be simply demonstrated within framework of one-period prediction problem, for more appropriate explanation see [10]. We would like to predict some random future value  $Y$ . We suppose that the prediction error is measured by the expected quadratic deviation and we solve two problems:

- Optimal non-anticipative prediction problem

$$\min_{\text{value } a} \mathbb{E}[(Y - a)^2],$$

where the optimal solution is the expectation  $a = \mathbb{E}(Y)$  and the optimal value is  $\text{var}(Y)$ .

- *Clairvoyant's optimal anticipative prediction problem*

$$\min_{\text{function } a(Y)} \mathbb{E}[(Y - a(Y))^2],$$

where the solution is  $a(Y) = Y$  and the optimal value is 0.

Thus, the variance (which is occasionally used as a risk measure) gets a new interpretation as the difference between the optimal value of the non-anticipative problem and the anticipative problem. Definition of new dynamic risk measure is based on this idea.

### 5.1.2 A Multi-period Measure

Let the planning horizon be the endpoint  $\tau$  of an interval  $[0, \tau]$  which is further covered by nonoverlapping time intervals (periods) indexed by  $t = 1, \dots, T$ . Suppose that  $I_1, I_2, \dots, I_T$  is a stream of random income variables on some probability space  $(\Omega, \mathcal{F}, P)$  with finite expectations  $\mathbb{E}[I_t] < \infty$ ,  $t = 1, \dots, T$ . Let  $\mathbb{F} = (\mathcal{F}_t, t = 1, \dots, T)$  be a filtration such that  $I_t$  is  $\mathcal{F}_t$ -measurable ( $I_t$  is a  $\mathcal{F}_t$ -adapted process),  $\mathcal{F}_0 := (\emptyset, \Omega)$ . We would like to maximize the expected consumption minus the expected shortfall costs. We denote

- $a_t \dots$  amount to be consumed during the period  $t$  (main decision variable);  $a_t$  is  $\mathcal{F}_{t-1}$  measurable, because the decision about  $a_t$  must be made at the end of the time period  $t - 1$ ,
- $r \dots$  fixed (technical) interest rate,
- $c_t = (1 + r)^{-t} \dots$  net present value (NPV),
- $q_t = q_0(1 + r)^{-t} \geq 0 \dots$  shortfall costs,
- $d = d_0(1 + r)^{-T} \geq 0 \dots$  discount factor for final surplus,
- $d_0$  and  $q_0 \dots$  constants satisfying  $d_0 \leq 1 \leq q_0$ ,
- $S_t = [S_{t-1} + I_t - a_t]^+ \dots$  surplus carried from the period  $t$  to  $t + 1$ ,  $S_0 = 0$ ,
- $M_t = [S_{t-1} + I_t - a_t]^- \dots$  shortfall at the end of the period  $t$ .

*Non-anticipative problem* can be written as

$$\begin{aligned} \mathcal{U}_{\mathbb{F}}(I_1, I_2, \dots, I_T) = \max \mathbb{E} \left[ \sum_{t=1}^T (c_t a_t - q_t M_t) + d S_T \right] \\ \text{s.t. } a_t \text{ is } \mathcal{F}_{t-1} \text{-measurable for } t = 1, \dots, T \end{aligned} \quad (5.1)$$

and *clairvoyant's problem* (where we are given complete information  $\mathcal{F}_T$ ) as

$$\begin{aligned} \mathcal{U}_{\mathcal{F}_T}(I_1, I_2, \dots, I_T) &= \max \mathbb{E} \left[ \sum_{t=1}^T (c_t a_t - q_t M_t) + dS_T \right] \\ \text{s.t. } a_t &\text{ is } \mathcal{F}_T \text{ - measurable for } t = 1, \dots, T, \end{aligned} \quad (5.2)$$

that can be reduced to ( $a_t = I_t$ )

$$\mathcal{U}_{\mathcal{F}_T}(I_1, I_2, \dots, I_T) = \sum_{t=1}^T c_t \mathbb{E}(I_t). \quad (5.3)$$

The **dynamic risk measure** is then defined as the difference between the optimal value of the non-anticipative problem and the anticipative problem, i.e.

$$\mathcal{R}(I_1, I_2, \dots, I_T) = \mathcal{U}_{\mathcal{F}_T}(I_1, I_2, \dots, I_T) - \mathcal{U}_{\mathbb{F}}(I_1, I_2, \dots, I_T). \quad (5.4)$$

Because of this definition minimization of the risk measure  $\mathcal{R}$  is equivalent to maximization of the expected utility  $\mathcal{U}_{\mathbb{F}}$ . According to [32], *the dual problem* to (5.1) has the following form

$$\begin{aligned} \min \mathbb{E} \sum_{t=1}^T \lambda_t I_t \\ \text{s.t.} \\ \mathbb{E}_{t-1} \lambda_t &= c_t, \quad t = 1, \dots, T, \\ \lambda_t &\leq q_t, \quad t = 1, \dots, T, \\ \lambda_T &\geq d, \\ \lambda_t &\geq \mathbb{E} \lambda_{t+1}, \quad t = 1, \dots, T-1. \end{aligned} \quad (5.5)$$

Multiplier process  $\{\lambda_t\}$  is a submartingale. Under our assumptions the problem (5.1) and its dual form (5.5) has the same finite optimal value. Basic principles of duality in stochastic programming can be found in [37].

**Theorem 5.1.1.** (*Properties of the expected utility and the dynamic risk measure*) [31]

1.  $\mathcal{U}_{\mathbb{F}}$  is monotonic, i.e.  $\mathcal{U}_{\mathbb{F}}(I_1^{(1)}, \dots, I_T^{(1)}) \leq \mathcal{U}_{\mathbb{F}}(I_1^{(2)}, \dots, I_T^{(2)})$ , where  $\{I_t^{(1)}\}, \{I_t^{(2)}\}$  are two  $\mathcal{F}_t$ -adapted processes such that  $I_t^{(1)} \leq I_t^{(2)}$ ,  $t = 1, \dots, T$ ,
2.  $(I_t^{(1)} | \mathcal{F}_{t-1}) \prec_{SSD} (I_t^{(2)} | \mathcal{F}_{t-1})$ ,  $t = 1, \dots, T \Rightarrow \mathcal{U}_{\mathbb{F}}(I_1^{(1)}, \dots, I_T^{(1)}) \leq \mathcal{U}_{\mathbb{F}}(I_1^{(2)}, \dots, I_T^{(2)})$ ,
3.  $\mathcal{R}$  is convex, i.e. for two streams  $(I_1^{(1)}, \dots, I_T^{(1)}), (I_1^{(2)}, \dots, I_T^{(2)})$  with finite expectations and every  $\lambda \in (0, 1)$  it holds

$$\begin{aligned} \mathcal{R}(\lambda(I_1^{(1)}, \dots, I_T^{(1)}) + (1-\lambda)(I_1^{(2)}, \dots, I_T^{(2)})) &\leq \\ &\leq \lambda \mathcal{R}(I_1^{(1)}, \dots, I_T^{(1)}) + (1-\lambda) \mathcal{R}(I_1^{(2)}, \dots, I_T^{(2)}). \end{aligned}$$

### 5.1.3 Mean - Dynamic Risk Model

In [31] a mean-dynamic risk model for random income streams  $(I_t^j, t = 1, \dots, T)$ ,  $j = 1, \dots, n$  is introduced. Its objective is to get a fixed mix portfolio with random income stream  $(I_t(x) = \sum_{j=1}^n x_j I_t^j, t = 1, \dots, T)$ , which maximizes the expected net present value  $\mu(x) = \sum_{i=1}^n x_i \sum_{t=1}^T c_t \mathbb{E}(I_t^i)$  and minimizes the risk measure  $\mathcal{R}(I_1(x), \dots, I_T(x))$  defined in (5.4). Thus, we solve the following optimization problem for some value of the parameter  $m$ .

$$\begin{aligned} \min \quad & \mathcal{R}(I_1(x), \dots, I_T(x)) \\ \text{s.t.} \quad & \mu(x) = m, \\ & \sum_{i=1}^n x_i = 1, \\ & x_i \geq 0, \quad i = 1, \dots, n, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \min \quad & \mathbb{E} \left[ \sum_{t=1}^T \left( c_t \left( \sum_{i=1}^n x_i I_t^i - a_t \right) + q_t M_t \right) - dS_T \right] \\ \text{s.t.} \quad & S_t - M_t = S_{t-1} + \sum_{i=1}^n x_i I_t^i - a_t, \quad t = 1, \dots, T, \\ & \sum_{i=1}^n x_i \sum_{t=1}^T c_t \mathbb{E}(I_t^i) = m, \\ & \sum_{i=1}^n x_i = 1, \\ & x_i \geq 0, \quad i = 1, \dots, n, \\ & S_t \geq 0, \quad M_t \geq 0, \quad t = 1, \dots, T. \end{aligned}$$

Multistage stochastic programming model for a scenario tree of income streams can be formulated as follows. Let the nodes of the tree are numbered  $0, 1, 2, \dots, K$ , with 0 being the root,  $\mathcal{K} = \{1, 2, \dots, K\}$  be set of the nodes without the root. We suppose that the decisions about consumption are made at the non-terminal nodes, the portfolio is mixed at the root and is fixed during our horizon. Let  $t_k$  denote the time stage of the node  $k \in \mathcal{K}$  and  $k-$  its predecessor. Let  $\mathcal{K}_t \subset \mathcal{K}$ ,  $t = 1, \dots, T$  be disjoint sets of numbers of the nodes at the layer  $t$ , thus for a node  $k$  it holds  $k \in \mathcal{K}_{t(k)}$ . The terminal nodes we denote  $\mathcal{K}_T = \{K_0, \dots, K\}$ . For probabilities of

nodes  $p_k$ ,  $k = 1, \dots, K$  it holds  $\sum_{k \in \mathcal{K}_t} p_k = 1$ .

$$\begin{aligned}
\min \quad & \sum_{k \in \mathcal{K}} p_k c_{t(k)} \left( \sum_{i=1}^n x_i I_k^i - a_{k-} \right) + \sum_{k \in \mathcal{K}} p_k q_{t(k)} M_k - \sum_{k \in \mathcal{K}_T} p_k d S_k \\
\text{s.t.} \quad & S_k + a_{k-} - M_k = \sum_{i=1}^n x_i I_k^i, \quad t(k) = 1, \quad k \in \mathcal{K}, \\
& S_k - S_{k-} + a_{k-} - M_k = \sum_{i=1}^n x_i I_k^i, \quad t(k) > 1, \quad k \in \mathcal{K}, \\
& \sum_{i=1}^n x_i \sum_{k \in \mathcal{K}} p_k c_{t(k)} I_k^i = m, \\
& M_k \geq 0, \quad S_k \geq 0, \quad k \in \mathcal{K}, \\
& x_i \geq 0, \quad i = 1, \dots, n.
\end{aligned}$$

A crucial question is how to generate the input for the previous decision model. Some methods for scenario generation are described in [10], [16].

## 5.2 Drawdown Measures

In this section we focus on applications of drawdown functional in portfolio optimization. This topic is discussed i.a. in [6], [5], [21]. The drawdown measures are usually applied to a sample path or fan scenario trees of uncompounded portfolio rate of returns, not to random loss variables or processes.

We suppose that the considered time interval  $[0, \tau]$  is divided into  $T \in \mathbb{N}$  nonoverlapping periods (intervals) indexed by  $t = 1, \dots, T$ . The portfolio absolute drawdown is defined as the drop of the current portfolio value comparing to its maximum achieved in the past up to current moment.

**Definition 5.2.1.** (*Drawdown function*) [5]

Denote by  $w = (w_t, t = 1, \dots, T)$  the sample path of uncompounded cumulative portfolio rate of return, where

$$w_t = \sum_{i=1}^N \left( \sum_{j=1}^t r_{ij} x_{ij} \right), \quad t = 1, \dots, T, \quad (5.6)$$

$r_{ij}$  is the logarithmic rate of return of  $i$ -th instrument in  $j$ -th period,  $x_{ij}$  is the instrument weight,  $\sum_{i=1}^N x_{ij} = 1$ ,  $j = 1, \dots, T$ .

The portfolio absolute drawdown at the end of period  $t$  is defined as

$$AD_t = \max_{1 \leq k \leq t} w_k - w_t. \quad (5.7)$$

We have got the absolute drawdown function (time series)  $\mathbf{AD}(w) = (AD_t, t = 1, \dots, T)$ , or simply  $\mathbf{AD}$ .

The portfolio weights may be constant through time (in portfolio optimization problems without revisions). We can argue for using logarithmic returns instead of

rates of return in the previous definition as follows. Let  $W_t$  denote portfolio wealth at the end of period  $t$ , i.e.  $W_t = P_0 \cdot (1 + R_t)$ ,  $t = 1, \dots, T$ , where  $P_0$  is an initial wealth and  $R_t$  the portfolio return for the time horizon  $t$ . Then the portfolio *relative drawdown* is defined as

$$RD_t = \frac{\max_{1 \leq k \leq t} W_k}{W_t}, \quad t = 1, \dots, T.$$

It is easy to see, that  $RD_t > 1$  just when portfolio wealth decreases and  $RD_t = 1$  just when portfolio wealth does not decrease. Sometimes the relative drawdown is defined as  $\tilde{RD}_t = \max_{1 \leq k \leq t} W_k / W_t - 1$ ,  $t = 1, \dots, T$ . For logarithm of the relative drawdowns it holds

$$\begin{aligned} \ln RD_t &= \max_{1 \leq k \leq t} \ln W_k - \ln W_t = \\ &= \max_{1 \leq k \leq t} \ln(1 + R_k) - \ln(1 + R_t). \end{aligned}$$

By using the relation  $\ln(1 + R_k) = r_t$ , where  $r_t$  is the logarithmic return, and the multiplicative effect for logarithmic returns, see Subsection 7.1.1, we get the absolute drawdown, i.e.  $\ln RD_t = AD_t$ ,  $t = 1, \dots, T$ . It is necessary to remark that the logarithmic returns for small values do not differ from corresponding rates of return too much. Basic properties of the absolute drawdown function do not change if we use rates of return in its definition.

Figure 5.1 illustrates an example of monthly time series of cumulative rate of return and corresponding drawdown function of Northrop Grumman Corp. When the uncomounded cumulative rate of return decreases (achieves its local minimum), the absolute drawdown function increases (achieves its local maximum).

**Remark 5.2.2.** (*Properties of the absolute drawdown function*) [6]

Let  $\mathbf{AD}(w)$  be the absolute drawdown function of an uncomounded cumulative portfolio rate of return  $w$ , then

1.  $AD_t \geq 0$ ,  $t = 1, \dots, T$  (*nonnegativity of the absolute drawdown function*).
2.  $AD_t = [AD_{t-1} - \sum_{i=1}^n r_{it} x_{it}]^+$ ,  $t = 1, \dots, T$ , where we set  $AD_0 = 0$  (*recursion*).
3.  $\mathbf{AD}(w + \text{const}) = \mathbf{AD}(w)$ ,  $\mathbf{AD}(\lambda w) = \lambda \mathbf{AD}(w)$ ,  $\forall \lambda \geq 0$  (*insensitivity to constant shift and positive homogeneity*).
4.  $\mathbf{AD}(\lambda w_1 + (1 - \lambda)w_2) \leq \lambda \mathbf{AD}(w_1) + (1 - \lambda) \mathbf{AD}(w_2)$ ,  $\lambda \in [0, 1]$ , where  $w_1$ ,  $w_2$  are uncomounded cumulative portfolio rates of return (*convexity*).

**Definition 5.2.3.** (*Nonparametric drawdown risk measures*) [5]

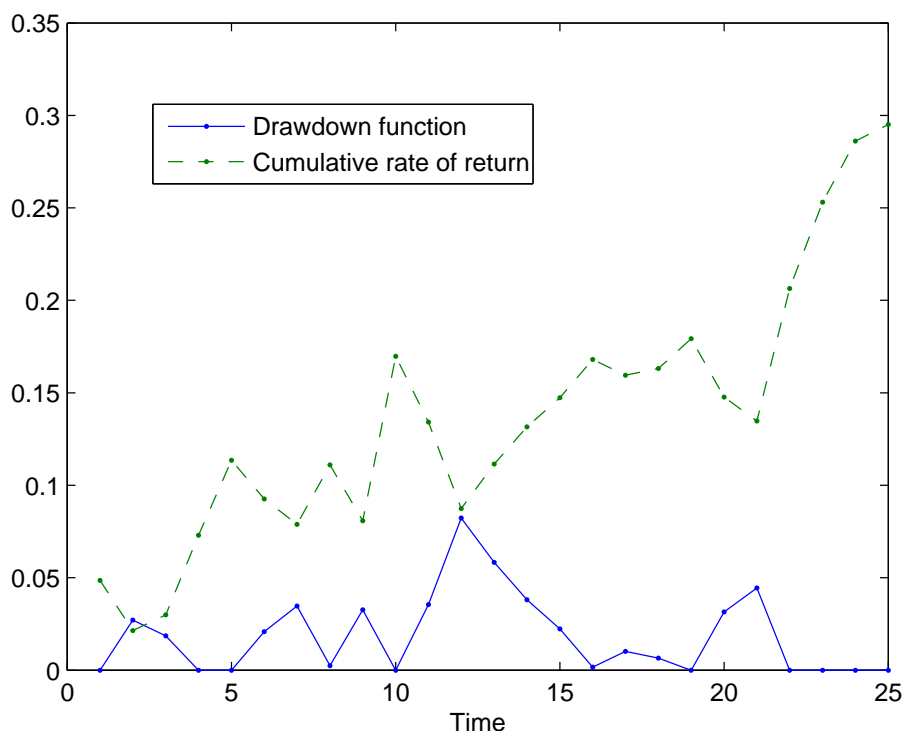
The Maximum Drawdown (*MaxDD*) on the interval  $[0, \tau]$  is the maximum of the drawdown function, i.e.

$$\text{MaxDD}(\mathbf{AD}(w)) = \max_{0 \leq t \leq T} AD_t.$$

The Average Drawdown (*AvDD*) is defined as

$$\text{AvDD}(\mathbf{AD}(w)) = \frac{1}{T} \sum_{t=1}^T AD_t.$$

Figure 5.1: Time series of cumulative rate of return and corresponding absolute drawdown function (Northrop Grumman Corp., 2.2.2004-1.2.2006, monthly).



The Maximum Drawdown is based on the worst case in the sample path, so it may be too conservative to minimize it while the Average Drawdown may mask large drawdowns. This leads us to introduce Conditional Drawdown at Risk (CDaR) which combines CVaR and drawdown approaches. CDaR is defined as the mean of the worst  $(1 - \alpha) \cdot 100\%$  absolute drawdowns. Conditional Drawdown at Risk represents a dynamic extension of Conditional Value at Risk, however, the absolute drawdowns are viewed, in fact, as independent realizations (with equal probability  $1/T$ ) of some discrete drawdown loss random variable dependent on decisions, even though they rather form a time series. In [6] CDaR is defined using the expression of CVaR as a weighted average of VaR and the mean of the losses strictly exceeding VaR under previous assumptions, see Subsection 2.3.2.

**Definition 5.2.4.** (*Conditional Drawdown at Risk (CDaR)*) [6]

Let  $\mathbf{AD}(w)$  be an absolute drawdown function,  $\alpha \in (0, 1)$ ,

$$\begin{aligned} \Pi_{\mathbf{AD}}(\xi) &= \frac{1}{T} \sum_{t=1}^T I(\mathbf{AD}_t \leq \xi), \quad \xi \in \mathbb{R}, \\ \Pi_{\mathbf{AD}}^{-1}(\alpha) &= \begin{cases} \min\{\xi : \Pi_{\mathbf{AD}}(\xi) \geq \alpha\} & \text{for } \alpha \in (0, 1) \\ 0 & \text{for } \alpha = 0, \end{cases} \\ \Xi_{\alpha} &= \{\mathbf{AD}_t : \mathbf{AD}_t > \Pi_{\mathbf{AD}}^{-1}(\alpha)\}, \quad \alpha \in [0, 1], \end{aligned}$$



where  $I(\cdot)$  is an indikator. Then the Conditional Drawdown at Risk (CDaR) is defined as

$$CDaR_\alpha(\mathbf{AD}(w)) = \left( \frac{\Pi_{\mathbf{AD}}(\Pi_{\mathbf{AD}}^{-1}(\alpha)) - \alpha}{1 - \alpha} \right) \Pi_{\mathbf{AD}}^{-1}(\alpha) + \frac{1}{(1 - \alpha)T} \sum_{AD_t \in \Xi_\alpha} AD_t. \quad (5.8)$$

The properties of CVaR showed in the Subsection 2.3.3 enables us to extend the minimization formula for CDaR which is useful in practical implementation.

**Theorem 5.2.5.** (Minimization formula for CDaR) [21]

Given an absolute drawdown function  $\mathbf{AD}(w)$ , computation of the Conditional Drawdown at Risk of the absolute drawdown function  $\mathbf{AD}(w)$  can be reduced to the following programming procedure

$$CDaR_\alpha(\mathbf{AD}(w)) = \min_{y \in \mathbb{R}} \left\{ y + \frac{1}{(1 - \alpha)T} \sum_{t=1}^T [AD_t - y]^+ \right\}, \quad (5.9)$$

for some  $\alpha \in (0, 1)$ ,

leading to a single optimal value equal to  $\Pi_{\mathbf{AD}}^{-1}(\alpha)$  if  $\Pi_{\mathbf{AD}}(\Pi_{\mathbf{AD}}^{-1}(\alpha)) > \alpha$ ,

and to a closed bounded interval with the left endpoint  $\Pi_{\mathbf{AD}}^{-1}(\alpha)$  if  $\Pi_{\mathbf{AD}}(\Pi_{\mathbf{AD}}^{-1}(\alpha)) = \alpha$ .

The optimization problem (5.9) can be rewritten using (5.6) and (5.7) as

$$CDaR_\alpha(\mathbf{AD}(w)) = \min_{y \in \mathbb{R}} \left\{ y + \frac{1}{(1 - \alpha)T} \sum_{t=1}^T \left[ \max_{1 \leq k \leq t} \left[ \sum_{i=1}^n \left( \sum_{j=1}^k r_{ij} x_{ij} \right) \right] - \sum_{i=1}^n \left( \sum_{j=1}^t r_{ij} x_{ij} \right) - y \right]^+ \right\}.$$

Basic properties of CDaR follow from basic properties of CVaR and from Theorem 5.2.2.

**Theorem 5.2.6.** (Properties of CDaR) [6]

Let  $\mathbf{AD}(w)$ ,  $\mathbf{AD}_1$ ,  $\mathbf{AD}_2$  be absolute drawdown functions,  $\alpha \in (0, 1)$ , then

1.  $CDaR_\alpha(\mathbf{AD}(w)) \geq 0$ ,  $t = 1, \dots, T$  (nonnegativity).
2.  $CDaR_\alpha(\mathbf{AD}(w) + \text{const}) = CDaR_\alpha(\mathbf{AD}(w)) + \text{const}$  (constant translation).
3.  $CDaR_\alpha(\lambda \mathbf{AD}(w)) = \lambda CDaR_\alpha(\mathbf{AD}(w))$ ,  $\forall \lambda \geq 0$  (positive homogeneity).
4.  $CDaR_\alpha(\lambda \mathbf{AD}_1 + (1 - \lambda) \mathbf{AD}_2) \leq \lambda CDaR_\alpha(\mathbf{AD}_1) + (1 - \lambda) CDaR_\alpha(\mathbf{AD}_2)$ ,  $\lambda \in [0, 1]$  (convexity).

**Remark 5.2.7.** [6]

CDaR includes the Average Drawdown and the Maximal Drawdown as its limiting cases, i.e.

$$\begin{aligned} CDaR_\alpha(\mathbf{AD}) &\xrightarrow{\alpha \rightarrow 0^+} AvDD(\mathbf{AD}), \\ CDaR_\alpha(\mathbf{AD}) &\xrightarrow{\alpha \rightarrow 1^-} MaxDD(\mathbf{AD}). \end{aligned}$$

Multi-scenario Conditional Drawdown at Risk (MCDaR) is defined for fan scenario tree  $\mathbf{ADS} = (\mathbf{AD}^s, s = 1, \dots, S)$  (in [6] called "drawdown surface"), where  $\mathbf{AD}^s = (AD_t^s, t = 1, \dots, T)$  is a drawdown function and  $s$  denotes one of  $S$  scenarios with probability  $p_s$ ,  $\sum_{s=1}^S p_s = 1$ . MCDaR can be defined as the mean of the worst  $(1 - \alpha) \cdot 100\%$  absolute drawdowns on the drawdown surface.

**Definition 5.2.8.** (*Multi-scenario Conditional Drawdown at Risk*) [6]  
Let  $\mathbf{ADS}$  be a drawdown surface,  $\alpha \in (0, 1)$ ,

$$\begin{aligned}\Pi_{\mathbf{ADS}}(\xi) &= \frac{1}{T} \sum_{t=1}^T \sum_{s=1}^S p_s I(AD_t^s \leq \xi), \quad \xi \in \mathbb{R}, \\ \Pi_{\mathbf{ADS}}^{-1}(\alpha) &= \begin{cases} \min\{\xi : \Pi_{\mathbf{ADS}}(\xi) \geq \alpha\} & \text{for } \alpha \in (0, 1] \\ 0 & \text{for } \alpha = 0, \end{cases} \\ \Xi_\alpha &= \left\{ AD_t^s : AD_t^s > \Pi_{\mathbf{ADS}}^{-1}(\alpha) \right\}, \quad \alpha \in [0, 1],\end{aligned}$$

where  $I(\cdot)$  is an indikator. Then the Multi-scenario Conditional Drawdown at Risk (MCDaR) is defined as

$$MCDaR_\alpha(\mathbf{ADS}) = \left( \frac{\Pi_{\mathbf{ADS}}(\Pi_{\mathbf{ADS}}^{-1}(\alpha)) - \alpha}{1 - \alpha} \right) \Pi_{\mathbf{ADS}}^{-1}(\alpha) + \frac{1}{(1 - \alpha)T} \sum_{AD_t^s \in \Xi_\alpha} p_s AD_t^s. \quad (5.10)$$

MCDaR has similar properties as CDaR, the minimization formula can be extended for MCDaR, for details see [6]. There is also solved a real-life portfolio optimization problem using drawdown measures. A numerical comparison of CDaR and CVaR approaches in portfolio optimization can be found in [21].

**Risk profile**  $\chi$  (discrete version), defined by

1.  $\chi(\alpha_l) \geq 0$ ,  $\alpha_l \in (0, 1)$ ,  $l = 1, \dots, L$ ,
2.  $\sum_{l=1}^L \chi(\alpha_l) = 1$ ,

enables us to assign specific weights for  $\alpha_l$ -CDaRs with predetermined confidence levels and to create a convex combination of them. The mixed CDaR is then defined as

$$CDaR_\chi(x) = \sum_{l=1}^L \chi(\alpha_l) CDaR_{\alpha_l}(x).$$

A portfolio manager can express his/her risk preferences by shaping the risk profile. This approach is similar to risk shaping with CVaR, see theorem 2.3.14.

# Chapter 6

## Conclusion

In this thesis we have focused on sensitivity and dynamics of risk measures of random losses. First, basic properties, advantages and disadvantages of two frequently discussed risk measures Value at Risk (VaR) and Conditional Value at Risk (CVaR) have been summarized. Then we have applied contamination techniques in stress testing for VaR and CVaR and for optimization problems with these risk criteria. Using the contamination techniques we have derived computable bounds which can provide a deeper insight into behaviour of these risk measures. We have focused on portfolio optimization problem with the relative VaR objective function, where we have considered correlation and volatility shocks. The correlation shocks have been involved in the numerical study, where we have supposed that the correlations between assets of companies in the same industry and sector will increase. It has turned out that the contamination bounds are relatively close to the optimal value of the relative VaR optimization problem. However, the optimal value increases significantly with increasing correlations which reflects instability of relative VaR with respect to correlation matrix estimation. A heuristic algorithm for stressing of large correlation matrices has been introduced. Next, we have studied sensitivity of VaR and CVaR through their derivatives with respect to the portfolio allocation. Assumptions under which it is possible to get close expressions for first and second order derivatives have been found. As a new result we have derived Hessians of VaR and CVaR which enables us to discuss their convexity with respect to the whole portfolio allocation vector. Convexity of CVaR has been confirmed, however, no conditions that would ensure convexity of VaR have been found. A challenge for future research is to find some simpler sufficient conditions which imply the assumptions for the derivative formulas. The last chapter has dealt with dynamic risk measures for multi-period discrete time decision models for a finite time horizon. A risk measure for income streams has been introduced and into multi-period mean-risk model for a scenario tree incorporated. Another approach how to define new dynamic risk measure is an extension of an one-period measure. Such extension of CVaR Conditional Drawdown at Risk (CDaR) represents. We have summarized basic properties of absolute drawdown function and some drawdown measures. However, many interesting topics, such as dual risk measures or general deviation risk measures, have had to be skipped.

# Chapter 7

## Appendix

### 7.1 Modelling of Returns

#### 7.1.1 Returns

This section uses knowledge from [10] and [41]. If we denote  $P_{it}$  price of the underlying asset  $i$  (security) at time  $t$ , we can define the *rate of return* by the relation

$$R_{it} := \frac{P_{it+1} - P_{it}}{P_{it}}.$$

Taking into account the multiplicative effect and an analogy with the force of interest, we can define another measure  $r_{it}$  as a *logarithmic rate of return* (logarithmic return) by

$$1 + R_{it} = \frac{P_{it+1}}{P_{it}} =: \exp(r_{it}).$$

If we define *logarithmic prices*  $p_{it} := \ln P_{it}$ , we can rewrite previous relation as

$$r_{it} = \ln(1 + R_{it}) = p_{it+1} - p_{it}.$$

Note that for small values of  $r_{it}$  it does not differ from  $R_{it}$  too much (which follows from Taylor expansion).

The *rate of return*  $R_i^T$  for the time horizon  $T$  is then defined as

$$1 + R_i^T = \prod_{t=1}^T (1 + R_{it}) = \exp\left(\sum_{t=1}^T r_{it}\right) = \frac{P_{iT}}{P_{i0}}.$$

If  $D_{it}$  is a dividend paid for the time interval  $[t, t + 1]$ , then

$$R_{it} := \frac{P_{it+1} + D_{it} - P_{it}}{P_{it}} = \frac{D_{it}}{P_{it}} + \frac{P_{it+1} - P_{it}}{P_{it}},$$

where  $D_{it}/P_{it}$  represents the *dividend (coupon) yield*.

## 7.1.2 Diffusion Processes and Returns

**Definition 7.1.1.** (*Wiener process*)

A stochastic process  $(W_t, t \geq 0)$  is a Wiener process (a standard Brownian motion) if it satisfies

1.  $W_0 = 0$  a.s.,
2.  $W_t$  has independent increments, i.e.  $W_{t_2} - W_{t_1}$  and  $W_{t_4} - W_{t_3}$  are independent r.v. for all  $0 \leq t_1 < t_2 < t_3 < t_4 < \infty$ ,
3.  $\mathcal{L}(W_t - W_s) = \mathcal{N}(0, |t - s|)$  for all  $s, t \geq 0$ ,
4. the process  $W_t$  has continuous trajectories.

**Remark 7.1.2.**

i)  $\mathbb{E}W_t = 0, \forall t \geq 0$ ,

ii) Wiener process can be written in the form (useful for simulation)

$$dW_t = \varepsilon \sqrt{dt},$$

where  $\varepsilon \sim \mathcal{N}(0, 1)$

**Definition 7.1.3.** (*Generalized Wiener process*)

The generalized Wiener process  $(X_t, t \geq 0)$  with the drift rate  $\mu$  and the rate of variance change  $\sigma^2$  satisfies the following conditions

1.  $X_0 = 0$  a.s.,
2. the process  $X_t$  has independent increments, i.e.  $X_{t_2} - X_{t_1}$  and  $X_{t_4} - X_{t_3}$  are independent r.v. for all  $0 \leq t_1 < t_2 < t_3 < t_4 < \infty$ ,
3.  $\mathcal{L}(X_t - X_s) = \mathcal{N}(|t - s|\mu, |t - s|\sigma^2)$  for all  $s, t \geq 0$ ,
4. the process  $X_t$  has continuous trajectories.

**Remark 7.1.4.**

The generalized Wiener proces may be written in continuous or discrete form

$$\begin{aligned} dX_t &= \mu dt + \sigma dW_t, \\ X_t - X_0 &= \mu t + \sigma \varepsilon \sqrt{t}. \end{aligned}$$

**Definition 7.1.5.** (*Ito's Process*)

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t.$$

Under the *random walk hypothesis* the logarithmic prices follow the model

$$p_{t+1} - p_t = \mu + \varepsilon_{t+1}, \quad t = 0, 1, \dots,$$

where  $\mu$  is a drift and  $\varepsilon_t$ 's are uncorrelated (or even independent) r.v. with  $\mathbb{E}\varepsilon_t = 0$  and  $\text{var}\varepsilon_t = \sigma^2$ . Then for the prices  $P_t$  we have

$$P_{t+1} = P_t e^{\mu + \varepsilon_{t+1}}, \quad t = 0, 1, \dots$$

More appropriate model suppose that the rates of return follow generalized Wiener process (we will present only discrete versions of processes), i.e.

$$\frac{P_{t+\Delta t} - P_t}{P_t} = \mu\Delta t + \sigma\varepsilon\sqrt{\Delta t},$$

where  $\mathcal{L}(\varepsilon) = \mathcal{N}(0, 1)$ . It can be also written in the form

$$P_{t+\Delta t} - P_t = \mu P_t \Delta t + \sigma P_t \varepsilon \sqrt{\Delta t}, \quad (7.1)$$

which is called *geometric Brownian motion*. Using Itô's lemma, see [10], we get

$$p_{t+\Delta t} - p_t = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}. \quad (7.2)$$

It holds  $\mathcal{L}(p_{t+\Delta t} - p_t) = \mathcal{N}\left((\mu - 1/2\sigma^2)\Delta t, \sigma^2\Delta t\right)$  and  $\mathcal{L}(P_{t+\Delta t}/P_t) = \mathcal{LN}\left((\mu - 1/2\sigma^2)\Delta t, \sigma^2\Delta t\right)$ , where  $\mathcal{LN}(\cdot, \cdot)$  denotes log-normal distribution.

### Parameters Estimation

Assume that we have  $T + 1$  observations of stock prices  $P_t$  at equally spaced time intervals  $\Delta$ . Let  $r_t$ ,  $t = 1, \dots, T$  denote corresponding logarithmic returns, that follow the model (7.2), thus they are normally distributed and uncorrelated. Let  $\bar{r}$  and  $s_r$  denote sample mean and standart deviation of  $r_t$ 's. Then the estimates of  $\mu$  and  $\sigma$  are

$$\begin{aligned} \hat{\sigma} &= \frac{s_r}{\sqrt{\Delta}}, \\ \hat{\mu} &= \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = \frac{\bar{r}}{\Delta} + \frac{s_r^2}{2\Delta}. \end{aligned}$$

### Jump Diffusion Models

The stochastic diffusion models based on Brownian motion fails to explain some characteristics of asset returns. In [41] a simple jump diffusion model is proposed. The returns implied by the model are leptokurtotic and assymetric with respect to zero. Let  $P_t$  follow the model

$$\frac{dP_t}{P_t} = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{n_t} (J_i - 1)\right),$$

where  $(W_t, t \geq 0)$  is a Wiener process,  $(n_t, t \geq 0)$  is a Poisson process with rate  $\lambda t$  and  $\{J_i\}_{i=1}^{n_t}$  is a sequence of independent and identically distributed nonnegative random variables such that  $X = \ln J$  has a double exponential distribution;  $W_t$ ,  $n_t$  and  $J_i$ ,  $\forall i, t$  are independent. The model consist of two parts - the geometric Brownian motion and a jump Poisson process. We may use the discrete-time approximation

$$\frac{P_{t+\Delta t} - P_t}{P_t} \approx \mu\Delta t + \sigma\varepsilon\sqrt{\Delta t} + \sum_{i=n_t+1}^{n_{t+\Delta t}} X_i,$$

where  $X_i = \ln J_i$ . For  $\Delta t$  small enough

$$\frac{P_{t+\Delta t} - P_t}{P_t} \approx \mu\Delta t + \sigma\varepsilon\sqrt{\Delta t} + I \times X,$$

where  $X$  is a double exponential variable and  $I$  is a Bernoulli random variable with  $P(I = 1) = \lambda\Delta t$  and  $P(I = 0) = 1 - \lambda\Delta t$ .

### 7.1.3 Stable Distributions

Distributions of financial returns have heavier tails, a higher peak than a normal distribution (i.e. they are leptokurtotic) and are typically right skewed. They are sometimes supposed to be stable distributed. In this section we summarize briefly basic facts about the stable distribution, whose density and distribution function does not have close form with the exception of some special cases. They are characterized by the characteristic function

$$\begin{aligned} \Phi_X(t) &= E(\exp^{itX}) = \\ &= \exp \left\{ -\gamma^\alpha |t|^\alpha \left( 1 - i\beta \operatorname{sign}(t) \operatorname{tg} \frac{\pi\alpha}{2} \right) + i\delta t \right\}, \text{ if } \alpha \neq 1, \end{aligned} \quad (7.3)$$

$$= \exp \left\{ -\gamma |t| \left( 1 - i\beta \frac{2}{\pi} \operatorname{sign}(t) \operatorname{ln} t \right) + i\delta t \right\}, \text{ if } \alpha = 1, \quad (7.4)$$

Formal definitions can be found in [14], [18]. Stable distributions are represented by four parameters, therefore they are denoted by  $S_\alpha(\delta, \beta, \gamma)$ , where

- $\alpha \in (0, 2]$  ... index of stability - if it is small, the distribution has a high peak and heavy tails,
- $\beta \in [-1, 1]$  ... skewness parameter - if  $\beta > 0$  ( $\beta < 0$ ), the distribution is skewed to the right (left),
- $\gamma > 0$  ... scale parameter, which generalizes the notion of standard deviation, variation  $\gamma^\alpha$  is generalized variance,
- $\delta \in \mathbb{R}$  ..., or location parameter.

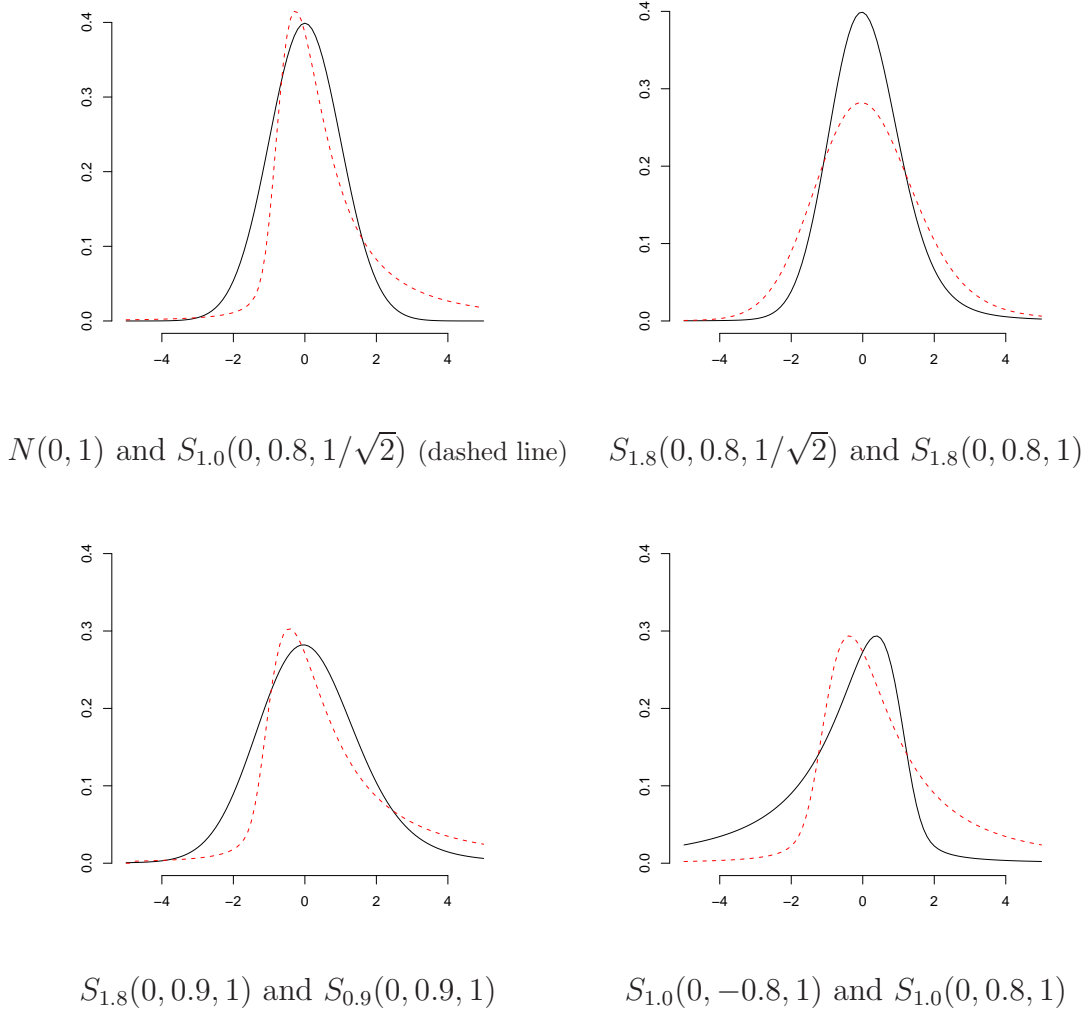
If we set

- $\alpha = 2$  and  $\beta = 0$ , we get normal (Gaussian) distribution,
- $\delta = 0$  and  $\gamma = 1/\sqrt{2}$ , we get standard stable distribution,
- $\beta = 1$  ( $-1$ ), we get stable distribution totally skewed to the right (left).

Financial models assume that  $\alpha \in (1, 2]$ , because for stable distributed random variable it holds:  $p$ -th absolute moment  $\mathbb{E}[|X|^p]$  is finite just when  $p > \alpha$ . Then it is possible to discuss expected value  $\mathbb{E}[X]$ .

In [4] some methods of parameters estimation are summarized: Method of Moments, Quantile Estimation Method, Regression-type Method, for details see [20], Maximum Likelihood Method.

Figure 5.1: Densities of stable distributions



## 7.2 How to Increase Correlations

In this section the heuristic algorithm from [12] for systematic adjustment (increasing) of correlations of chosen assets is extended. It is guaranteed that the new correlation matrix is positive (semi-)definite. We believe that the following algorithm is applicable to very large correlation matrices, because it is noniterative and it does not need to compute eigenvalues of the new matrix to verify its positive semidefiniteness.

We consider  $n$  assets and their random (logarithmic) returns  $R_i$ ,  $i = 1, \dots, n$  with the correlation matrix  $C = (\rho_{i,j})_{i,j=1,\dots,n}$ , that we want to stress. Let  $I_k \subset \{1, \dots, n\}$ ,  $\text{card}(I_k) = m_k$ ,  $k = 1, \dots, K$  be nonempty disjoint sets of assets. We want to increase the correlation within the set  $I_k$ ,  $\forall k$  so as the correlation between assets in the set and out of it does not increase significantly.

1. Define new random variables

$$\hat{R}_i = \begin{cases} (1 - \theta_k)R_i + \theta_k R_k^{Avg} & \text{if } i \in I_k, k = 1, \dots, K, \\ R_i & \text{otherwise,} \end{cases}$$



where  $R_k^{Avg}$  denotes "average" of random variables in  $I_k$ , i.e.

$$R_k^{Avg} = \frac{1}{m_k} \sum_{i \in I_k} R_i.$$

Parameters  $\theta_k \in [0, 1]$ ,  $k = 1, \dots, K$  specify the proportion of increase in the set  $k$ . We can express new random variables in matrix notation by  $\hat{R} = A \cdot R$ , where

$$\begin{aligned} R^T &= (R_1, \dots, R_n), \\ \hat{R}^T &= (\hat{R}_1, \dots, \hat{R}_n), \\ A_{ij} &= \begin{cases} 1 - \theta_k + \theta_k/m_k & \text{if } i = j, i \in I_k, \\ 1 & \text{if } i = j, i \notin \bigcup_{k=1}^K I_k, \\ \theta_k/m_k & \text{if } i \neq j, i, j \in I_k, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

2. We get the covariance matrix of  $\hat{R}$ 's, i.e.  $\hat{\Sigma} = A \cdot C \cdot A'$ , which we need to normalize using the matrix  $\hat{S} = (\hat{S}_{ij})_{i,j=1,\dots,n}$  defined by

$$\hat{S}_{ij} = \begin{cases} 1/\sqrt{\hat{\Sigma}_{ij}} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

3. Finally, we compute the new correlation matrix

$$\hat{C} = \hat{S} \hat{\Sigma} \hat{S}.$$

### 7.3 RiskMetrics

Volatility is an important term in risk management, where its modeling provides simple approach to calculating VaR. A special feature of stock volatility is that it is not directly observable, see [41]. We assume that

- $p_t = \ln(P_t)$ ,  $r_t = p_t - p_{t-1}$ ,
- $\mathcal{F}_{t-1}$  is the information set available at time  $t - 1$ ,
- $\mathcal{L}(r_t | \mathcal{F}_{t-1}) = \mathcal{N}(\mu_t, \sigma_t^2)$ ,
- $\mu_t = 0 \Rightarrow r_t = a_t$  follows an IGARCH(1,1) process without a drift  $\alpha_0 = 0$ :

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t, \quad \varepsilon_t \text{ is a Gaussian white noise process,} \\ \sigma_t^2 &= \alpha \sigma_{t-1}^2 + (1 - \alpha) r_{t-1}^2, \quad \alpha \in (0, 1), \quad \left( \text{often } \alpha \in (0.9, 1) \right). \end{aligned}$$

We denote  $r_t[k] = r_{t+1} + \dots + r_{t+k}$   $k$ -horizon log return from time  $t + 1$  to  $t + k$ ,  $\mathcal{L}(r_t[k] | \mathcal{F}_t) = \mathcal{N}(0, \sigma_t^2[k])$ , where

$$\sigma_t^2[k] = \text{Var}(r_t[k] | \mathcal{F}_t) \stackrel{\varepsilon_t \text{ i.i.d.}}{=} \sum_{i=1}^k \text{Var}(a_{t+i} | \mathcal{F}_t) = \sum_{i=1}^k \mathbb{E}(\sigma_{t+i}^2 | \mathcal{F}_t). \quad (7.5)$$

Using  $r_{t-1} = \sigma_{t-1}\varepsilon_{t-1}$  we obtain from (7.5)

$$\begin{aligned}\sigma_t^2 &= \sigma_{t-1}^2 + (1 - \alpha)\sigma_{t-1}^2(\varepsilon_{t-1}^2 - 1), \quad \forall t, \\ \sigma_{t+i}^2 &= \sigma_{t+i-1}^2 + (1 - \alpha)\sigma_{t+i-1}^2(\varepsilon_{t+i-1}^2 - 1), \quad i = 2, \dots, k, \\ \mathbb{E}(\sigma_{t+i}^2 | \mathcal{F}_t) &= \mathbb{E}(\sigma_{t+i-1}^2 | \mathcal{F}_t) \quad \text{for } i = 2, \dots, k,\end{aligned}\tag{7.6}$$

if  $\mathbb{E}(\varepsilon_{t+i-1}^2 - 1 | \mathcal{F}_t) = 0$  for  $i = 2, \dots, k$ . From (7.5) and (7.6) we have the conditional variance of  $r_t[k]$

$$\sigma_t^2[k] = k\sigma_{t+1}^2.\tag{7.7}$$

Thus, for  $k$ -period VaR it holds:  $VaR(k) = u_\alpha\sqrt{k}\sigma_{t+1} = \sqrt{k}VaR$ . It is easy to extend this model for CVaR measuring thanks to the relation between parametric VaR and CVaR showed in Section 2.4.

**Multiple Positions** under RiskMetrics model: If

$$\rho_{ij} = \frac{\text{Cov}(r_{it}, r_{jt})}{\sqrt{\text{Var}(r_{it})\text{Var}(r_{jt})}}, \quad i, j = 1, \dots, m,$$

is the correlation coefficient between returns of  $i$ -th and  $j$ -th asset and  $VaR_i$  is VaR of  $i$ -th asset, then

$$VaR = \sqrt{\sum_{i=1}^m VaR_i^2 + 2 \sum_{i < j} \rho_{ij} VaR_i VaR_j}.$$

## 7.4 Software

Table 7.1: Used software.

Software	Version	Using
Matlab	5.0	optimization, graphs drawing
SPSS	12.0	estimation, statistical testing
MikTeX	2.2	typesetting
TeXnicCenter	beta	typesetting

## 7.5 Numerical results

Normality tests and correlation matrices used in the numerical study in Subsection 3.2.4 are proposed. Source codes and numerical results can be found on enclosed CD.

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