Report on the diploma thesis by Norbert Požár

The thesis with the title 'Selected properties of stationary axially symmetric fields in general relativity' presents a critical review of the use of methods from algebraic geometry in the context of solutions to the stationary axisymmetric Einstein equations in vacuum.

It summarizes the axiomatic approach on the construction of solutions in terms of hyperelliptic theta functions as given by Korotkin and in later works by Klein and Richter. This alone would be an achievement since the subject has by far not reached the level of textbook knowledge. In the moment only scientific papers and a monograph based on a series of papers exist. The fact that a diploma student could provide an own compilation of such a subject would already be remarkable. However, Požár's thesis does not stop here, since he adds critical remarks to the proofs and complements them with own ideas. In addition he presents an own implementation of a numerical code based on spectral methods by Frauendiener and Klein within Mathematica.

I will now discuss the contents of the thesis in detail and add some remarks where appropriate. In the introduction Požár describes aspects of general relativity which distinguishes it from Newtonian theory and notices the difficulty to construct physically interesting solutions to Einstein's equations. It should be noted in this context that an expanding universe is also possible in a Newtonian theory (see the works of Heckmann and Schücking). The main experimental problem of Newtonian theory was the failure to describe the perihelion advance of Mercury. This is correctly reproduced by general relativity which is up to now in accordance with all experiments.

In section 2 the author presents a short review of general relativity and the Einstein equations. In the stationary axisymmetric case he reproduces Ernst's approach to obtain the equations in the form of the complex valued Ernst equation from an action principle. He discusses the 2-soliton solution of the Ernst equation, the Kerr solution, which gives the gravitational field of the stationary axisymmetric black hole. He also presents the derivation of the stationary axisymmetric Einstein-Maxwell equations in the Ernst form.

In section 3 he gives a short account of the theory of Riemann surfaces including the topology, the homology, differentials and integrals, the notion of divisors and hyperelliptic surfaces as a special case. Theta functions as building elements to describe meromorphic functions on Riemann surfaces are discussed in section 4. The theorem 4.2 should not be called Riemann theorem, Riemann has too many theorems; Riemann vanishing theorem (for the Riemann theta function) would be better. In section 5 the collected mathematical facts are used to solve a (scalar) Riemann-Hilbert problem on a Riemann surface as was done by Zverovich. It would be appropriate to cite

Zverovich's review in Russian Mathematical Surveys, not only Rodin's book, since the former's contributions were very important for the subject. Požár also addresses more recent developments on the solution of Hilbert's 21st problem by Korotkin, who solved the latter for quasi-permutation matrices by using the theory of Hurwitz spaces. In this context it should be noted that this problem which is also known under the name Riemann-Hilbert problem is not always solvable in contrast to the former. There are explicit counter-examples known thanks to Bolybrukh and Kostov. Also the problem considered by Korotkin is not solvable for the so-called Malgrange divisor. In the case of the Ernst equation this happens for the ergo-sphere.

In section 6 the author presents the construction of hyperelliptic solutions to the nonlinear Schrödinger equation via the so-called Baker-Akhiezer function, a generalization of the exponential function to Riemann surfaces. In section 7 he studies the linear system for the Ernst equation which is defined on the Riemann sphere with branch points depending on the physical coordinates. Notice that the used form of the linear system is not the most common in literature, it has just the advantage that the Ernst potential can be read off directly at infinity. Its disadvantage is that the symmetry of the Ernst equation cannot be seen immediately. It is later mentioned in the thesis that such a symmetry exists, but one should add that it is an $SU(1,1)$ symmetry. Notice also that there is an evolutionary form of the Ernst equation denoted by the term hyperbolic Ernst equation. Thus the difference with respect to integrable evolution equations as NLS is really the non-autonomous linear system and not the fact that the elliptic Ernst equation is elliptic.

The author gives a detailed account on how the linear system can be used to construct solutions to the Ernst equation. He even uses the Riemann-Roch theorem in this context though people working in the field typically consider it as obvious that a function on a surface of genus 0 can only depend on the local parameter and the square-root. The form of the logarithmic derivative of the matrix of the linear system follows from this. The solution of the scalar Riemann-Hilbert problem is then used to construct the full matrix of the linear system on a four-sheeted Riemann surface. Again the construction is given in more detail than in the literature. The author produces on his own what is shown in section 2.2 of reference [1], that Riemann surfaces factorized with respect to an automorphism define again a Riemann surface. There it is also shown that fixed points of the automorphism lead to branch points of the quotient surface. It is remarkable that the author can reproduce this, but it is a bit exaggerated to write of a gap in a proof which refers to known literature.

In section 8 the author describes the spectral approach for a numerical treatment of hyperelliptic theta functions by Frauendiener and Klein. It should be noted that this code was already briefly described in a publication in 2000. Thus the statement in the abstract appears a bit surprising that a similar code as in the thesis has in the meantime be published. This spectral code has later been generalized to hyperelliptic surfaces with real branch points of arbitrary genus. The only open question in this context appears to be the inclusion of different cut systems for higher genus Riemann surfaces in the case of Ernst potentials which is appropriate to avoid the crossing of cuts on the surface. However, Požár gives again a critical discussion of this code and uses own ideas and a slightly different analytic treatment of the almost degenerate cases on the Riemann surface.

The thesis is well written in a generally good English. The weakest part seems to be the abstract which should be reread. In general the author mixes the comparative 'than' with 'then' which describes an order in time. I will list several typos I found below (p: page, l: line).
- p4, l3 from below: 'find' instead of 'found'
- p12, 2nd paragraph: 'WConsider' instead of 'We consider'
- p13, after Def. 3.7: isomorphic to ('to' is missing)
- p31, 1st paragraph: representation

To sum up, this is a well written thesis where the author not only could summarize a complex and cluttered scientific field in a comprehensive and impressive way, but where he critically complemented proofs in technically complicated algebraic-geometric context, and where he added own ideas in a state of the art numerical code. The thesis obviously deserves the best available mark (1 or A).

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