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DIPLOMOVÁ PRÁCE

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Modely úrokových měr ve spojitém čase

Katedra pravděpodobnosti a matematické statistiky

Vedoucí diplomové práce: Prof. RNDr. Tomáš Cipra, DrSc.

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Děkuji Prof. RNDr. Tomášovi Ciprovi, DrSc. za vedení diplomové práce, poskytnutí studijních materiálů a cenné rady a připomínky. Zároveň děkuji svým rodičům za pomoc a podporu po dobu celého studia.

Prohlašuji, že jsem svou diplomovou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

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DIPLOMA THESIS

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Interest rates models in continuous time

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Abstrakt

Název práce: Modely úrokových měr ve spojitém čase

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Abstrakt: Jadrom práce je predstaviť pravdepodobnostné metódy aplikované v bežne používaných finančných modeloch a formulovať časovú štruktúru úrokových mier v spojitom čase, bez arbitrážnych možností. Stochastické procesy v tejto práci sú reverzné, pretože v dlhšom časovom horizonte majú tendenciu návratu k priemerným dlhodobým úrovniam. Všetky predstavené modely úrokových mier sú Itôove procesy založené na Brownovom pobybe a každý z nich definuje parametre, pomocou ktorých sa snažia čo najviac priblížiť reálnemu vývoju úrokových mier. Na znázornenie výsledkov sú poskytnuté príklady a grafy.

Kľúčové slová: Martingal, Itôovo lemma, Stochastický proces, Časová štruktúra úrokových mier

Abstract

Title: Interest rates models in continuous time

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Abstract: The core of this work is to introduce the probabilistic techniques used in widely applied financial models and to formulate the term structure of interest rates using the continuous-time no-arbitrage framework. Stochastic processes in this work are mean-reverting, because over the long time horizon, interest rates have the tendency to revert to their average long-term levels. All the short rate models explained are Itô processes based on the Brownian motion, which one-by-one define the parameters to best represent the real behavior of interest rates in continuous time. Examples and graphs are provided for illustration of the key results.

Keywords: Martingale, Itô lemma, Stochastic process, Term structure of interest rates

Chapter 1

Introduction

The objective of this work is to give an introduction to the probabilistic techniques required to understand the most widely used financial models. Recently, financial analysts are using sophisticated mathematical instruments as martingales and stochastic integration for the description of financial-markets behavior and for deriving the computing methods.
Chapter 2 describes the best known stochastic process - the Wiener process, used for modelling Brownian motion and provides the mathematical explanation of martingales, stochastic integrals and differentials, accompanied by examples. Itô's formula is one of the most used and simultaneously one of the most powerful theorems in Stochastic calculus. Girsanov theorem states the conditions for Martingale representation theorem and tells how the stochastic processes change under changes in measure. It's used mostly to convert from probability measure to risk-neutral measure, which ensures arbitrage-free pricing.
Black-Scholes formula from Chapter 4 prices European call and put options on no-dividend paying stocks and derives computing methods used in financial models. It defines the replicating strategy for the price of claims. This formula is frequently used by practitioners, although the model has simplifying character. The practical model for pricing the options is included.
Heath, Jarrow and Morton present the general framework for modelling the evolution of interest rates, mainly the forward rates. HJM models capture the full dynamics of the entire forward rate curve, while the short-rate models capture just the dynamics of a point on the yield curve. The core of this work are short-rate models presented in chapter 7. They are compared with HJM model and one-by-one theoretically derived. These mathematical models describe the future evolution of interest rates by describing the evolution of the short rate. The practical demonstration of the most used models is included. Chapter 7 speaks about interest rate instruments, where the short-rate models are applied. Multi-dimensional models from Chapter 8 are preferred in praxis, because they are based on several Brownian motions and contain more sources of uncertainty.

Chapter 2

Introduction to continuous-time models

Securities belong among the most common and most attractive applications of financial mathematics. Financial market instruments are divided into two basic types: basic market securities, such as stocks, bonds, currencies, commodities and their derivatives - claims (payments made in the future according to the contract) that are contingent on stock's behavior. Derivatives can both reduce or magnify the risk, depends on the fix price of a future transaction.
Both types depend on each other and the connection between them is complex and uncertain, the stock and the claims apparently have a random nature. To be random doesn't necessarily mean to be without inner structure - things are often random in non-random ways. One of the means how to cope with randomness is the study of expectation and probability.

2.1 Continuous processes

The value of continuous process is a real number and it can change at any time and from instant to instant, but it can't make instantaneous jumps.

Family $\{X_t; t \in \mathbf{R}\}$ of random variables defined on probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with values in measurable space (E, ε) is called a **continuous-time stochastic process**.

For the probability space $(\Omega, \mathcal{A}, \mathbf{P})$, an increasing family of σ-algebras included in $\mathcal{A}$ is called a **filtration** $\{\mathcal{F}_t; t \geq 0\}$.
Filtration $\mathcal{F}_t$ represents the information available at time t. Random process $\{X_t; t \geq 0\}$ is adapted to filtration $\mathcal{F}_t$, if for any t, $\{X_t; t \geq 0\}$ is $\mathcal{F}_t$-measurable.

Later on, we'll use this kind of filtration:

$$\text{If } A \in \mathcal{A} \text{ and if } \mathbf{P}(A) = 0, \text{ then for any } t,\ A \in \mathcal{F}_t.$$

Filtration $\mathcal{F}_t$ contains all the **P**-null sets of $\mathcal{A}$. If $X = Y$ **P**a.s. and Y is $\mathcal{F}_t$-measurable, then it could be shown that X is also $\mathcal{F}_t$-measurable.
Filtration $\mathcal{F}_t$ generated by process $\{X_t; t \geq 0\}$ is $\sigma(X_s; s \leq t)$. But if we want filtration satisfying the condition mentioned above, we'll have σ-algebra generated by both $\mathcal{F}_t$ and $\mathcal{N}$ (σ-algebra generated by **P**-null sets of $\mathcal{A}$), called **natural filtration** of process $\{X_t; t \geq 0\}$.

Stopping time $\tau \in [0, \infty]$ with respect to filtration $(\mathcal{F}_t; t \geq 0)$ is a random variable, such that for any $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$.
At any given time t we know, if the stopping time is smaller than t. To associate all together: $\mathcal{F}_\tau = \{A \in \mathcal{A}$, for any $t \geq 0, A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$ is σ-algebra representing the information available before the random time τ.

2.2 Brownian motion - Wiener process

Browian motion was discovered by a botanist Robert Brown. It's based on the zigzagging of microscopic particles under the continuous buffeting of gas.
One century after this discovery, mathematical model - Wiener process was built on the idea of Brownian motion. Brownian motion is an effective component to build a continuous process with, it is the core of most financial models concerning stocks, currencies or interest rates.

Brownian motion - Wiener process
Process W_t, $W_t : t \geq 0$ is a P-Brownian motion if and only if

1. $W_0 = 0$
2. $\{W_t, t \geq 0\}$ has continuous trajectory, so the map $s \mapsto W_s(\omega)$ is continuous
3. for arbitrary time instants $0 \leq t_1 < t_2 < ... < t_n$, increments W_{t1}, $W_{t2} - W_{t1}, ..., W_{tn} - W_{t(n-1)}$ are independent random variables
4. for arbitrary time instants $0 \leq s < t$ the distribution of increments $(W_t - W_s)$ is $N(0, \sigma^2(t - s))$ where σ is a positive constant.

Brownian motion is a real-valued, continuous stochastic process.

$EW_t = ?$
$0 = E(W_t - W_s) = E(W_t - W_0) = EW_t - EW_0 = EW_t \Rightarrow EW_t = 0$

$varW_t = ?$
$varW_t = E(W_t)^2 - (EW_t)^2 = E(W_t)^2 = E[(W_t - W_s) + (W_s - W_0)]^2 = E(W_t - W_s)^2 + 2E(W_t - W_s)(W_s - W_0) + E(W_s - W_0)^2 = \sigma^2 t$ for $t > s$
because $(W_t - W_s)$ and $(W_s - W_0)$ are independent increases and $var(W_t - W_s) = \sigma^2|t - s|$.

Brownian motion is centered, gaussian random process with independent, stationary and orthogonal increments $(W_t - W_s)$.

- Independent increments: for $0 \leq t_1 < t_2 < ... < t_n$ $(W_{t2} - W_{t1}), ..., (W_{tn} - W_{t(n-1)})$ are independent.
- Stationary increments: for every $s, t \in T, s < t$ the distribution of $W_t - W_s$ depends just on $t - s$.
- Orthogonal increments: for $t_1 < t_2 < t_3 < t_4, (t_1, t_2] \cap (t_3, t_4] = \emptyset$ $E(W_{t2} - W_{t1})(W_{t4} - W_{t3}) = 0$.

Brownian motion with respect to filtration $\mathcal{F}_t$:
A real-valued continuous stochastic process is an $\mathcal{F}_t$-Brownian motion if:

- for any $t \geq 0$, W_t is $\mathcal{F}_t$-measurable
- if $s \leq t$, $W_t - W_s$ is independent of σ-algebra $\mathcal{F}_s$
- if $s \leq t$, $W_t - W_s$ and if $W_{t-s} - W_0$ are equally distributed

Brownian motion is a Markovov random process, which means the future value is dependent just on the present value, irrespective of the values before.

General characteristics of Brownian motion defined in [2]:
- W is continuous everywhere but differentiable nowhere
- W hits every real value (it doesn't matter how large or negative it is) and after hitting the value for the first time, it hits it infinitely often
- W doesn't depend on the scale, it looks the same at every scale.

2.3 Continuous-time martingales

We are still working on probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with filtration $(\mathcal{F}_t, t \geq 0)$.

A family of integrable random variables $\{M_t, t \geq 0\}$ where $\mathbf{E}(|M_t|) < \infty$ for any time t is

- a **martingale** if, for any $s \leq t$,
$$\mathbf{E}(M_t|\mathcal{F}_s) = M_s \tag{2.1}$$
- a **supermartingale** if, for any $s \leq t$, $\mathbf{E}(M_t|\mathcal{F}_s) \leq M_s$.
- a **submartingale** if, for any $s \leq t$, $\mathbf{E}(M_t|\mathcal{F}_s) \geq M_s$.

If $\{M_t, t \geq 0\}$ is a martingale, then $\mathbf{E}(M_t) = \mathbf{E}(M_0)$ for any t.
To explain the definition of a martingale, the expected future value conditional on its present value and past history, is just the present value.

Lets focus the attention to examples of martingales for $\{W_t; t \geq 0\}$, a standard $\mathcal{F}_t$-Brownian motion:

1. constant process $W_t = c$ is $\mathcal{F}_t$-martingale
2. W_t is $\mathcal{F}_t$-martingale
3. $(W_t^2 - t)$ is $\mathcal{F}_t$-martingale
4. $exp(\sigma W_t - (\frac{\sigma^2}{2})t)$ is $\mathcal{F}_t$-martingale

Proof:
1. constant process $W_t = c$:
$\mathbf{E}(W_t|\mathcal{F}_t) = c = W_s$ for all $s \leq t$.

2. Brownian motion W_t:
For $s \leq t$, the increment $(W_t - W_s)$ is independent of σ-algebra $\mathcal{F}_s \Rightarrow \mathbf{E}(W_t - W_s|\mathcal{F}_s) = \mathbf{E}(W_t - W_s)$ and we know that for Brownian motion, $\mathbf{E}(W_t - W_s) = 0$.

3. The third example of martingale is a little bit more difficult:
First,
$\mathbf{E}(W_t^2 - W_s^2|\mathcal{F}_s) = \mathbf{E}((W_t - W_s)^2 + 2W_s(W_t - W_s)|\mathcal{F}_s) = \mathbf{E}((W_t - W_s)^2|\mathcal{F}_s) + 2W_s(W_t - W_s|\mathcal{F}_s)$.
According to 2., where we proved that W_t is a martingale, $\mathbf{E}(W_t - W_s) = 0 \Rightarrow$
$\mathbf{E}(W_t^2 - W_s^2|\mathcal{F}_s) = \mathbf{E}((W_t - W_s)^2|\mathcal{F}_s)$
and from the assumption of stationary and independent increments:
$\mathbf{E}((W_t - W_s)^2|\mathcal{F}_s) = \mathbf{E}(W_{t-s}^2) = t - s$, because $W_t \sim N(0, t)$.
And to finish, $\mathbf{E}(W_t^2 - t|\mathcal{F}_s) = W_s^2 - s$ for $s < t$.

4. To prove that the last assertion is a martingale, we need to use the normal distribution of a random variable X.
The probability density function of a standard normal variable $X \sim N(0,1)$ is $f(x) = \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2})$ and for any complex number c, $\mathbf{E}(e^{cX}) = e^{\frac{c^2}{2}}$.
We want to show that $exp(\sigma W_t - (\frac{\sigma^2}{2})t)$ is a martingale. If $s < t$

$$\mathbf{E}(e^{\sigma W_t - (\frac{\sigma^2 t}{2})}|\mathcal{F}_s) = e^{\sigma W_s - (\frac{\sigma^2 t}{2})}\mathbf{E}(e^{\sigma(W_t - W_s)}|\mathcal{F}_s),$$

because W_s is $\mathcal{F}_s$-measurable. We have to consider that the increment $(W_t - W_s)$ is independent of $\mathcal{F}_s$, what helps us with the second part of the expression:

$$\mathbf{E}(e^{\sigma(W_t - W_s)}|\mathcal{F}_s) = \mathbf{E}(e^{\sigma(W_t - W_s)}) = \mathbf{E}(e^{\sigma W_{t-s}})$$

To finish the proof, we'll consider an equation for a standard random variable Y:

$$\mathbf{E}(e^{\lambda Y}) = \int_{-\infty}^{+\infty} e^{\lambda x} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = e^{\frac{\lambda^2}{2}}.$$

According to this, we get:

$$\mathbf{E}(e^{\sigma W_{t-s}}) = e^{\frac{1}{2}\sigma^2(t-s)}$$

and

$$\mathbf{E}(e^{\sigma W_t - (\frac{\sigma^2 t}{2})} | \mathcal{F}_s) = e^{\sigma W_s - (\frac{\sigma^2 t}{2})} e^{\frac{1}{2}\sigma^2(t-s)} = e^{\sigma W_s - \frac{1}{2}\sigma^2 s}, \text{ so it's a martingale.}$$

As we will see later, central to pricing derivatives is to produce martingales including a time line.
Consider claim X that depends just on events up to time T. If $\mathbf{E}(|X|) < \infty$, we should show that $N_t = \mathbf{E}(X|\mathcal{F}_t)$ is a martingale. Lets prove that $\mathbf{E}(N_t|\mathcal{F}_s) = N_s$:

$$\mathbf{E}(N_t|\mathcal{F}_s) = \mathbf{E}(\mathbf{E}(X|\mathcal{F}_t)|\mathcal{F}_s) = \mathbf{E}(X|\mathcal{F}_s) \tag{2.2}$$

This equation shows us the so called **tower law of conditional expectation** saying that conditioning firstly on information up to time t and then on information up to time s is the same as if we begin with conditioning up to time s.

2.4 Stochastic integral

Brownian motion is usually used as a global model for stock behavior. But it can't be used just on its own. The common character of the stock is that it grows at some rate, because the prices rise (due to many factors, important one is inflation). So, we'll form Brownian motion with drift $S_t = W_t + \mu t$ by adding constant μ which reflects a nominal growth. To make the Brownian motion less or more noisy, we scale it by a constant factor σ: $S_t = \sigma W_t + \mu t$. The prices of the stock can't be negative, that's why we have to consider a process that never goes negative. The resultant process describing stock behavior the most realistic is exponential Brownian motion with drift:

$$X_t = \exp(\sigma W_t + \mu t).$$

We would like to model stock prices. In praxis, there are used models that are functions of one or more Brownian motions. But, one of the characteristics of Brownian motion is, that its path is not differentiable at any point. That's the main reason why we need to define an integral with respect to Brownian motion - **stochastic integral**.
In continuous time, for building a stock we'll consider 2 building blocks: Newtonian functions and Brownian motion. Stochastic process X has both: a Newtonian component based on dt and a Brownian component based on dW_t (infinitesimal increment of W):

$$dX_t = \sigma_t dW_t + \mu_t dt$$

where μ is a drift rate and σ is a diffusion rate or noisiness.
The drift μ_t depends on time t, or can be random and depend on values that X or W took up until time t itself. And for noisiness σ_t, we have the same. Processes like X and σ are called **adapted** to the filtration $\mathcal{F}$, if their value at time t depends on the history $\mathcal{F}_t$, but not the future.

Uniqueness of volatility and drift:

- If 2 processes agree at time $t = 0$ and they have identical σ_t and μ_t, then they are equal. That means X is unique for given σ_t, μ_t and X_0.
- For given X there is just one pair of volatility σ_t and drift μ_t which satisfies the equation $X_t = X_0 + \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds$ for all t.

2.5 Stochastic differentials

We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t; t \geq 0\}, \mathbf{P})$ and an $\mathcal{F}_t$-Brownian motion $\{W_t; t \geq 0\}$. **Stochastic process** X is a continuous process $\{X_t : t \geq 0\}$ such that X_t can be written as

$$X_t = X_0 + \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds \tag{2.3}$$

where X_0 is $\mathcal{F}_0$-measurable, σ, μ are random $\mathcal{F}$-previsible processes such that $\int_0^t (\sigma_s^2 + |\mu_s|) ds$ is finite for all times t (with probability 1). The differential form of this equation is:

$$dX_t = \sigma_t dW_t + \mu_t dt. \tag{2.4}$$

The following proposition defined in [2] holds:
If $\{M_t; 0 \leq t \leq T\}$ is a continuous martingale such that

$$M_t = \int_0^t \mu_s ds, \text{ with } \mathbf{P}\text{a.s. } \int_0^T |\mu_s| ds < +\infty,$$
then
$$\mathbf{P}\text{a.s. } \forall t \leq T,\ M_t = 0.$$

Thanks to this proposition, Itô process decomposition is unique.

2.5.1 Itô formula for continuous martingales

Let $\{X_t; 0 \leq t \leq T\}$ be a stochastic Itô process satisfying (2.3) and (2.4) and f be a deterministic twice continuously differentiable function. Then $Y_t := f(X_t)$ is also a stochastic process and it's given by:

$$\mathbf{dY_t = (\sigma_t f'(X_t))dW_t + (\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t))dt}. \tag{2.5}$$

This equation can be found in [6], [2] or [8].
Examples:

- $\mathbf{Y_t = f(W_t)}$
  This is the simplest but very important derivation. We'll use two methods and see if we get the same result.

  - Taylor extension of $f(W_t)$ for a smooth function f:
    $df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)(dW_t)^2 + \frac{1}{3!}f'''(W_t)(dW_t)^3 + ...$
    Brownian motion (BM) is odd and it's important to know how does $(dW_t)^2$ look in the Taylor expansion. Lets model the integral of $(dW_t)^2$ using an approximation

    $$\int_0^t (dW_t)^2 = \sum_{i=1}^n (W(\tfrac{ti}{n}) - W(\tfrac{t(i-1)}{n}))^2$$

    Each increment of BM $(W(\frac{ti}{n}) - W(\frac{t(i-1)}{n}))$ is a normal variable with distribution $N(0, \frac{t}{n})$. We'll use it to get a set of IID normals $N(0,1)$ by setting for each n

    $$Z_{n,i} = \frac{(W(\frac{ti}{n}) - W(\frac{t(i-1)}{n}))}{\sqrt{t/n}}$$

    According to this, we can approximate the integral with

    $$\int_0^t (dW_t)^2 \approx t \sum_{i=1}^n \frac{Z_{n,i}^2}{n} \to t$$

    because $\sum_{i=1}^n \frac{Z_{n,i}^2}{n}$ converges towards the constant expectation of $Z_{n,i}^2$ what is 1 according to the Weak Law of Large Numbers. We get $\int_0^t (dW_t)^2 = t$ or $d(W_t)^2 = dt$. The sizes of $(dW_t)^3, (dW_t)^4, ...$ are really small in comparison to dW_t and $(dW_t)^2$, so we assume they are 0. And the Taylor expansion looks:

    $$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt$$

  - Itô formula
    We simply substitute the values $\mu_t = 0$, $\sigma_t = 1$ and $Y_t = f(W_t)$ to the formula:

    $$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt,$$

    so the results are really the same.

- $\mathbf{X_t = W_t}$ **and** $\mathbf{f(x) = x^2}$
  In this case $Y_t = W_t^2$, $\mu = 0$ and $\sigma = 1$. Using Itô formula we get:

  $$W_t^2 = 2\int_0^t W_s dW_s + t,$$

  and

  $$d(W_t^2) = 2W_t dW_t + dt.$$

It turns out that $W_t^2 - t = 2\int_0^t W_s dW_s$ and since $\mathbf{E}\int_0^t W_s^2 ds < \infty$, we've just confirmed the fact of $W_t^2 - t$ being a martingale. This example also shows the main difference between standard and Itô differentials.

- $\mathbf{X_t = W_t}$ and $\mathbf{f(x) = e^x}$
  We have $Y_t = e^{W_t}$, $\mu = 0$ and $\sigma = 1$. Using Itô formula we get:

$$d(e^{W_t}) = e^{W_t}dW_t + \tfrac{1}{2}e^{W_t}dt = Y_t dW_t + \tfrac{1}{2}Y_t dt.$$

Stochastic Differential Equation (SDE)

The most immediate use of Itô is to generate SDEs from a functional expression of the process. Consider $X_t = \sigma_t W_t + \mu_t dt$. In special case when σ and μ depend on W only trough X_t, the equation

$$dX_t = \sigma(X_t, t)dW_t + \mu(X_t, t)dt$$

is called SDE: a **stochastic differential equation** for X. These equations don't need to have a solution and if they have, the solution might not be unique. To work with differential equation, we can't use simple integration. Itô formula is the right way to go.

- $\mathbf{Y_t = exp(\sigma W_t + \mu t)}$
  Denote $X_t = \sigma W_t + \mu t$ and $f(x) = x^2$. The differential of X_t is $dX_t = \sigma dW_t + \mu dt$ and with application of Itô to $Y_t = f(X_t)$ we get:

$$df(X_t) = (\sigma f'(X_t))dW_t + (\mu f'(X_t) + \tfrac{1}{2}\sigma^2 f''(X_t))dt$$

  Exponential function is very pleasant what concerns differentials:
  $f(X_t) = f'(X_t) = f''(X_t) = ...$
  and we can rewrite the equation for Y_t:

$$df(X_t) = \sigma f(X_t))dW_t + (\mu f(X_t) + \tfrac{1}{2}\sigma^2 f(X_t))dt) \Rightarrow$$

$$df(X_t) = f(X_t)(\sigma dW_t + (\mu + \frac{1}{2}\sigma^2)dt) \tag{2.6}$$

  For exponential function X_t, variable σ is called *log volatility* (it is volatility of process $\log X_t$) and variable μ is called *log drift* of the process.

Itô formula is used not only to create SDEs from processes, but also the opposite way - to convert SDEs to processes or simply, to solve them. SDEs are sometimes too difficult to solve, but we'll focus the attention to some special cases. A solution of SDE is called a **diffusion**. SDEs are useful to model most financial assets - stocks or interest rates processes.

Process from SDE - Dóleans exponential of Brownian motion
We are supposed to solve

- $dX_t = \sigma X_t dW_t$
  Using (2.6) modified to $dX_t = \sigma X_t dW_t + (\mu + \frac{1}{2}\sigma^2)X_t dt$,
  we can see that we get this result if we denote $\mu = -\frac{1}{2}\sigma^2$ and the second part of equation will be zero. We've got one of just a few soluble equations, called **Dóleans exponential of Brownian motion**.

- $\mathbf{dX_t} = (\sigma \mathbf{dW_t} + \mu \mathbf{dt})$
  We've got to the point, where we can just play with the equations in order to get something we know already from the exercises before. We denote $\lambda = \mu - \frac{1}{2}\sigma^2$ and we get SDE $e^{\sigma W_t + \lambda t}$ which has the solution:
  $$X_t = e^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t}.$$

- $\mathbf{dX_t} = \mathbf{X_t}(\sigma \mathbf{dW_t} + \mu_\mathbf{t} \mathbf{dt})$ for μ_t a general bounded integrable function.
  We can inspire ourselves by the last example, the only problem is μ_t, because it depends on time. This is why the solution is:
  $$X_t = X_0 e^{\sigma W_t + \int_0^t \mu_s ds - \frac{1}{2}\sigma^2 t}.$$

2.5.2 Integration by parts formula

- X_t and Y_t are two stochastic processes adapted to the same BM:
  $$dX_t = \sigma_t dW_t + \mu_t dt,$$
  $$dY_t = \rho_t dW_t + \nu_t dt$$
  After applying the Itô formula (2.5) to $X_t Y_t = \frac{1}{2}((X_t + Y_t)^2 - X_t^2 - Y_t^2)$ we get:
  $$d(X_t Y_t) = X_t dY_t + Y_t dX_t + \sigma_t \rho_t dt \tag{2.7}$$
  Lets prove the formula:
  $$(X_t + Y_t)^2 = (X_0 + Y_0)^2 + 2\int_0^t (X_s + Y_s) d(X_s + Y_s) + \int_0^t (\sigma_s + \rho_s)^2 ds$$
  $$(X_t)^2 = X_0^2 + 2\int_0^t X_s dX_s + \int_0^t \sigma_s^2 ds$$
  $$(Y_t)^2 = Y_0^2 + 2\int_0^t Y_s dY_s + \int_0^t \rho_s^2 ds$$
  And after substraction it turns out that
  $$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \int_0^t \sigma_s \rho_s ds$$
  or in differential form:
  $$d(X_t Y_t) = X_t dY_t + Y_t dX_t + \sigma_t \rho_t dt.$$

- X_t and Y_t are two stochastic processes adapted to two different and independent Brownian motions:
  For σ_t, ρ_t the respective volatilities of X and Y, μ_t, ν_t the drifts and W, W' two different Brownian motions denote

$$X_t = \sigma_t dW_t + \mu_t dt,$$
$$Y_t = \rho_t dW_t' + \nu_t dt,$$

then the integration by parts formula is the same as in Newtonian calculus:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t \tag{2.8}$$

and the proof is similar to the one above.

Chapter 3

Change of probability measure

3.1 Change of probability

Brownian motions change in easy and pleasant ways under changes in measure. By extension through their differentials, the stochastic processes do the same.

Equivalence of probabilities
Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. Probability measures $\mathbf{P}$ and $\mathbf{Q}$ operating on the same probability space $(\Omega, \mathcal{A})$ are equivalent if, for A being an event in the sample space $\mathcal{A}$:

$$\forall A \in \mathcal{A} : \mathbf{P}(A) > 0 \Leftrightarrow \mathbf{Q}(A) > 0 \tag{3.1}$$

That means if A is possible under $\mathbf{P}$, then it is also possible under $\mathbf{Q}$ and vice versa.

3.1.1 Continuous Radon-Nikodym derivative

We have equivalent measures $\mathbf{P}$, $\mathbf{Q}$ and a dense division $\{t_1, ..., t_n\}$ of time interval $[0, T]$. Denote $x_i = W_{t_i}(\omega)$, then the derivative $\frac{d\mathbf{Q}}{d\mathbf{P}}$ up to time T is defined as a limit of the likelihood ratios

$$\frac{d\mathbf{Q}}{d\mathbf{P}}(\omega) = \lim_{n\to\infty} \frac{f^n_{\mathbf{Q}}(x_1, ..., x_n)}{f^n_{\mathbf{P}}(x_1, ..., x_n)},$$

and the continuous time derivative $\frac{d\mathbf{Q}}{d\mathbf{P}}$ satisfies

- for all claims X_T known by time T:

$$\mathbf{E}_{\mathbf{Q}}(X_T) = \mathbf{E}_{\mathbf{P}}(\frac{d\mathbf{Q}}{d\mathbf{P}} X_T) \tag{3.2}$$

- for $s \le t \le T$:

$$\mathbf{E}_{\mathbf{Q}}(X_T|\mathcal{F}_s) = \zeta_s^{-1} \mathbf{E}_{\mathbf{P}}(\zeta_t X_T|\mathcal{F}_s) \tag{3.3}$$

where X_t is any process adapted to history $\mathcal{F}_t$ and $\zeta_t = \mathbf{E_P}(\frac{d\mathbf{Q}}{d\mathbf{P}}|\mathcal{F}_t)$.

Equivalent definition of Radon-Nikodym theorem:

$\mathbf{Q}$ is absolutely continuos relative to $\mathbf{P}$ if and only if there exists a non-negative random variable Z on $(\Omega, \mathcal{A})$,such that

$$\forall A \in \mathcal{A} : \mathbf{Q}(A) = \int_A Z(\omega) d\mathbf{P}(\omega)$$

where Z is the density of $\mathbf{Q}$ relative to $\mathbf{P}$ denoted by $\frac{d\mathbf{Q}}{d\mathbf{P}}$.

Switching from measure $\mathbf{P}$ to measure $\mathbf{Q}$ changes the relative likelihood of path being chosen. All that measure can change in Brownian motion is the drift μ. The processes we are interested in are representable as instantaneous differentials made up of Brownian motion and drift.

3.1.2 Cameron-Martin-Girsanov theorem

The Girsanov theorem or enlarged Cameron-Martin-Girsanov theorem tells how stochastic processes change under changes in measure. The main contribution of the theorem is the process of converting from the physical measure which describes the probability that an underlying instrument (share price or interest rate) will take a particular value or values to the risk-neutral measure which is a very useful tool for evaluating the value of derivatives on the underlying. This theorem will help us to utilize measure $\mathbf{Q}$ in order to turn a $\mathbf{P}$-Brownian motion into a Brownian motion with some specified drift.

Consider a probability space $(\Omega, \{\mathcal{F}_t\}, \mathbf{P})$ on the time interval $[0, T]$ with the filtration of standard Brownian motion $\{W_t; t \in [0, T]\}$.

Cameron-Martin-Girsanov theorem, (see [2] or [11]):
Let W_t be a $\mathbf{P}$-Brownian motion and $\{\theta_t\}$ is an $\mathcal{F}$-previsible adapted process satisfying the condition $\mathbf{E_P}(e^{\frac{1}{2}\int_0^T \theta_s^2 ds}) < \infty$, then there exists a measure $\mathbf{Q}$ such that

- $\mathbf{Q}$ is equivalent to $\mathbf{P}$
- $\frac{d\mathbf{P}}{d\mathbf{Q}} = e^{-\int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T \theta_s^2 ds}$ is a martingale
- a standard $\mathbf{Q}$-Brownian motion is

$$\widetilde{W_t} = W_t + \int_0^t \theta_s ds \tag{3.4}$$

W_t is a drifting Brownian motion with drift $(-\theta_t)$ at time t.

Cameron-Girsanov converse
If W_t is a $\mathbf{P}$-Brownian motion and $\mathbf{Q}$ is a measure equivalent to $\mathbf{P}$ (3.1) then there exists an $\mathcal{F}$-previsible process θ_t such that

$$\widetilde{W_t} = W_t + \int_0^t \theta_s ds \tag{3.5}$$

is a $\mathbf{Q}$-Brownian motion (that means W_t plus drift θ_t is a $\mathbf{Q}$-Brownian motion). Radon-Nikodym derivative of $\mathbf{Q}$ with respect to $\mathbf{P}$ (according to (3.2)) at time T is $\exp(-\int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T \theta_s^2 ds)$.

Aplication of CMG to stochastic differentials
Suppose W is a Brownian motion and X is a stochastic process from (2.3) with increment

$$dX_t = \sigma_t dW_t + \mu_t dt$$

We would like to know, if there is a measure $\mathbf{Q}$ that changes the drift of process X from $\mu_t dt$ to $\nu_t dt$. Process can be rewritten as

$$dX_t = \sigma_t(dW_t + (\tfrac{\mu_t - \nu_t}{\sigma_t})dt) + \nu_t dt$$

Denote $\theta_t = \frac{\mu_t - \nu_t}{\sigma_t}$. We have a new measure $\mathbf{Q}$ with which $\widetilde{W_t} = W_t + \int_0^t (\frac{\mu_s - \nu_s}{\sigma_s})ds$ is a $\mathbf{Q}$-Brownian motion after satisfying the condition $\mathbf{E_P}(e^{\frac{1}{2}\int_0^T \theta_s^2 ds}) < \infty$.

And how does the differential of X under $\mathbf{Q}$ look?

$$dX_t = \sigma_t d\widetilde{W_t} + \nu_t dt,$$

and a $\mathbf{Q}$-Brownian motion gives X the drift ν_t we wanted.

One important note: the change of measure only changes the Brownian-motion to a Browian-motion plus drift, the volatility of the process stays the same.

Now show the CMG theorem on few examples:

- $\mathbf{X_t} = \sigma \mathbf{W_t} + \mu$
  is a $\mathbf{P}$-Brownian motion, σ, μ are constants. We apply the CMG theorem (3.4) to $\theta_t = \frac{\mu}{\sigma}$. Then there exists an equivalent measure $\mathbf{Q}$ defined in (3.1) under which

  $\widetilde{W_t} = W_t + \frac{\mu}{\sigma}t$, $\widetilde{W_t}$ is a $\mathbf{Q}$-Brownian motion up to time T
  and
  $X_t = \sigma\widetilde{W_t}$ is a scaled $\mathbf{Q}$-Brownian motion

  Compare $\mathbf{E_P}(X_t^2) = \mu^2 t^2 + \sigma^2 t$ and $\mathbf{E_Q}(X_t^2) = \sigma^2 t$. Different measures give rise to different expectations.

- Exponential Brownian-motion $\mathbf{X_t}$
  It's given a **P**-Brownian motion W with stochastic differential equation

  $$dX_t = X_t(\sigma dW_t + \mu dt).$$

  How can we change the measure, to get a new SDE

  $$dX_t = X_t(\sigma dW_t + \nu dt)$$

  for X with arbitrary constant drift ν? Using the same procedure as before, we denote $\theta_t = \frac{\mu-\nu}{\sigma}$. There exists a measure **Q** under which $\widetilde{W}_t = W_t + \frac{(\mu-\nu)t}{\sigma}$ is **Q**-Brownian motion and X has SDE

  $$dX_t = X_t(\sigma d\widetilde{W}_t + \nu dt).$$

3.2 Martingale representation theorem

The Martingale Representation Theorem in [2] shows the existence of a hedging strategy. The process is expected to stay the same under martingale measure. Consider two processes M_t and N_t, we would like to represent changes in N_t by scaled changes in M_t, another non-trivial martingale.

Martingale representation theorem
Suppose M_t is a **Q**-martingale process with non-zero volatility σ_t. Then if N_t is another **Q**-martingale, there exists an $\mathcal{F}$-previsible process ϕ, that $\mathbf{P}(\int_0^T \phi_t^2\sigma_t^2 dt < \infty) = 1$ and

$$N_t = N_0 + \int_0^t \phi_s dM_s, \tag{3.6}$$

where ϕ is unique (it's the ratio of volatilities of M_t and N_t).

Martingales and drifts
Consider stochastic process X with volatility σ_t and SDE $dX_t = \sigma_t dW_t + \mu_t dt$ which satisfies condition $\mathbf{E}[(\int_0^t \sigma_s^2 ds)^{\frac{1}{2}}] < \infty$ then

$$X \text{ is a martingale} \Leftrightarrow X \text{ is driftless } (\mu_t \equiv 0).$$

Exponential martingales
For exponential martingales, control of the technical condition from above could be too difficult. That's why for this kind of martingales it's better to use the test:
If $dX_t = \sigma_t X_t dW_t$ for some previsible process σ_t,

$$\mathbf{E}(e^{\frac{1}{2}\int_0^T \sigma_s^2 ds}) < \infty \tag{3.7}$$

$$\Rightarrow X \text{ is a martingale.}$$

And the solution to the SDE is $X_t = X_0 e^{\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds}$.

Chapter 4

Black-Scholes model

Black-Scholes model is a mathematical formula to calculate the theoretical value of financial instruments, especially stock options. The main aim is to describe the behavior of markets and derive computing methods used in financial models.

Options
Option gives its holder right but not obligation to buy or sell a certain amount of the financial asset, by a certain date, for a certain strike price K. The writer of an option specifies:

- the type of option:
  - call option is the option to buy
  - put option is the option to sell
- the underlying asset (a stock, a bond, a currency, etc.)
- the amount of underlying asset to be sold or purchased
- the expiration date
  - *American option* can be exercised at any time before maturity
  - *European option* can be exercised just at maturity
- the exercise price at which the transaction is done if the option is exercised

The price of an option is called *premium*. The premium is usually quoted by the organized market.

Black-Scholes model
Black and Scholes are considered to be the first who successfully used martingales (2.1) and stochastic integration (2.3) and created a model for computing the stock prices. They tackled the problem of pricing and hedging the (put or call) European option on a non-dividend paying stock in their work [3]. Their simplest model is a continuous-time model considering a market that consists of one random security (risky asset)

and a riskless cash account bond. The basic assumption commonly accepted in every financial market is the **absence of arbitrage opportunity**, that means it's not possible to make riskless profit. At the time when the option is written, the price is not known.

We'll denote ϕ_t a process that indicates the number of units of security and ψ_t that indicates the number of units of bond, both hold at time t. Security process should depend just on history up to time t, but not t itself (it should be $\mathcal{F}$ -previsible). **Portfolio (ϕ, ψ)** is a pair of these processes that can take both positive and negative values. Pair of variables (ϕ_t,ψ_t) is a dynamic strategy that tells us the amount of each component held at each time instant. A portfolio is **self-financing** if and only if the change of its value depends just on the change of the asset prices.

Stochastic differential equations.
The value V_t of portfolio (ϕ_t, ψ_t) for stock price S_t and bond price B_t is

$$V_t = \phi_t S_t + \psi_t B_t. \tag{4.1}$$

In the next time instant, portfolio changes the value due to the change of prices S_t and B_t. If there are used just money from profits and losses and no extra money are required, the portfolio is **self-financing** $\Leftrightarrow$

$$dV_t = \phi_t dS_t + \psi_t dB_t. \tag{4.2}$$

But to be self-financing is not an automatic property of portfolio, we always have to check it by using the Itô formula (2.5).

Replicating strategy
We have the equation for the value of portfolio (4.1) and we have to create a strategy that ties down the price of claim X not just at payoff but everywhere. On a market we described, suppose that a risky security S has a volatility σ_t and X is a claim on events up to time T. **Replicating strategy** for X is a self-financing portfolio such that:

$$\int_0^T \sigma_t^2 \phi_t^2 dt < \infty \tag{4.3}$$

and

$$V_T = \phi_T S_T + \psi_T B_T = X, \tag{4.4}$$

which means that the value of portfolio at time T is exactly X.

4.1 Basic Black-Scholes model

The model suggested by Black and Scholes describes the behavior of prices in the continuous time. The bond price B_t and stock price S_t are denoted:

$$B_t = e^{rt}, \quad S_t = S_0 e^{\sigma W_t + \mu t} \tag{4.5}$$

with deterministic constants: r indicating the riskless interest rate, σ the stock volatility and μ the stock drift. We suppose no transaction costs and freely and instantaneously tradable instruments at the quoted prices.

Now, we'll concentrate on the model consisting of a riskless constant-interest rate cash bond and a risky tradable stock following an exponential Brownian motion.

4.2 Model with zero interest rate

This model is the simplest to show the procedure on, because it's simplified by the fact that $B_t = e^{rt} = e^0 = 1$. The process of finding the replication strategy (4.3), (4.4) consists of 3 steps:

- Use the Cameron-Martin-Girsanov theorem (3.4) and SDE to find a measure $\mathbf{Q}$ under which S_t is a martingale:
  $S_t = e^{\sigma W_t + \mu t}$, denote $Z_t = log(S_t)$, then $Z_t = \sigma W_t + \mu t$ and the SDE for Z_t is: $dZ_t = \sigma dW_t + \mu dt$. From this, using the Itô formula (2.5), it's easy to write the SDE for $S_t = \exp(Z_t)$:

  $$dS_t = \sigma S_t dW_t + (\mu + \tfrac{1}{2}\sigma^2) S_t dt.$$

  And in order to find a martingale measure $\mathbf{Q}$: we have a Brownian motion $d\widetilde{W}_t = dW_t + \theta t$ which we gained from the equation above by determining $\theta = (\mu + \frac{1}{2}\sigma^2)/\sigma$. The SDE for S_t using $\widetilde{W}_t$ is

  $$dS_t = \sigma S_t d\widetilde{W}_t.$$

  S_t has the martingale measure $\mathbf{Q}$, because the technical condition for S_t being a martingale under $\mathbf{Q}$ is satisfied due to the fact that σ is a constant.

- Convert the claim X in order to form the process $\mathbf{E_Q}(X|\mathcal{F}_t) = E_t$ from (2.2). This is the work we've done already in the theoretic preparation.

- Find a previsible process ϕ_t such that $dE_t = \phi_t dS_t$. According to the martingale theorem (3.6), with the satisfied condition that the volatility of S_t is always positive, we get:

  $$\begin{aligned} E_t &= \mathbf{E_Q}(X|\mathcal{F}_t) = \mathbf{E_Q}(X) + \textstyle\int_0^t \phi_s dS_s \\ dE_t &= \phi_t dS_t. \end{aligned} \tag{4.6}$$

The last step is to find a replicating strategy (4.4) such that the portfolio will be worth E_t at each time-instant t. The strategy should be self-financing, that means $V_t = \phi_t S_t + \psi_t B_t = E_t$ according to (4.1), (4.2).

We'll suppose the strategy:

- hold ϕ_t units of stock at time t and
- hold $\psi_t = E_t - \phi_t S_t$ units of the bond at time t.

This strategy is really self-financing thanks to the fact that $B_t = 1$ and $dV_t = dE_t = \phi_t dS_t$. The terminal value is $E_T = X$ and we've got an arbitrage price for X at all time instants. The value of portfolio at time $t = 0$ is the price $E_0 = \mathbf{E_Q}(X)$ and the price of the claim X is expected value under the measure which makes process S_t a martingale and S_t is given by: $S_t = e^{\sigma \widetilde{W}_t - \frac{1}{2}\sigma^2 t}$, where σ is constant volatility and $(-\frac{1}{2}\sigma^2)$ is constant drift.

4.3 Non-zero interest rate model

In the case when r was zero, we derived a martingale measure under which S_t was a martingale. Unfortunately, we can't use the result in the case when $r \neq 0$. We have to consider that with a non-zero instantaneous interest rate, the cash grows.

- the riskless asset B_t is determined by $B_t = e^{rt}$ for $t > 0$, ($B_t = 1$ for $t = 0$) and differential equation: $dB_t = rB_t dt$.
- Stock price is $S_t = S_0 e^{\sigma W_t + \mu t}$ (S_0 is the spot price at time $t = 0$) and it's behavior is determined by SDE: $dS_t = S_t(\sigma dW_t + \mu dt)$.
- The validity of the model is in the time horizon $[0, T]$, where T is the maturity.

The solution of the SDE for S_t is following:

$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t}.$$

The law of S_t is log-normal, and process S_t is a solution of the SDE above just in case when $\log(S_t)$ is a Brownian motion.
Here we get to three conditions or hypotheses necessary for Black-Scholes model:

- continuity of paths
- independence of relative increments
- stationarity of relative increments.

Lets get back to the model and the non-zero interest rate. As we know already, the cash grows due to interest rate r. But, to remove the growth of cash, we'll consider a **discount process**. We'll call B_t^{-1} the discount process and denote

- $\widetilde{S}_t = B_t^{-1} S_t = e^{-rt} S_t$ a discounted stock
- $\widetilde{X} = B_t^{-1} X = e^{-rt} X$ a discounted claim.

Let's focus the attention to the discounted stock $\widetilde{S}_t$. We would like to prove that there exists a probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ satisfying (3.1) under which $\widetilde{S}_t$ is a martingale. Lets write down an SDE for $\widetilde{S}_t$:

$$\begin{aligned}
\widetilde{S}_t &= e^{-rt}S_t = S_0 e^{-rt+\mu t+\sigma W_t} \\
L_t &= \log(\widetilde{S}_t) = (\mu - r)t + \sigma W_t \\
dL_t &= (\mu - r)dt + \sigma dW_t \\
d\widetilde{S}_t &= \widetilde{S}_t(\sigma dW_t + (\mu - r + \tfrac{1}{2}\sigma^2)dt).
\end{aligned}$$

And now, we'll repeat the 3 steps to find the replicating strategy (4.4):

- Use the Cameron-Martin-Girsanov theorem (3.4) and SDE to find a measure $\mathbf{Q}$ under which the discounted stock price $\widetilde{S}_t$ is a martingale. Denote $\theta = (\mu - r + \frac{1}{2}\sigma^2)/\sigma$, the SDE for $\widetilde{S}_t$ using $\widetilde{W}_t$ is $d\widetilde{S}_t = \sigma\widetilde{S}_t d\widetilde{W}_t$. There exists martingale measure $\mathbf{Q}$, under which $\widetilde{W}_t$ is a $\mathbf{Q}$-Brownian motion, $\widetilde{S}_t$ is driftles and a martingale:

$$d\widetilde{S}_t = \sigma\widetilde{S}_t d\widetilde{W}_t.$$

- Convert the claim X in order to form the process $\mathbf{E_Q}(\widetilde{X}|\mathcal{F}_t) = E_t$ from (2.2), which is a discounted claim and also a $\mathbf{Q}$-martingale.

- Find a previsible process ϕ_t such that $dE_t = \phi_t d\widetilde{S}_t$. As we know from above, stock price $\widetilde{S}_t$ and conditional expectation process of the discounted claim E_t are $\mathbf{Q}$-martingales.

How will our holding at time T look? We'll have ϕ_T units of stock and ψ_T units of bond worth $X = \phi_T S_T + \psi_T B_T = B_T E_T$ and we'll suppose replicating strategy:

- hold ϕ_t units of stock at time t and
- hold $\psi_t = E_t - \phi_t\widetilde{S}_t$ units of the bond at time t.

The last question is if the portfolio strategy is self-financing according to (4.2). At time t, the value V_t of portfolio (ϕ, ψ) is $V_t = \phi_t S_t + \psi_t B_t = B_t E_t$, $dV_t = B_t dE_t + E_t dB_t = \psi_t B_t d\widetilde{S}_t + E_t dB_t$, we derive that $E_t = \psi_t\widetilde{S}_t + \phi_t$ and dV_t is given by equation

$$dV_t = \phi_t B_t d\widetilde{S}_t + (\phi_t\widetilde{S}_t + \psi_t)dB_t = \phi_t(B_t d\widetilde{S}_t + \widetilde{S}_t dB_t) + \psi_t dB_t.$$

To make it simpler, we know that $d(B_t\widetilde{S}_t) = B_t d\widetilde{S}_t + \widetilde{S}_t dB_t$ and $S_t = B_t\widetilde{S}_t$ and so

$$dV_t = \phi_t dS_t + \psi_t dB_t.$$

and the portfolio (ϕ, ψ) is self-financing.
The strategy in case of non-zero interest rate is self financing if

$$dV_t = \phi_t dS_t + \psi_t dB_t$$

or equivalently, as in (4.6)

$$dE_t = \phi_t d\widetilde{S}_t.$$

To explain the last equations: the strategy is self-financing if the changes of the value are influenced just by the changes in asset values or, in the language of discounted values, changes in discounted value are caused just by changes in the discounted values of assets.

4.4 Summary of Black-Scholes model

We showed how does the model work for both zero and non-zero interest rates. Just summarize it briefly:
We suppose the existence of constants r, μ and σ, a continuously tradable stock $S_t = S_0 e^{\sigma W_t + \mu t}$ and a bond $B_t = e^{rt}$. Then all claims X (we suppose they are integrable), knowable by some time horizon T, have associated replicating strategies (ϕ_t, ψ_t). The price of claim X is an arbitrage price and it's given (for martingale measure $\mathbf{Q}$ and discounted stock $B_t^{-1} S_t$) by

$$V_t = B_t \mathbf{E_Q}(B_T^{-1} X | \mathcal{F}_t) = e^{-r(T-t)} \mathbf{E_Q}(X | \mathcal{F}_t). \tag{4.7}$$

This expression is also the option value at time t, for any option defined by a non-negative, $\mathcal{F}$-measurable random variable, square integrable. This option is replicable and V_t is the value of any replicating portfolio at time t.
The Black-Scholes model is practically used, showing also the Put-Call Parity, on the CD enclosed.

4.5 European call option

Black and Scholes suggested a model to derive an explicit price for European call option paying no dividends. Thanks to this model, the writer of the option can hedge himself and by following this strategy until maturity T, the call premium is exactly the amount of money needed at time 0 to replicate the payoff. The formula depends just on *volatility*, a parameter that is non-directly observable.

Lets define **European call option** by an $\mathcal{F}$-measurable, non-negative random variable C. Call option is the right but not obligation to buy a unit of stock for a predetermined amount K (the exercise price or the *strike* of an option) at particular maturity date T. S_t is the *price of the option* at time t. In case $S_t < K$, the holder of the option doesn't have any interest to exercise the option. But if $S_t > K$, the holder makes a profit $S_t - K$ by exercising the option, it means he buys the stock for price K and sells it on the market for price S_T.

The *value of call option* at maturity time is $(S_T - K)_+ = \max(S_T - K, 0)$. This value depends just on one, maturity time T. If the option is exercised, the writer must

deliver it at price K, it means he must generate an amount $(S_T - K)_+$ at maturity time. But S_T is unknown at time of writing the option.
The replicating strategy at time $t = 0$ under measure $\mathbf{Q}$ for discounted stock $B_t^{-1}S_t$, according to (4.7) is:

$$\mathbf{E_Q}((X|\mathcal{F}_0)) = \mathbf{E_Q}((S_T - K)_+), \quad V_0 = e^{-rT}\mathbf{E_Q}((S_T - K)_+)$$

To find $\mathbf{E_Q}((S_T - K)_+)$, we should find the marginal distribution of S_T under martingale measure $\mathbf{Q}$ due to the fact that $((S_T - K)_+)$ depends on the stock price just at single time-point, maturity time T. For $\mathbf{Q}$-Brownian motion we have:

$$\begin{aligned} d(\log S_t) &= (\sigma d\widetilde{W}_t + (r - \tfrac{1}{2}\sigma^2)dt) \\ \log S_t &= \log S_0 + \sigma\widetilde{W}_t + (r - \tfrac{1}{2}\sigma^2)t \\ S_t &= S_0 \exp(\sigma\widetilde{W}_t + (r - \tfrac{1}{2}\sigma^2)t) \end{aligned}$$

In brackets, we have a normally distributed random variable with mean $(r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$. For $Y \sim N(-\frac{1}{2}\sigma^2 T, \sigma^2 T)$ we get $S_T = S_0 e^{Y+rT}$ and

$$\begin{aligned} V_0 &= e^{-rT}\mathbf{E_Q}((S_T - K)_+) = \\ &= e^{-rT}\mathbf{E_Q}((S_0 e^{Y+rT} - K)_+) = \\ &= \tfrac{1}{\sqrt{2\pi\sigma^2 T}} \int_{log(\frac{K}{S_0})-rT}^{\infty} (S_0 e^x - K e^{-rT}) \exp(-\tfrac{(x+\frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}) dx \end{aligned}$$

Using normal distribution $N(0,1)$ with density function $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and characteristic function $\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, we transform the formula above. For $z = -\frac{(x+\frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}}$ ($\Rightarrow x = -z\sigma\sqrt{T} - \frac{1}{2}\sigma^2 T$) we get:

$$\begin{aligned} V_0 &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{*} (S_0 e^{-z\sigma\sqrt{T} - \frac{1}{2}\sigma^2 T} - K e^{-rT}) e^{-\frac{z^2}{2}} dz \\ * &= \tfrac{1}{\sigma\sqrt{T}} (log\tfrac{S_0}{K} + (r - \tfrac{1}{2}\sigma^2)T) \\ e^{-z\sigma\sqrt{T} - \frac{1}{2}\sigma^2 T} e^{-\frac{z^2}{2}} &= e^{-\frac{1}{2}(\sigma^2 T + 2z\sigma\sqrt{T} + z^2)} = e^{-\frac{1}{2}(\sigma\sqrt{T} + z)^2}) \\ V_0 &= \tfrac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{*+\sigma\sqrt{T}} e^{-\frac{1}{2}z^2} dz - \tfrac{K e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{*} e^{-\frac{z^2}{2}} dz \\ &= S_0 \Phi(* + \sigma\sqrt{T}) - K e^{-rT} \Phi(*) \end{aligned}$$

The Black-Scholes formula for pricing European call options:

$$V_0 = V(S_0, T) = S_0 \Phi(\frac{log(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) - K e^{-rT} \Phi(\frac{log(\frac{S_0}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}). \quad (4.8)$$

Lets find some relations between the prices K and S_0. If the exercise price K is much bigger than the value of the stock price S_0, then the probability of exercising the option is very small (the value of Black-Scholes formula (4.8) is small). Option is out of money. Conversely, if K is much bigger than the value of S_0, the exercising of the option is much more probable and the option becomes a forward. The writer of the option will lose some money in the future so he has to price the option in order to get the expected future loss. The option price is $S_0 - Ke{-}rT$, what is the value of a forward for time T, struck at price K.

For time to maturity, there is also a dependence. As the time to maturity gets smaller, the changes of the option price are less and less volatile and they converge to the claim value $(S_0 - K)_+$ (taken at the current price). The probability of exercising the option are getting smaller also. The value of the option gets larger for longer times. If the time of the maturity is very long (lets say infinite), the discounting formula gets big due to the time t and the value of the option approaches S_0, because the current cost of K converges to zero.
And last important thing: the worth of the option gets bigger with bigger volatility of the stock.

4.6 Black-Scholes model for pricing the options

Black-Scholes model enables the analytical determination of pricing the options. We'll assume the knowledge of the parameters S_0, r, dt, X and σ and we would like to determine the price of call and put option, graphically illustrate the dependence of realization price X on the prices of the option and verify the validity of Put-Call Parity.
The price of **European call option** (C) from (4.8) can be simplified to the form

$$C = S_0 N(c_1) - e^{-rT} X N(c_2) \tag{4.9}$$

for $c_1 = \frac{log(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $c_2 = c_1 - \sigma\sqrt{T}$. $N(c_i)$ means the cumulative standardized normal distribution.
The price of **European put option** (P) is expressed, using the same c_1 and c_2 as follows:

$$P = e^{-rT} X N(-c_2) - S_0 N(c_1) \tag{4.10}$$

From the graphs (4.1) and (4.2) we see that the influence of realization price X on the price of call option is nondirectly proportional and on the price of put option it's directly proportional. Last thing is the adjustment of **Put-Call Parity**:

$$C + e^{-rT} X = P + S_0, \tag{4.11}$$

shown in more detail on the CD enclosed.
On internet, there can be found online calculators for pricing the options:

- Black–Scholes Option Calculator on tradingtoday.com
- Option Valuation and Calculators on DerivativeOne.com
- Black–Scholes Pricing Analysis - including dividends on hoadley.net

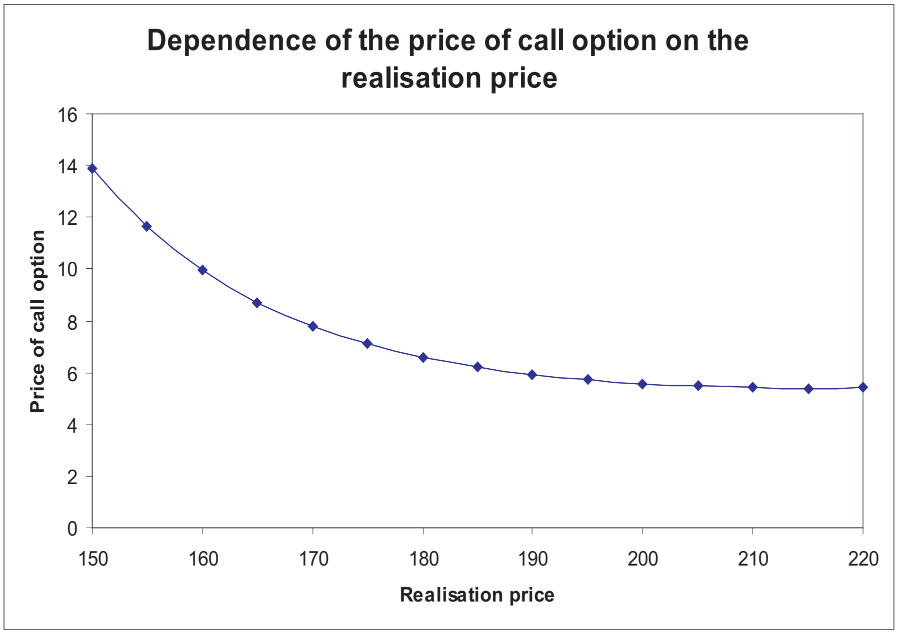

Figure 4.1: Price of the call option

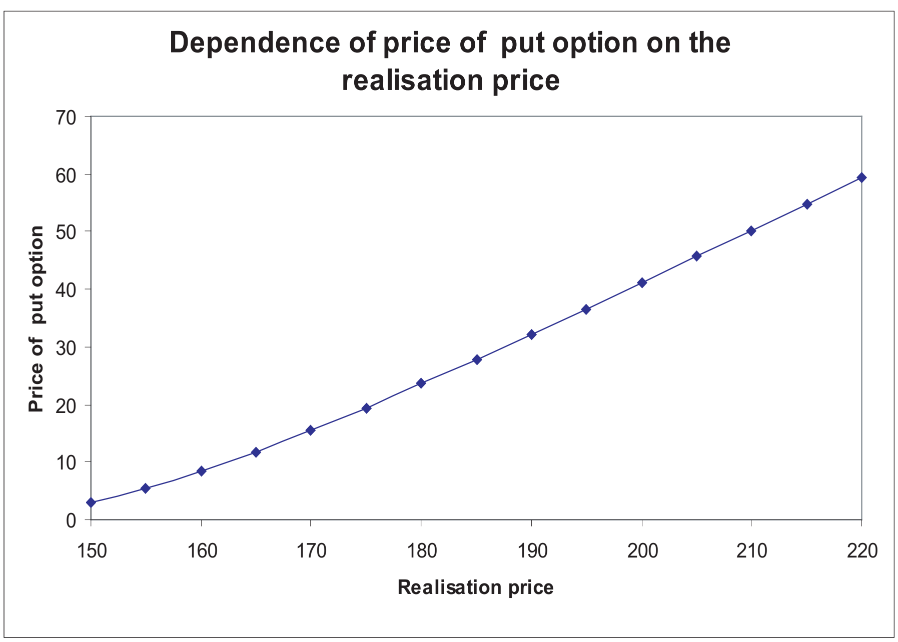

Figure 4.2: Price of the put option

Chapter 5

Interest rate models in continuous time

5.1 Interest rates

Interest rate market is an institution where price of money is set. The price of money depends on two important factors: length of the term and instantaneous fluctuations of the interest rate market. To describe the behavior of money, Brownian motion will be used. We'll present the features of interest rate modelling, following essentially the works [1], [2] and [11].
We'll introduce interest rate models which price and hedge bonds and bond options. They, together with interest rate swaps, exotic contracts and many others are the derivatives of interest rate market. This market is much wider comparing to stock market products.
The main aim is to calculate prices of contracts on the risk-free hedging basis.

Bond is a contract in which an issuer undertakes to make payment to an owner or beneficiary on a predetermined dates specified in the contract. Bond is a dept instrument in which a borrower (corporation, government) receives an advance of contracts or funds to make payments of principal sum and payments of interest in the future according to the contract. These securities bearing interest can make either lump sum payment (an amount of money given in a single payment) or regular interest payments. Bonds usually have long specified maturities (or none maturies in case they are consols). Some bonds allow their holders to convert bonds to shares of the issuer's common stock at conversion value that is specified. Black and Scholes discussed about the determining the values of such options.

Bond option is an option to buy or sell a bond at certain date in the future.

Interest rate contract is an agreement to pay an amount of money now for a promise of receiving a larger sum in the set time in the future. We require a knowledge of 2 basic facts in the contract:

- the length or maturity T
- the ratio of the size of the later payment to the initial payment, $P(0,T)$.

Interest rate depends both on

- the date t of loan emission
- the date T of maturity of the loan.

Discount bond is a bond that has no coupons. It doesn't provide any cash-flow and it's retired at maturity with a payment that is the same as its face value, or par value that is higher than the price of issuing the bond.
The prices of bond are different at different times to maturity. The price at time zero is $P(0,T)$ and at any other time $t \leq T$, **the price of bond** is $P(t,T)$.
$P(t,T)$ is a process in time. The worth of the bond at maturity T is $P(T,T) = 1$. Bonds with the same starting time $t = 0$ but different times to maturity are correlated, because the bonds move in similar ways. That's the reason why the exact start and end of the bond need to be determined - bonds prices are function of two time variables.

Graph of bond price against maturity is a graph for all the assets starting at time $t = 0$ with different maturities. Current worth of bond decreases with longer time to maturity.
Graph of bond price against time is a graph of one particular bond and it develops as a noisy stochastic process. The right-most point at time T shows value one at maturity.

For constant interest rate, the price of the bond (with maturity time T) at time $t < T$ would be

$$B_t = e^{-r(T-t)}. \tag{5.1}$$

For real interest rate, we get:

$$P(t,T) = e^{-r(T-t)} \Rightarrow r = \frac{-\log P(t,T)}{T-t} = R(t,T)$$

Yield $R(t,T)$ is the average interest rate offered by a bond. It denotes continuously compounded interest rate for a zero coupon bond sold at t to be repaid at T.

Term structure of interest rates concerns the relationship among the yields of default-free securities differing only with respect to their term to maturity. The relationship is shown as the shape of yield curve.

Yield curve is a graph of yield $R(t,T)$ against maturity T (for fixed time t). Due to uncertainty about the development of interest rates in a longer time period, the curve has increasing or decreasing character. More rare case of decreasing curve appears in a situation when the current rates are expected to fall. The curve is useful to declare about the average return of the bond.

Current cost of borrowing is denoted by **instantaneous rate** r which is the rate of interest paid on a very short term loan. It is a spot rate of interest at time t for a loan that will be repaid an instant later. To express the instantaneous rate, we'll consider a borrowing paid back almost instantly:

$$R(t, t+\Delta t) = \tfrac{-\log P(t,t+\Delta t)}{\Delta t}$$

For very small time instants, the value converges to $R(t,t)$ and the instantaneous rate r is a *process in time* given by

$$\begin{aligned} r_t &= \lim_{T\to t} R(t,T) \\ R(t,t) = r_t &= -\tfrac{\delta}{\delta T}\log P(t,t). \end{aligned}$$

Instantaneous rate is very important process in the interest rate market and many models are based on its behavior.

Problem is, the mapping of r_t from the discount price $P(t,T)$ and yield $R(t,T)$ is not one-to-one. We need to find an extension of r_t that fulfils both the assumptions: one-to-one mapping and instantaneousness.
We get it from the forward contract model by setting the future time of paying and time of receiving a payment in return very close to each other. The sought function of time and maturity is the **forward rate** of instantaneous borrowing at time T:

$$f(t,T) = -\frac{\delta}{\delta T}\log P(t,T). \tag{5.2}$$

Now, we are able to recover $P(t,T)$ and $R(t,T)$ and also the current instantaneous rate for $T = t$:

$$f(t,t) = r_t. \tag{5.3}$$

Do bond yield curve and forward rate curve have something in common? They have the same left-most point and there's a coherence:

$$\begin{aligned} R(t,T) &= -\tfrac{\log P(t,T)}{T-t} \\ (T-t)\tfrac{\delta}{\delta T}R(t,T) + R(t,T) &= -\tfrac{\delta}{\delta T}\log P(t,T) \\ (T-t)\tfrac{\delta}{\delta T}R(t,T) + R(t,T) &= f(t,T) \end{aligned}$$

For increasing yield curve, the forward rate curve is higher and for inverted yield curve, the forward rate curve is lower.
Forward rate as a function of time starts with initial value $f(0,T)$, evolves as a stochastic process and finishes at maturity T with value r_T.

An investor who has funds to invest until time T could also buy a zero-coupon bond with maturity T and the annualized return of $R(t,T)$. Alternatively, investor can roll over a set of shorter bonds or buy a longer bond and sell it at time T. These action should have the same final return no matter which one the investor chooses - this is the sign, that there's no arbitrage.

$$\begin{aligned} [P(t,T)]^{-1} &= \exp(\int_t^T r_s ds) \\ R(t,T) &= \frac{1}{T-t}\int_t^T r_s ds \end{aligned}$$

is a continuously compounded long rate which is an average of the instantaneous rates.

5.2 Market of default-free zero coupon discount bonds

We consider continuous trading economy.

- $[0,\tau]$ is a *trading interval* for fixed $\tau > 0$
- *Probability space* $(\Omega, \mathcal{F}, P)$ characters the uncertainty in economy:
  Ω is the state place, $\mathcal{F}$ is σ-algebra indicating measurable events and P is probability measure
- Information evolves over $[0,\tau]$ according to right continuous, augmented, complete filtration $\mathcal{F}_t$. *Filtration* $\mathcal{F}_t : t \in [0,\tau]$ is generated by $n \geq 1$ independent Brownian motions $\{W_1(t), W_2(t), ...W_n(t) : t \in [0,\tau]\}$ initialized at zero.
- $\mathbf{E}(.)$ is *expectation* with respect to probability measure $\mathbf{P}$

Discount bond is a bond which promises to make a single sum payment at a future date, but until that time it's worth less than its face value. If the bond is default free, there's no chance that the bond issuer won't meet his financial undertakings.

We have a set of default-free discount bonds. This bonds trade with different maturities T from trading interval $[0,\tau]$.

- $P(t,T)$ the *price* of the T-bond at time t that pays off 1$ at time T.
  We require:
  1. $P(T,T) = 1$ for all $T \in [0,\tau]$
     (which normalizes the bond's payoff to be 1$ at maturity).
  2. $P(t,T) > 0$ for all $T \in [0,T]$ and $t \in [0,T]$
     (which excludes arbitrage opportunity)
  3. $\frac{\delta \log P(t,T)}{\delta T}$ exists for all $T \in [0,T]$ and $t \in [0,T]$
     (which guarantees that forward rates are well-defined)
- $R(t,T)$ the *average yield* of the bond over its remaining lifetime
- $f(t,T)$ the *instantaneous forward rate* at time t for date $T > t$
  (the price now of instantaneous borrowing at time T)

- $r(t)$ the *spot rate*, the instantaneous forward rate at time t for date t, $r(t) = f(t,t)$ for all $t \in [0, \tau]$

The forward rates and the yield can be written in terms of the bond prices as:

$$f(t,T) = -\frac{\delta}{\delta T} \log P(t,T)$$

$$R(t,T) = -\frac{\log P(t,T)}{T-t} \tag{5.4}$$

Conversely, the bond prices can be given in terms of the forward rates or the yields by:

$$P(t,T) = e^{-\int_t^T f(t,u)du}, \qquad P(t,T) = e^{-(T-t)R(t,T)} \tag{5.5}$$

We get it by solving the differential equation of $f(t,T)$ defined in (5.2).

5.3 Simple model of interest rates

Denote σ a constant volatility. Drift α is a bounded deterministic function of time t and maturity T. The forward rate curve evolves as:

$$\mathbf{d_t f(t,T) = \sigma dW_t + \alpha(t,T)dt} \tag{5.6}$$

The forward curve is a function of time and it starts with an initial T-integrable forward rate curve $f(0,T)$ (which is given) and evolves as a stochastic process, finishing with value r_T at time T.
This is the SDE for the forward rate. The **forward rate** itself is

$$f(t,T) = f(0,T) + \sigma W_t + \int_0^t \alpha(s,T)ds \tag{5.7}$$

Forward rate is normally distributed. Given different maturities T and S,

$$f(t,T) = f(0,T) + \sigma W_t + \int_0^t \alpha(s,T)ds$$
$$f(t,S) = f(0,S) + \sigma W_t + \int_0^t \alpha(s,S)ds$$

the difference is purely deterministic:

$$f(t,T) - f(t,S) = [f(0,T) - f(0,S)] + \left(\int_0^t \alpha(s,T)ds - \int_0^t \alpha(s,S)ds\right)$$

Brownian motion W_t, a process over time, is the only random component.
The **spot rate** is determined from instantaneous interest rate $f(t,t)$ as:

$$r_t = f(t,t) = f(0,t) + \sigma W_t + \int_0^t \alpha(s,t)ds \tag{5.8}$$

The SDE for r_t is:

$$dr_t = d_t f(t,t) + \frac{\delta}{\delta T} f(t,t) \tag{5.9}$$

The spot rate is normally distributed and it's a random process.

Instantaneous rate is a random process. Considering a time interval $(t, t+\delta t)$, it's possible to borrow at rate r_t. Denote B_t a riskless asset

$$B_t = e^{\int_0^t r_s ds} \tag{5.10}$$

where $\int_0^T |r_t| dt < \infty$ and consider risky assets: zero-coupon bonds with maturity less or equal T. For each maturity S less than T, we define process $P(t, S)$ for $0 \leq t \leq S$ satisfying $P(S, S) = 1$. We can see the price of zero-coupon bond with maturity S is a function of time.

We'll try to find an arbitrage-complete model, where all claims (payments that will be made in the future according to a contract) can be hedged by underlying bonds. The replicating strategy, as a self-financing portfolio trading strategy, will hedge the claim (protect it against the risk of market movements).

We'll eliminate the presence of arbitrage opportunities by using a probability **Q** equivalent to **P** (3.1), under which the prices of discounted assets are martingales.

Analysis of discounted bond price process, using (5.5), (5.7), (5.8), (5.10):
cash bond

$$B_t = e^{\int_0^t r_s ds} = e^{\int_0^t f(0,u)du + \sigma \int_0^t W_s ds + \int_0^t \int_s^t \alpha(s,u) du ds},$$

price of T-maturity bond for fixed T

$$P(t,T) = e^{-\int_t^T f(t,u)du} = e^{-[\int_t^T f(0,u)du + \sigma(T-t)W_t + \int_0^t \int_t^T \alpha(s,u) du ds]},$$

discounted bond price

$$Z_t = B_t{}^{-1} P(t,T)$$

$$\begin{aligned} Z_t = e^{-(\int_0^t f(0,u)du + \int_t^T f(0,u)du + \sigma(T-t)W_t + \sigma \int_0^t W_s ds + \int_0^t \int_s^t \alpha(s,u) du ds + \int_0^t \int_t^T \alpha(s,u) du ds)} = \\ = e^{-(\int_0^T f(0,u)du + \sigma(T-t)W_t + \sigma \int_0^t W_s ds + \int_0^t \int_s^T \alpha(s,u) du ds)}. \end{aligned}$$

Step-by-step, we'll construct a **replicating strategy** for discounted T-bond, using the following:

Itô formula to get SDE for discounted bond price Z_t:

- Let $\{X_t; 0 \leq t \leq T\}$ be a stochastic Itô process

$$\begin{aligned} X_t &= X_0 + \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds, \\ dX_t &= \sigma_t dW_t + \mu_t dt \end{aligned}$$

  and f a deterministic twice continuously differentiable function. Then $Z_t := f(X_t)$ is also a stochastic process and it's given as:

$$\mathbf{dZ_t} = (\sigma_\mathbf{t}\mathbf{f'}(\mathbf{X_t}))\mathbf{dW_t} + (\mu_\mathbf{t}\mathbf{f'}(\mathbf{X_t}) + \tfrac{\mathbf{1}}{\mathbf{2}}\sigma_\mathbf{t}^\mathbf{2}\mathbf{f''}(\mathbf{X_t}))\mathbf{dt}.$$

Using this notation and inspired by (2.6) in Itô formula for continuous martingales, we get:

$$\begin{aligned} X_t &= \int_0^T f(0,u)du + \sigma(T-t)W_t + \sigma\int_0^t W_s ds + \int_0^t\int_s^T \alpha(s,u)duds, \\ dX_t &= \sigma(T-t)dW_t - \int_t^T \alpha(t,u)dudt + \tfrac{1}{2}\sigma^2(T-t)^2 dt, \\ Z_t &= f(X_t) = e^{-X_t} \end{aligned}$$

which implies

$$dZ_t = Z_t[-\sigma(T-t)dW_t - \int_t^T \alpha(t,u)dudt + \frac{1}{2}\sigma^2(T-t)^2 dt].$$

As we can see, both B_t and $P(t,T)$ are adapted to the same Brownian motion W_t.

Cameron-Martin-Girsanov theorem (3.4) will help us to turn a **P**-Brownian motion into a motion with no drift-term using measure **Q**

- *Let W_t be a $\mathbf{P}$-Brownian motion and $\{\theta_t\}$ is an $\mathcal{F}$-previsible adapted process satisfying the condition $\mathbf{E_P}(e^{\frac{1}{2}\int_0^T \theta_s^2 ds}) < \infty$, then there exists a measure $\mathbf{Q}$ such that*
  - *$\mathbf{Q}$ is equivalent to $\mathbf{P}$*
  - *$\frac{d\mathbf{P}}{d\mathbf{Q}} = e^{-\int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T \theta_s^2 ds}$ is a martingale*
  - *$\widetilde{W}_t = W_t + \int_0^t \theta_s ds$ is a standard $\mathbf{Q}$-Brownian motion*

  W_t is a drifting Brownian motion with drift $-\theta_t$ at time t.

By applying the CMG theorem (3.4) to

$$\theta_t = \tfrac{1}{\sigma(T-t)}[\int_t^T \alpha(t,u)du + \tfrac{1}{2}\sigma^2(T-t)^2] = \tfrac{1}{2}\sigma(T-t) + \tfrac{1}{\sigma(T-t)}\int_t^T \alpha(t,u)du,$$

where the technical conditions are satisfied, because θ_t is bounded up to time $S < T$, we'll find equivalent measure **Q** under which $\widetilde{W}_t = W_t + \int_0^t \theta_s ds$ is a **Q**-Brownian motion. Using θ_t, Z_t is *driftless* and its SDE is:

$$dZ_t = -\sigma Z_t(T-t)d\widetilde{W}_t$$

We have found the measure **Q** under which the T-bond discounted by the cash bond $Z_t = B_t^{-1}P(t,T)$ is a martingale.

Second step is to convert claim X in order to form the process from (4.6)

$$E_t = \mathbf{E_Q}(B_S^{-1}X|\mathcal{F}_t), \tag{5.11}$$

that is the conditional expectation of the discounted claim $B_S^{-1}X$. E_t is a **Q**-martingale. Third step of replicating strategy is to find a previsible process ϕ_t such that $dE_t = \phi_t dZ_t$. We already know that both Z_t and E_t are **Q**-martingales and we'll use it in Martingale representation theorem.

- **Martingale representation theorem**
  Inspired by (3.6) suppose Z_t is a $\mathbf{Q}$-martingale process with non-zero volatility $\sigma(T-t)$. Then if E_t is another $\mathbf{Q}$-martingale, there exists $\mathcal{F}$-previsible process ϕ, that $\mathbf{P}(\int_0^T \phi_t^2\sigma^2(T-t)^2 dt < \infty) = 1$ and

  $$E_t = E_0 + \int_0^t \phi_s dZ_s = \mathbf{E_Q}(B_S^{-1}X) + \int_0^t \phi_s dZ_s,$$

  where ϕ is unique (it's the ratio of volatilities of Z_t and E_t).

And finally, this is the **trading strategy** at time t:

- hold ϕ_t units of T-bond $P(t,T)$
- hold $\psi_t = E_t - \phi_t Z_t$ units of the cash bond B_t^0

What is the value of portfolio at time t without discounting?

$$V_t = \phi_t P(t,T) + \psi_t B_t = B_t E_t$$

If no extra money are required and just money from profits and losses are used, the portfolio is self-financing $\Leftrightarrow$ $dV_t = \phi_t d_t P(t,T) + \psi_t dB_t$, so our portfolio really is self-financing and the terminal value $V_S = X$ hedges the claim. Here, the arbitrage is present.
Option price formula for interest rates:

$$V_t = B_t \mathbf{E_Q}(B_S^{-1}X|\mathcal{F}_t) \tag{5.12}$$

The last problem in this model is that the change of measure was found for one particular bond. How does it look for bonds with different maturities?
For $S < T$, we want to price S-bond at time t: $P(t,S)$. We can do it both:

- from the SDE
- hedging the claim $X = P(S,S) = 1$ via cash bond and T-bond

To avoid the arbitrage, the results of both of them must be the same.

For discounted S-bond:

$$\begin{aligned}
Y_t &= B_t^{-1}P(t,S) \\
dY_t &= Y_t[-\sigma(S-t)dW_t - (\int_t^S \alpha(t,u)du)dt + \tfrac{1}{2}\sigma^2(S-t)^2 dt] \\
\theta_t^S &= \tfrac{1}{\sigma(S-t)}[\int_t^S \alpha(t,u)du + \tfrac{1}{2}\sigma^2(S-t)^2] \\
&= \tfrac{1}{2}\sigma(S-t) + \tfrac{1}{\sigma(S-t)}\int_t^S \alpha(t,u)du \\
&\Downarrow \\
dY_t &= -\sigma Y_t(S-t)(d\widetilde{W}_t + (\theta_t^S - \theta_t)dt)
\end{aligned}$$

We need Y_t to be a martingale, that's why the drift must be zero according to the Martingale representation theorem. This condition is satisfied in case when:

$$\theta_t^S = \theta_t$$

This is what we were looking for: the choice of T can't affect process θ_t.
(θ_t must be independent of T, $\frac{\delta\theta}{\delta T} = 0$).

Restriction on the drift
In an arbitrage-free market, the drift $\alpha(t,T)$ satisfies:

$$\begin{aligned}
\sigma(T-t)\theta_t &= \int_t^T \alpha(t,u)du + \tfrac{1}{2}\sigma^2(T-t)^2 \\
\tfrac{\delta\theta}{\delta T} : \theta_t\sigma &= -\sigma^2(T-t) + \alpha(t,T) \\
\Rightarrow \alpha(t,T) &= \sigma^2(T-t) + \sigma\theta_t
\end{aligned}$$

where $\sigma^2(T-t)$ is a function and $\sigma\theta_t$ is a process independent of T. This is the proof that we can restrict the forward-rate drifts to avoid the arbitrage.
SDE for $P(t,T)$ under measure $\mathbf{P}$ is:

$$\begin{aligned}
P(t,T) &= e^{-X_t} \\
X_t &= \sigma(T-t)W_t + \int_t^T f(0,u)du + \int_0^t \int_t^T \alpha(s,u)duds \\
dX_t &= \sigma(T-t)dW_t - (\sigma W_t + f(0,t) + \int_0^t \alpha(s,t)ds)dt + \\
&\quad + \int_t^T \alpha(s,u)dudt - \tfrac{1}{2}\sigma^2(T-t)^2dt = \\
&= \sigma(T-t)dW_t - r_tdt + (\tfrac{1}{2}\sigma^2(T-t)^2 + \sigma\theta_t(T-t) - \tfrac{1}{2}\sigma^2(T-t)^2)dt = \\
&= \sigma(T-t)dW_t - r_tdt + \sigma\theta_t(T-t)dt \\
d_tP(t,T) &= e^{-X_t}(-dX_t) = P(t,T)[-\sigma(T-t)dW_t + (r_t - \sigma(T-t)\theta_t)dt]
\end{aligned}$$

5.3.1 Market price of risk

Process is called tradable if its discounted price is a martingale under martingale measure $\mathbf{Q}$. Market price of risk defined in [6] is the rate of extra return per 1 unit of risk:

$$q = \frac{\mu - r}{\sigma} \tag{5.13}$$

where μ is a growth rate of the tradable, r is the growth rate of the riskless bond and σ is a measure of the risk of the asset. *All tradable securities should instantaneously have the same market price of risk.* If it wasn't so, the tradables would produce a non-martingale process which means arbitrage.
Market price of risk is the drift change of the Brownian motion which we get after using the CMG theorem (3.4). Measure $\mathbf{Q}$ is called risk-neutral measure.
To get back to our example, every security has to have the same market price of risk, that's why θ_t doesn't depend on the maturity T. Measure $\mathbf{Q}$ makes a martingale form *each* discounted bond. So it's not important, which bond we chose.

SDE for $f(t,T)$ in terms of $\mathbf{Q}$-Brownian motion $\widetilde{W}_t$ is:

$$\begin{aligned}
d_tf(t,T) &= \sigma dW_t + \alpha(t,T)dt = \\
&= \sigma dW_t + \sigma\theta_tdt + \sigma^2(T-t)dt \\
&= \sigma d\widetilde{W}_t + \sigma^2(T-t)dt
\end{aligned}$$

The drift $\alpha(t,T)$ vanished, but it must be recoverable by a change of measure θ_t independent on T.

This simple model with stochastic interest rates is arbitrage-complete. All claims can be hedged by the underlying bonds. Replication provides the price.

Bonds and rates in terms of the Q-Brownian motion $\widetilde{W}_t$, as expressed in [2]:

The bond prices:

$$\begin{aligned}
\mathbf{P(t,T)} &= \exp -[\sigma(T-t)W_t + \int_t^T f(0,u)du + \int_0^t \int_t^T \alpha(s,u)duds] = \\
&= \exp -[\sigma(T-t)W_t + \int_t^T f(0,u)du + \int_0^t [\tfrac{1}{2}\sigma^2(T^2-t^2) - \\
&\quad -\sigma^2 s(T-t) + \sigma\theta_s(T-t)]ds] = \\
&= \exp -[\sigma(T-t)(W_t + \int_0^t \theta_s ds) + \int_t^T f(0,u)du + \tfrac{1}{2}\sigma^2(T^2t - Tt^2)] = \\
&= \mathbf{exp} - [(\sigma(\mathbf{T}-\mathbf{t})\widetilde{\mathbf{W}}_\mathbf{t} + \int_\mathbf{t}^\mathbf{T} \mathbf{f(0,u)du} + \tfrac{\mathbf{1}}{\mathbf{2}}\sigma^\mathbf{2}\mathbf{T}(\mathbf{T}-\mathbf{t})\mathbf{t})]
\end{aligned}$$

Cash bond:

$$\begin{aligned}
\mathbf{B_t} &= \exp[\sigma \int_0^t W_s ds + \int_0^t f(0,u)du + \int_0^t \int_s^t \alpha(s,u)duds] = \\
&= \exp[\sigma \int_0^t W_s ds + \int_0^t f(0,u)du + \int_0^t (\tfrac{1}{2}\sigma^2(t-s)^2 + \sigma\theta_s(t-s))] = \\
&= \exp[\sigma \int_0^t (W_s + \int_0^t \theta_v dv)ds + \int_0^t f(0,u)du + \tfrac{1}{6}\sigma^2 t^3 \\
&= \mathbf{exp}(\sigma \int_\mathbf{0}^\mathbf{t} \widetilde{\mathbf{W}}_\mathbf{s}\mathbf{ds} + \int_\mathbf{0}^\mathbf{t} \mathbf{f(0,u)du} + \tfrac{\mathbf{1}}{\mathbf{6}}\sigma^\mathbf{2}\mathbf{t^3})
\end{aligned}$$

forward rate:

$$\begin{aligned}
\mathbf{f(t,T)} &= \sigma W_t + f(0,T) + \int_0^t \alpha(s,T)ds = \\
&= \sigma(W_t + \int_0^t \theta_s ds) + f(0,T) + \int_0^t \sigma^2(T-s)ds = \\
&= \sigma\widetilde{\mathbf{W}}_\mathbf{t} + \mathbf{f(0,T)} + \sigma^\mathbf{2}(\mathbf{T} - \tfrac{\mathbf{1}}{\mathbf{2}}\mathbf{t})\mathbf{t}
\end{aligned}$$

short rate:

$$\begin{aligned}
\mathbf{r_t} &= \sigma W_t + f(0,t) + \int_0^t \alpha(s,t)ds = \\
&= \sigma(W_t + \int_0^t \theta_s ds) + f(0,t) + \sigma^2 \int_0^t (t-s)ds = \\
&= \sigma\widetilde{\mathbf{W}}_\mathbf{t} + \mathbf{f(0,t)} + \tfrac{\mathbf{1}}{\mathbf{2}}\sigma^\mathbf{2}\mathbf{t^2}
\end{aligned}$$

Chapter 6

Short rate models

6.1 Single-factor HJM interest rate model

Interest rate model created by Heath, Jarrow, Morton and described in [9] and [15] is based on $f(t,T)$, the instantaneous forward rates (5.7).
Denote $\sigma(t,T)$ volatilities and $\alpha(t,T)$ drifts, both dependent on the history of Brownian motion W_t and on the rates up to time t. The forward rate curve evolves as:

$$\begin{aligned} d_t f(t,T) &= \sigma(t,T)dW_t + \alpha(t,T)dt \\ f(t,T) &= f(0,T) + \int_0^t \sigma(s,T)dW_s + \int_0^t \alpha(s,T)ds, \quad 0 \leq t \leq T \end{aligned} \tag{6.1}$$

The forward curve is a function of time and it starts with an initial T-integrable forward rate curve $f(0,T)$ (which is given) and evolves as a stochastic process, finishing with value r_T at maturity time T. For fixed T, every forward rate evolves according to its own volatility $\sigma(t,T)$ and drift $\alpha(t,T)$. The correlation of the incremental changes of forward rates ($\Rightarrow$ also yields and bond prices) is perfect ($= 1$). To ensure that the forward rates $f(t,T)$ are well defined by their SDE, these constraints need to be taken:

- for each T, process $\sigma(t,T)$ is previsible, depends only on history of Brownian motion up to time t and $\int_0^T \sigma^2(t,T)dt < \infty$
- for each T, process $\alpha(t,T)$ is previsible, depends only on history of Brownian motion up to time t and $\int_0^T |\alpha(t,T)|dt < \infty$
- initial forward curve $f(0,T)$ is deterministic and $\int_0^T |f(0,u)|du < \infty$

Fubini's theorem
Multiple integral consists of a number of integrals taken with respect to different variables: $\int ... \int f(x_1, ..., x_n)dx_1...dx_n$.
Repeated integral is an integral taken multiple times over single variable: $\int_0^x f(t)dt$.
Fubini's theorem (see [8] or Wikipedia) establishes a connection between multiple and repeated integral:

If $f(x,y)$ is continuous on $\Omega_1 \times \Omega_2$, $x \in \Omega_1$ and $y \in \Omega_2$ then

$$\int_{\Omega_1 \times \Omega_2} f(x,y)d(x,y) = \int_{\Omega_1} \left(\int_{\Omega_2} f(x,y)dy \right) dx \tag{6.2}$$

In case of HJM model, the constraints for $f(t,T)$ (from above) hold and the constraints for drift and volatility are:

- drift α has finite integral $\int_0^T \int_0^u |\alpha(t,u)| dtdu$
- volatility σ has finite expectation $\mathbf{E} \int_0^T [\int_0^u \sigma(t,u)dW_t]du$.

All the constraints that we need so far are determined, lets get back to the model.

Instantaneous rate $r_t = f(t,t)$:

$$r_t = f(0,t) + \int_0^t \sigma(s,t)dW_s + \int_0^t \alpha(s,t)ds \tag{6.3}$$

Cash product: bond or account starting with 1$ at $t = 0$ and reinvesting continually at rate r_t:

$$\begin{aligned} B_0 &= 0 \\ B_t &= e^{\int_0^t r_s ds} \\ dB_t &= r_t B_t dt \end{aligned} \tag{6.4}$$

Using the constraint of finite volatility expectation, we can overwrite integral $\int_0^t [\int_0^s \sigma(s,u)dW_s]du$ to $\int_0^t [\int_u^t \sigma(s,u)du]dW_s$ and we get:

$$B_t = \exp^{(\int_0^t [\int_s^t \sigma(s,u)du]dW_s + \int_0^t f(0,u)du + \int_0^t [\int_s^t \alpha(s,u)du]dW_s)}$$

Tradable assets: bonds $P(t,T)$ continuous in times t, T:

$$\begin{aligned} P(t,T) &= \exp -(\int_t^T f(t,u)du) = \\ &= \exp -(\int_0^t [\int_t^T \sigma(s,u)du]dW_s + \int_t^T f(0,u)du + \\ &\quad + \int_0^t [\int_t^T \alpha(s,u)du]dW_s) \\ P(0,T) &= e^{-\int_0^T f(0,u)du} \\ P(T,T) &= e^0 = 1 \end{aligned} \tag{6.5}$$

Discounted asset price for fixed maturity date T:

$$\begin{aligned} Z(t,T) &= B_t^{-1} P(t,T) = \\ &= \exp -(\int_0^t [\int_s^T \sigma(s,u)du]dW_s + \int_0^T f(0,u)du + \int_0^t [\int_s^T \alpha(s,u)du]dW_s) \end{aligned}$$

We use **Itô formula** (2.5) to get the SDE for $Z(t,T)$:

$$d_t Z(t,T) = -Z(t,T) \left((\int_t^T \alpha(t,u)du - \tfrac{1}{2} V_L^2(t,T))dt + \int_t^T \sigma(t,u)dudW_t \right),$$

the log-volatility of $P(t,T)$ is $V_L(t,T) = \Big(-\int_t^T \sigma(t,u)du\Big)$.

The next step is to make martingale form discounted bond. In order to change the measure from $\mathbf{P}$ to $\mathbf{Q}$, we need to find the change of measure drift to get rid off $\Big(-\int_t^T \alpha(t,u)du\Big)$:

$$\theta_t = \tfrac{1}{2}V_L(t,T) - \tfrac{1}{V_L(t,T)}\int_t^T \alpha(t,u)du$$

and we get $d_tZ(t,T)$ which is driftless under measure $\mathbf{Q}$:

$$d_tZ(t,T) = Z(t,T)V_L(t,T)(dW_t + \theta_t) = Z(t,T)V_L(t,T)d\widetilde{W_t}$$

The only technical condition for this mathematical operation is the condition for exponential martingales: $\mathbf{E_Q}[e^{\frac{1}{2}\int_0^T V_L^2(t,T)dt}] < \infty$.

The bond price $P(t,T)$ under martingale measure $\mathbf{Q}$:

$$\begin{aligned} P(t,T) &= \exp(-\textstyle\int_t^T f(t,u)du) \\ d_tP(t,T) &= P(t,T)\tfrac{\delta}{\delta t}(-\textstyle\int_t^T f(t,u)du) = \\ &= P(t,T)[f(t,t)dt - d_tf(t,T)] = \\ &= P(t,T)[r_tdt - d_tf(t,T)] \end{aligned}$$

Using the relations (6.1) and

$$V_L(t,T)\theta_t = \tfrac{1}{2}V_L^2(t,T) - \int_t^T \alpha(t,u)du,$$

we get the equation for $d_tP(t,T)$:

$$\begin{aligned} d_tP(t,T) &= P(t,T)r_tdt + P(t,T)V_L(t,T)dW_t + [\tfrac{1}{2}V_L^2(t,T) - \textstyle\int_t^T \alpha(t,u)du]dt = \\ &= P(t,T)[r_tdt + V_L(t,T)(dW_t + \theta_t)] = \\ &= P(t,T)[r_tdt + V_L(t,T)d\widetilde{W_t}] \end{aligned}$$

And this is the place where we find out, that HJM is a Black-Scholes model, so the price $P(t,T)$ under measure $\mathbf{Q}$ doesn't depend on the drift α, just on $V_L(t,T)$ - function of volatility $\sigma(t,T)$.

Replicating strategies for claims X that pay off at time $S < T$ will hedge the claims with discount bond maturing at T. Now we follow as in the simple model:

- define E_t which is a conditional $\mathbf{Q}$-expectation of discounted claim $B_S^{-1}X$, using (5.11): $E_t = \mathbf{E_Q}(B_S^{-1}X|\mathcal{F}_t)$ is the discounted claim process under measure $\mathbf{Q}$
- Matingale representation theorem

Due to martingale representation theorem (3.6), we need to ensure that the bond volatility $V_L(t,T)$ is never zero before maturity T. If this condition is satisfied, we apply the theorem on $Z(t,T)$ and E_t, which both are processes under the same measure $\mathbf{Q}$, for $\mathcal{F}$-previsible process ϕ:

$$E_t = E_0 + \int_0^T \phi_s dZ(s,T)$$

- Trading strategy:
  - hold ϕ_t units of T-bond at time t
  - hold $\psi_t = E_t - \phi_t Z(t,T)$ units of cash bond at time t

The value of portfolio:

$$\begin{aligned} V_t &= \phi_t P(t,T) + \psi_t B_t = \\ &= \phi_t P(t,T) + [E_t - \phi_t Z(t,T)]B_t = \\ &= \phi_t P(t,T) + [E_t - \phi_t P(t,T) B_t^{-1}]B_t = \\ &= E_t B_t = \\ &= B_t \mathbf{E_Q}(B_S^{-1} X | \mathcal{F}_t) \end{aligned}$$

The **portfolio is self-financing**, because it satisfies the condition (4.2):

$$dV_t = \phi_t d_t P(t,T) + \psi_t dB_t$$
or
$$dE_t = \phi_t d_t Z(t,T)$$

and $V_t = B_t \mathbf{E_Q}(B_T^{-1} X|\mathcal{F}_t) = \mathbf{E_Q}(\exp(-\int_t^T r_s ds)X|\mathcal{F}_t)$ is value of payoff X at time t.

Before, we've already mentioned the market price of risk. To avoid the arbitrage, the market price of risk has to be the same for all bonds with different maturities. Deriving from $\theta_t = \frac{1}{2}V_L(t,T) - \frac{1}{V_L(t,T)}\int_t^T \alpha(t,u)du$, we get the equation for the drift term

$$\int_t^T \alpha(t,u)du = \tfrac{1}{2}V_L^2(t,T) - V_L(t,T)\theta_t$$

and after differentiating it with respect to T we get

$$\alpha(t,T) = \sigma(t,T)[\theta_t - V_L(t,T)]$$

and using the risk-neutral measure we derive:
Forward rates

$$\begin{aligned} d_t f(t,T) &= \sigma(t,T)dW_t + \alpha(t,T)dt = \\ &= \sigma(t,T)dW_t + \sigma(t,T)[\theta_t - V_L(t,T)]dt = \\ &= \sigma(t,T)[dW_t + \theta_t dt] - \sigma(t,T)V_L(t,T)dt = \\ &= \sigma(t,T)d\widetilde{W}_t - \sigma(t,T)V_L(t,T)dt \end{aligned}$$

Short rates

$$\begin{aligned} r_t &= f(0,t) + \int_0^t \sigma(s,t)dW_s + \int_0^t \alpha(s,t)ds = \\ &= f(0,t) + \int_0^t \sigma(s,t)[dW_s + \theta_s ds] - \int_0^t \sigma(s,t)V_L(s,t)ds = \\ &= f(0,t) + \int_0^t \sigma(s,t)d\widetilde{W}_s - \int_0^t \sigma(s,t)V_L(s,t)ds. \end{aligned}$$

6.2 HJM and short rate models

Short rate models

HJM models are, using the proper transformations, a useful basis for all short rate models. Using the short rates models allows us to price derivatives which depend just on one underlying bond. This models are based on risk-neutral measure $\mathbf{Q}$ and short-rate process r_t, which is the rate of instantaneous borrowing for infinitesimal period. The aim of these models is to denote the value of $P(t,T)$, the time structure of rates r_t and coefficients $\sigma(t,T)$ and $\Sigma(t,T)$. The knowledge of these models has both theoretical and practical value, because thanks to these models, yield curves can be constructed (using the parametrical estimates) from reduced number of observations. Consider r_t is the Markov diffusion with volatility $v(r_t,t)$ and drift $u(r_t,t)$:

$$\mathbf{dr_t = u(r_t,t)dt + v(r_t,t)dW_t} \tag{6.6}$$

where $u(x,t)$ and $v(x,t)$ are deterministic functions of space and time, $u(x,t)$ measures the expected change in the interest rate per unit of time and $v(x,t)$ measures the standard deviation of changes in the interest rate per unit of time, dW_t is the increment to Brownian motion. $P(t,T)$ denotes the price of zero coupon bond at time t (maturing at T) with a face value of 1\$ where r_t is the short rate, so the value of bond is fully estimated by value r_t for any maturity time T. We suppose there are no arbitrage possibilities which would allow the investor to make a profit without any risk. For deterministic function $q(x,t,T)$ denote

$$\begin{aligned} q(r_t,t,T) &= \int_t^T f(t,u)du = -\log P(t,T) \\ q(x,t,T) &= -\log \mathbf{E_Q}(\exp(-\int_t^T r_s ds)|r_t = x) \end{aligned} \tag{6.7}$$

Then by applying Itô (2.5) to $f(t,T) = -\frac{\delta}{\delta T}\log P(t,T) = \frac{\delta q}{\delta T}(r_t,t,T)$ we get the equation for $d_t f(t,T)$:

$$d_t f(t,T) = \frac{\delta^2 q}{\delta x \delta T}(u(r_t,t)dt + v(r_t,t)dW_t) + \frac{\delta^2 q}{\delta t \delta T}dt + \frac{1}{2}\frac{\delta^3 q}{\delta t \delta T}v^2(r_t,t)dt$$

from which the volatility term must match $\sigma(t,T)$ (shown in the tabular below). The initial forward rate curve is

$$f(0,T) = \frac{\delta q}{\delta T}(r_0,0,T)$$

For short rate r_t under measure $\mathbf{Q}$, the initial curve and structure of volatility identify HJM model for this market.

Conversely, HJM models are short rate models also. We can identify if from

$$r_t = f(0,t) + \int_0^t \sigma(s,t)d\widetilde{W}_s - \int_0^t \sigma(s,t)\Sigma(s,t)ds$$

Equivalence of description of single factor HJM model and short rates model using the mathematical transformation:

	single factor HJM model	Short rates model
B_t	$e^{\int_0^t r_s ds}$	$e^{\int_0^t r_s ds}$
$P(t,T)$	$B_t \mathbf{E_Q}(B_T^{-1} \vert \mathcal{F}_t)$	$\mathbf{E_Q}(\exp(-\int_t^T r_s ds) \vert \mathcal{F}_t)$
V_t	$B_t \mathbf{E_Q}(B_T^{-1} X \vert \mathcal{F}_t)$	$\mathbf{E_Q}(\exp(-\int_t^T r_s ds) X \vert \mathcal{F}_t)$
$\sigma(t,T)$	$\sigma(t,T)$	$v(r_t,t)\frac{\delta^2 q}{\delta x \delta T}(r_t,t,T)$
$\Sigma(t,T)$	$\Sigma(t,T)$	$-v(r_t,t)\frac{\delta q}{\delta x}(r_t,t,T)$

In the following text, important examples of short rate models will be presented.

6.3 Ho and Lee model

In Ho and Lee model (see [2]), the short rate r_t is given by the SDE

$$\mathbf{dr_t} = \mathbf{u_t dt} + \sigma \mathbf{dW_t} \tag{6.8}$$

for u_t deterministic and bounded and constant σ.

First, we'll find the deterministic function $q(x,t,T)$ from (6.7) via Itô formula (2.5):

$$\mathbf{q}(\mathbf{x},\mathbf{t},\mathbf{T}) = x(T-t) - \tfrac{1}{6}\sigma^2(T-t)^3 + \int_t^T (T-s)u_s ds$$

The volatility surface $\sigma(t,T)$ from HJM model (6.1) has a simple prescription in this model, it depends neither on time nor on maturity:

$$\sigma(\mathbf{t},\mathbf{T}) = \sigma \tfrac{\delta^2 q}{\delta x \delta T}(r_t,t,T) = \sigma$$

(because $\frac{\delta q}{\delta x}(x,t,T) = T-t$ and $\frac{\delta^2 q}{\delta x \delta T}(x,t,T)$=1).
Short rate:

$$dr_t = u_t dt + \sigma dW_t$$
$$\Downarrow$$
$$r_t = f(0,t) + \int_0^t u_s ds + \sigma W_t$$

Forward rate:

$$f(t,T) = \tfrac{\delta q}{\delta T}(r_t,t,T) = r_t - \tfrac{1}{2}\sigma^2(T-t)^2 + \int_0^t u_s ds$$
$$\Downarrow$$
$$d_t f(t,T) = dr_t + \sigma^2(T-t) - u_t dt = \sigma dW_t + \sigma^2(T-t)$$

Initial forward rate:

$$f(0,T) = \tfrac{\delta q}{\delta T}(r_0,0,T) = \tfrac{\delta}{\delta T}(r_0 T - \tfrac{1}{6}\sigma^2 T^3 + \int_0^T (T-s)u_s ds) = r_0 - \tfrac{1}{2}\sigma^2 T^2 + \int_0^T u_s ds$$

Ho & Lee model is the single-factor model with constant volatility σ and drift u_t. The volatility of all forward rates is set by σ and drift u_t has an important meaning when we want to match to any initial forward curve (we use here the equation for $f(0,T)$).

	Ho&Lee model	generalization to a deterministic short-rate volatility
$\sigma(t,T)$	σ	σ_t
dr_t	$\sigma dW_t + u_t dt$	$\sigma_t dW_t + u_t dt$
$d_t f(t,T)$	$\sigma dW_t + \sigma^2(T-t)dt$	$\sigma_t dW_t + \sigma_t^2(T-t)dt$
$f(0,T)$	$r_0 - \frac{1}{2}\sigma^2 T^2 + \int_0^T u_s ds$	$r_0 - \int_0^T \sigma_s^2(T-s)ds + \int_0^T u_s ds$

The problem of Ho & Lee model is that the forward rates and short rate can be negative and can go to infinity for longer terms under measure **Q** or any equivalent measure **P**. The restriction of Ho & Lee model lies in constant volatility.
The cash bond price and bond price are log-normally distributed ⇒ the Black-Scholes formula (4.8) holds. It's possible for short rates to have many different drifts under the measure **P**.

6.4 Vasicek model

Ornstein-Uhlenbeck process

OUP is given as the only solution of the equation:

$$\begin{aligned} U_0 &= u \\ dU_t &= -\alpha U_t dt + \sigma dW_t \end{aligned} \tag{6.9}$$

Let V_t be a function of U_t and exponential function $e^{\alpha t}$:

$$V_t = U_t e^{\alpha t} \tag{6.10}$$

Then, using the integration by parts formula (2.7) we get

$$\begin{aligned} dV_t &= dU_t e^{\alpha t} + U_t d(e^{\alpha t}) = \\ &= -\alpha e^{\alpha t} U_t dt + \sigma e^{\alpha t} dW_t + \alpha e^{\alpha t} U_t dt = \\ &= \sigma e^{\alpha t} dW_t \end{aligned}$$

$$V_t = U_t e^{\alpha t}$$
$$\Downarrow$$
$$\mathbf{U_t} = V_t e^{-\alpha t} = u e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s$$

The mean of U_t:

$$\begin{aligned} \mathbf{E}(\mathbf{U_t}) &= E(ue^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s) = \\ &= ue^{-\alpha t} + \sigma e^{-\alpha t} E(\int_0^t e^{\alpha s} dW_s) = \\ &= ue^{-\alpha t}, \end{aligned}$$

because the martingale $\int_0^t e^{\alpha s} dW_s$ is null at $t = 0$ and the condition $E(\int_0^t (e^{\alpha s})^2 ds) < \infty$ is satisfied.

The volatility of U_t:

$$\begin{aligned}
\mathbf{var(U_t)} &= E[(U_t - E(U_t))^2] = \\
&= E(ue^{-\alpha t} + \sigma e^{-\alpha t} \textstyle\int_0^t e^{\alpha s} dW_s - ue^{-\alpha t})^2 = \\
&= E(\sigma e^{-\alpha t} \textstyle\int_0^t e^{\alpha s} dW_s)^2 = \\
&= \sigma^2 e^{-2\alpha t} E(\textstyle\int_0^t e^{2\alpha s} dW_s)) = \\
&= \sigma^2 e^{-2\alpha t} \tfrac{1}{2c}(e^{2\alpha t} - 1) = \\
&= \tfrac{\sigma^2}{2c}(1 - e^{-2\alpha t}).
\end{aligned}$$

U_t is a normal variable and process $U_t, t \geq 0$ is Gaussian, because:
there exist the real numbers $a_1, a_2, ..., a_n$ that for $0 \leq t_1 < t_2 < ... < t_n$, the random variable $a_1 U_{t_1} + a_2 U_{t_2} + ... + a_n U_{t_n}$ is normal. To show it on this example, we have

$$\begin{aligned}
\mathbf{U_{t_i}} &= ue^{-\alpha t_i} + \sigma e^{-\alpha t_i} \textstyle\int_0^\infty I_{s \leq t_i} e^{\alpha s} dW_s = \\
&= u_i + \textstyle\int 0^t f_i(s) dW_s
\end{aligned}$$

and

$$a_1 U_{t_1} + a_2 U_{t_2} + ... + a_n U_{t_n} = \textstyle\sum_{i=1}^n a_i u_i + \int_0^t (\sum_{i=1}^n a_i f_i(s)) dW_s,$$

is a normal random variable, because we have here a stochastic integral of a deterministic function of time.

The Vasicek model

Vasicek model is the second simplest short rate model. It's similar to Ho & Lee model (6.8) with one diversity: the short rate's drift depends on its current value.

$$\mathbf{dr_t} = \sigma \mathbf{dW_t} + \alpha(\beta - \mathbf{r_t})\mathbf{dt} \tag{6.11}$$

for constant α, υ and σ. Drift υ is called restoring drift, it pushes the process up when $r_t < \upsilon$ and pushes it down when $r_t > \upsilon$. Vasicek model is often interchanged with Hull-White model. The difference between this models is in the fact, which parameters in the model are time-dependent. The most commonly accepted hierarchy of the models is:

- the **Vasicek model** .... α is constant
- the **Hull-White model** .... α is time-dependent
- the **extended Vasicek model** .... α and β are both time-dependent

These models are defined in works [2], [6] and [10]. We'll focus the attention to Vasicek model. The model is based on Ornstein-Uhlenbeck process *OUP* (6.9) which is used to explain the form of the drift-term. To show this, denote

$$U_t = r_t - \beta$$

which is a solution of differential equation

$$dU_t = -\alpha U_t dt + \sigma dW_t.$$

This equation is nothing else that the equation for OUP. We can now write the equation for r_t:

$$r_t = r_0 e^{-\alpha t} + \beta(1 - e^{-\alpha t}) + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s,$$

and we can see that r_t follows a normal law with mean $E(r_t) = r_0 e^{-\alpha t} + \beta(1 - e^{-\alpha t})$ and variance $var(r_t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$.

What is the idea of Vasicek model?
We apply the Itô formula (2.5) on $P(r_t, t, T)$ which is a function of diffusion process r_t and we get P in form of diffusion process:

$$dP = \mu P dt + \nu P dW_t,$$

where, as in the one-factor model showed before,

$$\begin{aligned} \mu(r_t, t, T)P(r_t, t, T) &= \frac{\delta P}{\delta t} + \alpha(\beta - r_t)\frac{\delta P}{\delta r} + \frac{1}{2}\sigma^2 \frac{\delta^2 P}{\delta r^2} \\ \nu(r_t, t, T)P(r_t, t, T) &= \sigma \frac{\delta P}{\delta r} \end{aligned}$$

Denote $q(r_t, t)$ from (6.7) the market price of risk defined in (5.13). Then,

$$q(r_t, t) = \frac{\mu(r_t, t, T) - r_t}{\nu(r_t, t, T)}$$

where ν is the rate of risk of the given bond and q is the price of one unit of risk. Rewritten equation says that the expected instantaneous profit μ with a value bigger than r_t compensates the risk (which price is) $q\nu$:

$$\mu - r_t = q\nu$$

We would like to get the price $P(t, T)$, which is the result of the equation $\mu - r_t = q\nu$, using the terms for μ and ν.

$$\begin{aligned} \frac{1}{P}\left[\frac{\delta P}{\delta t} + \alpha(\beta - r_t)\frac{\delta P}{\delta r} + \frac{1}{2}\sigma^2 \frac{\delta^2 P}{\delta r^2}\right] - r_t &= q\frac{1}{P}\sigma\frac{\delta P}{\delta r} \\ r_t P &= \frac{\delta P}{\delta t} + \alpha(\beta - r_t)\frac{\delta P}{\delta r} + \frac{1}{2}\sigma^2 \frac{\delta^2 P}{\delta r^2} - \sigma q \frac{\delta P}{\delta r} \\ r_t P &= \frac{\delta P}{\delta t} + [\alpha(\beta - r_t) - \sigma q]\frac{\delta P}{\delta r} + \frac{1}{2}\sigma^2 \frac{\delta^2 P}{\delta r^2} \end{aligned}$$

This, together with $P(r_t, T, T) = 1$ is the partial differential equation for the price P. Vasicek described two possible ways of solving it, depending on the value of market price of risk q defined in (5.13).

1. $\mathbf{q = 0}$
   Using the equation $\mu - r_t = q\nu$, we get $\mu = r_t$ which means that in any time instant, the mean of bond yield μ equals to the instantaneous, risk-free measure r_t. In this case, we get the bond's value just by discounting its 1$ nominal value in the time interval (t,T), which is the same than to count it according to the equation using the mean in time interval $< t, T >$ for all r_t:

$$P(r_t,t,T)=\mathbf{E_Q}[e^{-\int_t^T r_s ds}|\mathcal{F}_t]$$

2. $\mathbf{q}=-\theta$ (q is constant)

We can rewrite the equation for dr_t:

$$\begin{aligned} dr_t &= \sigma dW_t+\alpha(\beta-r_t)dt \\ dr_t &= \sigma d\widetilde{W}_t+\theta\sigma dt+\alpha(\beta-r_t)dt \\ dr_t &= \sigma d\widetilde{W}_t+\alpha(\widetilde{\beta}-r_t)dt \end{aligned}$$

where $\widetilde{W}_t=W_t+\theta t$ and $\widetilde{\beta}=\beta-\frac{\theta\sigma}{\alpha}$. According to this equation, we'll calculate the zero-coupon bond price under probability $\mathbf{Q}$:

$$\begin{aligned} P(r_t,t,T) &= \mathbf{E_Q}(e^{-\int_t^T r_s ds}|\mathcal{F}_t)= \\ &= e^{-\widetilde{\beta}(T-t)}\mathbf{E_Q}(e^{-\int_t^T \widetilde{U}_s ds}|\mathcal{F}_t) \end{aligned}$$

where $\widetilde{U}_t=r_t-\widetilde{\beta}$. $\widetilde{U}_t$ is OUP (the solution of diffusion equation):

$$\begin{aligned} U_0 &= u \\ dU_t &= -\alpha U_t dt+\sigma d\widetilde{W}_t \end{aligned}$$

Thanks to this, we express

$$\mathbf{E_Q}(e^{-\int_t^T \widetilde{U}_s ds}|\mathcal{F}_t)=F(T-t,r_t-\widetilde{\beta})$$

as a function of two arguments, which we want to calculate. For U_t^* as the only solution of the differential equation above, the Laplace transformation enables us to write

$$\mathbf{E_Q}(e^{-\int_t^T U_s^* ds})=\exp[-\mathbf{E_Q}(\int_0^{T-t}U_s^* ds)+\tfrac{1}{2}var(\int_0^{T-t}U_s^* ds)]$$

and if we apply the expression for mean and variance of OUP, we get:

$$\mathbf{E_Q}(\int_0^{T-t}U_s^* ds)=\tfrac{u}{\alpha}(1-e^{-\alpha(T-t)})$$

and

$$var(\int_0^{T-t}U_s^* ds)=\tfrac{\sigma^2(T-t)}{\alpha^2}-\tfrac{\sigma^2}{\alpha^3}(1-e^{-\alpha(T-t)})-\tfrac{\sigma^2}{2alpha^3}(1-e^{-\alpha(T-t)})^2$$

Finally we get

$$\mathbf{P(r_t,t,T)}=$$

$$=\exp[\tfrac{1}{\alpha}(1-e^{-\alpha(T-t)})(\beta-\tfrac{\theta\sigma}{\alpha}-\tfrac{\sigma^2}{2\alpha^2}-r_t)-(\beta-\tfrac{\theta\sigma}{\alpha}-\tfrac{\sigma^2}{2\alpha^2})(T-t)-\tfrac{\sigma^2}{4\alpha^3}(1-e^{-\alpha(T-t)})^2]$$

with parameters μ and ν:

$$\mu(\mathbf{r_t},\mathbf{t},\mathbf{T})=r_t+\tfrac{\theta\sigma}{\alpha}(1-e^{-\alpha(T-t)})$$

and

$$\nu(\mathbf{r_t},\mathbf{t},\mathbf{T})=\tfrac{\sigma}{\alpha}(1-e^{-\alpha(T-t)}).$$

We can see the equation $\mu - r_t = q\nu$ really holds in this case and ν increases with increasing maturity T. The yield curve evolves according to:

$$\begin{aligned} R(t,0) &= r_t \\ \mathbf{R(t,T)} &= \beta - \tfrac{\theta\sigma}{\alpha} - \tfrac{\sigma^2}{2\alpha^2} - \tfrac{1}{\alpha T}(1-e^{-\alpha T})(\beta - \tfrac{\theta\sigma}{\alpha} - \tfrac{\sigma^2}{2\alpha^2} - r_t) + \tfrac{\sigma^2}{4\alpha^3 T}(1-e^{-\alpha T})^2, \end{aligned}$$

increases in case $r_t \leq \beta - \frac{\theta\sigma}{\alpha} - \frac{3\sigma^2}{4\alpha^2}$ and decreases for $r_t \geq \beta - \frac{\theta\sigma}{\alpha}$

The last, we should analyze what happens if $q = 0$ ($\theta = 0$)?

$$\begin{aligned} \mathbf{P(r_t,t,T)} &= \exp[\tfrac{1}{\alpha}(1-e^{-\alpha(T-t)})(\beta - \tfrac{\sigma^2}{2\alpha^2} - r_t) - \\ &\quad -(\beta - \tfrac{\sigma^2}{2\alpha^2})(T-t) - \tfrac{\sigma^2}{4\alpha^3}(1-e^{-\alpha(T-t)})^2] = \\ &= \exp[-r_t\tfrac{1}{\alpha}(1-e^{-\alpha(T-t)})]\exp[(\beta - \tfrac{\sigma^2}{2\alpha^2})\tfrac{1}{\alpha}(1-e^{-\alpha(T-t)}) - \\ &\quad -(\beta - \tfrac{\sigma^2}{2\alpha^2})(T-t) - \tfrac{\sigma^2}{4\alpha^3}(1-e^{-\alpha(T-t)})^2] \end{aligned}$$

and, to make it simpler:

$$\begin{aligned} V(t,T) &= \tfrac{1-e^{-\alpha(T-t)}}{\alpha} \\ U(t,T) &= \exp[(\beta - \tfrac{\sigma^2}{2\alpha^2})(V(t,T) - (T-t)) - \tfrac{1}{4\alpha}\sigma^2 V(t,T)^2] \end{aligned}$$

and the price P for the null market price of risk is:

$$\mathbf{P(r_t,t,T)} = U(t,T)\exp[-r_t V(t,T)]$$

Last, to define the proper HJM model for Vasicek model, we use Itô for $q(x,t,T)$ and we get the equations:

$$\begin{aligned} \sigma(t,T) &= \sigma e^{-\alpha(T-t)} \\ f(0,T) &= r_0 e^{-\alpha T} + \beta(1-e^{-\alpha T}) - \tfrac{\sigma^2}{2\alpha^2}(1-e^{-\alpha T})^2 \end{aligned}$$

Vasicek model introduces dependence of volatility surface on maturity. Volatility is derived from short rate's drift and short rate's volatility under measure $\mathbf{Q}$. Vasicek model is mean-reverting and constant α allows much more drifts (under measure $\mathbf{P}$) than Ho&Lee model.
Bond prices $P(r_t,t,T)$ are log-normally distributed. Problems of this model are, that instantaneous rate and also the forward rates can be negative.
Vasicek model includes just drift of the short rate. What interests us, is the volatility of the entire curve.

6.5 Cox-Ingersoll-Ross model

Cox-Ingersoll-Ross model deals with the idea to ensure that the process never goes negative. The stochastic process for the interest rate is:

$$\mathbf{dr_t} = \sigma\sqrt{\mathbf{r_t}}\mathbf{dW_t} + (\beta - \alpha\mathbf{r_t})\mathbf{dt} \tag{6.12}$$

for σ and β non-negative and $\alpha \in \mathbf{R}$, $q_t = -b\sqrt{r_t}$ for $b \in \mathbf{R}$. The square root function is defined only on $\mathbf{R}^+$ and is not Liptschitz.

For α_t, β_t and σ_t, deterministic functions of time, and model

$$\mathbf{dr_t} = \sigma_\mathbf{t}\sqrt{\mathbf{r_t}}\mathbf{dW_t} + (\beta_\mathbf{t} - \alpha_\mathbf{t}\mathbf{r_t})\mathbf{dt},$$

the process is called autoregressive. The interest rate is attracted toward its mean value $\frac{\beta_t}{\alpha_t}$ by the drift term and influenced also by a noise term. The variance of noise term is proportional to the prevailing level of interest rate. This ensures that the interest rate never goes negative, because the volatility term gets smaller when r_t comes closer to zero and that's why the importance of β rises and doesn't allow r_t to become negative. The process stays positive as long as $\beta_t \geq \frac{1}{2}\sigma^2$.

We'll focus on the simpler model with non-negative constants α, β, σ, inscribed in [6], [8], [7], [14] and [11]. For a standard Brownian motion $\{\widetilde{W}_t, 0 \leq t \leq T\}$ under probability $\mathbf{Q}$ and $q_t = -b\sqrt{r_t}$ we derive:

$$\begin{aligned} dr_t &= \sigma\sqrt{r_t}dW_t + (\beta - \alpha r_t)dt \\ dr_t &= \sigma\sqrt{r_t}d\widetilde{W}_t + [\beta - (\alpha + \sigma b)r_t]dt \end{aligned}$$

The procedure of zero-coupon bond price at time $t = 0$ under probability $\mathbf{Q}$ can be found in [11] and gives the result

$$\begin{aligned} P(r_t, 0, T) &= \mathbf{E_Q}\left(e^{-\int_0^T r_s ds}|\mathcal{F}_t\right) = \\ &= e^{-\beta n(T) - r_0 m(T)} \end{aligned}$$

where functions n and m are given for $\alpha^* = \alpha + \sigma b$ and $\gamma^* = \sqrt{(\alpha^*)^2 + 2\sigma^2}$ by this equations:

$$\begin{aligned} \mathbf{m(t)} &= \frac{2(e^{\gamma^* t} - 1)}{\gamma^* - \alpha^* + e^{\gamma^* t}(\gamma^* + \alpha^*)} \\ \mathbf{n(t)} &= \frac{-2}{\sigma^2}\log\left(\frac{2\gamma^* e^{\frac{1}{2}t(\gamma^* + \alpha^*)}}{\gamma^* - \alpha^* + e^{\gamma^* t}(\gamma^* + \alpha^*)}\right) \end{aligned}$$

And the price $P(r_t, t, T)$ at time t is:

$$\mathbf{P(r_t, t, T)} = \exp[-\beta n(T - t) - r_t m(T - t)]$$

Zero coupon yield has a limit of

$$R_\infty = \frac{2b\beta}{\beta + \gamma + \sqrt{(b + \gamma)^2 + 2\sigma^2}}$$

regardless of the current short rate. If $r_t < R_\infty$ the yield curve is sloping upwards, if $r_t > \beta$ it's downward sloping and between this two, the yield curve has one single hump.

How can we express **Cox-Ingersoll-Ross model in HJM terms**?

Define $B(t, T)$ as a solution of differential equation $P(r_t, t, T) = A(t, T)[B(t, T)r_t]$. After differentiating this function, we get the ODS for B

$$\begin{aligned} \frac{\delta \mathbf{B}}{\delta \mathbf{t}} &= -1 + \alpha B(t, T) + \frac{1}{2}\sigma^2 B^2(t, T) \\ B(T, T) &= 0 \end{aligned}$$

(Basic of Riccati DE which is used in CIR model is the decomposition of $B(t,T)$: $B(t,T) = L_1 + L_2\frac{g'(T-t)}{g(T-t)}$ for specified function g.)

Then we can write the function $q(r_t, t, T) = (-\log P(t,T)|r_t = x)$, this way:

$$\mathbf{q}(\mathbf{x}, \mathbf{t}, \mathbf{T}) = xB(t,T) + \int_t^T \beta B(s,T)ds$$

The volatility structure of CIR in HJM terms is as follows:

$$\begin{aligned} P(t,T) &= e^{-q(r_t,t,T)} \\ \sigma(t,T) &= \sigma\sqrt{r_t}\frac{\delta B}{\delta T}(t,T) \\ \Sigma(t,T) &= -\sigma\sqrt{r_t}B(t,T) \\ f(0,T) &= r_0\frac{\delta B}{\delta T}(0,T) + \int_0^T \beta\frac{\delta B}{deltaT}(s,T)ds. \end{aligned}$$

6.6 Black-Derman-Toy model

This model, defined in [17], is practically identical to Ho&Lee model, the only difference is that we don't use single r but it's logarithm $\log r$:

$$d\log r = u_t dt + \sigma dW_t$$

and using the Itô formula we get the transformation

$$dr = [u_t + \tfrac{1}{2}\sigma^2]rdt + \sigma r dW_t.$$

6.7 Black-Karasinski model

Black-Karasinski model, see [2] or [17], keeps the short rate positive by using the exponential. Process U_t is considered to be the Ornstein-Uhlenbeck process from Vasicek model.

$$\mathbf{dU_t} = \sigma_\mathbf{t}\mathbf{dW_t} + \alpha_\mathbf{t}(\upsilon_\mathbf{t} - \mathbf{U_t})\mathbf{dt}$$

for σ_t, α_t and υ_t deterministic functions of time and

$$r_t = e^{U_t}$$

and, the rate drifts towards the mean υ_t and is always positive and logarithm of the rate drifts toward the mean υ_t. U_t is normally distributed what ensures that r_t is log-normal.

6.8 Comparison of short rate models

The comparison of short rate models is the most schematic on the graphs, where the trend of interest rates for particular models can be seen. Again, we get inspired by [17] and show the development of 4 most used models: Ho&Lee model (6.8), Vasicek model (6.11), slightly reconditioned Cox-Ingersol-Ross model (6.12) and Hull-White model, described in the section about Vasicek model.
The simulation of the random development of short interest rate in Excel is based on the generation of pseudo-random numbers from normal distribution $N(0,1)$. Then the rates are calculated according to the short rate models formulas and the results are graphically represented. The short rates for particular models are counted according to the same Brownian motion. Graph by graph, we can see the limitations of particular models.

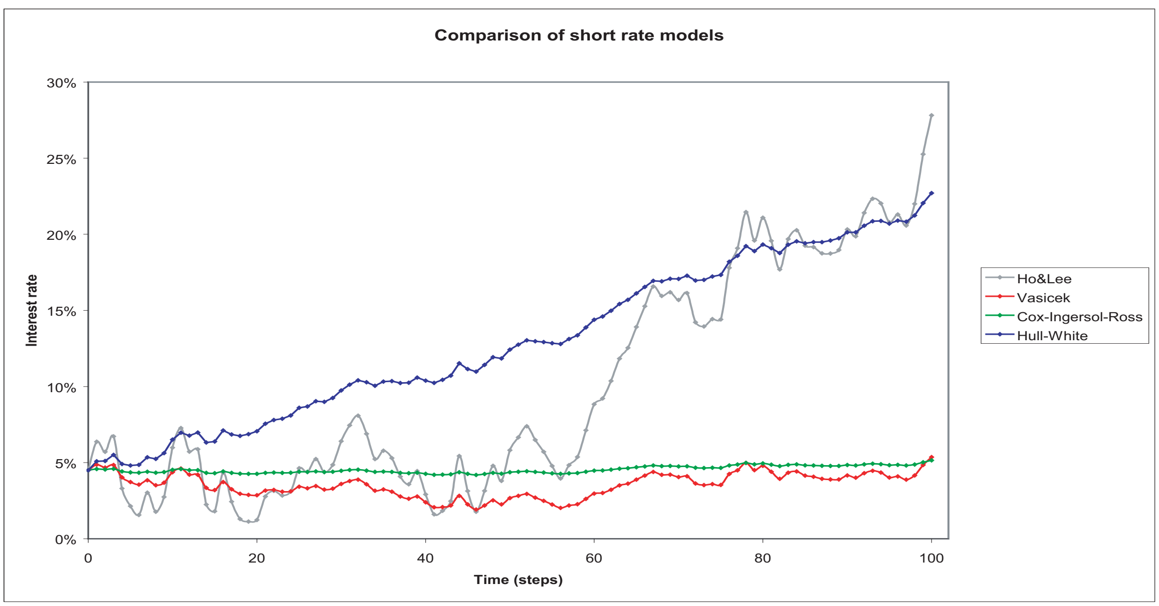

Figure 6.1: Short rate models: Ho&Lee, Vasicek, Cox-Ingersoll-Ross, Hull-White

Due to the graphs (6.2) and (6.3), the best decision is to take into consideration just 3 short rate models without Ho&Lee model which seems too unrealistic for multiple generated pseudo-random numbers. Of course, the models can be reclaimed by using different parameters. Graphs (6.4), (6.5) and (6.6) demonstrate 3 short rate models without Ho&Lee model.

Figure 6.2: Short rate models: negative Ho&Lee, Vasicek, Cox-Ingersoll-Ross, Hull-White

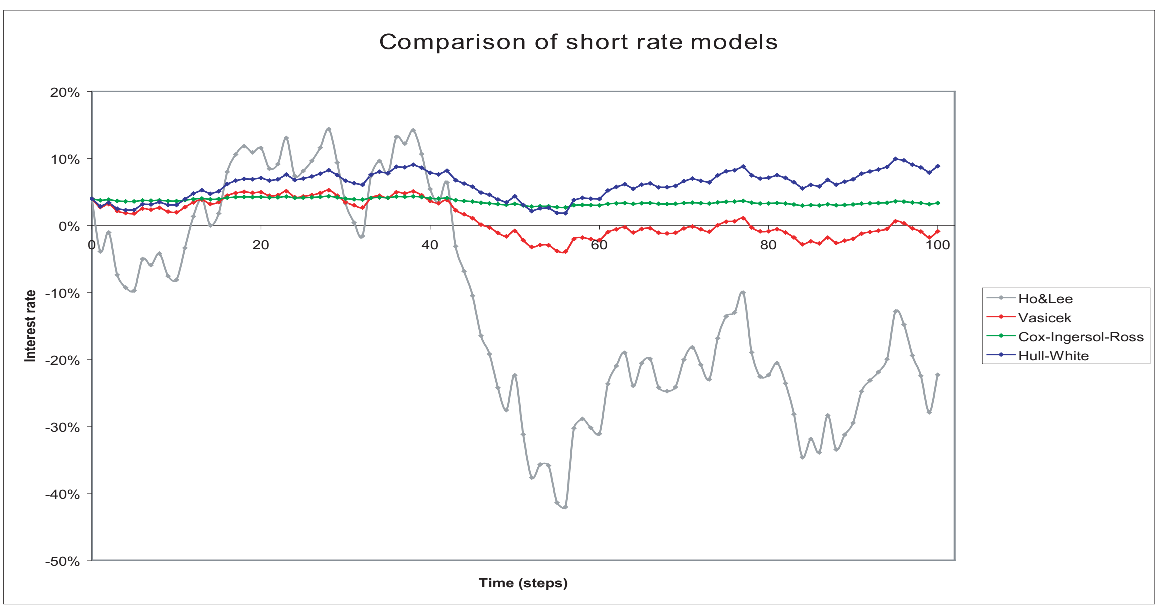

Figure 6.3: Short rate models: negative Ho&Lee, negative Vasicek, Cox-Ingersoll-Ross, Hull-White

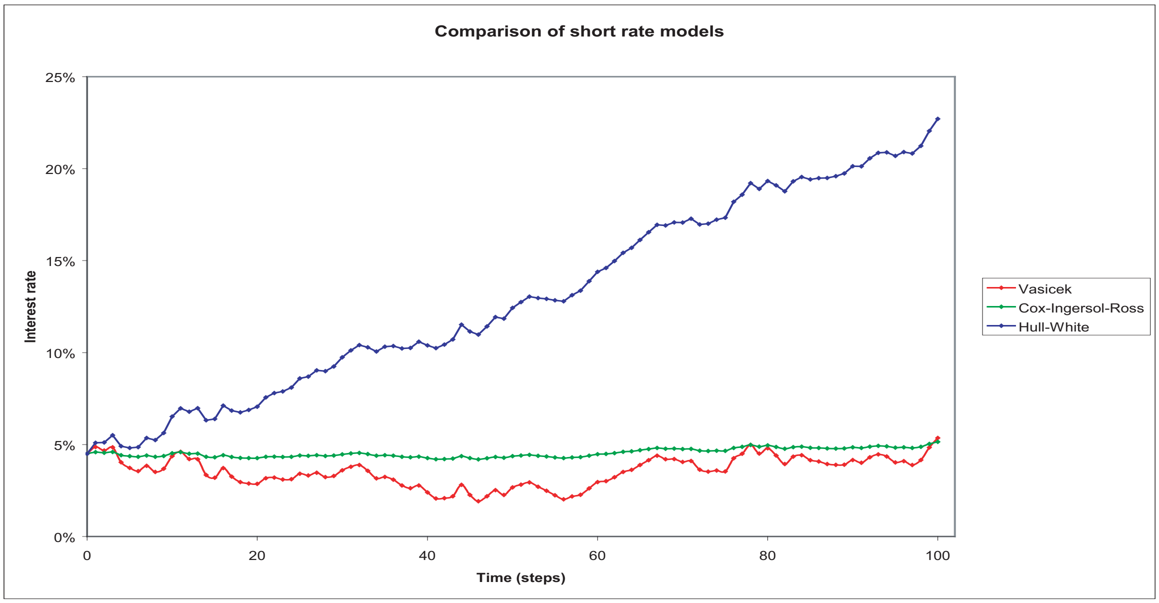

Figure 6.4: Short rate models: Vasicek, Cox-Ingersoll-Ross, Hull-White

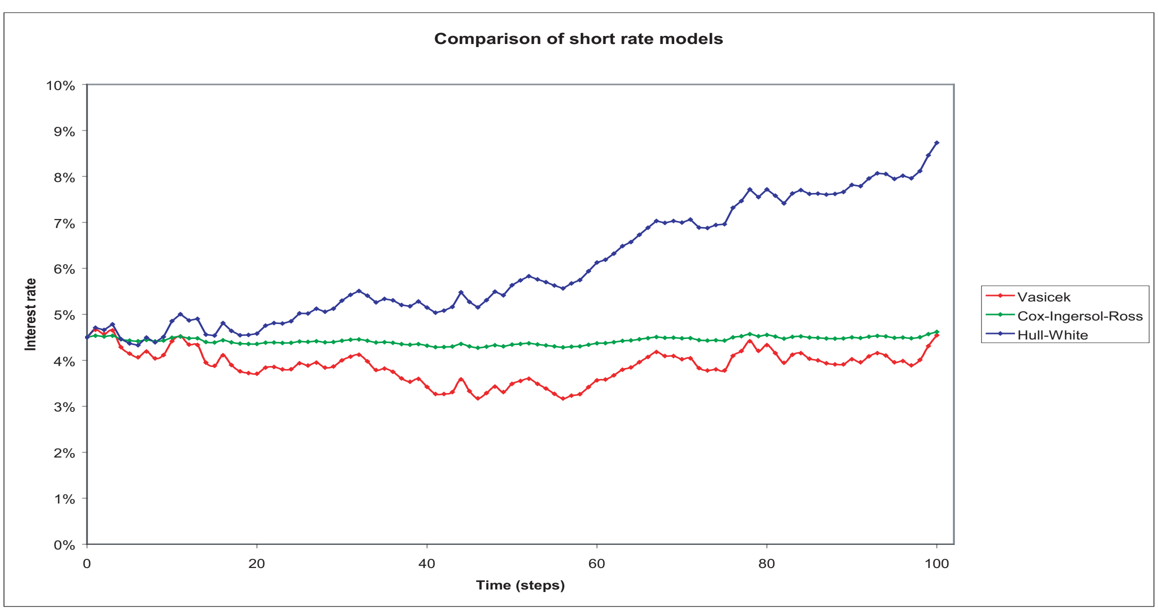

Figure 6.5: Short rate models: Vasicek, Cox-Ingersoll-Ross, Hull-White

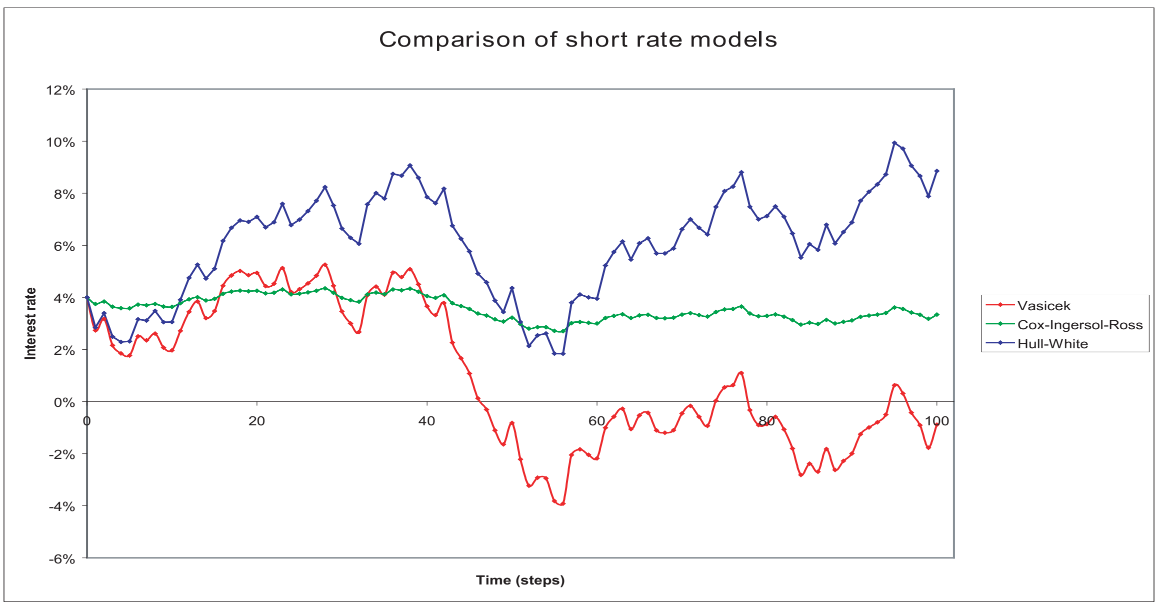

Figure 6.6: Short rate models: negative Vasicek, Cox-Ingersoll-Ross, Hull-White

Chapter 7

User's manual

CD with program for Short interest rate models is enclosed to this work. It's a simple program created in Excel 2003 or Excel 97 which is able to count the evolution of interest rates using the chosen parameters. Four models are presented: Ho and Lee model, Vasicek model, Hull-White model and Cox-Ingersol-Ross model. The program works as follows: parameters r_0, α, β, σ, θ and Δt have to be inserted (or user can work with parameters given in the table). In the first green cell (C11), the user puts in the number of steps. The time interval Δt should be $\frac{1}{C11}$ for each model. The second green cell (C12) is the possibility to chose which from 3 pages should be shown first after the program is activated. Page 1 contains the calculations, 3 short rate interest models without Ho and Lee model are plotted on the second page and all 4 short interest models are plotted on page number 3.

The program itself is activated by pressing ALT+F8.

The task is to present the simulation of random evolution of short interest rate according to 4 short interest rate models and verify the models. Program firstly generates the random variables[1] in column M. These variables have normal distribution $N(0, 1)$. Using random variables and parameters chosen, the interest rates for each model are computed and plotted in the graphs. The graph of 3 instead of 4 models is plotted on the second page due to unrealistic evolution of Ho&Lee model for most of the parameters.

The conclusion after running the program for several times certifies the mean reverting character of the models and shows also the limitations of single models, for example the negative values of interest rates in Ho&Lee and Vasicek models.

[1]The generator of random variables isn't up to standard on the professional quality, but the results can be considered to be correct and trustworthy. The values of random variables can be generated also by using the procedure of inverse transformation explained in [17].

Chapter 8

Interest rate instruments

The largest market in the world is generated by interest rate derivatives, where the underlying asset represents the right to receive or pay defined sum of money at interest rate that is given. The market expanded in latest years, mainly the over-the-counter markets which contain hundreds of traded claims. Stocks, bonds and mutual funds are the most popular assets and they also get most of the market's attention. But the investors are also interested in more sophisticated and more complex investment opportunities including options, futures, and currency.

According to classification shown and explained in [6], the interest rate derivatives can be basically subdivided into two groups according to the position of the contract participants:

- **unconditional derivatives**
  The participants are obligatory to accomplish the contract at the date of maturity, no matter what the price of the basic document is. The entrance to a contract doesn't require any payment in advance, the buyer of basic document is in the long position, the seller is in the short possition.
  The main types of unconditional derivatives:
  - Forwards
  - Futures
  - Swaps
- **conditional derivatives**
  One of the participants (for an extra payment) has the right but not obligation to accomplish the contract at the date of maturity. The other participant is in the passive position. The active participant who bought an option is in the long position, the passive one (the seller) is in the short possition.
  The main types of conditional derivatives:
  - options
  - warrants

  - caps and floors
  - exotic options
  - swaptions, captions, etc.

Most of the interest rate derivatives are based on the interest rate called LIBOR. It's the London Interbank Offered Rate, used daily among the banks which offer to lend the unsecured funds at the London money market. LIBOR rates are widely used as reference rates for:

- forward rate agreements
- interest rate swaps
- floating rate notes
- number of currencies, etc.

For Euro, the rates are determined by the European Banking Federation and are called EURIBOR.

To share some facts about LIBOR, it is fixed by the British Bankers Association (BBA) at about 11:00 each day (London time). LIBOR is a significant reference rate for US dollar, Pound Sterling, the Swiss Franc, the Yen, the Canadian dollar and the Danish Krone.

Description of the most common types of interest rate derivatives follows.

8.1 Forward contract

Forward contract is an agreement (at time t) between two parties to buy or sell an asset at a pre-determined future date T_2. The trade date T_1 and delivery date T_2 are separated. Lets say, the contract is set up for receiving one dollar at delivery date T_2.

In the contract, one party agrees to buy, the other to sell, for a forward price k agreed in advance. In a forward transaction, no cash changes hands. Spot price is the price at which the asset changes hands. The difference between the spot and the forward price is the forward premium or forward discount.

A standardized forward contract that is traded on an exchange is called a futures contract. If there is no allowance for credit risks, then the forward price will equal the futures price.

For an interest rate r_t which changes in time, the price of a cash bond B_t at time $t < T$, from (5.10) is:

$$B_t = \int_0^t r_s ds.$$

Then the value of a claim at time t under martingale measure $\mathbf{Q}$ (using the pricing formula (4.7)) is:

$$V_t = B_t \mathbf{E}_{\mathbf{Q}}(B_{T_2}^{-1}|\mathcal{F}_t) - B_t \mathbf{E}_{\mathbf{Q}}(k B_{T_1}^{-1}|\mathcal{F}_t) = P(t, T_2) - kP(t, T_1),$$

because $B_t \mathbf{E}_{\mathbf{Q}}(B_T^{-1}|\mathcal{F}_t) = P(t, T)$. If we want to set the worth of this contract now to be zero, then the current payment k is worth the ratio of the price of T_2-bond and T_1-bond. The forward yield from T_1 to T_2, if we select the time period between T_1 and T_2 to be very short, approximates the instantaneous forward rate (5.2) for borrowing:

$$f(t, T) = -\frac{\delta}{\delta T} \log P(t, T).$$

The hedging strategy in case of forward contract is static and the results don't depend on which particular interest model we chose.

8.1.1 Forward rate agreement

FRA is a forward contract in which one party pays a fixed interest rate (what is the rate at which the contract is agreed), and receives a floating interest rate (usually LIBOR or Euribor). The net differential between this rates is calculated and payed on the termination date. The floating interest rate is fixed one or two days before the termination date. FRAs are over-the counter derivatives. A swap is a combination of FRAs. The payer of the fixed interest rate is also known as the borrower or the buyer, whilst the receiver of the fixed interest rate is the lender or the seller.

8.2 Bond with coupons

In finance, coupons are "attached" to bonds, either physically (used for older bonds) or electronically. Each coupon represents a predetermined payment promised to the holder of the bond in return for his loan of money to the issuer of the bond. The bond-holder is typically not the original lender, but receives this payment for effectively lending the money. The coupon rate (the amount promised per dollar of the face value of the bond) helps determine the interest rate or yield on the bond.

The notation that will be used in this and many following cases is as follows:

- $PD = \{T_0, T_1, ...T_n\}$ the coupon payments dates, T_n is the last coupon payment
- Δ is the length of the time interval between two coupon payments, we consider the payments are regular, so Δ is constant
- $T_i = T_0 + i\Delta$, $i = 1, ..., n$
- r_i is the interest rate in the market applicable from time T_i to time T_{i+1}, r_c is a constant rate

- the face value of the bond is 1$
- $P(T_i, T_{i+1})$ is the price of bond at time T_{i+1} expected at time T_i

Consider a bond that pays n regular coupon payments $r_c\Delta$ (r_c is the constant rate and Δ is the time interval same for all the coupon payments) at predetermined dates and pays off 1$ at maturity time T_n. In these payments the holder receives the same amount of money as if he owned one T_n-bond and $r_c\Delta$ units of the bonds maturing at times $T_i = T_0 + i\Delta$, $i = 1, ..., n$. The price of initial payment is:

$$P(T_0, T_n) + r_c\Delta \sum_{i=1}^{n} P(T_0, T_i)$$

and we determine the coupon rate in case of starting with bond's face value by:

$$\begin{aligned} 1 &= P(T_0, T_n) + r_c\Delta \sum_{i=1}^{n} P(T_0, T_i) \\ &\Downarrow \\ r_c &= \frac{1 - P(T_0, T_n)}{\Delta \sum_{i=1}^{n} P(T_0, T_i)}. \end{aligned}$$

8.3 Floating rate bond

The bonds with variable coupon rates are called Floating Rate Bonds. This means the coupons of bonds aren't fixed, but dependent on the current interest rates. The standard method of bond-valuation is to discount all cash flows of the bond at a rate that is based on the yield available on a comparable instrument in the market. For example, we'll consider an interest paid in the time interval $[V, T]$ which equals the yield of the T-bond that we bought at time V. The valuation of a floating rate bond is defined as finding the expected bond price at time T_0.

The bond coupon payments in this case aren't of the same amount in all the time intervals, but change according to the rates in the market. If we consider LIBOR - rate set at T_{i-1} which determines the amount of payment at time T_i then:

$$L(T_{i-1}) = \frac{1}{\Delta}\left(\frac{1}{P(T_i, T_{i-1})} - 1\right), \tag{8.1}$$

and the payment at time T_i is:

$$\Delta L(T_{i-1}) = \left(\frac{1}{P(T_i, T_{i-1})} - 1\right).$$

We get the same payment as we would obtain as an interest from T_i-bond (with 1$ face value) bought at time T_{i-1}.
The value of the variable coupon bond is

$$\begin{aligned} 1 = V_0 &= P(T_0, T_n) + \sum_{i=1}^{n} B_{T_0} \mathbf{E}_{\mathbf{Q}}(B_{T_{i-1}}^{-1} - B_{T_i}^{-1} | \mathcal{F}_{T_0}) = \\ &= P(T_0, T_n) + \sum_{i=1}^{n} (P(T_0, T_{i-1} - P(T_0, T_i)). \end{aligned}$$

Variable coupon bond has the same cash-flow as if we bought T_1-bonds for 1$, took their interest at time T_1 (and considered it to be a coupon), bought T_2-bond for 1$,...and continued like this until we own 1$ at time T_n.

8.4 Swap

In the market of derivatives, interest rate swap is one of the popular forms of swaps. It's based on the exchange of streams of interest and used mostly by companies, which re-allocate their exposure to interest-rate fluctuations, typically by exchanging fixed-rate obligations for floating rate obligations. Trading an interest-rate swap is one of the most common forms of over-the-counter derivatives. Swap is a widely traded derivative, but it doesn't have the properties to "change hands" so easily, which makes it not possible to be traded through a futures exchange.

Swap is a contract according to which side A receives a series of regular payments of fixed amounts in exchange for paying payments that depend on the dominant interest rates (and vice versa for side B). In praxis, not all the amounts are payed, just the net differences between them.

The swap present value is computed as the present value of its components. The actual payment rates of the swap are known just in the future, but they can be approximated by the yield curve of zero-coupon bonds with different maturities. Each payment of variable rate is determined from the forward rate for considered payment date. The use of these forward rates leads to series of cash flows, from which each cash flow is discounted at the date of payment by zero-coupon rate. The information is contained also in the yield curve data that are available in the market. The main reason why zero-coupon rates are used is that these are the rates of the bonds that pay just one cash flow. Calculating according to the procedure that was described, we get the present value of the swap.

In the time of entering the contract, no side has any advantage and swap contract doesn't require any payment in advance.

We'll show this on forward swap agreement. We are the side that receives the fixed payments at rate r_c at times $T_i = T_0 + i\Delta$, $i = 1, ..., n$. Then the value of the swap at T_0 is:

$$X = P(T_0, T_n) + r_c\Delta \sum_{i=1}^{n} P(T_0, T_i) - 1,$$

because it is like a portfolio that is short a variable coupon bond that costs 1$ and long a fixed coupon bond that is worth $P(T_0, T_n) + r_c\Delta \sum_{i=1}^{n} P(T_0, T_i)$. If we consider $t < T_0$, then the present value of X (from (4.4)) is:

$$\begin{aligned} V_t &= B_t \mathbf{E}_{\mathbf{Q}}(B_{T_0}^{-1} X | \mathcal{F}_{T_0}) = \\ &= P(t, T_n) + r_c\Delta \sum_{i=1}^{n} P(t, T_i) - P(t, T_0) \end{aligned}$$

and we get the forward swap rate r_c by giving the forward swap a null initial value:

$$\begin{aligned} r_c &= \frac{P(t,T_0) - P(t,T_n)}{\Delta \sum_{i=1}^{n} P(t,T_i)} = \\ &= \frac{1 - F_t(T_0,T_n)}{\Delta \sum_{i=1}^{n} F(t,T_i)}, \end{aligned}$$

where $F_t(T_0, T_i) = \frac{P(t,T_i)}{P(t,T_0)}$ is the forward price for which we buy a T_i-bond at time T_0.

8.5 Options on bonds

Bond options give the right to buy a bond at the future date for a price that is given. Under the martingale measure $\mathbf{Q}$, the current worth at time t of an option on a T-bond, struck at rate r_c at time t is:

$$\mathbf{E}_{\mathbf{Q}}(B_t^{-1}(P(t,T)-r)^+)$$

Ho & **Lee model**:

- forward rates and instantaneous short rate normally distributed
- T-bond and discount bond log-normally distributed
- $F=\frac{P(0,T)}{P(0,t)}$ current forward price for $P(t,T)$
- $(\sigma(T-t))^2t$ is the log-variance of $P(t,T)$
- The option price is:

$$V_0 = P(0,t)(F\Phi(\frac{\log\frac{F}{r_c}+\frac{1}{2}(\sigma(T-t))^2t}{\sigma(T-t)\sqrt{t}}) - r_c\Phi(\frac{\log\frac{F}{r_c}-\frac{1}{2}(\sigma(T-t))^2t}{\sigma(T-t)\sqrt{t}}))$$

Vasicek model:

- log-normal bond prices
- $F=\frac{P(0,T)}{P(0,t)}$ current forward price for $P(t,T)$
- the variance depends on σ_t and ϕ_t, the deterministic processes

8.6 Caps and floors

For an investor who is borrowing at a floating rate it may be important to provide against the ineligible increase to interest rates. There exists a derivative called **interest rate cap** in which the investor receives money if the interest rate exceeds the strike price at the end of each period. Interest rate cap is considered to be a series of *caplets* (which is an individual payment at each T_i) or series of European call options. The payoff of one capelet struck at r is

$$\begin{aligned}\Delta(L(T_{i-1})-r)^+ &= P(T_{i-1},T_i)^{-1}[1-(1+r\Delta)P(T_{i-1},T_i)]^+ = \\ &= (1+r\Delta)P(T_{i-1},T_i)^{-1}[(1+r\Delta)^{-1}-P(T_{i-1},T_i)]^+ \quad = \quad X\end{aligned}$$

where $L(T_{i-1})$ is the LIBOR rate as mentioned earlier in (8.1). The price of interest rate cap is just the addition of all caplet values. The value of caplet at time t is:

$$(1+r\Delta)B_t\mathbf{E}_{\mathbf{Q}}(B_{T_{i-1}}^{-1}[(1+r\Delta)^{-1}-P(T_{i-1},T_i)]^+|\mathcal{F}_t).$$

Second possible is the valuation of caplets via Black model. The assumptions of this model are: the rate is log-normally distributed with volatility σ. The mapping between volatility and present value of the option is 1-to-1. In the market, the price of the caplet is determined by its volatility, which is called *Black vol*.

In the inverse situation the investor promises (for a premium) never to pay less than is the rate r. this is **Interest rate floor** on a specified rate (LIBOR). It means he pays at each time T_i:

$$\Delta(r - L(T_{i-1}))^+,$$

this individual payment is called *floorlet*. An interest rate floor can be considered as a series of European put options or floorlets.

Caps and floors can be valued using the short rate models. The interest rate models for valuing the bond puts are used to value an interest rate cap on a LIBOR from t to T (which is equivalent to a multiple of a t-maturity put on a T-maturity bond). A floor is equivalent to a certain bond call. The **Hull-White model** has this degree of tractability.

8.7 Swaptions

Swaption is another financial instrument which guarantees the owner of an option to enter some interest rate swap. Usually, the swaps exchange fixed rate flows for variable rate flows but there are also cases, when two (mostly irregular) fixed rate cash flows are exchanged. The main use of swaptions is when the swap is done to reduce risk. One side of the contract wants to do the contract just in the case if some specific market condition is reached. There is a liquid swaption market on the LIBOR rates of all the world's major currencies. On the market, there are 3 main swaptions according to the time of exercising the option:

- American Swaption - the owner can enter the swap in any time of an agreed time interval
- Bermudan Swaption - the owner can enter the swap on a sequence of dates
- European Swaption - the owner can enter the swap on one specified day

The worth of the option at time T_0 is the same as a call option, struck at 1, on a T_n-coupon bond:

$$(P(T_0, T_n) + r_c \Delta \textstyle\sum_{i=1}^{n} P(T_0, T_i) - 1).$$

Chapter 9

Multi-factor models

9.1 Multi-factor HJM interest-rate model

We have introduced the simple HJM model (6.1). The idea of multi-factor HJM model from [2] differs just in actual fact that the model doesn't contain just one, but a whole set of Brownian motions $W_1(t), ..., W_n(t)$. These motions enable us the pricing dependent on two points on the yield curve. In single factor model, the correlation of two increments in the bond prices was always perfect. But to approach the real word, we want to find the models where the bonds have different correlations with other different bonds.
Now, we'll continue just in the same way as in single-factor HJM to make it easy to find out the differencies.

Denote $\sigma_i(t,T)$ the volatilities for each $W_i(t)$ and $\alpha(t,T)$ drifts, both dependent on the history of Brownian motions $W_i(t)$ and on the rates up to time t.
The *forward rate curve* evolves as:

$$f(t,T) = f(0,T) + \sum_{i=1}^{n} \int_0^t \sigma_i(u,T)dW_i(u) + \int_0^t \alpha(u,T)du, \quad 0 \leq t \leq T \tag{9.1}$$

We have n independent Brownian motions which determine the stochastic fluctuation of the entire forward rate curve. The forward rate curve is a function of time, it starts with an initial T-integrable forward rate curve $f(0,T)$ (which is given). The change of forward rate (for each particular maturity) to each Brownian motion is driven by volatility coefficients (which differ). The volatility coefficients $\sigma_i(t,T)$ can depend on the entire past of the Brownian motions. What concerns the economical restrictions, the forward rate processes have continuous sample paths and across the entire forward rate curve, these processes depend only on a finite number of random shocks.

The *total square volatility* of forward rate $f(t,T)$ defined in (9.1) is

$$\textstyle\sum_{i=1}^{n} \sigma_i^2(t,T)$$

and the *covariance* of two forward rates with different maturities T and V is

$\sum_{i=1}^{n} \sigma_i(t,T)\sigma_i(t,V)$.

To ensure that the forward rates $f(t,T)$ are well defined by their SDE, these constraints need to be taken on volatility and drift:

- for each T process $\sigma_i(t,T)$ is previsible, depends only on the history of Brownian motion up to time t and $\int_0^T \sigma_i^2(t,T) < \infty$
- for each T, process $\alpha(t,T)$ is previsible, depends only on the history of Brownian motion up to time t and $\int_0^T |\alpha(t,T)| < \infty$
- initial forward curve $f(0,T)$ is deterministic and $\int_0^T |f(0,u)|du < \infty$
- for each $1 \leq i \leq n$, the expectation of each volatility σ_i is $E\int_0^T [\int_0^s \sigma_i(t,s)dW_i(t)]ds < \infty$

The dynamics of the spot rate process are determined by *instantaneous rate* $r_t = f(t,t)$:

$$r_t = f(0,t) + \sum_{i=1}^{n} \int_0^t \sigma_i(u,t)dW_i(u) + \int_0^t \alpha(u,t)du, \quad \forall t \in [0,T] \tag{9.2}$$

The spot rate and forward rate processes are similar, the only difference is that, for short rate process, time and maturity arguments vary simultaneously.

Cash product: bond or money market account initialized at $t = 0$ with an investment of 1\$, reinvesting continually at rate r_t:

$$\begin{aligned} B_0 &= 0, \\ B_t &= e^{\int_0^t r_u du}, \quad 0 < B_t < \infty \\ dB_t &= r_t B_t dt \end{aligned}$$

We would like to **find measure Q equivalent to P,** which makes the discounted bond prices into martingales as in (3.1). This needs a sort of theoretical preparation, because we need to ensure that all the technical requirements are satisfied. We'll start with the dynamics of bond price process.

For $t \in [0,T]$, $T \in [0,\tau]$, $i = 1,...,n$

$$\int_0^t [\int_u^t \sigma_i(u,y)dy]^2 du < \infty,$$
$$\int_0^t [\int_t^T \sigma_i(u,y)dy]^2 du < \infty$$
and
$$t \to \int_t^T [\int_0^t \sigma_i(u,y)dW_i(u)]dy \text{ continuous,}$$

the dynamics of the bond price process are

$$\begin{aligned} \mathbf{lnP(t,T)} &= -\int_t^T f(t,u)du = \\ &= \log P(0,T) + \int_0^t [r_u + A(u,T)]du - \\ &\quad -\tfrac{1}{2}\sum_{i=1}^{n}\int_0^t \Sigma_i^2(u,T)du + \sum_{i=1}^{n}\int_0^t \Sigma(u,T)dW_i(u) \end{aligned}$$

where, to simplify the equation:

$$\begin{aligned} A(t,T) &= -\int_t^T \alpha(t,u)du + \frac{1}{2}\sum_{i=1}^n \Sigma_i^2(u,T)du \\ \Sigma(t,T) &= -\int_t^T \sigma_i(t,u)du \end{aligned}$$

We apply the Itô formula (2.5) on $P(t,T)$ and the strong solution to this differential equation is following:

$$d_tP(t,T) = P(t,T)[r_u + A(t,T)]dt + P(t,T)\sum_{i=1}^n \Sigma_i(t,T)dW_i(t) \tag{9.3}$$

Here we can see, that the drift term $r_u + A(t,T)$ and the volatility coefficients $\Sigma_i(t,T)$ can depend on the history of Brownian motions. This is why the bond price process is not Markov.

Transformation

$$Z(t,T) = B_t^{-1}P(t,T) \tag{9.4}$$

eliminates the drift of the bond due to the spot rate process. We define the relative price for a T-maturity bond using the Itô formula on $Z(t,T)$ as follows:

$$\begin{gathered} \log Z(t,T) = \\ = \log Z(0,T) + \int_0^t A(u,T)du - \frac{1}{2}\sum_{i=1}^n \int_0^t \Sigma_i^2(u,T)du + \sum_{i=1}^n \int_0^t \Sigma_i(u,T)dW_i(u). \end{gathered}$$

The relative bond price, as we mentioned also for the bond price, can't be written as a function using just the current values, because it depends on the drift term $A(t,T)$ and the volatility coefficients $\Sigma_i(t,T)$.

We need the necessary and sufficient conditions on the forward rate process which ensure the existence of unique, equivalent martingale probability measure. The **uniqueness of the equivalent martingale probability measure** is given by the non-singularity of the matrix of volatilities

$$M_t = (\Sigma_i(t,T_j))_{i,j=1}^n = \begin{pmatrix} \Sigma_1(t,T_1) & ... & \Sigma_n(t,T_1) \\ . & & . \\ . & & . \\ \Sigma_1(t,T_n) & ... & \Sigma_n(t,T_n) \end{pmatrix}$$

for maturities $0 < T_1 < ... < T_n \leq \tau$. Define $\lambda_i(t,T_1,...T_n)$ the market prices of risk inspired by the one-dimensional (5.13). Using the matrix M_t, we get:

$$A(t,T) = \sum_{i=1}^n \sigma_i(t,T)[-\lambda_i(t,T_1,...T_n)]$$

The solutions of this equation depend on the set of bond maturities $T_1,...T_n$.

Uniqueness of the martingale measure for all bonds

Now, we should have all the technical conditions for the existence of unique equivalent martingale probability measure for bonds $Z(t,T_1),...Z(t,T_n)$. The martingale measure and the market prices of risk depend on the set of bond maturities $T_1,...T_n$:

- measure **Q**, equivalent to the measure **P**, is the unique probability measure such that $Z(t,T)$ is a martingale for all $T \in [0,\tau]$ and $t \in [0,T_1]$
- consider 2 sets of maturities $\{T_i\}$ and $\{V_i\}$ which satisfy all the needed constraints. Then $\lambda_i(t, T_1, ...T_n) = \lambda_i(t, V_1, ...V_n)$
- there exists a previsible process $\xi_i(t)$ $(i = 1, .., n)$ such that $\alpha(t,T) = \sum_{i=1}^n \sigma_i(t,T)(\xi_i(t) - \Sigma_i(t,T))$

The first condition says that

$$P(t,T) = B_t E[\tfrac{1}{B_T} exp(\textstyle\sum_i^n \int_0^T \xi_i(t) dW_i(t) - \frac{1}{2}\sum_{i=1}^n \int_0^T \xi_i^2(t)dt)|\mathcal{F}_t]$$

depends on forward rate drifts & volatilities and the initial forward rate curve.
The second one, called standard finance condition for arbitrage free pricing, is necessary if we want to ensure the absence of arbitrage.
Third one is the forward rate drift restriction, which shows what restriction on the family of drift processes is needed.

We already know that the equivalent measure to **P** exists and is unique, so using the multi-dimensional Cameron-Martin-Girsanov theorem, we'll find the equivalent measure **Q**, under which we get independent **Q**- Brownian motions $\widetilde{W}_1, ..., \widetilde{W}_n$:

$$\widetilde{W}_i(t) = W_i(t) + \textstyle\int_0^t \xi_i(u)du$$

and rewrite the equation for $Z(t,T)$ in (9.4), $d_tP(t,T)$ in (9.3) and $d_tf(t,T)$ from (9.1) as follows:

$$\begin{aligned} d_tZ(t,T) &= Z(t,T)\textstyle\sum_{i=1}^n \Sigma_i(t,T)d\widetilde{W}_i(t) \\ d_tP(t,T) &= P(t,T)(r_tdt + \textstyle\sum_{i=1}^n \Sigma_i(t,T)d\widetilde{W}_i(t)) \\ d_tf(t,T) &= \textstyle\sum_{i=1}^n \sigma_i(t,T)d\widetilde{W}_i(t) - \sum_{i=1}^n \sigma_i(t,T)\Sigma_i(t,T)dt. \end{aligned}$$

The replicating strategy
Finding the equivalent measure **Q** refers the first step of the replicating strategy.

Second step is the option pricing formula:
Let X be the payoff of a derivative at time T, then for the value V_t:

$$V_t = B_t\mathbf{E_Q}(B_T^{-1}X|\mathcal{F}_t) = \mathbf{E_Q}(\exp(-\textstyle\int_t^T r_sds)X|\mathcal{F}_t),$$

which is the same form as (5.12) in single-factor HJM. We've converted claim X in order to form the process $E_t = \mathbf{E_Q}(B_S^{-1}X|\mathcal{F}_t)$ from (5.11), that is the conditional expectation of the discounted claim $B_T^{-1}X$. E_t is a **Q**-martingale.
Third step of replicating strategy is to find a previsible process ϕ_t such that $dE_t = \rho_t dZ_t$. We already know that both Z_t and E_t are **Q**-martingales and we'll use it in Martingale representation theorem.

n-factor Martingale representation theorem
Denote

- $\widetilde{W}$ n-dimensional **Q**-Brownian motion
- N_t n-dimensional **Q**-martingale process, $N_t = (N_1(t), ..., N_n(t))$
- $n_{ij}(t)$ volatility matrix of N_t, $dN_j(t) = \sum_i n_{ij}(t)d\widetilde{W}_i(t)$ and the matrix is non-singular with probability one

if O_t is one-dimensional **Q**-martingale process, there exists an n-dimensional $\mathcal{F}$-previsible process $\varrho_t = (\varrho_1(t), .., \varrho_n(t))$ such that $\int_0^T (\sum_j n_{ij}(t)\varrho_j(t))^2 < \infty$ and the martingale O can be written as:

$$N_t = N_0 + \sum_{j=1}^n \int_0^t \varrho_j(s)dN_j(s)$$

where ϱ is unique.

To hedge the claim X, we chose the set of bonds $T_1, ..., T_n$ with maturities longer than T. The **self-financing strategy** is the combination of n-vector holdings (in the bonds) and cash bond B_t:

$$(\varrho_1(t), .., \varrho_n(t), \psi_t)$$

the value of portfolio at time t:

$$\begin{aligned} V_t &= B_t \mathbf{E_Q}(B_T^{-1}X|\mathcal{F}_t) = \\ &= \sum_{j=1}^n \varrho_j(t)P(t, T_j) + \psi_t B_t, \end{aligned}$$

for the discounted value $E_t = V_t B_t^{-1}$:

$$\begin{aligned} E_t &= \sum_{j=1}^n \varrho_j(t)Z(t, T_j) + \psi_t, \\ dE_t &= \sum_{j=1}^n \varrho_j(t)d_tZ(t, T_j) \end{aligned}$$

and finally, using the representation theorem, there is an n-vector of previsible processes ϱ_t such that

$$E_t = \mathbf{E_Q}\left(B_T^{-1}X + \sum_{j=1}^n \int_0^t \varrho_j(s)dZ(s, T_j)\right),$$

which gives the self-financing strategy ϱ. According to the self-financing strategy we hold

- $\varrho_j(t)$ units of the bond with maturity T_j
- $E_t - \sum_{j=1}^n \varrho_j(t)Z(t, T_j) = \psi_t$ units of the cash bond
- we buy the portfolio for $\mathbf{E_Q}(B_T^{-1}X)$
- by time T, the portfolio is worth X.

9.2 Two-factor models

Two-factor models are the simples multi-dimensional models which deal with bonds that are not perfectly correlated and yield curves which can be dependent on different parameters. Still, we will use similar techniques of the bond pricing as we used before. We introduce the two-factor model defined in [8] that depends on 2 state variables: r_t - short interest rate and another state variable, lets say v. Both variables have following dynamics:

$$dr = \alpha(r, v, t)dt + \sigma(r, v, t)dW_1$$
$$dv = \beta(r, v, t)dt + \upsilon(r, v, t)dW_2$$

where α and β measure the expected changes of r and v per time-unit and σ and υ measure standard deviations of changes in r and v per time-unit, dW_1 and dW_2 are the changes in two different Brownian motions.
We continue in analyzing this model in a similar way that in one-factor interest rate model and we get to the partial differential equation for bond pricing:

$$rP = \tfrac{1}{2}\sigma^2\tfrac{\delta^2 P}{\delta r^2} + \alpha^*\tfrac{\delta P}{\delta r} + \tfrac{1}{2}\upsilon^2\tfrac{\delta^2 P}{\delta v^2} + \beta^*\tfrac{\delta P}{\delta v} + \rho\sigma\upsilon\tfrac{\delta^2 P}{\delta r\delta v} + \tfrac{\delta P}{\delta r}$$

where α^*, β^* are the drift terms modified under martingale measure $\mathbf{Q}$, the first two terms in the equation are related to r, another two terms to v and ρ indicates the correlation between r and v. To solve this differential equation, we need to specify what exactly the state variable is, because we need to know how the joint stochastic process looks and what is the exact modification of martingale measure. There are few possibilities, we'll mention some of them:

- v is price of an asset:
  In this case, $ru = \beta - q_2\upsilon$, where q_2 is the market price of risk (5.13)

  $$\Rightarrow \beta^* = ru$$

  what is Black-Scholes model for pricing the options (4.8).

- v is yield-to-maturity for a zero coupon bond with maturity T
  First, the price of the bond is counted from the pricing equation:

  $$P(r, u, t, T) = e^{-v(T-t)},$$

  then

  $$\begin{aligned} rP &= \tfrac{1}{2}\sigma^2\tfrac{\delta^2 P}{\delta r^2} + \alpha^*\tfrac{\delta P}{\delta r} + \tfrac{1}{2}\upsilon^2\tfrac{\delta^2 P}{\delta v^2} + \beta^*\tfrac{\delta P}{\delta v} + \rho\sigma\upsilon\tfrac{\delta^2 P}{\delta r\delta v} + \tfrac{\delta P}{\delta r} \\ re^{-v(T-t)} &= \tfrac{1}{2}\upsilon^2\tfrac{\delta^2 P}{\delta v^2} + \beta^*\tfrac{\delta P}{\delta v} + \rho\sigma\upsilon\tfrac{\delta^2 P}{\delta r\delta v} + \tfrac{\delta P}{\delta r} \\ r &= \tfrac{1}{2}\upsilon^2(T-t)^2 + \beta^*[-(T-t)] + v \\ &\Downarrow \\ \beta^* &= \tfrac{1}{t-T}(r - v - \tfrac{1}{2}\upsilon^2(T-t)^2) \end{aligned}$$

Many studies were devoted to the case, when the state variable v is not related to marketed asset. Still, q_2 (the market price of risk of the second state variable) can be determined. The problem is treated in following works:

- **Brennan, Schwartz** [5]
  application of a finite difference numerical approximation
- **Dobson, Sutch, Vanderford** [16]
  forecasting models using geometrically smoothed averages of past short rates as predictors of future rates
- **Malkiel** [12]
  the short rate returning to normal level, measured by geometric average of past interest rates
- **Cox, Ingersoll, Ross** [7]
  model when the variance is $\sigma\sqrt{r}$ for constant σ
- **Richard and Cox, Ingersoll, Ross** [13] and [7]
  real interest rates and expected inflation are the two-factor model state variables

9.2.1 Ho & Lee two-factor interest rate model

Here we have an extension of Ho & Lee model (6.8), which is again HJM consistent, so it's just a nice illustration of using the specified 2-dimensional HJM model. The trend of forward rate is denoted by:

$$d_t f(t,T) = \sigma_a dW_a(t) + \sigma_b e^{-\beta(T-t)} dW_b(t) + \alpha(t,T)$$

where σ_a, σ_b, β are constant and α is a deterministic function of time and maturity. We can determine the drift α using the term derived in n-factor HJM by setting $n = 2$ and presenting ξ_a and ξ_b to be $\mathcal{F}$-previsible processes subsequently:

$$\alpha(t,T) = \textstyle\sum_{i=1}^{2} \sigma_i(t,T)(\xi_i(t) - \Sigma_i(t,T))$$

where $-\Sigma_i(t,T) = \int_t^T \sigma_i(t,u)du$

$$\begin{aligned} \alpha(t,T) &= \sigma_a\xi_a(t) + \sigma_b e^{-\beta(T-t)}\xi_b + \sigma_a \textstyle\int_t^T \sigma_a(t,u)du + \sigma_b \int_t^T \sigma_b(t,u)du = \\ &= \sigma_a\xi_a(t) + \sigma_b e^{-\beta(T-t)}\xi_b + \sigma_a^2(T-t) + \sigma_b^2 \tfrac{1}{\beta}[1 - e^{-\beta(T-t)}]e^{-\beta(T-t)} \end{aligned}$$

The forward rate under martingale measure $\mathbf{Q}$:

$$\begin{aligned} f(t,T) &= f(0,T) + \textstyle\int_0^t \sigma_a dW_a(u) + \int_0^t \sigma_b e^{-\beta(T-u)} dW_b(u) + \int_0^t \alpha(u,T)du = \\ &= f(0,T) + \sigma_a W_a(t) + \sigma_b e^{-\beta T} \textstyle\int_0^t e^{\beta u} dW_b(u) + \int_0^t \alpha(u,T)du, \end{aligned}$$

because under martingale measure, ξ_a and ξ_b are null. As can be seen from the prescription, the forward rates are normally distributed. As in the one-factor Ho & Lee, the rates can go negative.

$$\begin{aligned}-\log P(t,T) &= \int_t^T f(t,u)du = \\ &= \int_t^T f(0,u)du + \sigma_a(T-t)W_a(t) + \sigma_b\frac{1}{\beta}[e^{-\beta t} - e^{-\beta T}]\int_0^t e^{\beta u}dW_b(u) + \\ &\quad + \int_0^t\int_t^T \alpha(u,s)dsdu\end{aligned}$$

and the short interest rate $r_t = f(t,t)$:

$$r_t = f(0,t) + \sigma_a W_a(t) + \sigma_b e^{-\beta t}\int_0^t e^{\beta u}dW_b(u) + \int_0^t \alpha(u,t)du.$$

The instantaneous interest rate r_t is no more carrying all the information about bond prices.

Last, we would like to estimate the price of an option, struck at k and exercised at time t. To do this, we'll use the Black-Scholes formula (4.7), which allows us to price caps and floors as well as options on the discount T-bonds.
We have derived the logarithmic price $\log P(t,T)$. To substitute it correctly to Black-Scholes formula, we need the log-price's variance:

$$\sigma^2_{\log P}(t,T) = \sigma_a^2(T-t)^2 + (\sigma_b\frac{1}{\beta}[1-e^{-\beta(T-t)}])^2\frac{1}{2\beta}(1-e^{-2\beta t})$$

and the log-normally distributed discounted bond $B_t = \int_0^t r_u du$

and, including $F = \frac{P(0,T)}{P(0,t)}$ the forward price of T-bond, we have everything to write the Black-Scholes formula as in (4.8):

$$V_0 = P(0,T)\left(F\Phi(\frac{\log\frac{F}{k} + \frac{1}{2}\sigma^2_{\log P}(t,T)}{\sigma_{\log P}(t,T)}) - k\Phi(\frac{\log\frac{F}{k} - \frac{1}{2}\sigma^2_{\log P}(t,T)}{\sigma_{\log P}(t,T)})\right).$$

9.3 Multi-factor normal model

Multi-factor interest rate model in [2] is a nice generalization of the two factor model, deriving the Black-Scholes option pricing formula and using the normally distributed forward rates.

We consider n-factor complete model, where the forward rates evolve in a way:

$$\mathbf{d_t f(t,T)} = \sum_{\mathbf{i=1}}^{\mathbf{n}} \sigma_{\mathbf{i}}(\mathbf{t},\mathbf{T})\mathbf{dW_i(t)} + \alpha(\mathbf{t},\mathbf{T})\mathbf{dt}$$

where the volatilities are stated to be a product of two deterministic functions $f_i(t)$ and $g_i(T)$:

$$\sigma_i(t,T) = f_i(t)g_i(T)$$

The explanation of the significance of the functions is, that f_i determines the size of shocks at time t and g_i controls how the shocks behave at different maturities. We suppose

- there exist $\mathcal{F}$-previsible processes $\xi_1, ..., \xi_n$ that:

$$\alpha(t,T) = \sum_{i=1}^{2} f_i(t)g_i(T)[\xi_i(t) + f_i(t)\int_t^T g_i(u)du]$$

- there exists a non-singular matrix $M_t = (G_j(t,T_i))_{i,j=1}^n$
(where $G_j(t,T_i) = \int_t^{T_i} g_i(u)du$), for all $t < T_1$ and $0 < T_1, ..., T_n < \tau$

To show an example of such a volatility: $\sigma_i(t,T) = \sigma(i)e^{-\beta_i(T-t)}$, for distinct functions β_i and deterministic functions $\sigma_i(t)$.

For $\sigma_i(t,T)$, the short rate and forward rates are normally distributed, prices are log-normally distributed and Black-Scholes for pricing the options can be used. As was shown in two-factor model:

$$\sigma^2_{\log P} = \frac{1}{t}\sum_{i=1}^{n} G_i^2(t,T)\int_0^t f_i^2(u)du$$

and the value at $t = 0$ of call option on the T-bond, struck at k, exercised at time t according to Black-Sholes (4.8):

$$V_0 = P(0,T)\left(F\Phi(\frac{\log\frac{F}{k}+\frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}) - k\Phi(\frac{\log\frac{F}{k}-\frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}})\right)$$

This model is really just the generalization of simpler models, because we can use it for

- ***Vasicek model***
where $f(t) = \sigma_t e^{\int_0^t \alpha_u du}$ and $g(T) = e^{-\int_0^T \alpha_u du}$

- ***Ho&Lee model***
where $f(t) = \sigma$ and $g(T) = 1$

9.4 Brace-Gatarek-Musiela model

In the examples of interest rates derivatives, we showed few of them that are influenced by Δ-period LIBOR rates. BGM model is multi-factor interest rate model that works this rates expressed by:

$$L(t,T) = \frac{1}{\Delta}(\frac{P(t,T)}{P(t,T+\Delta)} - 1),$$

which is a rate for borrowing at time T. The forward volatilities $\sigma_i(t,T)$ in this model follow:

$$\int_T^{T+\Delta} \sigma_i(t,u)du = \frac{\Delta L(t,T)}{1+L(t,T)}\theta_i(t,T) = [1 - \frac{P(t,T+\Delta)}{P(t,T)}]\theta_i(t,T)$$

for some n-dimensional deterministic function θ, absolutely continuous with respect to T. $L(t,T)$ in this case is both $\mathbf{P}_{T+\Delta}$-martingale and log-normal. This is the basic assumption for pricing caps and swaptions. All about this non-linear model based on the HJM model can be found in [4], where the authors defined conditions under which a measure for risk-free dynamics exists.

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