UNIVERZITA KARLOVA V PRAZE
MATEMATICKO - FYZIKÁLNÍ FAKULTA

DIPLOMOVÁ PRÁCE

MARTINA ŠAJTAROVÁ

Úlohy o dopravních sítích - stochastické varianty a aplikace

Katedra pravděpodobnosti a matematické statistiky

Vedoucí diplomové práce: Prof. RNDr. Jitka Dupačová, DrSc.
Studijní program: Matematika
Studijní obor: Pravděpodobnost, matematická statistika a ekonometrie
Studijní plán: Ekonometrie

Prohlašuji, že jsem svou diplomovou práci napsal(a) samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 12. dubna 2006

Martina Šajtarová

Šajtarová

Charles University in Prague
Faculty of Mathematics and Physics

DIPLOMA THESIS

Martina Šajtarová

Transportation Network Problems - Stochastic Variations and Applications

Department of Probability and Mathematical Statistics

Supervisor: Prof. RNDr. Jitka Dupačová, DrSc.
Study Program: Mathematics
Study Branch: Probability, Mathematical Statistics and Econometrics
Study Plan: Econometrics

Contents

Název práce: Úlohy o dopravních sítích - stochastické varianty a aplikace
Autor: Martina Šajtarová
Katedra (ústav): Katedra pravděpodobnosti a matematické statistiky
Vedoucí diplomové práce: Prof. RNDr. Jitka Dupačová, DrSc., Katedra pravděpodobnosti a matematické statistiky
e-mail vedoucího: **dupacova@karlin.mff.cuni.cz**
Abstrakt: *V této práci se zabýváme dvěma typy stochastických úloh o dopravních sítích - dopravním problémem s náhodnými poptávkami nebo náhodnou kapacitou hran a dopravním problémem s náhodnými časy dopravy a obsluhy. Na začátku tohoto textu jsou představeny tři speciální typy mnohostupňových stochastických programovacích problémů, které používáme k formulaci dopravních problémů s náhodnými požadavky nebo náhodnými kapacitami hran. Dále uvádíme několik aproximací pro úplnou síťovou kompenzaci, které nám umožňují řešit tyto stochastické dopravní problémy přímo, a jedna z nich je aplikována na problém distribuce peněz. Ve druhé polovině této práce jsou definovány dvě odlišné verze stochastického dopravního problému s náhodnými časy dopravy a obsluhy a jsou shrnuty některé z jejich vlastností. Dále je představen DESVRP algoritmus (algoritmus založený na deterministických ekvivalentech k stochastickému problému směrování dopravních prostředků, zavedený Kenyonem a Mortonem v roce 2003) pro obě verze tohoto stochastického dopravního problému. Na konci tohoto textu je tento algoritmus aplikován opět na problém distribuce peněz.*
Klíčová slova: dopravní sítě, stochastický dopravní problém, náhodné poptávky a náhodné kapacity hran, náhodné časy dopravy a obsluhy

Title: Transportation Network Problems - Stochastic Variations and Applications
Author: Martina Šajtarová
Department: Department of Probability and Mathematical Statistics
Supervisor: Prof. RNDr. Jitka Dupačová, DrSc., Department of Probability and Mathematical Statistics
Supervisor's e-mail address: **dupacova@karlin.mff.cuni.cz**
Abstract: *In our work we deal with two types of stochastic transportation network problem - with transportation problem with random demands or random arc capacities and with transportation problem with random travel and service times. At the beginning of this text, there are introduced three special types of multistage stochastic programming problems that we use for formulation of transportation problems with random demands or random arc capacities. Further we present some approximations to full network recourse that allow us to solve these stochastic transportation problems directly and one of them is applied on the money distribution problem. In the second half of this work, two different versions of stochastic transportation problem with random travel and service times are defined and some of their properties are presented. Then, we introduce the DESVRP (deterministic equivalents to stochastic vehicles routing problem) algorithm, given by Kenyon and Morton in 2003, for both of version of this stochastic transportation problem. In the end of this text, this algorithm is applied again to the money distribution problem.*
Keywords: Transportation networks, stochastic transportation problem, random demands and random arc capacities, random travel and service times

Preface

In this diploma thesis we deal with stochastic transportation problem which is one special example of the transportation network problems. Under the name of stochastic transportation problem we understand various extensions of the classic transportation problem (see for example [5]) in the presence of some uncertainties.

For this work, we choose two types of stochastic transportation problems – stochastic transportation problem with random demands or random arc capacities (these two problems are closely related to each other since random demands can be modelled as random arc capacities) and stochastic transportation problem with random travel and service times.

In transportation and logistics there arise very often specific stochastic network problems that can be formulated as stochastic transportation problems with random arc capacities. These problems can be written as the multistage stochastic programs with network recourse. Since the full network recourse can have very complicated shape it is useful to find some approximations that allow us to solve these stochastic transportation problems directly using classical optimizing methods.

Two different approach to search of approximations to stochastic problems exist. The first one uses sampling a small number of scenarios that describe future possible outcomes. As a result, a much larger optimization problem is created that forces the model to recommend a single first-stage decision. A disadvantage of this approach is that the original problem network structure can be lost. This attitude can be for instance found in [9] or [18].

The second approach is based on evolving approximations to the network recourse function and is presented for example in [4], [7], [9] or [19]. In this work, we will concern only with this approach to solving the stochastic transportation problems with network recourse.

The stochastic transportation problem with random travel and service times can be formulated as a network problem, too. We can imagine it as a vehicle routing problem on a network whose arcs have random travel times and whose nodes have random service times. Vehicles routes are selected before random parameters are known and no reoptimizations are allowed. This problem can be viewed as stochastic multiple traveling salesman problem (TSP - see for instance [15]) and is for example presented in [12], [13] or [14]. These problems are mostly solved with so called branch-and-cut approach that was for the first time introduced in [3] for the TSP problem.

In this work, chapter 2 provides three specific transportation problems formulated as network flow problems with random demands or random arc capacities. In chapter 3 we introduce some approximations to full network recourse that can be used for solving stochastic transportation problems mentioned in chapter 2. In chapter 4 we study stochastic transportation problem with random travel and service times and present an algorithm for solving this problem.

Chapter 1

Stochastic Transportation Problem

Under the name of stochastic transportation problem we understand various extensions of the classic transportation problem (see for example [5]) in the presence of forecasting uncertainties which arise in dynamic fleet management, dynamic vehicle routing and logistics. In all of these cases we have to make some decision in the face of uncertain future events. For example, we do not know future demands in individual demand points; travel and service times for individual points can be random or there are random vehicle availabilities in individual depots. For that reason our decision need not be either acceptable or optimal.

We usually assume that the future process has a known (mostly discrete) distribution, i.e. there exist many possible options (several possible scenarios) for the future course. For example, all alternatives and their probabilities can be estimated from past dates.

It is very useful to use the multistage stochastic programming (presented for instance in [6]) for formulating these stochastic transportation problems. Some typical multistage stochastic programming problem (in this case only two-stage one) can be written as

$$\min_x \{c^T x + \overline{Q}(x)\} \tag{1.1}$$

subject to $Ax = b, x \geq 0$, where $\overline{Q}(x)$ is the *expected recourse function* defined by

$$\overline{Q}(x) = E_\xi[\min_y \{q^T y | Wy = \xi - Tx,\ y \geq 0\}]. \tag{1.2}$$

In this two-stage stochastic programming problem, x and y are the first and the second stage decisions, c and q are vectors of costs of the respective stage, ξ is the vector of random variables, T is the technology matrix and W is the recourse matrix. First, a decision in the first stage must be made, afterwards the random vector ξ is realized and in the end we can improve or correct our decision using the optimal solution y in the second-stage optimizing problem.

Unfortunately, we very often meet the problem that we are not capable to write $\overline{Q}(x)$ analytically as a function of x if we deal with transportation networks since the full networks recourse can have very complicated structure. For that reason we want to find an approximate function $\Phi(x, \xi)$ which satisfies

$$\Phi(x, \xi) \geq Q(x, \xi) = \min_y \{q^T y | Wy = \xi - Tx,\ y \geq 0\}, \tag{1.3}$$

where $\Phi(x, \xi)$ is obtained by heuristic optimizing the recourse function $Q(x, \xi)$ (see [9]). Especially, $\Phi(x, \xi)$ should reasonably approximate $Q(x, \xi)$ and should have a sufficient structure in order to its expectation can be easily found. Furthermore, we want the approximation $\Phi(x, \xi)$ to be piecewise linear, separable and convex in ξ.

Then we can replace the original general expected recourse function $\overline{Q}(x)$ with the function $\overline{\Phi}(x)$ where

$$\overline{\Phi}(x) = E_{\xi}\Phi(x, \xi)$$

and solve the recourse problem (1.1) using standard techniques.

Chapter 2

Stochastic Transportation Problem with Random Arc Capacities or Random Demands

Some stochastic transportation problems can be formulated as networks with random arc capacities or random demands. In this part we show how it can be done. We present three special types of multistage stochastic programming problems (the two-stage transportation problem with random demands, the N-stage transportation problem with random arc capacities and the general N-stage network with random arc capacities) that will be explained in detail in further parts of this chapter and that can be found for example in [7]. We will not deal with such stochastic transportation problems that cannot be formulated as any from these three particular stochastic programs.

In all by us considered stochastic dynamic network problems there are flows between different points in space and time. We call these points in space for simplicity cities but they can represent regions, countries, terminals, airports, ports, warehouses, etc. We naturally assume that the flows between cities move only forward in time and that in all cases each stage consists only of a single time period. All random variables in a given stage are realized simultaneously and after all random variables from all previous stages.

Now let us introduce notation we use for the formulation of the three above mentioned stochastic transportation problems. The set of all cities is denoted by $\mathbf{R}$ and $t = 1, \ldots, P$ are the time periods where P is the planning horizon. We do not consider problems related to the truncated horizon option. Next let (i, t) denote a node in the network representing a given city $i \in \mathbf{R}$ in time t. If t_{ij} is the travel time from city $i \in \mathbf{R}$ to city $j \in \mathbf{R}$ then we obtain notation (i, t, j) for the link from a point (i, t) to another point $(j, t + t_{ij})$. In some situations we permit the travel time between two different cities in the network to be zero.

Decision variables $x_{ij}(t)$ give the flows from city $i \in \mathbf{R}$ to city $j \in \mathbf{R}$ in time period t. In the N-stage stochastic transportation problem and in the general N-stage stochastic network random variables $\xi_{ij}(t)$ denote the random arc capacity for the link (i, t, j). In the two-stage transportation problem with random demands random variables $\eta_j(t)$ give the random demands in city $j \in \mathbf{R}$ at time t.

Further we denote as $S_i(t)$ the flow through node (i, t) which is determined by our decisions made before time period t. Finally, $R_i(t)$ represents the exogenous demands on the network (flows entering or leaving the network) at point (i, t) and $c_{ij}(t)$ is cost for the route from city $i \in \mathbf{R}$ to city $j \in \mathbf{R}$ in the t-th period .

In this work we will also use the following vectors:

$$\begin{aligned}
S(t) &= (S_i(t) : i \in \mathbf{R}) \\
R(t) &= (R_i(t) : i \in \mathbf{R}) \\
x(t) &= (x_{ij}(t) : i, j \in \mathbf{R}) \\
c\,(t) &= (c_{ij}(t) : i, j \in \mathbf{R}) \\
\eta(t) &= (\eta_j(t) : j \in \mathbf{R}) \\
\xi(t) &= (\xi_{ij}(t) : i, j \in \mathbf{R}).
\end{aligned}$$

Now we can proceed to present our three basic stochastic dynamic network problems with random demands or random arc capacities.

2.1 The Two-Stage Transportation Problem with Random Demands

In this network programming problem an initial vector of supplies $R(1)$ must be allocated from one set of cities to another set of cities (it is evidently possible that these sets can be for simplicity identical). Thus we have to move flows from supply points to demand points in the first stage before random demands $\eta(2)$ in the second stage are revealed. Then we can again allocate flows from city to city.

We can imagine this situation very easily. Let the set of cities contain plants, warehouses and customers. Thus in the first stage we move goods from plants to warehouses and in the second stage after random demands were observed we transport this goods from warehouses to customers.

This problem can be written as

$$\min_{x(1),S(1)} \{c\,(1)^T x(1) + \overline{Q}(S(1))\} \tag{2.1}$$

subject to

$$\sum_{j\in\mathbf{R}} x_{ij}(1) = R_i(1) \qquad \forall i \in \mathbf{R} \tag{2.2}$$

$$\sum_{i\in\mathbf{R}} x_{ij}(1) = S_j(1) \qquad \forall j \in \mathbf{R} \tag{2.3}$$

$$x_{ij}(1) \geq 0 \qquad \forall i, j \in \mathbf{R}. \tag{2.4}$$

$\overline{Q}(S(1))$ is the expected recourse function defined by

$$\overline{Q}(S(1)) = E_{\eta(2)}[Q(S(1), \eta(2))], \tag{2.5}$$

where

$$Q(S(1), \eta(2)) = \min_{x(2)} c\,(2)^T x(2) \tag{2.6}$$

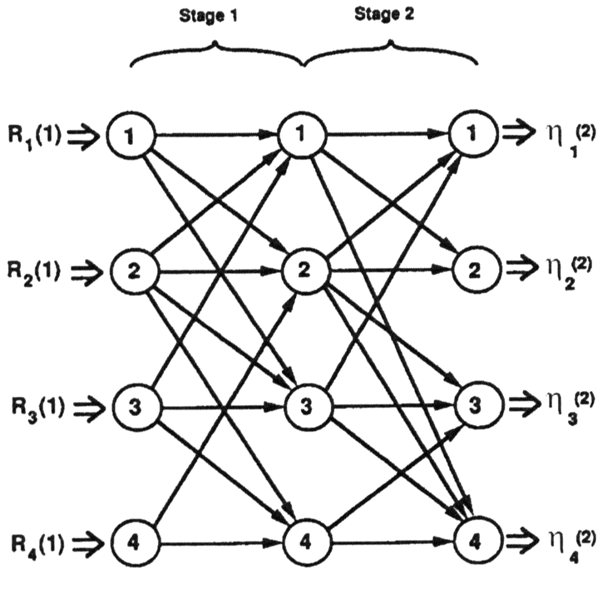

Figure 2.1: The two-stage stochastic transportation problem

subject to

$$\sum_{j \in \mathbf{R}} x_{ij}(2) = S_i(1) \qquad \forall i \in \mathbf{R} \tag{2.7}$$

$$\sum_{i \in \mathbf{R}} x_{ij}(2) \geq \eta_j(2) \qquad \forall j \in \mathbf{R} \tag{2.8}$$

$$x_{ij}(2) \geq 0 \qquad \forall i, j \in \mathbf{R}. \tag{2.9}$$

The decision variables $x(2)$ for the conditional recourse function $Q(S(1), \eta(2))$ are conditional on $\eta(2)$ because we choose $x(2)$ after $\eta(2)$ is realized. We want to find one vector $x(2)$ for each possible realization of $\eta(2)$.

In the Figure 2.1 we can see this transportation problem for four cities. Set of supply points and set of demand points are in this case identical.

In the first stage we have to move the whole supply $R_i(1)$ from city $i \in \mathbf{R}$. This gives us the equation (2.2). The relation (2.3) defines a flow $S_j(1)$ through node $(j, 1)$ as a sum of flows coming to this node in the first stage. The equation (2.7) says that in the second time period we have to move from city $i \in \mathbf{R}$ the whole flow coming to this node during the first time period. The inequality (2.8) says that the total flow to city $j \in \mathbf{R}$ has to meet or exceed the random demand in this city $\eta_j(2)$. Of course, flows from city i to city j are non-negative variables in both stages.

2.1.1 The Two-Stage Transportation Problem with Random Demands and with Random Arc Capacities

We can very easily extend the two-stage transportation problem with random demands by including random arc capacities in addition to random demands. If $\xi_{ij}(2)$ denotes with respect to prior mentioned notation the random arc capacity for link between cities i and j from set $\mathbf{R}$ in the second stage and $u_{ij}(1)$ denotes the arc capacity for the link $(i, 1, j)$ then we can write the two-stage transportation problem with random demands and random arc capacities as

$$\min_{x(1),S(1)} \{c(1)^T x(1) + \overline{Q}(S(1))\} \tag{2.10}$$

subject to

$$\sum_{j\in\mathbf{R}} x_{ij}(1) = R_i(1) \qquad \forall i \in \mathbf{R} \tag{2.11}$$

$$\sum_{i\in\mathbf{R}} x_{ij}(1) = S_j(1) \qquad \forall j \in \mathbf{R} \tag{2.12}$$

$$x_{ij}(1) \geq 0 \qquad \forall i,j \in \mathbf{R} \tag{2.13}$$

$$x_{ij}(1) \leq u_{ij}(1) \qquad \forall i,j \in \mathbf{R}. \tag{2.14}$$

$\overline{Q}(S(1))$ is again the expected recourse function defined analogically by

$$\overline{Q}(S(1)) = E_{\xi(2)}[\,E_{\eta(2)}[\,Q(S(1),\eta(2),\xi(2))\,]\,] \tag{2.15}$$

where

$$Q(S(1),\eta(2),\xi(2)) = \min_{x(2)} c\,(2)^T x(2) \tag{2.16}$$

subject to

$$\sum_{j\in\mathbf{R}} x_{ij}(2) = S_i(1) \qquad \forall i \in \mathbf{R} \tag{2.17}$$

$$\sum_{i\in\mathbf{R}} x_{ij}(2) \geq \eta_j(2) \qquad \forall j \in \mathbf{R} \tag{2.18}$$

$$x_{ij}(2) \geq 0 \qquad \forall i,j \in \mathbf{R} \tag{2.19}$$

$$x_{ij}(2) \leq \xi_{ij}(2) \qquad \forall i,j \in \mathbf{R}. \tag{2.20}$$

Most equations have the same meaning as corresponding relations in the two-stage transportation problem with random demands (problem (2.1) - (2.9)). The inequality (2.14) restricts flows in the first stage with deterministic arc capacities $u_{ij}(1)$ and the inequality (2.20) states the flow restriction in the second stage given with random arc capacities $\xi_{ij}(2)$.

2.2 The N-Stage Transportation Problem with Random Arc Capacities

This problem arises for example in dynamic fleet management, where the number of vehicles that can be send loaded between two cities is limited by forecasted (thus uncertain) demands. These demands are modelled as random arc capacities $\xi_{ij}(t)$. We again assume that sets of cities are identical for all N stages.

In the Figure 2.2 it is very understandably depicted how we can imagine this network problem. There are no demands at any nodes and hence flows do not leave the network at any time. Every arc from city i to city j in the t-th stage is associated partly with cost $c_{ij}(t)$, partly with random capacity $\xi_{ij}(t)$.

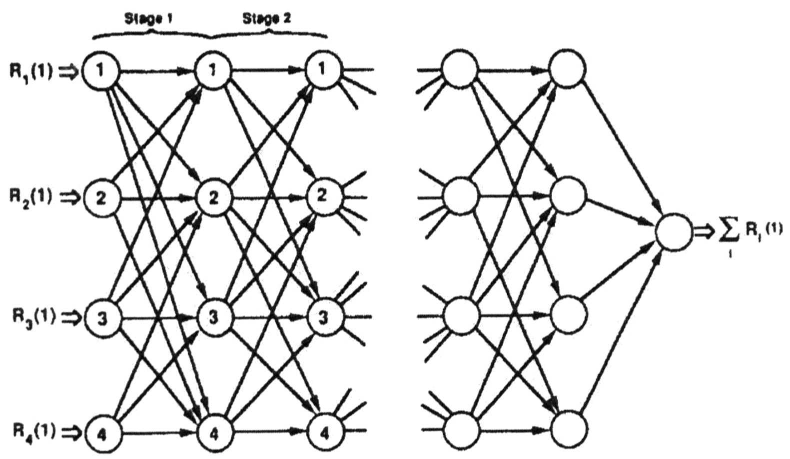

Figure 2.2: The N-stage stochastic transportation problem

The N-stage transportation problem with random arc capacities can be formulated as

$$\min_{x(1),S(1)} \{c(1)^T x(1) + \overline{Q}(S(1))\} \tag{2.21}$$

subject to

$$\sum_{j\in\mathbf{R}} x_{ij}(1) = R_i(1) \qquad \forall i \in \mathbf{R} \tag{2.22}$$

$$\sum_{i\in\mathbf{R}} x_{ij}(1) = S_j(1) \qquad \forall j \in \mathbf{R} \tag{2.23}$$

$$x_{ij}(1) \geq 0 \qquad \forall i,j \in \mathbf{R} \tag{2.24}$$

$$x_{ij}(1) \leq u_{ij}(1) \qquad \forall i,j \in \mathbf{R} \tag{2.25}$$

where

$$\overline{Q}(S(t-1)) = E_{\xi(t)}[Q(S(t-1),\xi(t))] \qquad t = 2,\ldots,P \tag{2.26}$$

$$\overline{Q}(S(P)) = 0. \tag{2.27}$$

The function $\overline{Q}(S(t-1))$ is the expected recourse function defined by the following recursion

$$Q(S(t-1),\xi(t)) = \min_{x(t),S(t)} \{c(t)^T x(t) + \overline{Q}(S(t))\} \tag{2.28}$$

subject to

$$\sum_{j\in\mathbf{R}} x_{ij}(t) = S_i(t-1) + R_i(t) \qquad \forall i \in \mathbf{R} \tag{2.29}$$

$$\sum_{i\in\mathbf{R}} x_{ij}(t) = S_j(t) \qquad \forall j \in \mathbf{R} \tag{2.30}$$

$$x_{ij}(t) \geq 0 \qquad \forall i,j \in \mathbf{R} \tag{2.31}$$

$$x_{ij}(t) \leq \xi_{ij}(t) \qquad \forall i,j \in \mathbf{R}. \tag{2.32}$$

Let us explain the individual equations in this model. Again, in the first stage we have to move the whole supply $R_i(1)$ from all cities i. This is expressed in (2.22). The equations (2.23) and (2.30) define the flow $S_j(t)$ through node (j,t) as the sum of flows from all cities coming to this node in the t-th stage. The equation (2.29) gives that the sum of flows coming from node (i,t) to all cities in the t-th stage has to be equal to the sum of flows through this city i in the $(t-1)$-th stage, $S_i(t-1)$, and of an exogenous demand on the network in the t-th stage $R_i(t)$. Hence the right hand side of this relation states a flow which can be transported in the t-th stage from city i. Flows from city i to city j are non-negative variables in all P stages; moreover in this model they are limited by arc capacities $u_{ij}(1)$ for $t=1$ and by random arc capacities $\xi_{ij}(t)$ for the rest $t=2,\ldots,P$.

2.3 General N-Stage Network with Random Arc Capacities

This problem is the most general of the here presented models. It is similar to N-stage transportation problem with random arc capacities but moreover it contains constraints on flows conservation at transshipment within each stage. In this problem we assume that movement times between individual cities are not identical in this problem.

Unlike the previous two models, now we need to distinguish between flows within a stage and flows between stages. Therefore we have to modify slightly our notation. Let decision variable $x_{ij}(t,t')$ denote flow from city i in stage t to city j in stage t' (where we assume $t' \geq t$). Then $S_i(t)$ gives the flow into city i from decisions made in stage t or earlier which can be transported in stage $t+1$.

The N-stage dynamic network with transshipment networks in each stage can be formulated as

$$\min_{x(1),S(1)} \{c(1)^T x(1) + \overline{Q}(S(1))\} \tag{2.33}$$

subject to

$$\sum_{t'\geq 1}\sum_{j\in\mathbf{R}} x_{ij}(1,t') - \sum_{k\in\mathbf{R}} x_{ki}(1,1) = R_i(1) \qquad \forall i\in\mathbf{R} \tag{2.34}$$

$$\sum_{i\in\mathbf{R}} x_{ij}(1,2) = S_j(1) \qquad \forall j\in\mathbf{R} \tag{2.35}$$

$$x_{ij}(1) \leq u_{ij}(1) \qquad \forall i,j\in\mathbf{R} \tag{2.36}$$

$$x_{ij}(1) \geq 0 \qquad \forall i,j\in\mathbf{R}. \tag{2.37}$$

The expected recourse function $\overline{Q}(S(t-1))$ is given by

$$\overline{Q}(S(t-1)) = E_{\xi(t)}[Q(S(t-1),\xi(t))], \tag{2.38}$$

where

$$Q(S(t-1),\xi(t)) = \min_{x(t),S(t)} \{c(t)^T x(t) + \overline{Q}(S(t))\} \tag{2.39}$$

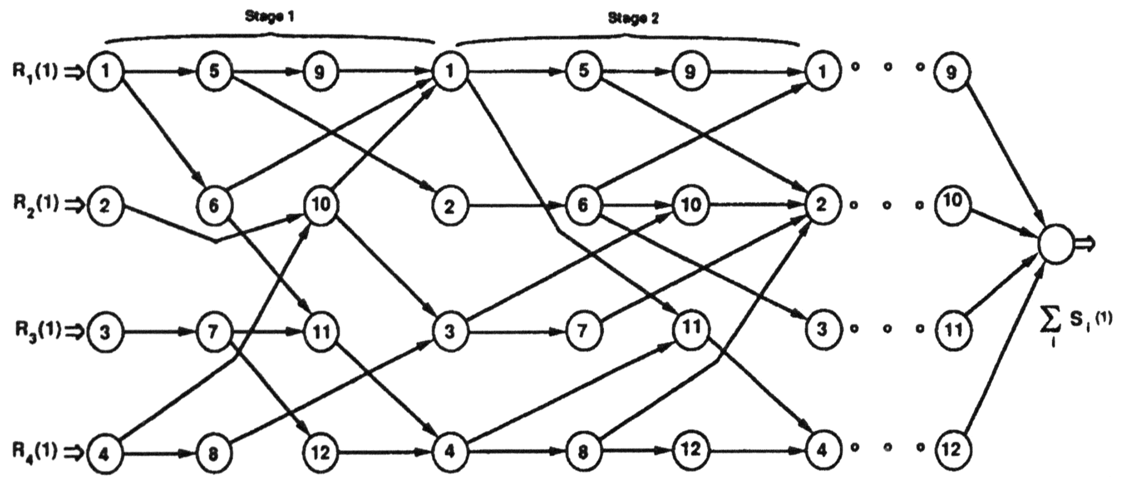

Figure 2.3: The general N-stage stochastic transportation problem

subject to

$$\sum_{t' \geq t} \sum_{j \in \mathbf{R}} x_{ij}(t, t') - \sum_{k \in \mathbf{R}} x_{ki}(t, t) = R_i(t) + S_i(t-1) \qquad \forall i \in \mathbf{R} \qquad (2.40)$$

$$\sum_{t' \leq t} \sum_{i \in \mathbf{R}} x_{ij}(t', t+1) = S_j(t) \qquad \forall j \in \mathbf{R} \qquad (2.41)$$

$$x_{ij}(t) \leq \xi_{ij}(t) \qquad \forall i, j \in \mathbf{R} \qquad (2.42)$$

$$x_{ij}(t) \geq 0 \qquad \forall i, j \in \mathbf{R}. \qquad (2.43)$$

This general transportation problem is depicted for twelve cities in the Figure 2.3.

Again, let us explain the individual equations in this model again. The first one, (2.34), says that in the first stage we move from city i flows whose sum has to be equal to the sum of the supply $R_i(1)$ for this city i and of flows which come to this city i in the first stage. Analogically, the equation (2.40) gives the flow conservation constraint for each node within a stage for the stages $t = 2, \ldots, P$. Relations (2.35) and (2.41) define $S_j(t)$ – the flow through city j which can be moved from this city in time period $t+1$. Again flows from city i to city j are non-negative variables and are limited by arc capacities $u_{ij}(1)$ for $t = 1$ and by random arc capacities $\xi_{ij}(t)$ for $t = 2, \ldots, P$.

Large problems without some specific network structure are unfortunately intractable with exact methods and therefore we will present various specialized approximations to the recourse function that are not as general as network recourse but allow us to solve these large problems with classical optimization methods. The above mentioned stochastic transportation problems (with the exception of the general N-stage network with random arc capacities) will further be used for the demonstration of the individual recourse strategies.

Chapter 3

Restricted Recourse Strategies for Dynamic Networks with Random Arc Capacities

3.1 A Strategy for Approximating Recourse Function

A great problem of stochastic programs is the shape of the recourse function because an optimization problem is contained within an expectation. We would like to find recourse function expressed directly as a function of $x(t)$. However, this goal is generally difficult because of high number of dimensions of both the decision vector $x(t)$ and the random vector $\xi(t)$ (respectively $\eta(t)$). Moreover $Q(x, \xi)$ (or $Q(x, \eta)$) is very often a nonseparable function of both variables.

Search of an approximate recourse function consists of two steps. First, we have to replace the imbedded optimization with a much simpler restricted optimization problem. Thus, instead of using full network recourse we would use restricted recourse strategies that would approximate a network optimization problem. Second, the probabilistic structure of the resulting optimal solution conditioned on the random vector ξ respectively η must allow to compute its expectation easily.

In the following sections we will deal with individual approximate recourse strategies for stochastic networks.

3.2 Simple Approximation

The simple approximation (see [7]) arises in cases where no recourse action is effective once the random vector ξ respectively η is realized. We only adjust our penalties.

The base of the simple approximation is the inclusion of a set of *recourse variables* which absorb the effect of randomness and therefore eliminate the interaction between random variables. The classical two-stage stochastic transportation problem with random demands (2.1) - (2.9) can be approximated using the simple approximation by replacing the constraint (2.8):

$$\sum_{i \in \mathbf{R}} x_{ij}(2) \geq \eta_j(2) \qquad \forall j \in \mathbf{R}$$

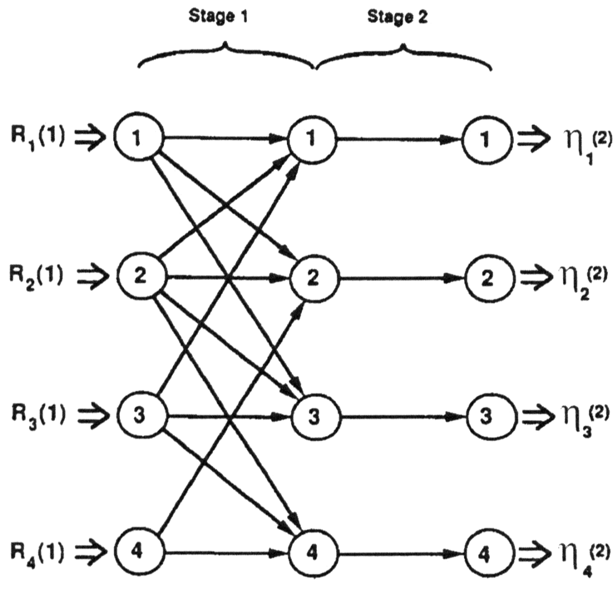

Figure 3.1: The two-stage stochastic transportation problem with the simple approximation

with

$$x_{ii}(2) + x_{ii}^{-}(2) - x_{ii}^{+}(2) = \eta_i(2) \tag{3.1}$$

$$x_{ij}(2) = 0 \qquad i \neq j \tag{3.2}$$

where $x_{ii}^{+}(2)$ and $x_{ii}^{-}(2)$ are the *recourse variables* and are given by

$$x_{ii}^{+}(2) = \max\left[x_{ii}(2) - \eta_i(2), 0\right] \tag{3.3}$$

$$x_{ii}^{-}(2) = \max\left[\eta_i(2) - x_{ii}(2), 0\right]. \tag{3.4}$$

The simple approximation for the two-stage transportation problem with random demands with four cities is illustrated in the Figure 3.1.

Let q^+ be the vector of the salvage values of excess supply and q^- be the vector of penalty for unsatisfied demand. Then we get the following recourse function:

$$\begin{aligned}\Phi(S(1), \eta(2)) = &\sum_{i \in \mathbf{R}} c_{ii}(2)\, S_i(1) \\ &+ \sum_{i \in \mathbf{R}} q_i^{+} \max\left[S_i(1) - \eta_i(2), 0\right] + \sum_{i \in \mathbf{R}} q_i^{-} \max\left[\eta_i(2) - S_i(1), 0\right].\end{aligned} \tag{3.5}$$

The right-hand side of (3.5) is a separable function of $\eta_i(2)$ and hence it is fairly easy to take expectation of $\Phi(S(1), \eta(2))$. With respect to this the two-stage stochastic transportation problem with simple approximation can be written as

$$\min_{x(1), S(1)} \{c\,(1)^T x(1) + E_{\eta(2)}[\Phi(S(1), \eta(2))]\} \tag{3.6}$$

subject to

$$\sum_{j\in\mathbf{R}} x_{ij}(1) = R_i(1) \qquad \forall i \in \mathbf{R} \tag{3.7}$$

$$\sum_{i\in\mathbf{R}} x_{ij}(1) = S_j(1) \qquad \forall j \in \mathbf{R} \tag{3.8}$$

$$x_{ij}(1) \geq 0 \qquad \forall i, j \in \mathbf{R}. \tag{3.9}$$

If $\eta(2)$ is a continuous (discrete) random variable then

$$\overline{\Phi}(S(1)) = E_{\eta(2)}[\Phi(S(1), \eta(2))]$$

becomes a nonlinear (piecewise linear) separable function of $x(1)$.

Analogically we can rewrite the N-stage transportation problem with random arc capacities. First, we put

$$x_{ij}(t) + x_{ij}^-(t) - x_{ij}^+(t) = \xi_{ij}(t), \tag{3.10}$$

where again

$$x_{ij}^+(t) = \max\left[x_{ij}(t) - \xi_{ij}(t), 0\right] \tag{3.11}$$

$$x_{ij}^-(t) = \max\left[\xi_{ij}(t) - x_{ij}(t), 0\right]. \tag{3.12}$$

We assume that $x_{ij}(t)$ must be chosen before the realization of random variable $\xi_{ij}(t)$ is observed while $x_{ij}^+(t)$ and $x_{ij}^-(t)$ must be computed for a given realization of $\xi_{ij}(t)$ using relations (3.11) and (3.12). $x^-(t)$ can be interpreted as a lost demand and is associated with a penalty q^- while $x^+(t)$ represents a movement of flow which does not produce any income and is connected with a lost profit q^+.

This notation implies that in the dynamic vehicle allocation problem $x_{ij}^+(t)$ represents vehicles which are moving empty at a positive cost because of insufficient demand. Thus $x_{ij}(t)$ is the total number of vehicles allocated to move from city i to city j in the t-th stage and $x_{ij}(t) - x_{ij}^+(t)$ are vehicles moving loaded and producing profit.

Let $\overline{\phi}_{ij}(x(t))$ be the expected recourse function for stage $t = 2, \ldots, P$ for the flows between cities $i \in \mathbf{R}$ and $j \in \mathbf{R}$, defined by

$$\overline{\phi}_{ij}(x(t)) = E_{\xi_{ij}(t)}[q_{ij}^- \, x_{ij}^-(t) + q_{ij}^+ \, x_{ij}^+(t)], \tag{3.13}$$

where $x^+(t)$ and $x^-(t)$ are given by (3.11) and (3.12).

If we use this relation we can rewrite the N-stage transportation problem with random arc capacities (2.21) - (2.32) as follows

$$\min_{x(1), S(1)} \; \{c(1)^T x(1) + \overline{\Phi}(S(1))\} \tag{3.14}$$

subject to

$$\sum_{j\in\mathbf{R}} x_{ij}(1) = R_i(1) \qquad \forall i \in \mathbf{R} \tag{3.15}$$

$$\sum_{i\in\mathbf{R}} x_{ij}(1) = S_j(1) \qquad \forall j \in \mathbf{R} \tag{3.16}$$

$$x_{ij}(1) \geq 0 \qquad \forall i, j \in \mathbf{R} \tag{3.17}$$

$$x_{ij}(1) \leq u_{ij}(1) \qquad \forall i, j \in \mathbf{R}, \tag{3.18}$$

with

$$\overline{\Phi}(S(t-1)) = E_{\xi(t)}[\Phi(S(t-1), \xi(t))] \qquad t = 2, \ldots, P \tag{3.19}$$

$$\overline{\Phi}(S(P)) = 0. \tag{3.20}$$

The function $\overline{\Phi}(S(t-1))$ is the expected recourse function for the t-th stage and is for this case of recourse defined by the following recursion:

$$\Phi(S(t-1), \xi(t)) = \min_{x(t), S(t)} \left\{ \sum_{i \in \mathbf{R}} \sum_{j \in \mathbf{R}} \{c_{ij}(t)\, x_{ij}(t) + \overline{\phi}_{ij}(x(t))\} + \overline{\Phi}(S(t)) \right\} \tag{3.21}$$

subject to

$$\sum_{j \in \mathbf{R}} x_{ij}(t) = S_i(t-1) + R_i(t) \qquad \forall i \in \mathbf{R} \tag{3.22}$$

$$\sum_{i \in \mathbf{R}} x_{ij}(t) = S_j(t) \qquad \forall j \in \mathbf{R} \tag{3.23}$$

$$x_{ij}(t) \geq 0 \qquad \forall i, j \in \mathbf{R}. \tag{3.24}$$

We can notice that (3.21) - (3.24) are no longer functions of $\xi(t)$ because it has already been included into the functions $\overline{\phi}(x_{ij}(t))$. This implies that

$$\overline{\Phi}(S(t-1)) = \Phi(S(t-1), \xi(t)).$$

Thus the whole problem (3.14) - (3.24) can be rewritten as a single optimization problem

$$\min_{x(1), \ldots, x(P)} \left\{ c\,(1)^T x(1) + \sum_{t=2}^{P} \left(c\,(t)^T x(t) + \sum_{i \in \mathbf{R}} \sum_{j \in \mathbf{R}} \overline{\phi}_{ij}(x(t)) \right) \right\} \tag{3.25}$$

subject to (3.18) and (3.22) - (3.24) where $S_i(0) = 0$ for all $i \in \mathbf{R}$. We get now a classical convex, nonlinear network flow problem which can be solved using standard techniques (see [11]).

Using this approach we can replace a complex nonseparable recourse function with a separable one. But this approach does not correspond with common realistic situations. Thus it is unlikely that simple approximation could succeed in more practical applications.

3.2.1 Application of Simple Approximation for Two-Stage Transportation Problem with Random Demands

We will consider the following stochastic transportation problem – there is a security agency that transports daily takings from supermarkets to banks. We naturally do not know these daily takings exactly in advance, we know only their (discrete) distributions that has been estimated from past dates. Our goal is to allocate vehicles on individual routes to minimize our costs. We will define this problem as the two-stage

	1	2	3	4	5	6	7	8	9	10
dp1	3000	3100	3200	3300	3500	3700	3900	4000		
dp2	6000	6200	6500	7000						
dp3	2400	2500	2600	2800	2900	3000	3100	3200	3500	3700
dp4	1500	1800	2000	2500	3000	4400				
dp5	6800	7000	7200	7300	7400	7700	7800			
dp6	1900	2000	2100	2300	2500					
dp7	3200	3300	3500							
dp8	6100	6200	6400	6500	6600					
dp9	7800	7900	8000	8100	8400	8500	8700			
dp10	2200	2400	2500	2800						
dp11	4000	4400								
dp12	7600	7700	7800	8000	8100					
dp13	1300	1400	1500	1600	1700	1800	1900	2000	2100	2200
dp14	8000	8200	8500	8600						
dp15	5200	5300	5500	5600	5800	6000				
dp16	6500	6600	6800	7000	7500					
dp17	1900	2000	2200	2500						
dp18	800	1000	1100	1200	1500	1700				
dp19	600	700	800	1000	1200	1500				
dp20	6700	6900	7000	7700						

Table 3.1: Individual possible realizations of the random demands $\eta_j(2)$

transportation problem with random demands and will use the simple approximation to solve it.

Our security agency has three depots ($\mathbf{I} = \{a, b, c\}$) and has to transport money from twenty supermarkets ($\mathbf{J} = \{dp1, \ldots, dp20\}$). Thus the set of all cities $\mathbf{R}$ has been divided into two disjunctive sets $\mathbf{I}$ and $\mathbf{J}$. We assume that the distances between banks and depots are insignificant in comparison with other distances and hence we will consider transportation costs only for transporting money from supermarkets to depots.

The amount of units of money in individual demand points (supermarkets) is a random variable. Let us denote this random variable for supermarket $j \in \mathbf{J}$ as $\eta_j(2)$ in agreement with the introduced notation. Thus in this stochastic transportation problem random demands will not concern a number of vehicles but an amount of units of money that should be transported and will be related to vehicles capacity. We made a study, to find out what realization of the random variables $\eta_j(2)$ can arise. Denoted the set of possible scenarios for the future values of these random variables as $\mathbf{H}$, where $\mathbf{H} = \{1, \ldots, 10\}$. Hence $\eta_{j,h}(2)$ denotes individual realizations of this random variable and states an amount of units of money that should be transported from demand point $j \in \mathbf{J}$ for scenario $h \in \mathbf{H}$. A probability of a realization of $\eta_{j,h}(2)$ is denoted as $\lambda_{j,h}(2)$. The tables 3.1 and 3.2 give the values of $\eta_{j,h}(2)$ and $\lambda_{j,h}(2)$.

In depots there are three types of vehicles ($\mathbf{K} = \{veh1, veh2, veh3\}$) at our disposal of different capacities. We denote the capacity of the vehicle of the type $k \in \mathbf{K}$ on the route from the depot $i \in \mathbf{I}$ to the demand point $j \in \mathbf{J}$ or back as B_{ijk}.

	1	2	3	4	5	6	7	8	9	10
dp1	0.05	0.05	0.3	0.1	0.1	0.25	0.08	0.07		
dp2	0.2	0.4	0.1	0.3						
dp3	0.05	0.1	0.1	0.08	0.12	0.05	0.2	0.15	0.05	0.1
dp4	0.15	0.15	0.3	0.2	0.1	0.1				
dp5	0.2	0.4	0.1	0.05	0.15	0.05	0.05			
dp6	0.2	0.1	0.1	0.1	0.5					
dp7	0.4	0.3	0.3							
dp8	0.3	0.1	0.1	0.1	0.4					
dp9	0.15	0.15	0.15	0.15	0.15	0.15	0.1			
dp10	0.2	0.3	0.1	0.4						
dp11	0.6	0.4								
dp12	0.05	0.15	0.25	0.35	0.2					
dp13	0.05	0.08	0.08	0.07	0.15	0.07	0.1	0.2	0.1	0.1
dp14	0.25	0.25	0.25	0.25						
dp15	0.1	0.05	0.05	0.4	0.2	0.2				
dp16	0.15	0.1	0.25	0.15	0.35					
dp17	0.4	0.2	0.2	0.2						
dp18	0.05	0.1	0.15	0.35	0.25	0.1				
dp19	0.15	0.15	0.2	0.3	0.1	0.1				
dp20	0.2	0.15	0.15	0.5						

Table 3.2: Probabilities $\lambda_{j,h}(2)$ of realizations of the random demands $\eta_j(2)$

	veh1	veh2	veh3
a	4	12	3
b	6	16	2
c	6	8	3

Table 3.3: The vehicles availability in individual depots – $A_{ik}(1)$

For simplicity we will suppose that a given vehicle has the same capacity on all routes. A vehicle of the first type has the capacity of 1000 units, of the second type 1500 units and of the third type 3000 units of money.

We denote number of vehicles of the type $k \in \mathbf{K}$ in the depot $i \in \mathbf{I}$ as $A_{ik}(1)$. Thus the vector

$$A(1) = (A_{ik}(1) : i \in \mathbf{I},\ k \in \mathbf{K})$$

corresponds to an initial vector of supplies that has been in the past denoted as $R(1)$. However we decided to use different notation since it is evident that in this case no all vehicles have to leave depots. The table 3.3 gives this types of vehicles availability in individual depots.

Tables 3.4, 3.5 and 3.6 state operating costs per a vehicle of individual types on the routes between individual depots and demand points. Again with respect to prior mentioned notation we denote these costs for a vehicle of the type $k \in \mathbf{K}$ on the route from the depot $i \in \mathbf{I}$ to the demand point $j \in \mathbf{J}$ as $c_{ijk}(1)$.

Integer decision variables $x_{ijk}(1)$ represent in this stochastic transportation problem

	a	b	c
dp1	18	42	16
dp2	21	56	42
dp3	16	48	29
dp4	10	34	35
dp5	17	26	26
dp6	6	35	42
dp7	26	31	38
dp8	12	20	15
dp9	19	42	4
dp10	20	26	44
dp11	8	27	19
dp12	14	13	21
dp13	15	56	33
dp14	18	48	40
dp15	32	39	27
dp16	17	49	10
dp17	43	40	51
dp18	36	34	12
dp19	23	26	30
dp20	15	8	16

Table 3.4: The operating costs per a vehicle of the first type – $c_{ij1}(1)$

a number of vehicles of the type $k \in \mathbf{K}$ that are allocated to route from the depot $i \in \mathbf{I}$ to the supermarket $j \in \mathbf{J}$. Then the variable

$$y_j(1) = \sum_{i \in \mathbf{I}} \sum_{k \in \mathbf{K}} B_{ijk}\, x_{ijk}(1)$$

gives the total vehicles capacity that is at our disposal in the demand point $j \in \mathbf{J}$ at the beginning of the second stage. These variables have analogical meaning to the variables $S_i(1)$ earlier mentioned but the variables $S_i(1)$ state flows of vehicles through node $(i, 1)$ and variables $y_j(1)$ give flows of vehicles capacity through node $(j, 1)$. Since we want to find one vector

$$x(1) = (x_{ijk}(1) : i \in \mathbf{I},\ j \in \mathbf{J},\ k \in \mathbf{K})$$

for each possible realization of the vector

$$\eta(2) = (\eta_j(2) : j \in \mathbf{J}),$$

it is obvious that the variables $y_j(1)$ cannot depend on $\eta(2)$, too.

Now we can proceed to define the *recourse variables* for our problem needed for the simple approximation:

$$y_j^+(1) = \max\,[\eta_j(2) - y_j(1), 0]$$

$$y_j^-(1) = \max\,[y_j(1) - \eta_j(2), 0].$$

	a	b	c
dp1	24	47	22
dp2	28	60	47
dp3	22	53	34
dp4	16	40	40
dp5	24	33	33
dp6	13	40	47
dp7	33	36	43
dp8	18	27	22
dp9	25	47	10
dp10	27	50	49
dp11	15	30	25
dp12	20	19	28
dp13	22	60	38
dp14	24	53	45
dp15	38	44	32
dp16	24	55	16
dp17	48	44	55
dp18	42	41	18
dp19	27	33	33
dp20	22	15	22

Table 3.5: The operating costs per a vehicle of the second type – $c_{ij2}(1)$

If we send in the first stage to some supermarket vehicles whose total capacity does not suffice to future random demand, we will have to leave some units of money in this supermarket. The recourse variables $y_j^+(1)$ state the amount of units of money we refused to transport from individual demand points because of lack of capacity in vehicles. This situation in the supermarket $j \in \mathbf{J}$ is connected with an additional cost q_j^+ since we have to pay some extra security guards to guard this higher cash.

If we send in the first stage to some supermarket vehicles whose total capacity exceeds future random demand, some free place stays in these vehicles. The recourse variables $y_j^-(1)$ give the size of this free space for individual demand points. This situation in the supermarket $j \in \mathbf{J}$ is connected with a lost profit q_j^- since we could send smaller vehicles to this demand point and use the bigger vehicles somewhere else.

The table 3.7 gives the values of the additional costs q_j^+ and of the lost profits q_j^- for individual demand points.

Further we denote the individual possible values of the variables $y_j^+(1)$ and $y_j^-(1)$ for individual realizations of the random demand $\eta_{j,h}(2)$ as $y_{j,h}^+(1)$ and $y_{j,h}^-(1)$. These variables are given by

$$y_{j,h}^+(1) = \max\left[\eta_{j,h}(2) - y_j(1), 0\right]$$

$$y_{j,h}^-(1) = \max\left[y_j(1) - \eta_{j,h}(2), 0\right].$$

With respect to this notation it is obvious that variable $y_j(1) - y_{j,h}^-(1)$ states an amount of units of money that is transported from the supermarket $j \in \mathbf{J}$ for the scenario

	a	b	c
dp1	28	50	25
dp2	30	65	50
dp3	25	58	39
dp4	19	44	46
dp5	28	36	36
dp6	17	45	50
dp7	36	41	49
dp8	22	30	25
dp9	28	55	15
dp10	30	36	54
dp11	19	35	28
dp12	24	23	30
dp13	25	65	43
dp14	28	58	49
dp15	40	48	36
dp16	28	60	19
dp17	53	47	60
dp18	46	46	22
dp19	30	36	40
dp20	25	19	25

Table 3.6: The operating costs per a vehicle of the third type – $c_{ij3}(1)$

$h \in \mathbf{H}$.

Now we can deal with solution of our stochastic transportation problem. We want to allocate our vehicles on individual routes between depots and supermarkets to serve all demand points and to minimize our costs. Our costs can be divided into an operating expense for routes of individual vehicles (costs of the first stage) and into an expected value of penalty for insufficient supply of vehicles and an expected lost profit for excessive supply of vehicles (costs of the second stage).

Thus the first-stage costs can be defined as

$$\sum_{i\in\mathbf{I}}\sum_{j\in\mathbf{J}}\sum_{k\in\mathbf{K}} c_{ijk}(1)\, x_{ijk}(1)$$

and the second-stage costs are given with respect to the shape of (3.5) by

$$E_{\eta(2)}\left[\sum_{j\in\mathbf{J}} q_j^+\, y_j^+ + \sum_{j\in\mathbf{J}} q_j^-\, y_j^-\right] =$$

$$= E_{\eta(2)}\left[\sum_{j\in\mathbf{J}} q_j^+ \max\left[\eta_j(2) - y_j(1), 0\right] + \sum_{j\in\mathbf{J}} q_j^- \max\left[y_j(1) - \eta_j(2), 0\right]\right]$$

since we assume $c_{jj}(2) = 0$ for all $j \in \mathbf{J}$.

For the discrete distribution of the random demand $\eta_j(2)$ in the supermarket $j \in \mathbf{J}$ this relation can be rewritten as

$$\sum_{h\in\mathbf{H}}\sum_{j\in\mathbf{J}} \lambda_{j,h}(2)\ \left[q_j^+\ \max\left[\eta_{j,h}(2) - y_j(1), 0\right] + q_j^-\ \max\left[y_j(1) - \eta_{j,h}(2), 0\right]\right]$$

individual demand points				q_j^+	q_j^-
dp1	dp12	dp14	dp18	0.02	0.006
dp2	dp5	dp11	dp17	0.002	0.0007
dp3	dp4	dp9	dp20	0.005	0.014
dp6	dp7	dp13	dp16	0.008	0.003
dp8	dp10	dp15	dp19	0.01	0.004

Table 3.7: The additional costs q_j^+ and the lost profits q_j^- for individual demand points

which is

$$\sum_{h\in\mathbf{H}}\sum_{j\in\mathbf{J}}\lambda_{j,h}(2)\ \left[q_j^+\, y_{j,h}^+(1) + q_j^-\, y_{j,h}^-(1)\right].$$

Hence we will minimize the following objective function:

$$\sum_{i\in\mathbf{I}}\sum_{j\in\mathbf{J}}\sum_{k\in\mathbf{K}} c_{ijk}(1)\, x_{ijk}(1)$$

$$+\sum_{h\in\mathbf{H}}\sum_{j\in\mathbf{J}}\lambda_{j,h}(2)\ \left[q_j^+\ \max\left[\eta_{j,h}(2) - y_j(1), 0\right] + q_j^-\ \max\left[y_j(1) - \eta_{j,h}(2), 0\right]\right] \tag{3.26}$$

with respect to $x(1)$ subject to the following restrictions:

$$\sum_{j\in\mathbf{J}} x_{ijk}(1) \le A_{ik}(1) \qquad \forall i\in\mathbf{I},\ \forall k\in\mathbf{K} \tag{3.27}$$

$$y_j(1) = \sum_{i\in\mathbf{I}}\sum_{k\in\mathbf{K}} B_{ijk}\, x_{ijk}(1) \qquad \forall j\in\mathbf{J} \tag{3.28}$$

$$x_{ijk} \ge 0 \qquad \forall i\in\mathbf{I},\ \forall j\in\mathbf{J},\ \forall k\in\mathbf{K}. \tag{3.29}$$

The relation (3.27) gives a restriction on a number of vehicles of given type we can allocate to route from given depot to all demand points and the equation (3.28) defines the total amount of units of money that maximally can be transported from individual demand points at the beginning of the second stage. Of course, a number of vehicles we allocate to move on individual routes between depots and supermarkets has to be a non-negative variable. Moreover, as we already said above, we will even demand the decision variables $x_{ijk}(1)$ to be integer non-negative variables. It is obvious from the relation (3.28) that if $x_{ijk}(1)$ are non-negative variables than the variables $y_j(1)$ have to be non-negative, too, since the vehicles capacities B_{ijk} are positive.

We used the model system GAMS and solver MINOS 5.51 to solving this stochastic transportation problem. The source program can be found on the appendant CD in the file *Two Stage.gms* and the reduced program output in the file *Two Stage.txt*.

Here are presented only the values of the decision variables. The following table gives the numbers of vehicles of individual types that are allocated to route from individual depots to individual supermarkets. The missing values are equal to zero.

The objective function value of this solution is 619.82 units – the individual vehicles routes in the first stage cost 278 units and the expected value of penalty for insufficient supply of vehicles is 341.82 units. No lost profit for excessive supply of vehicles is expected with this solution.

```
VARIABLE x1.L  number of vehicles of type k routing from depot i
                                to demand point j

               veh1         veh2         veh3
a.dp6          2.000
a.dp14                                   3.000
b.dp12                      1.000        2.000
c.dp1                                    1.000
c.dp8                                    2.000
c.dp9          6.000
c.dp18                      1.000
```

3.2.2 Application of Simple Approximation for the N-Stage Transportation Problem with Random Arc Capacities

In this part we will consider similar stochastic transportation problem as in the previous section but now it will be defined as the N-stage transportation problem with random arc capacities. Again, we will use the simple approximation to solve it.

In this case our security agency delivers every day money from three bigger branches of some bank to twenty local branches of this bank (set $\mathbf{J}$). We assume that the distances between depots and bigger branches are insignificant to other distances and hence we will consider only the routes between depots (set $\mathbf{I}$) and local branches (demand points).

The most of the variables and sets have the same meaning and values as in the previous section thus we will present here only these variables and sets meaning of which has changed or that have not been presented yet.

For simplicity we suppose that N is only so high that we exactly know money demands of individual local branches. This is obviously possible since if someone wants to withdraw here bigger cash he has to inform this small branch several days in advance. Money demands for the local branch $j \in \mathbf{J}$ in the t-th stage we denote as $m_j(t)$. Our goal is to allocate vehicles on individual routes to supply with money all demand points and to minimize costs if individual arcs have random capacities.

We can imagine this situation very easily – individual routes from depots to local branches have restricted capacities because of security reason. Thus there exist restrictions on total amount of units of money that can be transported on the given arc in the given time period and these restrictions are random with discrete distributions. With respect to prior mentioned notation we denote these random arc capacities for the depot $i \in \mathbf{I}$ and the demand point $j \in \mathbf{J}$ in the t-th stage as $\xi_{ij}(t)$. For simplicity we will in this part assume that the vectors

$$\xi(t) = (\xi_{ij}(t) : i \in \mathbf{I},\ j \in \mathbf{J})$$

have the same distribution for all time periods $t = 2, \ldots, P$ where P is the planning horizon and is equal to the number of stages N since we assume that every stage consists only of a single time period. (For the first stage we anyway have known arc capacities $u_{ij}(1)$ as has been mentioned in the presentation of the N-stage transportation problem with random arc capacities (2.21) - (2.32).) Because of this we will use only the notation

$$\xi = (\xi_{ij} : i \in \mathbf{I},\ j \in \mathbf{J})$$

for random vectors without stating the stage in the rest of this section.

Further we will assume that

$$\xi_{ij} = \xi_{ji}.$$

This implies that the same random restrictions will hold for routes of empty vehicles, too. Thus individual arcs from demand points to depots (with the exception of the first stage) are limited with the same corresponding random arc capacities and in this case the random variable ξ_{ji} restricts the total capacity of all vehicles allocated to move from the demand point $j \in \mathbf{J}$ to the depot $i \in \mathbf{I}$ in the given stage.

We again denote the set of possible scenarios for the future values of the random variables (for this once ξ_{ij} and ξ_{ji}) as $\mathbf{H}$, where $\mathbf{H} = \{1, \ldots, 10\}$. Since we assume that the vectors $\xi(t)$ have the same distribution for all stages and that $\xi_{ij} = \xi_{ji}$ we do not need more than this one set $\mathbf{H}$. Further $\xi_{ij,h}$ denotes individual realizations of the random variable ξ_{ij} (analogically $\xi_{ji,h}$ denotes individual realizations of the random variable ξ_{ji}) and states an amount of units of money that can be maximally transported from the depot $i \in \mathbf{I}$ to the local branch $j \in \mathbf{J}$ by the scenario $h \in \mathbf{H}$, possibly states the total capacity of all vehicles allocated to route from the local branch $j \in \mathbf{J}$ to the depot $i \in \mathbf{I}$ by the scenario $h \in \mathbf{H}$ since again

$$\xi_{ij,h} = \xi_{ji,h}$$

for all $h \in \mathbf{H}$. We denote as $\lambda_{ij,h}$ the probability of a realization of $\xi_{ij,h}$ and again

$$\lambda_{ij,h} = \lambda_{ji,h}$$

for all $h \in \mathbf{H}$. The tables 3.8, 3.9, 3.10 and 3.11, 3.12, 3.13 give the values of $\xi_{ij,h}$ and $\lambda_{ij,h}$.

Further we will assume that the operating costs $c_{ijk}(t)$ per vehicle of the given type $k \in \mathbf{K}$ from the depot $i \in \mathbf{I}$ to the demand point $j \in \mathbf{J}$ in the t-th stage, where $t = 2, \ldots, P$, are equal to $c_{ijk}(1)$ and that the operating cost does not depend on the direction of the route of the given vehicle; hence

$$c_{ijk}(t) = c_{jik}(t) = c_{ijk}(1) = c_{jik}(1).$$

The values of the operating costs $c_{ijk}(1)$ are the same as in the previous section and are stated in the tables 3.4, 3.5 and 3.6.

The decision variables $x_{ijk}(t)$ (for odd stages) logically represent in this stochastic transportation problem the number of vehicles of the type $k \in \mathbf{K}$ that are allocated to route from the depot $i \in \mathbf{I}$ to the local branch $j \in \mathbf{J}$ and the decision variables $x_{jik}(t)$ (for even stages) represent the number of vehicles of the type $k \in \mathbf{K}$ that are allocated to move from the demand point $j \in \mathbf{J}$ to the depot $i \in \mathbf{I}$. We again want all decision variables to be integer. In this case we have two types of decision variables since in the odd stages we send our vehicles loaded from depots to branches and in the even stages all vehicles return empty from demand points back to depots. For simplicity we suppose that in odd stages vehicles move on individual arcs fully loaded and that they leave their whole loads in the demand points they have been allocated to route to. We do not consider some additional costs connected with excessive supply. Further we will use the vector $x(t)$ that has the following meaning:

$$x(t) = \begin{cases} (x_{ijk}(t) : i \in \mathbf{I},\ j \in \mathbf{J},\ k \in \mathbf{K}) & \text{for odd stages } t \\ (x_{jik}(t) : j \in \mathbf{J},\ i \in \mathbf{I},\ k \in \mathbf{K}) & \text{for even stages } t \end{cases}$$

	1	2	3	4	5	6	7	8	9	10
dp1	800	900	1000	1200	1400	1800				
dp2	1600	1700	1800	2000	2200	2400	2500			
dp3	400	600	700	900	1000	1300				
dp4	1500	1700	2000							
dp5	2500	2600	2700	2800	2900	3000	3100	3200	3300	3400
dp6	100	200	400	500	600					
dp7	2500	2600	2800							
dp8	3000	3200	3300	3600						
dp9	3200	3500	3600	3700	4000					
dp10	900	1100	1500							
dp11	1200	1300	1400	1500	1800	1900				
dp12	3000	3200	3500	3700						
dp13	200	400	500	600	700	900	1000			
dp14	2500	2600	2700	2800	2900	3100	3300	3400	3600	
dp15	1600	1700	1900	2100	2200					
dp16	2400	2600	2700	2800	3000	3200	3500	3600	3800	4000
dp17	500	600	800	900						
dp18	100	200	300	400	500	600				
dp19	200	500	600	900						
dp20	2800	3000	3300							

Table 3.8: Individual possible realizations of the random arc capacities ξ_{ij} for arcs leading from the depot a – values of $\xi_{aj,h}$

and the vector of the operating costs for the t-th stage $c(t)$ is defined as

$$c(t) = \begin{cases} (c_{ijk}(t) : i \in \mathbf{I},\ j \in \mathbf{J},\ k \in \mathbf{K}) & \text{for odd stages } t \\ \\ (c_{jik}(t) : j \in \mathbf{J},\ i \in \mathbf{I},\ k \in \mathbf{K}) & \text{for even stages } t. \end{cases}$$

Analogically to prior mentioned notation the variables $S_{ik}(t)$ and $S_{jk}(t)$ state the flow of vehicles of the given type $k \in \mathbf{K}$ in the stage t through the depot $i \in \mathbf{I}$ and through the demand point $j \in \mathbf{J}$, respectively. These flows can be routed from nodes in the following stage. Further in this part the following vectors will appear:

$$S(t) = \begin{cases} (S_{ik}(t) : i \in \mathbf{I},\ k \in \mathbf{K}) & \text{for odd stages } t \\ \\ (S_{jk}(t) : j \in \mathbf{J},\ k \in \mathbf{K}) & \text{for even stages } t. \end{cases}$$

Now we need to set up new variable $L_{ik}(t)$ for the odd stages t. This variable gives a number of vehicles of the given type $k \in \mathbf{K}$ that have not been allocated to route in the t-th stage from the depot $i \in \mathbf{I}$ and that can be used in the $(t+1)$-stage. In odd stages we send loaded vehicles from depots to demand points and we do not need to use all vehicles we have at our disposal in individual depots. Than in even stages all vehicles must leave branches and return empty to some depot. These vehicles determine the flow through node $i \in \mathbf{I}$ and their number must be increased for the number of vehicles that have stayed in this depot $i \in \mathbf{I}$ in the past. These all vehicles can be allocated to routes in the following stage.

	1	2	3	4	5	6	7	8	9	10
dp1	800	1000	1200	1300						
dp2	2800	2900	3200	3500	3800					
dp3	1200	1300	1500	1600	1800	2000	2200	2400	2500	
dp4	200	300	500	600	800					
dp5	2000	2100	2300	2500	2800					
dp6	2500	2600	2800	2900	3100	3200	3400	3500		
dp7	100	300	400	600						
dp8	2800	2900	3100	3500	3600					
dp9	1300	1400	1600	1700	2000					
dp10	800	900	1100	1300						
dp11	1200	1300	1700	1900	2100					
dp12	3800	3900	4100	4300	4400	4700	5000	5200		
dp13	1300	1500	1700	1800	2300					
dp14	1000	1100	1300	1500	1700					
dp15	2600	2700	2900	3100	3300	3400	3700	3900	4100	4200
dp16	2500	2700	2800	3000	3300	3500				
dp17	2000	2300	2600	2700	2900	3100	3300			
dp18	400	500	700	800	1000	1100	1500			
dp19	1400	1500	1800	2100						
dp20	2800	2900	3300							

Table 3.9: Individual possible realizations of the random arc capacities ξ_{ij} for arcs leading from the depot b – values of $\xi_{bj,h}$

For our illustrative example of the use of the simple approximation for the N-stage transportation problem with the random arc capacities we will consider the planning horizon P (and consequently the number of stages N as well) equal to 4 (hence we want to plan two transports of money). It will be obvious how our example can be extended for longer planning horizon. With respect to the previous definition of the N-stage transportation problem with random arc capacities and to the introduced notation our stochastic transportation problem can be written as

$$\min_{x(1),S(1)} \{c(1)^T x(1) + \overline{Q}(S(1))\} \tag{3.30}$$

subject to

$$\sum_{j\in\mathbf{J}} x_{ijk}(1) \leq A_{ik}(1) \qquad \forall i \in \mathbf{I},\ \forall k \in \mathbf{K} \tag{3.31}$$

$$\sum_{i\in\mathbf{I}} x_{ijk}(1) = S_{jk}(1) \qquad \forall j \in \mathbf{J},\ \forall k \in \mathbf{K} \tag{3.32}$$

$$\sum_{i\in\mathbf{I}}\sum_{k\in\mathbf{K}} B_{ijk}\, x_{ijk}(1) \geq m_j(1) \qquad \forall j \in \mathbf{J} \tag{3.33}$$

$$\sum_{k\in\mathbf{K}} B_{ijk}\, x_{ijk}(1) \leq u_{ij}(1) \qquad \forall i \in \mathbf{I},\ \forall j \in \mathbf{J} \tag{3.34}$$

$$x_{ijk}(1) \geq 0 \qquad \forall i \in \mathbf{I},\ \forall j \in \mathbf{J},\ \forall k \in \mathbf{K}. \tag{3.35}$$

	1	2	3	4	5	6	7	8	9	10
dp1	1400	1500	1800	1900	2200					
dp2	1500	1800	1900	2100	2300	2400	2700			
dp3	900	1100	1200	1300	1600	1900				
dp4	600	700	800	900	1000	1300	1400	1600		
dp5	2900	3000	3200	3300	3500	3700				
dp6	100	200	300							
dp7	1200	1500	1600	2000						
dp8	1100	1500								
dp9	2700	2900	3000	3500	3600					
dp10	1100	1300	1400	1700						
dp11	900	1000	1300	1400	1600	1700				
dp12	800	900	1100	1300	1500	1700	1900			
dp13	100	300	400	700						
dp14	5500	5600	5800	5900	6000	6300	6400	6500	6600	6800
dp15	1400	1500	1700	1900	2100	2200				
dp16	1500	1700	1800	1900	2200	2500	2600			
dp17	400	500	700	800	1200					
dp18	1500	1900	2000	2200	2300	2500				
dp19	200	500								
dp20	1000	1100	1300	1500	1800	1900	2100			

Table 3.10: Individual possible realizations of the random arc capacities ξ_{ij} for arcs leading from the depot c – values of $\xi_{cj,h}$

The relation (3.31) gives a restriction on a number of vehicles of given type we can maximally allocate to route from given depot to all demand points in the first stage (values of the variables $A_{ik}(1)$ are in the table 3.3) and the equation (3.32) defines the total number of vehicles of the given type that arrived to the demand point $j \in \mathbf{J}$ during the first stage from all depots and that can set off from this demand points at the beginning of the second stage. The inequality (3.33) says that we have to fulfil or exceed money demands $m_j(1)$ of individual local branches (money demands of individual local branches in the first stage $m_j(1)$ are in the table 3.14) and the (3.34) states the restriction on the total amount of units of money that can be transported on the arc from the depot $i \in \mathbf{I}$ to the local branch $j \in \mathbf{J}$ in the first stage. This amount is limited by arc capacity $u_{ij}(1)$ that is given for individual arcs in the table 3.15. Of course, numbers of vehicles we allocate to move on individual routes between depots and demand points in the first stage has to be non-negative variables. Non-negativity of the decision variables will run for the rest of the stages, too.

It is obvious that

$$L_{ik}(1) = A_{ik}(1) - \sum_{j \in \mathbf{J}} x_{ijk}(1) \qquad \forall i \in \mathbf{I}, \forall k \in \mathbf{K}.$$

The expected recourse function $\overline{Q}(S(1))$ is defined as

$$\overline{Q}(S(1)) = E_{\xi}\left[Q(S(1), \xi \right] \tag{3.36}$$

	1	2	3	4	5	6	7	8	9	10
dp1	0.2	0.1	0.5	0.3	0.5	0.3				
dp2	0.5	0.15	0.2	0.3	0.5	0.15	0.1			
dp3	0.1	0.3	0.15	0.5	0.25	0.15				
dp4	0.2	0.5	0.3							
dp5	0.5	0.1	0.15	0.5	0.5	0.25	0.15	0.1	0.5	0.5
dp6	0.3	0.35	0.5	0.15	0.15					
dp7	0.4	0.25	0.35							
dp8	0.2	0.25	0.25	0.3						
dp9	0.25	0.15	0.4	0.2						
dp10	0.5	0.5	0.45							
dp11	0.1	0.2	0.5	0.5	0.15	0.45				
dp12	0.3	0.35	0.35							
dp13	0.15	0.25	0.5	0.2	0.15	0.1	0.1			
dp14	0.5	0.15	0.5	0.1	0.35	0.1	0.1	0.5	0.5	
dp15	0.3	0.2	0.15	0.15	0.2					
dp16	0.5	0.5	0.1	0.1	0.2	0.5	0.2	0.5	0.1	0.1
dp17	0.5	0.4	0.25	0.3						
dp18	0.3	0.1	0.1	0.15	0.5	0.3				
dp19	0.25	0.25	0.25	0.25						
dp20	0.6	0.3	0.1							

Table 3.11: Probabilities $\lambda_{ij,h}$ of realizations of the random arc capacities ξ_{ij} for arcs leading from the depot a – values of $\lambda_{aj,h}$

where

$$Q(S(1),\xi) = \min_{x(2),S(2)} \{c(2)^T x(2) + \overline{Q}(S(2))\} \tag{3.37}$$

subject to

$$\sum_{i\in \mathbf{I}} x_{jik}(2) = S_{jk}(1) \qquad \forall j \in \mathbf{J},\ \forall k \in \mathbf{K} \tag{3.38}$$

$$\sum_{j\in \mathbf{J}} x_{jik}(2) = S_{ik}(2) \qquad \forall i \in \mathbf{I},\ \forall k \in \mathbf{K} \tag{3.39}$$

$$\sum_{k\in \mathbf{K}} B_{jik}\, x_{jik}(2) \le \xi_{ji} \qquad \forall j \in \mathbf{J},\ \forall i \in \mathbf{I} \tag{3.40}$$

$$x_{jik}(2) \ge 0 \qquad \forall j \in \mathbf{J},\ \forall i \in \mathbf{I},\ \forall k \in \mathbf{K}. \tag{3.41}$$

The equation (3.38) gives the number of vehicles of given type that have to be allocated to move from individual branches in the second stage, the relation (3.39) defines the total number of vehicles of the given type that arrived to the depot $i \in \mathbf{I}$ during the second stage from all demand points and the (3.40) states the restriction on the total capacity of all vehicles that have been allocated to route on the given arc in the second stage (again $B_{jik} = B_{ijk}$).

Analogically we get the expected second-stage recourse function

$$\overline{Q}(S(2)) = E_\xi\,[\,Q(S(2),\xi\,] \tag{3.42}$$

	1	2	3	4	5	6	7	8	9	10
dp1	0.2	0.4	0.3	0.1						
dp2	0.25	0.35	0.2	0.5	0.15					
dp3	0.5	0.1	0.25	0.5	0.15	0.5	0.1	0.15	0.1	
dp4	0.4	0.15	0.2	0.1	0.15					
dp5	0.2	0.45	0.15	0.1	0.1					
dp6	0.5	0.15	0.1	0.2	0.15	0.5	0.2	0.1		
dp7	0.5	0.15	0.5	0.3						
dp8	0.7	0.5	0.1	0.1	0.5					
dp9	0.4	0.2	0.15	0.15	0.1					
dp10	0.15	0.2	0.5	0.6						
dp11	0.25	0.2	0.15	0.35	0.5					
dp12	0.15	0.2	0.1	0.5	0.1	0.5	0.3	0.5		
dp13	0.2	0.4	0.1	0.5	0.25					
dp14	0.3	0.15	0.25	0.2	0.1					
dp15	0.5	0.15	0.1	0.5	0.2	0.15	0.5	0.1	0.5	0.1
dp16	0.2	0.15	0.5	0.25	0.2	0.15				
dp17	0.15	0.1	0.1	0.5	0.25	0.1	0.25			
dp18	0.25	0.1	0.3	0.5	0.1	0.15	0.5			
dp19	0.5	0.15	0.6	0.2						
dp20	0.75	0.2	0.5							

Table 3.12: Probabilities $\lambda_{ij,h}$ of realizations of the random arc capacities ξ_{ij} for arcs leading from the depot b – values of $\lambda_{bj,h}$

where

$$Q(S(2),\xi) = \min_{x(3),S(3)} \{c(3)^T x(3) + \overline{Q}(S(3))\} \tag{3.43}$$

subject to

$$\sum_{j\in\mathbf{J}} x_{ijk}(3) \le S_{ik}(2) + L_{ik}(1) \qquad \forall i \in \mathbf{I},\ \forall k \in \mathbf{K} \tag{3.44}$$

$$\sum_{i\in\mathbf{I}} x_{ijk}(3) = S_{jk}(3) \qquad \forall j \in \mathbf{J},\ \forall k \in \mathbf{K} \tag{3.45}$$

$$\sum_{i\in\mathbf{I}}\sum_{k\in\mathbf{K}} B_{ijk}\, x_{ijk}(3) \ge m_j(3) \qquad \forall j \in \mathbf{J} \tag{3.46}$$

$$\sum_{k\in\mathbf{K}} B_{ijk}\, x_{ijk}(3) \le \xi_{ij} \qquad \forall i \in \mathbf{I},\ \forall j \in \mathbf{J} \tag{3.47}$$

$$x_{ijk}(3) \ge 0 \qquad \forall i \in \mathbf{I},\ \forall j \in \mathbf{J},\ \forall k \in \mathbf{K}. \tag{3.48}$$

The inequality (3.44) says that the number of vehicles of given type which can be allocated to route from given depot to all branches in the third stage is restricted with the sum of the vehicles which arrived to this depot during the second stage and of the vehicles which stayed here from the first stage. The remaining relations have similar meanings as in the previous stages.

	1	2	3	4	5	6	7	8	9	10
dp1	0.1	0.15	0.2	0.45	0.1					
dp2	0.2	0.15	0.35	0.5	0.15	0.5	0.5			
dp3	0.5	0.2	0.4	0.1	0.15	0.1				
dp4	0.2	0.1	0.5	0.15	0.3	0.5	0.1	0.5		
dp5	0.3	0.2	0.15	0.1	0.2	0.5				
dp6	0.5	0.8	0.15							
dp7	0.6	0.25	0.15							
dp8	0.8	0.2								
dp9	0.15	0.1	0.25	0.15	0.35					
dp10	0.7	0.5	0.1	0.15						
dp11	0.1	0.1	0.5	0.5	0.1	0.15				
dp12	0.4	0.5	0.2	0.15	0.5	0.1	0.5			
dp13	0.2	0.6	0.15	0.5						
dp14	0.5	0.1	0.5	0.15	0.2	0.25	0.5	0.5	0.5	0.5
dp15	0.3	0.1	0.15	0.1	0.25	0.1				
dp16	0.35	0.1	0.25	0.5	0.1	0.5	0.1			
dp17	0.4	0.1	0.15	0.25	0.1					
dp18	0.2	0.1	0.25	0.25	0.1	0.1				
dp19	0.3	0.7								
dp20	0.1	0.2	0.1	0.25	0.1	0.15	0.1			

Table 3.13: Probabilities $\lambda_{ij,h}$ of realizations of the random arc capacities ξ_{ij} for arcs leading from the depot c – values of $\lambda_{cj,h}$

Finally,

$$\overline{Q}(S(3)) = E_\xi\,[\,Q(S(3),\xi\,] \tag{3.49}$$

where again

$$Q(S(3),\xi) = \min_{x(4),S(4)} \{c\,(4)^T x(4) + \overline{Q}(S(4))\} \tag{3.50}$$

subject to

$$\sum_{i\in\mathbf{I}} x_{jik}(4) = S_{jk}(3) \qquad \forall j \in \mathbf{J},\ \forall k \in \mathbf{K} \tag{3.51}$$

$$\sum_{j\in\mathbf{J}} x_{jik}(4) = S_{ik}(4) \qquad \forall i \in \mathbf{I},\ \forall k \in \mathbf{K} \tag{3.52}$$

$$\sum_{k\in\mathbf{K}} B_{jik}\, x_{jik}(4) \leq \xi_{ji} \qquad \forall j \in \mathbf{J},\ \forall i \in \mathbf{I} \tag{3.53}$$

$$x_{jik}(4) \geq 0 \qquad \forall j \in \mathbf{J},\ \forall i \in \mathbf{I},\ \forall k \in \mathbf{K} \tag{3.54}$$

with

$$\overline{Q}(S(4)) = 0.$$

It is obvious that we will not need to use the equation (3.52) in our optimization. The rest of the relations have again similar meanings as before.

	$m_j(1)$	$m_j(3)$
dp1	3000	1200
dp2	6000	7500
dp3	2400	2000
dp4	1500	1500
dp5	6800	6200
dp6	1900	2300
dp7	3200	2900
dp8	6100	6500
dp9	7500	8000
dp10	2200	2600
dp11	4000	3600
dp12	7600	8500
dp13	1300	1000
dp14	8000	7000
dp15	5200	6000
dp16	6500	6300
dp17	1900	2300
dp18	800	1200
dp19	600	300
dp20	6700	7700

Table 3.14: The money demands $m_j(1)$ and $m_j(3)$ of individual local branches in the first and in the third stage

Now we can proceed to introduce the simple approximation for our stochastic transportation problem. First we need to define

$$y_{ij}(t) = \sum_{k \in \mathbf{K}} B_{ijk}\, x_{ijk}(t) \qquad \forall i \in \mathbf{I},\, \forall j \in \mathbf{J},\, t = 3$$

$$y_{ji}(t) = \sum_{k \in \mathbf{K}} B_{jik}\, x_{jik}(t) \qquad \forall j \in \mathbf{J},\, \forall i \in \mathbf{I},\, t = 2, 4.$$

The variable $y_{ij}(t)$ states the total amount of units of money that are transported on the arc from the depot $i \in \mathbf{I}$ to the demand point $j \in \mathbf{J}$ in the t-th stage (we assume that all vehicles are moving fully loaded) and $y_{ji}(t)$ gives the total capacity of the vehicles allocated to move on the arc from the local branch $j \in \mathbf{J}$ to the depot $i \in \mathbf{I}$ in the t-th stage.

In this our stochastic transportation problem we will have the following *recourse variables*:

$$y_{ij}^{+}(t) = \max\left[\, y_{ij}(t) - \xi_{ij}, 0\,\right] \qquad \forall i \in \mathbf{I},\, \forall j \in \mathbf{J},\, t = 3$$

$$y_{ji}^{+}(t) = \max\left[\, y_{ji}(t) - \xi_{ji}, 0\,\right] \qquad \forall j \in \mathbf{J},\, \forall i \in \mathbf{I},\, t = 2, 4.$$

If we send in the t-th stage ($t = 2, 3, 4$) on the given arc vehicles with total capacity higher than the random capacity of this arc (the variables $y_{ij}^{+}(t)$ and $y_{ji}^{+}(t)$ define how much we exceeded the random arc capacities) there arise an additional costs q_{ij}^{+} or q_{ji}^{+}

	a	b	c
dp1	1000	1000	1600
dp2	2000	3000	1800
dp3	1000	1500	1200
dp4	1700	600	1000
dp5	3000	2300	3200
dp6	300	3000	200
dp7	2700	400	1600
dp8	3300	3100	1100
dp9	3600	1500	3200
dp10	1100	1000	1300
dp11	1600	1500	1200
dp12	3300	4600	1200
dp13	700	1700	400
dp14	2800	1400	6200
dp15	1800	3400	1700
dp16	3000	3100	1800
dp17	700	3200	600
dp18	400	800	1900
dp19	600	1700	200
dp20	3100	3400	1400

Table 3.15: The arc capacities $u_{ij}(1)$ for arcs between individual depots and demand points in the first stage

(we assume that the additional costs do not depend on the stage) since we have to pay some extra security guards to guard this higher cash moving on the given arc. We will again suppose that the additional cost does not depend on the direction of the route hence that

$$q_{ij}^{+} = q_{ji}^{+}.$$

Values of additional costs for individual arcs are given in the table 3.16. We do not suppose additional costs connected with total capacity of vehicles moving on the given arc lower than is the random capacity for this arc.

Further we denote the individual possible values of the variables $y_{ij}^{+}(t)$ and $y_{ji}^{+}(t)$ for individual realizations of the random arc capacities $\xi_{ij,h}$ and $\xi_{ji,h}$ as $y_{ij,h}^{+}(t)$ and $y_{ji,h}^{+}(t)$ where

$$y_{ij,h}^{+}(t) = \max\left[\, y_{ij}(t) - \xi_{ij,h}, 0\,\right] \qquad \forall i \in \mathbf{I},\ \forall j \in \mathbf{J},\ \forall h \in \mathbf{H},\ t = 3$$

$$y_{ji,h}^{+}(t) = \max\left[\, y_{ji}(t) - \xi_{ji,h}, 0\,\right] \qquad \forall j \in \mathbf{J},\ \forall i \in \mathbf{I},\ \forall h \in \mathbf{H},\ t = 2, 4.$$

These variables state how much we exceeded the random arc capacities in the t-th stage by the scenario $h \in \mathbf{H}$.

The expected recourse functions for the arc from the depot $i \in \mathbf{I}$ to the local branch $j \in \mathbf{J}$ and for the link from the demand point $j \in \mathbf{J}$ to the depot $i \in \mathbf{I}$ for the t-th stage, where $t = 2, \ldots, 4$, are denoted according to prior mentioned notation as $\overline{\phi}_{ij}(x(t))$ and

	a	b	c
dp1	0.005	0.008	0.002
dp2	0.02	0.01	0.04
dp3	0.003	0.004	0.001
dp4	0.08	0.02	0.05
dp5	0.006	0.009	0.004
dp6	0.002	0.008	0.003
dp7	0.007	0.004	0.009
dp8	0.07	0.05	0.02
dp9	0.03	0.06	0.04
dp10	0.09	0.06	0.06
dp11	0.001	0.003	0.002
dp12	0.004	0.002	0.01
dp13	0.06	0.02	0.03
dp14	0.08	0.04	0.03
dp15	0.002	0.02	0.008
dp16	0.05	0.002	0.006
dp17	0.01	0.06	0.005
dp18	0.06	0.04	0.09
dp19	0.005	0.005	0.008
dp20	0.004	0.003	0.007

Table 3.16: The additional costs q_{ij}^{+} connected with excessive transport on arcs between individual depots and local branches

as $\overline{\phi}_{ji}(x(t))$ and are defined by

$$\overline{\phi}_{ij}(x(t)) = E_{\xi_{ij}}[q_{ij}^{+}\, y_{ij}^{+}(t)] \qquad \forall i \in \mathbf{I},\ \forall j \in \mathbf{J},\ t = 3$$

$$\overline{\phi}_{ji}(x(t)) = E_{\xi_{ji}}[q_{ji}^{+}\, y_{ji}^{+}(t)] \qquad \forall j \in \mathbf{J},\ \forall i \in \mathbf{I},\ t = 2, 4.$$

These expectations can be rewritten thanks to the discrete distribution of the random arc capacities ξ_{ij} and ξ_{ji} as

$$\begin{aligned}
\overline{\phi}_{ij}(x(t)) &= q_{ij}^{+} \sum_{h \in \mathbf{H}} \lambda_{ij,h}\, y_{ij,h}^{+}(t) = \\
&= q_{ij}^{+} \sum_{h \in \mathbf{H}} \lambda_{ij,h} \max\left[y_{ij}(t) - \xi_{ij,h}, 0 \right] = \\
&= q_{ij}^{+} \sum_{h \in \mathbf{H}} \lambda_{ij,h} \max\left[\sum_{k \in \mathbf{K}} B_{ijk}\, x_{ijk}(t) - \xi_{ij,h}, 0 \right]
\end{aligned}$$

and

$$\begin{aligned}\overline{\phi}_{ji}(x(t)) &= q_{ji}^{+} \sum_{h\in\mathbf{H}} \lambda_{ji,h}\, y_{ji,h}^{+}(t) = \\ &= q_{ji}^{+} \sum_{h\in\mathbf{H}} \lambda_{ji,h} \max\left[\, y_{ji}(t) - \xi_{ji,h}, 0\,\right] = \\ &= q_{ji}^{+} \sum_{h\in\mathbf{H}} \lambda_{ji,h} \max\left[\sum_{k\in\mathbf{K}} B_{jik}\, x_{jik}(t) - \xi_{ji,h}, 0\right].\end{aligned}$$

Now we can proceed to deal with our stochastic transportation problem. We already know from the section about simple approximation that the objective function for the N-stage transportation problem with random arc capacities (2.21) - (2.32) with simple approximation can be written as

$$c\,(1)^T x(1) + \sum_{t=2}^{P} \left(c\,(t)^T x(t) + \sum_{i\in\mathbf{R}} \sum_{j\in\mathbf{R}} \overline{\phi}_{ij}(x(t)) \right).$$

Hence our stochastic transportation problem with simple approximation reads

$$\min_{x(1),\,\dots,\,x(4)} \left\{ \sum_{t=1}^{4} c\,(t)^T x(t) + \sum_{i\in\mathbf{I}} \sum_{j\in\mathbf{J}} \left\{ \overline{\phi}_{ij}(x(3)) + \overline{\phi}_{ji}(x(2)) + \overline{\phi}_{ji}(x(4)) \right\} \right\} \tag{3.55}$$

subject to:

$$\sum_{j\in\mathbf{J}} x_{ijk}(1) \le A_{ik}(1) \qquad \forall i \in \mathbf{I},\ \forall k \in \mathbf{K} \tag{3.56}$$

$$\sum_{k\in\mathbf{K}} B_{ijk}\, x_{ijk}(1) \le u_{ij}(1) \qquad \forall i \in \mathbf{I},\ \forall j \in \mathbf{J} \tag{3.57}$$

$$\sum_{j\in\mathbf{J}} x_{ijk}(3) \le S_{ik}(2) + L_{ik}(1) \qquad \forall i \in \mathbf{I},\ \forall k \in \mathbf{K} \tag{3.58}$$

$$\sum_{i\in\mathbf{I}} x_{jik}(t+1) = S_{jk}(t) \qquad \forall j \in \mathbf{J},\ \forall k \in \mathbf{K},\ t = 1,3 \tag{3.59}$$

$$\sum_{i\in\mathbf{I}} x_{ijk}(t) = S_{jk}(t) \qquad \forall j \in \mathbf{J},\ \forall k \in \mathbf{K},\ t = 1,3 \tag{3.60}$$

$$\sum_{j\in\mathbf{J}} x_{jik}(2) = S_{ik}(2) \qquad \forall i \in \mathbf{I},\ \forall k \in \mathbf{K} \tag{3.61}$$

$$\sum_{i\in\mathbf{I}} \sum_{k\in\mathbf{K}} B_{ijk}\, x_{ijk}(t) \ge m_j(t) \qquad \forall j \in \mathbf{J},\ t = 1,3 \tag{3.62}$$

$$x_{ijk}(t) \ge 0 \qquad \forall i \in \mathbf{I},\ \forall j \in \mathbf{J},\ \forall k \in \mathbf{K},\ t = 1,3 \tag{3.63}$$

$$x_{jik}(t) \ge 0 \qquad \forall j \in \mathbf{J},\ \forall i \in \mathbf{I},\ \forall k \in \mathbf{K},\ t = 2,4. \tag{3.64}$$

It is obvious that the conditions (3.58) - (3.61) can be replaced with

$$\sum_{j\in\mathbf{J}} x_{ijk}(3) + \sum_{j\in\mathbf{J}} x_{ijk}(1) \leq \sum_{j\in\mathbf{J}} x_{jik}(2) + A_{ik}(1) \qquad \forall i \in \mathbf{I},\ \forall k \in \mathbf{K} \tag{3.65}$$

$$\sum_{i\in\mathbf{I}} x_{jik}(t+1) = \sum_{i\in\mathbf{I}} x_{ijk}(t) \qquad \forall j \in \mathbf{J},\ \forall k \in \mathbf{K},\ t = 1,3 \tag{3.66}$$

in order to elimination of variables $S_{ik}(t)$, $S_{jk}(t)$ and $L_{ik}(1)$.

We again used model system GAMS to solving this stochastic transportation problem. Unfortunately, none of the solvers we have at our disposal was able to settle up with this problem with integer decision variables. Therefore, we have to content with the solution where decision variables need not be integer.

The source program of this problem can be found on the appendant CD in the file *NStage.gms* and the reduced program output obtained with MINOS 5.51 Solver in the file *NStage.txt*.

Since the results of this problem are extensive we will not present in this place the values of the decision variables as in the previous application. The objective function value of the found solution is 6320.082 units - the individual vehicles routes in all stages form the cost to the extent of 4917.867 units and the expected value of penalty for overrunning the random arc capacities is 1402.215 units.

3.3 Null Recourse

Null recourse (see [7]) is a generalization of simple approximation but still works under very strong assumptions. Simple approximation for N-stage transportation problem with random arc capacities can be imagined as replacing each arc with flow $x_{ij}(t)$ with two other links with flows $x_{ij}(t) - x^+_{ij}(t)$ and $x^+_{ij}(t)$. If $\xi_{ij}(t) < x_{ij}(t)$ then flow will move on the overflow arc. For the dynamic vehicle allocation problem it would mean that if a vehicle cannot move loaded over an arc, it will move over this arc empty.

A better strategy might be to let the overflow fall onto an (unbounded) inventory link instead of onto the overflow arcs. This means that if a vehicle cannot move loaded from some city over the arc it has been allocated to it will stay in this given city till the next stage. Thus null recourse can be viewed as a process when flow spills from a bounded arc onto an inventory link. We can see this type of recourse in the Figure 3.2. Every arc is described with the flow over it, its capacity and with the revenue from one unit of the flow moving on it.

The total spilled flow on the inventory link is given by

$$x^+_{ii}(t) = \sum_{j\in\mathbf{R}} \max\left[x_{ij}(t) - \xi_{ij}(t), 0\right] \tag{3.67}$$

and is connected with a cost q^+_{ii}. With the use of this equation the original expected recourse function for the N-stage transportation problem with random arc capacities (problem (2.21) - (2.32))

$$\overline{Q}(S(t-1)) = E_{\xi(t)}\left[\min_{x(t),S(t)} \{c(t)^T x(t) + \overline{Q}(S(t))\}\right] \tag{3.68}$$

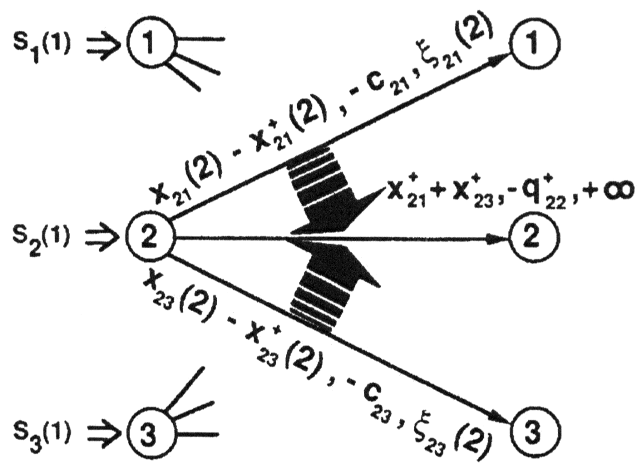

Figure 3.2: The stochastic transportation problem with the null recourse

subject to

$$\sum_{j\in\mathbf{R}} x_{ij}(t) = S_i(t-1) + R_i(t) \qquad \forall i \in \mathbf{R} \tag{3.69}$$

$$\sum_{i\in\mathbf{R}} x_{ij}(t) = S_j(t) \qquad \forall j \in \mathbf{R} \tag{3.70}$$

$$x_{ij}(t) \geq 0 \qquad \forall i,j \in \mathbf{R} \tag{3.71}$$

$$x_{ij}(t) \leq \xi_{ij}(t) \qquad \forall i,j \in \mathbf{R} \tag{3.72}$$

can be replaced by the null recourse approximation

$$\overline{\Phi}(S(t-1)) = E_{\xi(t)}\left[\min_{x(t),S(t)} \left\{\overline{\phi}(x(t)) + \overline{\Phi}(S(t))\right\}\right] \qquad t = 2,\dots,P$$

$$\overline{\Phi}(S(P)) = 0$$

subject to (3.69) - (3.71), where

$$\overline{\phi}(x(t)) = E_{\xi(t)}\left[\sum_{i\in\mathbf{R}}\sum_{j\in\mathbf{R}} \min\left[x_{ij}(t), \xi_{ij}(t)\right] c_{ij}(t) + \sum_{i\in\mathbf{R}}\sum_{j\in\mathbf{R}} \max\left[x_{ij}(t) - \xi_{ij}(t), 0\right] q^+_{ii}\right]. \tag{3.73}$$

We again get problem where

$$\overline{\Phi}(S(t-1)) = \Phi(S(t-1), \xi(t)) = \min_{x(t),S(t)} \left\{\overline{\phi}(x(t)) + \overline{\Phi}(S(t))\right\} \qquad t = 2,\dots,P$$

since $\xi(t)$ have already been included into the functions $\overline{\phi}(x(t))$. Thus the N-stage transportation problem with random arc capacities (2.21) - (2.32) with the null recourse can be written as a single optimization problem

$$\min_{x(1),\dots,x(P)} \left\{c(1)^T x(1) + \sum_{t=2}^{P} \overline{\phi}(x(t))\right\} \tag{3.74}$$

subject to (3.18) and (3.69) - (3.71) where again $S_i(0) = 0$ for all $i \in \mathbf{R}$.

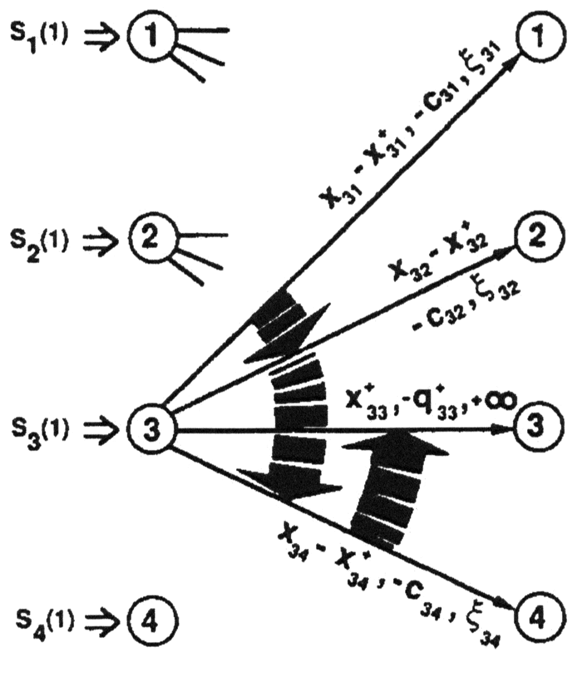

Figure 3.3: The stochastic transportation problem with the nodal recourse

Now, the total flow moving from city i to city j in the t-th stage is $x_{ij}(t) - x^{+}_{ij}(t)$ which is a random variable. Thus $S_i(t)$ is a random variable for all t and for all $i \in \mathbf{R}$ and therefore with this type of recourse a three or more stage transportation problems with random arc capacities are difficult to solve exactly.

3.4 Nodal Recourse

Nodal recourse (this notion was introduced in [8]) generalizes both simple approximation and null recourse. These two strategies exploit an overflow option to deal with realizations of the random vector $\xi(t)$. The overflow option is the first example of so called hierarchical recourse strategies. We are trying to put flow on one link but if it is not possible because of random arc capacity then we put the flow on an overflow link. Thus we have a two-level hierarchy for each arc.

Nodal recourse can be viewed as a multilevel hierarchical recourse policy because it offers multiple overflow options. This restrictive recourse strategy can be very well explained on the N-stage transportation problem with random arc capacities (problem (2.21) - (2.32)) but it does not extend easily to more general stochastic networks.

For nodal recourse we have to set up a new notation to be able to explain it. Let a vector $\delta_i(t)$ denote the hierarchy (order) of options for a unit of flow out of a city i at time t. Thus we can imagine this vector in the following way:

$$\delta_i(t) = \{\delta_{i1}(t)\,,\, \delta_{i2}(t)\,, \ldots,\, \delta_{i(M-1)}(t)\},$$

where $M-1$ gives the total number of options for a unit of flow out of the city i at time t and $\delta_{im}(t)$ represents the m-th option for a unit of flow out of the city i at time t which is used if the first $m-1$ options are impracticable. To guarantee feasibility we have to add to these $M-1$ options an extra overflow option.

Let us mark

$\Delta_{ij}(t)$ ... the option to move over the link (i, t, j)

$D_{ii}(t)$ ... the option to move over the overflow arc for the link (i, t, i).

A cost $c_{ij}(t)$ is connected with the movement over the link (i, t, j) while the overflow option means for us a penalty $q_{ii}^{+}(t)$.

With this notation a simple nodal recourse strategy can be represented as

$$\delta_3(t) = \{\Delta_{31}(t)\,,\,\Delta_{32}(t)\,,\,\Delta_{34}(t)\,,\,D_{33}(t)\},$$

what can be read as

- move over the link from city 3 to city 1 in time t if it is possible
- otherwise, move over the link from city 3 to city 2 in time t if it is possible
- otherwise, move over the link from city 3 to city 4 in time t if it is possible
- otherwise, move over the overflow arc from city 3 to city 3 in time t .

For better understanding, this situation is depicted in the Figure 3.3 . Again every arc is described with the flow over it, its capacity and with the revenue from one unit of the flow moving on it.

If we want to use the simple nodal recourse to solve our optimization problem we have first to find a suitable permutation of options for a unit of flow in the vector $\delta_i(t)$. A simple method for doing this is to find a set of values $\omega_{im}(t)$ which represent conditional marginal values of a unit of flow if we use the m-th option of movement out of node (i, t). These values can be calculated using the following relations:

$$\omega_{in}(t) = \begin{cases} c_{ij}(t) + p_j(t+1) & \text{if the } n\text{-th option is to move over the link } (i,t,j) \\ q_{ii}^{+}(t) + p_i(t+1) & \text{if the } n\text{-th option is to move over the overflow arc,} \end{cases}$$

where $p_j(t+1)$ is an estimated cost for moving the unit of flow from city j in time $t+1$.

Let us choose the vector $\delta_i(t)$ so that

$$\omega_{i1}(t) \le \omega_{i2}(t) \le \cdots \le \omega_{iM}(t).$$

Thus using the criterion of conditional marginal values of a unit of flow we obtain a suitable permutation of options over which a unit of flow can be moved out of city i at time t. Now we are moving down the list of options and adding units of flow in order to gain as big profit as possible.

Further we need to determine the probabilities that the k-th unit of flow will be moved out of a city i at time t over a given m-th option. These probabilities $d_{im}^{k}(t)$ depend on a capacity which is available after the first $k-1$ units of flow have been transported. Let $U_{im}(t)$ be a capacity of the m-th option in city i in the t-th stage given by

$$U_{im}(t) = \begin{cases} \xi_{ij}(t) & \text{if } \delta_{im}(t) = \Delta_{ij}(t) \\ +\infty & \text{if } \delta_{im}(t) = D_{ii}(t). \end{cases}$$

Now we denote $Y_{im}^{k-1}(t)$ flow allocated to the m-th option in city i at time t after k units of flow have been moved through node (i,t) and $V_{im}^{k-1}(t)$ capacity which is available for the m-th option in city i at time t after k units of flow have been allocated. It is obvious that

$$V_{im}^{k-1}(t) = U_{im}(t) - Y_{im}^{k-1}(t)$$

since the capacity we have at our disposal for the m-th option in city i in the t-th stage after $k-1$ units of flow have already been moved in this stage through city i we get as difference between total capacity $U_{im}(t)$ of the m-th option in city i at time t and flow already allocated to move with the m-th option from city i at time t after $k-1$ units of flow have been transported.

Let us define the following vectors:

$$\begin{aligned} U_i(t) &= (U_{i1}(t)\,, \,\dots\,, U_{iM}(t))^T \\ Y_i^{k-1}(t) &= (Y_{i1}^{k-1}(t)\,, \,\dots\,, Y_{iM}^{k-1}(t))^T \\ V_i^{k-1}(t) &= (V_{i1}^{k-1}(t)\,, \,\dots\,, V_{iM}^{k-1}(t))^T. \end{aligned}$$

Using this notation we get the following relation for the probability that the k-th unit of flow will be moved out of a city i at time t using the m-th option:

$$d_{im}^{\,k}(t) = P\left[\left(\sum_{l=1}^{m-1} V_{il}^{k-1}(t) = 0\right) \cap \left(V_{im}^{k-1}(t) > 0\right)\right].$$

Thus the k-th unit of flow is transported with the m-th option if there is a sufficient capacity for the m-th option but no capacity for the first $m-1$ options. The critical value here is the vector of the residual capacities $V_i^{k-1}(t)$ which states the capacities remaining on individual options after $k-1$ units of flow have been allocated.

Since the vector

$$V_i^{k-1}(t) = U_i(t) - Y_i^{k-1}(t)$$

can have very complex probabilistic structure many problems are computationally intractable.

Chapter 4

Stochastic Transportation Problem with Random Travel and Service Times

In this chapter we will deal with stochastic transportation problem with random travel and service times that can be found for instance in [12]. We use networks whose arcs have nonnegative random travel times and whose nodes have nonnegative random service times in the process. We suppose that the distributions of both random travel and service times are known.

Our goal is to allocate our fleet of vehicles to route through this network and to find optimal vehicles routes in order to we service each node in the presence of these random travel and service times. For simplicity we assume that all our vehicles are uncapacitated. In this case random demands of individual nodes are modelled via random node service times.

A route of a vehicle is defined as the set of arcs it moves on and the set of nodes it services and we suppose that it begins and ends at a specific depot node for each vehicle. These routes are selected before the random travel and service times are realized and in that way that every node will be visited by some vehicle. After realizations of individual random travel and service times are observed we compute the real time that is necessary to finish each route; we do not suppose that any route reoptimizations are allowed.

The time at which the last vehicle returns to the depot after all nodes have been serviced is called the *completion time*. We will consider two problems with different objectives: The first one minimizes the expected completion time and the second one maximizes the probability that the whole transportation operation is complete on or before a prespecified target time T.

4.1 The Vehicles Routing Problem with Random Travel and Service Times

Let $\mathbf{G} = (\mathbf{N}, \mathbf{A})$ be a directed graph (hence we assume that there can exist some one-way roads in our network) where $\mathbf{N}$ is the set of nodes and $\mathbf{A}$ is the set of arcs. The set of vehicles at our disposal is denoted as $\mathbf{L}$. The routes of all vehicles must start and end in the node 1 - in the depot. Each node in $\mathbf{N}\backslash\{1\}$ is a location that must be serviced by some vehicle. We do not require the graph to be complete but we assume that it is possible to depart from the depot, visit every node in the network and return back to the depot.

If τ_{ijl} is the random travel time for the vehicle $l \in \mathbf{L}$ moving on the arc from the node $i \in \mathbf{N}$ to the node $j \in \mathbf{N}$ (we do not in general assume the random travel times to be symmetric since the random travel times can depend on direction of the vehicle route) and σ_{il} denotes the random service time for the vehicle $l \in \mathbf{L}$ in the node $i \in \mathbf{N}$ than

$$\zeta = \begin{pmatrix} \tau \\ \sigma \end{pmatrix}$$

where

$$\tau = (\tau_{ijl} : (i,j) \in \mathbf{A},\, l \in \mathbf{L})$$

and

$$\sigma = (\sigma_{il} : i \in \mathbf{N},\, l \in \mathbf{L})$$

contains all of the random elements of this stochastic transportation problem. The travel and service times are nonnegative random variables and the components of ζ can be dependent.

In this vehicle routing problem with random travel and service times we have two types of the decision variables - the decision variables for the routing (x_{ijl}) and the decision variables for the servicing (v_{il}). Both of them are binary decision variables with the following values:

$$x_{ijl} = \begin{cases} 1 & \text{if the arc } (i,j) \in \mathbf{A} \text{ is a part of the route of the vehicle } l \in \mathbf{L} \\ 0 & \text{otherwise} \end{cases}$$

and

$$v_{il} = \begin{cases} 1 & \text{if the node } i \in \mathbf{N} \text{ is serviced by the vehicle } l \in \mathbf{L} \\ 0 & \text{otherwise.} \end{cases}$$

Further in some places of this chapter we will use these vectors of decision variables:

$$x = (x_{ijl} : (i,j) \in \mathbf{A},\, l \in \mathbf{L})$$

$$v = (v_{il} : i \in \mathbf{N},\, l \in \mathbf{L}).$$

We would like to optimize the expectation of the completion time. Our transportation project is complete when all vehicles have returned to the depot after finishing their routes and after all nodes have been serviced. The completion time $h(x, v, \zeta)$ is thus the maximum of $|\mathbf{L}|$ random variables (where $|\mathbf{L}|$ denotes the power of the set of all our vehicles $\mathbf{L}$) – the individual vehicles completion times. Hence

$$h(x, v, \zeta) = \max_{l \in \mathbf{L}} \left\{ \sum_{(i,j) \in \mathbf{A}} \tau_{ijl}\, x_{ijl} + \sum_{i \in \mathbf{N}} \sigma_{il}\, v_{il} \right\} \tag{4.1}$$

As already has been said we will consider two problems that differ only in their objective functions. The first one minimizes the expected completion time thus minimizes

$$\mathrm{E}h(x,v,\zeta)$$

and the second one maximizes the probability of completing the operation by the given deadline T thus maximizes

$$\mathrm{P}(h(x,v,\zeta)\leq T)=\mathrm{E}\mathrm{I}(h(x,v,\zeta)\leq T),$$

where I(.) denotes the *indicator function* which takes the value of 1 if its argument is true and zero otherwise.

There exist a lot of situations when we are rather interested in the completion time than in total travel of all vehicles time or some other criterions. For example we can lease the vehicles and return them after completing the operation as a group. When the leasing costs are our principal costs and are proportional to time then it is very important for us that the completion time is as small as possible or that the probability that we complete our project before the in advance given target time T is as big as possible.

Both problems are subject to the following constraints

$$\sum_{(i,j)\in\mathbf{RS}(j)} x_{ijl}=\sum_{(j,i)\in\mathbf{FS}(j)} x_{jil} \qquad \forall j\in\mathbf{N},\ \forall l\in\mathbf{L} \tag{4.2}$$

$$\sum_{l\in\mathbf{L}} v_{il}=1 \qquad \forall i\in\mathbf{N}\backslash\{1\} \tag{4.3}$$

$$v_{il}\leq\sum_{(i,j)\in\mathbf{FS}(i)} x_{ijl} \qquad \forall i\in\mathbf{N}\backslash\{1\},\ \forall l\in\mathbf{L} \tag{4.4}$$

$$\sum_{(1,i)\in\mathbf{FS}(1)} x_{1il}\geq 1 \qquad \forall l\in\mathbf{L} \tag{4.5}$$

$$x_{ijl}\in\{0,1\} \qquad \forall (i,j)\in\mathbf{A},\ \forall l\in\mathbf{L} \tag{4.6}$$

$$v_{il}\in\{0,1\} \qquad \forall i\in\mathbf{N},\ \forall l\in\mathbf{L} \tag{4.7}$$

$$\sum_{\substack{i\in\tilde{\rho}\\ j\in\rho\\ (i,j)\in\mathbf{A}}} x_{ijl}\geq\frac{1}{|\mathbf{A}_\rho|}\sum_{(i,j)\in\mathbf{A}_\rho} x_{ijl} \qquad \begin{array}{l}\forall\rho\subset\mathbf{N}\backslash\{1\},\\ 2\leq|\rho|\leq|\mathbf{N}|-2,\\ \forall l\in\mathbf{L}.\end{array} \tag{4.8}$$

where $\mathbf{FS}(i)$ represents the set of arcs leading off the node $i\in\mathbf{N}$ and $\mathbf{RS}(i)$ denotes the set of arcs entering the node $i\in\mathbf{N}$. Further

$$\mathbf{A}_\rho=\{(i,j):(i,j)\in\mathbf{A},\ i,j\in\rho\}$$

and $\tilde{\rho}$ is the complement of the index set ρ.

Let us now explain individual relations. The constraint (4.2) conserves the flow of each vehicle at each node (if some vehicle arrive in some node it has to leave this node, too) and the relation (4.3) guarantees that every node will be serviced exactly once.

The inequality (4.4) gives that a node can be serviced by a vehicle only if this node is on the route of the given vehicle. But every vehicle can drive through nodes without servicing them. The constraint (4.5) states that all the vehicles routes must start in the depot and the relations (4.6) and (4.7) define all our decision variables to be binary. The inequality (4.8) then eliminates subtours that are isolated from the depot.

For arbitrary index sets ρ and $\tilde{\rho}$ (ρ, $\tilde{\rho} \subset \mathbf{N}\backslash\{1\}$, ρ is the complement of $\tilde{\rho}$) that have at least two elements must be fulfilled that the number of the arcs on which the given vehicle $l \in \mathbf{L}$ is coming to the index set ρ from the index set $\tilde{\rho}$ is greater or equal to the number of the arcs on which this vehicle is moving in the index set ρ divided by the total number of arcs among nodes in this index set.

Now we can proceed to formulating our prior mentioned problems. If we denote the set of the decision vectors (x, v) that satisfy (4.2) - (4.8) as $\mathbf{XV}$, than the first stochastic transportation problem can be written as

$$\begin{aligned} z_E^* = &\min_{x,v} \; \mathrm{E}h(x, v, \zeta) \\ &(x, v) \in \mathbf{XV} \end{aligned} \tag{E}$$

and the second one as

$$\begin{aligned} z_P^* = &\max_{x,v} \; \mathrm{P}(h(x, v, \zeta) \leq T) \\ &(x, v) \in \mathbf{XV}. \end{aligned} \tag{P}$$

Besides these two problems we will use further the following auxiliary deterministic model

$$\begin{aligned} z_M^* = &\min_{x,v} \; h(x, v, \mathrm{E}\zeta) \\ &(x, v) \in \mathbf{XV} \end{aligned} \tag{M}$$

that minimizes the completion time under the assumption that all random travel and service times take their mean values.

Let us show some interesting properties of problems (E), (P) and (M) and mutual relations that hold among these three transportation problems. They are resumed in the following proposition that can be found in [12].

PROPOSITION 1:

(a) *If $\boldsymbol{L} = \{1\}$, an optimal solution to (E) may be obtained by solving (M).*

(b) *$H(x, v) = Eh(x, v, \zeta)$ is a convex function on co($\boldsymbol{XV}$), where co($\boldsymbol{XV}$) is the convex hull of $\boldsymbol{XV}$.*

(c) $z_M^* \leq z_E^*$.

(d) *If $T > 0$, then* $z_P^* \geq 1 - \dfrac{z_E^*}{T}$.

PROOF:

(a) If $\mathbf{L} = \{1\}$, we get

$$\begin{aligned} h(x, v, \zeta) &= \max_{l \in \mathbf{L}} \left\{ \sum_{(i,j) \in \mathbf{A}} \tau_{ijl}\, x_{ijl} + \sum_{i \in \mathbf{N}} \sigma_{il}\, v_{il} \right\} = \\ &= \sum_{(i,j) \in \mathbf{A}} \tau_{ij1}\, x_{ij1} + \sum_{i \in \mathbf{N}} \sigma_{i1}\, v_{i1}. \end{aligned}$$

The optimal solution of problem (E) on the set **XV** is given as

$$\begin{aligned} z_E^* &= \min_{x,v} \mathrm{E}h(x,v,\zeta) = \\ &= \min_{x,v} \mathrm{E}\left\{ \sum_{(i,j)\in \mathbf{A}} \tau_{ij1}\, x_{ij1} + \sum_{i\in \mathbf{N}} \sigma_{i1}\, v_{i1} \right\} = \\ &= \min_{x,v} \left\{ \sum_{(i,j)\in \mathbf{A}} x_{ij1}\, \mathrm{E}\,\tau_{ij1} + \sum_{i\in \mathbf{N}} v_{i1}\, \mathrm{E}\,\sigma_{i1} \right\} = \\ &= \min_{x,v} h(x,v,\mathrm{E}\zeta) = z_M^*. \end{aligned}$$

We could make this modification thanks to the linearity of $h(x,v,\zeta)$ in ζ.

(b) As $h(.,\zeta)$ is the maximum of a collection of linear (thus convex) functions, it is a convex function on $co(\mathbf{XV})$. Hence

$$h(\lambda x^1 + (1-\lambda)x^2, \lambda v^1 + (1-\lambda)v^2, \zeta) \le \lambda h(x^1, v^1, \zeta) + (1-\lambda)h(x^2, v^2, \zeta),$$

where $0 \le \lambda \le 1$ and (x^1, v^1), $(x^2, v^2) \in co(\mathbf{XV})$ are arbitrary.
If we take expectation on both sides we get

$$\mathrm{E}h(\lambda x^1 + (1-\lambda)x^2, \lambda v^1 + (1-\lambda)v^2, \zeta) \le \lambda \mathrm{E}h(x^1, v^1, \zeta) + (1-\lambda)\mathrm{E}h(x^2, v^2, \zeta),$$

what can be rewritten as

$$H(\lambda x^1 + (1-\lambda)x^2, \lambda v^1 + (1-\lambda)v^2) \le \lambda H(x^1, v^1, \zeta) + (1-\lambda)H(x^2, v^2, \zeta).$$

Thus $H(x,v)$ is a convex function on $co(\mathbf{XV})$.

(c) Since $h(x,v,.)$ is for fixed $(x,v) \in \mathbf{XV}$ the maximum of a collection of linear (thus convex) functions on $co(\Xi)$, where Ξ is the support of ζ and $co(\Xi)$ is the convex hull of Ξ, it is a convex function on $co(\Xi)$. Thus we can use the *Jensen's inequality* [1] that gives us

$$h(x,v,\mathrm{E}\zeta) \le \mathrm{E}h(x,v,\zeta) \qquad \forall (x,v) \in \mathbf{XV}. \tag{4.9}$$

Further it is obvious that

$$\min_{(x,v)\in \mathbf{XV}} h(x,v,\mathrm{E}\zeta) \le h(x,v,\mathrm{E}\zeta) \qquad \forall (x,v) \in \mathbf{XV}. \tag{4.10}$$

[1] **Jensen's inequality:** Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space, such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ-integrable, and if φ is convex function on the range of g, then

$$\varphi\left(\int_\Omega g\, d\mu\right) \le \int_\Omega \varphi \circ g\, d\mu.$$

In the terminology of probability theory, μ is a probability measure. The function g is replaced by a real-valued random variable X. The integral of any function over the space Ω with respect to the probability measure μ becomes an expected value. The inequality then says that if φ is any convex function, then

$$\varphi\left(\mathrm{E}\{X\}\right) \le \mathrm{E}\{\varphi(X)\}.$$

Jensen's inequality can be found for instance in [1] and its proof in [16].

If we combine the inequalities (4.9) and (4.10) we get

$$z_M^* = \min_{(x,v)\in\mathbf{XV}} h(x,v,\mathrm{E}\zeta) \le h(x,v,\mathrm{E}\zeta) \le \mathrm{E}h(x,v,\zeta) \qquad \forall (x,v)\in\mathbf{XV}.$$

Now we minimize the right-hand side of this relation over $(x,v) \in \mathbf{XV}$ which implies

$$z_M^* = \min_{(x,v)\in\mathbf{XV}} h(x,v,\mathrm{E}\zeta) \le \min_{(x,v)\in\mathbf{XV}} \mathrm{E}h(x,v,\zeta) = z_E^*$$

and this is the desired inequality.

(d) Let (x_E^*, v_E^*) and (x_P^*, v_P^*) denote the optimal solutions of the problems (E) and (P), respectively. With using a simple variant of *Markov's inequality* [2] that can be found for example in [17] we get

$$\mathrm{P}(h(x_E^*, v_E^*, \zeta) \le T) \ge 1 - \frac{\mathrm{E}h(x_E^*, v_E^*, \zeta)}{T}. \tag{4.11}$$

Since (x_P^*, v_P^*) is the optimal solution of (P) and (x_E^*, v_E^*) is a feasible solution for (P) (as (x_E^*, v_E^*) belongs to $\mathbf{XV}$), we have

$$\mathrm{P}(h(x_P^*, v_P^*, \zeta \le T) \ge \mathrm{P}(h(x_E^*, v_E^*, \zeta) \le T). \tag{4.12}$$

The relations (4.11) and (4.12) together imply

$$z_P^* = \mathrm{P}(h(x_P^*, v_P^*, \zeta \le T) \ge 1 - \frac{\mathrm{E}h(x_E^*, v_E^*, \zeta)}{T} = 1 - \frac{z_E^*}{T}.$$

□

Part (a) of the proposition 1 implies that the stochastic transportation problem (E) with only one vehicle is reduced to deterministic transportation problem where random travel and service times are replaced by their population means - thus we get the problem (M) with $\mathbf{L} = \{1\}$. It turn out, however, that its generalization to more than one vehicle is not possible. Indeed when we have at our disposal more than one vehicle, $h(x,v,\zeta)$ is the maximum of a collection of functions over all vehicles and thus it is a nonlinear function in ζ. Therefore

$$\mathrm{E}h(x,v,\zeta) \ne h(x,v,\mathrm{E}\zeta),$$

what can be rewritten as

$$\mathrm{E}\left[\max_{l\in\mathbf{L}}\left\{\sum_{(i,j)\in\mathbf{A}} \tau_{ijl}\, x_{ijl} + \sum_{i\in\mathbf{N}} \sigma_{il}\, v_{il}\right\}\right] \ne \max_{l\in\mathbf{L}}\left\{\sum_{(i,j)\in\mathbf{A}} \mathrm{E}\left[\tau_{ijl}\right] x_{ijl} + \sum_{i\in\mathbf{N}} \mathrm{E}\left[\sigma_{il}\right] v_{il}\right\}.$$

[2]**Markov's inequality:** If $(\Omega, \mathcal{A}, \mu)$ is a measurable space, f is a measurably extended real-valued function, and $t > 0$, then

$$\mu(\{x \in \Omega \mid |f(x)| \ge t\}) \le \frac{1}{t}\int_\Omega f\, d\mu.$$

For the special case where the space has measure 1 (i.e., it is a probability space), X is any random variable and $t > 0$, then

$$\mathrm{P}(|X| \ge t) \le \frac{\mathrm{E}(|X|)}{t}.$$

We can imagine the above relation in this simplified form:

$$\mathrm{E} \max_{l \in \mathbf{L}} X_l \neq \max_{l \in \mathbf{L}} \mathrm{E} X_l$$

where X_l are random variables. In the following example we show that in general equality does not hold true if $|\mathbf{L}| > 1$.

Let X_1 and X_2 are independent real-value random variables with the densities

$$f_1(x) = \begin{cases} 1 & \text{for } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_2(x) = \begin{cases} x & \text{for } x \in [0,1] \\ 2-x & \text{for } x \in [1,2] \\ 0 & \text{otherwise.} \end{cases}$$

Let us compute the means of these random variables.

$$\mathrm{E}X_1 = \int_0^1 x \, \mathrm{d}x = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2},$$

$$\mathrm{E}X_2 = \int_0^1 x^2 \, \mathrm{d}x + \int_1^2 (2-x)x \, \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^1 + \left[x^2\right]_1^2 - \left[\frac{x^3}{3}\right]_1^2 = 1.$$

Hence

$$\max_{l \in \{1,2\}} \mathrm{E}X_l = \mathrm{E}X_2 = 1.$$

For the random variable $X = \max_{l \in \{1,2\}} X_l$ we have the following distribution function:

$$\begin{aligned} \mathrm{F}_{\max}(x) &= \mathrm{P}(\max(X_1, X_2) \leq x) = \mathrm{P}(X_1 \leq x, X_2 \leq x) = \\ &= \mathrm{P}(X_1 \leq x) \, \mathrm{P}(X_2 \leq x) = \mathrm{F}_1(x) \, \mathrm{F}_2(x), \end{aligned}$$

where

$$F_1(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0] \\ x & \text{for } x \in [0,1] \\ 1 & \text{for } x \in [1, +\infty] \end{cases}$$

and

$$F_2(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0] \\ \frac{x^2}{2} & \text{for } x \in [0,1] \\ -\frac{x^2}{2} + 2x - 1 & \text{for } x \in [1,2] \\ 1 & \text{for } x \in [1, +\infty]. \end{cases}$$

It is obvious, that

$$F_{\max}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0] \\ \frac{x^3}{2} & \text{for } x \in [0, 1] \\ -\frac{x^2}{2} + 2x - 1 & \text{for } x \in [1, 2] \\ 1 & \text{for } x \in [1, +\infty] \end{cases}$$

and

$$f_{\max}(x) = \begin{cases} \frac{3}{2}x^2 & \text{for } x \in [0, 1] \\ 2 - x & \text{for } x \in [1, 2] \\ 0 & \text{otherwise.} \end{cases}$$

Now we can already compute EX.

$$\mathrm{E}X = \int_0^1 \frac{3}{2}x^3 \,\mathrm{d}x + \int_1^2 (2-x)x \,\mathrm{d}x = \left[\frac{3}{8}x^4\right]_0^1 + \left[x^2\right]_1^2 - \left[\frac{x^3}{3}\right]_1^2 = \frac{25}{24}.$$

We can see, that

$$\mathrm{E} \max_{l \in \{1,2\}} X_l = \frac{25}{24} \neq 1 = \max_{l \in \{1,2\}} \mathrm{E}X_l.$$

Hence if we have more than one vehicle, we can not simplify our stochastic transportation problem (E) to the deterministic one.

If we replace all random elements with their population means and solve the deterministic problem (M) we gain the optimal solution of this problem - (x^*_M, v^*_M). This solution is feasible for the problem (E) but in general is not optimal for it. If we are able to compute $\mathrm{E}h(x^*_M, v^*_M, \zeta)$ than we can express so called *value of stochastic solution* that is defined as $\mathrm{E}h(x^*_M, v^*_M, \zeta) - z^*_E$ (see for example [2]). This difference states how much we improve our solution if we solve the stochastic transportation problem (E) instead of the mean-based approximation (M). In case that this value is sufficiently small, we can assume that (x^*_M, v^*_M) is enough good solution for the problem (E). However, in majority of problems this difference is quite large and therefore it is mostly useful to solve the stochastic transportation problem (E).

Part (d) of the proposition 1 gives us the lower bound on the optimal objective function value of model (P). As this bound is distribution-free, probably it will be only very crude for most of the problems.

Unlike the problem (E), the stochastic transportation problem (P) can not be solved with standard cutting-plane methods, as the function

$$G(x, v) = \mathrm{P}(h(x, v, \zeta) \leq T)$$

is in general not a concave function. A convexity result can be obtained for one vehicle stochastic transportation problem (P) where the travel and service times are assumed to be normally distributed. Since the normal distribution of ζ is however contradictory with nonnegative travel and random times, the normal distribution can give us an appropriate probabilistic model only if the probabilities of negative times are very small. The convexity result is introduce in the following proposition (from [12]).

PROPOSITION 2: *Consider the special case of (P), where* $\mathbf{L} = \{1\}$ *and* $\zeta = (\tau, \sigma)$ *has a normal multivariate distribution with mean* $\mu = E\zeta$ *and positive definite covariance matrix* V. *Let* $y = (x, v)$. *Then, solving (P) is equivalent to finding the largest* $\alpha \in \Re$ *for which the following system*

$$y \in \mathbf{XV} \tag{4.13}$$

$$\mu^T y - T + \alpha\sqrt{y^T V y} \leq 0 \tag{4.14}$$

is feasible.

PROOF: Since $\zeta = (\tau, \sigma)$ is normally multivariate distributed, the function

$$h(x, v, \zeta) = \max_{l \in \mathbf{L}} \left\{ \sum_{(i,j)\in\mathbf{A}} \tau_{ijl}\, x_{ijl} + \sum_{i\in\mathbf{N}} \sigma_{il}\, v_{il} \right\} = \sum_{(i,j)\in\mathbf{A}} \tau_{ij1}\, x_{ij1} + \sum_{i\in\mathbf{N}} \sigma_{i1}\, v_{i1}$$

is a linear combination of normal distributed random variables and therefore it is normally distributed random variable with mean $\mu^T y$ and variance $y^T V y$. From this follows that

$$\max_{(x,v)\in\mathbf{XV}} \mathrm{P}(h(x, v, \zeta) \leq T) = \max_{(x,v)\in\mathbf{XV}} \mathrm{P}\left(\left\{ \sum_{(i,j)\in\mathbf{A}} \tau_{ij1}\, x_{ij1} + \sum_{i\in\mathbf{N}} \sigma_{i1}\, v_{i1} \right\} \leq T\right) =$$

$$= \max_{(x,v)\in\mathbf{XV}} \mathrm{P}\left(\{\tau^T x + \sigma^T v\} \leq T\right) = \max_{y\in\mathbf{XV}} \mathrm{P}(\zeta^T y \leq T)$$

is equivalent to

$$\max_{y\in\mathbf{XV}} \mathrm{P}\left(X \leq \frac{T - \mu^T y}{\sqrt{y^T V y}}\right) \tag{4.15}$$

where $X \sim N(0, 1)$ (i.e., X is a normal random variable with zero mean and unit variance). Solving this problem is equivalent to solving

$$\max_{y\in\mathbf{XV}} \left[\alpha = \frac{T - \mu^T y}{\sqrt{y^T V y}}\right] \tag{4.16}$$

since the probability in (4.15) will be maximal if the upper bound will be as large as possible.

The problem (4.16) can be as well read as find the largest $\alpha \in \Re$ such that the system (4.13) - (4.14) is feasible.

□

The Proposition 2 is related to convexity results for chance-constrained stochastic programs. In [10] is showed that $y^T V y$ is convex function and therefore the set of y's satisfying (4.14) is convex provided that $\alpha \geq 0$ (thus if we have at least a 50% chance of not exceeding the deadline) as $\mu^T y$ is a linear function of y and $-T$ is a constant function.

It is not possible to generalize the result of Proposition 2 to the multivehicle stochastic transportation problem (P). The objective function of this problem can be written as

$$\begin{aligned} \mathrm{P}(h(x,v,\zeta) \le T) &= \mathrm{P}\left(\max_{l \in \mathbf{L}} \left\{ \sum_{(i,j)\in\mathbf{A}} \tau_{ijl}\, x_{ijl} + \sum_{i\in\mathbf{N}} \sigma_{il}\, v_{il} \right\} \le T \right) = \\ &= \mathrm{P}\left(\max_{l\in\mathbf{L}} \left\{ \tau_l^T x_l + \sigma_l^T v_l \right\} \le T \right) = \mathrm{P}\left(\max_{l\in\mathbf{L}} \zeta_l^T y_l \le T \right) = \\ &= \mathrm{P}\left(\zeta_1^T y_1 \le T, \ldots, \zeta_C^T y_C \le T \right), \end{aligned}$$

where $C = |\mathbf{L}|$ and for all $l \in \mathbf{L}$ we define vectors

$$\begin{aligned} x_l &= (x_{ijl} : (i,j) \in \mathbf{A}) \\ v_l &= (v_{il} : i \in \mathbf{N}) \\ \zeta_l &= (\tau_{ijl} : (i,j) \in \mathbf{A}\,;\, \sigma_{il} : i \in \mathbf{N}) \\ y_l &= (x_{ijl} : (i,j) \in \mathbf{A}, v_{il}\,;\, i \in \mathbf{N}). \end{aligned}$$

Proposition 2 implies that for $\mathbf{L} = \{1\}$, hence for $|\mathbf{L}| = C = 1$, the set

$$\mathbf{Y} = \left\{ y = (y_1, \ldots, y_C) : \mathrm{P}\left(\max_{l\in\mathbf{L}} \zeta_l^T y_l \le T \right) \ge p \right\}$$

is convex for $p \ge \frac{1}{2}$. The following counterexample shows that this is not true for stochastic transportation problem (P) with $|\mathbf{L}| > 1$.

Let $T = 1$, $p = \frac{1}{2}$ and $|\mathbf{L}| = 2$. In order to the set

$$\mathbf{Y} = \left\{ y = (y_1, y_2) : \mathrm{P}\left(\max_{l\in\{1,2\}} \zeta_l^T y_l \le 1 \right) \ge \frac{1}{2} \right\}$$

be convex, for all $y \in \mathbf{Y}$ and all $z \in \mathbf{Y}$ and for all $a \in [0,1]$ the points $ay + (1-a)z$ must belong to the set $\mathbf{Y}$, too.

Consider vectors

$$y_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$z_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad z_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and let the random vector ζ_l have the following discrete distribution (same for both of the vehicles)

$$\zeta_l = \begin{cases} \begin{pmatrix} 1 \\ 3 \end{pmatrix} & \text{with probability } \frac{1}{2} \\ \begin{pmatrix} 3 \\ 1 \end{pmatrix} & \text{with probability } \frac{1}{2}. \end{cases}$$

Now we check if the vectors y and z are really from the set $\mathbf{Y}$.

$$\zeta_l^T y_1 = \zeta_l^T y_2 = \zeta_l^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{cases} (1,3)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 & \text{with probability } \frac{1}{2} \\ (3,1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3 & \text{with probability } \frac{1}{2}, \end{cases}$$

$$\zeta_l^T z_1 = \zeta_l^T z_2 = \zeta_l^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{cases} (1,3)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3 & \text{with probability } \frac{1}{2} \\ (3,1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 & \text{with probability } \frac{1}{2}. \end{cases}$$

Thus

$$\mathrm{P}\left(\max_{l\in\{1,2\}} \zeta_l^T y_l \leq 1 \right) = \mathrm{P}\left(\zeta_l^T y_1 \leq 1\right) = \mathrm{P}\left(\zeta_l^T y_2 \leq 1\right) = \frac{1}{2},$$
$$\mathrm{P}\left(\max_{l\in\{1,2\}} \zeta_l^T z_l \leq 1 \right) = \mathrm{P}\left(\zeta_l^T z_1 \leq 1\right) = \mathrm{P}\left(\zeta_l^T z_2 \leq 1\right) = \frac{1}{2},$$

and therefore both of the vectors y and z belong to the set $\mathbf{Y}$. Now let us choose the vector $w = ay + (1-a)z$, where $a = \frac{1}{2}$. It is obvious, that

$$w = (w_1, w_2)$$

where

$$w_1 = \frac{1}{2}y_1 + \frac{1}{2}z_1$$
$$w_2 = \frac{1}{2}y_2 + \frac{1}{2}z_2.$$

Hence $w_1 = w_2$ and therefore

$$\mathrm{P}\left(\max_{l\in\{1,2\}} \zeta_l^T w_l \leq 1 \right) = \mathrm{P}\left(\zeta_l^T w_1 \leq 1\right) = \mathrm{P}\left(\zeta_l^T w_2 \leq 1\right) =$$
$$= \mathrm{P}\left(\zeta_l^T \left(\frac{1}{2}y_2 + \frac{1}{2}z_2\right) \leq 1\right) = \mathrm{P}\left(\frac{1}{2}\zeta_l^T \left(y_2 + z_2\right) \leq 1\right) =$$
$$= \mathrm{P}\left(\zeta_l^T \left(y_2 + z_2\right) \leq 2\right) = \mathrm{P}\left(\zeta_l^T \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq 2\right) = 0,$$

since

$$\zeta_l^T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4 \quad \text{with probability 1.}$$

Thus the vector $w = \frac{1}{2}y + \frac{1}{2}z$ does not belong to the set $\mathbf{Y}$ and therefore the set $\mathbf{Y}$ is not a convex set. Consequently we can not generalize the result of Proposition 2 to the multivehicle stochastic transportation problem (P).

4.2 DESVRP Algorithm

For solving stochastic transportation problems with random travel and service times with small cardinality of the sample space so called DESVRP (Deterministic Equivalents to Stochastic Vehicles Routing Problem) algorithm (see [12]) can be used. This method solves the *deterministic equivalents models* for the stochastic vehicle routing problems (E) and (P) and is based on a branch-and-cut approach. The branch-and-cut method was for the first time introduced in the work [3] and lies in solving a sequence of relaxations of the given problem.

If we are able to enumerate the individual realizations of the travel and service times vector ζ by $h \in \mathbf{H}$ where the cardinality of the set $\mathbf{H}$ is not very large, than the deterministic equivalent of the stochastic vehicle routing problem (E) can be written as

$$z_E^* = \min_{x,v,\theta} \sum_{h \in \mathbf{H}} \lambda_h \theta_h \tag{4.17}$$

subject to

$$\theta_h \geq \sum_{(i,j) \in \mathbf{A}} \tau_{ijl,h}\, x_{ijl} + \sum_{i \in \mathbf{N}} \sigma_{il,h}\, v_{il} \qquad \forall l \in \mathbf{L}, \forall h \in \mathbf{H} \tag{4.18}$$

$$(x, v) \in \mathbf{XV} \tag{4.19}$$

where λ_h denotes the probabilities of individual realizations of the vector ζ, hence

$$\lambda_h = \mathrm{P}(\zeta = \zeta_h) = \mathrm{P}\left(\begin{pmatrix} \tau \\ \sigma \end{pmatrix} = \begin{pmatrix} \tau_h \\ \sigma_h \end{pmatrix}\right),$$

with

$$\tau_h = (\tau_{ijl,h} : (i,j) \in \mathbf{A},\, l \in \mathbf{L})$$

$$\sigma_h = (\sigma_{il,h} : i \in \mathbf{N},\, l \in \mathbf{L}),$$

and the nonnegative decision variable θ_h states the length of the longest route under the scenario $h \in \mathbf{H}$.

The deterministic equivalent of the stochastic vehicle routing problem (P) can be formulated in a similar way as

$$z_P^* = \max_{x,v,\vartheta} \sum_{h \in \mathbf{H}} \lambda_h \vartheta_h \tag{4.20}$$

subject to

$$\sum_{(i,j) \in \mathbf{A}} \tau_{ijl,h}\, x_{ijl} + \sum_{i \in \mathbf{N}} \sigma_{il,h}\, v_{il} \leq T + M_{lh}(1 - \vartheta_h) \qquad \forall l \in \mathbf{L}, \forall h \in \mathbf{H} \tag{4.21}$$

$$(x, v) \in \mathbf{XV} \tag{4.22}$$

$$\vartheta_h \in \{0, 1\} \qquad \forall h \in \mathbf{H}. \tag{4.23}$$

The binary decision variable ϑ_h gives whether we manage to complete our transportation project under the scenario $h \in \mathbf{H}$ by the deadline T, therefore,

$$\vartheta_h = \begin{cases} 1 & \text{if the longest route under the scenario } h \in \mathbf{H} \text{ takes at most } T \\ 0 & \text{otherwise.} \end{cases}$$

The auxiliary parameter M_{lh} guarantees that the longest route need not be less or equal T for all of possible scenarios $h \in \mathbf{H}$. This parameter should be chosen in that way that the equation (4.21) stays in validity even if the variable ϑ_h takes the value of 0 hence if we do not complete our project by a deadline. It is obvious, that a sufficiently large value for this parameter is

$$M_{lh} = \sum_{(i,j)\in \mathbf{A}} \tau_{ijl,h} + \sum_{i\in \mathbf{N}} \sigma_{il,h} - T.$$

Now we can already deal with the algorithm. It is based on solving a sequence of relaxations of problems (4.17) - (4.19) and (4.20) - (4.23). We start to handle problems without constraints of type (4.8) (since there is an exponential number of them) and gradually add in our problem only such constraints of this type that seem to be needful for finding the optimal solution.

DESVRP Algorithm for problem (E):

Step 0: Select $\delta > 0$ and let $\overline{z} = +\infty$. Form the relaxation (RE) of the problem (4.17) - (4.19) that contains no constraint of type (4.8).

Step 1: Solve (RE) to obtain solution (x_R, v_R) with objective function value $\underline{z}$.

Step 2: If no subtours isolated from the depot exist in x_R than STOP: (x_R, v_R) is the optimal solution to the problem (4.17) - (4.19).

Step 3: Let $v'_F = v_R$. Join the isolated subtours of x_R and construct a solution x'_F so that $(x'_F, v'_F) \in \mathbf{XV}$. Let $z' = \sum_{h\in\mathbf{H}} \lambda_h\, h(x'_F, v'_F, \zeta_h)$.

Step 4: If $z' < \overline{z}$, then let $\overline{z} = z'$ and $(x_F, v_F) = (x'_F, v'_F)$.

Step 5: If $\dfrac{\overline{z} - \underline{z}}{\underline{z}} \leq \delta$, then STOP: (x_F, v_F) is a solution with objective function value $\overline{z}$ within $(100 \cdot \delta)\%$ of z^*_E. Otherwise, add to (RE) such constraints of type (4.8) that remove subtours from the solution (x_R, v_R). Go to Step 1.

Since we obtain $\underline{z}$ in Step 1 as a objective function value of problem (RE), which represents a relaxation of problem (4.17) - (4.19), it is obvious that $\underline{z} \leq z^*_E$. Hence we have the lower bound for the objective function value of the optimal solution. The upper bound for the optimal objective function value follows from Step 3 where we construct a feasible solution to problem (4.17) - (4.19). Therefore $z^*_E \leq \overline{z}$.

If we decide to consider the solution (x_F, v_F) with the objective function value $\overline{z}$ to be sufficiently good, the fraction $\dfrac{\overline{z} - \underline{z}}{\underline{z}} \cdot 100$ states by maximally how much percents is the objective function value of the found solution (x_F, v_F) worse than the optimal one.

In Step 3 we join the isolated subtours to the subtour that contains the depot node on the basis of mean values of the random travel times. We choose the cheapest connection that eliminates the isolated subtours and that still services all nodes that have been serviced in the original solution with the isolated subtours.

Similarly, this algorithm for the problem (P) can be formulated. There are only minor changes that are due to the fact that we solve maximizing problem whereas we minimize the objective function in problem (E).

DESVRP Algorithm for problem (P):

Step 0: Select $\delta > 0$ and let $\underline{z} = 0$. Form the relaxation (RE) of the problem (4.20) - (4.23) that contains no constraint of type (4.8).

Step 1: Solve (RE) to obtain solution (x_R, v_R) with objective function value $\overline{z}$.

Step 2: If no subtours isolated from the depot exist in x_R than STOP: (x_R, v_R) is the optimal solution to the problem (4.20) - (4.23).

Step 3: Let $v'_F = v_R$. Join the isolated subtours of x_R and construct a solution x'_F so that $(x'_F, v'_F) \in \mathbf{XV}$. Let $z' = \sum\limits_{h \in \mathbf{H}} \lambda_h \, \mathrm{I}(h(x'_F, v'_F, \zeta_h) \leq T)$.

Step 4: If $z' > \underline{z}$, then let $\underline{z} = z'$ and $(x_F, v_F) = (x'_F, v'_F)$.

Step 5: If $\dfrac{\overline{z} - \underline{z}}{\underline{z}} \leq \delta$, then STOP: (x_F, v_F) is a solution with objective function value $\overline{z}$ within $(100 \cdot \delta)\%$ of z^*_P. Otherwise, add to (RE) such constraints of type (4.8) that remove subtours from the solution (x_R, v_R). Go to Step 1.

Since we must in Step 1 solve programming problem with $(|\mathbf{A}| + |\mathbf{N}|) \cdot |\mathbf{L}|$ binary decision variables (number of decision variables x_{ijl} and v_{il}) and with $|\mathbf{H}|$ nonnegative decision variables (variables θ_h) for the problem (E) or with $(|\mathbf{A}| + |\mathbf{N}|) \cdot |\mathbf{L}| + |\mathbf{H}|$ binary decision variables (variables x_{ijl}, v_{il} and ϑ_h) for the problem (P), respectively, the DESVRP Algorithm can be applied only to problems with small cardinalities of all mentioned sets.

4.3 Application of DESVRP Algorithm for Problem (E)

Consider again a security agency as in the previous computational applications. In this case, our security agency transports daily takings from supermarkets to a bank and the random demands of individual supermarkets (amount of money that must be taken from them) are modelled as random service times with known discrete distribution. The travel times between individual nodes (supermarkets, depot) are random, too. Further we assume, that the travel time between the bank and our depot is insignificant to other travel times therefore we will consider only money transportation to the depot. Our goal is to allocate our vehicles on individual routes to service all demand points (supermarkets) and to minimize the expected completion time of money transportation; in the process all vehicles routes must start and end in the depot node. Hence our stochastic transportation problem will be formulated as the problem (E).

In Figure 4.1 a twelve-nodes network ($\mathbf{N} = \{n1, \ldots, n12\}$) is depicted that symbolically represents the system we are working with. The node 1 in figure denotes the depot (node $n1$), the remainder are locations to service. The distances between

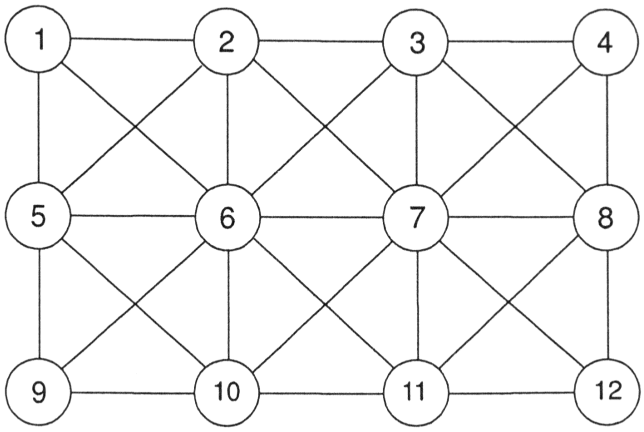

Figure 4.1: Network for vehicles routing problem with random travel and service times

	n1	n2	n3	n4	n5	n6	n7	n8	n9	n10	n11	n12
n1	0	1	0	0	1	1	0	0	0	0	0	0
n2	1	0	1	0	1	1	1	0	0	0	0	0
n3	0	1	0	1	0	1	1	1	0	0	0	0
n4	0	0	1	0	0	0	1	1	0	0	0	0
n5	1	1	0	0	0	1	0	0	1	1	0	0
n6	1	1	1	0	1	0	1	0	1	1	1	0
n7	0	1	1	1	0	1	0	1	0	1	1	1
n8	0	0	1	1	0	0	1	0	0	0	1	1
n9	0	0	0	0	1	1	0	0	0	1	0	0
n10	0	0	0	0	1	1	1	0	1	0	1	0
n11	0	0	0	0	0	1	1	1	0	1	0	1
n12	0	0	0	0	0	0	1	1	0	0	1	0

Table 4.1: The arcs in our stochastic vehicles routing problem

individual nodes do not correspond to reality and we suppose that there exist no one-ways thus our graph is not directed. The table 4.1 presents the set of all arcs **A** in our network.

We have three vehicles at our disposal ($\mathbf{L} = \{veh1, veh2, veh3\}$) and we assume them to be identical in this way that the random travel and service times do not depend on the vehicle. Therefore,

$$\tau_{ij,veh1} = \tau_{ij,veh2} = \tau_{ij,veh3} = \tau_{ij}$$

$$\sigma_{i,veh1} = \sigma_{i,veh2} = \sigma_{i,veh3} = \sigma_i.$$

With respect to the prior mentioned notation we denote the set of possible scenarios for these random times as **H** where $\mathbf{H} = \{1, \ldots, 5\}$ and λ_h stands for the probabilities of individual scenarios $h \in \mathbf{H}$. These probabilities one by one take the following values: $0.2, 0.35, 0.15, 0.25$ and 0.05.

The possible realizations of random service times for individual nodes ($\sigma_{i,h}$) are resumed in the table 4.2; we assume the service time for the node $n1$ to be deterministic and zero since the depot is not a location to service hence there exist no random

	1	2	3	4	5	Eσ_i
n1	0	0	0	0	0	0
n2	1	2	3	4	5	2.6
n3	2	2.5	3	3.5	4	2.8
n4	0.5	2	2.5	3	3.5	2.1
n5	0.5	1.5	2	2.5	3.5	1.7
n6	1	2	2.5	3.5	4	2.4
n7	1	1.5	2	3	3.5	2
n8	2	2.5	3.5	4.5	5	3.2
n9	0.5	1.5	2.5	3.5	4.5	2.1
n10	2	2.5	3.5	4	4.5	3
n11	1.5	2	2.5	3.5	4	2.5
n12	1.5	2.5	3	3.5	4.5	2.7

Table 4.2: The possible realizations of random service times for individual nodes - $\sigma_{i,h}$

	n1	n2	n3	n4	n5	n6	n7	n8	n9	n10	n11	n12
n1	0	3.1	0	0	6.5	13.2	0	0	0	0	0	0
n2	3.1	0	8.4	0	9.5	17.4	5.1	0	0	0	0	0
n3	0	8.4	0	10.9	0	14.9	23	17.4	0	0	0	0
n4	0	0	10.9	0	0	0	15.4	10.1	0	0	0	0
n5	6.5	9.5	0	0	0	7.5	0	0	12.9	19.9	0	0
n6	13.2	17.4	14.9	0	7.5	0	15.2	0	18.6	9.4	4.5	0
n7	0	5.1	23	15.4	0	15.2	0	7.8	0	10.7	20.5	19.5
n8	0	0	17.4	10.1	0	0	7.8	0	0	0	11.8	4.1
n9	0	0	0	0	12.9	18.6	0	0	0	10	0	0
n10	0	0	0	0	19.9	9.4	10.7	0	10	0	5.5	0
n11	0	0	0	0	0	4.5	20.5	11.8	0	5.5	0	8.5
n12	0	0	0	0	0	0	19.5	4.1	0	0	8.5	0

Table 4.3: The expected travel times for individual arcs - Eτ_{ij}

demands. Furthermore, the expected service time for individual nodes - Eσ_i, can be found in this table.

Since the table of possible realizations of random travel times for individual arcs is too large, it can be found on the appendant CD. In table 4.3 only the expected travel times on individual arcs are introduced. If zero values occur somewhere in this table it means that no arc exists between given two nodes. The travel time from some city to the same city is logically zero, too. For simplicity we suppose that all travel times are symmetric, thus

$$\tau_{ij} = \tau_{ji}$$

$$\tau_{ij,h} = \tau_{ji,h} \qquad \forall h \in \mathbf{H}.$$

As already have been said previously, the binary decision variables for movement are denoted as x_{ijl} and the binary decision for servicing are represented with v_{il}. Then the length of the route of the vehicle $l \in \mathbf{L}$ under the travel and service times scenario $h \in \mathbf{H}$ can be computed as

$$\theta_{l,h} = \sum_{(i,j)\in\mathbf{A}} \tau_{ij,h}\, x_{ijl} + \sum_{i\in\mathbf{N}} \sigma_{i,h}\, v_{il}.$$

It is obvious that for the length of the longest route under the travel and service times scenario $h \in \mathbf{H}$, for θ_h

$$\theta_h \geq \theta_{l,h} \qquad \forall l \in \mathbf{L}.$$

is fulfilled.

Further we denote as ι_{ij} the identifier of the existence of the arc (i,j). Hence

$$\iota_{ij} = \begin{cases} 1 & \text{if the arc } (i,j) \text{ is in our set } \mathbf{A} \\ 0 & \text{otherwise.} \end{cases}$$

The values of this parameter are given in table 4.1. This identifier allows us to formulate our stochastic transportation problem without using the set of all arcs $\mathbf{A}$ and without using sets of arcs entering and leading off individual nodes (the sets $\mathbf{RS}(i)$ and $\mathbf{FS}(i)$).

Therefore, our transportation problem with random travel and service times can be written as

$$\min_{x,\,v,\,\theta} \sum_{h\in\mathbf{H}} \lambda_h \theta_h \tag{4.24}$$

subject to

$$\sum_{i\in\mathbf{N}} x_{ijl}\, \iota_{ij} = \sum_{i\in\mathbf{N}} x_{jil}\, \iota_{ji} \qquad \forall j \in \mathbf{N},\ \forall l \in \mathbf{L} \tag{4.25}$$

$$\sum_{l\in\mathbf{L}} v_{il} = 1 \qquad \forall i \in \mathbf{N}\backslash\{n1\} \tag{4.26}$$

$$v_{il} \leq \sum_{j\in\mathbf{N}} x_{ijl}\, \iota_{ij} \qquad \forall i \in \mathbf{N}\backslash\{n1\},\ \forall l \in \mathbf{L} \tag{4.27}$$

$$\sum_{j\in\mathbf{N}} x_{n1,jl}\, \iota_{n1,j} \geq 1 \qquad \forall l \in \mathbf{L} \tag{4.28}$$

$$x_{ijl} \in \{0,1\} \qquad \forall i,j \in \mathbf{N},\ \forall l \in \mathbf{L} \tag{4.29}$$

$$v_{il} \in \{0,1\} \qquad \forall i \in \mathbf{N},\ \forall l \in \mathbf{L} \tag{4.30}$$

$$\sum_{i\in\tilde{\rho}} \sum_{j\in\rho} x_{ijl}\, \iota_{ij} \geq \frac{1}{|\mathbf{A}_\rho|} \sum_{(i,j)\in\mathbf{A}_\rho} x_{ijl} \qquad \forall \rho \subset \mathbf{N}\backslash\{n1\},\ 2 \leq |\rho| \leq 10,\ \forall l \in \mathbf{L}. \tag{4.31}$$

$$\theta_{l,h} = \sum_{i\in\mathbf{N}} \sum_{j\in\mathbf{N}} \tau_{ij,h}\, x_{ijl}\, \iota_{ij} + \sum_{i\in\mathbf{N}} \sigma_{i,h}\, v_{il} \qquad \forall l \in \mathbf{L},\ \forall h \in \mathbf{H} \tag{4.32}$$

$$\theta_h \geq \theta_{l,h} \qquad \forall l \in \mathbf{L},\ \forall h \in \mathbf{H}. \tag{4.33}$$

Now we can already proceed to application of the DESVRP Algorithm to our stochastic transportation problem. We will again use the model system GAMS with the solver MINOS 5.51. Let $\delta = 0.05$ and $\overline{z} = +\infty$. The relaxation (RE) of our stochastic vehicle routing problem can be formulated as (4.24) - (4.33) without the constraints of type (4.31). From solver we get the following solution (x_R, v_R) of this relaxation with the objective function value $\underline{z} = 46.3$:

VARIABLE v.L states if vehicle l is servicing location i

	veh1	veh2	veh3
n2			1.000
n3			1.000
n4			1.000
n5	1.000		
n6	1.000		
n7		1.000	
n8	1.000		
n9		1.000	
n10		1.000	
n11	1.000		
n12	1.000		

VARIABLE x.L states if vehicle l is moving on arc (i, j)

	veh1	veh2	veh3
n1 .n2		1.000	1.000
n1 .n5	1.000		
n2 .n1		1.000	1.000
n2 .n7		1.000	
n3 .n4			1.000
n4 .n3			1.000
n5 .n1	1.000		
n6 .n11	1.000		
n7 .n2		1.000	
n8 .n12	1.000		
n9 .n10		1.000	
n10.n9		1.000	
n10.n11			1.000
n11.n6	1.000		
n11.n10			1.000
n12.n8	1.000		

For better understanding, see Figure 4.2. The route of the vehicle $veh1$ is depicted with full lines, of the of the vehicle $veh2$ with dash lines and of the vehicle $veh3$ with dotted lines. The nodes serviced with $veh1$ are dark with black numbers, the dark nodes with white numbers belong to the service set of $veh2$ and white nodes (except

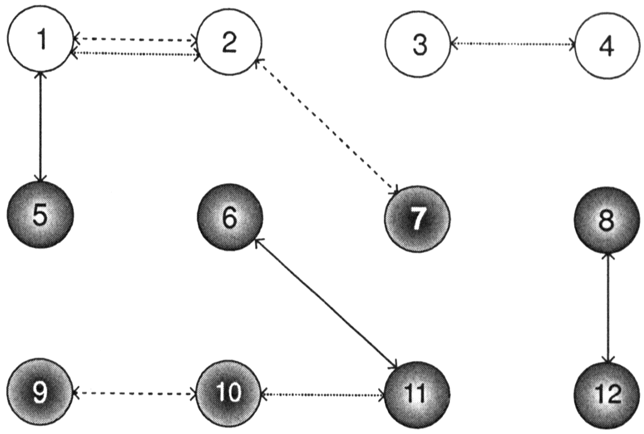

Figure 4.2: Routes and nodes to service for individual vehicles obtained as the solution of the problem (RE)

the depot - the node 1) are served with $veh3$.

Notice that all vehicles routes contain subtours isolated from depot. Therefore, (x_R, v_R) is not an optimal solution to our transportation problem. We keep servicing of individual demand points with individual vehicles without changes and join the isolated subtours of vehicles to their main tours - to the tours that contain the depot node $n1$. For every vehicle we choose the cheapest one from all possible routes in order to set of nodes it services stays unchanged and the acquired solution must lie in the set $\mathbf{XV}$. We gain the solution (x'_F, v'_F) where the tracks of individual vehicles are given in the following output:

	veh1	veh2	veh3
n1.n2	1	1	1
n1.n5	0	0	0
n2.n1	0	0	1
n2.n3	0	0	1
n2.n7	1	1	0
n3.n2	0	0	1
n3.n4	0	0	1
n4.n3	0	0	1
n5.n1	1	1	0
n6.n5	1	0	0
n7.n8	1	0	0
n7.n10	0	1	0
n8.n12	1	0	0
n9.n5	0	1	0
n10.n9	0	1	0
n11.n6	1	0	0
n12.n11	1	0	0

For better understanding these routes are depicted in Figures 4.3 - 4.5. The nodes that are serviced with individual vehicles are for this once in figures dark with black numbers.

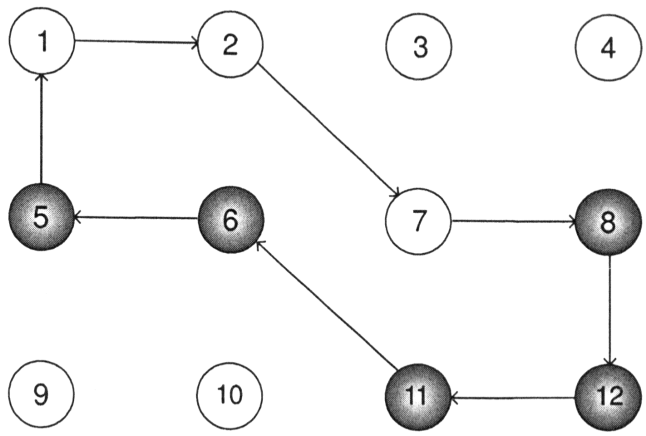

Figure 4.3: Nodes to service and route without subtours isolated from depot for the vehicle $veh1$

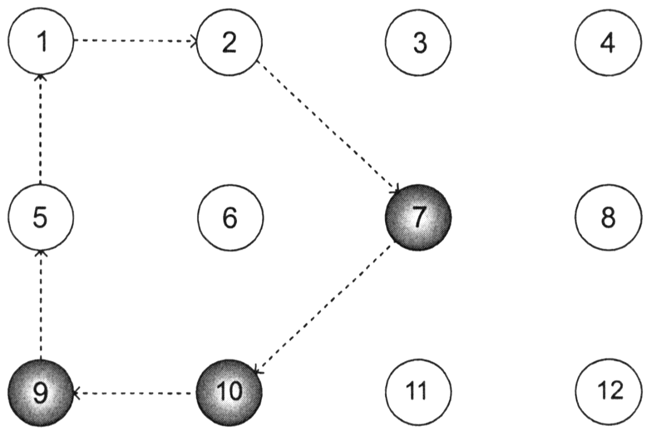

Figure 4.4: Nodes to service and route without subtours isolated from depot for the vehicle $veh2$

Let us compute the objective function value of this solution.

$$
\begin{aligned}
z' &= \sum_{h \in \mathbf{H}} \lambda_h \, h(x'_F, v'_F, \zeta_h) = \\
&= \sum_{h \in \mathbf{H}} \lambda_h \cdot \max_{l \in \mathbf{L}} \left\{ \sum_{(i,j) \in \mathbf{A}} \tau_{ij,h} \, x'_{F_{ijl}} + \sum_{i \in \mathbf{N}} \sigma_{i,h} \, v'_{F_{il}} \right\} = \\
&= \sum_{h \in \mathbf{H}} \lambda_h \cdot \max_{l \in \mathbf{L}} \{\theta_{l,h}\} .
\end{aligned}
$$

Since

```
PARAMETER thetal  length of route of vehicle l under scenario h

          veh1        veh2        veh3
1       33.500      35.500      29.500
2       51.500      46.500      46.500
3       64.500      59.000      56.500
4       79.500      73.500      68.500
5      101.000      89.500      84.500
```

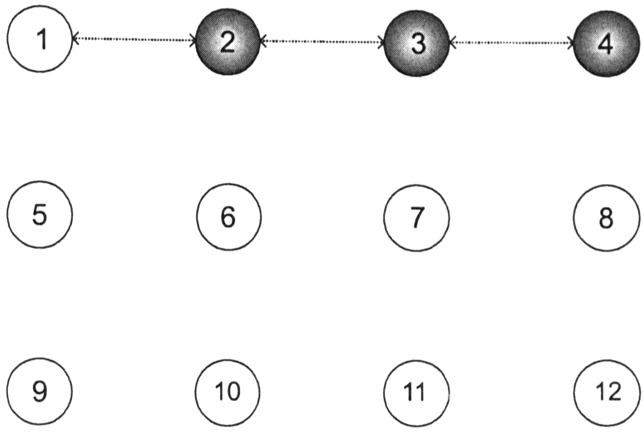

Figure 4.5: Nodes to service and route without subtours isolated from depot for the vehicle $veh3$

we get

$$z' = 0.2 \cdot 35.5 + 0.35 \cdot 51.5 + 0.15 \cdot 64.5 + 0.25 \cdot 79.5 + 0.05 \cdot 101 = 59.725.$$

As $z' < \overline{z} = +\infty$ we replace $\overline{z} = z' = 59.725$ and let $(x_F, v_F) = (x'_F, v'_F)$.

$$\frac{\overline{z} - \underline{z}}{\underline{z}} = \frac{59.725 - 46.3}{46.3} \doteq 0.3 > \delta = 0.05. \tag{4.34}$$

The STOP criterion from the Step 5 of the DESVRP Algorithm is not fulfilled therefore we add to the relaxation (RE) (problem (4.24) - (4.33) without the constraints of type (4.31)) the constraints of type (4.31) that will eliminate the isolated subtours $(8-12-8)$, $(6-11-6)$, $(9-10-9)$, $(3-4-3)$ and $(10-11-10)$ of the solution (x_R, v_R) and solve this relaxation of the original stochastic transportation problem.

Unfortunately, none of the GAMS solvers we have at our disposal was able to settle up with this problem. Therefore, we have to content with the solution (x_F, v_F). As results from the relation (4.34) the objective function value of this solution can maximally differ from the optimal objective function value by 30%.

4.4 Application of DESVRP Algorithm for Problem (M)

For the record we apply the DESVRP Algorithm to the problem (M). We use the same transportation problem as in the previous section but we assume that all random travel and service times take their mean values. Thus we get the problem

$$\min_{x,v,\theta} \theta \tag{4.35}$$

subject to (4.25) - (4.31) and

$$\theta_l = \sum_{i\in\mathbf{N}}\sum_{j\in\mathbf{N}} \mathrm{E}\tau_{ij}\, x_{ijl}\, \iota_{ij} + \sum_{i\in\mathbf{N}} \mathrm{E}\sigma_i\, v_{il} \qquad \forall l \in \mathbf{L} \tag{4.36}$$

$$\theta \geq \theta_l \qquad \forall l \in \mathbf{L} \tag{4.37}$$

where θ_l is the length of the route of the vehicle $l \in \mathbf{L}$ and θ is the length of the longest route.

The here presented results are very brief; further information and figures can be found on the appendant CD. We again use the model system GAMS and the solver MINOS 5.51 to solve. The relaxation (RE) can be in this case written as (4.35) - (4.37) plus (4.25) - (4.31) without the constrains of type (4.31). We get the solution (x_R, v_R) with objective function value $\underline{z} = 45.375$ that contains isolated subtours. After their removing the solution (x'_F, v'_F) with the following objective function value is obtained:

$$\begin{aligned} z' &= h(x'_F, v'_F, \mathrm{E}\zeta) = \\ &= \max_{l \in \mathbf{L}} \left\{ \sum_{(i,j) \in \mathbf{A}} \mathrm{E}\tau_{ij}\, x'_{F_{ijl}} + \sum_{i \in \mathbf{N}} \mathrm{E}\sigma_i\, v'_{F_{il}} \right\} = \\ &= \max_{l \in \mathbf{L}} \{\theta_l\} = \theta = 59.450. \end{aligned}$$

The STOP criterion from the Step 5 of the DESVRP algorithm is not fulfilled but again, after adding constraints of type (4.31) to the relaxation (RE), none of GAMS solver is able to settle up with this problem. Therefore, we have to content with the solution whose objective function value can maximally differ from the optimal objective function value by 30%.

4.5 Application of DESVRP Algorithm for Problem (P)

We apply the DESVRP Algorithm to the problem (P), too. We use the same stochastic transportation problem as in the section 4.3 but in this case we want to maximize the probability that the whole transportation operation is complete by the deadline $T = 100$. The results of this application can again be found on the appendant CD.

The objective function value of the solution (x'_F, v'_F) obtained by the Algorithm for this problem is 0.7 and can maximally differ from the optimal objective function value by 40%. This value is high, but unfortunately, we are not able to find with the assistance of this Algorithm better solution since none of GAMS solvers settles up with the problem with added constraints of type (4.31).

For the record let us notion that if we decide to use the solution (x'_F, v'_F) from the section 4.3 for this stochastic transportation problem we obtain the probability 0.95 that the whole transportation problem is complete before the target time T since

$$\begin{aligned} z' &= \sum_{h \in \mathbf{H}} \lambda_h \cdot \mathrm{I}(h(x'_F, v'_F, \zeta_h) \leq T) = \\ &= \sum_{h \in \mathbf{H}} \lambda_h \cdot \mathrm{I}\left(\max_{l \in \mathbf{L}} \left\{ \sum_{(i,j) \in \mathbf{A}} \tau_{ij,h}\, x'_{F_{ijl}} + \sum_{i \in \mathbf{N}} \sigma_{i,h}\, v'_{F_{il}} \right\} \leq T \right) = \\ &= \sum_{h \in \mathbf{H}} \lambda_h \cdot \mathrm{I}\left(\max_{l \in \mathbf{L}} \{\theta_{l,h}\} \leq T \right) = \\ &= 0.2 \cdot 1 + 0.35 \cdot 1 + 0.15 \cdot 1 + 0.25 \cdot 1 + 0.05 \cdot 0 = 0.95. \end{aligned}$$

Chapter 5

Conclusion

Let us summarize what we have in this diploma thesis done.

In the chapter 2 we have resumed three special stochastic transportation problems that can be formulated as network flow problems with random demands or random arc capacities – the two-stage transportation problem with random demands, the N-stage transportation problem with random arc capacities and the general N-stage network with random arc capacities. We have unitized the notation in these problems, particularly in recourse functions. We have extended the two-stage transportation problem with random demands in comparison with available literature to the two-stage transportation problem with random demands and with random arc capacities.

In the following chapter, in 3, we have written down some restrictive recourse strategies that can be used for solving the problems presented in the second chapter. The simple approximation has been applied to money distribution problem formulated as the the two-stage transportation problem with random demands and as the N-stage transportation problem with random arc capacities. The null recourse has been simplified to the single-optimization problem and the nodal recourse has been corrected.

In the chapter 4 we have presented the stochastic transportation problem with random travel and service times in two versions (the first one minimizes the expected completion time – problem (E), the second one maximizes the probability of transportation project completion by the deadline – problem (P)).

Some important properties of these stochastic transportation problems have been summarized in this chapter, too. We have supplemented the proofs of two propositions from [12]. Further we have added two counterexamples. The first one has showed that if we have more than one vehicle we cannot simplify the stochastic transportation problem (E) to the deterministic one and the second one has proved that we cannot generalize the convexity result of the second proposition to the multivehicle stochastic transportation problem (P).

In the end, we have described the DESVRP Algorithm and have applied it on three different money distribution problem formulated as the problem (E), as the problem (P) and as the deterministic transportation problem.

Bibliography

[1] Anděl J. (2005): Základy matematické statistiky. Matfyzpress, Praha.

[2] Birge J. R. (1982): The Value of theStochastic Solution in Stochastic Linear programs with Fixed Recourse. *Math. Programming* **24**, 314-325.

[3] Dantzig G. B., Fulkerson D. R., Johnson S. M. (1954): Solution of a Large-Scale Zero-One linear Programming Problems. *Oper. Res.* **2**, 393-410.

[4] Dantzig G. B. (1963): Linear Programming and Extensions. Princeton University Press, Princeton, New Jersey.

[5] Dupačová J. (1986): Stochastické programování. Ministerstvo školství, mládeže a tělovýchovy ČSR, Praha.

[6] Dupačová J., Hurt J., Štěpán J. (2002): Stochastic Modeling in Economics and Finance. Kluwer Academic Publishers, Preprint.

[7] Frantzeskakis L. F., Powell W. B. (1994): Restricted Recourse Strategies for Dynamic Networks with Random Arc Capacities. *Transportation Sci.* **28**, 3-23.

[8] Frantzeskakis L. F., Powell W. B. (1990): A Successive Linear Approximation Procedure for Stochastic Dynamic Vehicle Allocation Problems. *Transportation Sci.* **24**, 40-57.

[9] Kall P., Wallace S. W. (1994): Stochastic Programming. John Wiley & Sons, New York.

[10] Kataoka S. (1963): A Stochastic Programming Model. *Econometrica* **31**, 181-196.

[11] Kennington J. L., Helgason R. V. (1980): Algorithms for Network Programming. John Wiley & Sons, New York.

[12] Kenyon A. S., Morton D. P. (2003): Stochastic Vehicle Routing with Random Travel Times. *Transportation Sci.* **37**, 69-82.

[13] Lambert V., Laporte G., Louveaux F. (1993): Designing Collection Routes through Bank Branches. *Comput. Oper. Res.* **20**, 783-791.

[14] Laporte G., Louveaux F., Mercure H. (1992): The Vehicle Routing Problem with Stochastic Travel Times. *Transportation Sci.* **26**, 161-170.

[15] Laporte G., Louveaux F., Mercure H. (1994): A priori Optimization of the Probabilistic Traveling Salesman Problem. *Oper. Res.* **42**, 543-549.

[16] Lehmann E. L. (1991): Theory of Point Estimation. Wadsworth, Inc., Belmot, California.

[17] Loéve M. (1963): Probability Theory. D. Van Nostrand Company, Inc., Princeton, New Jersey.

[18] Rockafellar T. R., Wets R. J.-B. (1991): Scenarios and Policy Aggregation in Optimization Under Uncertatinty. *Math. Oper. Res.* **16**, 119-147.

[19] Wallace S. W. (1987): A Piecewise Linear Upper Bound on the Network Recourse Function. *Math. Programming* **38**, 133-146.