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Funkce kontinua na singulárních kardinálech

The Continuum Function on Singular Cardinals

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Prohlašuji, že jsem bakalářskou práci vypracovala samostatně a že jsem řádně citovala všechny použité prameny a literaturu a že práce nebyla využita v rámci jiného vysokoškolského studia či k získání jiného nebo stejného titulu.

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Abstrakt

Bakalářská práce se zabývá chováním funkce kontinua na singulárních kardinálech v teorii ZFC. Práce je rozdělena na dvě části. První část se soustředí na Silverovu větu a rozebírá dva různé důkazy této věty, původní Silverův a čistě kombinatorický důkaz dle Baumgartnera a Přikrého. Druhá část je věnována hypotéze singulárních kardinálů, která ovlivňuje chování funkce kontinua. V práce je ukázáno, za předpokladu velkých kardinálů, že hypotéza singulárních kardinálů je nedokazatelná nad teorií ZFC. Pomocí Eastonova a Přikrého forcingu je nalezen model ZFC ve kterém hypotéza singulárních kardinálů neplatí.

Klíčová slova

Singulární kardinál, SCH, Funkce kontinua.

Abstract

Bachelor thesis studies the behaviour of the continuum function on singular cardinals in theory ZFC. The work is divided into two part. The focus of the first part is on the Silver's Theorem and it analyzes two different proofs of this Theorem, Silver's original proof and the second, purely combinatorial, proof by Baumgartner and Prikry. The second part is devoted to the Singular Cardinal Hypothesis, which influences the behaviour of the continuum function. In the thesis it is shown that, in the presence of large cardinals, Singular Cardinal Hypothesis is not provable in ZFC. Using Easton and Prikry forcing a model is found where the Singular Cardinal Hypothesis does not hold.

Keywords

Singular cardinal, SCH, Continuum Function.

Obsah

1 Introduction

In 1871, Cantor proved that for any set X the set of all subsets of X has strictly greater size than X. A natural question is how many subsets there are? In modern set theory we can rephrase this question using continuum function, which each cardinal κ assigns 2^κ. This means that for any set X of cardinality κ, we can say how many subsets it has if we know the value of continuum function on κ. Now the question is what are we able to prove about the continuum function in ZFC or at least what behaviour of continuum function is consistent with ZFC.

In 1970, Easton showed that if GCH holds in V and $F : REG \to CARD$ is a class function from the class of regular cardinals to the class of all cardinals such that

$$\forall\kappa \in \mathrm{REG}\ F(\kappa) > \kappa,$$

$$\forall\kappa, \lambda \in \mathrm{REG}\ \kappa < \lambda \Rightarrow F(\kappa) \leq F(\lambda),$$

$$\forall\kappa \in \mathrm{REG}\ \mathrm{cf}(F(\kappa)) > \kappa,$$

then there exists a cofinality preserving generic extension $V[G]$ of V such that

$$V[G] \models \forall\kappa \in \mathrm{REG}\ 2^\kappa = F(\kappa).$$

This theorem says that values of continuum function on regular cardinals are quite independent of other values and that we can define continuum function almost arbitrarily. During the next years, it was generally believed that the domain of Easton function can be extended to singular cardinals. But the case of singular cardinals is more complicated. In 1974, Silver proved that if GCH holds at a stationary subset of singular strong limit cardinal κ of uncountable cofinality, then GCH holds also at κ. The original proof of Silver's theorem used forcing and the method of ultrapowers. Two years later, it was reworked by Baumgartner and Prikry to the proof which used only the knowledge of infinite combinatorics.

After this, a natural question rises. Could anything stronger be proved about the continuum function on singular cardinals? In the 70's Singular Cardinal Hypothesis (SCH) was formulated. It says that if κ is singular, then $\kappa^{\mathrm{cf}(\kappa)} = \max(2^{\mathrm{cf}(\kappa)}, \kappa^+)$. And so if κ is strong limit singular cardinal, then $2^\kappa = \kappa^+$. If SCH holds, then cardinal arithmetic is determined by the continuum function on the regular cardinals.

By Silver's theorem, SCH can not fail for the first time at singular cardinal with uncountable cofinality, but it is known that SCH can fail for the first time at singular cardinal with countable cofinality. We will show how to find a model of ZFC in which SCH fails. The role of large cardinals is important in this context. Jensen showed that if $0^\sharp$ does not exist, then SCH holds (see for instance [Kan03] or [Jec03]). This means that if we want to find a model in which SCH fails, we need a stronger theory than ZFC (i.e. ZFC with some large cardinals). Work of Gitik ([Git89] and [Git91]) shows that the consistency strength of failure of SCH is exactly a measurable cardinal κ with $o(\kappa) = \kappa^{++}$ (see for instance [Kan03] or [Jec03]).

2 Preliminaries

In this section we present some definitions and lemmas important in proofs in chapters three and four. For details and proofs see [Jec03] or [Kan03].

2.1 Stationary Sets and Closed Unbounded Filter

Stationary sets play a fundamental role in modern set theory. The idea of stationary sets first appeared in 1950's and it is ascribed to G. Bloch. Basic theorems about stationary sets were proved by Fodor. In this chapter we present some basic definitions and claims that are important for understanding both proofs of Silver's theorem.

Definition 2.1 Let A be a set. A system $F \subseteq P(A)$ is called a *filter* if:

(i) $A \in F$,

(ii) if $X \in F$ and $X \subseteq Y$ then $Y \in F$,

(iii) if $X \in F$ and $Y \in F$ then $X \cap Y \in F$.

A filter F is called *proper* iff $\emptyset \notin F$.

Definition 2.2 Let A be a set. A proper filter $F \subseteq P(A)$ is called an *ultrafilter* if for all $X \subseteq A$, either $X \in F$ or $A \setminus X \in F$.

Definition 2.3 Let A be a set. If κ is a regular uncountable cardinal and F is a filter on A, then F is called *κ-complete* if F is closed under intersection of less than κ sets, i.e. if $\{X_\alpha | \alpha < \gamma\}$ is a family of elements of F for $\gamma < \kappa$, then

$$\bigcap_{\alpha<\gamma} X_\alpha \in F.$$

Definition 2.4 Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a sequence of subsets of κ. The *diagonal intersection* of X_α, where $\alpha < \kappa$, is defined as follows:

$$\Delta_{\alpha<\kappa} X_\alpha = \left\{ \beta < \kappa : \beta \in \bigcap_{\alpha<\beta} X_\alpha \right\}.$$

Definition 2.5 Let F be a filter on a cardinal κ. F is *normal* if it is closed under diagonal intersections:

$$\text{if } X_\alpha \in F \text{ for all } \alpha < \kappa, \text{ then } \Delta_{\alpha<\kappa} X_\alpha \in F.$$

Definition 2.6 Let κ be a limit ordinal. A set $C \subseteq \kappa$ is *closed unbounded* if:

(i) $\sup C = \kappa$,

(ii) all limit ordinals $\alpha < \kappa$,

$$\sup(C \cap \alpha) = \alpha \rightarrow \alpha \in C.$$

Definition 2.7 If $\mathrm{cf}(\kappa) > \omega$, then

$$Club(\kappa) = \{X \subseteq \kappa : (\exists C \subseteq X)(C \text{ is closed unbounded in } \kappa)\}$$

is closed unbounded filter on κ.

The dual of the closed unbounded filter is the ideal of nonstationary sets, the nonstationary ideal I_{NS}.

Definition 2.8 Let κ be a cardinal such that $\mathrm{cf}(\kappa) > \omega$. A set $S \subseteq \kappa$ is *stationary* if $S \notin I_{NS}$.

In previous definitions we considered just cardinals with uncountable cofinality. It is beacause when $\mathrm{cf}(\kappa) = \omega$, then every cofinal subset of κ of order type ω is closed unbounded set. It is easy to find two such sets which are also disjoint and so we cannot consider filter generated by closed unbounded sets at cardinals with countable cofinality.

We can show that $Club(\kappa)$ is not an ultrafilter and so the property to be a stationary set is not the same as to be an element of $Club(\kappa)$.

It is obvious that a set $X \subseteq \kappa$ is stationary if and only if for every closed unbounded subset of κ, its intersection with X is nonempty. This characterization is very useful when we want to show that some set is stationary.

Fact 2.9 *$Club(\kappa)$ is a normal and κ-complete filter.*

Definition 2.10 An ordinal function on a set S is *regressive* if $f(\alpha) < \alpha$ for every $\alpha \in S$.

The following frequently used lemma describes stationary sets using regressive functions. Every regressive function at stationary set is constant at stationary many points.

Lemma 2.11 (Fodor) *If f is regressive function on a stationary set $S \subseteq \kappa$, then there is a stationary set $S' \subseteq S$ and some $\alpha < \kappa$ such that $f(\xi) = \alpha$ for all $\xi \in S'$.*

Proof: Assume for contradiction that for each $\alpha < \kappa$, the set $\{\xi \in S : f(\xi) = \alpha\}$ is not stationary and so $C_\alpha = \kappa \backslash \{\xi \in S : f(\xi) = \alpha\}$ is in $Club(\kappa)$. Note that for each $\xi \in S \cap C_\alpha$, $f(\xi) \neq \alpha$. Since $Club(\kappa)$ is normal, $C = \Delta_{\alpha<\kappa} C_\alpha$ is in $Club(\kappa)$ and so $S \cap C$ is stationary. If $\xi \in S \cap C$, then $f(\xi) \neq \beta$ for each $\beta < \xi$. Hence $f(\xi) \geq \xi$ and this is a contradiction. □

Fodor's Lemma uses the normality of $Club(\kappa)$. Next lemma shows that using regressive functions, we can define normal filter.

Lemma 2.12 *For a κ-complete ultrafilter U over $\kappa > \omega$ the following are equivalent:*

(i) U is normal.

(ii) Whenever $f : \kappa \to \kappa$ and $\{\xi < \kappa : f(\xi) < \xi\} \in U$, there is some $\alpha < \kappa$ such that $\{\xi < \kappa : f(\xi) = \alpha\} \in U$.

Proof: (i)⇒(ii) Let $f : \kappa \to \kappa$ be such that $\{\xi < \kappa : f(\xi) < \xi\} \in U$ and for every $\alpha < \kappa, \{\xi < \kappa : f(\xi) = \alpha\} \notin U$. Since U is an ultrafilter,

$$X_\alpha = \{\beta < \kappa : f(\beta) \neq \alpha\} \in U$$

for every $\alpha < \kappa$. Since U is normal,

$$\Delta_{\alpha<\kappa} X_\alpha = \left\{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi} \{\beta < \kappa : f(\beta) \neq \alpha\}\right\} \in U \Leftrightarrow$$

$$\{\xi < \kappa : \forall \alpha < \xi (f(\xi) \neq \alpha)\} \in U \Leftrightarrow \{\xi < \kappa : f(\xi) \geq \xi\} \in U.$$

So we have $\{\xi < \kappa : f(\xi) \geq \xi\} \in U$ and $\{\xi < \kappa : f(\xi) < \xi\} \in U$, which is a contradiction.

(ii)⇒(i) Suppose $\langle X_\alpha : \alpha \in \kappa \rangle$ is a sequence in U such that the diagonal intersection $\Delta_{\alpha<\kappa} X_\alpha \notin U$. Since U is ultrafilter, $\kappa \backslash \Delta_{\alpha<\kappa} X_\alpha \in U$. Let $f : \kappa \backslash \Delta_{\alpha<\kappa} X_\alpha \to \kappa$ be such that $f(\alpha) = \min\{\beta < \alpha : \alpha \in (\kappa \backslash X_\beta)\}$. Since f is regressive, there is $\beta < \kappa$ such that $\{\xi < \kappa : f(\xi) = \beta\} \in U$ but $\{\xi < \kappa : f(\xi) = \beta\} \subseteq \kappa \backslash X_\beta$, which contradicts the assumption that $X_\beta \in U$. □

The following lemma shows that it is enough to consider just the stationary sets at regular cardinals strictly greater than ω.

Lemma 2.13 *Let κ be a singular cardinal such that $\xi = cf(\kappa) > \omega$. If $f : \xi \to \kappa$ is a normal function converging to κ, then for any $A \subseteq \kappa$ the following holds:*

(i) $A \in Club(\kappa) \leftrightarrow \{\alpha < \xi : f(\alpha) \in A\} \in Club(\xi)$,

(ii) A *is stationary in* $\kappa \leftrightarrow \{\alpha < \xi : f(\alpha) \in A\}$ *is stationary in* ξ.

Proof: Ad (i). Let $A \in Club(\kappa)$ be given. Since $Rng(f)$ is closed unbounded in κ, then $Rng(f) \cap A$ is also closed unbounded in κ, and so

$$\{\alpha < \kappa : f(\alpha) \in A\} = f^{-1}[Rng(f) \cap A] \in Club(\xi).$$

Conversely, it is obvious that image of closed unbounded set is closed unbounded.

Ad (ii). Let A be stationary in κ and let C be closed unbounded in ξ. We want to show that $\{\alpha < \kappa : f(\alpha) \in A\} \cap C \neq \emptyset$ and so $\{\alpha < \kappa : f(\alpha) \in A\}$ is stationary. If C is closed unbounded in ξ, then $f[C]$ is closed unbounded in κ. Since A is stationary in κ, $A \cap f[C] \neq \emptyset$ and so $A \cap Rng(f) \cap f[C] \neq \emptyset$. It follows that

$$f^{-1}[Rng(f) \cap A] \cap C \neq \emptyset \Leftrightarrow \{\alpha < \kappa : f(\alpha) \in A\} \cap C \neq \emptyset.$$

Converse direction is analogous. □

2.2 Ultrapowers and Elementary Embedding

Definition 2.14 Let $\{A_i : i \in I\}$ be a family of non-empty sets, we define the *product* $\prod_{i \in I} A_i$ as follows:

$$\prod_{i \in I} A_i = \left\{ f : I \to \bigcup_{i \in I} A_i : (\forall i \in I) f(i) \in A_i \right\}.$$

Definition 2.15 Let $f, g \in \prod_{i \in I} A_i$. Let U be an ultrafilter on I. We say that f, g are *U-equivalent*, in symbols $g \equiv_U f$, if

$$\{i \in I : f(i) = g(i)\} \in U.$$

Definition 2.16 We say that $\prod_U \mathfrak{A}_i$ for a language L is an *ultraproduct* of $\{\mathfrak{A}_i : i \in I\}$ by U if:

(i) The domain of $\prod_U \mathfrak{A}_i$ is the set $\prod_U A_i = \left\{[f]_U : f \in \prod_{i\in I} A_i\right\}$.

(ii) If $\dot{c}$ is a constant, then the realization $c_{\prod_U \mathfrak{A}_i}$ of $\dot{c}$ is the equivalence class $[c]$, where c is the function such that $c(i) = c_{\mathfrak{A}_i}$ for each i in I.

(iii) If $\dot{f}$ is an n-ary function symbol, then the realization $f_{\prod_U \mathfrak{A}_i}$ of $\dot{f}$ is defined as follows. For $[f_1]_U, ..., [f_n]_U \in \prod_U A_i$, we define

$$f_{\prod_U \mathfrak{A}_i}([f_0]_U, ..., [f_n]_U) = [F],$$

where $F(i) = f_{\mathfrak{A}_i}(f_1(i), ..., f_n(i))$ for every $i \in I$.

(iv) If $\dot{P}$ is an n-ary relation symbol, then the interpretation $P_{\prod_U \mathfrak{A}_i}$ of $\dot{P}$ is defined as follows. For every $[f_1]_U, ..., [f_n]_U \in \prod_U A_i$, we define

$$P([f_1]_U, ..., [f_n]_U) \Leftrightarrow \{i \in I : P_{\mathfrak{A}_i}(f_1(i), ..., f_n(i))\} \in U.$$

Definition 2.17 Let $\mathfrak{A}$ be a structure. If in previous definition for each $i \in I$, $\mathfrak{A}_i = \mathfrak{A}$, then we call this special ultraproduct an *ultrapower*.

Definition 2.18 Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures for a language L. We say that a 1-1 function $f : \mathfrak{A} \to \mathfrak{B}$ is an *elementary embedding* if for every formula $\varphi(x_1, ..., x_n)$ and elements $a_1, ..., a_n$ in A (evaluating all free variables in $\varphi(x_1, ..., x_n)$)

$$\mathfrak{A} \models \varphi(a_1, ..., a_n) \Leftrightarrow \mathfrak{B} \models \varphi(f(a_1), ..., f(a_n)).$$

Theorem 2.19 (Łos) *Let U be an ultrafilter over I and let $\mathfrak{A}$ be the ultraproduct of $\{\mathfrak{A}_i : i \in I\}$ by U. If φ is a fomula, then for every $f_1, ..., f_n \in \prod_{i\in I} A_i$,*

$$\mathfrak{A} \models \varphi([f_1], ..., [f_n]) \textit{ iff } \{i \in I : \mathfrak{A}_i \models \varphi(f_1(i), ..., f_n(i))\} \in U.$$

Proof: See [Jec03], Theorem 12.3, page 159.

We will now extend the ultrapower construction to the proper class. This method was introduced by Dana Scott when he proved in [Sco61] that if there is a measurable cardinal, then $V \neq L$, where L is class of all constructible sets.

Let U be an ultrafilter over a set I and let f and g be functions from I to V. We define

$$f =^* g \text{ iff } \{i \in I : f(i) = g(i)\} \in U,$$

$$f \in^* g \text{ iff } \{i \in I : f(i) \in g(i)\} \in U.$$

Let $Ult_U(V)$ be the class off all $[f]$, where f is a function on I, and consider the model $(Ult_U(V), \in^*)$. The problem is that $[f]$ is a proper class but we can use the axiom of foundation and reduce the class to a set. By the Theorem of Łos, $(Ult_U(V), \in^*)$ is elementary equivalent to the universe $(V, \in)$ and we can define elementary embedding $j : V \to (Ult_U(V), \in^*)$ this way: $j(a) = [c_a]$, where c_a is a constant function defined for every set a. Another problem is that $(Ult_U(V), \in^*)$ may not be well-founded but at least the following fact holds.

Fact 2.20 *If U is σ-complete ultrafilter, then $(Ult_U(V), \in^*)$ is well-founded.*

So $(Ult_U(V), \in^*)$ is a model of ZFC which may not be transitive but since $\in^*$ is well-founded, set-like and extensional, we can use Mostowski's Collapsing Theorem (See [Jec03], Theorem 6.15, page 69.) to obtain a transitive model ZFC which is isomorphic with $(Ult_U(V), \in^*)$.

Definition 2.21 We say that d is a *diagonal function* on κ if for every $\alpha < \kappa$, $d(\alpha) = \alpha$.

Lemma 2.22 *Let U be a κ-complete normal ultrafilter over κ and let d be a diagonal function on κ. Let $j : V \to M \cong Ult_U(V)$ be an elementary embedding derived from U. Then:*

(i) for every $\alpha < \kappa$, $j(\alpha) = \alpha$,

(ii) $j(\kappa) \neq \kappa$,

(iii) $\kappa = [d]$.

Proof: Ad (i). Assume for contradiction that $\alpha < \kappa$ is the least ordinal such that $\alpha < j(\alpha)$. Let $[f] = \alpha$. Then

$$[f] = \alpha < j(\alpha) = [c_\alpha] \Leftrightarrow \{\xi < \kappa : f(\xi) < c_\alpha(\xi)\} \in U \Leftrightarrow \{\xi < \kappa : f(\xi) < \alpha\} \in U$$

and so by κ-completness there is $\beta < \alpha$ such that $\{\xi < \kappa : f(\xi) = \beta\} \in U$. But then $\alpha = [f] = \beta$ is a contradiction.

Ad (ii). If U is κ-complete, then for every $\alpha < \kappa$, $\alpha \notin U$. Thus we have $\alpha < d(\beta)$ for almost all β and so $j(\alpha) = \alpha < [d]$ for every $\alpha < \kappa$. It follows that $\kappa \leq [d]$. Since $\alpha = d(\alpha) < \kappa$ for all $\alpha < \kappa$ and so, by definition of j,$[d] < j(\kappa)$. So we have

$$\kappa \leq [d] < j(\kappa).$$

Ad (iii). Since we have proved that $\kappa \leq [d]$, it suffices to show that $[d] \leq \kappa$. Let $[f] < [d]$ be given. Hence

$$\{\xi < \kappa : f(\xi) < d(\xi)\} \in U \Leftrightarrow \{\xi < \kappa : f(\xi) < \xi\} \in U.$$

Since U is normal, there is $\beta < \kappa$ such that the set $\{\xi < \kappa : f(\xi) = \beta\} \in U$. Thus we have $[f] = \beta < \kappa$, and so $\kappa = [d]$. □

2.3 Forcing

Forcing was invented by Paul Cohen. It was first used, in 1963, to prove independence of the Continuum Hypothesis (see [Coh63]). The main idea of forcing is to extend a transitive model of ZFC by adjoining a new set. The new set is approximated by a forcing notion which is a partially ordered set in the ground model.

Definition 2.23 Let P be a set and let $\leq_P$ be a binary relation on P. We say that $\leq_P$ is a *preorder* if it is reflexive and transitive.

Definition 2.24 Let P be preordered by $\leq_P$. We say that $E(P)$ is a *quotient version* of P if the domain $E(P)$ is the set of equivalence classes under $p \leftrightarrow q$ iff $p \leq_P q$ and $q \leq_P p$. The partial ordering $\leq_{E(P)}$ on $E(P)$, we define $[p] \leq_{E(P)} [q]$ iff $p \leq_P q$.

Note that $E(P)$ is always a partial order.

Definition 2.25 Let P, Q be two preorders. We say that $i : P \to Q$ is a *dense embedding* if:

(i) $\forall p, p' \in P(p \leq_P p' \to i(p) \leq_Q i(p'))$,
(ii) $\forall p, p' \in P(p \perp p' \leftrightarrow i(p) \perp i(p'))$,
(iii) $i''P$ is dense in Q.

In the Definition 2.25 and in the definition of forcing notion it is more common to consider ordered sets. However, we frequently use preorders as sets of forcing condition.

Definition 2.26 Let P, Q be two forcing notions. P and Q are *forcing equivalent* if they have the same generic extensions.

Fact 2.27 *Let $i : P \to Q$ be a dense embedding. Then:*
(i) If G is P-generic, then $H = \{q \in Q : \exists p \in G(i(p) \leq_Q q)\}$ is Q-generic.
(ii) If H is Q-generic, then $i^{-1}{''}H$ is P-generic.
(iii) In (i) and (ii), $M[G] = M[H]$.

Proof: See [Kun80], Theorem 7.11, page 221.

Lemma 2.28 *If P is preorder and $E(P)$ is a quotient version of P, then P and $E(P)$ are forcing equivalent.*

Proof: By Fact 2.27, it suffices to show that there is dense embedding $i : P \to E(P)$. Define i so that it assigns to $p \in P$ an equivalence class $[p]$. We need to check the following:
(i) $\forall p, q \in P(p \leq_P q \to i(p) \leq_{E(P)} i(q))$,
(ii) $\forall p, q \in P(p \perp q \leftrightarrow i(p) \perp i(q))$,
(iii) $i''P$ is dense in $E(P)$.

Ad (i). Let $p \leq_P q \in P$. Then by definition of $\leq_{E(P)}$, $[p] \leq_{E(P)} [q]$ and so $i(p) \leq_{E(P)} i(q)$.

Ad (ii). ($\Leftarrow$) Let p and q be compatible, then there is $r \in P$ such that $r \leq_P p$ and $r \leq_P q$ and so $[r] \leq_{E(P)} [p]$ and $[r] \leq_{E(P)} [q]$.

($\Rightarrow$) Let $[p]$ and $[q]$ be compatible, then there is $[r] \in E(P)$ such that $[r] \leq_{E(P)} [p]$ and $[r] \leq_{E(P)} [q]$ and so by the definition $\leq_{E(P)}$, $r \leq_P q$ and $r \leq_P p$. Hence p and q are compatible.

Ad (iii). Let $[p]$ be an element of $E(P)$, we want to show that there is $[q] \in i''P = \{[p] \in E(P) : \exists r \in P(i(r) = [p])\}$ such that $[q] \leq_{E(P)} [p]$. Since i is onto, $[p] \in i''P$ and we can take $[q] = [p]$. $\square$

3 Silver's Theorem

The original proof of Silver's Theorem, published in [Sil75], used the forcing and the ultrapower constructions. Two years later, it was reworked by Baumgartner and Prikry ([BP76] and [BP77]) to the proof which uses only the knowledge of infinite combinatorics. Both proofs can be found in [Jec03].

3.1 The Original Proof

In this proof, a generic elementary embedding was used for the first time. Let M be a ground model and let κ be a regular uncountable cardinal. Let I be an ideal on κ and let us consider generic extension of M given by the completion of the Boolean algebra $P(\kappa)/I$. Let G be generic filter on $P(\kappa)/I$ and let us consider in $M[G]$ the ultrapower $Ult_G(M)$. Then we call the canonical embedding $j : M \to Ult_G(M)$ generic elementary embedding.

Definition 3.1 Let M be a transitive model of ZFC and let κ be a cardinal in M. We say that U is *M-Ultrafilter* on κ if:

(i) $\kappa \in U$ and $\emptyset \notin U$,

(ii) if $X \in U$ and $Y \in U$, then $X \cap Y \in U$,

(iii) if $X \in U$ and $Y \in M$ is such that $X \subseteq Y$, then $Y \in U$,

(iv) for every $X \subseteq \kappa$ such that $X \in M$, either X or $\kappa \backslash X$ is in U.

We say that U is M-κ-complete if whenever $\alpha < \kappa$ and $\{X_\xi : \xi < \alpha\} \in M$ is such that $X_\xi \in U$ for all $\xi < \alpha$, then $\bigcap_{\xi<\alpha} X_\xi \in U$.
We say that U is M-normal if whenever $f \in M$ is a regressive function on $X \in U$, then f is constant on some $Y \in U$.

Definition 3.2 Let κ be a regular cardinal. Two functions, f and g, on κ are *almost disjoint* if $|\{\alpha : f(\alpha) = g(\alpha)\}| < \kappa$.

Theorem 3.3 (Silver) *Let κ be a singular cardinal such that $cf(\kappa) > \omega$. If $2^\alpha = \alpha^+$ for all cardinals $\alpha < \kappa$, then $2^\kappa = \kappa^+$.*

Proof: To simplify notation we shall consider the special case when $\mathrm{cf}(\kappa) = \omega_1$. The general case is proved in a similar way. Let κ be a singular cardinal of cofinality ω_1 and assume that $2^\alpha = \alpha^+$ for all $\alpha < \kappa$. We want to prove that $2^\kappa = \kappa^+$. Let I_{NS} be the ideal of nonstationary subsets of ω_1. Let us consider a forcing notion $(P, \leq)$, where $P = \{S \subseteq \omega_1 : S \notin I_{NS}\}$ and $S \leq S'$ iff $S \backslash S' \in I_{NS}$.

Note that $P(\omega_1)/I_{NS} = E(P)$ for the notion of forcing $(P, \leq)$, where $P(\omega_1)/I_{NS}$ is ω_1-complete boolean algebra.

Let G be a P-generic filter over M. Let us work in $M[G]$. The filter G is:

(i) M-Ultrafilter on ω_1 extending the $(Club(\omega_1))^M$,

(ii) M-ω_1-complete M-Ultrafilter,

(iii) M-normal.

Ad (i). We first show that G is an M-Ultrafilter. It suffices to show that if $X \in M$ and $X \subseteq \omega_1$, then

$$\{Y \in P : Y \leq X \text{ or } Y \leq \omega_1 \backslash X\}$$

is dense. Let S be given, we want to show that $S \cap X$ or $S \cap \omega_1 \backslash X$ is stationary. Assume for contradiction that $S \cap X \in I_{NS}$ and $S \cap \omega_1 \backslash X \in I_{NS}$. Then $(S \cap X) \cup (S \cap \omega_1 \backslash X) \in I_{NS}$ and so $S \in I_{NS}$. This is a contradiction.

Now we show that G extends $(Club(\omega_1))^M$. Let $C \in (Club(\omega_1))^M$, then the set $\{S \in P : S \subseteq C\}$ is dense. Let $S' \in P$ be given. Then $S' \cap C \in P$ and clearly $S' \cap C \leq S$.

Ad (ii). If $\{X_n : n \in \omega\} \in M$ is a partition of ω_1, then $\{Y \in P : \exists n \in \omega (Y \leq X_n)\}$ is dense in P. Let $S \in P$ be given. Assume for contradiction that for each n, it holds that $X_n \cap S \in I_{NS}$. Then by ω_1-completeness of I_{NS}, $\bigcup_{n<\omega} (S \cap X_n) \in I_{NS}$. Hence $S \in I_{NS}$, which is a contradiction.

Ad (iii). If $X \in G$ and if $f \in M$ is a regressive function on X, then $\{Y \leq X : f \text{ is constant on } Y\}$ is dense below X by Fodor's Lemma. Since f is constant on some $Y \in G$, G is normal.

Let us consider in $M[G]$ the ultrapower $N = Ult_G(M)$. Let $j : M \to N$ be the canonical elementary embedding.

Let $\langle \kappa_\alpha : \alpha < \omega_1 \rangle$ be in M an increasing continuous sequence of cardinals converging to κ. Let e be the cardinal number in N represented by function $e(\alpha) = \kappa_\alpha$.

For each $X \subseteq \kappa$ in M, let f_X be the function on ω_1^M defined by $f_X(\alpha) = X \cap \kappa_\alpha$. Each f_X represents in N a subset of e and if $X \neq Y$, then there is $\beta < \omega_1$ such that $f_X(\alpha) \neq f_Y(\alpha)$ for all $\alpha > \beta$. It follows that f_X and f_Y represent in N distinct subsets of e and hence $M[G] \models |P^M(\kappa)| \leq |P^N(e)|$. Since $M \models 2^{\kappa_\alpha} = \kappa_\alpha^+$ for all α,

$$\left\{\alpha \in \omega_1 : 2^{\kappa_\alpha} = \kappa_\alpha^+\right\} \in G \Leftrightarrow \left\{\alpha \in \omega_1 : 2^{e(\alpha)} = e(\alpha)^+\right\} \in G \Leftrightarrow N \models 2^e = e^+.$$

The ordinal numbers of the model N make a linearly ordered class, which is not necessarily well-founded, but since $N \models 2^e = e^+$ and N satisfies ZFC, there is F in $M[G]$ which is a bijection between $P^N(e) = \{x \in N : N \models x \subseteq e\}$ and $\left\{x \in N : x <^N e^+\right\}$.

And so

$$|P^M(\kappa)| \leq^{M[G]} |P^N(e)| =^{M[G]} |\left\{x \in N : x <^N e^+\right\}|. \tag{3.1}$$

Next, we observe that $e = \sup\left\{j(\kappa_\gamma) : \gamma < \omega_1^M\right\}$. By definition of supremum, it suffices to show two things:

(i) That e is the upper bound of $\left\{j(\kappa_\gamma) : \gamma < \omega_1^M\right\}$.

(ii) That e is the least upper bound.

Ad (i). We want to show that for all $j(\kappa_\alpha)$, $j(\kappa_\alpha) <^N e$. Let $j(\kappa_\alpha)$ be given. Let c_{κ_α} be a function such that $c_{\kappa_\alpha}(\beta) = \kappa_\alpha$ for all β, then by definition of j:

$$j(\kappa_\alpha) = [c_{\kappa_\alpha}]$$

and so for all $\beta > \alpha$, it holds that $e(\beta) = \kappa_\beta > \kappa_\alpha = c_{\kappa_\alpha}(\beta)$. It follows that $\{\alpha \in \omega_1 : c_{\kappa_\alpha}(\alpha) < e(\alpha)\} \in G$, i.e. $j(\kappa_\alpha) < e$.

Ad (ii). Let $[f] < e$, we want to show that $[f] < j(\kappa_\gamma)$ for some γ.

$$[f] < e \Leftrightarrow \{\alpha < \omega_1 : f(\alpha) < \kappa_\alpha\} \in G$$

Since the set of all limit ordinals α less than ω_1 is closed unbounded set in ω_1, there is a set of limit ordinals $X \in G$ such that $f(\alpha) < \kappa_\alpha$ for all $\alpha \in X$. Hence we can choose $g : \omega_1 \to \omega_1$ such that $f(\alpha) < \kappa_{g(\alpha)}$ where $g(\alpha) < \alpha$ and by normality of G, there is a set $Y \subseteq X$ in G such that g is constant on Y, i.e. there is γ such that:

$$\{\alpha < \omega_1 : f(\alpha) < \kappa_\gamma\} \in G \Leftrightarrow [f] < j(\kappa_\gamma).$$

Since for each $\alpha < \omega_1^M$, $|\{x \in N : x \in_N j(\kappa_\alpha)\}| \leq |(\kappa_\alpha^{\aleph_1})^M| < \kappa$,

$$M[G] \models |\left\{x \in N : x <^N e\right\}| \leq \kappa.$$

If $x <^N e^+$, then there is in N a one-to-one mapping of x into e, and therefore $|\left\{y \in N : y <^N x\right\}| \leq |\left\{y \in N : y <^N e\right\}| \leq \kappa$. Thus $\left\{x \in N : x <^N e^+\right\}$ is a linearly ordered set whose each initial segment has size at most κ and so

$$|\left\{x \in N : x <^N e^+\right\}| \leq \kappa^+.$$

By (3.1) we have

$$M[G] \models |P^M(\kappa)| \leq \kappa^+.$$

We have proved that $|P^M(\kappa)|^{M[G]} \leq (\kappa^+)^{M[G]}$. But since $|P| = 2^{\aleph_1} < \kappa$ (in M), all cardinals greater than κ and especially κ^+ remain cardinals in $M[G]$. This is because P satisfies the $2^{\aleph_1}$-chain condition. Hence $M[G] \models |P^M(\kappa)|^M \leq (\kappa^+)^M$ and since M is model of ZFC and $M \subseteq M[G]$, we have $M \models 2^\kappa = \kappa^+$. □

Corollary 3.4 *Let κ be a singular cardinal such that cf(κ) $> \omega$. If the set $A = \{\alpha < \kappa : 2^\alpha = \alpha^+\}$ is stationary, then $2^\kappa = \kappa^+$.*

Proof: By Lemma 2.13, it suffices (in proof of Silver's Theorem) to guarantee that for the set $\{\alpha < \mathrm{cf}(\kappa) : e(\alpha) \in A\}$ there is some generic G such that $\{\alpha < \mathrm{cf}(\kappa) : e(\alpha) \in A\}$ is in G. But this follows from Rasiowa-Sikorski's Lemma, which says that if M is a transitive countable model of ZF and $P \in M$ is a forcing notion, then there exist a generic filter G over M below every $p \in P$. □

3.2 The Combinatorial Proof

This proof originated from the first proof by replacing the forcing technique by a purely combinatorial argument.

Lemma 3.5 *Let $\langle \kappa_\alpha : \alpha < \omega_1 \rangle$ be an increasing continuous sequence of cardinals converging to κ. Assume that $\kappa_\alpha^{\aleph_1} < \kappa$ for all $\alpha < \omega_1$. Let A_α be some set and let F be an almost disjoint family of functions*

$$F \subseteq \prod_{\alpha < \omega_1} A_\alpha$$

such that the set

$$\{\alpha < \omega_1 : |A_\alpha| \leq \kappa_\alpha\}$$

is stationary. Then $|F| \leq \kappa$.

Proof: We can assume that each A_α is a set of ordinals and that $A_\alpha \subseteq \kappa_\alpha$ for all α in some stationary subset of $\aleph_1$. Let

$$S' = \{\alpha < \omega_1 : \alpha \text{ is limit ordinal and } A_\alpha \subseteq \kappa_\alpha\}.$$

Let $f \in F$ and $\alpha \in S'$. By definition of S', $A_\alpha \subseteq \kappa_\alpha$ and so $f(\alpha) < \kappa_\alpha$. Since α is limit, there is $\beta < \alpha$ such that $f(\alpha) < \kappa_\beta$. Now we define a function g, so that it assigns to $\alpha \in S'$ this β. The function g is regressive on S' and by Fodor's Lemma there is a stationary set $S = \{\alpha \in S' : g(\alpha) = \gamma\}$. Hence $f(\alpha) < \kappa_\gamma$ for all $\alpha \in S$.

So we can assign to each f a pair $(S, f \restriction S)$ where $S \subseteq S'$ is a stationary set and $f \restriction S$ is bounded function. For any S, if $f \restriction S = g \restriction S$, then $f = g$ since any two distinct functions in F are almost disjoint. So we have one to one correspondence between f and $(S, f \restriction S)$.

For a given S, the number of bounded function on S is at most

$$\sum_{\alpha < \omega_1} \kappa_\alpha^{|S|} = \sup_{\alpha < \omega_1} \kappa_\alpha^{\aleph_1} = \kappa.$$

Since $|P(\omega_1)| = 2^{\aleph_1} < \kappa$, the number of pairs $(S, f \restriction S)$ is at most κ and so $|F| \leq \kappa$.
□

Lemma 3.6 *Let $\langle \kappa_\alpha : \alpha < \omega_1 \rangle$ be an increasing continuous sequence of cardinals converging to κ. Assume that $\kappa_\alpha^{\aleph_1} < \kappa$ for all $\alpha < \omega_1$. Let A_α be some set and let H be an almost disjoint family of functions*

$$H \subseteq \prod_{\alpha < \omega_1} A_\alpha$$

such that the set

$$\{\alpha < \omega_1 : |A_\alpha| \le \kappa_\alpha^+\}$$

is stationary. Then $|H| \le \kappa^+$.

Proof: Let U be an ultrafilter on ω_1 such that it extends the closed unbounded filter on ω_1.

We may assume that each A_α is subset of $\kappa_{\alpha+1}$. We can define

$$f < g \text{ iff } \{\alpha < \omega_1 : f(\alpha) < g(\alpha)\} \in U.$$

We claim that the relation $f < g$ is a linear ordering of H. We need to check that the relation is:

(i) antireflexive and transitive,

(ii) linear.

Ad (i). For antireflexivity, the set $\{\alpha \in \omega_1 : f(\alpha) \not< f(\alpha)\}$ is in U. For transitivity, let $f < g$ and $g < h$. This is equivalent to $A = \{\alpha \in \omega_1 : f(\alpha) < g(\alpha)\} \in U$ and $B = \{\alpha \in \omega_1 : g(\alpha) < h(\alpha)\} \in U$ and since $A \cap B \in U$, we have $f < h$.

Ad (ii). Let $f, g \in H$. If f, g are distinct, then since f and g are almost disjoint, $\{\alpha \in \omega_1 : f(\alpha) = g(\alpha)\} \notin U$. If $\{\alpha \in \omega_1 : f(\alpha) < g(\alpha)\} \notin U$, then since U is ultrafilter, the set $\omega_1 \setminus \{\alpha \in \omega_1 : f(\alpha) = g(\alpha)\} \cup \{\alpha \in \omega_1 : f(\alpha) < g(\alpha)\} \in U$.

Let $H_f = \{g \in H : \exists T(\text{T is stationary and } g(\alpha) < f(\alpha) \text{ for all } \alpha \in T)\}$. The H_f satisfies the assumption of Lemma 3.5, and so $|H_f| \le \kappa$. If $g < f$, then $g \in H_f$ and so $|\{g \in H : g < f\}| \le \kappa$. It follows that $|H| \le \kappa^+$. □

Proof: Proof of Silver's Theorem: By Lemma 2.13, the assumption of the Theorem is equivalent to $\{\alpha < \omega_1 : 2^{\kappa_\alpha} = \kappa_\alpha^+\} = S$ is stationary in ω_1. It follows that for all $\alpha < \omega_1$, $\kappa_\alpha^{\omega_1} < \kappa$. For every $\alpha < \kappa$, we denote φ_α the bijection between $P(\kappa_\alpha)$ and κ_α^+. To every subset $X \subseteq \kappa$, we assign a function $f_X \in \prod_{\alpha<\omega_1} \kappa_\alpha^+$ such that

$$f_X(\alpha) = \begin{cases} \varphi_\alpha(X \cap \kappa_\alpha) & \text{if } \alpha \in S, \\ 0 & \text{otherwise.} \end{cases}$$

If $X \neq Y$, then there is $\beta < \omega_1$ such that $X \cap \kappa_\alpha \neq Y \cap \kappa_\alpha$ for all $\alpha > \beta$ and since φ_α is one to one function, it follows that f_X and f_Y are almost disjoint. So we have a 1-1 function from $P(\kappa)$ to the system

$$\{f_X : X \subseteq \kappa\} \subseteq \prod_{\alpha<\omega_1} \kappa_\alpha^+,$$

which satisfies the assumption of Lemma 3.6 and so $|P(\kappa)| = 2^\kappa = \kappa^+$. $\square$

Note the similarity between the original proof and the combinatorial proof of Silver's Theorem. Elements of N are equivalence classes of functions determined by ultrafilter and so if $[f], [g] \in N$ and $[f] \neq [g]$, then f and g are almost disjoint. So we can understand $\left\{x \in N : x <^N e\right\}$ as an almost disjoint family of function and note that it satisfies the assumptions of Lemma 3.5. $|\{x \in N : x < e^+\}| \leq \kappa^+$ follows from $|\left\{x \in N : x <^N e\right\}| \leq \kappa$ and this proof is analogous to the proof of Lemma 3.6 from Lemma 3.5. Both proofs are completed by the observation that there is 1-1 function from $P(\kappa)$ to the almost disjoint family of functions. In the first case $\left\{x \in N : x <^N e^+\right\}$, in the second case H.

4 The Singular Cardinal Hypothesis

The Singular Cardinal Hypothesis (SCH) says that if κ is singular, then $\kappa^{\mathrm{cf}(\kappa)} = \max(2^{\mathrm{cf}(\kappa)}, \kappa^+)$. Hence if κ is strong limit singular cardinal, then $2^\kappa = \kappa^+$. In this section we show that there is model of ZFC in which the Singular Cardinal Hypothesis fails. First we construct a generic extension in which κ is measurable and $2^\kappa = \kappa^{++}$. Then we extend the model further to make κ a singular cardinal. We show that the new model satisfies $2^\kappa \geq \kappa^{++}$ and κ is a strong limit cardinal.

4.1 Violating GCH at a Measurable Cardinal

Now we construct a model such that there is measurable cardinal κ and $2^\kappa = \kappa^{++}$. The consistency strength of this is more than measurability. In Theorem 4.11 we assume κ^{++}-supercompact cardinal, but the argument we have given is not quite optimal. Work of Gitik ([Git89] and [Git91]) shows that the consistency strength is exactly a measurable cardinal κ with $o(\kappa) = \kappa^{++}$.

Definition 4.1 An uncountable cardinal κ is *measurable* if there is a κ-complete ultrafilter U on κ.

Definition 4.2 Let κ be a measurable cardinal. If U_0, U_1 are measures on κ, then

$$U_0 < U_1 \text{ iff } U_0 \in \mathrm{Ult}(V, U_1).$$

The relation $U_0 < U_1$ is called the *Mitchell order*.

The Mitchell order is transitive and irreflexive. Moreover, it is well-founded.

Definition 4.3 Let U be a normal measure on κ. Let $o(U)$, the *order* of U, denote the rank of U in $<$. Let $o(\kappa)$ denote the height of $<$.

Definition 4.4 A cardinal κ is λ-*supercompact* iff there is an elementary embedding $j : V \to M$ such that crit(j)=κ, $\lambda < j(\kappa)$ and ${}^\lambda M \subseteq M$.

Lemma 4.5 *Let $j : M \to N$ be an elementary embedding between transitive models of ZFC. Let P be a notion of forcing, let G be P-generic over M and let H be $j(P)$-generic over N. Let $j''G \subseteq H$. Then there exists an elementary embedding $j^* : M[G] \to N[H]$ such that $j = j^* \restriction M$ and $j^*(G) = H$.*

Proof: Let $j''G \subseteq H$. We can define $j^* : M[G] \to N[H]$ as follows:

$$j^*(i_G(\dot{\tau})) = i_H(j(\dot{\tau}))$$

We need to check that j^* is function and elementary. Let $i_G(\dot{\sigma}) = i_G(\dot{\tau})$ and fix $p \in G$ such that $p \Vdash_P^M \dot{\sigma} = \dot{\tau}$. By elementarity of j, $j(p) \Vdash_{j(P)}^N j(\dot{\sigma}) = j(\dot{\tau})$ and since $j(p) \in H$, we have $i_H(j(\dot{\sigma})) = i_H(j(\dot{\tau}))$. A similar proof shows that j^* is elementary. Let $M[G] \models \varphi(a_1...a_n)$ and let $\dot{a_1}...\dot{a_n}$ be P-names for $a_1...a_n$. So there is $p \in G$ such that $p \Vdash_P^M \varphi(\dot{a_1}...\dot{a_n})$. By elementarity of j, $j(p) \Vdash_{j(P)}^N \varphi(j(\dot{a_1})...j(\dot{a_n}))$ and since $j(p) \in H$, we have $N[H] \models \varphi(j(a_1)...j(a_n))$.

It is easy to check that $j = j^* \restriction M$. If $x \in M$ and $\check{x}$ is the P-name for x, then $j(\check{x})$ is $j(P)$-name for $j(x)$ and so

$$j^*(x) = j^*(i_G(\check{x})) = i_H(j(\check{x})) = j(x).$$

Lastly, we need to check that $j^*(G) = H$. If $\dot{G}$ is P-name for the P-generic filter, then $j(\dot{G})$ is $j(P)$-name for the $j(P)$-generic filter, and so $j^*(G) = H$. $\square$

Definition 4.6 We denote by $\mathrm{Add}(\kappa, \kappa^{++})$ the *Cohen forcing*. $\mathrm{Add}(\kappa, \kappa^{++})$ is the collection of all functions $p : \mathrm{dom}(p) \to 2$, where $\mathrm{dom}(p) \subseteq \kappa \times \kappa^{++}$ and $|\mathrm{dom}(p)| < \kappa$. The ordering is by reverse inclusion: $p \leq q \leftrightarrow p \supseteq q$, the greatest element is $\emptyset$.

Definition 4.7 A notion of forcing $(P, <)$ is λ-*directed closed* if whenever $D \subseteq P$ is such that $|D| < \lambda$ and for any $d_1, d_2 \in D$ there is some $e \in D$ with $e \leq d_1$ and $e \leq d_2$, then there exists a $p \in P$ such that $p \leq d$ for all $d \in D$.

Definition 4.8 Let $\alpha \geq 1$ and let P_α be an iterated forcing of length α. P_α is an iteration with *Easton support* if for every limit ordinal $\gamma \leq \alpha$, P_γ is a direct limit if γ is regular, and inverse limit otherwise.

For more details about iterated forcing see [Kun80] or [Bau83].

Let P_α be an iteration of length α and let $\beta < \alpha$, then forcing with P_α is the same as forcing with P_β followed by an (α-β)-iteration in $V[G_\beta]$. We denote this forcing $R_{\beta,\alpha}$. Now we present some facts about $R_{\beta,\alpha}$. For definition of $R_{\beta,\alpha}$ and the proof of the following facts see [Bau83], Section 5.

Fact 4.9 *P_α is isomorphic to a dense subset of $P_\beta * \dot{R}_{\beta,\alpha}$.*

Fact 4.10 *$\Vdash_\beta R_{\beta,\alpha}$ is isomorphic to an (α-β)-iteration.*

The following proof is based on [Jec03] and [Bau83].

Theorem 4.11 (Silver) *If there exists a κ^{++}-supercompact κ, then there is a generic extension in which κ is a measurable cardinal and $2^\kappa = \kappa^{++}$.*

Proof: Let A be the set of inaccessible cardinals less then κ. Let $P = P_{\kappa+1}$ be the iteration of length $\kappa + 1$ with Easton support in which $\dot{Q}_\alpha$ names $\mathrm{Add}(\alpha, \alpha^{++})_{V^{P_\alpha}}$ if $\alpha \in A \cup \{\kappa\}$ and names the trivial forcing otherwise. Let G_κ be P_κ-generic over V, let g_κ be Q_κ-generic over $V[G_\kappa]$ and let $G = G_\kappa * g_\kappa$. We shall prove that κ is a measurable cardinal in $V[G]$ and that $V[G] \models 2^\kappa = \kappa^{++}$. We need several lemmas:

Fact 4.12 *P_κ is κ-c.c. with size κ.*

Fact 4.13 *P is κ^+-c.c. with size κ^{++}.*

Since P_κ is κ-c.c., κ is a regular cardinal in $V[G_\kappa]$. In $V[G_\kappa]$, Q_κ is a notion of forcing that adjoins κ^{++} subsets of κ and preserves all cardinals and so

$$V[G] \models 2^\kappa = \kappa^{++}.$$

Lemma 4.14 *Let P be a notion of forcing in V. If ${}^{\kappa^{++}}M \cap V \subseteq M$, $P \in M$, P is κ^{+++}-c.c. and G is P-generic, then ${}^{\kappa^{++}}M[G] \cap V[G] \subseteq M[G]$.*

Proof: It suffices to show that if $f \in V[G]$ is a function from κ^{++} into ordinals, then $f \in M[G]$. Let $\dot{f} \in V$ be the name for f and let $p_0 \in G$ be the condition that forces $\dot{f}$ is function from κ^{++} into ordinals. We need to find name $\sigma \in M$ such that $V[G] \models f = i_G(\sigma)$. By the κ^{+++}-c.c., there is a set D in V such that $|D| \leq \kappa^{++}$ and $p_0 \Vdash Rng(\dot{f}) \subseteq \check{D}$. For each $\alpha \leq \kappa^{++}$ and $\beta \in D$ consider the set

$$X_{\alpha,\beta} = \left\{ p \leq p_0 : p \Vdash \left\langle \check{\alpha}, \check{\beta} \right\rangle \in \dot{f} \right\}.$$

Let $A_{\alpha,\beta}$ be a maximal incompatible subset of $X_{\alpha,\beta}$. This is an application of Zorn's Lemma in V and so $A_{\alpha,\beta} \in V$. Since $|A_{\alpha,\beta}| \leq \kappa^{++}$, $A_{\alpha,\beta} \in M$.

Let $\sigma = \bigcup \{ \{ \langle \check{\alpha}, \check{\beta} \rangle \} \times A_{\alpha,\beta} : \alpha \leq \kappa^{++} \text{ and } \beta \in D \}$. Note that $\sigma \in M$ and $\langle \alpha, \beta \rangle \in i_G(\sigma) \leftrightarrow \exists a \in A_{\alpha,\beta}\ a \in G$.

We will argue that $V[G] \models f = i_G(\sigma)$. Let $V[G] \models \langle \alpha, \beta \rangle \in f$. There is $q \in G$ such that $q \Vdash \langle \check{\alpha}, \check{\beta} \rangle \in \dot{f}$. Since G is a filter, there is $r \leq p_0, q$ and $r \Vdash \langle \check{\alpha}, \check{\beta} \rangle \in \dot{f}$. Since $A_{\alpha,\beta}$ is a maximal antichain below r, $G \cap A_{\alpha,\beta} \neq \emptyset$ and so $\langle \alpha, \beta \rangle \in i_G(\sigma)$.

Conversely, let $V[G] \models \langle \alpha, \beta \rangle \in i_G(\sigma)$. By definition of σ, there is some $a \in A_{\alpha,\beta} \cap G$ and $a \Vdash \langle \check{\alpha}, \check{\beta} \rangle \in \dot{f}$. Since $a \in G$, $V[G] \models \langle \alpha, \beta \rangle \in f$.

Since $\sigma \in M$, we have $i_G(\sigma) \in M[G]$ and so $f \in M[G]$. □

Since $|P_\kappa| = \kappa$ and κ is κ^{++}-supercompact, $P_\kappa \in M$ and so we can consider the model $M[G_\kappa]$.

Corollary 4.15 ${}^{\kappa^{++}}M[G_\kappa] \cap V[G_\kappa] \subseteq M[G_\kappa]$

Proof: It follows from Fact 4.12 and Lemma 4.14. □

Since $|P| = \kappa^{++}$ and κ is κ^{++}-supercompact, $P \in M$ and so we can consider the model $M[G]$.

Corollary 4.16 ${}^{\kappa^{++}}M[G] \cap V[G] \subseteq M[G]$

Proof: It follows from Fact 4.13 and Lemma 4.14. □

Lemma 4.17 $j(P)_{\kappa+1} = P_{\kappa+1}$.

Proof: If $\alpha < \kappa$ then $P_\alpha \in V_\kappa$ and so $j(P)_\alpha = j(P)_{j(\alpha)} = j(P_\alpha) = P_\alpha$. κ is inaccessible in M and so a direct limit is taken at stage κ in the construction of $j(P)$. The direct limit construction is absolute so $j(P)_\kappa = P_\kappa$.

Since each condition in Q_κ in $V[G_\kappa]$ is a function $p : dom(p) \to 2$, where $\text{dom}(p) \subseteq \kappa \times \kappa^{++}$ and $|\text{dom}(p)| < \kappa$ and so, by Corollary 4.15, Q_κ is the same in $V[G_\kappa]$ and $M[G_\kappa]$. Hence $j(P)_{\kappa+1} = P_{\kappa+1}$.

□

Now consider $j(P)$. In M, $j(P)$ is an iteration of length $j(\kappa + 1) = j(\kappa) + 1$. By Lemma 4.17, we have $j(P)_{\kappa+1} = P_{\kappa+1}$. The first nontrivial step above $\kappa + 1$ in the iteration occurs at the least inaccessible cardinal in M above κ, thus the first nontrivial direct limit is taken far above κ^{++}. By the Fact 4.9, $j(P)$ is isomorphic to the two-step iteration in M:

$$j(P)_{\kappa+1} * \dot{R}_{\kappa+1,j(\kappa+1)},$$

where $R_{\kappa+1,j(\kappa+1)}$ is an iteration of length $j(\kappa+1)$ inside $M^{J(P)_{\kappa+1}}$.

By the Fact 4.10 is definable in $M[G]$, hence in $V[G]$.

Fact 4.18 *Let $R_{\kappa+1,j(\kappa+1)} = i_G(\dot{R}_{\kappa+1,j(\kappa+1)})$. Then*

$$V[G] \models R \textit{ is } \kappa^{+++}\textit{-directed closed.}$$

Let $p \in P$. Then $j(p)$ is represented by a pair $(s, \dot{q})$, where $s \in P$ and $\dot{q} \in M[G_\kappa]$ is in $\dot{R}_{\kappa+1,j(\kappa+1)}$. By the definition of P, $p = \langle p_\xi : \xi < \kappa+1\rangle$ and there is $\xi_0 < \kappa$ such that $p_\xi = 1$ for all ξ, $\xi_0 \leq \xi < \kappa$. Thus $j(p) = \left\langle p'_\xi : \xi < j(\kappa)+1\right\rangle$ and $p'_\xi = 1$ for all ξ, $\xi_0 \leq \xi < j(\kappa)$. In particular $p'_\kappa = 1$ and since $p'_\xi = p_\xi$ for all $\xi < \kappa$ and $s = j(p) \restriction (\kappa+1)$, we have $s = (p \restriction \kappa)^\frown 1$. This implies that if $p \in G$ and $j(p) = (s, \dot{q})$, then $s \in G$. Let

$$D = \left\{q \in R_{\kappa+1,j(\kappa+1)} : \exists p \in G(j(p) = (s, \dot{q}) \text{ and } q = i_G(\dot{q}))\right\}.$$

Lemma 4.19 *D is directed, i.e. if $q_1, q_2 \in D$ then there is $q \in D$ such that $q \leq q_1$ and $q \leq q_2$.*

Proof: Suppose $p_1, p_2 \in G$ and $j(p_1) = (s_1, \dot{q}_1)$ and $j(p_2) = (s_2, \dot{q}_2)$. Since G is filter, there is $p \in G$ such that $p \leq p_1$ and $p \leq p_2$. Let $j(p) = (s, \dot{q})$. Since $p \in P$, we have $(p, \dot{q}) \in j(P)$. Since $p \restriction \kappa = j(p) \restriction \kappa$ and $j(p)(\kappa) = 1$, we have $(p, \dot{q}) \leq (s, \dot{q})$. But by elementarity of j, $j(p) \leq j(p_1)$ and $j(p) \leq j(p_2)$ and so $(p, \dot{q}) \leq (p, \dot{q}_1)$ and $(p, \dot{q}) \leq (p, \dot{q}_2)$. And by definition of $R_{\kappa+1,j(\kappa+1)}$, this means that $q \leq q_1$ and $q \leq q_2$. □

It follows from Fact 4.18 that D has a lower bound $q_0 \in R_{\kappa+1,j(\kappa+1)}$. Let H be a $V[G]$-generic filter on $R_{\kappa+1,j(\kappa+1)}$ such that $q_0 \in H$ (we call q_0 a *master condition*). Since H is also $M[G]$-generic and $j(P) = P * R_{\kappa+1,j(\kappa+1)}$, there is an M-generic filter G' on $j(P)$ such that $M[G'] = M[G][H]$. Formally we define $G' = \{(s, \dot{q}) : s \in G \text{ and } i_G(\dot{q}) \in H\}$.

Now we work in $V[G][H]$ and extend the elementary embedding $j : V \to M$ to an embedding $j^* : V[G] \to M[G']$. If $p \in G$ and $j(p) = (s, \dot{q})$, then $p \leq j(p) \restriction \kappa + 1$, and $q \in H$ since $q_0 \in H$ and $q_0 \leq q$. Therefore $j(p) \in G'$ and this means that $j''G \subseteq G'$. And so we can use Lemma 4.5 and extend j.

Thus we have in $V[G][H]$ an elementary embedding $j^* : V[G] \to M[G']$ and we can define $V[G]$-ultrafilter on κ:

$$U = \{X \subseteq \kappa : \kappa \in j^*(X)\}.$$

U is nonprincipal and κ-complete. It suffices to show that U is in $V[G]$.

By Fact 4.18, every subset of κ in $V[G][H]$ is already in $V[G]$ and since $V[G] \models 2^\kappa = \kappa^{++}$, $V[G][H] \models |U| \leq \kappa^{++}$ and by Fact 4.18, $U \in V[G]$. Therefore $V[G] \models 2^\kappa = \kappa^{++}$ and κ is measurable. □

Remark 4.20 In this proof we showed that κ is a measurable cardinal in generic extension $V[G]$ and for the proof of failure of SCH, it suffices. But it can be shown that κ is κ^{++}-supercompact in $V[G]$. For the proof, see [Cum10].

4.2 Prikry Forcing

Now we construct a generic extension in which all cardinals are preserved, but the cofinality of former measurable cardinal is countable. The folowing definition of Prikry Forcing is taken from [Git10].

Definition 4.21 Let $f : [\kappa]^{<\omega} \to \gamma$ for $\gamma < \kappa$ be a function. The set $A \subseteq \kappa$ is *homogeneous* for f if for every $n < \omega$ and every $s, t \in [A]^n$, $f(s) = f(t)$.

Definition 4.22 An uncountable cardinal κ is *Ramsey* iff $\kappa \longrightarrow (\kappa)^{<\omega}_2$.

The notation $\kappa \longrightarrow (\alpha)^{<\omega}_\mu$ means that for each map $f : [\kappa]^{<\omega} \to \mu$ there is a set $A \subseteq \kappa$ such that the order-type of A is α and A is homogeneous for f.

Let κ be measurable cardinal and let U be a normal ultrafilter on κ.

Definition 4.23 We denote by $P_U(\kappa)$ the following forcing notion. A forcing condition is a pair (p, A) where

(i) p is a finite subset of κ,

(ii) $A \in U$,

(iii) $\min(A) > \max(p)$.

Let (p, A) and (q, B) be conditions. We say that (p, A) is stronger than (q, B) and denote this by $(p, A) \leq (q, B)$ iff

(i) p is an end extension of q, i.e. $p \cap (\max(q) + 1) = q$

(ii) $A \subseteq B$,

(iii) $p \backslash q \subseteq B$.

The forcing satisfying the previous definition is called Prikry forcing.

Lemma 4.24 *Let* $(P, \leq)$ *be the forcing* $P_U(\kappa)$ *and let* G *be a generic filter. Then the set* $\bigcup\{p : \exists A \in U(p, A) \in G\}$ *is an* ω*-sequence cofinal in* κ.

Proof: It suffices to show that for every $\alpha < \kappa$, the set

$$D_\alpha = \{(p, A) \in P : \max(p) > \alpha\}$$

is dense in $(P, \leq)$. Let $(s, C) \in P_U(\kappa)$. If $\max(s) > \alpha$, then $(s, C) \in D_\alpha$ and the proof is finished.

If $\max(s) \leq \alpha$ then either $\min(C) > \alpha + 1$ or $\min(C) \leq \alpha + 1$. In the first case consider the pair $(s^\frown \min(C), \{\beta : \beta > \min(C)\} \cap C)$. In the second case consider the pair $(s^\frown(\alpha + 1), \{\beta : \beta > \alpha + 1\} \cap C)$. $\square$

Lemma 4.25 $P_U(\kappa)$ *satisfies the* κ^+*-c.c.*

Proof: Note that if $(p, A), (p, B) \in P_U(\kappa)$, then $(p, A \cap B)$ is a condition such that $(p, A \cap B) \leq (p, B)$ and $(p, A \cap B) \leq (p, B)$. It follows that $(p, A), (p, B)$ are compatible. Thus in the antichain A there are just conditions with different first coordinate and so $|A| \leq \kappa$. $\square$

Definition 4.26 Let $(P, \leq) = P_U(\kappa)$ and let $(p, A), (q, B) \in P$ be forcing conditions. We say that (p, A) is a *direct (or Prikry)* extension of (q, B) and denote this by $(p, A) \leq^* (q, B)$ iff

(i) $p = q$,

(ii) $A \subseteq B$.

This order is important for showing that no new bounded subsets are added to κ after forcing with $P_U(\kappa)$.

Lemma 4.27 $\leq^* \subseteq \leq$.

Proof: This is obvious from the definitions. If $p = q$ then p is end extension of q and $p \backslash q = \emptyset$. $\square$

Lemma 4.28 *$P_U(\kappa)$ with the ordered $\leq^*$ is κ-closed.*

Proof: Let $\langle (p, A_\alpha) : \alpha < \lambda \rangle$ be a $\leq^*$-decreasing sequence of length λ for some $\lambda < \kappa$. Since U is κ-complete $(p, \bigcap_{\alpha<\kappa} A_\alpha)$ is condition in $P_U(\kappa)$ and it is stronger than each (p, A_α) in $\leq^*$. $\square$

Theorem 4.29 (Rowbottom) *Let κ be a measurable cardinal and let U be a normal ultrafilter over κ. Then if $f : [\kappa]^{<\omega} \to \gamma$ where $\gamma < \kappa$, then there is a set $H \in U$ that is homogeneous for f.*

Proof: Let U be a normal ultrafilter over κ and let $f : [\kappa]^{<\omega} \to \gamma$ where $\gamma < \kappa$. If for each $n \in \omega$ there are sets $H_n \in U$ such that f is constant on $[H_n]^n$, then since U is κ-complete, $\bigcap_{n\in\omega} H_n \in U$ would be as required.

By induction on n, we prove that for any $g : [\kappa]^n \to \gamma$, g is constant on $[H_n]^n$ for some $H_n \in U$.

If $n = 1$, it follows from the κ-completeness of U. For contradiction assume that for each $\alpha < \gamma$, $\bigcup g^{-1\prime\prime}\alpha \notin U$. But since U is ultrafilter, $\bigcup g^{-1\prime\prime}\alpha \in I$, where I is dual ideal to U. And since U is κ-complete, $\kappa \in I$ and this is a contradiction.

So assume that the assertion holds for n and we prove that it holds for $n+1$. Let $g : [\kappa]^{n+1} \to \gamma$ where $\gamma < \kappa$. For each $\alpha < \kappa$ define $g_\alpha : [\kappa \backslash \{\alpha\}]^n \to \gamma$ by $g_\alpha(s) = g(\{\alpha\} \cup s)$. By the induction hypothesis, there exists for each $\alpha < \kappa$ a set $X_\alpha \in U$ such that g_α is constant on $[X_\alpha]^n$, i.e. there is $\delta_\alpha < \gamma$ such that for each $s \in [X_\alpha]^n$, $g_\alpha(s) = \delta_\alpha$. Since U is normal, the diagonal intersection $X = \left\{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} X_\beta\right\}$ is in U and if $\beta < \alpha_1 < \alpha_2 < ... < \alpha_n$ are in X, then

$\{\alpha_1, ..., \alpha_n\} \in [X_\beta]^n$ and so $g(\{\beta, \alpha_1, ..., \alpha_n\} = g_\beta(\{\alpha_1, ..., \alpha_n\}) = \delta_\beta$. But since the number of the possible values is less than γ, by the κ-completeness of U, there exist $\delta < \gamma$ and $H \subseteq X$ in U such that $\delta_\beta = \delta$ for all $\beta \in H$. It follows that $g(s) = \delta$ for all $s \in [H]^{n+1}$. □

Remark 4.30 Note that Theorem 4.29 says that each measurable cardinal is Ramsey. Rowbottom also proved that the least Ramsey cardinal is not measurable. For more details see [Kan03].

Lemma 4.31 (The Prikry condition) *Let $(P, \leq)$ be a Prikry forcing. Let $(q, B) \in P$ and φ be a statement of the forcing language of $(P, \leq)$. Then there is a $(p, A) \leq^* (q, B)$ which decides φ, i.e. $(p, A) \Vdash \varphi$ or $(p, A) \Vdash \neg\varphi$.*

Proof: We define a partition $h : [B]^{<\omega} \to 2$ as follows:

$$h(s) = \begin{cases} 1 & \text{if there is a } C \text{ such that } (q \cup s, C) \Vdash \varphi, \\ 0 & \text{otherwise.} \end{cases}$$

Since U is normal ultrafilter, by Rowbottom's Theorem, there is an $A \subseteq B$ such that $A \in U$ an A is homogeneous for h. It follows that (q, A) decides φ. Otherwise, there would be a $(q \cup s_0, B_0) \leq (q, A)$ and $(q \cup s_1, B_1) \leq (q, A)$ such that $(q \cup s_0, B_0) \Vdash \varphi$ and $(q \cup s_1, B_1) \Vdash \neg\varphi$. We can assume that $|s_0| = |s_1|$ but then $h(s_0) = 1$ and $h(s_1) = 0$, which contradicts the homogenity of A. □

Lemma 4.32 *$P_U(\kappa)$ does not add new bounded subsets of κ.*

Proof: Let p be a condition, $\dot{a}$ be a name, $\lambda < \kappa$ and

$$p \Vdash \dot{a} \subseteq \check{\lambda}.$$

For every $\alpha < \lambda$ denote by φ_α the statement "$\check{\alpha} \in \dot{a}$". We define by recursion a $\leq^*$-decreasing sequence of conditions $\langle p_\alpha : \alpha < \lambda \rangle$ such that $p_\alpha || \varphi_\alpha$ (i.e. $p_\alpha \Vdash \varphi_\alpha$ or $p_\alpha \Vdash \neg\varphi_\alpha$) for each $\alpha < \lambda$. By Lemma 4.31, there is $p_0 \leq^* p$ such that p_0 decides φ_0. Suppose that $\langle p_\beta : \beta < \alpha \rangle$ is defined. We define p_α. First, by Lemma 4.28, there is p'_α such that $p'_\alpha \leq^* p_\beta$ for all $\beta < \alpha$. Then, by Lemma 4.31, choose a direct extension p_α

of p'_α deciding φ_α. So we have defined $\langle p_\alpha : \alpha < \lambda \rangle$. Now we use Lemma 4.28 again. Let p^* be direct lower bound of $\langle p_\alpha : \alpha < \lambda \rangle$. Then $p^* \leq p$ and $p^* \Vdash \dot{a} = \check{b}$, where $b = \{\alpha < \lambda : p^* \Vdash \alpha \in \dot{a}\}$. □

Theorem 4.33 *Let κ be a measurable cardinal. There is a generic extension in which cf(κ) $= \omega$ and all the cardinals are preserved.*

Proof: Let G be a generic filter on $P_U(\kappa)$, then by Lemma 4.24, $V[G] \models \mathrm{cf}(\kappa) = \omega$ and since $P_U(\kappa)$ is κ^+-c.c. and $\leq^*$ is κ-closed, all the cardinals are preserved. □

Corollary 4.34 *It is consistent, relative to the existence of a κ^{++}-supercompact cardinal, that there is a strong limit singular cardinal κ such that $2^\kappa \geq \kappa^{++}$.*

Proof: Suppose that there is κ^{++}-supercompact cardinal. By Theorem 4.11, there is a model in which κ is a measurable and $2^\kappa = \kappa^{++}$, and by Theorem 4.33, there is a generic extension in which κ is a strong limit and $2^\kappa \geq \kappa^{++}$. □

5 Conclusion

In this bachelor thesis we showed the basic properties of the continuum function on singular cardinals.

In the part concering Silver's Theorem, we showed and compared two different proofs of this Theorem. Original Silver's proof and purely combinatorial proof by Baumgartner and Prikry. We showed that these proofs have a similar structure but each uses different technique.

In the next section we focused on Singular Cardinal Hypothesis. We found a model such that there is a measurable cardinal κ and $2^\kappa = \kappa^{++}$. For this we used the iteration of Cohen forcing with Easton support, with the assumption of an existence of κ^{++}-supercompact cardinal. To show the failure of the SCH, it was enough to define Prikry forcing, which adds new cofinal ω-sesequence to a measurable cardinal κ, and it preserves cardinals.

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