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DOCTORAL THESIS

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Generating Methods in GR and Properties of the Resulting Solutions

Institute of Theoretical Physics

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Generující metody v OTR a vlastnosti získaných řešení

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Abstrakt: Použití konformní transformace ke generování řešení Einstenových rovnic bylo zkoumáno především v případech, kde je původní prostoročas vakuový. Výsledným prostoročasem se pak často ukazují být *pp*-vlny. V této práci jsou zkoumány konformní elektrovakuové prostoročasy, tedy řešení provázaných Einsteinových a Maxwellových rovnic. Použitím konformní transformace se však lze řešení druhých zmíněných rovnic ve výsledném prostoročase vyhnout. Tato metoda je konkrétně zkoumána pro nulová Einsteinova-Maxwellova pole a ukazuje se, že přípustné jsou opět pouze *pp*-vlny. Při zobecnění této metody je však možné třídu konformních nulových Einsteinových-Maxwellových polí rozšířit na další Kundtovy prostoročasy.

Klíčová slova: konformní transformace, Maxwellovo pole, *pp*-vlny

Title: Generating Methods in GR and Properties of the Resulting Solutions

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Abstract: The use of conformal transformation as a method for generating solutions of Einstein's equations has been mainly studied in the cases where the original spacetime is vacuum. The generated spacetimes then frequently belong to the class of *pp*-waves. In the present work, the electrovacuum spacetimes are studied, i.e the solutions of coupled Einstein's and Maxwell's equations. By using the conformal transformation, it is possible to circumvent solving the later equations. This method is concretely studied for null Einstein-Maxwell fields and it turns out that the admissible spacetimes are *pp*-waves again. However, if the method is generalized, it is possible to enlarge the class of conformal null Einstein-Maxwell fields to a wider family of Kundt spacetimes.

Keywords: conformal transformation, Maxwell field, *pp*-waves

Contents

Preface

Exact solutions

Although Einstein's theory of general relativity will soon celebrate its hundredth birthday, one can argue that only few physically relevant exact solutions have been found so far and such a basic theoretical construction as the two body problem, which is easily solved and understood in Newtonian physics, still lacks a fully relativistic description. This is, of course, the price that has to be paid for having a realistic theory with back-reactions, governed by a set of ten non-linear partial differential equations. Therefore, exact solutions should be treasured all the more, even though they might not be as realistic as one would wish in astrophysical applications, for example. But even if the desire is to model a physical situation more accurately, the exact solutions find their use as backgrounds on which perturbations are taken to achieve better agreement with reality. They are also invaluable for studying qualitatively new non-Newtonian effects that occur in the presence of strong gravitational fields. A well arranged compendium of the known solutions containing hundreds of spacetimes and their families can be found in [1], which also served as the main source of information for this thesis. The large number of existing exact solutions is in sharp contrast to the small amount of the physically viable ones. The implication is not necessarily that most of the exact solutions are unphysical, but rather that the physical interpretation has not been found yet and so the effort should also be put into trying to understand the existing solutions better. This point of view is advocated and practised in [2].

Generating methods

Since Einstein's equations are so difficult to solve, numerous methods of generating new solutions have been invented or borrowed from other parts of physics in order to simplify this complicated system. Quite powerful generating methods work on the class of stationary electrovacuum fields for which two complex potentials can be introduced and certain transformations of these functions map solutions onto other solutions, for concrete examples of such transformations see Buchdahl [3] or Ehlers [4], an overview of the topic can be found in the first chapter of this work. In the case of stationary axially-symmetric vacuum fields, the generating methods are even more fruitful and such an interesting solution as the superposition of two Kerr black holes can be obtained from the most basic Minkowski spacetime by two successive Bäcklund transformations [5]. In context of these successful generating methods, the one that will be closely examined in the present thesis is falling behind. It is the well known conformal transformation. Although it plays an important role in electrodynamics, which is a conformally invariant theory in vacuum, it found very little use as a generating method in general relativity so far. Despite the fact that there are many existing results that limit the use of conformal transformation, ranging from the first theorem by Brinkmann [6] in 1925, through the series of papers by Van den Bergh [7] in 1980s to the generalization of Brinkmann's results in 1998 by Daftardar-Gejji [8], there are still some possibilities left to be explored, some of which are addressed in this thesis.

The aim and arrangement of the thesis

In the diploma thesis [9], a particular method based on conformal invariance of the Maxwell field was used in attempt to generate solutions of Einstein-Maxwell equations. A solution was generated that belonged to the same family of solutions as the seed spacetime, i.e. to the class of *pp*-waves. In the present work, the method will be further studied and the equivalence problem will be addressed in order to exclude the possibility that the found solution is isometric to the original one. The method will then be generalized for arbitrary null Einstein-Maxwell fields and all the admissible spacetimes will be revealed.

In order to comprehensibly achieve the goals that have been set in the previous paragraph, the contents of this thesis are organized as follows. The first chapter will bring a general overview of generating techniques used in general relativity and some of their applications. In the second chapter, the notion of conformal transformation will be introduced as well as its important properties and the relevant existing theorems will be mentioned. Since some of the results will be obtained by the use of spinor calculus and Newman-Penrose formalism, the third chapter will recall these apparatuses. The fourth chapter should familiarize the reader with the equivalence problem and its solution via Cartan scalars and invariants constructed from the Riemann tensor. Chapter five is dedicated to the Maxwell fields as these serve as the sole source of gravitation in this work. The generating method based on conformal invariance of the Maxwell field that was used in the diploma thesis [9] will be briefly introduced in the sixth chapter and the new results in this area will be presented. In the last chapter, the theorem concerning conformal null Einstein-Maxwell fields will be proved. It generalizes the results from the sixth chapter and allows for a wider family of spacetimes to be generated than just the usual *pp*-waves. The appendices will recall basic notions and conventions that are used in the present work. Some of the lengthier equations that would distract the reader in the main chapters will also be moved to the appendices.

Part I

Overview

1. Generating techniques

Most of the generating techniques in and outside general relativity use the symmetries of the underlying equations to construct new solutions. This chapter should, therefore, familiarize the reader with the general concepts of symmetries possessed by partial differential equations (PDEs) and the related generating techniques. Once the general apparatus is established, its application in general relativity will be discussed.

1.1 Symmetries of PDEs

A system of partial differential equations of order k can be generally written in the form

$$H_A\left(x^\mu, u^i, {u^i}_{,\mu}, \ldots, {u^i}_{,\mu_1\cdots\mu_k}\right) = 0 \qquad A = 1, \ldots, N\ ,\ i = 1, \ldots, K \tag{1.1}$$

where x^μ is the n-tuple of independent variables (coordinates on manifold M) and $u^i, {u^i}_{,\mu_1}, \ldots, {u^i}_{,\mu_1\cdots\mu_k}$ are the dependent variables (functions of x^μ), resp. their derivatives. Clearly N is the number of equations in the system and K is the number of dependent variables. The K-dimensional space in which the dependent variables take values will be denoted by U. Generally, systems with $N > K$ are overdetermined and have a solution only if the integrability conditions are fulfilled. The notorious Laplace's equation on $M = \mathbb{R}^2$

$$u_{,xx} + u_{,yy} = 0$$

will be used as a model example.
The systems of partial differential equations, their solutions and symmetries are usually tackled within the framework of *jet bundles*.

Definition (Jet bundle)**.** *Suppose the system of partial differential equations is given by* (1.1)*. The* ***jet bundle*** $J^k(M, U)$ *is the fibre bundle, where the space of independent variables* M *is the base manifold and the fibre at* $x \in M$ *is the set of all possible values of the dependent variables and their derivatives up to the* k*th order at* x*.*

To put it more illustratively, $J^k(M, U)$ (or J^k for short) is the space in which one would draw graphs of functions u^i and their derivatives. The coordinates on J^k consist of x^μ on the base manifold and $z^i, {z^i}_{\mu_1}, {z^i}_{\mu_1\cdots\mu_k}$ on the fibre, where z has been used instead of u to emphasize that these are *independent* coordinates in context of J^k, much like q and $\dot{q}$ are independent in context of Lagrangian mechanics.
For the Laplace's equation, the relevant jet bundle is $J^2 = J^2(\mathbb{R}^2, \mathbb{R})$ and the possible coordinates are

$$(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy})\,.$$

Thus, $J^2(\mathbb{R}^2, \mathbb{R})$ is evidently eight-dimensional.
Each set of functions

$$u(x) \equiv \{u^i(x)\}_{i=1}^K\ ,\ u^i(x) \in \mathcal{F}M$$

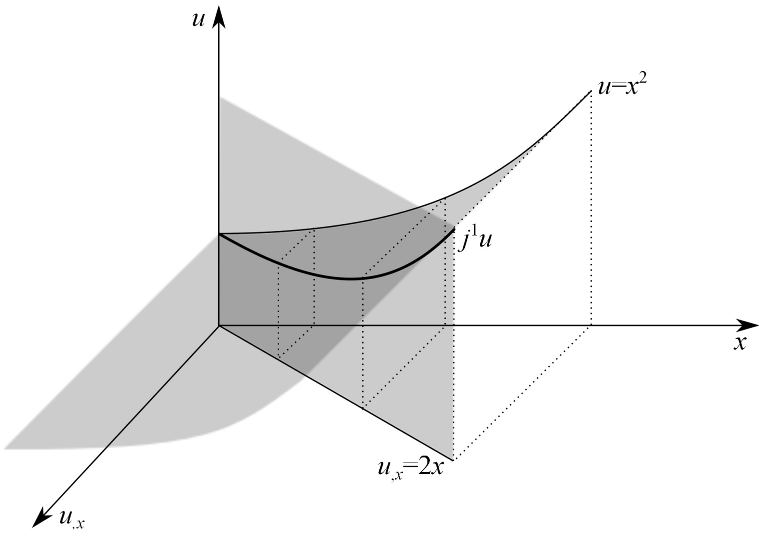

Figure 1.1: Jet bundle section: *The figure depicts a simple section j^1u in $J^1(\mathbb{R},\mathbb{R})$ generated by $u = x^2$.*

can be lifted to J^k by setting

$$z^i = u^i(x)\,, z^i{}_\mu = u^i{}_{,\mu}(x)\,, z^i{}_{\mu\nu} = u^i{}_{,\mu\nu}(x), \ldots$$

Thus, an n-dimensional surface is generated in J^k, which is called the *section generated by* u and denoted j^ku. Explicitly

$$j^ku := \left\{ \left(x^\mu, u^i(x), u^i{}_{,\mu}(x)\,, \ldots, u^i{}_{,\mu_1\cdots\mu_k}(x)\right), x^\mu \in M \right\}\,.$$

The section j^ku can be thought of as a generalized graph of functions u^i, plotting also their derivatives.

In the model case of J^2, suppose for example $u(x,y) = x^2 - y^2$. Such function generates the section

$$j^2u := \left\{ \left(x, y, x^2 - y^2, 2x, 2y, 2, 0, -2\right), (x,y) \in \mathbb{R}^2 \right\}\,. \tag{1.2}$$

The system (1.1) defines another subspace $Y \subset J^k$ by the algebraic equations for the jet bundle coordinates

$$Y := \left\{ H_A\left(x^\mu, z^i, z^i{}_\mu, \ldots, z^i{}_{\mu_1\cdots\mu_k}\right) = 0 \right\}.$$

Every section generated by a solution of eq. (1.1) must evidently lie in this subspace. The converse is also true — every section that lies in Y corresponds to a solution of (1.1), i.e.

$$H_A\left(x^\mu, u^i(x), u^i{}_{,\mu}(x), \ldots, u^i{}_{,\mu_1\cdots\mu_k}(x)\right) = 0 \iff j^ku \subset Y$$

The subspace Y for the model Laplace equation is given by $z_{xx} + z_{yy} = 0$ and so it forms a seven-dimensional plane in J^2. The section j^2u from (1.2) lies in Y, because $u = x^2 - y^2$ is a solution of the Laplace equation.

In this geometrical language, symmetry of the system of partial differential equations can be naturally defined.

Definition (Symmetry I). *A mapping $\Phi : Y \mapsto Y$ is a* ***symmetry*** *of system* (1.1) *iff it maps sections into sections.*

Therefore, one would like know whether an n-dimensional surface $\Sigma \subset J^k$ parameterized by x^μ is a section generated by some function $f \in \mathcal{F}M$. This question can be answered in terms of differential forms[1] on J^k. Suppose the following set of one-forms

$$\begin{aligned} \boldsymbol{\omega}^i &= \mathrm{d}z^i - z^i_\alpha \mathrm{d}x^\alpha \,, \\ \boldsymbol{\omega}^i_\alpha &= \mathrm{d}z^i_\alpha - z^i_{\alpha\beta} \mathrm{d}x^\beta \,, \\ &\vdots \\ \boldsymbol{\omega}^i_{\alpha_1\cdots\alpha_{k-1}} &= \mathrm{d}z^i_{\alpha_1\cdots\alpha_{k-1}} - z^i_{\alpha_1\cdots\alpha_{k-1}\alpha_k} \mathrm{d}x^{\alpha_k} \,. \end{aligned}$$

Clearly, all these one-forms vanish when restricted (pulled back) onto any section $j^k f$. The converse is also true — every n-dimensional surface $\Sigma \subset J^k$ is a section if the aforementioned forms vanish on Σ. Such set of one-forms is denoted CJ^k and called the *contact module* of J^k, i.e.

$$CJ^k := \{\boldsymbol{\omega}^i, \boldsymbol{\omega}^i_\alpha, \ldots, \boldsymbol{\omega}^i_{\alpha_1\cdots\alpha_{k-1}}\} \,.$$

Thus, J^k equipped with the contact module 'knows' about the sections. It remains to encode the information about the system (1.1). This task could also be accomplished by either

a) restricting the contact module to Y, or

b) if the system of PDEs is linear in highest order derivatives, one can attach additional forms to the contact module of J^{k-1} (not J^k!) that characterize the solution surface.

These notions may be illustrated on our model example of Laplace's equation. The contact module CJ^2 contains these one-forms

$$\begin{aligned} \boldsymbol{\omega} &= \mathrm{d}z - z_x \mathrm{d}x - z_y \mathrm{d}y \,, \\ \boldsymbol{\omega}_x &= \mathrm{d}z_x - z_{xx} \mathrm{d}x - z_{xy} \mathrm{d}y \,, \\ \boldsymbol{\omega}_y &= \mathrm{d}z_y - z_{xy} \mathrm{d}x - z_{yy} \mathrm{d}y \end{aligned}$$

If the method a) is to be applied, the contact module should be restricted to the surface $Y : z_{xx} + z_{yy} = 0$. This could be done, for example, by eliminating the z_{yy} coordinate using the substitution $z_{yy} = -z_{xx}$. The restricted contact module is then

$$\begin{aligned} \boldsymbol{\omega} &= \mathrm{d}z - z_x \mathrm{d}x - z_y \mathrm{d}y \,, \\ \boldsymbol{\omega}_x &= \mathrm{d}z_x - z_{xx} \mathrm{d}x - z_{xy} \mathrm{d}y \,, \\ \boldsymbol{\omega}_y &= \mathrm{d}z_y - z_{xy} \mathrm{d}x + z_{xx} \mathrm{d}y \,. \end{aligned}$$

[1]For a brief review of differential forms, see appendix C

Alternatively, following the option b), the Laplace's equation can be encoded into the following forms on J^1

$$\boldsymbol{\omega}^1 = \mathrm{d}z - z_x \mathrm{d}x - z_y \mathrm{d}y \,, \tag{1.3}$$
$$\boldsymbol{\omega}^2 = \mathrm{d}z_x \wedge \mathrm{d}y - \mathrm{d}z_y \wedge \mathrm{d}x \tag{1.4}$$

Indeed, suppose that the surface Σ is given by the mapping

$$\Phi : (x, y) \mapsto (z = u(x, y), z_x = v(x, y), z_y = w(x, y)) \ .$$

Then the vanishing of $\boldsymbol{\omega}_A$ on Σ yields

$$0 = \Phi^* \boldsymbol{\omega}^1 = \mathrm{d}u - v\mathrm{d}x - w\mathrm{d}y \Longrightarrow u_{,x} = v, u_{,y} = w \,,$$
$$0 = \Phi^* \boldsymbol{\omega}^2 = \mathrm{d}v \wedge \mathrm{d}y - \mathrm{d}w \wedge \mathrm{d}x \Longrightarrow v_{,x} + w_{,y} = u_{,xx} + u_{,yy} = 0 \,.$$

Obviously, the first form represents the contact module CJ^1, but it does not 'know' about the Laplace's equation, which is encoded in the second form $\boldsymbol{\omega}^2$. The system of partial differential equations does not determine the set $\{\boldsymbol{\omega}^A\}$ uniquely. It is obvious that any $\boldsymbol{\alpha} \wedge \boldsymbol{\omega}^A$ also vanishes on the solution surface for any differential form $\boldsymbol{\alpha}$. Therefore, the entire *ideal* generated by $\boldsymbol{\omega}^A$ carries the same information about the equation.

Definition (Ideal)**.** *The **ideal** $\mathcal{I}$ generated by $\{\boldsymbol{\omega}^A\}$ is a subspace of ΩJ^k given by the condition*

$$\boldsymbol{\alpha} \in \mathcal{I} \Longleftrightarrow \boldsymbol{\alpha} = \boldsymbol{\eta}_A \wedge \boldsymbol{\omega}^A \ , \ \boldsymbol{\eta}_A \in \Omega J^k$$

It is further required that the ideal generated by $\boldsymbol{\omega}^A$ is *closed* — the exterior derivative of each $\boldsymbol{\alpha} \in \mathcal{I}$ must remain in $\mathcal{I}$, or symbolically $d\mathcal{I} \subset \mathcal{I}$. This requirement ensures that the underlying system of PDEs is integrable, i.e. it has a solution.
Notice that the ideal generated by (1.3), (1.4) is not closed. The closure is accomplished by attaching a new form

$$\boldsymbol{\omega}^3 := \mathrm{d}\boldsymbol{\omega}^1 = -\mathrm{d}z_x \wedge \mathrm{d}x - \mathrm{d}z_y \wedge \mathrm{d}y$$

to the other two generators of the ideal.
Since the solutions of a given system of PDEs are determined by the ideal of $\{\boldsymbol{\omega}^A\}$, the definition of symmetry may be refined.

Definition (Symmetry II)**.** *Symmetry is a mapping $\Phi : J^k \mapsto J^k$ that preserves the ideal generated by $\{\boldsymbol{\omega}^A\}$, i.e. $\Phi^*(\mathcal{I}) \subset \mathcal{I}$.*

The symmetries in this chapter will always depend at least on one continuous parameter ε or an arbitrary function. Such symmetries form a Lie group which allows one to work with infinitesimal versions of the transformations. Different classes of symmetry transformations will be considered.

1.1.1 Lie point transformation

In this section, the ansatz for the symmetry transformation is such that the new independent and dependent variables are just functions of the old independent and dependent variables, not of their derivatives, i.e.

$$\tilde{x}^\mu = f^\mu(x, z^i; \varepsilon) \,,$$
$$\tilde{z}^i = g^i(x, z^j; \varepsilon) \,.$$

The infinitesimal version of the transformation is

$$\begin{aligned} \tilde{x}^\mu &= x^\mu + \varepsilon \xi^\mu(x, z^i)\,, \\ \tilde{z}^i &= z^i + \varepsilon \eta^i(x, z^j)\,, \end{aligned}$$

where, as usually in the infinitesimal calculus, the equalities holds up to terms containing higher powers of ε, which are neglected. In order for the transformation to preserve the sections, it must be prolonged to the other jet coordinates. The prolongation generally reads

$$\begin{aligned} \tilde{z}^i_\alpha &= z^i_\alpha + \varepsilon \eta^i_\alpha(x, z^j, z^j_\beta)\,, \\ \tilde{z}^i_{\alpha_1\alpha_2} &= z^i_{\alpha_1\alpha_2} + \varepsilon \eta^i_{\alpha_1\alpha_2}(x, z^j, z^j_\beta, z^j_{\beta_1\beta_2})\,, \\ &\vdots \\ \tilde{z}^i_{\alpha_1\cdots\alpha_k} &= z^i_{\alpha_1\cdots\alpha_k} + \varepsilon \eta^i_{\alpha_1\cdots\alpha_k}(x, z^j, z^j_\beta, \ldots, z^j_{\beta_1\cdots\alpha_k})\,. \end{aligned}$$

From the infinitesimal transformation one can deduce the components of $\boldsymbol{V}$ — the vector field, whose flow yields the finite transformation.

$$\boldsymbol{V} := \xi^\mu \frac{\partial}{\partial x^\mu} + \eta^i \frac{\partial}{\partial z^i} + \eta^i_\mu \frac{\partial}{\partial z^i_\mu} + \cdots + \eta^i_{\mu_1\cdots\mu_k} \frac{\partial}{\partial z^i_{\mu_1\cdots\mu_k}}\,.$$

The first two components ξ^α, η^i are prescribed, the others can be derived from the requirement that the infinitesimal transformation preserves sections. Since all the information about sections is contained in the ideal $\mathcal{I}_C$ generated by the contact module C, the transformation has to preserve this ideal. Mathematically, this condition reads

$$\pounds_{\boldsymbol{V}} \mathcal{I}_\mathcal{C} \subset \mathcal{I}_\mathcal{C}\,,$$

where $\pounds$ is the Lie derivative, see page 82. The resulting formulas for the other components of $\boldsymbol{V}$ are

$$\begin{aligned} \eta^i_\alpha &= \frac{D\eta^i}{Dx^\alpha} - z^i_\mu \frac{D\xi^\mu}{Dx^\alpha}\,, \\ \eta^i_{\alpha\beta} &= \frac{D\eta^i_\alpha}{Dx^\beta} - z^i_{\alpha\mu} \frac{D\xi^\mu}{Dx^\beta}\,, \\ &\vdots \end{aligned}$$

where

$$\frac{D}{Dx^\alpha} = \frac{\partial}{\partial x^\alpha} + z^i_\alpha \frac{\partial}{\partial z^i} + z^i_{\alpha\beta} \frac{\partial}{\partial z^i_\beta} + \cdots$$

With such $\boldsymbol{V}$, section will be mapped onto sections and it only remains to ensure that the system (1.1) remains satisfied. This is accomplished by

$$\boldsymbol{V} H_A = \xi^\alpha \frac{\partial H_A}{\partial x^i} + \eta^i \frac{\partial H_A}{\partial z^i} + \eta^i_\mu \frac{\partial H_A}{\partial z^i_\mu} + \ldots + \eta^i_{\mu_1\cdots\mu_k} \frac{\partial H_A}{\partial z^i_{\mu_1\cdots\mu_k}} = 0 \quad (\mathrm{mod}\ H_A = 0)\,.$$

This is a system of linear partial differential equations for $\xi^\mu(x, z^i)$ and $\eta^i(x, z^j)$ and it has to hold for *all* values of the jet bundle coordinates $x^\mu, z^i, z^i_\alpha, \ldots$ The

addendum $\mathrm{mod}\, H_A = 0$ suggests that the relation $H_A = 0$ is used whenever it appears. Having solved these equations, the infinitesimal symmetry is known and the finite symmetry can be found by integrating $\boldsymbol{V}$.
The Lie point symmetries for Einstein's equations in vacuum are obtained from the condition

$$\boldsymbol{V} R_{\mu\nu} = 0 \qquad (\mathrm{mod}\ R_{\mu\nu} = 0)$$

and were first found by Ibragimov [10]. They read

$$\boldsymbol{V} = \xi^\mu \frac{\partial}{\partial x^\mu} - \left(\xi^\rho{}_{,\mu} g_{\rho\nu} + \xi^\rho{}_{,\nu} g_{\mu\rho} - 2a g_{\mu\nu}\right) \frac{\partial}{\partial g_{\mu\nu}} \quad , \quad \xi^\mu = \xi^\mu(x)\,. \tag{1.5}$$

Evidently, the Lie group depends on one real parameter a and an arbitrary vector $\boldsymbol{\xi} = \xi^\mu(x)\partial_\mu$ on M. The finite transformation corresponding to a is the scaling of metric (*homothety*, see page 26) and the vector $\boldsymbol{\xi}$ generates a diffeomorphism Φ on M via

$$\left.\frac{d\Phi^\mu(x,\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \xi^\mu(x)\,,$$

resp. its pullback Φ^* on the metric tensor, i.e.

$$\begin{aligned} \boldsymbol{\xi} = 0\,,\ a \in \mathbb{R}: &\quad \tilde{x} = x\ ,\ \tilde{\boldsymbol{g}} = e^{2\varepsilon a}\boldsymbol{g}\,, \\ \boldsymbol{\xi} = \xi^\mu(x)\partial_\mu\,,\ a = 0: &\quad \tilde{x}^\mu = \Phi^\mu(x,\varepsilon)\ ,\ \tilde{\boldsymbol{g}} = \Phi^*\boldsymbol{g}\,. \end{aligned}$$

Unfortunately, these symmetries are not very useful in context of generating new solutions. Diffeomorphisms, of course, do not yield new solutions, only different coordinate representations of the original metric. The scaling by a constant factor may result in non-isometric geometries, but even in those cases, the two spacetimes are usually related in a quite trivial way — the transformation only changes the values of parameters in the original solution, e.g. mass, charge, etc. Despite this negative result for general vacuum spacetimes, the symmetries may be richer in some special subclasses of the vacuum solutions.

Similarity reduction. Although the Lie point symmetries in general relativity can not be used straightforwardly to construct new solutions, they can be used for *similarity reduction*. This method looks for solutions that are invariant under symmetry transformations, because in these cases, the equations may be simplified. The invariance under the action of symmetry means that the surface $z^i - u^i(x) = 0$ remains the same, i.e.

$$\boldsymbol{V}(z^i - u^i(x)) = \eta^i - \xi^\mu u^i_{,\mu} = 0\,.$$

If the relativistic generator of Lie point symmetries (1.5) is inserted into the above equation, one obtains

$$\xi^\rho{}_{,\mu} g_{\rho\nu} + \xi^\rho{}_{,\nu} g_{\mu\rho} - 2a g_{\mu\nu} + \xi^\rho g_{\mu\nu,\rho} = 0\,, \tag{1.6}$$

or, rewritten in terms of Lie derivative

$$\pounds_{\boldsymbol{\xi}} \boldsymbol{g} = 2a\boldsymbol{g}$$

and so the sought similarity solutions in general relativity are spacetimes admitting a *homothetic vector*, for definition see page 25. The homothetic vector can be set to $\boldsymbol{\xi} = \partial_3$ by an appropriate[2] choice of coordinates x^μ and then the generator reads simply

$$\boldsymbol{V} = \frac{\partial}{\partial x^3} + 2ag_{\mu\nu}\frac{\partial}{\partial g_{\mu\nu}}\,.$$

The condition (1.6) then becomes

$$2ag_{\mu\nu} = g_{\mu\nu,3} \implies g_{\mu\nu}(x^0,\ldots,x^3) = e^{2ax^3}\hat{g}_{\mu\nu}(x^0,x^1,x^2)$$

and for a Killing vector ($a=0$) the x^3 dependence completely vanishes. Einstein's equations reduce to a set of PDEs for just $\hat{g}_{\mu\nu}(x^0,x^1,x^2)$, which yields the desired simplification.

1.1.2 Lie-Bäcklund and potential symmetry

The Lie point transformation restricted the allowed $\tilde{x},\tilde{z}$ to depend only on x,z, not on $z_\mu, z_{\mu\nu},\ldots$. This assumption is now dropped and $\tilde{x},\tilde{z}$ may depend on an arbitrary and even infinite number of derivative coordinates $z_\mu, z_{\mu\nu}$, etc.
In the case of the vacuum Einstein equations and finite number of dervative coordinates, the ansatz does not yield any new symmetries besides (1.5), as was shown by Anderson and Tore [11].
If the the new dependent and independent coordinates are functions of infinitely many derivative coordinates, the problem is usually translated to the previously examined Lie point symmetries by the use of newly defined variables — *potentials.* These symmetries are, therefore, sometimes called *potential symmetries.*
To construct a potential, one attaches and extra variable φ (the potential) to the other coordinates of the jet bundle and defines a one-form

$$\boldsymbol{\Theta} := -d\boldsymbol{\varphi} + \Phi_\mu(x^\nu, z, z_\nu,\ldots)\mathrm{d}x^\mu\,,$$

which is added to the set generators $\{\boldsymbol{\omega}^A\}$ of the original ideal $\mathcal{I}$ to form a prolonged ideal $\mathcal{I}'$. Then, the requirement is imposed that $\mathrm{d}\boldsymbol{\Theta} \in \mathcal{I}'$, i.e. the prolonged ideal is closed. This condition is equivalent to the integrability condition $\mathrm{dd}\varphi = 0$. Thus, with $z^i = u^i(x)$, the system of PDEs is equivalent to the integrability conditions for the potential.
This procedure may be illustrated on the two-dimensional Laplace's equation again. In this case, the ideal $\mathcal{I}$ of the original equation is best written in terms of these generators, which are all two-forms

$$\begin{aligned}\boldsymbol{\omega}^1 &= \mathrm{d}z\wedge\mathrm{d}x + z_y\mathrm{d}x\wedge\mathrm{d}y\,,\\ \boldsymbol{\omega}^2 &= \mathrm{d}z\wedge\mathrm{d}y - z_x\mathrm{d}x\wedge\mathrm{d}y\,,\\ \boldsymbol{\omega}^3 &= \mathrm{d}z_x\wedge\mathrm{d}y - \mathrm{d}z_y\wedge\mathrm{d}x\,.\end{aligned}$$

The ansatz for $\boldsymbol{\Theta}$ is

$$\boldsymbol{\Theta} := -d\varphi + \Phi_x(x,y,z,z_x,z_y)\mathrm{d}x + \Phi_y(x,y,z,z_x,z_y)\mathrm{d}y\,,$$

And its exterior derivative is

$$\begin{aligned}\mathrm{d}\boldsymbol{\Theta} = &\left(\frac{\partial\Phi_y}{\partial x} - \frac{\partial\Phi_x}{\partial y} + z_x\frac{\partial\Phi_y}{\partial z} - z_y\frac{\partial\Phi_x}{\partial z}\right)\mathrm{d}x\wedge\mathrm{d}y + \left(\frac{\partial\Phi_y}{\partial z_x} + \frac{\partial\Phi_x}{\partial z_y}\right)\mathrm{d}z_x\wedge\mathrm{d}y\\ &+ \frac{\partial\Phi_x}{\partial z_x}\mathrm{d}z_x\wedge\mathrm{d}x + \frac{\partial\Phi_y}{\partial z_y}\mathrm{d}z_y\wedge\mathrm{d}y \qquad (\mathrm{mod}\ \mathcal{I}')\,,\end{aligned}$$

[2]The intergral curves of $\boldsymbol{\xi}$ are chosen to be the coordinate lines of x^3

where the terms proportional to $\mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}z \wedge \mathrm{d}y, \mathrm{d}z_y \wedge \mathrm{d}x$ were eliminated by subtracting suitable forms from $\mathcal{I}'$. The absence of these terms allows one to claim that $\mathrm{d}\boldsymbol{\Theta} \in \mathcal{I}'$ if and only if $\mathrm{d}\boldsymbol{\Theta} = 0$. The resulting equations are

$$\frac{\partial \Phi_y}{\partial x} - \frac{\partial \Phi_x}{\partial y} + z_x \frac{\partial \Phi_y}{\partial z} - z_y \frac{\partial \Phi_x}{\partial z} = 0 \,,$$
$$\frac{\partial \Phi_y}{\partial z_x} + \frac{\partial \Phi_x}{\partial z_y} = 0 \ , \ \frac{\partial \Phi_x}{\partial z_x} = 0 \ , \ \frac{\partial \Phi_y}{\partial z_y} = 0 \,.$$

A common simplification is to assume that Φ_x, Φ_y do not depend on x, y explicitly. The simplified set

$$z_x \frac{\partial \Phi_y}{\partial z} - z_y \frac{\partial \Phi_x}{\partial z} = 0 \ , \ \frac{\partial \Phi_y}{\partial z_x} + \frac{\partial \Phi_x}{\partial z_y} = 0 \ , \ \frac{\partial \Phi_x}{\partial z_x} = 0 \ , \ \frac{\partial \Phi_y}{\partial z_y} = 0$$

is easily solved and the potential φ fulfils

$$\mathrm{d}\varphi = (az_y + b)\mathrm{d}x + (-az_x + c)\mathrm{d}y \quad (\mathrm{mod}\ \mathcal{I}') \quad a, b, c \in \mathbb{R} \,.$$

When the above formula is restricted to the solution surface $z = u(x, y)$, one obtains

$$\mathrm{d}\varphi = (au_{,y} + b)\mathrm{d}x + (-au_{,x} + c)\mathrm{d}y \,.$$

The integrability conditions for φ are equivalent to the Laplace's equation and so the the existence of potential φ is guaranteed as long as u solves the Laplace's equation.
The potential symmetry for the general Einstein's vacuum equations is not known, but if the spacetime admits at least one Killing vector, i.e. for the larger set of equations

$$R_{\mu\nu} = 0 \ , \ \pounds_{\boldsymbol{\xi}} \boldsymbol{g} = 0$$

the potential symmetries can and will be constructed in section 1.3.1.

1.1.3 Prolongation and pseudopotentials

The idea of potentials can be enriched by considering a more general ansatz

$$\boldsymbol{\Theta}^A := -\mathrm{d}\varphi^A + \Phi^A_\mu \mathrm{d}x^\mu \,, \tag{1.7}$$

where φ^A are the *pseudopotentials* and the functions Φ^A_μ are allowed to depend on them, i.e. $\Phi^A_\mu = \Phi^A(x, z, z_\mu, \ldots, \varphi)$. Again, the forms $\boldsymbol{\Theta}^A$ are added to the original ideal $\mathcal{I}$ to form the prolonged ideal $\mathcal{I}'$ to which the exterior derivatives $\mathrm{d}\boldsymbol{\Theta}^A$ are required to belong.

Sine-Gordon equation. The construction of pseudopotentials will be illustrated on a simpler example of sine-Gordon equation

$$u_{,xy} = \sin u \qquad u = u(x, y) \,,$$

which can be equivalently described by the ideal $\mathcal{I}$ generated by

$$\boldsymbol{\omega}^1 = \mathrm{d}z \wedge \mathrm{d}y - z_x \mathrm{d}x \wedge \mathrm{d}y \,, \tag{1.8}$$
$$\boldsymbol{\omega}^2 = \mathrm{d}z_x \wedge \mathrm{d}x + \sin z \, \mathrm{d}x \wedge \mathrm{d}y \,. \tag{1.9}$$

The forms $\mathbf{\Theta}^A$ explicitly read

$$\mathbf{\Theta}^A := -\mathrm{d}\varphi^A + \Phi_x^A(x,y,z,z_x,\varphi^B)\mathrm{d}x + \Phi_y^A(x,y,z,z_x,\varphi^B)\mathrm{d}y\,,$$

with an unspecified number of pseudopotentials φ^A. The exterior derivative reads

$$\mathrm{d}\mathbf{\Theta}^A = \left(\frac{\partial \Phi_y^A}{\partial x} - \frac{\partial \Phi_x^A}{\partial y} + \frac{\partial \Phi_y^A}{\partial \varphi^B}\Phi_x^B - \frac{\partial \Phi_x^A}{\partial \varphi^B}\Phi_y^B + \frac{\partial \Phi_y^A}{\partial z} z_x - \sin z \frac{\partial \Phi_x^A}{\partial z_x}\right)\mathrm{d}x\wedge\mathrm{d}y + \\ + \frac{\partial \Phi_x^A}{\partial z}\mathrm{d}z\wedge\mathrm{d}x + \frac{\partial \Phi_y^A}{\partial z_x}\mathrm{d}z_x\wedge\mathrm{d}y \qquad (\mathrm{mod}\ \mathcal{I}')\,, \tag{1.10}$$

where the equations (1.8),(1.9),(1.1.3) have been used to eliminate terms containing $\mathrm{d}\varphi^A, \mathrm{d}z\wedge\mathrm{d}y, \mathrm{d}z_x\wedge\mathrm{d}x$. The condition $\mathrm{d}\mathbf{\Theta}^A \in \mathcal{I}'\{\boldsymbol{\omega}^A, \mathbf{\Theta}^B\}$ is simply $\mathrm{d}\mathbf{\Theta}^A = 0$ for the particular form of $\mathrm{d}\mathbf{\Theta}^A$ and so the following equations are obtained.

$$\frac{\partial \Phi_y^A}{\partial x} - \frac{\partial \Phi_x^A}{\partial y} + \frac{\partial \Phi_y^A}{\partial \varphi^B}\Phi_x^B - \frac{\partial \Phi_x^A}{\partial \varphi^B}\Phi_y^B + z_x\frac{\partial \Phi_y^A}{\partial z} - \sin z \frac{\partial \Phi_x^A}{\partial z_x} = 0\,,$$
$$\frac{\partial \Phi_x^A}{\partial z} = 0\ ,\ \frac{\partial \Phi_y^A}{\partial z_x} = 0\,,$$

A more compact form of the first equation from the above set is achieved by introducing vector fields

$$\mathbf{\Phi}_x := \Phi_x^A \frac{\partial}{\partial \varphi^A}\ ,\ \mathbf{\Phi}_y := \Phi_y^A \frac{\partial}{\partial \varphi^A}\,,$$

whose commutator appears in that equation. Explicitly,

$$\left[\mathbf{\Phi}_x + \frac{\partial}{\partial x}, \mathbf{\Phi}_y + \frac{\partial}{\partial y}\right]^A + z_x\frac{\partial \Phi_y^A}{\partial z} - \sin z\frac{\partial \Phi_x^A}{\partial z_x} = 0\,,$$

The simplifying assumption that Φ_x^A, Φ_y^A do not depend on x, y will be applied once again and the equation reduces to

$$[\mathbf{\Phi}_x, \mathbf{\Phi}_y]^A + z_x\frac{\partial \Phi_y^A}{\partial z} - \sin z\frac{\partial \Phi_x^A}{\partial z_x} = 0\ ,\ \Phi_x^A = \Phi_x^A(z_x,\varphi^B)\ ,\ \Phi_y^A = \Phi_y^A(z,\varphi^B)\,. \tag{1.11}$$

After some manipulations, the dependencies of Φ_x^A, Φ_y^A on variables z, z_x, φ^A can be separated and the following ansatz for the solution is obtained[3]

$$\mathbf{\Phi}_x = z_x\boldsymbol{X}_1 + \boldsymbol{X}_2\,,$$
$$\mathbf{\Phi}_y = \sin z\boldsymbol{X}_3 + \cos z\boldsymbol{X}_4\,,$$

where the components of vectors $\boldsymbol{X}_i$ depend only on the pseudopotentials, i.e.

$$\boldsymbol{X}_i = X_i^A(\varphi^B)\frac{\partial}{\partial \varphi^A}\,.$$

The ansatz for $\mathbf{\Phi}_x, \mathbf{\Phi}_y$ must now be inserted into (1.11), the result is

$$[z_x\boldsymbol{X}_1 + \boldsymbol{X}_2, \sin z\boldsymbol{X}_3 + \cos z\boldsymbol{X}_4] + z_x(\cos z\boldsymbol{X}_3 - \sin z\boldsymbol{X}_4) - \sin z\boldsymbol{X}_1 = 0\,. \tag{1.12}$$

[3]For more detailed calculation, see Harrison [12].

By comparing the terms with the same dependence on z and z_x, one obtains the commutator relations

$$[\boldsymbol{X}_1, \boldsymbol{X}_3] = \boldsymbol{X}_4 \ , \ [\boldsymbol{X}_1, \boldsymbol{X}_4] = -\boldsymbol{X}_3 \ , \ [\boldsymbol{X}_2, \boldsymbol{X}_3] = \boldsymbol{X}_1 \ , \ [\boldsymbol{X}_2, \boldsymbol{X}_4] = 0 \, .$$

The vectors $\boldsymbol{X}_i$ form an incomplete algebra — the *prolongation structure*, because the commutators $[\boldsymbol{X}_1, \boldsymbol{X}_2]$ and $[\boldsymbol{X}_3, \boldsymbol{X}_4]$ are missing. The sought functions Φ_x^A, Φ_y^A may be found, for example, by assuming that they are linear in pseudopotentials, i.e.

$$\Phi_x^A = F_B^A(z_x)\varphi^B \ , \ \Phi_y^A = G_B^A(z)\varphi^B \, ,$$

where matrices F, G depend on z, z_x. The vector commutator $[\boldsymbol{\Phi}_x, \boldsymbol{\Phi}_y]$ may now be rewritten in terms of the matrices

$$[\boldsymbol{\Phi}_x, \boldsymbol{\Phi}_y]^A := \Phi_x^B \frac{\partial \Phi_y^A}{\partial \varphi^B} - \Phi_y^B \frac{\partial \Phi_x^A}{\partial \varphi^B} = F_C^B \varphi^C G_B^A - G_C^B \varphi^C F_B^A = -[F, G]_C^A \varphi^C \, , \quad (1.13)$$

where the commutator in the last term is a matrix commutator but note the sign change. The solution of eq. (1.11) is then analogical

$$\begin{aligned} F &= z_x X_1 + X_2 \, , \\ G &= \sin z X_3 + \cos z X_4 \, , \end{aligned}$$

but X_i are constant matrices now with the following commutators

$$[X_1, X_3] = -X_4 \ , \ [X_1, X_4] = X_3 \ , \ [X_2, X_3] = -X_1 \ , \ [X_2, X_4] = 0 \, .$$

The opposite sign of the commutators is a consequence of the extra minus in (1.13). The last step that will be ommited here is to find a matrix representation of the commutator relations. Usually, the first[4] non-trivial representation is chosen.

Although the sine-Gordon equation was just a simple example, the procedure of constructing the pseudopotentials runs in a similar fashion for other, more complex systems of PDEs as well.
With the obtained set of pseudopotentials φ^A, the symmetries are sought in the infinitesimal form

$$\boldsymbol{V} := \xi^\mu(x^\mu, z^i, \ldots, \varphi^A) \frac{\partial}{\partial x^\mu} + \eta^i(x^\mu, z^j, \ldots, \varphi^A) \frac{\partial}{\partial z^i} + \cdots$$

and the condition on $\boldsymbol{V}$ to generate a (pseudo)potential symmetry is then

$$\pounds_{\boldsymbol{V}} \mathcal{I}' \subset \mathcal{I}' \, ,$$

where $\mathcal{I}'$ is the prolonged ideal generated by the set $\{\boldsymbol{\omega}^A, \boldsymbol{\Theta}^B\}$.

1.2 Harmonic maps

The method of *harmonic maps* is used for PDEs that can be obtained from the principle of the least action with a Lagrangian of the form

$$\mathcal{L}(x^\mu, z^i, z^i_\nu) = \gamma^{1/2} G_{ij} \gamma^{\mu\nu} z^i_\mu z^j_\nu \, , \quad (1.14)$$

[4] First in the sense of lowest dimension.

where $\boldsymbol{\gamma} = \gamma_{\mu\nu}(x^\mu)\mathrm{d}x^\mu\mathrm{d}x^\nu$ is the metric tensor on M, the space of independent variables (background space), γ is its determinant and $\boldsymbol{G}=G_{ij}(z^k)\mathrm{d}z^i\mathrm{d}z^j$ is the metric tensor on U, the space of dependent variables (potential space). Such form of Lagrangian can be obtained in many physical applications, namely for the case of source-free Einstein-Maxwell spacetimes with a non-null Killing vector, which will be discussed in section 1.3.1.

Definition (Harmonic map)**.** *A map $u : M \mapsto U$ is harmonic iff it satisfies the Euler-Lagrange equations for the Lagrangian* (1.14)*, i.e.*

$$\frac{\delta\mathcal{L}}{\delta u^i} \equiv \frac{\partial\mathcal{L}}{\partial z^i} - \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial z^i_\mu} = 0\,,$$

where the derivatives of $\mathcal{L}$ are evaluated at the point

$$(x^\mu, z^i, z^i_\nu) = \left(x^\mu, u^i(x), u^i{}_{,\nu}(x)\right)\,.$$

The explicit form of the Euler-Lagrange equation for the aforementioned Lagrangian is

$$\Box u^a + \gamma^{\mu\nu}\Gamma^a{}_{bc}u^b{}_{,\mu}u^c{}_{,\nu} = 0\,, \tag{1.15}$$

Where $\Box := \nabla^\rho\nabla_\rho$ is the d'Alembertian on the background space $(M, \boldsymbol{\gamma})$, but the Christoffel symbols $\Gamma^a{}_{bc}$ are calculated from the potential metric $\boldsymbol{G}$, i.e.

$$\Gamma^a{}_{bc} := \frac{1}{2}G^{ai}\left(\frac{\partial G_{bi}}{\partial z^c} + \frac{\partial G_{ic}}{\partial z^a} - \frac{\partial G_{bc}}{\partial z^i}\right)$$

The equations (1.15) are invariant under the infinitesimal transformation

$$\tilde{u}^a = u^a + \varepsilon V^a(u)$$

as long as $\boldsymbol{V}$ fulfils

$$V_{a;bc} = R_{abcd}V^d\,, \tag{1.16}$$

where the covariant derivative and the Riemann tensor are also both constructed from $\boldsymbol{G}$ and the vectors $\boldsymbol{V}$ are the *affine collineations* of the potential metric. The killing vectors of $\boldsymbol{G}$ satisfying $V_{(a;b)} = 0$ clearly belong to this class of transformations. Therefore, by studying the geometry of the potential space $(U, \boldsymbol{G})$, one can obtain the symmetries of the original equation (1.15).

1.3 Application in general relativity

As has been mentioned earlier, it is useful to study the similarity solutions of Einstein's equations, because the field equations simplify and may allow for new symmetries. The spacetimes which admit Killing vectors belong to the class of similarity solutions and so they will be the subject of interest in this section.

1.3.1 Stationary spacetimes

Stationary spacetimes are those solutions of Einstein's field equations which allow for a timelike Killing vector $\boldsymbol{\xi}$. As a consequence, one can choose the coordinate t such that $\boldsymbol{\xi} = \partial_t$ and the metric tensor does not depend on t. The spacelike three-surfaces $t = $ const. then foliate the whole spacetime, although the foliation is not unique. Suppose that the surface Σ is given by $t = 0$. Due to the presence of time translation symmetry, it is not surprising that the four-dimensional Einstein's equations reduce to a system of equations on Σ. The tensors $\boldsymbol{T}$ in the 4-dimensional spacetime M that 'effectively live' on Σ are precisely those that fulfil

$$\xi^i T^{a\cdots b}_{c\cdots i\cdots d} = 0 \ , \ \xi_j T^{a\cdots j\cdots b}_{c\cdots d} = 0 \ , \ \pounds_{\boldsymbol{\xi}} \boldsymbol{T} = 0$$

The quantities that will appear in the reduced equations are

- $F := \xi^\mu \xi_\mu \geq 0$ ... norm of $\boldsymbol{\xi}$
- $h_{\mu\nu} := g_{\mu\nu} - \xi_\mu \xi_\nu / F$ ... metric tensor on Σ
- $\gamma_{\mu\nu} := F h_{\mu\nu}$ ... conformally rescaled metric tensor on Σ
- $\omega^\mu := \epsilon^{\mu\nu\rho\sigma} \xi_\nu \xi_{\rho;\sigma}$ ... *twist vector*, tangent to Σ
- $\hat{R}_{\mu\nu}$ ... Ricci tensor calculated from metric tensor $\boldsymbol{\gamma}$ on Σ

For further discussion, it is necessary to specify the matter content — the spacetimes will be electrovacuum, i.e. the solutions of Einstein-Maxwell equations outside the sources. The stationary Maxwell field allows one to construct a complex potential Φ, the derivative of which is given by

$$\Phi_{,\nu} := G^{1/2} \xi^\mu F^*_{\mu\nu} \, ,$$

for definition of $\boldsymbol{F}^*$, see eq. C.1 in the appendix. The integrability conditions for Φ are satisfied thanks to Maxwell's equations $F^{*\mu\nu}{}_{;\nu} = 0$ and the stationarity of the Maxwell field, i.e. $\pounds_{\boldsymbol{\xi}} \boldsymbol{F} = 0$. Einstein's equation then guarantee the existence of another complex potential $\mathcal{E}$, the Ernst potential found in [13]

$$\mathcal{E}_{,\mu} := F_{,\mu} + i\omega_\mu - 2\Phi_{,\mu}\overline{\Phi} \, , \tag{1.17}$$

$$\operatorname{Re}\mathcal{E} := F - |\Phi|^2 \, . \tag{1.18}$$

Einstein-Maxwell equations can then be given, according to Harrison [14], Neugebauer and Kramer [15], in terms of the aforementioned quantities on Σ

Theorem 1.1 (Stationary Einstein-Maxwell fields)**.** *Einstein-Maxwell equations for stationary spacetimes reduce to*

$$2F^2 \hat{R}_{\mu\nu} = \operatorname{Re}\left(\mathcal{E}_{,(\mu}\overline{\mathcal{E}}_{,\nu)} + 2\overline{\mathcal{E}}_{,(\mu}\Phi_{,\nu)}\overline{\Phi} - 4\mathcal{E}\Phi_{(,\mu}\overline{\Phi}_{,\nu)}\right) \, , \tag{1.19}$$

$$0 = \Box\mathcal{E} + F^{-1}\gamma^{\mu\nu}\mathcal{E}_{,\mu}\left(\mathcal{E}_{,\nu} + 2\overline{\Phi}\Phi_{,\nu}\right) \, , \tag{1.20}$$

$$0 = \Box\Phi + F^{-1}\gamma^{\mu\nu}\Phi_{,\mu}\left(\mathcal{E}_{,\nu} + 2\overline{\Phi}\Phi_{,\nu}\right) \, , \tag{1.21}$$

where the covariant derivatives in the d'Alembertian $\Box$ are taken with respect to the three-metric $\boldsymbol{\gamma}$.

This is a set of equations for the metric tensor $\boldsymbol{\gamma}$ and two complex potentials $\Phi, \mathcal{E}$. The function F must be expressed in terms of $\Phi, \mathcal{E}$ from (1.18) as

$$F := \operatorname{Re}\mathcal{E} + |\Phi|^2 \,. \tag{1.22}$$

Certain values of the potentials describe various physical situations.

- Vacuum solutions are obtained by setting $\Phi = 0$.
- If $\mathcal{E}$ is real, the twist vector $\boldsymbol{\omega}$ vanishes. As a result, the Killing vector $\boldsymbol{\xi}$ is hypersurface orthogonal and so $g_{t\mu} = 0$ in suitable coordinates — the spacetime is *static*.
- For static solutions with $\operatorname{Im}\Phi = 0$ one can find a static orthonormal frame in which the magnetic component of the Maxwell field vanishes and the Einstein-Maxwell solution is called *electrostatic*.
- A static solution with $\operatorname{Re}\Phi = 0$ is *magnetostatic*, because the electric component can be transformed away by a spatial rotation of the static frame.
- If $\mathcal{E} = 0$, the space metric $\boldsymbol{\gamma}$ is conformally flat and the spacetime is called *conformastatic*.

The important fact is that the equations (1.19)–(1.21) can be obtained from the Lagrangian

$$\mathcal{L} = \gamma^{1/2}\left[\hat{R} + \frac{1}{2}F^{-2}\gamma^{\mu\nu}\operatorname{Re}\left(\mathcal{E}_{,(\mu}\overline{\mathcal{E}}_{,\nu)} + 2\overline{\mathcal{E}}_{,(\mu}\Phi_{,\nu)}\overline{\Phi} - 4\mathcal{E}\Phi_{(,\mu}\overline{\Phi}_{,\nu)}\right)\right] ,$$

concretely, eqs. (1.19),(1.20),(1.21) follow from $\delta\mathcal{L}/\delta\gamma_{\mu\nu} = 0$, $\delta\mathcal{L}/\delta\mathcal{E} = 0$, $\delta\mathcal{L}/\delta\Phi = 0$, respectively. Note that the second term in the square bracket is of the form (1.14) and so the symmetries of the potential space may be examined in the context of harmonic maps. The explicit form of the potential space metric $\boldsymbol{G}$ is

$$\boldsymbol{G} = \frac{1}{2}F^{-2}\left(\mathrm{d}\mathcal{E}\mathrm{d}\overline{\mathcal{E}} + \Phi\mathrm{d}\mathcal{E}\mathrm{d}\overline{\Phi} + \overline{\Phi}\mathrm{d}\overline{\mathcal{E}}\mathrm{d}\Phi - 4\operatorname{Re}\mathcal{E}\mathrm{d}\Phi\mathrm{d}\overline{\Phi}\right) .$$

Note again that F is a function of $\mathcal{E}, \Phi$ given by eq. (1.22). The metric $\boldsymbol{G}$ is four-dimensional with the signature $(+,+,-,-)$ and its affine collineations $\boldsymbol{V}$ defined by (1.16) have to be found in order to find the symmetries of (1.19)–(1.21). The result is that $\boldsymbol{G}$ has eight affine collineations, all of which are, in fact, Killing vectors. The corresponding finite mappings thus form an eight-dimensional isometry group of $\boldsymbol{G}$ and each such map can be composed from the following 'elementary' transformations

$$\tilde{\mathcal{E}} = |\alpha|^2\mathcal{E} \quad , \quad \tilde{\Phi} = \alpha\Phi \qquad \alpha \in \mathbb{C}\,, \tag{1.23}$$
$$\tilde{\mathcal{E}} = \mathcal{E} + ia \quad , \quad \tilde{\Phi} = \Phi \qquad a \in \mathbb{R}\,, \tag{1.24}$$
$$\tilde{\mathcal{E}} = \frac{\mathcal{E}}{1 + ib\mathcal{E}} \quad , \quad \tilde{\Phi} = \frac{\Phi}{1 + ib\mathcal{E}} \qquad b \in \mathbb{R}\,, \tag{1.25}$$
$$\tilde{\mathcal{E}} = \mathcal{E} - 2\bar{\beta}\Phi - |\beta|^2 \quad , \quad \tilde{\Phi} = \Phi + \beta \qquad \beta \in \mathbb{C}\,, \tag{1.26}$$
$$\tilde{\mathcal{E}} = \frac{\mathcal{E}}{1 - 2\bar{\gamma}\Phi - |\gamma|^2\mathcal{E}} \quad , \quad \tilde{\Phi} = \frac{\Phi + \gamma\mathcal{E}}{1 - 2\bar{\gamma}\Phi - |\gamma|^2\mathcal{E}} \qquad \gamma \in \mathbb{C}\,. \tag{1.27}$$

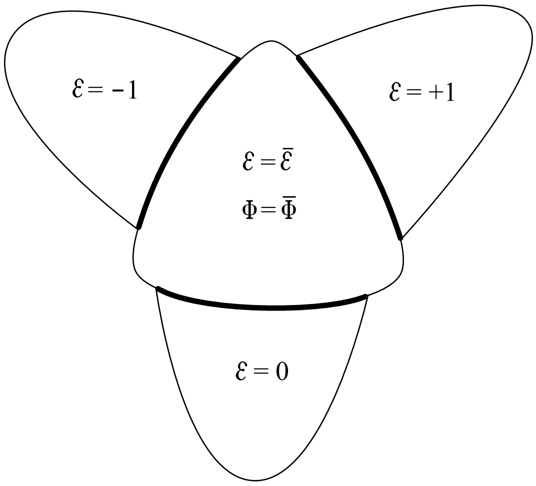

Figure 1.2: Inequivalent classes of stationary Einstein-Maxwell fields: *Each of the four separated regions can be generated by a $SU(2,1)$ transformation from the representative solutions written inside. One cannot get from the inside of one region to the inside of other by a $SU(2,1)$ transformation. The thick lines are mapped onto themselves under $SU(2,1)$ and represent solutions for which the potentials depend on each other.*

All these transformations and their combinations map solutions of (1.19)–(1.21) onto other solutions. Thus, one can generate new stationary Einstein-Maxwell fields from old ones. However, some of the transformations are trivial in the sense of 'physical' fields. The transformations (1.24), (1.26) are only gauge because they do not change $g_{\mu\nu}$ or $F_{\mu\nu}$. Also the transformation (1.23) is somewhat trivial, because it yields just a *duality rotation* of the Maxwell field, a symmetry possesed by all electrovacuum solutions according to Rainich's conditions from 1925, see [16]. By combining the aforementioned elementary transformations with suitably chosen parameters, the following discrete transformation may be obtained

$$\tilde{\mathcal{E}} = \mathcal{E}^{-1} \ , \ \tilde{\Phi} = \mathcal{E}^{-1}\Phi \, ,$$

called the *inversion*. The inversion was used by Buchdahl in [3] to map static vacuum solutions ($\mathcal{E} \in \mathbb{R}, \Phi = 0$) onto other solutions, obviously of the same type. The transformation of static vacuum fields to stationary vacuum spacetimes by the use of (1.25) is known as the Ehlers transformation, see [4]. Of course, the vacuum solutions can be mapped onto electrovacuum fields by (1.27). Such transformation was first used by Harrison in [14].
According to Kinnersley [17], the transformations (1.23)–(1.27) constitute a non-linear representation of $SU(2,1)$ — the group of unimodular linear operators in $\mathbb{C}^3$ that preserve the bilinear form $\boldsymbol{\eta} := \mathrm{diag}(1,1,-1)$. The action of $SU(2,1)$ divides the stationary Einstein-Maxwell fields into four equivalence classes. Each class can be generated from one of its elements by the action of the group but one cannot pass from one class to another by a $SU(2,1)$ transformation, unless the potentials depend on each other, see fig. 1.2. For example, the Reissner-Nordström solution can belong to either of the classes $\mathcal{E} = -1, 0, +1$, according to the sign of $Q^2 - M^2$ and the three branches are inequivalent in the sense of $SU(2,1)$ transformations.

1.3.2 Stationary axially-symmetric spacetimes

In addition to the timelike Killing vector $\boldsymbol{\xi}$, stationary axially symmetric spacetimes further contain a spacelike Killing vector $\boldsymbol{\eta}$, which commutes with $\boldsymbol{\xi}$. This allows one to construct coordinates t, φ such that $\boldsymbol{\xi} = \partial_t, \boldsymbol{\eta} = \partial_\varphi$ and the metric does not depend on these two coordinates. As in the previous section, the spacetimes of interest will be the Einstein-Maxwell solutions, whose metric can be written in the Weyl coordinates as

$$\boldsymbol{g} = e^{2U}(\mathrm{d}t + A d\varphi)^2 - e^{-2U}\left[e^{2k}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\varphi^2\right] ,$$

where U, A, k are functions of ρ, z only. Not surprisingly, Einstein-Maxwell field equations can be written in terms of potentials $\mathcal{E}, \Phi$ living on a two-dimensional surface endowed with the metric $\boldsymbol{\gamma} = e^{2k}(\mathrm{d}\rho^2 + \mathrm{d}z^2)$. The equations read

$$F\left(\rho\mathcal{E}_{,\mu}\right)^{;\mu} = \rho\,\gamma^{\mu\nu}\mathcal{E}_{,\mu}\left(\mathcal{E}_{,\nu} + 2\overline{\Phi}\Phi_{,\nu}\right) , \tag{1.28}$$
$$F\left(\rho\Phi_{,\mu}\right)^{;\mu} = \rho\,\gamma^{\mu\nu}\Phi_{,\mu}\left(\mathcal{E}_{,\nu} + 2\overline{\Phi}\Phi_{,\nu}\right) , \tag{1.29}$$

where $F = \mathrm{Re}\,\mathcal{E} + |\Phi|^2$ and the covariant derivative is taken with respect to the 2-dimensional metric $\boldsymbol{\gamma}$. These two equations are the equivalents of (1.20), (1.21). The equivalent of (1.19), where the Ricci tensor of $\boldsymbol{\gamma}$ comes into play is particularly simple in this case — it reduces to just one equation expressing k with respect to the potentials, the concrete formula will be given in the following theorem.

Theorem 1.2. *Einstein-Maxwell field equations for stationary axially-symmetric spacetimes reduce to two complex equations* (1.28), (1.29) *for* $\mathcal{E}, \Phi$. *The metric functions* U, A, k *are then obtained from*

$$e^{2U} = F , \tag{1.30}$$
$$F^2 A_{,\zeta} = \rho\left[i(\mathrm{Im}\,\mathcal{E})_{,\zeta} + \overline{\Phi}\Phi_{,\zeta} - \Phi\overline{\Phi}_{,\zeta}\right] , \tag{1.31}$$
$$4F^2 k_{,\zeta} = 2^{1/2}\rho\left(\mathcal{E}_{,\zeta}\overline{\mathcal{E}}_{,\zeta} + 2\Phi\overline{\Phi}_{,\zeta}\mathcal{E}_{,\zeta} + 2\overline{\Phi}\Phi_{,\zeta}\overline{\mathcal{E}}_{,\zeta} - 4\mathrm{Re}\mathcal{E}\Phi_{,\zeta}\overline{\Phi}_{,\zeta}\right) , \tag{1.32}$$

where $F = \mathrm{Re}\,\mathcal{E} + |\Phi|^2$ *and* $\zeta := 2^{-1/2}(\rho + iz)$ *is a complex coordinate.*

The integrability conditions for A and k are both satisfied thanks to (1.28), (1.29).
The generating techniques mostly use the vacuum stationary axisymmetric spacetimes. The static Einstein-Maxwell fields may then be created by the Bonnor transformation

Theorem 1.3 (Bonnor)**.** *Suppose a vacuum stationary axisymmetric spacetime is given by the triplet*

$$(\mathcal{E} = e^{2U} + i\psi, \Phi = 0, k) ,$$

then

$$(\mathcal{E}' = \mathcal{E}\overline{\mathcal{E}}, \Phi' = i\psi, k' = 4k)$$

represents a new static axisymmetric Einstein-Maxwell field.

The method of prolongation may also be used and the pseudopotentials can be successfully constructed for the discussed class of solutions, as was shown in [18]. Unfortunately, the involved calculations are rather lengthy and require deeper knowledge of Lie groups, algebras and their representations. Therefore, the curious reader is referred to chapter 34 of [1] and the references therein.

2. Conformal transformation

2.1 Conformal map

Conformal map is intuitively described as a function $\Phi : M \mapsto \tilde{M}$ that locally preserves angles. To formalize this vague definition, the domain M and the image $\tilde{M}$ of the map should be specified. In order to be able to measure angles, it is supposed that both M and $\tilde{M}$ are manifolds equipped with a metric tensor[1] $\boldsymbol{g}$, resp. $\tilde{\boldsymbol{g}}$. The following definition formally captures the local angle preservation:

Definition (Conformal map)**.** *Suppose the metric tensors $\boldsymbol{g}$, $\tilde{\boldsymbol{g}}$ are defined on M, resp. $\tilde{M}$. A diffeomorphism $\Phi : M \mapsto \tilde{M}$ is then called a **conformal map** iff*

$$\Phi^* \tilde{\boldsymbol{g}} = e^{2U} \boldsymbol{g}$$

for some function $U \in \mathcal{F}M$, where Φ^* is the pullback induced by Φ.

Specially, if $U = \text{const.}$, the map is called a *homothety* and if further $U = 0$, the map is an *isometry.*
Conformal maps appear frequently in mathematics and physics. One of the best known conformal maps is the *stereographic projection* $\mathcal{S}$ which maps a unit sphere $\mathbb{S}^2$ without the north pole onto its equatorial plane $\mathbb{E}^2$, see fig 2.1. To describe the map, suppose that the standard spherical coordinates (θ, φ) are prescribed on the unit sphere equipped with the natural metric $\boldsymbol{g} = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$ and the polar coordinates (ρ, ϕ) cover the equatorial plane with the Euclidean metric $\tilde{\boldsymbol{g}} = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2$. The stereographic projection S is then given by

$$\mathcal{S} : (\theta, \varphi) \longmapsto (\rho, \phi) = \left(\frac{\sin\theta}{1 - \cos\theta}, \varphi \right) \ .$$

To confirm that this is a conformal map, the pullback of the metric $\tilde{g}$ should be calculated and indeed it yields

$$\mathcal{S}^* \tilde{\boldsymbol{g}} = \frac{1}{(1 - \cos\theta)^2} \boldsymbol{g} \ .$$

[1] This introduces metric structure to both manifolds, although a weaker notion — *conformal structure* — is sufficient for measuring the angles.

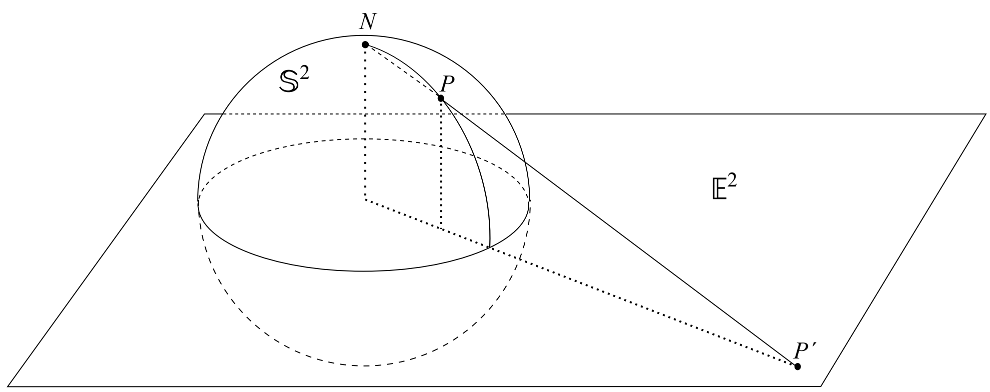

Figure 2.1: Stereographic projection: *The point P from the sphere $\mathbb{S}^2$ is mapped to P' on its equatorial plane $\mathbb{E}^2$.*

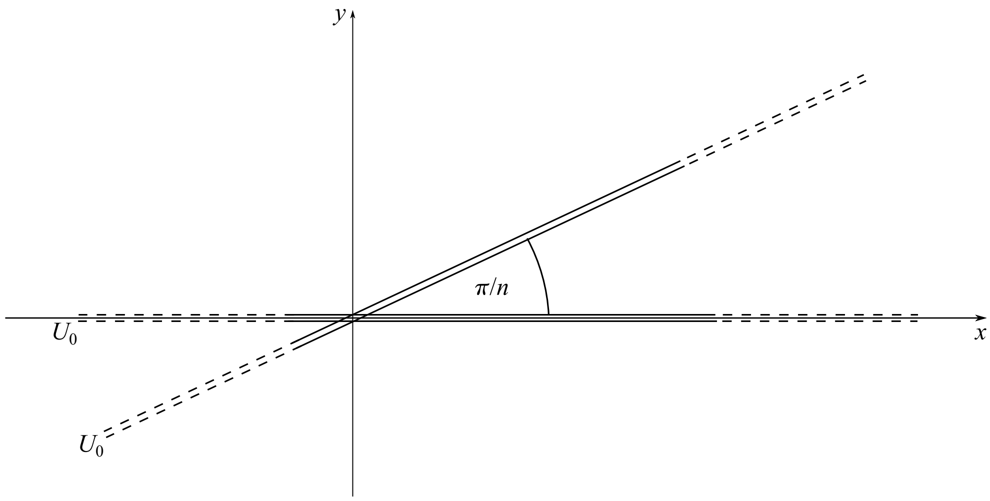

Figure 2.2: Intersecting planes

The stereographic projection is also used in complex analysis as a map between the Riemann sphere and the extended Gauss plane. Here, a point (infinity) is formally added to the Gauss plane as an image of the north pole of the Riemann sphere so that the projection is a genuine bijection. The coordinate representation of the map is also very simple in this case. Suppose that the unit sphere in $\mathbb{R}^3$ is given in standard Cartesian coordinates by $x^2 + y^2 + z^2 = 1$ and the Gauss plane is labeled by complex numbers $\zeta = X + iY$. The map is then given by

$$\mathcal{S} : \underbrace{(x, y, z)}_{x^2+y^2+z^2=1} \longmapsto \zeta = \frac{x + iy}{1 - z} \ .$$

The connection between conformal maps and complex analysis goes, however, much deeper. Every holomorphic function $h(\zeta)$, in fact, defines a conformal map on the domain where its derivative is non-vanishing and the converse is also true — every conformal map on the complex plane is holomorphic.

Conformal transformation finds its use in physics namely thanks to the fact that the solutions of the Laplace's equation $\Delta\varphi = 0$ are conformally invariant in two-dimensional spaces. This property can be used, for instance, in electrostatics, where the source-free Maxwell's equations reduce precisely to the Laplace's equation, provided that the examined problem is effectively two-dimensional due to the presence of symmetry.

Such approach may be applied, for example, in the case of two intersecting conducting planes with a prescribed value of potential on them, see fig. 2.2. Suppose the angle between the two planes is π/n for some $n \in \mathbb{N}$ and the potential has the value U_0 on the planes. Moreover, let one of the planes be given by $y = \mathrm{Im}(\zeta) = 0$, the other plane is then prescribed by $y \cos \pi/n = x \sin \pi/n$. Instead of solving the Laplace's equation $\Delta\varphi = 0$ in this geometry, the holomorphic function $h(\zeta) = \zeta^n$ is used to map the two planes effectively into one[2], i.e. their angle becomes π . Therefore, the potential is effectively prescribed on a single plane $y = 0$ and the Laplace's equation is readily solved in the transformed space

[2]The angle between the planes at $\zeta = 0$ changes, because the map is not conformal at this point — the derivative is vanishing here.

by $\hat{\varphi} = -E_o \mathrm{Im}(\zeta) + U_o$. The solution φ of the original setup is then just the pull-back of $\hat{\varphi}$, thanks to the conformal invariance of the two-dimensional Laplace's equation

$$\varphi = h^* \hat{\varphi} = -E_o \, \mathrm{Im}(\zeta^n) + U_o \ .$$

2.1.1 Conformal motions

A special class of conformal maps are the Conformal motions.

Definition (Conformal motion)**.** ***Conformal motion*** *on M with metric $\boldsymbol{g}$ is a continuous one-parameter group of conformal maps Φ_t, i.e.:*
$\Phi_t : M \longmapsto M$ is a conformal map for each $t \in \mathbb{R}$ ($\Phi_t^* \boldsymbol{g} = e^{2U_t} \boldsymbol{g}$) ,
Φ_0 is an identity on M,
$\Phi_t \circ \Phi_s = \Phi_{t+s}$ for each $t, s \in \mathbb{R}$.

Conformal motions form a Lie group. The vector field $\boldsymbol{\xi}$ that generates conformal motion Φ_t then satisfies

$$\pounds_{\boldsymbol{\xi}} \boldsymbol{g} = \phi \, \boldsymbol{g} \ , \tag{2.1}$$

where $\phi = 2 \frac{d}{dt}|_{t=0} U_t$. The conformal motion is *proper* when $\mathrm{d}\phi \neq 0$ and $\boldsymbol{\xi}$ is called a *conformal Killing vector.* If $\phi = \text{const.}$, $\boldsymbol{\xi}$ is a *homothetic vector* and further for $\phi = 0$, $\boldsymbol{\xi}$ is a *Killing vector.* The dimension of the group of conformal motions (or conformal group for short) strongly depends on the dimension of the manifold it acts on. For a two-dimensional manifold with an arbitrary metric tensor, the conformal group is infinitely dimensional, since every holomorphic function generates a conformal Killing vector. In general relativity, however, only few spacetimes admit proper conformal motions (see [1], p. 565–570) and the conformal group is at most 15-dimensional[3]. While the conformal Killing vectors find a limited use in classical general relativity, they play an important role in an alternative theory of gravity — the conformal gravity. This theory tries to avoid the problems with quantization of general relativity and also aspires to explain the dark energy, while its most palpable problem is the fact that its field equations contain fourth order derivatives. For a review of conformal gravity, see for example [19].

2.2 Conformal transformation

Unlike the conformal map, which 'moves' the points on the manifold and the metric tensor stretches/shrinks as a result of this deformation, the *conformal transformation* does not involve any point transformation, the metric is just changed 'by hand'.

Definition (Conformal transformation)**.** *Suppose the metric tensor $\boldsymbol{g}$ is defined on M, the redefinition of the metric tensor on M to $\tilde{\boldsymbol{g}} = e^{2U} \boldsymbol{g}$ for some real function U is then called* ***conformal transformation****.*

[3]This maximal dimension is achieved in conformally flat spacetimes

If a conformal transformation exists between $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$, these metric tensors are said to be *conformal.* If U = const., the transformation is a *homothety*, otherwise the conformal transformation is called to be *proper.* Angles between intersecting curves remain unchanged by conformal transformation and if the metric has indefinite signature, the null cones are also invariant. This property is used in general relativity, where the preservation of the causal structure is used for constructing conformal diagrams.

The fundamental tensor in context of conformal transformation is the *Weyl tensor* $C^{\mu}{}_{\nu\rho\sigma}$, which is defined in terms of the Riemann curvature tensor $R^{\mu}{}_{\nu\rho\sigma}$ and its contractions. In an arbitrary dimension greater that two, the Weyl tensor is defined as

$$\begin{aligned} C^{\mu}{}_{\nu\rho\sigma} := R^{\mu}{}_{\nu\rho\sigma} &+ \frac{1}{(n-1)(n-2)} R(\delta^{\mu}_{\rho} g_{\nu\sigma} - \delta^{\mu}_{\sigma} g_{\nu\rho}) \\ &- \frac{1}{n-2}(\delta^{\mu}_{\rho} R_{\nu\sigma} + R^{\mu}_{\rho} g_{\nu\sigma} - g_{\mu\sigma} R_{\nu\rho} - R^{\mu}_{\sigma} g_{\nu\rho}) \end{aligned} \tag{2.2}$$

The main property of this tensor is its conformal invariance in the sense that

$$\tilde{C}^{\mu}{}_{\nu\rho\sigma} = C^{\mu}{}_{\nu\rho\sigma} \ ,$$

where $\tilde{C}^{\mu}{}_{\nu\rho\sigma}$ is constructed from $\tilde{\boldsymbol{g}} = e^{2U}\boldsymbol{g}$ and $C^{\mu}{}_{\nu\rho\sigma}$ from $\boldsymbol{g}$. Spaces with vanishing Weyl tensor are conformal to the flat space and so are called *conformally flat.* As can be inferred from the form of the prefactors in (2.2), lower-dimensional cases have to be treated separately.

The one-dimensional case is trivial, because the curvature is always zero.

In a two-dimensional space, every metric is conformally flat, i.e. can be written in the form

$$\boldsymbol{g} = e^{2U}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \ ,$$

so there is no non-trivial equivalent of the Weyl tensor.

If the dimension of the space is three, the Weyl tensor, as defined above, always vanishes as well. However, in this case it does not imply that every three-dimensional space is flat. Instead, a different tensor called the *Cotton tensor* must be constructed

$$\tilde{C}^{\mu}{}_{\nu\rho} = R^{\mu}{}_{\nu;\rho} - R^{\mu}{}_{\rho;\nu} + \frac{1}{4}R_{,\nu}\delta^{\mu}_{\rho} - \frac{1}{4}R_{,\rho}\delta^{\mu}_{\nu} \ .$$

This tensor has the desired property that it vanishes if and only if the space is conformally flat.

From now on, only the four-dimensional spacetimes of general relativity will be discussed. The other quantities like the connection and curvature do change under the conformal rescaling and they follow these transformation rules

$$\tilde{g}_{\mu\nu} = e^{2U} g_{\mu\nu} \ , \tag{2.3}$$

$$\tilde{\Gamma}^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\nu\rho} + \delta^{\mu}_{\nu} U_{\rho} + \delta^{\mu}_{\rho} U_{\nu} - g_{\rho\nu} U^{,\mu} \ , U^{,\mu} := g^{\mu\nu} U_{\nu} \ , \tag{2.4}$$

$$Y^{\mu}_{\nu} := U^{,\mu}{}_{;\nu} - U^{,\mu} U_{,\nu} + \delta^{\mu}_{\nu} \|\mathrm{d}U\|^2/2 \ , \tag{2.5}$$

$$e^{2U}\tilde{R}^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} + 4Y^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]} \ , \tag{2.6}$$

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + 2U_{;\mu\nu} - 2U_{,\mu}U_{,\nu} + g_{\mu\nu}(\Box U + 2\|\mathrm{d}U\|^2) \ , \tag{2.7}$$

$$e^{2U}\tilde{R} = R + 6(\Box U + \|\mathrm{d}U\|^2) \ . \tag{2.8}$$

Note that the indices of the quantities with tilde are always shifted with $\tilde{\boldsymbol{g}}$ and the quantities on the right-hand sides of the above equations use $\boldsymbol{g}$ for this purpose.

In principle, conformal transformation can be used to construct new solutions of Einstein's equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu} \ ,$$

However, if one tries to generate spacetimes with a physically reasonable stress-energy tensor $T_{\mu\nu}$, there are some restrictions given by the following theorems.

Theorem 2.1 (Brinkmann, 1925)**.** *Two distinct conformally related Einstein spaces* ($R_{\mu\nu} = R g_{\mu\nu}/4$) *are either both vacuum pp-waves or Minkowski and* $(A)dS$.

This theorem implies that vacuum *pp*-waves are the only non-trivial vacuum spacetimes that can be generated by conformal transformation from vacuum solutions. This old theorem was generalised in 1998 by Daftardar-Gejji

Theorem 2.2 (Daftardar-Gejji, 1998)**.** *If two distinct conformally related spacetimes have equal Einstein tensors, i.e.* $\tilde{G}_{\mu\nu} = G_{\mu\nu}$*, they are both pp-waves.*
If the Einstein tensors of two conformally related spacetimes differ by a cosmological constant term, i.e. $\tilde{G}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu}$*, a larger class of solutions is admissible, including perfect fluids.*

The possibilities of constructing perfect fluids from vacuum spacetimes were studied in several papers by Van den Bergh (see [20] and references therein) but since this thesis concentrates on Einstein-Maxwell fields, the following theorem by the same author is more relevant.

Theorem 2.3 (Van den Bergh, 1986)**.** *Conformally vacuum Einstein-Maxwell field is null (algebraically special) if and only if the gradient of the conformal factor is null (light-like) and the spacetime is a pp-wave.*

There are also other theorems regarding conformal transformation, but the above mentioned are the most relevant.

3. Spinors and the NP formalism

Similarly to the other applications of this calculus, the use of spinors in context of conformal transformation simplifies many equations and helps to understand their structure better. The interconnection between spinors and the conformal transformation is perhaps even deeper than in the generic case, because spinors define the null cone structure, which is invariant under conformal transformation. In fact, it is possible to introduce spinors on a manifold that is equipped with just a conformal structure[1] instead of the metric. The basic notions of the spinor calculus will be outlined in this chapter. For a deeper insight into this subject, the reader is referred to [21] or [22].

3.1 Spinor algebra

In this first part of the spinor overview, the vector space on which spinors live will be introduced. Such space will be "glued" to each point of the manifold later, but for now, the subjects of interest are "spinors at a point".

Definition (Spinor algebra)**.** *Spinor algebra S is a two-dimensional vector space over $\mathbb{C}$ with the antisymmetric bilinear product $\boldsymbol{\epsilon}$, i.e. for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S$:*
$\boldsymbol{\epsilon}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = -\boldsymbol{\epsilon}(\boldsymbol{\beta}, \boldsymbol{\alpha})$,
$\boldsymbol{\epsilon}(c\boldsymbol{\alpha}, \boldsymbol{\beta}) = c\boldsymbol{\epsilon}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \quad \forall\, c \in \mathbb{C}$.

The elements of S are then called *spinors* and $\boldsymbol{\epsilon}$ plays an analogous role to the Minkowski metric in special relativity. Connection of the abstract spinor algebra to physics is provided by the fact that the linear transformations L on S which preserve $\boldsymbol{\epsilon}$, i.e. $\boldsymbol{\epsilon}(L\boldsymbol{\alpha}, L\boldsymbol{\beta}) = \boldsymbol{\epsilon}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, form a group that can be mapped two-to-one to the group of proper Lorentz transformations.
In some calculations and especially in the Newmann-Penrose formalism, it is convenient to work in a fixed basis. Therefore, the *spin basis* is introduced.

Definition (Spin basis)**.** *A set o two spinors ($\boldsymbol{e}_0 = \boldsymbol{o}, \boldsymbol{e}_1 = \boldsymbol{\iota}$) forms a **spin basis** in S if and only if $\boldsymbol{\epsilon}(\boldsymbol{o}, \boldsymbol{\iota}) = 1$.*

Such basis is, of course, not unique. In fact, every two non-proportional spinors $\boldsymbol{\eta}, \boldsymbol{\xi}$ can be made into a spin basis by just rescaling one of them. The spin basis, as will be mentioned later, is a spinor equivalent of the null basis[2]. Therefore, the linear transformations $L : S \mapsto S$ that map spin bases to other spin bases, i.e. $\boldsymbol{\epsilon}(L\boldsymbol{o}, L\boldsymbol{\iota}) = 1$, are of special interest, because they also preserve null bases in Minkowski vector space. Each such L can be composed from these three fundamental transformations of the spin basis:

- *spin-boosts*:
$$(\boldsymbol{o}', \boldsymbol{\iota}') = (\lambda\boldsymbol{o}, \lambda^{-1}\boldsymbol{\iota}) \ , \ \lambda \in \mathbb{C}, \lambda \neq 0 \tag{3.1}$$

[1]Conformal structure on a manifold is given by an equivalence class of metric tensors, where the metric tensors are assumed to be equivalent iff they are conformal.

[2]basis of four null vectors $(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n})$ in the Minkowski vector space in which the Minkowski metric looks like $g_{\mu\nu} = 2k_{(\mu}l_{\nu)} - 2m_{(\mu}\bar{m}_{\nu)}$

- *null rotations about $\boldsymbol{k}$*:

$$(\boldsymbol{o}', \boldsymbol{\iota}') = (\boldsymbol{o}, \boldsymbol{\iota} + a\boldsymbol{o}) \ , \ a \in \mathbb{C} \tag{3.2}$$

- *null rotations about $\boldsymbol{l}$*:

$$(\boldsymbol{o}', \boldsymbol{\iota}') = (\boldsymbol{o} + b\boldsymbol{\iota}, \boldsymbol{\iota}) \ , \ b \in \mathbb{C} \tag{3.3}$$

These three transformations are the spinor equivalents of Lorentz transformations. In total, there are three complex parameters, that corespond to six real parameters of the Lorentz group. The terminology for the last two transformations should be more understandable after reading the next section.

The convention is that the indices of spinors from S are written in capital latin letters in the upper position, e.g. α^A. Of course, the dual space S^* can be constructed, which is the space of linear maps $S \mapsto S$ and the indices of its elements shall be written in lower positon, e.g. β_A. In this context, $\boldsymbol{\epsilon}$ is an element of $S^{*2} := S^* \otimes S^*$ and so it reads ϵ_{AB} when written with indices. The antisymmetric product $\boldsymbol{\epsilon}$ also provides a natural isomorphism between S and S^* in the similar way the metric does in relativity, i.e.

$$S^* \ni \boldsymbol{\beta}(\,.\,) = \boldsymbol{\epsilon}(\boldsymbol{\alpha},\,.\,), \boldsymbol{\alpha} \in S\,.$$

Conventionally, the isomorphic elements in S and S^* are denoted by the same letter and the distinction is made only by the index position, e.g. $\alpha_A = \epsilon_{AB}\alpha^B$ — this is the index lowering and attention must be paid to the correct order of indices on $\boldsymbol{\epsilon}$. Much like in the special relativity, the index raising is defined through $\epsilon^{AB} := o^A\iota^B - \iota^A o^B$, which is, up to a sign, an inverse of ϵ_{AB}. The index is therefore raised as $\beta^A = \epsilon^{AB}\beta_B$, note again that the indices on ϵ^{AB} should not be swapped.

The fact that S is only two-dimensional has an important consequence — antisymmetric spinors from S^2, i.e. spinors for which $\tau_{AB} = -\tau_{BA}$, form just a one-dimensional subspace in S^2 and so every antisymmetric spinor has to be proportional to $\boldsymbol{\epsilon} \in S^{*2}$. Therefore, the antisymmetric part of an arbitrary spinor from S^2 may be reduced to

$$\tau_{[AB]} := \frac{1}{2}\epsilon_{AB}\tau_C{}^C \ .$$

This is the foundation of the popular motto "only symmetric spinors matter", because the antisymmetric part of a spinor of an arbitrary rank can always be decomposed into the tensor product of $\boldsymbol{\epsilon}$ with a lower rank spinor, and the same can be repeated for the lower rank spinor. After a finite number of such steps, the original spinor is decomposed into products of $\boldsymbol{\epsilon}$s and totally symmetric spinors.

The complex conjugate of a spinor from S, e.g. $\bar{\boldsymbol{\alpha}}$, lives in space $\bar{S}$ that is not isomorphic[3] to S. Therefore, to distinguish between spinors from S and $\bar{S}$, the indices will be dotted in the later case, e.g. $\beta^{\dot{A}} \in \bar{S}$. Because S and $\bar{S}$ are not isomorphic, the dotted indices on spinors from $S^m \otimes \bar{S}^n$ may be swapped with

[3]it is, in fact, anti-isomorphic, because the scalar multiplication $\boldsymbol{\beta} = c\boldsymbol{\alpha}$ conjugates to $\bar{\boldsymbol{\beta}} = \bar{c}\bar{\boldsymbol{\alpha}}$, while an isomorphism should satisfy $\bar{\boldsymbol{\beta}} = c\bar{\boldsymbol{\alpha}}$.

undotted, but the relative order of both dotted and undotted indices must remain the same, e.g.

$$\tau^{AB\dot{C}} = \tau^{A\dot{C}B} \neq \tau^{B\dot{C}A} \ .$$

The last thing to note in this section is that the complex conjugate of ϵ_{AB} is usually written without the bar, i.e. $\overline{\epsilon_{AB}} = \epsilon_{\dot{A}\dot{B}}$, since the components of $\boldsymbol{\epsilon}$ in spin bases are always real[4] anyway.

3.2 Spinors and vectors

In this section, the link between spinors and vectors will be outlined.

Definition (Hermitian spinor). *Spinor $\boldsymbol{\tau} \in S \otimes \bar{S}$ is called **Hermitian** iff $\tau^{A\dot{B}} = \bar{\tau}^{B\dot{A}}$.*

It can be shown that the space H of Hermitian spinors is four-dimensional if viewed as a vector space over $\mathbb{R}$. A bijection can now be constructed between the Hermitian spinors and the traditional four-vectors from Minkowski vector space M_4.

Theorem 3.1 (Spinor-vector isometry). *Let $(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}})$ be a null basis in M_4 and let $(\boldsymbol{o}, \boldsymbol{\iota}), (\overline{\boldsymbol{o}}, \overline{\boldsymbol{\iota}})$ be spin bases in S, resp. $\bar{S}$. The bijection between $M_4 \leftrightarrow H$ (and their dual spaces) given by*

$$k^a \leftrightarrow o^A \bar{o}^{\dot{A}} \ , \quad l^a \leftrightarrow \iota^A \bar{\iota}^{\dot{A}} \ , \quad m^a \leftrightarrow o^A \bar{\iota}^{\dot{A}} \ , \quad \bar{m}^a \leftrightarrow \iota^A \bar{o}^{\dot{A}} \ , \tag{3.4}$$

$$k_a \leftrightarrow o_A \bar{o}_{\dot{A}} \ , \quad l_a \leftrightarrow \iota_A \bar{\iota}_{\dot{A}} \ , \quad m_a \leftrightarrow o_A \bar{\iota}_{\dot{A}} \ , \quad \bar{m}_a \leftrightarrow \iota_A \bar{o}_{\dot{A}} \ . \tag{3.5}$$

is an isometry.

Instead of the sign '$\leftrightarrow$', the equality '$=$' is usually written for convenience. Since every (co)vector has its spinor counterpart, the whole tensor algebra can be mirrored into spinor algebra. The equivalents of specific real tensors used in general relativity are listed in table 3.1.

3.3 Petrov classification

Petrov classification is based on an interesting property of symmetric spinors

Theorem 3.2 (Symmetric spinor decomposition). *Every totally symmetric spinor $\tau_{AB\cdots C}$ of rank n can be decomposed into a symmetrized tensor product of n spinors of rank one, i.e.*

$$\tau_{AB\cdots C} = \alpha_{(A}\beta_B \cdots \gamma_{C)} \ .$$

$\boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots, \boldsymbol{\gamma}$ are called the principal spinors of $\boldsymbol{\tau}$ and the corresponding vectors are the principal null directions of $\boldsymbol{\tau}$.

[4] concretely $\epsilon_{00} = \epsilon_{11} = 0 \, , \epsilon_{01} = -\epsilon_{10} = 1$

Table 3.1: Spinor equivalents of tensor quantities

Tensor	Properties	Spinor equivalent	Properties
null vector	$n^a n_a = 0$	$n^a = \pm\eta^A \bar{\eta}^{\dot{A}}$	+ future, − past
metric	$g_{ab} = g_{ba}$	$g_{ab} = \epsilon_{AB}\epsilon_{\dot{A}\dot{B}}$	
Maxwell tensor	$F_{ab} = F_{[ab]}$	$F_{ab} = \epsilon_{AB}\bar{\phi}_{\dot{A}\dot{B}} + \epsilon_{\dot{A}\dot{B}}\phi_{AB}$	$\phi_{AB} = \phi_{(AB)}$
Riemann tensor	$R_{abcd} = R_{[ab][cd]}$ $R_{abcd} = R_{cdab}$ $R_{a[bcd]} = 0$	$R_{abcd} = \epsilon_{AB}\epsilon_{CD}\bar{\chi}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ $+\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}}\chi_{ABCD}$ $+\epsilon_{AB}\epsilon_{\dot{C}\dot{D}}\Phi_{CD\dot{A}\dot{B}}$ $+\epsilon_{\dot{A}\dot{B}}\epsilon_{CD}\Phi_{AB\dot{C}\dot{D}}$	$\Phi_{AB\dot{C}\dot{D}} = \overline{\Phi_{CD\dot{A}\dot{B}}}$ $\Phi_{AB\dot{C}\dot{D}} = \Phi_{(AB)(\dot{C}\dot{D})}$ $\chi_{ABCD} = \Psi_{ABCD}$ $-2\Lambda\epsilon_{(A(C}\epsilon_{D)B)}$ $\Psi_{ABCD} = \Psi_{(ABCD)}$ $\Lambda = R/24$
Weyl tensor	$C_{abcd} = C_{[ab][cd]}$ $C_{abcd} = C_{cdab}$ $C^k{}_{akb} = 0$ $C_{a[bcd]} = 0$	$C_{abcd} = \epsilon_{AB}\epsilon_{CD}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ $+\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}}\Psi_{ABCD}$	
Traceless Ricci tensor	$S_{ab} = S_{ba}$ $S^a_a = 0$	$S_{ab} = -2\Phi_{AB\dot{A}\dot{B}}$	

The Petrov types of spinors are then determined by the multiplicities of the principal spinors. If no principal spinor is a scalar multiple of some other, the classified spinor is said to be *algebraically general*. If there are some colinear principal spinors, then the symmetric spinor is *algebraically special*. The two physical symmetric spinors which were listed in table 3.1 are the Maxwell spinor ϕ_{AB} and the Weyl spinor Ψ_{ABCD}.

Maxwell spinor. ϕ_{AB} is a rank two spinor and so the Petrov classification is very simple. According to theorem 3.2, the Maxwell spinor can be decomposed into

$$\phi_{AB} = \alpha_{(A}\beta_{B)}$$

and only two cases are admissible:

1. $\boldsymbol{\beta}$ is not proportional to $\boldsymbol{\alpha}$, the Maxwell spinor is then algebraically general. The corresponding electromagnetic field is called *non-null*.
2. $\boldsymbol{\beta} = c\boldsymbol{\alpha}$ for some $c \in \mathbb{C}$, this is the algebraically special case and the electromagnetic field is called *null*.

Weyl spinor. The classification is richer for the rank four Weyl tensor, for which the decomposition reads

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)} \ .$$

The list of possibilities is adequately longer

1. None of the four principal spinors are proportional and so the Weyl spinor is algebraically general, the spacetime is said to be of *type I.*
2. Precisely two principal spinors are proportional, the Weyl spinor is algebraically special and so will be all other following cases. The spacetime is of *type II.*
3. There are two different pairs of proportional spinors. This is a Petrov *type D* spacetime.
4. Three principal null directions coincide and the spacetime is called *type III.*
5. All four principal spinors are aligned in the same direction. This is the most special case and the spacetime is of *type N.*

3.4 Spinor fields on a manifold

Up to now, all the spinors lived on the spinor algebra "at a point", but naturally, one would like to introduce notions like covariant derivative, curvature, etc. and for this purpose, the spinor algebra is glued to each point of the manifold M, much like the tangent space of vectors. A *spinor field* is then a smooth map that assigns a spinor $\boldsymbol{\tau}(x)$ to each point $x \in M$. In further text, spinor fields will be often called just spinors for short, as the true meaning should be obvious from context.

The *spinor covariant derivative* can be defined as a consistent extension of classical tensorial covariant derivative to spinor fields. An exact definition will not be given here, just the important properties will be listed.
Spinor covariant derivative $\nabla_{A\dot{A}}$ is a map $\boldsymbol{\tau} \mapsto \nabla_{A\dot{A}}\boldsymbol{\tau}$ that obeys all the rules for classical tensorial torsion-free covariant derivative like linearity and Leibniz rule, but additionally:

- $\nabla_{A\dot{A}}\epsilon_{BC} = 0$; this conditition ensures that the covariant derivative annihilates metric.
- $\nabla_{A\dot{A}}\overline{\boldsymbol{\tau}} = \overline{\nabla_{A\dot{A}}\boldsymbol{\tau}}$

With the covariant derivative in hands, the Riemann tensor can now be constructed from the Ricci identity

$$2\nabla_{[a}\nabla_{b]}V_c = -R_{abcd}V^d$$

The spinor equivalent of the commutator $2\nabla_{[a}\nabla_{b]}$ is

$$2\nabla_{[a}\nabla_{b]} = \epsilon_{AB}\Box_{\dot{A}\dot{B}} + \epsilon_{\dot{A}\dot{B}}\Box_{AB} \ , \ \Box_{\dot{A}\dot{B}} := \nabla_{\dot{A}(C}\nabla^{C}{}_{\dot{B})} \ , \ \Box_{AB} := \nabla_{\dot{C}(A}\nabla^{\dot{C}}{}_{B)}$$

and since the equivalent of the Riemann tensor reads

$$R_{abcd} = \epsilon_{AB}\epsilon_{CD}\bar{\chi}_{\dot{A}\dot{B}\dot{C}\dot{D}} + \epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}}\chi_{ABCD} + \epsilon_{AB}\epsilon_{\dot{C}\dot{D}}\Phi_{CD\dot{A}\dot{B}} + \epsilon_{\dot{A}\dot{B}}\epsilon_{CD}\Phi_{AB\dot{C}\dot{D}} \ ,$$

where

$$\chi_{ABCD} = \Psi_{ABCD} - 2\Lambda\epsilon_{(A(C}\epsilon_{D)B)} \ ,$$

one can readily obtain the spinor equivalent of Ricci identities

$$\Box_{AB}\tau_C = \Psi_{ABCD}\tau^D - 2\Lambda\tau_{(A}\epsilon_{B)C}\ ,\ \Box_{AB}\bar{\tau}_{\dot{C}} = \Phi_{AB\dot{C}\dot{D}}\tau^{\dot{D}}\ .$$

Naturally, the Bianchi identities $R_{ab[cd;e]}$ also have their spinor counterpart, which will be very important in context of the last chapters

$$\nabla^X_{\dot{D}}\Psi_{ABCX} = \nabla^{\dot{X}}_{(A}\Phi_{BC)\dot{D}\dot{X}}\ , \tag{3.6}$$

$$\nabla^{X\dot{X}}\Phi_{AX\dot{A}\dot{X}} = -3\nabla_{A\dot{A}}\Lambda \tag{3.7}$$

One of the main advantages of the spinor formalism is the transparent way in which the equations for zero rest mass fields in vacuum can be written

Theorem 3.3 (Massless fields)**.** *Every zero rest mass field of spin $s = n/2$, $n \in \mathbb{N}_0$ can be represented by a totally symmetric rank n spinor $\boldsymbol{\phi}$ and its field equations in vacuum read*

$$\nabla^{C\dot{D}}\phi_{AB\cdots C} = 0$$

The most prominent fields in this thesis are of spin 1 and 2. The former represents the Maxwell field and the latter is the free gravitational field. Indeed, Maxwell's equations in vacuum $F_{ab}{}^{;b} = 0$, $F_{[ab;c]} = 0$, when translated into spinor language with the help of table 3.1, read simply

$$\nabla^{B\dot{B}}\phi_{AB} = 0\ . \tag{3.8}$$

The fact that the gravitational field in vacuum satisfies

$$\nabla^X_{\dot{D}}\Psi_{ABCX} = 0$$

follows immediately from the Bianchi identities.

3.5 The Newman-Penrose formalism

Although the Newman-Penrose (NP) formalism can be utilized without any knowledge of spinors, the quantities that will be defined in this section can be understood more easily in the spinor language. The main idea behind the NP formalism is to work with scalars that are independent on the choice of coordinates, unlike the components of tensors. For this purpose, the projections of certain physical spinors onto the spin basis $(\boldsymbol{o}, \boldsymbol{\iota})$ are constructed[5], and the only ambiguity that may occur is the freedom in choice of the spin basis, which is given by transformations (3.1)-(3.3).

The projections of covariant derivatives onto the spin (null) basis will be denoted by

$$D := o^A\bar{o}^{\dot{A}}\nabla_{A\dot{A}} = k^a\nabla_a\ ,\ \Delta := \iota^A\bar{\iota}^{\dot{A}}\nabla_{A\dot{A}} = l^a\nabla_a\ ,$$
$$\delta := o^A\bar{\iota}^{\dot{A}}\nabla_{A\dot{A}} = m^a\nabla_a\ ,\ \bar{\delta} := \iota^A\bar{o}^{\dot{A}}\nabla_{A\dot{A}} = \bar{m}^a\nabla_a\ ,$$

In order to fully capture the effect of covariant derivative on an arbitrary spinor, it is sufficient to know how it acts on a spin basis. For this purpose, the 12 complex

Table 3.2: Newman-Penrose spin coefficients

$\cdot$	$o^A \nabla_{B\dot{B}} o_A$	$o^A \nabla_{B\dot{B}} o_A = \iota^A \nabla_{B\dot{B}} o_A$	$\iota^A \nabla_{B\dot{B}} \iota_A$
$o^B \bar{o}^{\dot{B}}$	κ	ε	π
$\iota^B \bar{\iota}^{\dot{B}}$	τ	γ	ν
$o^B \bar{\iota}^{\dot{B}}$	σ	β	μ
$\iota^B \bar{o}^{\dot{B}}$	ρ	α	λ

scalars — *NP spin coefficients* — are introduced in table 3.2. The Maxwell spinor ϕ_{AB} defines three complex scalars by

$$\phi_0 := \phi_{AB} o^A o^B = \frac{1}{2}(E_1 - B_2) - \frac{i}{2}(E_2 + B_1) \ , \tag{3.9}$$

$$\phi_1 := \phi_{AB} o^A \iota^B = -\frac{1}{2}(E_3 + iB_3) \ , \tag{3.10}$$

$$\phi_2 := \phi_{AB} \iota^A \iota^B = -\frac{1}{2}(E_1 + B_2) - \frac{i}{2}(E_2 - B_1) \ , \tag{3.11}$$

where E_i and B_i are the components of electric, resp. magnetic fields in the orthonormal frame

$$\boldsymbol{e}_0 = 2^{-1/2}(\boldsymbol{k} + \boldsymbol{l}) \, , \ \boldsymbol{e}_1 = 2^{-1/2}(\boldsymbol{m} + \overline{\boldsymbol{m}}) \, , \ \boldsymbol{e}_2 = 2^{-1/2} i(\boldsymbol{m} - \overline{\boldsymbol{m}}) \, , \ \boldsymbol{e}_3 = 2^{-1/2}(\boldsymbol{k} - \boldsymbol{l}). \tag{3.12}$$

Thus the spinor ϕ_{AB} may be written in terms of ϕ_0, ϕ_1, ϕ_2 and the spin basis as

$$\phi_{AB} := \phi_2 o_A o_B - 2\phi_1 o_{(A} \iota_{B)} + \phi_0 \iota_A \iota_B$$

With the given component decompositions of covariant derivative and the Maxwell spinor, vacuum Maxwell's equations in terms of NP scalars can be easily obtained. They read

$$D\phi_1 - \bar{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2 \ , \tag{3.13}$$

$$D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2 \ , \tag{3.14}$$

$$\Delta\Phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2 \ , \tag{3.15}$$

$$\Delta\Phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2 \ . \tag{3.16}$$

The *NP curvature scalars* are obtained by contractions of (the spinor equivalents of) traceless Ricci tensor $\Phi_{AB\dot{A}\dot{B}}$ and Weyl tensor Ψ_{ABCD}.

$$\begin{aligned}
\Phi_{00} &:= \Phi_{AB\dot{A}\dot{B}} o^A o^B \bar{o}^{\dot{A}} \bar{o}^{\dot{B}} \ , \\
\Phi_{01} &:= \Phi_{AB\dot{A}\dot{B}} o^A o^B \bar{o}^{\dot{A}} \bar{\iota}^{\dot{B}} \ , \\
\Phi_{02} &:= \Phi_{AB\dot{A}\dot{B}} o^A o^B \bar{\iota}^{\dot{A}} \bar{\iota}^{\dot{B}} \ , \\
\Phi_{10} &:= \Phi_{AB\dot{A}\dot{B}} o^A \iota^B \bar{o}^{\dot{A}} \bar{o}^{\dot{B}} \ , \\
\Phi_{11} &:= \Phi_{AB\dot{A}\dot{B}} o^A \iota^B \bar{o}^{\dot{A}} \bar{\iota}^{\dot{B}} \ , \\
\Phi_{12} &:= \Phi_{AB\dot{A}\dot{B}} o^A \iota^B \bar{\iota}^{\dot{A}} \bar{\iota}^{\dot{B}} \ ,
\end{aligned}$$

[5]If one wants to establish NP formalism without spinors, the tensors should be projected onto the null tetrad which is equivalent to the spin basis.

$$
\begin{aligned}
\Phi_{20} &:= \Phi_{AB\dot{A}\dot{B}} o^A \iota^B \bar{\iota}^{\dot{A}} \bar{o}^{\dot{B}} \ , \\
\Phi_{21} &:= \Phi_{AB\dot{A}\dot{B}} \iota^A \iota^B \bar{o}^{\dot{A}} \bar{\iota}^{\dot{B}} \ , \\
\Phi_{22} &:= \Phi_{AB\dot{A}\dot{B}} \iota^A \iota^B \bar{\iota}^{\dot{A}} \bar{\iota}^{\dot{B}} \ .
\end{aligned}
$$

This set of scalars clearly satisfies $\Phi_{ij} = \overline{\Phi_{ji}}$ and so it contains nine independent real quantities which is the appropriate amount for the traceless Ricci tensor. The Weyl tensor that has ten independent real components yields these five complex scalars

$$
\begin{aligned}
\Psi_0 &:= \Psi_{ABCD} o^A o^B o^C o^D \ , \\
\Psi_1 &:= \Psi_{ABCD} o^A o^B o^C \iota^D \ , \\
\Psi_2 &:= \Psi_{ABCD} o^A o^B \iota^C \iota^D \ , \\
\Psi_3 &:= \Psi_{ABCD} o^A \iota^B \iota^C \iota^D \ , \\
\Psi_4 &:= \Psi_{ABCD} \iota^A \iota^B \iota^C \iota^D \ .
\end{aligned}
$$

The last NP curvature scalar is the quantity $\Lambda = R/24$, defined in table 3.1. Of course, the curvature is linked to the connection by the Ricci identities. In NP formalism, the curvature is represented by NP curvature scalars, while NP spin coefficients represent the connection. The equivalent of Ricci equations that relates the NP curvature scalars to the NP spin coefficients are called the *NP field equations*, which are a bit lengthy and so they will not be listed here and the same applies to the NP Bianchi identities.

4. The equivalence problem

The fundamental and desirable property of general relativity, the coordinate independence, comes hand in hand with the equivalence problem — how does one determine whether two metric tensors $\boldsymbol{g}$, $\boldsymbol{g}'$ given in different coordinate systems are equivalent? This problem naturally arises in context of generating methods — is the generated solution really distinct from the original one, or is it just the same spacetime disguised in different coordinates? The straightforward approach to this problem is to look for the coordinate transformation $x' = x'(x)$ that satisfies

$$g_{\rho\sigma} = g'_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} \ .$$

However, the group of general coordinate transformations is too complicated to work with. Therefore, the problem is usually reformulated and a different question is posed — how to uniquely describe the local geometry in a coordinate-independent way? In this thesis, two solutions to this problem will be presented. The first approach is based on the work of Cartan [23] that utilizes successive fixing of frames in which the components of the Riemann tensor and its derivatives are calculated. The other solution uses the scalar curvature invariants (SCIs), this description is unique except for a special class of spacetimes [24]. It should be emphasized that both these methods are local, i.e. sufficient to prove the equivalence in neighbourhood of some point, the global structure may be generally different[1].

4.1 Cartan scalars

This section is dedicated to the unique local description of geometry via the set of scalars that are obtained by projections of the Riemann tensor and its derivaties onto a fixed frame. This set is then called the *Cartan scalars*. To describe the geometry $\boldsymbol{g}$ of a spacetime, rather than the coordinate components $g_{\mu\nu}$, Cartan uses the set of four linearly independent one-forms $\boldsymbol{\theta}^i = \theta^i_{\mu}(x)\mathrm{d}x^{\mu}$ called the *frame* in which $\boldsymbol{g} = g_{ij}\boldsymbol{\theta}^i\boldsymbol{\theta}^j$ for some fixed matrix g_{ij} [2]. The frame for a given geometry is, however, not unique. If, for example, the frame is chosen to be orthonormal, i.e. $g_{ij} = \eta_{ij} = \mathrm{diag}(1, -1, -1, -1)$, then a point-wise Lorentz transformation gives other frames that yield the same geometry. The trick to get rid of this ambiguity is to consider the frame $\{\boldsymbol{\theta}^i\}$ not on the manifold M, but on the *Lorentz bundle* ΛM, which is the fibre bundle of orthonormal frames (or any other rigid frames) over M. For those unfamiliar with frame bundles an informal definition will follow.

Definition (Lorentz bundle)**.** *Each element $\boldsymbol{\theta}$ of ΛM is a specific orthonormal frame $\{\boldsymbol{\theta}^i(x)\}$ at a given point $x \in M$. Each orthonormal frame at given $x \in M$ can be obtained from a chosen reference frame $\{\boldsymbol{\theta}^i_0(x)\}$ by a unique Lorentz transformation and so the subspace of all orthonormal frames at a given point*

[1] An elementary example is the two-dimensional cone which is locally flat, but has different global structure than the Euclidean plane.

[2] g_{ij} is usually chosen to be constant and the corresponding frames are called *rigid frames*, e.g. orthonormal or null frame.

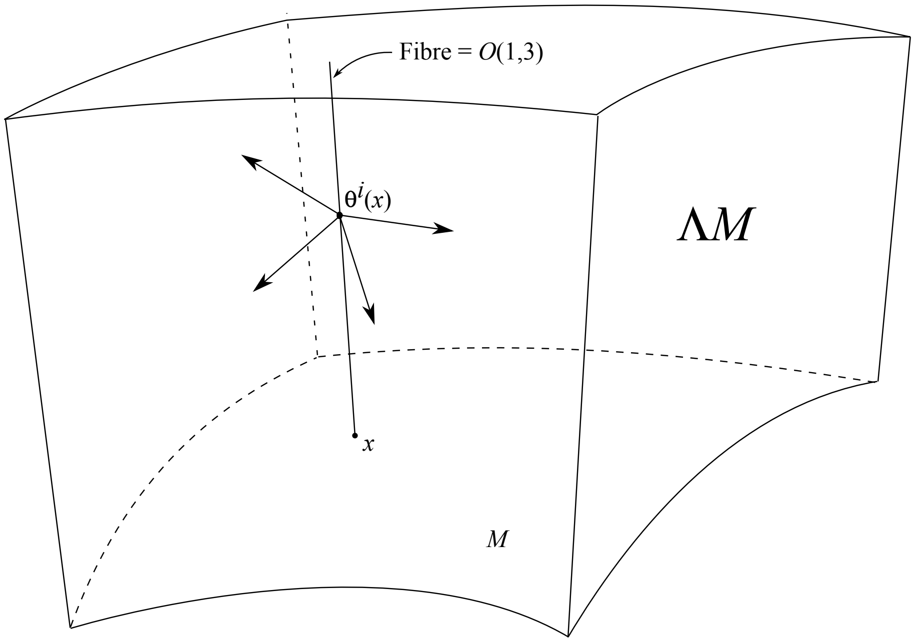

Figure 4.1: Lorentz bundle ΛM

(this is the fibre) is isomorphic to the Lorentz group $O(1,3)$. The dimension of ΛM is, therefore, $10 = 4 + 6$. The standard coordinates of an element $\{\boldsymbol{\theta}^i(x)\}$ are $(x^\mu, \Lambda^a{}_b)$, where:
x^μ are the coordinates of $x \in M$ and
$\Lambda^a{}_b$ transforms the reference frame $\{\boldsymbol{\theta}_0^i(x)\}$ to $\{\boldsymbol{\theta}^i(x)\}$, i.e. $\boldsymbol{\theta}^i(x) = \Lambda^i{}_j \boldsymbol{\theta}_0^j(x)$.

The advantage of working on ΛM is that its cotangent space possesses a unique basis of one-forms, unlike the original M. At each point $\boldsymbol{\theta} \in \Lambda M$, four one-forms $\hat{\boldsymbol{\theta}}^i$ are naturally introduced as those of the frame which the point $\boldsymbol{\theta}$ represents. Six more independent one-forms are needed to complete the basis. They will be denoted $\hat{\boldsymbol{\omega}}^i{}_j$ and can be defined via

$$\mathrm{d}\hat{\boldsymbol{\theta}}^i =: -\hat{\boldsymbol{\omega}}^i{}_j \wedge \hat{\boldsymbol{\theta}}^j \quad , \quad \hat{\boldsymbol{\omega}}_{ij} = -\hat{\boldsymbol{\omega}}_{ji} \quad , \quad \hat{\boldsymbol{\omega}}_{ij} := \eta_{ik}\,\hat{\boldsymbol{\omega}}^k{}_j \ .$$

Note that the exterior derivative in the above formula consists of the usual derivative with respect to x^μ, but it also contains the derivative with respect to the fibre coordinates $\Lambda^a{}_b$. It can be verified that the set $(\hat{\boldsymbol{\theta}}^i, \hat{\boldsymbol{\omega}}^i{}_j)$ is independent and therefore forms a uniquely defined basis in the cotangent space of ΛM. The equivalence problem is then resolved by Cartan's theorem

Theorem 4.1 (Cartan)**.** *The set of n linearly independent one-forms $\boldsymbol{\omega}^i$ on M is uniquely determined by the set of scalars*

$$\begin{aligned}
c^i{}_{jk}(x)\ &,\ \mathrm{d}\boldsymbol{\omega}^i =: c^i{}_{jk}\boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k\ ,\\
c^i{}_{jk|l_1}(x)\ &,\ \mathrm{d}c^i{}_{jk} =: c^i{}_{jk|l_1}\boldsymbol{\omega}^{l_1}\ ,\\
&\ \vdots\\
c^i{}_{jk|l_1\cdots l_{p-1}}(x)\ &,\\
c^i{}_{jk|l_1\cdots l_p}(x)\ &,\ \mathrm{d}c^i{}_{jk|l_1\cdots l_{p-1}} =: c^i{}_{jk|l_1\cdots l_p}\boldsymbol{\omega}^{l_p}\ ,
\end{aligned}$$

where the penultimate line is the last line that contains scalars functionally independent on quantities from previous lines. The scalars in the last line are, therefore, all functionally dependent on scalars from the previous lines.

p, the number of derivatives of $c^i{}_{jk}$ one has to calculate is clearly limited by the dimension of the manifold, since there cannot be more than n functionally independent scalars on an n-dimensional manifold. The nth derivative could be required only if one new functionally independent scalar appeared for each derivative, but generally, the sufficient number of derivatives is lower, i.e. $p \leq n$. This theorem can be used to construct an invariant characterization of geometry, because the metric $\boldsymbol{g}$ uniquely determines its Lorentz bundle ΛM, which possesses a canonically defined basis of one-forms $(\boldsymbol{\theta}^i, \boldsymbol{\omega}^i{}_j)$. The theoretical maximal number of derivatives one has to calculate is ten since it is the dimension of ΛM. This application of Cartan's theorem in general relativity was first introduced by Brans [25]. Using the Cartan's structure equations (see appendix C.4)

$$\begin{aligned} \mathrm{d}\boldsymbol{\theta}^i &= -\boldsymbol{\omega}^i{}_j \wedge \boldsymbol{\theta}^j \ , \\ \mathrm{d}\boldsymbol{\omega}^i{}_j &= -\boldsymbol{\omega}^i{}_k \wedge \boldsymbol{\omega}^k{}_j + \frac{1}{2} R^i{}_{jkl} \boldsymbol{\theta}^k \wedge \boldsymbol{\theta}^l \ , \end{aligned}$$

one can find that the set of scalars $\{c^i{}_{jk}, c^i{}_{jk|l_1}, \ldots, c^i{}_{jk|l_1 \cdots l_{p-1}}, c^i{}_{jk|l_1 \cdots l_p}\}$ for the basis of one-forms $(\boldsymbol{\theta}^i, \boldsymbol{\omega}^i{}_j)$ is equivalent to the set of covariant derivatives of the Riemann tensor on ΛM, i.e. the set

$$R^p := \{\hat{R}^i{}_{jkl}(x,\Lambda), \hat{R}^i{}_{jkl;m_1}(x,\Lambda), \ldots, \hat{R}^i{}_{jkl;m_1 \cdots m_p}(x,\Lambda)\}$$

gives the desired invariant description of local geometry, where the scalar quantities $\hat{R}^i{}_{jkl;m \cdots n}(x,\Lambda)$ on ΛM can be interpreted as the components of $R^i{}_{jkl;m \cdots n}$ at $x \in M$ in basis determined by Λ. To decide whether two metric elements $\boldsymbol{g}$ and $\boldsymbol{g}'$ are isometric, one has to construct the sets R^p, R'^p for both spacetimes and compare them

$$\begin{aligned} \hat{R}^i{}_{jkl}(x,\Lambda) &= \hat{R}'^i{}_{jkl}(x',\Lambda'), \\ \hat{R}^i{}_{jkl;m_1}(x,\Lambda) &= \hat{R}'^i{}_{jkl;m_1}(x',\Lambda'), \\ &\vdots \\ \hat{R}^i{}_{jkl;m_1 \cdots m_p}(x,\Lambda) &= \hat{R}'^i{}_{jkl;m_1 \cdots m_p}(x',\Lambda'). \end{aligned}$$

The two geometries are then equivalent if and only if all the above equations are consistent as relations between x, Λ and x', Λ'. Theoretically, the equivalence problem is solved, but in practice, the set R^p contains too many scalars, because the Riemann tensor itself has generally twenty independent components and with each additional derivative, the number of scalars multiplies. Therefore, the attempt is to decrease the number of scalars in R^p in the following ways:

- reduce the dimension of the fibre (i.e. restrict the frames) before each differentiation by casting the scalars into a canonical form.
- discard from R^p the scalars that must be dependent on others due to Biachi and Ricci identities.

A practical procedure exists which addresses the first task in a guise that only works with scalars on M and their isotropy groups.

4.1.1 Algorithm for obtaining Cartan scalars

The algorithm consists of several repetitions of the following steps.

Zeroth derivative

0. Choose an arbitrary orthonormal[3] frame $\{\boldsymbol{\theta}_0^i\}$.

1. Calculate the components of the Riemann tensor in this frame, this provides the set of scalars $R^0 = \{R^0_{ijkl}(x)\}$

2. Count the number of independent functions in R^0 and label this number t_0.
In context of ΛM, this step determines the dependence of $\hat{R}_{ijkl}(x, \Lambda)$ on x.

3. Find the isotropy group I_0 that leaves scalars in R^0 invariant.
*I_0 is a subgroup of the Lorentz group $O(1,3)$. For example, in vacuum spacetimes of Petrov type D, scalars R^0_{ijkl} are invariant under spin-boost transformations, i.e. they form a two-dimensional isotropy group I_0. In context of ΛM, this step determines the dependence of $\hat{R}_{ijkl}(x, \Lambda)$ on Λ. To be precise, I_0 represents those Λ on which $\hat{R}_{ijkl}$ does **not** depend.*

4. Construct a new frame $\{\boldsymbol{\theta}_1^i\}$ by an $O(1,3)$ transformation of the old frame $\{\boldsymbol{\theta}_0^i\}$ so that R^1_{ijkl} assumes the canonical form.
This frame is fixed up to transformations in I_0. For example, in a vacuum Petrov type D spacetime, R^1_{ijkl} has only one independent non-zero component that corresponds to Ψ_2, the other components vanish. Such a canonical form exists for all other Petrov types, see appendix D. Of course, in non-vacuum spacetimes, there are additional canonical forms for the Ricci tensor based on its Segre type.

First derivative

1. Calculate the components of the first covariant derivative of the Riemann tensor in the new frame $\{\boldsymbol{\theta}_1^i\}$, this provides the set of scalars
$R^1 = \{R^1_{ijkl}(x), R^1_{ijkl;m_1}(x)\}$.

2. Count the number of independent functions in R^1 and label this number t_1.
Clearly, $t_1 \geq t_0$. If $t_1 = t_0$ then there are no new functionally independent components in the sense of x dependence.

3. Find the isotropy group I_1 that leaves $R^1_{ijkl;m_1}$ invariant.
I_1 is evidently a subgroup of I_0 and if $I_1 = I_0$ then there are no new functionally independent components in the sense of Λ dependence.

4. Construct a new frame $\{\boldsymbol{\theta}_2^i\}$ by an I_0 transformation of the old frame $\{\boldsymbol{\theta}_1^i\}$ so that $R^2_{ijkl;m_1}$ assumes the canonical form.

[3] in practise, the null frames are rather used

If both $t_1 = t_0$ and $I_1 = I_0$, then all the scalars $R^2_{ijkl;m_1}$ are functionally dependent on R^2_{ijkl} in the ΛM sense and so no more derivatives are required, according to Cartan's theorem. The spacetime is then fully described by the set of scalars

$$R^1 = \{R^2_{ijkl}(x), R^2_{ijkl;m_1}(x)\} \ .$$

If $t_1 > t_0$ or $\dim I_1 < \dim I_0$, the procedure continues in the same fashion, i.e. the second derivatives are calculated, t_2 and I_2 are found as well as the canonical frame and the procedure repeats until both $t_p = t_{p-1}$ and $I_p = I_{p-1}$ and the geometry is uniquely defined by

$$R^p := \{R^{p+1}_{ijkl}(x), R^{p+1}_{ijkl;m_1}(x), \dots, R^{p+1}_{ijkl;m_1\cdots m_p}(x)\} \ .$$

This procedure was first presented by Karlhede and MacCallum [26] and it provides, besides the invariant characterization of a given geometry, the isotropy group of the spacetime $I = I_p$ and the dimension r of isometry group G_r, which is given by

$$r = 4 + \dim I - t_p. \tag{4.1}$$

The procedure was implemented in the CLASSI program developed by Aman [27].

If the Bianchi and Ricci identities are considered, some of the scalars in R^p must be algebraically related and thus may be left out. The minimal set was obtained by MacCallum and Aman [28] in the spinor formalism. The minimal set contains the components of spinors $\Psi_{ABCD}, \Phi_{AB\dot{A}\dot{B}}, \Lambda$, their totally symmetrized spinorial covariant derivatives and further the following quantities:

- totally symmetrized covariant derivatives of $\Xi_{ABC\dot{A}} := \nabla^X{}_{\dot{A}}\Psi_{ABCX}$.
- d'Alembertian $\nabla^{A\dot{A}}\nabla_{A\dot{A}}$ on all previously calculated quantities.

4.1.2 Example: Schwarzschild spacetime

The procedure will be illustrated on the oldest known solution of Einstein's equations — the Schwarzschild spacetime

$$\boldsymbol{g} = F(r)\mathrm{d}t^2 - F(r)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \quad , \quad F(r) := 1 - \frac{2M}{r} \ .$$

Zeroth derivative

0. The null frame is chosen

$$\boldsymbol{\theta}^0 := F^{1/2}\mathrm{d}t + F^{-1/2}\mathrm{d}r \ , \boldsymbol{\theta}^1 := F^{1/2}\mathrm{d}t - F^{-1/2}\mathrm{d}r \ ,$$
$$\boldsymbol{\theta}^2 := r\mathrm{d}\vartheta + ir\sin\vartheta\mathrm{d}\varphi \ , \ \boldsymbol{\theta}^3 := r\mathrm{d}\vartheta - ir\sin\vartheta\mathrm{d}\varphi \ .$$

In such frame, $\boldsymbol{g} = \boldsymbol{\theta}^0\boldsymbol{\theta}^1 - \boldsymbol{\theta}^2\boldsymbol{\theta}^3$.

1. Since Schwarzschild spacetime is a vacuum solution, only the Weyl tensor is non-zero and its components read

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \ , \ \Psi_2 = -\frac{M}{r^3} \ .$$

These conditions imply that the solution is of Petrov type D. The set R^0 thus contains only the single non-zero scalar,

$$R^0 = \left\{ \Psi_2 = -\frac{M}{r^3} \right\}.$$

2. Evidently, there is just one independent function in R^0 and it is a function of r.

3. The isotropy group I_0 contains the following transformations and their compositions:

   - spin-boosts:

$$\begin{pmatrix} \boldsymbol{o}' \\ \boldsymbol{\iota}' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{o} \\ \boldsymbol{\iota} \end{pmatrix} , \ a \in \mathbb{C},$$

   - exchange of $k \leftrightarrow l$:

$$\begin{pmatrix} \boldsymbol{o}' \\ \boldsymbol{\iota}' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{o} \\ \boldsymbol{\iota} \end{pmatrix} .$$

4. The Weyl tensor is already in the canonical form for Petrov type D spacetimes, so there is no need to choose a new frame.

First derivative

1. Only the components of $\nabla_{\dot{A}(A}\Psi_{BCDE)}$ need to be calculated, because $\Phi_{AB\dot{A}\dot{B}} = 0 = \Lambda$ and also $\Xi_{ABC\dot{A}} := \nabla^X{}_{\dot{A}}\Psi_{ABCX} = 0$ due to Bianchi identities.

$$\nabla\Psi_{20} := \nabla_{\dot{A}(A}\Psi_{BCDE)}o^A o^B o^C \iota^D \iota^E \bar{o}^{\dot{A}} = \frac{3M}{2^{1/2}r^4}F^{1/2} ,$$
$$\nabla\Psi_{31} := \nabla_{\dot{A}(A}\Psi_{BCDE)}o^A o^B \iota^C \iota^D \iota^E \bar{\iota}^{\dot{A}} = -\frac{3M}{2^{1/2}r^4}F^{1/2} ,$$

   and the other components vanish. Thus,

$$R^1 = \left\{ \Psi_2 = -\frac{M}{r^3}, \nabla\Psi_{20} = \frac{3M}{2^{1/2}r^4}F^{1/2}, \nabla\Psi_{31} = -\frac{3M}{2^{1/2}r^4}F^{1/2} \right\}.$$

2. All scalars in R^1 are functions of just one coordinate r and so $t_1 = t_0 = 1$, no new independent functions appeared. It is to be expected that no other coordinate dependencies will appear since Schwarzschild is static (t independent) and spherically symmetric (ϑ, φ independent).

3. It is easily verified that I_1 is a proper subgroup of I_0, since it contains only spin transformations

$$\begin{pmatrix} \boldsymbol{o}' \\ \boldsymbol{\iota}' \end{pmatrix} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} \boldsymbol{o} \\ \boldsymbol{\iota} \end{pmatrix} , \ \phi \in \mathbb{R}.$$

4. The other transformations from I_0 change the ratios between $\nabla\Psi_{20}$ and $\nabla\Psi_{31}$, resp. swap them. The requirement $\nabla\Psi_{20} = -\nabla\Psi_{31}$ and $\nabla\Psi_{20} > 0$ fixes the frame up to I_1 transformations and says that the present frame is already canonical.

The procedure has to continue, since this step uncovered new dependency on the Λ coordinate in ΛM, that reflected in the non-invariance of R^1 under I_0.

Second derivative

1. The quantities that shall be computed are $\nabla_{(A(\dot{A}}\nabla_{\dot{B})B}\Psi_{CDEF)}$ and $\Box\Psi_{ABCD}$.

$$\begin{aligned}
\Box\Psi_2 &= \Box\Psi_{ABCD}o^A o^B \iota^C \iota^D = -\frac{6M^2}{r^6} \; , \\
\nabla^2\Psi_{20} &:= \nabla_{(A(\dot{A}}\nabla_{\dot{B})B}\Psi_{CDEF)}o^A o^B o^C o^D \iota^E \iota^F \bar{o}^{\dot{A}}\bar{o}^{\dot{B}} = -\frac{6M}{r^5}F \; , \\
\nabla^2\Psi_{31} &:= \nabla_{(A(\dot{A}}\nabla_{\dot{B})B}\Psi_{CDEF)}o^A o^B o^C \iota^D \iota^E \iota^F \bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}} = \frac{6M}{r^5}F - \frac{3M^2}{2r^6} \; , \\
\nabla^2\Psi_{42} &:= \nabla_{(A(\dot{A}}\nabla_{\dot{B})B}\Psi_{CDEF)}o^A o^B \iota^C \iota^D \iota^E \iota^F \bar{\iota}^{\dot{A}}\bar{\iota}^{\dot{B}} = -\frac{6M}{r^5}F
\end{aligned}$$

and the other components vanish. Thus,

$$\begin{aligned}
R^2 = \Bigg\{ &\Psi_2 = -\frac{M}{r^3}, \nabla\Psi_{20} = \frac{3M}{2^{1/2}r^4}F^{1/2}, \nabla\Psi_{31} = -\frac{3M}{2^{1/2}r^4}F^{1/2}, \\
&\nabla^2\Psi_{20} = -\frac{6M}{r^5}F, \nabla^2\Psi_{31} = \frac{6M}{r^5}F - \frac{3M^2}{2r^6}, \nabla^2\Psi_{42} = -\frac{6M}{r^5}F \Bigg\} .
\end{aligned}$$

2. All scalars in R^2 are functions of r again and so $t_2 = t_1 = 1$.

3. R^2 is invariant under I_1, so $I_2 = I_1$

4. Canonical form is already acquired.

The procedure now terminates, because $t_2 = t_1$ and $I_2 = I_1$. R^2 is the set of Cartan scalars that fully and invariantly describes the Schwarzschild spacetime. Eq. (4.1) gives the correct dimension of the isometry group $r = 4$, because Schwarzschild spacetime is static (one-dimensional Abelian group) and spherically symmetric (three-dimensional group of rotations).

4.1.3 Application of Cartan scalars

Besides the equivalence problem, Cartan scalars found also other uses in general relativity. Since they uniquely characterize the geometry, the reverse application was considered by Karlhede [29] and others that determines the metric tensor from the given set of Cartan scalars. This generating method mainly reconstructed some old solutions, but new ones were found as well, for example see Marklund [30] and references therein.

The physical properties should also be extractable from the Cartan scalars, but only some attempts were made in this direction. Herrera [31] proposed that the active gravitational mass m_a defined by Whittaker [32] is related to the Cartan scalars in the Reissner-Nordström spacetime. His argument is based on the comparison of Cartan scalar Ψ_2 for the Reissner-Nordström and Schwarzschild spacetime. Namely, he observes that Ψ_2 has the same form for both these solutions when written in terms of the active gravitational mass.

- In Schwarzschild spacetime: $m_a(r) = M_S$, $\Psi_2 = -m_a(r)/r^3$, where M_S is the Schwarzchild mass parameter.
- In Reissner-Nordström spacetime: $m_a(r) = M_{RN} - Q^2/r$, $\Psi_2 = -m_a(r)/r^3$, where M_RN is the Reissner-Nordström mass parameter and Q is the charge.

However, it is easy to show that the analogy between Cartan scalars ends with Ψ_2, other Cartan scalars can not be expressed in such a simple way in terms of m_a, nor do the expressions agree for the two spacetimes. Also, for other spherically symmetric spacetimes Ψ_2 does not generally fulfill the condition $\Psi_2 = -m_a(r)/r^3$. Therefore, should the notion of mass be hidden somewhere in the set of Cartan scalars, other candidates than just Ψ_2 need to be considered.

4.2 Scalar curvature invariants

The alternative description of the local geometry can be obtained from the set of scalar curvature invariants (SCIs)

$$\mathcal{I} := \{R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, R_{\mu\nu\rho\sigma;\alpha}R^{\mu\nu\rho\sigma;\alpha}, R_{\mu\nu\rho\sigma;\alpha\beta}R^{\mu\nu\rho\sigma;\alpha\beta}, \ldots\}$$

The advantage of this approach to the equivalence problem is that the calculation of the curvature invariants is straightforward and algorithmic, unlike the Cartan scalars. On the other hand, the problem of this method is that certain non-isometric spacetimes have the same set of curvature invariants $\mathcal{I}$, these spacetimes are called *$\mathcal{I}$-degenerate* and the equivalence problem for them must be solved otherwise. Fortunatelly, Coley, Hervik and Pelavas [24] proved that the class of $\mathcal{I}$-degenerate spacetimes is not too large. Concretely, only a subclass of the Kundt spacetimes given by

$$\begin{aligned} \boldsymbol{g} &= 2\mathrm{d}u\left(H\mathrm{d}u + \mathrm{d}v + W\mathrm{d}\zeta + \bar{W}\mathrm{d}\bar{\zeta}\right) - 2P^{-2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta} \qquad , \\ H &:= H(u,v,\zeta,\bar{\zeta})\ ,\ W := W(u,v,\zeta,\bar{\zeta})\ ,\ W_{,vv} = 0\ ,\ P = P(u,\zeta,\bar{\zeta})\ , \end{aligned}$$

is $\mathcal{I}$-degenerate and Cartan scalars have to be used to distinguish non-isometric spacetimes from this family. For $\mathcal{I}$-non-degenerate spacetimes, the use of scalar curvature invariants is advised over Cartan scalars.

5. Maxwell fields

The electromagnetic interaction is one of the four fundamental forces in nature. Just like gravity, electromagnetism works on large scales, which separates these two forces from the weak and strong interactions. It is thus natural and physically reasonable to study the mutual propagation of the EM field and gravity in the framework of general relativity. In this entire chapter only the vacuum electromagnetic field (*electrovacuum*) will be studied, because this case is more relevant in context of calculations in the following chapters.

5.1 Maxwell's equations

The classical EM field is governed by the set of eight equations that were found in the nineteenth century by J. C. Maxwell, whose important discoveries in the subject led to an alternative appellation of EM field as the Maxwell field. The aforementioned equations in vacuum read

$$\operatorname{rot} \boldsymbol{E} + \partial_t \boldsymbol{B} = 0 \ , \tag{5.1}$$

$$\operatorname{rot} \boldsymbol{B} - \partial_t \boldsymbol{E} = 0 \ , \tag{5.2}$$

$$\operatorname{div} \boldsymbol{E} = 0 \ , \tag{5.3}$$

$$\operatorname{div} \boldsymbol{B} = 0 \ , \tag{5.4}$$

where $\boldsymbol{E}, \boldsymbol{B}$ are vectors in the three-dimensional space representing the electric, resp. magnetic field, rot and div are the well-known vector operators given by $(\operatorname{rot}\boldsymbol{V})^i = \epsilon^{ijk}\partial_j V_k$, $\operatorname{div}\boldsymbol{V} = \partial_i V^i$ in Cartesian coordinates. Note that the units are chosen such that the speed of light is 1. The electromagnetic field is better understood in context of special relativity, where the electric and magnetic fields are both incorporated into a single quantity — the Maxwell tensor $\boldsymbol{F}$,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \ .$$

The Maxwell tensor is clearly antisymmetric and so $\boldsymbol{F}$ can be viewed as a differential two-form. The amazing property of the Maxwell tensor is that Maxwell's equations can be written in a compact form

$$F^{\mu\nu}{}_{;\nu} = 0 \ , \tag{5.5}$$

$$F_{[\mu\nu;\rho]} = 0 \ . \tag{5.6}$$

Even more compact notation is achieved in the exterior calculus, by the use of exterior derivative d and the Hodge dual operator $*$, for a review of exterior calculus, see appendix C.

$$\mathrm{d}{*}\boldsymbol{F} = 0 \ ,$$

$$\mathrm{d}\boldsymbol{F} = 0 \ .$$

The latter equation suggests that $\boldsymbol{F} = \mathrm{d}\boldsymbol{A}$ for some one-form $\boldsymbol{A}$, according to the Poincaré's lemma. $\boldsymbol{A}$ is the called the *four-potential* and the second equation is automatically satisfied for this ansatz, while the first becomes

$$\Box A_\mu - A_{\nu;\mu}{}^{\nu} = 0 \ . \tag{5.7}$$

There is an obvious arbitrariness in the choice of the four-potential, namely $\boldsymbol{A}' = \boldsymbol{A} + \mathrm{d}f$ for some function f, because $\boldsymbol{A}'$ yields the same Maxwell tensor $\boldsymbol{F}$. Such arbitrariness is called the *gauge* freedom and this notion plays an important role in formulation of more general Yang-Mills theories. For convenience, the gauge is sometimes fixed by imposing an additional condition on $\boldsymbol{A}$. Usually, the covariant Lorenz gauge is used in which $A^\mu{}_{;\mu} = 0$ and eq. (5.7) simplifies to

$$\Box_{dR} A_\mu = 0 \quad , \quad \Box_{dR} A_\mu := \Box A_\mu - R^\nu_\mu A_\nu$$

where $\Box_{dR}$ is the Laplace-de Rham operator. Note also the Maxwell's equations (3.8) in spinor formalism and (3.13)-(3.16) in NP formalism, since these were mostly used for calculations in this thesis.

5.2 Maxwell's action

Maxwell's equations can also be derived through the principle of least action. Maxwell's action reads

$$\begin{aligned} S[\boldsymbol{g}, \boldsymbol{A}] &= \int \mathcal{L}(\boldsymbol{g}, \boldsymbol{A}) \mathrm{d}^4 x \ , \\ \mathcal{L}(\boldsymbol{g}, \boldsymbol{A}) &:= \frac{1}{8\pi} F_{\mu\nu} F^{\mu\nu} (-g)^{1/2} \ . \end{aligned}$$

where $g := \det g_{\mu\nu}$. This action must be varied with respect to the one-form $\boldsymbol{A}$ in order to obtain the correct equations, namely (5.7). In context of this thesis, it is important to note that the action is invariant under conformal transformations and therefore Maxwell's equations must also have this property. The generating method proposed in [9] and further developed in this thesis is based on this observation.

The stress-energy tensor $T_{\mu\nu}$ of the Maxwell field can also be derived from the action, but the variation must be taken with respect to metric $\boldsymbol{g}$ this time. The defining equation is

$$(-g)^{1/2} T^{\mu\nu} := \frac{\delta \mathcal{L}(\boldsymbol{g}, \boldsymbol{A})}{\delta g_{\mu\nu}} \equiv \frac{\partial \mathcal{L}(\boldsymbol{g}, \boldsymbol{A})}{\partial g_{\mu\nu}} \tag{5.8}$$

and when Maxwell's action is inserted, one obtains

$$T_{\mu\nu} = \frac{1}{4\pi} \left(\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F_{\mu\alpha} F_\nu{}^\alpha \right) \ .$$

The evident fact that the stress-energy tensor of a vacuum Maxwell field is traceless is closely related to the conformal invariance of the action. To be precise, the following theorem holds (see [35]):

Theorem 5.1 (Traceless $T^{\mu\nu}$ and conformally invariant action)**.** *The action of a field* $\boldsymbol{\Phi}$ *is conformally invariant if and only if the stress-energy tensor* $T^{\mu\nu}$ *defined by* (5.8) *is traceless, i.e.*

$$S[e^{2U}\boldsymbol{g},\boldsymbol{\Phi}] = S[\boldsymbol{g},\boldsymbol{\Phi}] \iff T^{\mu}_{\mu} = 0.$$

As in many other cases, the formula for the stress-energy tensor of a Maxwell field is much simpler in the spinor formalism. It reads

$$T_{ab} = \frac{1}{2\pi}\phi_{AB}\overline{\phi}_{\dot{A}\dot{B}} \ .$$

5.3 Null Maxwell fields

The null Maxwell fields[1] can be defined via two invariants of the Maxwell tensor

$$\boldsymbol{F}\cdot\boldsymbol{F} = F_{\mu\nu}F^{\mu\nu} = 2(|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2) \ , \tag{5.9}$$

$$*\boldsymbol{F}\cdot\boldsymbol{F} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = 4\,\boldsymbol{E}\cdot\boldsymbol{B} \ . \tag{5.10}$$

Definition (Null Maxwell field I)**.** *A Maxwell field is called null iff* $\boldsymbol{F}\cdot\boldsymbol{F} = *\boldsymbol{F}\cdot\boldsymbol{F} = 0$.

Note that these conditions are conformally invariant and so the null Maxwell fields preserve their nullity under conformal transformations. The aforementioned definition for null Maxwell fields is fully equivalent to the criterion from the chapter dedicated to spinors.

Definition (Null Maxwell field II)**.** *A Maxwell field is called null iff* $\phi_{AB} = \phi\,\alpha_{(A}\alpha_{B)}$ *for some* $\alpha_A \in S^*$.

An *aligned basis* is typically used in which $o^A = \alpha^A$ and the basis is completed by an arbitrary independent spinor. The Maxwell spinor is then

$$\phi_{AB} = \phi\, o_{(A}o_{B)}$$

and its NP components are clearly $\phi_0 = \phi_1 = 0$, $\phi_2 = \phi$. The basis remains aligned under null rotations about $\boldsymbol{l}$ and spin-boosts. The boost freedom is usually fixed by the requirement that $\boldsymbol{k}$ is an affinely parametrized geodesic. NP Maxwell's equations (3.13)–(3.16) in the aligned basis become

$$0 = \kappa\phi_2 \ , \tag{5.11}$$

$$D\phi_2 = (\rho - 2\epsilon)\phi_2 \ , \tag{5.12}$$

$$0 = \sigma\phi_2 \ , \tag{5.13}$$

$$\delta\phi_2 = (\tau - 2\beta)\phi_2 \tag{5.14}$$

and the existence of a null Maxwell field thus puts constraints on the spacetime, namely $\kappa = \sigma = 0$. This restriction is known as the Mariot-Robinson theorem.

Theorem 5.2 (Mariot-Robinson)**.** *A null Maxwell field* $\phi_{AB} = \phi\, o_A o_B$ *generates a geodesic* ($\kappa = 0$) *and shear-free* ($\sigma = 0$) *null congruence* $k^a = o^A\bar{o}^{\dot{A}}$.

[1] which are sometimes also called *purely radiatative*, *degenerate* or *algebraically special*

This assertion also holds in the other direction and such variant is called the Robinson's theorem.

Theorem 5.3 (Robinson). *If a spacetime contains a geodesic shear-free null congruence, then there exists a null Maxwell field aligned with this congruence*[2].

Spacetimes with geodesic shear-free null congruences also appear in the Goldberg-Sachs theorem.

Theorem 5.4 (Goldberg-Sachs). *If a spacetime contains a null congruence $k^a = o^A \bar{o}^{\dot{A}}$ that is geodesic ($\kappa = 0$) and shear-free ($\sigma = 0$) and if $\Phi_{00} = \Phi_{01} = \Phi_{11} = 0$ then the gravitational field is algebraically special and k^a is the repeated principal null direction, i.e. $\Psi_0 = \Psi_1 = 0$.*

The aforementioned theorems have an important consequence for null Einstein-Maxwell fields. A null Einstein-Maxwell field is a solution of Einstein's equations

$$\Phi_{AB\dot{A}\dot{B}} = 2G\phi_{AB}\overline{\phi}_{\dot{A}\dot{B}} \ , \tag{5.15}$$

where ϕ_{AB} is a null Maxwell field that satisfies Maxwell's equations. In the aligned basis, Einstein's equations (5.15) imply

$$\Phi_{ij} = 0 \quad \forall \, i, j \neq 2$$

Therefore, in combination with the Mariot-Robinson theorem, all assumptions of Goldberg-Sachs are met and the following assertion clearly holds:

Theorem 5.5 (Alignment theorem). *The Weyl tensor of a null Einstein-Maxwell field is algebraically special and the repeated PND of the Weyl tensor coincides with the repeated PND of the null Maxwell field. In the aligned basis, this statement translates to $\kappa = \sigma = \Psi_0 = \Psi_1 = 0$.*

The physical meaning of null Maxwell fields is best illustrated on the Minkowski spacetime with the metric given in null coordinates

$$\boldsymbol{g} = 2\mathrm{d}u\mathrm{d}v - 2\mathrm{d}\zeta\mathrm{d}\bar{\zeta} \qquad u, v \in \mathbb{R} \ , \ \zeta \in \mathbb{C} \ .$$

According to Robinson's theorem, the null Maxwell field generates a null geodesic congruence $\boldsymbol{k}$. Every geodesic $\boldsymbol{k}$ has constant components with respect to null coordinates in a flat space and a Lorentz transformations can be used to set $\boldsymbol{k} = \partial_v$ without loss of generality. The null frame can then be completed by

$$(k_\mu, l_\mu, m_\mu, \bar{m}_\mu) = (\mathrm{d}u, \mathrm{d}v, \mathrm{d}\zeta, \mathrm{d}\bar{\zeta}) \ .$$

In such frame, all NP spin coefficients vanish and Maxwell's equations reduce to $D\phi_2 = \delta\phi_2 = 0$, which implies $\phi_2 = \phi(u, \bar{\zeta})$. An observer who measures the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ with respect to the orthonormal frame given by (3.12) will find out that the electromagnetic field propagates in the direction e_3, while the mutually orthogonal vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ oscillate in the plane perpendicular to e_3. Such Maxwell field is called a *plane electromagnetic wave* and physically describes electromagnetic radiation far from the source.

[2]The null Maxwell field is generally just a test field on the spacetime, i.e. it does not enter the right-hand side of Einstein's equations.

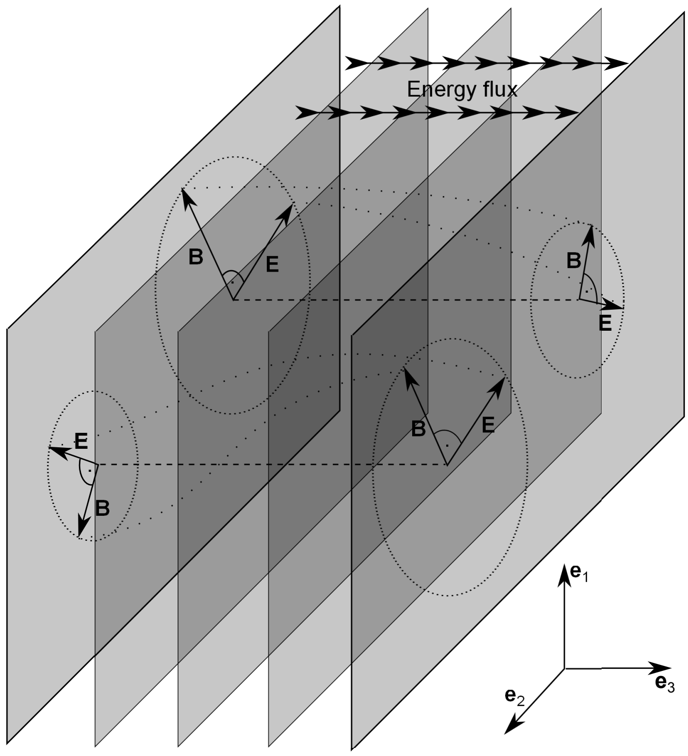

Figure 5.1: Plane electromagnetic wave

5.4 Non-null Maxwell field

Definition (Non-null Maxwell field)**.** *A Maxwell field is non-null if at least one of the invariants $\boldsymbol{F} \cdot \boldsymbol{F}$ and $*\boldsymbol{F} \cdot \boldsymbol{F}$ is non-zero.*

The Maxwell spinor of a non-null field decomposes into a symmetrized tensor product of two independent spinors, i.e.

$$\phi_{AB} = \chi\, \alpha_{(A}\beta_{B)}$$

An aligned basis for the non-null Maxwell field is constructed by taking

$$o^A = \alpha^A \ , \ \iota^B = \lambda^{-1}\beta^B \quad \text{where} \quad \lambda = \alpha_A \beta^A \ ,$$

so that

$$\phi_{AB} = \lambda\chi\, o_{(A}\iota_{B)}$$

In this basis, $\phi_0 = \phi_2 = 0$, $\phi_1 = -\lambda\chi/2$. The aligned basis is fixed up to spin-boosts, which leave ϕ_1 invariant. NP Maxwell's equations then read

$$\begin{aligned} D\phi_1 &= 2\rho\phi_1 \ , \\ \delta\phi_1 &= 2\tau\phi_1 \ , \\ \bar{\delta}\phi_1 &= -2\pi\phi_1 \ , \\ \Delta\phi_1 &= -2\mu\phi_1 \ . \end{aligned}$$

In this case, Maxwell's equations do not restrict the geometry — each spacetime allows for a test non-null Maxwell field. Physically, a non-null solution of Maxwell's equations describes the electromagnetic field of a bounded source. If

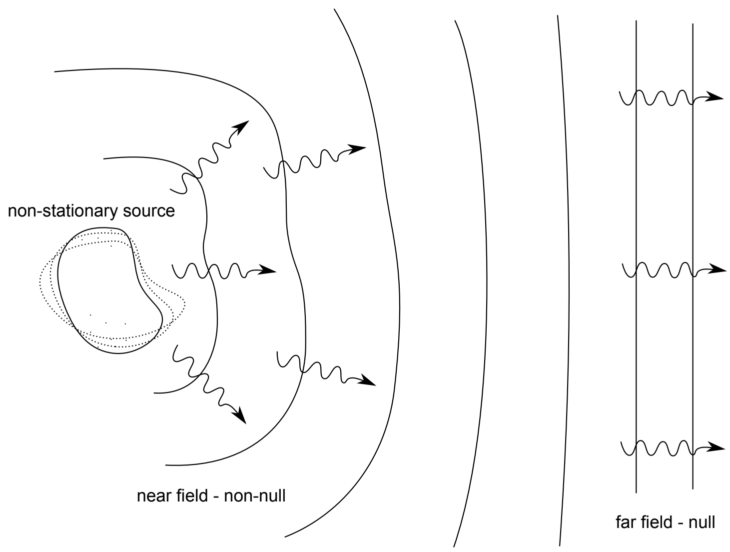

Figure 5.2: Peeling of the Maxwell field: *Near the non-stationary source, the electromagnetic field is algebraically general, but with the increasing distance from the source, the null component starts to dominate.*

the invariant $*\boldsymbol{F} \cdot \boldsymbol{F}$ vanishes, the electromagnetic field is non-radiative and is called either *purely electric* for $\boldsymbol{F} \cdot \boldsymbol{F} < 0$ or *purely magnetic* for $\boldsymbol{F} \cdot \boldsymbol{F} > 0$, because an orthonormal frame can always be found in which only the electric resp. magnetic component of the field survives. The interpretations of null and non-null Maxwell fields can be made more precise by the *peeling theorem.*

Theorem 5.6 (Peeling of the Maxwell field)**.** *Suppose there is*

- *a bounded source of a Maxwell field ϕ_{AB} in an asymptotically simple space-time,*
- *an affinely parametrized null geodesic $\gamma(r)$.*

Then the components of ϕ_{AB} behave like

$$\phi_2 \sim r^{-1} \ , \ \phi_1 \sim r^{-2} \ , \ \phi_0 \sim r^{-3}$$

along the geodesic as $r \to \infty$.

Far from the source, i.e. for large r, the component ϕ_2 dominates over ϕ_1 and ϕ_0. Thus each radiative field looks like null from distance and as one approaches the source, the non-null character of the field becomes apparent.

Part II

Results

6. Generating method

This chapter contains the author's results regarding the generating method that exploits the conformal invariance of the Maxwell field in order to simplify the solution of Einstein-Maxwell equations in conformally related spacetimes. The new spacetime automatically fulfills Maxwell's equations provided that the seed Maxwell field did. The included paper does not employ spinor formalism and so different conventions are used, namely those in Stephani [1]. The main deviations are in the metric signature, which is $(-,+,+,+)$ in the article and the Riemann and Ricci tensors are defined with an opposite sign with respect to the rest of the thesis.

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Conformally related non-vacuum spacetimes

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Abstract

We present an example of two non-trivially conformally related solutions of Einstein–Maxwell equations. To our knowledge, this is the first case of non-vacuum spacetimes related through a non-trivial conformal transformation representing thus an extension of the Brinkmann's and following theorems.

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1. General background

Finding exact solutions to Einstein equations is very difficult. Therefore, much attention has been paid to the methods of generating new solutions from existing ones. One of them is to use a conformal transformation.

Conformal transformations play an important role in higher-derivative theories of gravity where they serve to compare these theories to the classical GR [1–3]. They also arise naturally in quantum fields on a curved background and are discussed in connection with the AdS/CFT correspondence [4–6].

Our approach in this paper is very simple—we start from a seed metric, which does not even have to represent a physically viable solution to Einstein equations. We define a non-negative scalar conformal factor varying across the manifold and produce the resulting metric by multiplying the original metric by the conformal factor. We now require the new metric to satisfy Einstein equations. In our case, we actually used a solution of the Einstein–Maxwell equations (with a non-zero electromagnetic field) as our seed. This has the advantage that the resulting metric also represents an electrovacuum solution as Maxwell equations are conformally invariant. Therefore, we do not need to deal with the electromagnetic part of Einstein–Maxwell equations. However, it may still happen that we actually do not obtain a new solution but the seed spacetime in a new coordinate system. Therefore, we must carefully check our solutions to make sure they cannot be coordinate-transformed back into the original spacetime.

2. Existing theorems

There are a number of theorems on conformally related spacetimes. Brinkmann [7] dealt with conformally related Einstein spaces $R_{\mu\nu} = ag_{\mu\nu}$ with a a constant. He showed that any two distinct (properly) conformally related Einstein spaces are either two vacuum pp-waves, or Minkowski and (anti) de Sitter. His theorems were later generalized by Daftardar-Gejji [8] who extended this theorem to include the cases where the two Einstein tensors are equal and where they differ by a cosmological constant term. In the former case, both spaces are (not necessarily vacuum) pp-waves; in the latter, for perfect fluids with $\mu = 0$, both spaces are Robertson–Walker with equations of state $\mu+3p = 0$ or $\mu = p$. Van den Bergh [9] established that the only null Einstein–Maxwell fields obtainable by a conformal transformation of a Ricci-flat solution are pp-waves. In all cases, the seed metric is Ricci-flat. We are interested in a case where the two spacetimes are solutions of full Einstein–Maxwell equations. Physically, this means that the causal structure of both spacetimes is identical in the corresponding regions. Yet, in general, it turned out that generating new solutions via a conformal transformation does not produce any interesting, non-trivial results. In our paper, we present the first non-trivial explicit example of two, non-vacuum solutions to Einstein equations that are conformally related. Moreover, we show that the two spacetimes are not isometric.

3. The generating method

We exploit the fact that, in a four-dimensional spacetime, the action of a source-free Maxwell field is conformally invariant and we thus generate solutions of Einstein–Maxwell equations. The invariance tells us that if $F_{\mu\nu}$ is a solution of source-free Maxwell equations on a seed spacetime $(M, g_{\mu\nu})$, then it is also a solution on $(M, \Omega^2 g_{\mu\nu})$. It implies that if we begin with a solution of Einstein–Maxwell equation, then after applying a conformal transformation, it is only Einstein equations we have to worry about, Maxwell equations are satisfied automatically.

3.1. Apparatus

Let us consider a conformal transformation of the seed metric $g_{\mu\nu}$ [1]

$$\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{1}$$

where Ω is a function on M. Then the following equation holds for conformally rescaled Ricci tensor $\widetilde{R}_{\mu\nu}$ computed from the new metric $\widetilde{g}_{\mu\nu}$:

$$\widetilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{2}{\Omega}\Omega_{;\mu\nu} + \frac{4}{\Omega^2}\Omega_{,\mu}\Omega_{,\nu} - \frac{1}{\Omega}(\Box\Omega)g_{\mu\nu} - \frac{1}{\Omega^2}\|\mathrm{d}\Omega\|^2 g_{\mu\nu}, \tag{2}$$

where all covariant derivatives are taken with respect to the seed metric, $\Box\Omega := g^{\mu\nu}\Omega_{;\mu\nu}$ and $\|\mathrm{d}\Omega\|^2 := g^{\mu\nu}\Omega_{,\mu}\Omega_{,\nu}$.

The stress–energy tensor of a source-free Maxwell field has the form

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{3}$$

and so it transforms as

$$\widetilde{T}_{\mu\nu} = \Omega^{-2}T_{\mu\nu}\ . \tag{4}$$

Suppose we have a triplet $(M, g_{\mu\nu}, F_{\mu\nu})$ which is a solution of Einstein–Maxwell equations. We can now write Einstein equations[2] for $(M, \widetilde{g}_{\mu\nu}, F_{\mu\nu})$:

$$\widetilde{R}_{\mu\nu} = 8\pi\widetilde{T}_{\mu\nu} = 8\pi\Omega^{-2}T_{\mu\nu} = \Omega^{-2}R_{\mu\nu}, \tag{5}$$

[1] Our signature is (−,+,+,+).

[2] Note that the stress–energy tensor of a Maxwell field is traceless, so the Ricci scalar vanishes.

so the equations we will solve are $\Omega^2\widetilde{R}_{\mu\nu} - R_{\mu\nu} = 0$ or

$$(\Omega^2 - 1)R_{\mu\nu} - 2\Omega\Omega_{;\mu\nu} + 4\Omega_{,\mu}\Omega_{,\nu} - \Omega(\Box\Omega)g_{\mu\nu} - g_{\mu\nu}\|\mathrm{d}\Omega\|^2 = 0 \tag{6}$$

in detail. The trace of (6) yields a necessary condition $\Box\Omega = 0$, so Ω has to be a harmonic function and (6) simplifies to the final form

$$(\Omega^2 - 1)R_{\mu\nu} - 2\Omega\Omega_{;\mu\nu} + 4\Omega_{,\mu}\Omega_{,\nu} - g_{\mu\nu}\|\mathrm{d}\Omega\|^2 = 0. \tag{7}$$

This system is generally overdetermined, we have ten equations for a single function Ω, however, in a special case, we might (and will) be able to solve it[3].

3.2. Application

As could be expected, equation (7) does not allow for a non-trivial solution in most cases. We explicitly checked the Reissner–Nordström, Bonnor–Melvin, Bertotti–Robinson, Tariq–Tupper, and Ozsváth solutions and showed that none of these spacetimes are suitable seeds. However, let us now turn our attention to *pp*-waves, which are spacetimes admitting a covariantly constant null vector field k^μ [10, 11] (p 383 and 323, respectively). The metric can be written in the form

$$\mathrm{d}s^2 = -2H(u,\xi,\bar{\xi})\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}v + 2\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}. \tag{8}$$

where $\xi = \frac{1}{\sqrt{2}}(x + \mathrm{i}y)$ is a complex coordinate. In these coordinates, the covariantly constant null vector field is simply ∂/∂_v. Not all *pp*-waves carry a non-zero Maxwell field. In order for them to do so, the function H has to satisfy

$$H = f(\xi,u) + \bar{f}(\bar{\xi},u) + 8\pi F(\xi,u)\bar{F}(\bar{\xi},u) \tag{9}$$

and the Maxwell field is then $F_{\mu\nu} = k_{[\mu}F_{,\nu]} + c.c.$ The Ricci tensor of a *pp*-wave is given by $R_{\mu\nu} = 2H_{,\xi\bar{\xi}}k_\mu k_\nu$. Having all the necessary ingredients, we can try to solve (7) for Ω. It turns out that this generally overdetermined system has a non-trivial solution if and only if H can be written as

$$H = \xi\bar{\xi}\Phi^2(u) + (\xi + \bar{\xi})h(u) + g(u), \tag{10}$$

where Φ^2, h, and g are arbitrary real functions of u only. Thus, except for the flat case, the spacetimes we are interested in having a non-zero Ricci tensor. There is a transformation of coordinates [10] (p 383) that preserves the metric in the form (8) and allows us to get rid of the last two terms. A *pp*-wave with such a form of H is called a plane wave and is conformally flat. In fact, McLenaghan *et al* [12] proved that such a *pp*-wave is the only conformally flat null Einstein–Maxwell field. Then, Ω has to be a function of u only, too, and has to satisfy a second-order ODE

$$\Omega\ddot{\Omega} - 2\dot{\Omega}^2 + \Phi^2(1 - \Omega^2) = 0, \tag{11}$$

where $\dot{\Omega} := \mathrm{d}\Omega/\mathrm{d}u$. With Ω depending only on u, it can be shown, that the vector field ∂/∂_v is covariantly constant in the new spacetime $(M, \widetilde{g}_{\mu\nu})$ as well, so the generated spacetime is again a *pp*-wave, undoubtedly a conformally flat one, in agreement with McLenaghan's results. A question naturally arises, whether the generated wave is generally different from (non-isometric to) the original one or we just generate different coordinate expressions of the same wave. To answer this question, let us examine the equivalence problem for an explicit pair of the seed and the generated wave. A good choice, that will allow us to solve (11) particularly easily is to start with the following *pp*-wave

$$\mathrm{d}s^2 = -4\xi\bar{\xi}\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}v + 2\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}, \tag{12}$$

[3] Actually, there is always the solution $\Omega^2 \equiv 1$, but then the transformation is trivial.

so $H = 2\xi\bar{\xi}$ and $\Phi^2 = 2$. A particular solution of (11) is then $\Omega = \tanh(u)$ and so the generated wave has the following line element:

$$\widetilde{\mathrm{d}s}^2 = \tanh^2(u)(-4\xi\bar{\xi}\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}v + 2\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}). \tag{13}$$

In our case, the first covariant derivative of the Ricci tensor is sufficient to prove the non-equivalence. The original spacetime has $R_{\mu\nu} = 4k_\mu k_\nu$ and its covariant derivative vanishes, i.e. $R_{\mu\nu;\rho} = 0$. However, the new spacetime has $\tilde{R}_{\mu\nu} = 4\Omega^{-6}\tilde{k}_\mu\tilde{k}_\nu$ and the covariant derivative no longer vanishes because of the non-zero gradient of the conformal factor Ω, which proves the non-equivalence of $\mathrm{d}s^2$ and $\widetilde{\mathrm{d}s}^2$.

Are we bringing any news or can one draw our conclusions from the results given in [12]? We can obviously conformally transform every conformally flat seed *pp*-wave directly to any other solution of the McLenaghan type and the question is: Can we always keep the original Maxwell tensor or do we generally need a different Maxwell field to produce a solution of full Einstein–Maxwell equations? In the following paragraph, we will argue that the latter is the case.

Let us take a seed and multiply it by a general conformal factor depending on u only (see above) to get

$$\widetilde{\mathrm{d}s}^2 = \Omega^2(u)(-2\Phi^2(u)\xi\bar{\xi}\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}v + 2\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}). \tag{14}$$

Now apply the following coordinate transformation:

$$\xi = \frac{\xi'}{\Omega(u)}, \tag{15}$$

$$v = v' - \frac{\xi'\bar{\xi}'\dot{\Omega}(u)}{\Omega^3(u)}, \tag{16}$$

$$\Omega^2(u)\mathrm{d}u = \mathrm{d}u'. \tag{17}$$

This yields

$$\widetilde{\mathrm{d}s}^2 = -2\Phi'^2(u')\xi'\bar{\xi}'\,\mathrm{d}u'^2 - 2\,\mathrm{d}u'\,\mathrm{d}v' + 2\,\mathrm{d}\xi'\,\mathrm{d}\bar{\xi}', \tag{18}$$

with

$$\Phi'^2(u') = \frac{\Phi^2(u)}{\Omega^4(u)} - \frac{\ddot{\Omega}(u)}{\Omega^5(u)} + \frac{2\dot{\Omega}^2(u)}{\Omega^6(u)}, \tag{19}$$

where u has to be viewed as a function of u', so we arrive at the canonical form of the metric again.

Now we apply the above transformation to the seed Maxwell tensor and see whether it necessarily corresponds to the canonical form of the Maxwell tensor as given in Stephani *et al* [10]. The condition that the two fields be the same takes precisely the form (11) and is obviously not satisfied identically. We thus established that, in general, we cannot proceed from one *pp*-wave of the form (8) with (10) to another keeping the same Maxwell field and, therefore, results in [12] do not imply ours. Additionally, we gave an explicit example of two conformally related Einstein–Maxwell fields. All the *pp*-waves of the above special form thus split into equivalence classes defined by their Maxwell fields.

Moreover, (19) implicitly gives us the conformal factor that transforms the wave back to Minkowski if we require $\Phi'^2 = 0$. It also implies that if we take Minkowski in null coordinates

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + 2\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}, \tag{20}$$

then *all* conformal transformations with an *arbitrary* $\Omega = \Omega(u)$ result in *pp*-waves of the McLenaghan type.

4. Conclusions and outlook

To our knowledge, this is the only non-trivial example of applying a conformal transformation to a seed spacetime to produce a non-vacuum solution to Einstein equations. Furthermore, due to the properties of conformal transformations, it is clear that the source remains in the form of an (vacuum) electromagnetic field. The original and resulting spacetimes are explicitly shown not to be identical. Thus, our solutions form the only known pair of conformally related non-vacuum spacetimes. The prospects of using conformal transformations to produce new solutions to Einstein equations may be dim but at least we provided a specific example where this approach works although the resulting solution belongs to the same class as the seed spacetime.

It might be of interest to see whether there may be any other non-trivial examples that would not involve *pp*-waves. Note that the previous literature always relies exactly on these types of spacetimes. Are there any suitable seeds other than *pp*-waves? Can the method be generalized to be less restrictive?

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References

[1] Dabrowski M P, Garecki J and Blaschke D B 2009 Conformal transformations and conformal invariance in gravitation *Ann. Phys.* **18** 13–32 (arXiv:0806.2683v3)
[2] Faraoni V, Gunzig E and Nardone P 1999 Conformal transformations in classical gravitational theories and in cosmology *Fundam. Cosm. Phys.* **20** 121 (arXiv:gr-qc/9811047v1)
[3] Fujii Y and Maeda K I 2003 *The Scalar-Tensor Theory of Gravitation* (Cambridge: Cambridge University Press)
[4] Shapiro I L 2006 Local conformal symmetry and its fate at quantum level *5th Int. Conf. on Mathematical Methods in Physics* (Rio de Janeiro, Brazil) (Published in PoS IC2006:030,2006) (arXiv:hep-th/0610168v1)
[5] Shapiro I L and Takata H 1995 Conformal transformation in gravity *Phys. Lett.* B **361** 31–7 (arXiv:hep-th/9504162v1)
[6] Aharony O, Gubser S S, Maldacena J, Ooguri H and Oz Y 2000 Large N field theories, string theory and gravity *Phys. Rep.* **323** 183–386 (arXiv:hep-th/9905111v3)
[7] Brinkmann H W 1925 Einstein spaces which are mapped conformally on each other *Math. Ann.* **18** 119–45
[8] Daftardar-Gejji V 1998 A generalization of Brinkmann's theorem *Gen. Rel. Grav.* **30** 695–700
[9] Van den Bergh N 1986 Conformally Ricci flat Einstein–Maxwell solutions with a null electromagnetic field *Gen. Rel. Grav.* **18** 1105–10
[10] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* 2nd edn (Cambridge: Cambridge University Press)
[11] Griffiths J B and Podolský J 2009 *Exact Space-Times in Einstein's General Relativity* (Cambridge: Cambridge University Press)
[12] McLenaghan R G, Tariq N and Tupper B O J 1975 Conformally flat solutions of the Einstein–maxwell equations for null electromagnetic fields *J. Math. Phys.* **16** 829–31

Since the article was published, some of the prospects sketched in its conclusion have been studied. The main goal was to obtain the full set of seed solutions for which the generating method works. Of course, finding all suitable seeds can not be handled by a 'trial-and-error' method so a more systematic approach was used, namely the integrability conditions for the main equation. The results were obtained for null Einstein-Maxwell fields.

6.1 Integrability conditions

The fundamental equation for the generating method from the article is

$$f_{;\mu\nu} = 4\pi G f(f^2-1)T_{\mu\nu} + \frac{1}{2f} g_{\mu\nu}\|\mathrm{d}f\|^2 \,, \tag{6.1}$$

where $\boldsymbol{g}$ is the seed metric ($\tilde{\boldsymbol{g}} = f^{-2}\boldsymbol{g}$ is the new one), G is the gravitational constant, and the stress-energy tensor $T_{\mu\nu}$ is traceless. This is an overdetermined system of ten equations for a single function f so generally no solution will exist. The special spacetimes which allow a solution of (6.1) must fulfill certain integrability conditions.

Theorem 6.1 (Covariant integrability conditions)**.** *Suppose the following equation for the unknown function $f(x)$*

$$f_{;\mu\nu}(x) = A_{\mu\nu}(x, f, f_{,\mu}) \,. \qquad *$$

This overdetermined system has a solution iff

1. $A_{\mu\nu} = A_{\nu\mu}$
2. $A_{\mu[\nu;\rho]} = -\frac{1}{2}R^{\alpha}{}_{\mu\nu\rho}f_{\alpha} \mod * \,,$

where mod $*$ *means that whenever $f_{;\mu\nu}$ appears during the calculation of $A_{\mu[\nu;\rho]}$, it should be substituted by $A_{\mu\nu}$.*

This is a customized version of the Schouten theorem concerning covariant differential equations, see [33]. The main equation (6.1) undoubtedly fulfils the first symmetry condition and the second integrability condition reads

$$(3f^2-1)T_{\mu[\nu}f_{\rho]} + f(f^2-1)T_{\mu[\nu;\rho]} + (f^2-1)g_{\mu[\nu}T_{\rho]\alpha}f^{\alpha} = -\frac{1}{8\pi G}R_{\alpha\mu\nu\rho}f^{\alpha} \,.$$

In the actual calculations, the spinor version of these integrability conditions will be used which reads

$$6Gf^2\overline{\phi}_{\dot{D}\dot{X}}\phi_{(AB}\nabla^{\dot{X}}_{C)}f + 2Gf(f^2-1)\overline{\phi}_{\dot{D}\dot{X}}\nabla^{\dot{X}}_{(A}\phi_{BC)} + \Psi_{ABCX}\nabla^{X}_{\dot{D}}f = 0 \,, \tag{6.2}$$

where Maxwell's equations $\nabla^{A\dot{X}}\phi_{AB} = 0$ were already used. Interestingly, the same equation could be obtained from the Bianchi identities in the new spacetime

$$\widetilde{\nabla}^{X}_{\dot{D}}\widetilde{\Psi}_{ABCX} = \widetilde{\nabla}^{\dot{X}}_{(A}\widetilde{\Phi}_{BC)\dot{D}\dot{X}} \,,$$

where one has to insert the appropriate transformation properties of the spinors and the covariant derivative

$$\begin{aligned}
\widetilde{\epsilon}^{AB} &= f\epsilon^{AB} \,, \\
\widetilde{\Psi}_{ABCD} &= \Psi_{ABCD} \,, \\
\widetilde{\Phi}_{AB\dot{A}\dot{B}} &= f^2\Phi_{AB\dot{A}\dot{B}} \,, \\
\widetilde{\nabla}_{A\dot{B}}\xi_C &= \nabla_{A\dot{B}}\xi_C + \xi_A\nabla_{C\dot{B}}\ln f \,.
\end{aligned}$$

6.2 Null Einstein-Maxwell fields as seeds

Taking null Einstein-Maxwell field as a seed means that the main equation for the conformal factor reads

$$f_{;\mu\nu} = 2Gf(f^2-1)|\phi|^2 k_\mu k_\nu + \frac{1}{2f} g_{\mu\nu} \|\mathrm{d}f\|^2 \,, \tag{6.3}$$

where ϕ is a solution of Maxwell's equations (5.12), (5.14) and $\boldsymbol{k}$ is the alignment of the Maxwell field. Being inspired by the proof of Brinkmann's theorem in [6], the main equation for the conformal factor shall also be treated in two separate cases:

- null conformal transformation: $\|\mathrm{d}f\|^2 = 0$, but $\mathrm{d}f \neq 0$ (homotheties are discarded).
- non-null conformal transformation: $\|\mathrm{d}f\|^2 \neq 0$.

Null conformal transformation. In this case, the equation (6.3) simplifies to

$$f_{;\mu\nu} = 2Gf(f^2-1)|\phi|^2 k_\mu k_\nu \tag{6.4}$$

and the integrability conditions do not have to be used. It is sufficient to contract the above equation with $f^{,\mu}$ to obtain

$$0 = 2Gf(f^2-1)|\phi|^2 k_\mu f^{,\mu} k_\nu \,, \tag{6.5}$$

where the left-hand side vanished due to

$$2f_{;\mu\nu} f^{,\mu} = \|\mathrm{d}f\|^2_{;\nu} = 0 \,.$$

Equation (6.5) then implies $f_{,\mu} = \lambda k_\mu$. This relation may be inserted back into (6.4) and one obtains

$$k_{\mu;\nu} = 2Gf(f^2-1)|\phi|^2 k_\mu k_\nu - k_\mu \lambda_{,\nu} \,.$$

As a result of this equation, many NP connection coefficients must vanish, namely

$$\rho = k_{\mu;\nu} m^\mu \overline{m}^\nu = 0 \ , \ \ \kappa = k_{\mu;\nu} m^\mu k^\nu = 0 \ , \ \ \sigma = k_{\mu;\nu} m^\mu m^\nu = 0$$

A spacetime with $\rho = \kappa = \sigma = 0$ belongs to the well-known Kundt class

$$\boldsymbol{g} = 2\mathrm{d}u(H\mathrm{d}u + \mathrm{d}v + W\mathrm{d}\zeta + \overline{W}\mathrm{d}\overline{\zeta}) - 2P^{-2}\mathrm{d}\zeta\mathrm{d}\overline{\zeta} \ , \quad P_{,v} = 0 \ ,$$

with real functions H, P and complex W. For such metric, the repeated principal null direction of the Weyl tensor is $\boldsymbol{k} = \partial_v$, which is, according to the alignment theorem 5.5, also the principal null direction of the Maxwell field. The covariant null frame is easily completed by

$$\boldsymbol{k} = \mathrm{d}u \ , \ \boldsymbol{l} = H\mathrm{d}u + \mathrm{d}v + W\mathrm{d}\zeta + \overline{W}\mathrm{d}\overline{\zeta} \ , \ \boldsymbol{m} = P^{-1}\mathrm{d}\overline{\zeta} \ , \ \overline{\boldsymbol{m}} = P^{-1}\mathrm{d}\zeta \,.$$

Since $\mathrm{d}f = \lambda\boldsymbol{k} = \lambda\mathrm{d}u$, the conformal factor f depends exclusively on u. As a result, the non-trivial components of eq. (6.4) are

$$f_{,uu} + f_{,u}H_{,v} = 2Gf(f^2-1)|\phi|^2 \ , \tag{6.6}$$

$$f_{,u}W_{,v} = 0 \ . \tag{6.7}$$

According to [1] (pages 476,477), the metric functions of a Kundt Einstein-Maxwell solution with $W_{,v} = 0$ satisfy

$$P = 1 \ , \ W = W(u,\overline{\zeta}) \ , \ H = \frac{1}{2}\left(W_{,\overline{\zeta}} + \overline{W}_{,\zeta}\right) v + H^0(u,\zeta,\overline{\zeta}) \ , \tag{6.8}$$

$$H^0_{,\zeta\overline{\zeta}} - \mathrm{Re}\left[\left(W_{,\overline{\zeta}}\right)^2 + WW_{,\overline{\zeta}\,\overline{\zeta}} + W_{,u\overline{\zeta}}\right] = 2G|\phi|^2 \ , \ \phi = \phi(u,\overline{\zeta}) \ . \tag{6.9}$$

With this ansatz, the second derivative of (6.6) with respect to ζ and $\overline{\zeta}$ yields $\phi_{,\overline{\zeta}} = 0$. Thus, the right-hand side of (6.6) depends only on u and the same must hold for $W_{,\overline{\zeta}}$. This implies that W is at most linear in $\overline{\zeta}$, but then the coordinate transformation

$$v' = v + \zeta\overline{\zeta}W_{,\overline{\zeta}}(u)$$

leads to $W' = 0$ and such spacetime is a *pp*-wave, for which the result has already been obtained in the article.

Non-null conformal transformation. The more general case $\|\mathrm{d}f\|^2 \neq 0$ requires the use of the integrability conditions (6.2). Using the aligned basis in which $\kappa = \sigma = \Psi_0 = \Psi_1 = 0$, the integrability conditions contracted with spin basis vectors $\boldsymbol{o}, \boldsymbol{\iota}$ yield

$$6Gf^2|\phi|^2\overline{\delta}f + 2Gf(f^2-1)\overline{\phi}(\overline{\delta}\phi + 2\alpha\phi) - \Psi_3\Delta f + \Psi_4\delta f = 0 \ , \tag{6.10}$$

$$2G^2|\phi|^2Df + 2Gf(f^2-1)|\phi|^2\rho - \Psi_2\Delta f + \Psi_3\delta f = 0 \ , \tag{6.11}$$

$$\Psi_3\overline{\delta}f - \Psi_4 Df = 0 \ , \tag{6.12}$$

$$\Psi_2\overline{\delta}f - \Psi_3 Df = 0 \ , \tag{6.13}$$

$$\Psi_2\delta f = 0 \ , \tag{6.14}$$

$$\Psi_2 Df = 0 \ . \tag{6.15}$$

Because $\|df\|^2 = 2Df\Delta f - 2\delta f\overline{\delta}f \neq 0$, the equations (6.12)–(6.15) imply that $\Psi_2 = \Psi_3 = 0$ and either $\Psi_4 = 0$ or $Df = 0$. In the former case, the spacetime would be conformally flat and therefore a *pp*-wave, according to McLenaghan's results [34]. Furthermore, the article shows that *pp*-waves are related by a null conformal transformation so this case is ruled out. The examination of the later case $Df = k^\mu f_{,\mu} = 0$ implies that $\mathrm{d}f$ has be spacelike. Equation (6.3) contracted with $2f^{;\nu}$ then gives

$$f\|\mathrm{d}f\|^2_{,\mu} = f_{,\mu}\|\mathrm{d}f\|^2$$

and the integration yields

$$\|\mathrm{d}f\|^2 = -a^2 f \qquad a \in \mathbb{R} \, .$$

Additionally, equation (6.11) for $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = Df = 0$ implies that $\rho = 0$, so the Kundt class is obtained again, specifically of Petrov type N. According to [2], type N Kundt spacetimes are either *pp*-waves (which can be ignored) or the so called *Kundt waves*

$$\boldsymbol{g} \ = \ 2du(Hdu + Q^2dv) - 2d\zeta d\overline{\zeta} \, ,$$
$$Q := \zeta + \overline{\zeta} \quad , \quad H := -Q^2v^2 + Q\mathcal{H}(u,\zeta,\overline{\zeta}) \quad , \quad Q\mathcal{H}_{,\zeta\overline{\zeta}} = 4G|\phi|^2 \, ,$$

where $\mathcal{H}$ is a real function and Maxwell equations are solved by $\phi = \phi(u, \overline{\zeta})$ in the frame

$$\boldsymbol{k} = \mathrm{d}u \ , \ \boldsymbol{l} = H\mathrm{d}u + Q^2\mathrm{d}v \ , \ \boldsymbol{m} = \mathrm{d}\overline{\zeta} \ , \ \overline{\boldsymbol{m}} = \mathrm{d}\zeta \, .$$

The main equation (6.3) may now be directly approached. The easily solvable components yield $f = a^2\zeta\overline{\zeta}$, but the constant a^2 may be ignored because it represents a homothety[1] applied successively after the proper conformal transformation with $f = \zeta\overline{\zeta}$. The component $f_{;uu}$ of (6.3) now remains to be solved, although not for f, which is already determined, but for the functions $\mathcal{H}, \phi$. The equation reads

$$4G\zeta\overline{\zeta}(\zeta^2\overline{\zeta}^2 - 1)Q^{-1}|\phi|^2 + \zeta\mathcal{H}_{,\zeta} + \overline{\zeta}\mathcal{H}_{,\overline{\zeta}} - \mathcal{H} = 0 \ .$$

This formula may be differentiated with respect to both ζ and $\overline{\zeta}$, because then the resulting equation does not contain $\mathcal{H}$. With the substitution $\phi = \exp\overline{\chi}$, the differentiated equation reads

$$A\chi_{,\zeta}\overline{\chi}_{,\bar{\zeta}} + \overline{B}\chi_{,\zeta} + B\overline{\chi}_{,\bar{\zeta}} + C = 0 \quad , \quad \chi = \chi(u, \zeta) \, , \tag{6.16}$$

where A, B, C are polynomials in ζ and $\overline{\zeta}$, concretely

$$\begin{aligned} A &:= Q^2(f^2 - 1) \, , \\ B &:= Q(1 + 3Qf\overline{\zeta} - f^2) \, , \\ C &:= 2(f^2 + 3Q^2 f - 1) \, , \\ Q &= \zeta + \overline{\zeta} \ , \ f = \zeta\overline{\zeta} \, . \end{aligned}$$

The function $\overline{\chi}_{,\bar{\zeta}}$ may be expressed from (6.16) as

$$\overline{\chi}_{,\bar{\zeta}} = -\frac{\overline{B}\chi_{,\zeta} + C}{A\chi_{,\zeta} + B} \, ,$$

and since $\overline{\chi} = \overline{\chi}(u, \overline{\zeta})$, differentiating the above equation with respect to ζ yields

$$(\overline{B}\chi_{,\zeta} + C)_{,\zeta}(A\chi_{,\zeta} + B) = (\overline{B}\chi_{,\zeta} + C)(A\chi_{,\zeta} + B)_{,\zeta} \, .$$

This is an equation for $\chi(u, \zeta)$, but at the same time, it is a polynomial in $\overline{\zeta}$, too. By inserting the explicit formulas for A, B, C, one learns that the above equation contains all powers of $\overline{\zeta}$ up to $\overline{\zeta}^6$. Therefore, 7 different equations for $\chi(u, \zeta)$ are obtained as the coefficients in front of $\overline{\zeta}^k, k = 0, \ldots, 6$. It is not surprising that these equations are contradicting and there is no solution. As a result, the Kundt waves are not an acceptable seed for the generating method. Therefore, the chapter may be concluded by the following theorem

Theorem 6.2 (Acceptable null seeds)**.** *The only null Einstein-Maxwell fields which can be used as seeds for the generating method presented in this chapter are the pp-waves of McLenaghan type, i.e.*

$$\boldsymbol{g} = -4G\zeta\overline{\zeta}\phi(u)\overline{\phi}(u)du^2 - 2dudv + 2d\zeta d\overline{\zeta} \ .$$

[1]Note that homotheties generally do not solve (6.3).

7. Conformal null E-M fields

This chapter deals with another prospect outlined in the conclusion of the article in the previous chapter. It addresses the question whether the generating method can be generalized so that it is less restrictive. The answer is affirmative and the generalization is to let the new Maxwell field be independent on the original. The disadvantage is that Maxwell's equations in the new spacetime are no longer automatically fulfilled. Therefore, the goal it to find all Einstein-Maxwell fields which can be mapped onto others by a conformal transformation. There are some major difficulties when dealing with non-null E-M fields, mainly due to the absence of a Robinson-like theorem, which would grant algebraical speciality or some other simplification of the integrability conditions. Therefore, the discussion will be restricted to the null case again. The following article is about to be submitted to CQG and since it is a generalization of the method outlined in the previous chapter, it contains similar formulas, but allows for a richer variety of possible solutions.

Conformally related null Einstein-Maxwell fields

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Abstract. We present a class of pairs of non-trivially conformally related solutions of Einstein-Maxwell equations that are not *pp*-waves. To our knowledge, this is the first such case and thus an extension of theorems by Brinkmann, Daftardar-Gejji, and Van den Bergh concerning conformal transformations of solutions with null electromagnetic fields.

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1. Introduction

This is a follow-up article to our previous work on conformally related non-vacuum spacetimes, in which we found a class of non-isometric solutions belonging to the *pp*-wave class and endowed with an algebraically special (*null*) Maxwell field that were related through a conformal transformation [1]. One of the issues we raised in the conclusion was whether it is possible to find a non-trivial example of two non-vacuum conformally linked spacetimes that would not be *pp*-waves.

Conformal transformations were originally viewed as a method of generating new solutions from seed metrics while preserving null cones and thus also the local causal structure of spacetimes. However, this hope has failed and instead there is a number of rather restrictive theorems limiting classes of conformally linked metrics. Conformally related Einstein spacetimes were investigated by Brinkmann [2] who showed that these include vacuum *pp*-waves, or Minkowski and (anti) de Sitter. His results were later extended by Daftardar-Gejji [3] to the case where the Einstein tensors are equal in both spacetimes of the conformal pair. Van den Bergh [4] established that the only null Einstein-Maxwell fields obtainable by a conformal transformation of a Ricci-flat solution are *pp*-waves. Szekeres and Cahen and Leroy [5, 6] found the most general conformally flat space with a non-null electromagnetic field. Skea [7] studied conformally flat pure radiation spacetimes using invariant classification. Except for the rather general theorem by Daftardar-Gejji, all these results assume that the seed spacetime is vacuum.

Due to the conformal invariance of electromagnetic field action in four dimensions, any solution of Maxwell equations in the seed spacetime (including test solutions) is still a valid electromagnetic field in the resulting spacetime. However, in the present paper, we concentrate on conformally related solutions of non-vacuum Einstein equations with two generally different source-free null electromagnetic fields, which can propagate only in algebraically special spacetimes and are automatically aligned with the repeated principal null direction (PND) of the Weyl tensor. We describe an entire class of non-vacuum, conformally linked, Kundt spacetimes that are not *pp*-waves but contain null Maxwell fields.

2. General setup

Let us consider a conformal transformation between two null Einstein-Maxwell (E-M) fields

$$\widetilde{g}_{ab} = f^{-2} g_{ab} \ ,$$

where g_{ab} is the seed metric tensor, $\widetilde{g}_{ab}$ is the new one and scalar f is the conformal factor. The metric signature is $(+,-,-,-)$, because the spinor calculus will be used.
If we prescribe a traceless Ricci tensor $\widetilde{R}_{ab}$ in the new spacetime $\widetilde{g}_{ab}$, the conformal factor f has to satisfy

$$f_{;ab} = \frac{1}{2} f (R_{ab} - \widetilde{R}_{ab}) + \frac{1}{2f} g_{ab} \|\mathrm{d}f\|^2, \tag{1}$$

where the covariant derivative is taken with respect to g_{ab} and $\|\mathrm{d}f\|^2 := g^{ab} f_{,a} f_{,b}$.
In the following calculations, we will use spinor calculus and NP formalism. The spin basis shall be denoted (o^A, ι^A) and the corresponding null frame reads

$$k^a = o^A \bar{o}^{\dot{A}}, \quad l^a = \iota^A \bar{\iota}^{\dot{A}}, \quad m^a = o^A \bar{\iota}^{\dot{A}}, \quad \overline{m}^a = \iota^A \bar{o}^{\dot{A}}.$$

The corresponding directional derivatives are denoted

$$D := k^a \nabla_a \ , \ \Delta := l^a \nabla_a \ , \ \delta := m^a \nabla_a \ , \ \overline{\delta} := \overline{m}^a \nabla_a \ .$$

3. Null Maxwell fields

The spinor equivalent ϕ_{AB} of a null Maxwell field can be written as

$$\phi_{AB} = \phi \, \alpha_A \alpha_B$$

for some complex function ϕ and spinor α_A, the null Maxwell field is then said to be *aligned* with α^A. For practical purposes, it is convenient to work with an *aligned spin basis* in which $o^A = \alpha^A$.
The spinor form of Maxwell equations is

$$\nabla^{X\dot{A}} \phi_{XB} = 0 \ . \tag{2}$$

When contracted with the aligned spin basis, (2) yields four complex equations

$$\begin{aligned} D\phi &= (\rho - 2\varepsilon)\phi\ , \quad (3)\\ \delta\phi &= (\tau - 2\beta)\phi\ , \quad (4)\\ 0 &= \kappa\phi\ , \quad (5)\\ 0 &= \sigma\phi\ . \quad (6) \end{aligned}$$

While the first pair of equations constrains ϕ (i.e., the Maxwell field), the latter pair imposes conditions on the spacetime—the null congruence $k^a = o^A\overline{o}^{\dot{A}}$ must be geodesic ($\kappa = 0$) and shear-free ($\sigma = 0$). The Goldberg-Sachs theorem then implies that

$$\Psi_0 = \Psi_{ABCD}o^A o^B o^C o^D = 0\ ,$$
$$\Psi_1 = \Psi_{ABCD}o^A o^B o^C \iota^D = 0\ ,$$

where Ψ_{ABCD} is the spinor equivalent of the Weyl tensor. Therefore, the spacetime must be algebraically special and the Maxwell field is aligned with the repeated PND of the Weyl tensor. The stress-energy tensor of a null Maxwell field takes the form

$$T_{ab} = \frac{1}{2\pi}|\phi|^2 k_a k_b\ ,$$

where $|\phi|^2 := \phi\overline{\phi}$. Einstein equations yield

$$R_{ab} = -8\pi G T_{ab} = -4G|\phi|^2 k_a k_b\ ,$$

or, in spinor formalism

$$\Phi_{AB\dot{A}\dot{B}} = 2G\phi_{AB}\overline{\phi}_{\dot{A}\dot{B}} = 2G|\phi|^2 o_A\overline{o}_{\dot{A}}o_B\overline{o}_{\dot{B}}\ ,$$

where $\Phi_{AB\dot{A}\dot{B}}$ is (up to a constant factor) the spinor equivalent of the trace-free Ricci tensor. The following nine scalars are usually constructed from $\Phi_{AB\dot{A}\dot{B}}$

$$\begin{aligned} &\Phi_{00} = \Phi_{AB\dot{A}\dot{B}}o^A o^B\overline{o}^{\dot{A}}\overline{o}^{\dot{B}}, \Phi_{01} = \Phi_{AB\dot{A}\dot{B}}o^A o^B\overline{o}^{\dot{A}}\overline{\iota}^{\dot{B}}, \Phi_{02} = \Phi_{AB\dot{A}\dot{B}}o^A o^B\overline{\iota}^{\dot{A}}\overline{\iota}^{\dot{B}},\\ &\Phi_{10} = \Phi_{AB\dot{A}\dot{B}}\iota^A o^B\overline{o}^{\dot{A}}\overline{o}^{\dot{B}},\ \Phi_{11} = \Phi_{AB\dot{A}\dot{B}}\iota^A o^B\overline{o}^{\dot{A}}\overline{\iota}^{\dot{B}},\ \Phi_{12} = \Phi_{AB\dot{A}\dot{B}}\iota^A o^B\overline{\iota}^{\dot{A}}\overline{\iota}^{\dot{B}},\\ &\Phi_{20} = \Phi_{AB\dot{A}\dot{B}}\iota^A \iota^B\overline{o}^{\dot{A}}\overline{o}^{\dot{B}},\ \Phi_{21} = \Phi_{AB\dot{A}\dot{B}}\iota^A \iota^B\overline{o}^{\dot{A}}\overline{\iota}^{\dot{B}},\ \Phi_{22} = \Phi_{AB\dot{A}\dot{B}}\iota^A \iota^B\overline{\iota}^{\dot{A}}\overline{\iota}^{\dot{B}}, \end{aligned}$$

which obviously fulfil $\overline{\Phi_{ij}} = \Phi_{ji}$.
Through Einstein equations, the main equation (1) becomes

$$f_{;ab} = 2Gf\left(|\widetilde{\phi}|^2\widetilde{k}_a\widetilde{k}_b - |\phi|^2 k_a k_b\right) + \frac{1}{2f}g_{ab}\|\mathrm{d}f\|^2\ . \quad (7)$$

The original null Maxwell field is aligned with k^a, the new alignment is $\widetilde{k}^a$. The crucial property of the Weyl tensor is its conformal invariance, thus, according to the Goldberg-Sachs theorem, both k^a and $\widetilde{k}^a$ must be repeated PNDs of a single Weyl tensor. As a result, the process of solving (7) naturally splits into two cases

(i) The Weyl tensor is of Petrov type D and has two repeated PNDs k^a and l^a. The original Maxwell field is aligned with k^a, the new with l^a. This case shall be called *re-aligning.*

(ii) For other algebraically special Petrov types there is only one repeated PND and both Maxwell fields must be aligned with it, i.e., $k^a = \tilde{k}^a$. This case will be called *co-aligned.*

The re-aligning case shall be treated first.

4. Conformally related re-aligning null E-M fields

According to Van den Bergh [8], all Petrov type D null E-M fields belong to the Robinson-Trautman class and are given by

$$ds^2 = -\frac{2m(u)}{r}\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r - 2r^2\mathrm{d}\zeta\mathrm{d}\overline{\zeta} \ ,$$
$$m(u) = -2G\int h(u)\overline{h}(u)du \ , \quad \phi = \frac{h(u)}{r} \ .$$

For such a metric tensor, the repeated PNDs are

$$k^a = \frac{\partial}{\partial r} \ , \ l^a = \frac{\partial}{\partial u} + \frac{m}{r}\frac{\partial}{\partial r}$$

and the Maxwell field is aligned with k^a. For our purposes, it will be convenient to introduce a second null coordinate v,

$$v^2 = r^2 + 2M(u) \quad , \quad \dot{M} \equiv M_{,u} = -m \ .$$

Now the metric reads

$$ds^2 = \frac{2v}{r}\mathrm{d}u\mathrm{d}v - 2r^2\mathrm{d}\zeta\mathrm{d}\overline{\zeta} \ ,$$

where r is to be regarded as a function of u and v, specifically $r^2 = v^2 - 2M$. In these coordinates, the PNDs read

$$k^a = \frac{r}{v}\frac{\partial}{\partial v} \ , \ l^a = \frac{\partial}{\partial u} \ .$$

It is obvious that any conformally related spacetime with a null E-M field must be of this form as well. The null Maxwell field is then re-aligning iff it is aligned with $\tilde{k}^a = l^a$ in the new spacetime. The null frame in the new spacetime is completed by

$$\tilde{k}^a = \frac{\partial}{\partial u} \ , \ \tilde{l}^a = f^2\frac{r}{v}\frac{\partial}{\partial v} \ , \ m^a = \frac{f}{r}\frac{\partial}{\partial \zeta} \ , \ \overline{m}^a = \frac{f}{r}\frac{\partial}{\partial \overline{\zeta}}$$

We now calculate $\widetilde{\Phi}_{AB\dot{A}\dot{B}}$ and require $\widetilde{\Phi}_{AB\dot{A}\dot{B}} = 2G|\phi|^2\tilde{k}_a\tilde{k}_b$. The resulting component equations (up to irrelevant factors) read

$$\tilde{\Phi}_{00} \propto r^2 f_{,uu} - \dot{M}f_{,u} + \ddot{M}f = 0 \ , \tag{8}$$
$$\tilde{\Phi}_{01} \propto r^2 f_{,u\zeta} + \dot{M}f_{,\zeta} = 0 \ , \tag{9}$$
$$\tilde{\Phi}_{02} \propto f_{,\zeta\zeta} = 0 \ , \tag{10}$$
$$\tilde{\Phi}_{11} \propto r^2 f_{,uv} + \dot{M}f_{,v} - vf_{,u} + \frac{v}{r}f_{,\zeta\overline{\zeta}} = 0 \ , \tag{11}$$
$$\tilde{\Phi}_{12} \propto r^2 f_{,v\zeta} - vf_{,\zeta} = 0 \ , \tag{12}$$
$$\tilde{\Phi}_{22} = f^3v^{-3}\left(vr^2 f_{,vv} + 2Mf_{,v}\right) = 2G|\tilde{\phi}|^2 \ . \tag{13}$$

If we differentiate (9) with respect to v, (11) with respect to ζ, and (12) with respect to u and use (10), we obtain

$$
\begin{aligned}
r^2 f_{,uv\zeta} + \dot{M} f_{,v\zeta} + 2v f_{,u\zeta} &= 0 \ , \\
r^2 f_{,uv\zeta} + \dot{M} f_{,v\zeta} - v f_{,u\zeta} &= 0 \ , \\
r^2 f_{,uv\zeta} - 2\dot{M} f_{,v\zeta} - v f_{,u\zeta} &= 0 \ .
\end{aligned}
$$

From these three linear equations one quickly obtains $f_{,u\zeta} = f_{,v\zeta} = 0$ and then (9) implies that $f_{,\zeta} = 0$. Naturally, the same holds for the complex conjugated coordinate $\bar{\zeta}$ and so f depends only on u and v. The new Ricci scalar has to vanish as well and this condition for $f = f(u,v)$ reads

$$\tilde{R} \propto f\left(r^2 f_{,uv} - \dot{M} f_{,v} + v f_{,u}\right) - 2r^2 f_{,u} f_{,v} = 0 \ .$$

Equation (11) then simplifies the term in the bracket and results in

$$f f_{,uv} - f_{,u} f_{,v} = 0 \ ,$$

which suggests that f separates into $f = U(u)V(v)$ and (8) with (11) now read

$$r^2 U_{,uu} - \dot{M} U_{,u} + \ddot{M} U = 0 \ , \tag{14}$$

$$r^2 U_{,u} V_{,v} + \dot{M} U V_{,v} - v U_{,u} V = 0 \ . \tag{15}$$

The v-dependent part of (14) yields $U_{,uu} = 0$, or equivalently $U = a(u+b)$ and the v-independent part then implies that $\dot{M} = c^2(u+b)$. Therefore, $M = c^2(u+b)^2/2$, as the integration constant can be chosen to be zero, because only $\dot{M} = -m$ is 'physical'. Since both functions f and M depend on $u+b$, the coordinate may be shifted so that $b = 0$. Moreover, by rescaling the coordinates ($v = cv'$, $\zeta = \sqrt{c}\zeta'$), one can effectively set $c = 1$. Finally, (15) yields $f = auv$ and we can further get rid of the constant a, because the conformal transformation can be decomposed into two successive transformations $f_1 = uv$ and $f_2 = a$, the latter being a homothety that we choose to neglect. Thus, we managed to construct a pair of conformal null E-M fields

$$
\begin{aligned}
ds^2 &= \frac{2v}{\sqrt{v^2-u^2}} \mathrm{d}u\mathrm{d}v - 2(v^2-u^2)\mathrm{d}\zeta\mathrm{d}\bar{\zeta} \ , \\
\widetilde{ds}^2 &= \frac{1}{u^2v^2}\left[\frac{2v}{\sqrt{v^2-u^2}} \mathrm{d}u\mathrm{d}v - 2(v^2-u^2)\mathrm{d}\zeta\mathrm{d}\bar{\zeta}\right] \ .
\end{aligned}
$$

Unfortunately, the coordinate transformation

$$
\begin{aligned}
u' &= 1/v \ , \\
v' &= 1/u
\end{aligned}
$$

shows that $\widetilde{ds}^2$ is, in fact, isometric to ds^2. Therefore, we can conclude this section by claiming that there are no distinct re-aligning conformal null E-M spacetimes.

5. Conformally related co-aligned null E-M fields

If the new null Maxwell field remains aligned with the same k^a as the original field, the main equation for the conformal factor (7) becomes

$$f_{;ab} = 2Gf\left(|\tilde{\phi}|^2 - |\phi|^2\right) k_a k_b + \frac{1}{2f} g_{ab} \|\mathrm{d}f\|^2. \tag{16}$$

The articles of Brinkmann and others suggest that it is useful to further split the main equation into two cases according to the type of $f_{,a}$.

(a) $\|\mathrm{d}f\|^2 = 0$ but $f_{,a} \neq 0$, this conformal transformation will be called *null.*

(b) $\|\mathrm{d}f\|^2 \neq 0$, the corresponding conformal transformation will be called *non-null.*

5.1. Null conformal transformation

In this special case, the main equation reads

$$f_{;ab} = 2Gf\left(|\widetilde{\phi}|^2 - |\phi|^2\right) k_a k_b. \tag{17}$$

Contracting the above equation with $f^{,b}$ yields

$$0 = 2Gf\left(|\widetilde{\phi}|^2 - |\phi|^2\right) k_a k_b f^{,b}\ ,$$

where the right-hand side vanished, because $f_{;ab}f^{,b} = \|\mathrm{d}f\|^2_{,a}/2 = 0$. The above equation then implies that either $|\widetilde{\phi}|^2 = |\phi|^2$ or $k^a f_{,a} = 0$. The first case yields equal Einstein tensors and so it is covered by the theorem of Daftardar-Gejji [3], allowing only *pp*-waves. Therefore, we will further investigate the latter case that implies $f_{,a} = \lambda k_a$. We insert this expression into (17) to get

$$k_{a;b} = 2Gf\left(|\widetilde{\phi}|^2 - |\phi|^2\right) k_a k_b - k_a \lambda_{,b}\ .$$

As a result of this equation, some NP connection coefficients must vanish, namely $\rho = k_{a;b}m^a\overline{m}^b = 0$. Therefore, the null congruence k^a is not only geodesic and shear-free ($\kappa = \sigma = 0$), but also expansion-free and so the spacetime belongs to the well-known Kundt class

$$ds^2 = 2\mathrm{d}u(H\mathrm{d}u + \mathrm{d}v + W\mathrm{d}\zeta + \overline{W}\mathrm{d}\overline{\zeta}) - 2P^{-2}\mathrm{d}\zeta\mathrm{d}\overline{\zeta}\ , \quad P_{,v} = 0\ ,$$

with H, P real and W complex. The repeated PND is $k^a = \partial/\partial v$ and so the covariant null tetrad can be chosen as

$$k_a = \mathrm{d}u,\ l_a = H\mathrm{d}u + \mathrm{d}v + W\mathrm{d}\zeta + \overline{W}\mathrm{d}\overline{\zeta},\ m_a = -P^{-1}\mathrm{d}\overline{\zeta},\ \overline{m}_a = -P^{-1}\mathrm{d}\zeta.$$

The relation $f_{,a} = \lambda k_a = \lambda \mathrm{d}u$ implies that $f = f(u)$. As a result, the only non-trivial components of (17) are

$$f_{;uu} = \ddot{f} + \dot{f}H_{,v} = 2Gf\left(|\widetilde{\phi}|^2 - |\phi|^2\right)\ , \tag{18}$$

$$f_{;u\zeta} = \frac{1}{2}\dot{f}W_{,v} = 0\ , \tag{19}$$

where overdot denotes the derivative with respect to u. According to Stephani [9] (p. 476), metric functions of a Kundt E-M solution with $W_{,v} = 0$ satisfy

$$P = 1\ ,\ W = W(u, \overline{\zeta})\ ,\ H = \frac{1}{2}\left(W_{,\overline{\zeta}} + \overline{W}_{,\zeta}\right) v + H^0(u, \zeta, \overline{\zeta})\ ,$$

$$H^0_{,\zeta\overline{\zeta}} - \mathrm{Re}\left[\left(W_{,\overline{\zeta}}\right)^2 + WW_{,\overline{\zeta}\,\overline{\zeta}} + W_{,u\overline{\zeta}}\right] = 2G|\phi|^2\ . \tag{20}$$

Maxwell equations in the original spacetime are fulfilled iff $\phi = \phi(u, \bar{\zeta})$ and the same holds in the new spacetime, i.e., $\tilde{\phi} = \tilde{\phi}(u, \bar{\zeta})$.
With such metric functions, (18) reads

$$\ddot{f} + \frac{1}{2}\dot{f}\left(W_{,\bar{\zeta}} + \overline{W}_{,\zeta}\right) = 2Gf\left(|\tilde{\phi}|^2 - |\phi|^2\right) . \tag{21}$$

To determine the spatially-dependent part of this equation, we differentiate it with respect to ζ and $\bar{\zeta}$ to obtain

$$\tilde{\phi}_{,\bar{\zeta}}\bar{\tilde{\phi}}_{,\zeta} = \phi_{,\bar{\zeta}}\bar{\phi}_{,\zeta}$$

which implies

$$\tilde{\phi} = e^{i\varphi}(\phi + a) , \tag{22}$$

where $\varphi = \varphi(u)$ is real and $a = a(u)$ is complex. The phase factor $e^{i\varphi}$ is somewhat trivial, because it does not appear in Einstein equations, it represents a u-dependent duality rotation of the original null Maxwell field. The restriction on the possible form of $\tilde{\phi}$ can be inserted back into (21) and the resulting equation is

$$\ddot{f} + \frac{1}{2}\dot{f}\left(W_{,\bar{\zeta}} + \overline{W}_{,\zeta}\right) = 2Gf(\bar{a}\phi + a\bar{\phi} + |a|^2) .$$

Differentiation with respect to $\bar{\zeta}$ yields

$$\frac{1}{2}\dot{f}W_{,\bar{\zeta}\bar{\zeta}} = 2Gf\bar{a}\phi_{,\bar{\zeta}} .$$

This equation is solvable only if $W_{,\bar{\zeta}\bar{\zeta}} = b(u)\phi_{,\bar{\zeta}}$ holds in the original spacetime, i.e.,

$$W_{,\bar{\zeta}} = b(u)\phi + c(u), \tag{23}$$

and the spatially dependent part of (21) reduces to

$$\frac{1}{2}\dot{f}b = 2Gf\bar{a}. \tag{24}$$

As a result, only an ordinary differential equation for function $f(u)$ remains

$$\ddot{f} + \frac{1}{2}\dot{f}(c + \bar{c}) = \frac{\dot{f}^2|b|^2}{8Gf} . \tag{25}$$

Therefore, we can construct the seed spacetime that allows a solution of the equation for the conformal factor (17) by

1. choosing arbitrary complex functions $b(u), c(u), \phi(u, \bar{\zeta})$,
2. integrating (23) to obtain W,
3. integrating (20) to obtain H^0,

The conformal factor is then obtained from (25). One can also determine the new Maxwell field by expressing a from (24) and inserting the relation into (22).
If $a = 0$ then (24) implies that $b = 0$, $W(u, \bar{\zeta})$ is linear in $\bar{\zeta}$ and a coordinate transformation

$$v' = v + A(u)\zeta\bar{\zeta} + B(u)\zeta + \overline{B}(u)\bar{\zeta}$$

can set W to be zero and the spacetime is a *pp*-wave. This is in perfect agreement with Daftardar-Gejji's results, because $a = 0$ implies $|\widetilde{\phi}|^2 = |\phi|^2$ and so the Einstein tensors are equal in both spacetimes which thus have to be *pp*-waves.
On the other hand, for $b \neq 0$ we obtain an example of two non-vacuum spacetimes that are conformally related and do not belong to the *pp*-wave class.
The resulting spacetime

$$\widetilde{ds}^2 = f^{-2}\left[2\mathrm{d}u\left(H\mathrm{d}u + \mathrm{d}v + W\mathrm{d}\zeta + \overline{W}\mathrm{d}\overline{\zeta}\right) - 2\mathrm{d}\zeta\mathrm{d}\overline{\zeta}\right]$$

is again a member of the Kundt family. The coordinate transformation that brings the new metric into the canonical form is

$$\mathrm{d}u' = f^{-3}\mathrm{d}u\ ,\ v' = fv\ ,\ \zeta' = f^{-1}\zeta\ .$$

The metric functions W' and $H^{0\prime}$ in the new coordinates are

$$\begin{aligned} W'(u', \overline{\zeta'}) &= f^2\left[W(u, \overline{\zeta}) - \dot{f}\overline{\zeta'}\right]\ , \\ H^{0\prime} &= f^4\left[H^0 - \dot{f}^2\zeta'\overline{\zeta'} + \dot{f}\left(\zeta' W + \overline{\zeta' W}\right)\right]\ . \end{aligned}$$

5.2. *Non-null conformal transformation*

The full equation (16) requires a different treatment than its simpler version (17). We start with the Bianchi identities in the new spacetime

$$\widetilde{\nabla}^X_{\dot{D}}\widetilde{\Psi}_{ABCX} = \widetilde{\nabla}^{\dot{X}}_{(A}\widetilde{\Phi}_{BC)\dot{D}\dot{X}}\ . \tag{26}$$

Under a conformal transformation, spinor quantities in the above equation behave like

$$\widetilde{\nabla}_{A\dot{B}}\xi_C = \nabla_{A\dot{B}}\xi_C + \xi_A\nabla_{C\dot{B}}\ln f\ ,\ \widetilde{\varepsilon}^{AB} = f\varepsilon^{AB}\ ,\ \widetilde{\Psi}_{ABCD} = \Psi_{ABCD}\ ,$$

where $\widetilde{\varepsilon}^{AB}$ raises spinor indices in the new spacetime. Einstein and Maxwell equations in the new spacetime are

$$\widetilde{\Phi}_{AB\dot{C}\dot{D}} = 2G\widetilde{\Phi}_{AB}\overline{\widetilde{\Phi}}_{\dot{C}\dot{D}}\ ,\ \widetilde{\nabla}^X_{\dot{A}}\widetilde{\Phi}_{BX} = 0\ .$$

With the aforementioned transformation properties and Einstein-Maxwell equations, Bianchi identities (26) yield

$$4G\overline{\widetilde{\phi}}_{\dot{D}\dot{X}}\widetilde{\phi}_{(AB}\nabla^{\dot{X}}_{C)}f + 2Gf\left(\overline{\widetilde{\phi}}_{\dot{D}\dot{X}}\nabla^{\dot{X}}_{(A}\widetilde{\phi}_{BC)} - \overline{\phi}_{\dot{D}\dot{X}}\nabla^{\dot{X}}_{(A}\phi_{BC)}\right) + \Psi_{ABCX}\nabla^X_{\dot{D}}f = 0\,.$$

Keeping in mind that the null E-M fields fulfil $\kappa = \sigma = \Psi_0 = \Psi_1 = 0$ in the aligned basis, we calculate the non-trivial components of the last equation

$$4G|\widetilde{\phi}|^2\overline{\delta}f + 2Gf\left[\overline{\widetilde{\phi}}\overline{\delta}\widetilde{\phi} - \overline{\phi}\,\overline{\delta}\phi + 2\alpha\left(|\widetilde{\phi}|^2 - |\phi|^2\right)\right] - \Psi_3\Delta f + \Psi_4\delta f = 0\,, \tag{27}$$

$$2G|\widetilde{\phi}|^2 Df + 2Gf\left(|\widetilde{\phi}|^2 - |\phi|^2\right)\rho - \Psi_2\Delta f + \Psi_3\delta f = 0\,, \tag{28}$$

$$\Psi_3\overline{\delta}f - \Psi_4 Df = 0\,, \tag{29}$$

$$\Psi_2\overline{\delta}f - \Psi_3 Df = 0\,, \tag{30}$$

$$\Psi_2\delta f = 0\,, \tag{31}$$

$$\Psi_2 Df = 0\,. \tag{32}$$

Because $\|\mathrm{d}f\|^2 = 2Df\Delta f - 2\delta f\overline{\delta}f \neq 0$, (29)–(32) imply that $\Psi_2 = \Psi_3 = 0$ and either $\Psi_4 = 0$ or $Df = 0$. In the former case, the spacetime would be conformally flat and therefore covered by McLenaghan's results [10], which allow only a certain class of conformally flat *pp*-waves. Therefore, we examine the latter condition $Df = k^a f_{,a} = 0$, which tells us that f_a has to be spacelike. If we multiply (16) by $f^{,a}$ once again and use $k^a f_{,a} = 0$, we obtain

$$f\|\mathrm{d}f\|^2_{,a} = f_{,a}\|\mathrm{d}f\|^2$$

and this equation implies that $\|\mathrm{d}f\|^2 = -2a^2 f$, where a is a real constant. Since the spacetime is of Petrov type N, (28) implies that $\rho = 0$, so the seed spacetime is a type N Kundt solution. According to Griffiths and Podolský [11], a Kundt spacetime of Petrov type N that is not a *pp*-wave is a *Kundt wave* with the metric

$$ds^2 = 2\mathrm{d}u(H\mathrm{d}u + Q^2\mathrm{d}v) - 2\mathrm{d}\zeta\mathrm{d}\overline{\zeta}\ ,$$

where $Q := \zeta + \overline{\zeta},\ H := -Q^2v^2 + Q\mathcal{H}$ and $\mathcal{H} = \mathcal{H}(u, \zeta, \overline{\zeta})$ is a real function that has to satisfy

$$Q\mathcal{H}_{,\zeta\overline{\zeta}} = 2G\phi\overline{\phi} \tag{33}$$

The repeated PND is $k^a = \partial/\partial v$ and the covariant null frame is completed by

$$k_a = \mathrm{d}u\ ,\ l_a = H\mathrm{d}u + Q^2\mathrm{d}v\ ,\ m_a = -\mathrm{d}\overline{\zeta}\ ,\ \overline{m}_a = -\mathrm{d}\zeta\ ,$$

Maxwell equations require $\phi = \phi(u, \overline{\zeta})$. Assuming this form of the metric, the easily solvable components of (16) are

$$\begin{aligned} f_{,\zeta\zeta} &= 0\ ,\\ f_{,\zeta\overline{\zeta}} &= a^2\ ,\\ Qf_{,u\zeta} - f_{,u} &= 0\ ,\\ f_{,\zeta} + f_{,\overline{\zeta}} &= Qa^2 \end{aligned}$$

with the general solution $f = a^2\zeta\overline{\zeta}$. The constant a may be discarded, because it represents a homothety applied after a proper conformal transformation. The non-trivial component of (16) is the uu component

$$2G\zeta\overline{\zeta}\frac{|\widetilde{\phi}|^2 - |\phi|^2}{Q} = \mathcal{H} - \zeta\mathcal{H}_{,\zeta} - \overline{\zeta}\mathcal{H}_{,\overline{\zeta}}\ . \tag{34}$$

Maxwell equations in the new spacetime are solved by

$$\widetilde{\phi} = \zeta\overline{\zeta}^{-1}\widehat{\phi}(u, \overline{\zeta})\ .$$

We differentiate (34) with respect to ζ and $\overline{\zeta}$ to obtain the necessary and sufficient condition for a solution to exist

$$\left(\zeta\overline{\zeta}\frac{|\widehat{\phi}|^2}{Q}\right)_{,\zeta\overline{\zeta}} = \zeta\overline{\zeta}\left(\frac{|\phi|^2}{Q}\right)_{,\zeta\overline{\zeta}}\ .$$

Reparametrization $\phi = e^{\overline{\chi}},\ \overline{\zeta}\widehat{\phi} = \overline{F}e^{\overline{\chi}},\ F = F(\zeta)$ then leads to

$$a\chi_{,\zeta}\overline{\chi}_{,\overline{\zeta}} + \overline{b}\chi_{,\zeta} + b\overline{\chi}_{,\overline{\zeta}} + c = 0 \tag{35}$$

where

$$a = Q^2(F\overline{F} - \zeta\overline{\zeta}) \ , \ b = Q^2 F_{,\zeta}\overline{F} - \frac{a}{Q} \ , \ c = Q^2 F_{,\zeta}\overline{F}_{,\overline{\zeta}} - \frac{b+\overline{b}}{Q} \ .$$

The complex function $\overline{\chi}_{,\overline{\zeta}}$ may be expressed from (35) and we obtain

$$\overline{\chi}_{,\overline{\zeta}} = -\frac{\overline{b}\chi_{,\zeta} + c}{a\chi_{,\zeta} + b}$$

Since $\overline{\chi} = \overline{\chi}(\overline{\zeta})$, the above equation, when differentiated with respect to ζ, yields

$$(\overline{b}\chi_{,\zeta} + c)_{,\zeta}(a\chi_{,\zeta} + b) = (\overline{b}\chi_{,\zeta} + c)(a\chi_{,\zeta} + b)_{,\zeta} \, .$$

This equation roughly reads

$$A\chi_{,\zeta\zeta} + B(\chi_{,\zeta})^2 + C\chi_{,\zeta} + D = 0 \ .$$

In principle, one can express $\chi_{,\zeta\zeta}$, differentiate the resulting equation with respect to $\overline{\zeta}$ to obtain another equation with one less term containing χ. Continuing in similar fashion, the terms containing χ can be gradually eliminated, leaving only an equation for F and $\overline{F}$. However, each derivative with respect to $\overline{\zeta}$ yields more complicated equations and the final 'χ-free' equation is too horrendous to be solved or even decided whether it is solvable. However, we did manage to guess a simple solution of (35) which is given by $a = 0$. Sadly, the resulting $F = \zeta$ yields $|\widetilde{\phi}|^2 = |\phi|^2$, so the theorem of Daftardar-Gejji implies that the resulting conformal pair is not distinct, because the spacetimes are not *pp*-waves.

6. Conclusions

We investigated pairs of conformally related solutions of Einstein-Maxwell equations where the electromagnetic field is null. Generally, we used the integration conditions of the equation for the conformal factor to restrict the admissible spacetime metrics. The problem naturally splits into null and non-null conformal transformations. In the former case we succeeded in finding non-trivial pairs of non-vacuum spacetimes that did not belong to the *pp*-wave class. This is the first such example and it shows that the promise of conformal transformations as a generating tool is not as dim as it has seemed previously since the class of relevant spacetimes is much broader. On the other hand, the non-null conformal transformation is more difficult to handle and only an isometric conformal pair of null E-M fields was found to exist. Distinct null E-M spacetimes related by a non-null conformal transformation are yet to be found.

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References

[1] Hruška J and Žofka M 2011 Conformally related non-vacuum spacetimes *Class. Quantum Grav.* **28** 075002

[2] Brinkmann H W 1925 Einstein spaces which are mapped conformally on each other. *Math. Ann.* **18** 119–45

[3] Daftardar-Gejji V 1998 A generalization of Brinkmann's theorem *Gen. Rel. Grav.* **30** 695–700

[4] Van den Bergh N 1986 Conformally Ricci flat Einstein-Maxwell solutions with a null electromagnetic field. *Gen. Rel. Grav.* **18** 1105–10

[5] Szekeres P 1966 On the Propagation of Gravitational Fields in Matter Citation *J. Math. Phys.* **7** 751

[6] Cahen M and Leroy J 1966 Exact solutions of Einstein-Maxwell equations. *J. Math. Mech.* **16** 501

[7] Skea J E F 1997 The invariant classification of conformally flat pure radiation spacetimes *Class. Quantum Grav.* **14** 2393

[8] Van den Bergh N 1989 Einstein-Maxwell null fields of Petrov type D. *Class. Quantum Grav.* **6** 1871–1878

[9] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* 2nd edn (Cambridge University Press)

[10] McLenaghan R G, Tariq N and Tupper B O J 1975 Conformally flat solutions of the Einstein-Maxwell equations for null electromagnetic fields *J. Math. Phys.* **16** 829–31

[11] Griffiths J B and Podolský J 2009 *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press)

Conclusion

Since the spacetimes conformal to vacuum fields have been studied by several other authors, the aim of the present work was to explore the possibility of using solutions of Einstein-Maxwell equations as the seeds. The presence of a Maxwell field in the original spacetime naturally incite one to consider the same matter content in the new spacetime, even more so when the conformal invariance of vacuum Maxwell's equations is taken into acount, which implies that Maxwell's equations do not have to be solved in the new spacetime if the Maxwell field is left unchanged. The generating method from the sixth chapter exploits this fact and it is shown that the only null Maxwell fields that are suitable as seeds are the conformally flat *pp*-waves given by McLenaghan, the generated spacetimes then belong to the same family as the seeds.

A generalization of the aforementioned method is acquired by allowing the Maxwell field to change under conformal transformation. The price paid for such generalization is the necessity to separately solve Maxwell's equations in the new spacetime. The calculations were restricted to null Maxwell fields again and the family of admissible seeds was larger than just the usual *pp*-waves. Conformally related null Einstein-Maxwell spacetimes belong to a wider family of Kundt spacetimes of Petrov type III or N.

The challenge for the future is to deal with the integrability conditions for both the original and the generalized method with non-null Einstein-Maxwell fields, because these were rarely discussed in the literature. The first step in this direction might be to find the non-null Einstein-Maxwell fields conformal to vacuum spacetimes, because this problem is also yet to be solved and appears at least a little less complicated than already starting with an Einstein-Maxwell field.

Part III

Appendices

A. Basic quantities

The purpose of this appendix is to briefly introduce quantities that are used in this thesis.

A.1 Tensors on a manifold

The arena for classical physics is the *Manifold M*. The exact definition will not be given here, a curious reader is referred to [35] for example, provided that he can read Slovak. Only a vague but comprehensible depiction of will be provided

Definition (Manifold). *Manifold M is a topological space covered by a set of charts, which map pieces of M homeomorphically to $\mathbb{R}^n$.*

Therefore, a manifold looks locally like $\mathbb{R}^n$, but its global structure may be more complicated. The charts provide coordinates $x^i, i = 1, \dots, n$ on the n-dimensional manifold and allow for an $\mathbb{R}^n$-like calculus. In this thesis, the manifolds are predominantly four-dimensional.

Scalars. The most basic objects that live on a manifold are functions (also called *scalars*). These are smooth maps from M to $\mathbb{R}$ and the space of all functions on M will be denoted by $\mathcal{F}M$. For practical purposes, a scalar $f(p), p \in M$ is often substituted by its coordinate representation $f(x^i), x^i \in \mathbb{R}^n$,where x^i are the coordinates of p. Thus, one can differentiate scalars with respect to the coordinates and the notation will be various

$$\frac{\partial f}{\partial x^i} \equiv \partial_i f \equiv f_{,i} \ .$$

Curves and vectors. A parametrized curve γ on M is a smooth map from $\mathbb{R}$ to M. At each point of the parametrized curve a *vector $\boldsymbol{v}$* can be defined whose orientation is tangential to the curve and its length characterizes how quickly does the parameter change along its path. All vectors at a point p form the *tangent space T_pM* and the union of all tangent spaces at all points of M is denoted $\mathcal{T}M$. The vectors ∂_i tangential to the coordinate lines x^i form a *coordinate basis in TM* and so every vector can be decomposed into

$$\boldsymbol{v} = v^i \partial_i \ ,$$

where the functions v^i are the called *coordinate components* of $\boldsymbol{v}$ with respect to coordinates x^i. Einstein's summation convention is naturally used here and in the whole thesis. Of course, one may choose a different, non-coordinate bases of vectors $\{\boldsymbol{e}_a\}_{a=1}^n$ called frames and the the functions v^a in the decomposition

$$\boldsymbol{v} = v^a \boldsymbol{e}_a$$

are called *frame components.*
The vectors also naturally act on $\mathcal{F}M$ as

$$\boldsymbol{v} f = v^i \frac{\partial f}{\partial x^i} \ ,$$

i.e. the vectors are be identified with directional derivatives. Such interpretation allows one to define the commutator $[\boldsymbol{u}, \boldsymbol{v}]$ of two vectors, which is another vector defined through its action on an arbitrary function f as

$$[\boldsymbol{u}, \boldsymbol{v}]f := \boldsymbol{u}(\boldsymbol{v}f) - \boldsymbol{v}(\boldsymbol{u}f) = u^i v^j{}_{,i} f_{,j} - v^i u^j{}_{,i} f_{,j} ,$$

so the commutator itself is

$$[\boldsymbol{u}, \boldsymbol{v}] = (u^i v^j{}_{,i} - v^i u^j{}_{,i})\partial_j .$$

One-forms. The linear maps $T_pM \mapsto \mathbb{R}$ are called one-forms and they form the dual *cotangent space* T_p^*M. The union of all cotangent spaces at all points is denoted by $\mathcal{T}^*M$.
One of the basic one-forms is the gradient of a function $\mathrm{d}f$, that is defined through its action on an arbitrary vector $\boldsymbol{v}$ as

$$\mathrm{d}f(\boldsymbol{v}) = v^i \partial_i f .$$

The easily obtainable relation

$$\mathrm{d}x^i(\partial_j) = \partial_j x^i = \delta^i_j$$

shows that the gradients of coordinate functions $\{\mathrm{d}x^i\}_{i=1}^n$ form a *dual basis* to $\{\partial_j\}_{j=1}^n$. Thus, every one-form can be decomposed into

$$\boldsymbol{\alpha} = \alpha_i \mathrm{d}x^i ,$$

where functions α_i are the coordinate components of $\boldsymbol{\alpha}$. Alternatively, the frame components may be used as in the case of vectors. The action of an arbitrary $\boldsymbol{\alpha}$ on a vector $\boldsymbol{v}$ results in

$$\boldsymbol{\alpha}(\boldsymbol{v}) = \alpha_i \mathrm{d}x^i(\partial_j) v^j = \alpha_i \delta^i_j v^j = \alpha_i v^i .$$

This operation is called the *contraction* of $\boldsymbol{\alpha}$ with $\boldsymbol{v}$.

Tensors. From the basic spaces $\mathcal{T}M$ and $\mathcal{T}^*M$ one can construct their tensor products

$$\mathcal{T}^p_q M := \underbrace{\mathcal{T}M \otimes \cdots \otimes \mathcal{T}M}_{p} \otimes \underbrace{\mathcal{T}^*M \otimes \cdots \otimes \mathcal{T}^*M}_{q}$$

and the elements $\boldsymbol{T} \in \mathcal{T}^p_q M$ are called *tensors*. Tensors from $\mathcal{T}^p_0 M$ are called *contravariant* and those from $\mathcal{T}^0_q M$ are *covariant*. The basis of $\mathcal{T}^p_q M$ contains tensor products of ∂_i and $\mathrm{d}x^i$ and so each tensor $\boldsymbol{T} \in \mathcal{T}^p_q M$ can be decomposed into coordinate components

$$\boldsymbol{T} = T^{i_1 \cdots i_p}_{j_1 \cdots j_q} \partial_{i_1} \cdots \partial_{i_p} \mathrm{d}x^{j_1} \cdots \mathrm{d}x^{j_q} ,$$

or into frame components.

Metric tensor. The pivotal quantity of general relativity is the metric tensor $\boldsymbol{g} \in \mathcal{T}^0_2 M$, which is

1. symmetric: $g_{ab} = g_{ba}$
2. non-degenerate: $\det(g_{ab}) \neq 0$

Metric tensor defines

- *scalar product* of two vectors: $\boldsymbol{u} \cdot \boldsymbol{v} := g_{ab} u^a v^b$
- *norm* of a vector: $\|\boldsymbol{u}\|^2 := g_{ab} u^a u^b$
  For *indefinite* metric tensors[1], the vectors can be divided into three groups: *timelike* ($\|\boldsymbol{u}\|^2 > 0$), *null* ($\|\boldsymbol{u}\|^2 = 0$) and *spacelike* ($\|\boldsymbol{u}\|^2 < 0$).
- (generalized) *angle* between two non-null vectors: $\cos\varphi := \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$

The metric tensor also provides an isomorphism between $\mathcal{T}M$ and $\mathcal{T}^*M$ by index lowering, that maps a vector $\boldsymbol{v}$ linearly to a one-form $\flat_{\boldsymbol{g}}\boldsymbol{v}$

$$\flat_{\boldsymbol{g}}\boldsymbol{v} := g_{ab} v^a \mathrm{d}x^b \,.$$

For convenience, the sign $\flat_{\boldsymbol{g}}$ is often omitted and both the vector and the form are denoted by the same letter, the true meaning of this letter is to be decided according to the context.
Index raising is enabled by the inverse metric $\boldsymbol{g}^{-1} \in \mathcal{T}^2_0 M$, whose components are denoted g^{ab} and fulfil

$$g_{ab} g^{bc} = \delta^c_a \,.$$

The index raising of $\boldsymbol{\alpha} \in \mathcal{T}^*M$ is then given by

$$\sharp_{\boldsymbol{g}}\boldsymbol{\alpha} := g^{ab} \alpha_a \partial_b$$

and the sign $\sharp_{\boldsymbol{g}}$ is also usually omitted.

A.2 Tensorial operations

Symmetrization. Tensor symmetrization is denoted by round brackets and is defined via components as

$$T_{a\cdots b(i_1\cdots i_k)c\cdots d} := \sum_\pi \frac{1}{k!} T_{a\cdots b\,\pi(i_1)\cdots\pi(i_k)\,c\cdots d} \;,$$

where the sum goes over all permutations π of the k element set of indices $\{i_1, \cdots, i_k\}$. The indices that should be left unsymmetrized are put inside '$|\ldots|$', e.g. $T_{a(bc|de|f)g}$ denotes symmetrization over indices b, c, f.

Antisymmetrization. Tensor antisymmetrization is denoted by square brackets and is defined as

$$T_{a\cdots b[i_1\cdots i_k]c\cdots d} := \sum_\pi \frac{1}{k!} \mathrm{sign}(\pi) T_{a\cdots b\,\pi(i_1)\cdots\pi(i_k)\,c\cdots d} \;,$$

where $\mathrm{sign}(\pi) = \pm 1$, the plus sign is for permutations that are composed of an even number of elementary two-index exchanges, else the minus sign applies.

[1]those metric tensors which allow $\|\boldsymbol{u}\|^2 < 0$

Lie derivative. The Lie derivative along vector $\boldsymbol{v}$ is an operation $\pounds_{\boldsymbol{v}} : \mathcal{T}^p_q M \mapsto \mathcal{T}^p_q M$ that fulfils

1. Linearity: $\pounds_{\boldsymbol{v}}(\boldsymbol{A} + \lambda \boldsymbol{B}) = \pounds_{\boldsymbol{v}}\boldsymbol{A} + \lambda \pounds_{\boldsymbol{v}}\boldsymbol{B}\ ,\ \lambda \in \mathbb{R}$
2. Leibniz rule: $\pounds_{\boldsymbol{v}}(\boldsymbol{A} \otimes \boldsymbol{B}) = (\pounds_{\boldsymbol{v}}\boldsymbol{A}) \otimes \boldsymbol{B} + \boldsymbol{A} \otimes \pounds_{\boldsymbol{v}}\boldsymbol{B}$
3. Commutes with contractions: $\pounds_{\boldsymbol{v}}(\boldsymbol{\alpha}(\boldsymbol{u})) = \boldsymbol{\alpha}(\pounds_{\boldsymbol{v}}\boldsymbol{u}) + (\pounds_{\boldsymbol{v}}\boldsymbol{\alpha})(\boldsymbol{u})$
4. Acts simply on functions: $\pounds_{\boldsymbol{v}} f = \boldsymbol{v} f = v^i \partial_i f$
5. Yields commutator on vectors: $\pounds_{\boldsymbol{v}}\boldsymbol{w} = [\boldsymbol{v}, \boldsymbol{w}] = (u^i v^j{}_{,i} - v^i u^j{}_{,i})\partial_j$

By the rule 3. one obtains the Lie derivative of a one-form $\boldsymbol{\alpha}$

$$\pounds_{\boldsymbol{v}}\boldsymbol{\alpha} = (v^j \alpha_{i,j} + v^j{}_{,i}\alpha_j)\mathrm{d}x^i$$

and 2. gives the action on an arbitrary tensor $\boldsymbol{T} \in \mathcal{T}^p_q M$

$$(\pounds_{\boldsymbol{v}}\boldsymbol{T})^{a\cdots b}_{c\cdots d} = v^i T^{a\cdots b}_{c\cdots d,i} - v^a{}_{,k} T^{k\cdots b}_{c\cdots d} - \ldots - v^b{}_{,k} T^{a\cdots k}_{c\cdots d} + v^k{}_{,c} T^{a\cdots b}_{k\cdots d} + \ldots + v^k{}_{,d} T^{a\cdots b}_{c\cdots k}$$

Covariant derivative. Covariant derivative is an essential tool for parallel transport. In this thesis, only *metric* and *torsion-free* covariant derivative is used. Covariant derivative along vector $\boldsymbol{v}$ is an operation $\nabla_{\boldsymbol{v}} : \mathcal{T}^p_q M \mapsto \mathcal{T}^p_q M$ that fulfils the rules 1.–4. for Lie derivative and additionally

5. Is f-linear in the lower argument: $\nabla_{\boldsymbol{u}+f\boldsymbol{v}} = \nabla_{\boldsymbol{u}} + f\nabla_{\boldsymbol{v}}\ ,\ f \in \mathcal{F}M$
6. Is metric: $\nabla_{\boldsymbol{v}}\boldsymbol{g} = 0$
7. Is torsion-free: $\nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} = [\boldsymbol{u}, \boldsymbol{v}]$

Thanks to the commutation with contractions, it is sufficient to know how it acts on basis vectors

$$\nabla_{\partial_i}\partial_j \equiv \nabla_i \partial_j = \Gamma^k{}_{ij}\partial_k\ . \tag{A.1}$$

This relation defines the *Christoffel's symbols* $\Gamma^k{}_{ij}$ and the metricity together with vanishing torsion implies that

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{km}(g_{mi,j} + g_{mj,i} - g_{ij,m})\,.$$

From eq. (A.1) the covariant derivative of an arbitrary vector can be calculated

$$\nabla_{\boldsymbol{v}}\boldsymbol{w} = v^i \nabla_i (w^j \partial_j) = v^i (w^j{}_{,i} + \Gamma^j{}_{ik} w^k)\partial_j =: v^i w^j{}_{;i}\partial_j\,,$$

where the last equality defines the behavior of vector coordinate components under covariant differentiation, i.e. $w^j{}_{;i} := w^j{}_{,i} + \Gamma^j{}_{ik} w^k$. By the commutation with contractions, one easily obtains the covariant derivative of one-forms. The components of covariant derivative of $\boldsymbol{\alpha} \in \mathcal{T}^* M$ are

$$\alpha_{i;j} = \alpha_{i,j} - \Gamma^k{}_{ij}\alpha_k$$

and the Leibniz rule yields the formula for an arbitrary tensor $\boldsymbol{T} \in \mathcal{T}^p_q M$

$$T^{a\cdots b}_{c\cdots d;i} = T^{a\cdots b}_{c\cdots d,i} + \Gamma^a{}_{ik} T^{k\cdots b}_{c\cdots d} + \ldots + \Gamma^b{}_{ik} T^{a\cdots k}_{c\cdots d} - \Gamma^k{}_{ic} T^{a\cdots b}_{k\cdots d} - \ldots - \Gamma^k{}_{id} T^{a\cdots b}_{c\cdots k}$$

Covariant commutator. The commutator of two covariant derivatives $\nabla_a\nabla_b - \nabla_b\nabla_a$ is an f-linear map due to the vanishing torsion and on a general one-form α it yields

$$\alpha_{a;bc} - \alpha_{a;cb} =: -R^d{}_{abc}\alpha_d \,.$$

This equation is called the *Ricci identity* and defines the Riemann curvature tensor $R^d{}_{abc}$.

B. Used conventions

This appendix summarizes the conventions and the definitions of basic notions used in this thesis in order to avoid confusion about signs and normalization factors of certain quantities. The conventions are mostly adopted from the books of Stewart [21] and Penrose [22], because these are standard when working with spinors.
The metric signature is $(+,-,-,-)$, which is not the most frequent, but it is the traditional signature for spinor formalism.
The Christoffel symbols $\Gamma^{\mu}{}_{\nu\rho}$ are defined as

$$\Gamma^{\mu}{}_{\nu\rho} := \frac{1}{2} g^{\mu\sigma}(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\mu\rho,\sigma}).$$

The Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$ is given by

$$R^{\mu}{}_{\nu\rho\sigma} := \Gamma^{\mu}{}_{\nu\rho,\sigma} - \Gamma^{\mu}{}_{\nu\sigma,\rho} + \Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\alpha}{}_{\nu\rho} - \Gamma^{\mu}{}_{\alpha\rho}\Gamma^{\alpha}{}_{\nu\sigma}$$

and the Ricci tensor $R_{\mu\nu}$ is obtained by contraction

$$R_{\mu\nu} := R^{\alpha}{}_{\mu\alpha\nu} \ .$$

The Einstein tensor is

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$$

and Einstein's equations read

$$G_{\mu\nu} = -8\pi G T_{\mu\nu}$$

where G is the Newton's gravitational constant. The stress-energy tensors $T_{\mu\nu}$ of the most typical matter contents are

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} - F_{\mu\alpha}F_{\nu}{}^{\alpha}\right)$$

for the Maxwell field $F_{\mu\nu}$ and

$$T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - p g_{\mu\nu}$$

for the perfect fluid with energy density ρ and pressure p moving with four-velocity u^{μ}. Note that most of the physical fields have $T_{00} \geq 0$.
Since the bible for those working with exact solutions is [1], table B.1 summarizes the differences in conventions.

Quantity	Stephani *et al*/here
Metric tensor $\boldsymbol{g}$	-1
Christoffel symbols $\Gamma^{\mu}{}_{\nu\rho}$	$+1$
Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$	-1
Ricci tensor $R_{\mu\nu}$	-1
Ricci scalar R	$+1$
Einstein tensor $G_{\mu\nu}$	-1
Einstein's equations	$G_{\mu\nu} = GT_{\mu\nu} \quad / \quad G_{\mu\nu} = -8\pi GT_{\mu\nu}$

Table B.1: Convention comparison: *The second column shows the constant by which one has to multiply quantities in this thesis in order to obtain Stephani's conventions. For example, to obtain the metric of a spacetime in Stephani's convention, one has to multiply the metric element from this thesis by* -1*, but the Ricci scalar will be the same.*

C. Exterior calculus

The exterior calculus was developed by Cartan and utilizes differential forms as well as special operations acting on them.

Definition (Differential form)**.** *A differential k-form (or a form of degree k) is a totally antisymmetric covariant tensor, i.e. $T_{a_1\cdots a_k} = T_{[a_1\cdots a_k]}$. A differential zero-form is a function on M.*

The space of differential k-forms on a manifold M is denoted by $\Omega^k M$ and clearly

$$\dim \Omega^k M = \binom{n}{k} = \frac{n!}{k!(n-k)!} \;,$$

where n is the dimension of the manifold.

C.1 Exterior product

The tensor product of two differential forms is generally not a differential form. Therefore, slightly modified operation is defined

Definition (Exterior product)**.** *The* ***exterior product*** '$\wedge$' *(also called the* ***wedge product****) is an operation*

$$\wedge : \Omega^p M \otimes \Omega^q M \;\longmapsto\; \Omega^{p+q} M$$

that is given by

$$(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})_{i_1\cdots i_{p+q}} := \alpha_{[i_1\cdots i_p}\,\beta_{i_{p+1}\cdots i_{p+q}]}$$

Such operation inherits all the nice properties of tensor product, such as associativity, distributivity and linearity in both arguments. Additionally, the exterior product fulfills

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = (-1)^{pq} \boldsymbol{\beta} \wedge \boldsymbol{\alpha} \qquad \boldsymbol{\alpha} \in \Omega^p M \;,\; \boldsymbol{\beta} \in \Omega^q M \;.$$

A basis in $\Omega^p M$ can be constructed from the exterior products of p one-forms $\boldsymbol{\theta}^i$ which form the basis of $\Omega^1 M$. Thus, every differential p-form can be written as

$$\boldsymbol{\alpha} = \alpha_{i_1\cdots i_p} \boldsymbol{\theta}^{i_1} \wedge \cdots \wedge \boldsymbol{\theta}^{i_p}$$

With the multiplication given by '$\wedge$', an algebra ΩM of all differential forms on M can be defined

$$\Omega M = \bigcup_{p=1}^{n} \Omega^p M \,.$$

A general $\boldsymbol{\omega} \in \Omega M$ is a formal linear combination of forms of different degrees and is called an *inhomogenous* form. A *homogenous* form is linear combination of basis forms of the same degree.
One of the main reasons why differential forms are so important in geometry is the fact that they can be integrated over oriented manifolds.

Definition (Integration of differential forms)**.** *Suppose there is an n-form $\boldsymbol{\alpha}$ on n-dimensional oriented manifold M. Each such form can be written as $\boldsymbol{\alpha} = f(x^1, \dots, x^n)\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n$. The integral of $\boldsymbol{\alpha}$ over an n-dimensional area V in M is then defined as*

$$\int_V \boldsymbol{\alpha} = \int_V f(x^1, \dots, x^n)\,\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n := \int_V f(x^1, \dots, x^n)\,\mathrm{d}x^1 \cdots \mathrm{d}x^n\,,$$

where the last integral is to be understood as a standard Riemann integral over V.

Such definition is possible thanks to the fact that the component of an n-form changes under coordinate substitution in the same way as the integrand of an integral, i.e. the determinant of Jacobian matrix appears in both cases.
In context of the following paragraph, it is useful to understand scalar multiplication as an exterior product with a zero-form, i.e.

$$\lambda\boldsymbol{\alpha} = \lambda \wedge \boldsymbol{\alpha}\ , \lambda \in \Omega^0 M \equiv \mathcal{F}M, \boldsymbol{\alpha} \in \Omega^p M$$

C.2 Exterior derivative

The differential forms are naturally equipped with an operator that generalizes the concept of gradient.

Definition (Exterior derivative)**.** *The* ***exterior derivative*** *is a map*

$$\mathrm{d} : \Omega^p M \longmapsto \Omega^{p+1} M$$

defined by the following axioms

$$\begin{aligned}
\mathrm{d}(\boldsymbol{\alpha} + \boldsymbol{\beta}) &= \mathrm{d}\boldsymbol{\alpha} + \mathrm{d}\boldsymbol{\beta}\ , \\
\mathrm{d}f &= f_{,i}\mathrm{d}x^i\ ,\ f \in \mathcal{F}M\ , \\
\mathrm{dd} &= 0 \\
\mathrm{d}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) &= (\mathrm{d}\boldsymbol{\alpha}) \wedge \boldsymbol{\beta} + (-1)^p \boldsymbol{\alpha} \wedge \mathrm{d}\boldsymbol{\beta}\ ,\ \boldsymbol{\alpha} \in \Omega^p M\ ,\ \boldsymbol{\beta} \in \Omega^q M\ .
\end{aligned}$$

It can be verified that such definition determines the exterior derivative uniquely and the following coordinate formula can be obtained

$$\mathrm{d}\boldsymbol{\alpha} = \boldsymbol{\alpha}_{i_1 \cdots i_p, j} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p}\ ,\ \boldsymbol{\alpha} \in \Omega^p M\,.$$

The exterior derivative has also other important properties:

- It commutes with pullback, i.e. $\mathrm{d}f^*\boldsymbol{\alpha} = f^*\mathrm{d}\boldsymbol{\alpha}$. This property is crucial in the proof of Cartan's theorem 4.1.
- If $\mathrm{d}\boldsymbol{\alpha} = 0$ for some $\boldsymbol{\alpha} \in \Omega^p M$ then locally $\boldsymbol{\alpha} = \mathrm{d}\boldsymbol{\beta}$ for some $\boldsymbol{\beta} \in \Omega^{p-1} M$. This is the famous Poincaré's lemma used for construction of potentials. It is also useful in context of integrability conditions for partial differential equations.
- Perhaps the best known application of exterior derivative is the *Stokes' theorem*

  $$\int_V \mathrm{d}\boldsymbol{\alpha} = \int_{\partial V} \boldsymbol{\alpha}\ , \quad \boldsymbol{\alpha} \in \Omega^p M\,,$$

  where V is a $p+1$-dimensional area and ∂V is its p-dimensional boundary.

Interior product. The contraction of a vector with an arbitrary differential form (but not zero-form) is again a differential form. Such trivial observation is used to define the *interior product* $i_{\boldsymbol{v}} : \Omega^k M \mapsto \Omega^{k-1} M$

$$(i_{\boldsymbol{v}}\boldsymbol{\alpha})_{b\cdots c} := v^a \alpha_{ab\cdots c} \qquad \boldsymbol{v} \in \mathcal{T}M, \boldsymbol{\alpha} \in \Omega^p M$$

Usually, the vector is contracted with the first index of the differential form, but the conventions vary across the literature.
The interaction of inner and outer product yields

$$i_{\boldsymbol{v}}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) := (i_{\boldsymbol{v}}\boldsymbol{\alpha}) \wedge \boldsymbol{\beta} + (-1)^p \boldsymbol{\alpha} \wedge i_{\boldsymbol{v}}\boldsymbol{\beta} \qquad \boldsymbol{\alpha} \in \Omega^p M\,.$$

A useful formula relates the Lie derivative of a differential form to its exterior derivative by

$$\pounds_{\boldsymbol{v}}\boldsymbol{\alpha} := i_{\boldsymbol{v}} \mathrm{d}\boldsymbol{\alpha} + \mathrm{d} i_{\boldsymbol{v}}\boldsymbol{\alpha}\,.$$

Levi-Civita tensor. On an n-dimensional manifold, the space $\Omega^n M$ is one-dimensional and thus all differential n-forms are proportional. With the help of the metric, a prominent representative can be invariantly chosen, provided that the manifold is *orientable*[1]. It is the *Levi-Civita* tensor $\boldsymbol{\epsilon}$ defined in a positively oriented basis $\{\mathrm{d}x^i\}_{i=1}^n$ by the formula

$$\boldsymbol{\epsilon} := (|g|)^{1/2} \varepsilon_{a\cdots b}\, \mathrm{d}x^a \cdots \mathrm{d}x^b\,,$$

where $g := \det(g_{ab})$, $\varepsilon_{a\cdots b}$ is the *permutation symbol* which is totally antisymmetric and all its components are fixed by $\varepsilon_{012\cdots n} = 1$.

C.3 Hodge dual

On an n-dimensional manifold, *Hodge dual* '$*$' is a linear map $* : \Omega^p M \mapsto \Omega^{n-p} M$ that is defined as

$$(*\boldsymbol{\alpha})_{a\cdots b} = \frac{1}{p!} \alpha^{c\cdots d} \epsilon_{c\cdots d a\cdots b} \qquad \boldsymbol{\alpha} \in \Omega^p M\,,$$

where $\boldsymbol{\epsilon}$ is the Levi-Civita tensor.
Evidetnly, if $\boldsymbol{\alpha} \in \Omega^p M$, then also $*(*\boldsymbol{\alpha}) \in \Omega^p M$. Generally, $**$ is an identity up to a possible sign. In a four-dimensional spacetime with Lorenzian signature, $** = -\mathrm{id}$.
Hodge dual of a two-form $\boldsymbol{F}$ is especially interesting, because $*\boldsymbol{F}$ is also a two-form (provided that the dimension of the spacetime is four) and one can form linear combinations of $\boldsymbol{F}$ and $*\boldsymbol{F}$. Explicitly, if $\boldsymbol{F} \in \Omega^2 M$, then

$$(*\boldsymbol{F})_{ab} = \frac{1}{2} \epsilon_{abcd} F^{cd}\,.$$

It is usually advantageous to construct the following complex linear combination

$$\boldsymbol{F}^* = \boldsymbol{F} + i *\boldsymbol{F}\,. \tag{C.1}$$

For example, the two sets of vacuum Maxwell's equations $F^{\mu\nu}{}_{;\nu} = 0 = F_{[\mu\nu;\rho]}$ can be written in a compact form using $\boldsymbol{F}^*$, i.e.

$$F^{*\mu\nu}{}_{;\nu} = 0\,.$$

[1] On an orientable manifold, one can globally and consistently say which frame is positively, resp. negatively oriented.

C.4 Cartan's structure equations

The exterior calculus is also powerful for calculating frame components of the curvature. For a given frame $\{\boldsymbol{\theta}^i\}$ and metric $\boldsymbol{g} = g_{ij}\boldsymbol{\theta}^i\boldsymbol{\theta}^j$, the connection one-forms $\boldsymbol{\omega}^i{}_j$ are defined by

$$\begin{aligned} \mathrm{d}\boldsymbol{\theta}^i &= -\boldsymbol{\omega}^i{}_j \wedge \boldsymbol{\theta}^j \,, \\ \mathrm{d}g_{ij} &= \boldsymbol{\omega}_{ij} + \boldsymbol{\omega}_{ji} \qquad \boldsymbol{\omega}_{ij} := g_{ik}\boldsymbol{\omega}^k{}_j \,. \end{aligned}$$

The full information about the metric connection is encoded in $\boldsymbol{\omega}^i{}_j$. The curvature is represented by the curvature two-forms $\boldsymbol{\Omega}^i{}_j$, which are defined in terms of connection one-forms as

$$\boldsymbol{\Omega}^i{}_j = \mathrm{d}\boldsymbol{\omega}^i{}_j + \boldsymbol{\omega}^i{}_k \wedge \boldsymbol{\omega}^k{}_j \,. \tag{C.2}$$

The frame components of the Riemann tensor can then be extracted from the formula

$$\boldsymbol{\Omega}^i{}_j = \frac{1}{2} R^i{}_{jkl}\boldsymbol{\theta}^k \wedge \boldsymbol{\theta}^l \,.$$

The Bianchi identities in Cartan's formalism are simply obtained by applying the exterior derivative to (C.2) and they read

$$\mathrm{d}\boldsymbol{\Omega}^i{}_j = \boldsymbol{\Omega}^i{}_k \wedge \boldsymbol{\omega}^k{}_j - \boldsymbol{\omega}^i{}_k \wedge \boldsymbol{\Omega}^k{}_j \,.$$

D. Canonical forms of the Weyl scalars

The canonical forms of the Weyl scalars for different Petrov types are useful in context of Cartan scalars, because they help to reduce the isotropy group to minimal dimension by fixing the null frame as much as possible. The construction of the canonical form will be illustrated on the algebraically general spacetime

Petrov type *I* The Weyl spinor assumes the most generic form

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$$

and the choice

$$o_A = \alpha_A \ , \ \iota_A = \chi^{-1}\delta_A \ , \qquad \chi := \alpha_A\delta^A$$

fixes the alignment of the basis in which $\Psi_0 = \Psi_4 = 0$. The remaining freedom lies in spin-boosts

$$\begin{aligned}
(\boldsymbol{o}', \boldsymbol{\iota}') &= (\lambda\boldsymbol{o}, \lambda^{-1}\boldsymbol{\iota}) \ , \ \lambda \in \mathbb{C} \\
\Psi_0' &= \lambda^4\Psi_0 = 0 \ , \\
\Psi_1' &= \lambda^2\Psi_1 \ , \\
\Psi_2' &= \Psi_2 \ , \\
\Psi_3' &= \lambda^{-2}\Psi_3 \ , \\
\Psi_4' &= \lambda^{-4}\Psi_4 = 0 \ .
\end{aligned}$$

By a proper choice of λ (concretely $\lambda^4 = \Psi_3/\Psi_1$) one can further set $\Psi_1 = \Psi_3$. There frame is now completely fixed up to a finite number of discrete transformations and the canonical form of the Weyl scalars is

$$\Psi_0 = \Psi_4 = 0 \ , \ \Psi_1 = \Psi_3 \neq 0 \neq \Psi_2$$

The discussion is very similar for other Petrov types and the resulting canonical forms are summarized in table D.1.

Petrov type	Canonical form	Isotropy group I_0
I	$\Psi_0 = \Psi_4 = 0,\ \Psi_1 = \Psi_3 \neq 0 \neq \Psi_2$	discrete transformations $\dim I_0 = 0$
II	$\Psi_0 = \Psi_1 = \Psi_4 = 0,\ \Psi_2 = \Psi_3 \neq 0$	discrete transformations $\dim I_0 = 0$
III	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0,\ \Psi_3 = 1$	discrete transformations $\dim I_0 = 0$
D	$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0,\ \Psi_2 \neq 0$	spin boosts $\dim I_0 = 2$
N	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0,\ \Psi_4 = 1$	null rotations about $\boldsymbol{k}$ $\dim I_0 = 2$

Table D.1: Canonical form of the Weyl scalars

Bibliography

[1] Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C. and Herlt, E. (2003). *Exact Solutions of Einstein's Field Equations* 2nd edn (Cambridge University Press)

[2] Griffiths, J. B., Podolský, J. (2009). *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press)

[3] Buchdahl, H. A. (1954). Reciprocal static solutions of the equations $G_{\mu\gamma} = 0$. *Quart. J. Math. Oxford* **5**, 116.

[4] Ehlers, J. (1957). Konstruktionen und Charakterisierungen von Lösungen der Einsteinschen Gravitationsfeldgleichungen. Dissertation, Hamburg.

[5] Kramer, D. and Neugebauer, G. (1980). The superposition of two Kerr solutions. *Phys. Lett. A* **75**, 259

[6] Brinkmann, H. W. (1925). Einstein spaces which are mapped conformally on each other. *Math. Ann.* **18**, 119

[7] Van den Bergh, N. (1986a). Conformally Ricci flat Einstein–Maxwell solutions with a null electromagnetic field. *GRG* **18**, 1105.

[8] Daftardar-Gejji, V. (1998). A generalization of Brinkmann's theorem. *GRG* **30**, 695.

[9] Hruška, J. (2008). Variační principy v obecné teorii relativity. Dissertation, Prague.

[10] Ibragimov, N. H. (1985). *Transformation groups applied to mathematical physics* (Reidel, Boston).

[11] Anderson, I. M. and Torre, C. G. (1996). Classification of local generalized symmetries for the vacuum Einstein equations. *Commun. math. phys.* **176**, 479.

[12] Harrison, B. K. (1984). Prolongation structures and differential forms, in *Solutions of Einstein's equations: Techniques and results.* Lecture notes in physics, vol. 205, eds. C. Hoenselaers and W. Dietz, page 26 (Springer, Berlin).

[13] Ernst, F. J. (1968). New formulation of the axially symmetric gravitational field problem II. *Phys. Rev.* **168**, 1415.

[14] Harrison, B. K. (1968). New solutions of the Einstein–Maxwell equations from old. *JMP* **9**, 1744.

[15] Neugebauer, G. and Kramer, D. (1969). Eine Methode zur Konstruktion station¨arer Einstein–Maxwell-Felder. *Ann. Phys.* (*Germany*) **24**, 62.

[16] Rainich, G. Y. (1925). Electrodynamics in general relativity. *Trans. Amer. Math. Soc.* **27**, 106.

[17] Kinnersley, W. (1973). Generation of stationary Einstein–Maxwell fields. *JMP* **14**, 651.

[18] Harrison, B. K. (1978). Bäcklund transformations for the Ernst equation of general relativity. *Phys. Rev. Lett.* **41**, 1197.

[19] Mannheim, P. D. (2011). Making the Case for Conformal Gravity. arXiv:1101.2186v2 [hep-th].

[20] Van den Bergh, N. (1988). Conformally Ricci-flat perfect fluids II. *JMP* **29**.

[21] Stewart, J. (1991). *Advanced general relativity.* (Cambridge University Press)

[22] Penrose, R., Rindler, W. (1984). *Spinors and spacetimev volume 1.* (Cambridge University Press)

[23] Cartan, E. (1946). *Lec¸ons sur la Geometrie des Espaces de Riemann, 2nd edn.* (Paris, Gauthier- Villars)

[24] Coley, A., Hervik, A., Pelavas, N. (2009). Spacetimes characterized by their scalar curvature invariants. arXiv:0901.0791v2[gr-qc]

[25] Brans, C. H. (1965). Invariant approach to the geometry of spaces in general relativity. *JMP* **6**, 95.

[26] Karlhede, A. and MacCallum, M. A. H. (1982). On determining the isometry group of a Riemannian space. *GRG* **14**, 673.

[27] Aman, J. E. (2002). Classification programs for geometries in general relativity – manual for CLASSI, 4th edition. Report, Stockholm.

[28] MacCallum, M. A. H. and Aman, J. E. (1986). Algebraically independent n-th derivatives of the Riemannian curvature spinor in a general spacetime. *CQG* **3**, 1133.

[29] Karlhede, A. (1980). On a coordinate-invariant description of Riemannian manifolds. *GRG* **12**, 963.

[30] Marklund, M. and Bradley, M. (1999). Invariant construction of solutions to Einstein's field equations – LRS perfect fluids II. *CQG* **16**, 1577

[31] Herrera, L., Santos, N. O. and Skea, J. E. F. (2003). Active Gravitational Mass and The Invariant Characterization of Reissner-Nordström Spacetime. arXiv:gr-qc/0306005v1.

[32] Whittaker E. T. (1935). On the Gauss' Theorem and the concept of Mass in General Relativity. *Proc. R. Soc. Lond. A* **149**, 384.

[33] Schouten, J. A. (1925). On the Conditions of Integrability of Covariant Differential Equations, *Transactions of the American Mathematical Society* **27**, No. 4, 441.

[34] McLenaghan, R. G., Tariq, N. and Tupper, B. O. J. (1975). Conformally flat solutions of the Einstein–Maxwell equations for null electromagnetic fields. *JMP* **16**, 829.

[35] Fecko, M. (2004). *Diferenciálna geometria a Lieovy grupy pre fyzikov.* (Iris, Bratislava).