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MASTER THESIS

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Approximation of a non-increasing rearrangement of a function

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Bez cenných rad svého vedoucího profesora Luboše Picka a námětů od profesora Rona Kermana by tato práce nemohla vzniknout, za to jim patří mé velké díky.

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Název práce: Approximation of a non-increasing rearrangement of a function

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Katedra: Katedra matematické analýzy

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Abstrakt: Nerostoucí přerovnání meřitelné realné funkce definované na měřitelném prostoru má obrovský význam v takových disciplínách jako je teorie prostorů funkcí nebo teorie interpolací (mezi prostory funkcí) a jejich aplikace v parcialních diferencialních rovnicích. Ačkoliv má nerostoucí přerovnání dobré a široce uplatnitelné vlastnosti jako zobrazení, je bohužel témeř nemožné vypočítat nerostoucí přerovnání konkrétní funkce přesně. Z tohoto důvodu jsou numerické algoritmy pro aproximaci žádoucí. V této práci se budeme zabývat takovou metodou postavenou na interpolaci pomocí lineárních splinů. V první polovině této práce bude tato metoda popsána, zatímco odhady chyb budou předmětem druhé části.

Klíčová slova: nerostoucí přerovnání, approximace, metoda konečných prvků

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Abstract: The non-increasing rearrangement of a measurable real function defined on an appropriate measure space is of the enormous significance in disciplines such as theory of function spaces or interpolation theory and their applications in PDEs. Unfortunately, while it has good and widely applicable mapping properties, it is virtually impossible to calculate the non-increasing rearrangement of a concrete given function precisely. Numerical algorithms for approximation are desirable for this reason. Such method of approximation, based on interpolation by a linear spline, is presented in this thesis. In the first half of this thesis, the developed method is described, while the error estimates of the method are subject to the second part.

Keywords: non-increasing rearrangement, approximation, finite element method

Contents

Introduction

Given two finite sequences of nonnegative numbers, $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^n$, it is quite natural to say that one is a *rearrangement* of the other. The rearrangement in this case is realized via a permutation. Of particular significance and interest is the rearrangement of the finite sequence $\{a_j\}_{j=1}^n$ in a *non-increasing* order. It turns out that, if an appropriate care is exercised, one can do similar things with measurable functions even though, of course, no simple technique such as permutation is available here. Two measurable functions (not even necessarily defined on the same measure space) may be considered *equimeasurable* if they have the same distribution function (that is, the same measure of all level sets). Again, the function which is equimasurable to a given one and which is defined and non-increasing on an interval is of particular significance and interest. Such function, denoted f^*, is called the *non-increasing rearrangement* of the given function f.

The non-increasing rearrangement of a measurable real function defined on an appropriate measure space was defined and first studied as early as in 1880's by Steiner (see [11]). Then it was almost forgotten for half a century. It resurfaced again thanks to the efforts of Hardy, Littlewood and Pólya in 1930's and of Luxemburg, Lorentz and others in 1950's. It was the work of Hardy and Littlewood that first proved the enormous significance of the non-increasing rearrangement, and the rapid development of then new disciplines such as interpolation theory, function spaces, Sobolev-type inequalities and their applications in PDEs and mathematical physics only confirmed its absolute indispensability when fine and sharp description of properties of operators on function spaces was required. It also found important applications in symmetrization and isoperimetric inequalities.

The non-increasing rearrangement of a function is defined as a certain generalized inverse of the distribution function. The mapping $f \mapsto f^*$ is a rather crude operation that reduces phenomena occurring on a general measure space to the one-dimensional ones. However, while it has good and widely applicable mapping properties that lead to deep theorems concerning its action on function spaces, it is virtually impossible to calculate the non-increasing rearrangement of a concrete given function precisely. For this reason, it is quite desirable to have instead at least some numerical algorithms for approximation of f^* in order to remedy the lack of the precise formula.

Our main objective in this thesis is to develop such algorithms.

The idea of our method is to approximate the non-increasing rearrangement of a given function by the rearrangement of its linear interpolation. Basis for our work is the article [8] where our algorithm is introduced and most results originated.

The first chapter is of a preliminary nature. We introduce all the necessary background material on both rearrangement-invariant norms and spaces and the finite element method. Moreover, we specify the interpolation we use; that will take place in Section 1.4, where our method is considered within the context of general finite element methods. We also present some known results on the error of interpolation.

An outline of our algorithm and the algorithm for computing the non-increasing rearrangement of partially linear functions are described in Chapter 2. In the first three sections we focus on the case of domain of dimensions 1, 2 and 3, studying their intrinsic properties separately. In the case of one dimension splines of higher order can be used. This approach is resumed in Section 2.4. The older algorithm is decribed in Section2.5.

The last chapter contains error estimates of the described method of approximation. This chapter is divided in three sections. Supporting theorems are stated in Section 3.2, while the actual estimates are proved in Section 3.3. Yet another way of obtaining error estimates, based on the use of theorems on error of interpolation from theory of finite elements, is described in Section 3.1.

1. Preliminaries

1.1 Elementary definitions

Before we move on to a more interesting matter, let us recall some elementary definitions that will be used later.

Definition 1.1. Let (R, μ) be a measurable space. Then the set of all real-valued measurable functions from R will be denoted by $\mathfrak{M}$. The set of all measurable functions $R \to \mathbb{R}^+$ will be denoted by $\mathfrak{M}_+$.

Definition 1.2 (Multi-index)**.** Let be $n \in \mathbb{N}$, then a finite sequence of integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathbb{N}_0)^n$ is called a *multi-index.* The length of α is given by

$$|\alpha| := \sum_{i=1}^{n} \alpha_i.$$

Given an $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we define

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

We use the notation

$$D^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f,$$

for f sufficiently smooth. If $k \in \mathbb{N}$, we define

$$D^k f = \sum_{|\alpha|=k} D^\alpha f.$$

Notation 1.3 (Sobolev pseudonorm)**.** Let $f \in L_{1,\mathrm{loc}}(\Omega)$, $k \in \mathbb{N}$, $p \in [1, \infty]$. Suppose that the weak derivatives $D^\alpha f$ exist for all $|\alpha| = k$. Define the *Sobolev pseudonorm*

$$|f|_{k,p,\Omega} = \begin{cases} \left(\int_\Omega \sum_{|\alpha|=k} |D^\alpha f(x)|^p \, dx\right)^{\frac{1}{p}} & p < \infty, \\ \max_{|\alpha|=k} \operatorname{ess\,sup}_{x\in\Omega} |D^\alpha f| & p = \infty, \end{cases}$$

if the right-hand side of the equation exists.

Definition 1.4. The space of all polynomials on a set $E \subset \mathbb{R}$ of degree up to k will be denoted as $P_k(E)$, in other words

$$P_k(E) = \left\{ \sum_{|\alpha| \le k} c_\alpha x^\alpha;\; c_\alpha \in \mathbb{R} \right\}.$$

Notation 1.5. The measure of the unit ball in $\mathbb{R}^n$ will be denoted by γ_n. Then

$$\gamma_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)},$$

where Γ is the Gamma function.

Definition 1.6 (Domain)**.** An open connected subset of $\mathbb{R}^n$ will be called a *domain.*

Definition 1.7 (Simplex)**.** Let $a_1, a_2, \dots, a_{n+1} \in \mathbb{R}^n$, $a_j = \{a_{ij}\}_{i=1}^n$ be points in $\mathbb{R}^n$ such that the matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} & a_{n,n+1} \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

is regular, then the convex hull of the points $a_1, \dots, a_{n+1}$,

$$T = \left\{ x \in \mathbb{R}^n : x = \sum_{j=1}^{n+1} \lambda_j a_j \,, \lambda_j \in [0,1], \ \sum_{j=1}^{n+1} \lambda_j = 1 \right\}$$

is called a *simplex* in $\mathbb{R}^n$. Points $a_1, a_2, \dots, a_{n+1}$ are called vertices of simplex T. We say that the set

$$F_I = \left\{ x \in T : x = \sum_{j=1}^{n+1} \lambda_j a_j \,, \lambda_j \in [0,1], \ \sum_{j=1}^{n+1} \lambda_j = 1, \ \lambda_i = 0, \ \text{ for } \ i \in I \right\},$$

where I is a subset of $\{1, \dots, n\}$ of cardinality $m+1$, is an m-facet of the simplex T.

Remarks 1.8. Although this notion is well known, let us recall couple of facts about simplices.

- Simplices in the one-dimensional space $\mathbb{R}$ are closed intervals. Simplices in $\mathbb{R}^2$ are called triangles and a simplex in a three dimensional space $\mathbb{R}^3$ is called a tetrahedron.
- 0-facets are vertices, 1-facets are called edges and $(n-1)$-facets of a simplex in $\mathbb{R}^n$ are often called faces of simplex.
- Let T be a simplex in $\mathbb{R}^n$ with vertices $a_1, \dots, a_{n+1}$. If we denote $A \in \mathbb{R}^{n \times n}$ a matrix whose columns are vectors $a_i - a_{n+1}$, $i = 1, \dots, n$, then

$$|T| = \frac{1}{n!} \left| \det (A) \right|.$$

1.2 Non-increasing rearrangement

We start with focusing on the notion of the non-increasing rearrangement of a function. To illustrate the idea of this notion let us consider a step function $f\colon \mathbb{R}^+ \to \mathbb{R}$ defined as follows

$$f(x) = \sum_{i \in \{1,2,\dots,k\}} a_i \, \chi_{[i-1,i)}(x),$$

where $a_i \in \mathbb{R}^+$ and $a_i \neq a_j$ for $i, j \in \{1, 2, \dots, k\}$, $j \neq i$. The non-increasing rearrangement, f^*, is then a very similar step function, for which it holds that $x > y$ implies $f^*(x) \leq f^*(y)$. In principle, we just sort the values of f in non-increasing order. If $\pi : \{1, \dots, k\} \to \{1, \dots, k\}$ is a permutation such that it holds that if $l > s$, then $a_{\pi(l)} < a_{\pi(s)}$, then f^* could be written in the form

$$f^*(x) = \sum_{i \in \{1,\dots,k\}} a_{\pi(i)} \, \chi_{[i-1,i)}(x).$$

However this approach fails in more complex cases or higher dimensions. Therefore the non-increasing rearrangement is defined through a distribution function in way showed in the following definition.

Definition 1.9 (distribution function, non-increasing rearrangement)**.** Let E be a measurable subset of $\mathbb{R}^n$ and suppose $f\colon E \to \mathbb{R}$ is a measurable function. Then, the *distribution function*, μ_f, of f is given by

$$\mu_f(\lambda) := \left| \{ x \in E : |f(x)| > \lambda \} \right|,$$

where $\lambda \geq 0$. The non-increasing rearrangement, f^*, of f is defined on interval $(0, |E|)$ by

$$f^*(t) := \inf\{ \lambda \in \mathbb{R} : \lambda \geq 0, \mu_f(\lambda) \leq t \},$$

where $0 < t < \infty$. In particular, $\mu_{f^*} = \mu_f$.

Some basic properties of the non-increasing rearrangement are summarized in the following proposition.

Proposition 1.10. *Suppose f, g, and f_i, $i \in \mathbb{N}$, belong to $\mathfrak{M}(R, \mu)$ and let $a \in \mathbb{R}$. The decreasing rearrangement f^* is a nonnegative, decreasing right-continuous function on $[0, \infty)$. Furthermore,*

$$\begin{aligned} |g| \leq |f| \ \mu\text{-a. e. implies that } \ g^* &\leq f^*; \\ (af)^* &= |a| \, f^*; \\ (f+g)^*(t_1 + t_2) &\leq f^*(t_1) + g^*(t_2), \ (t_1, t_2 \geq 0); \\ |f| \leq \liminf_{n\to\infty} |f_n| \ \mu\text{-a. e. implies that } \ f^* &\leq \liminf_{n\to\infty} f_n^*; \end{aligned}$$

in particular,

$$\begin{aligned} |f_n| \uparrow |f| \ \ \mu\text{-a. e. implies that } \ f_n^* \uparrow f^*; \\ f^*(\mu_f(\lambda)) \le \lambda, \ (\mu_f(\lambda) < \infty); \\ \mu_f(f^*(t)) \le t, \ (f^*(t) < \infty); \\ f \ \text{ and } \ f^* \ \text{ are equimeasurable;} \\ (|f|^p)^* = (f^*)^p, \ (0 < p < \infty). \end{aligned}$$

Proof. The proof can be found in [2, Chapter 2, Proposition 1.7]. □

One of the basic tools to work with non-increasing rearrangements is the following theorem.

Theorem 1.11. *Let $\Omega \subset \mathbb{R}^n$, $f, g \in \mathfrak{M}(\Omega)$, then*

$$\int_\Omega |f(x)g(x)| \ dx \le \int_0^{|\Omega|} f^*(t)g^*(t) \ dt. \tag{1.1}$$

Proof. This is Theorem 2.2 from [2, Chapter 2]. □

We will often use that, in some sense, rearrangement is a non-expansive mapping with respect to Lebesgue norms, which is a consequence of the following result.

Theorem 1.12. *Let $\Phi : [0, \infty) \to \mathbb{R}$ be convex, non-negative, non-decreasing, and $u, v \in \mathfrak{M}(\mathbb{R}^n)$. Then*

$$\int_0^\infty \Phi\left(|u^*(s) - v^*(s)|\right) \ ds \le \int_{\mathbb{R}^n} \Phi\left(\big||u(x)| - |v(x)|\big|\right) \ dx.$$

Proof. This result is the main theorem in [4], the proof can be found there. □

Corollary 1.13. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $f, s \in L_p(\Omega)$, $p \in [1, \infty]$, then it holds that*

$$\|f^* - s^*\|_p \le \|f - s\|_p \, .$$

Proof. The case $p = \infty$ will be discussed in Remark 1.22 below. In case $p \in [1, \infty)$ we will use Theorem 1.2. First, we note that the function

$$C(t) := t^p,$$

where $p \in [1, \infty)$ and $t \geq 0$, is non-decreasing, non-negative and convex, hence it can play the role of the function Φ in Theorem 1.2. Let us define for $f \in \mathfrak{M}(\Omega)$ the zero extension as

$$f_0(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Now, we use Theorem 1.2 to obtain

$$\begin{aligned} \|f^* - s^*\|_{p,\Omega}^p &= \|f_0^* - s_0^*\|_{p,\mathbb{R}^n}^p = \int_0^\infty (|f_0^*(t) - s_0^*(t)|)^p \ dt \\ &\leq \int_{\mathbb{R}^n} (||f_0(x)| - |s_0(x)||)^p \ dx \\ &\leq \int_\Omega (||f(x)| - |s(x)||)^p \ dx \\ &\leq \int_\Omega (|f(x) - s(x)|)^p \ dx = \|f - s\|_{p,\Omega}^p . \end{aligned}$$

□

1.3 Rearrangement invariant spaces

The notion of the non-increasing rearrangement allows us to study a new class of spaces, the rearrangement invariant Banach function spaces. It is interesting that this notion covers the similarities between all Lebesgue spaces, Orlicz spaces and Lorentz spaces.

Definition 1.14 (rearrangement invariant norms)**.** A *rearrangement invariant Banach function norm*, ϱ, on the set, $\mathfrak{M}(E)$, of measurable function on the $E \subset \mathbb{R}^n$ satisfies the following seven axioms:

(A1) $\varrho(f) = \varrho(|f|) \geq 0$ with $\varrho(f) = 0$ if and only if $f = 0$ a. e. on E;

(A2) $\varrho(cf) = |c| \, \varrho(f)$, $c \in R$;

(A3) $\varrho(f + g) \leq \varrho(f) + \varrho(g)$;

(A4) $f_n \uparrow f$ implies $\varrho(f_n) \uparrow \varrho(f)$, $\{f_n\}$ is a sequence in $\mathfrak{M}(E)$;

(A5) $\varrho(\chi_F) < \infty$ for all $F \subset E$, $|F| < \infty$, where χ_F denotes the characteristic function of F;

(A6) $\int_F |f(x)| \ dx \leq C_F \varrho(f)$, for all measurable $F \subset E$, with $|F| < \infty$, where C_F is independent of f;

(A7) $\varrho(f) = \varrho(g)$ whenever $f^* = g^*$,

where $f, g \in \mathfrak{M}(E)$. Usually, the rearrangement invariant Banach function norms are called just r. i. norms. The collection of functions from $\mathfrak{M}(E)$ such that $\varrho(f) < \infty$ is called a *rearrangement invariant Banach function space* and we denote it as $X_\varrho(E)$. For $f \in X_\varrho(E)$ we define the norm

$$\|f\|_{X_\varrho} = \varrho(f).$$

Luxemburg has shown that corresponding to any r. i. norm ϱ on $\mathfrak{M}(E)$ there is a r. i. norm, $\overline{\varrho}$, on $\mathfrak{M}\,(0, |E|)$ for which

$$\varrho(f) = \overline{\varrho}(f^*),\ f \in \mathfrak{M}(E).$$

The classical Lebesgue norm, $\|.\|_p$, is rearrangement invariant since

$$\|f\|_p := \left[\int_E |f(x)|^p \ dx\right]^{\frac{1}{p}} = \left[\int_0^{|E|} (f^*(t))^p \ dt\right]^{\frac{1}{p}},\ 1 \le p < \infty,$$

and

$$\|f\|_\infty := \operatorname*{esssup}_{x \in E} |f(x)| = f^*(0+).$$

Observe that, even if E is a complicated domain, the second integral can be readily calculated once an approximation to f^* is known. The same is true for the norm of the more general Orlicz gauge norm, $\|.\|_\Phi$, which we now define.

Definition 1.15 (Young function, Orlicz gauge norm)**.** Let $E \subset \mathbb{R}^n$ and suppose Φ is a *Young function*, that is,

$$\Phi(x) := \int_0^t \phi(s)\ ds,$$

where $s > 0$ and $\phi(t)$ is increasing, $\phi(0+) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. Then, if $f : E \to \mathbb{R}$ is measurable, its Orlicz gauge norm is

$$\|f\|_\Phi := \inf\left\{\lambda > 0 : \int_E \Phi\left(\frac{|f(x)|}{\lambda}\right)\ dx \le 1\right\}.$$

One can show that

$$\|f\|_\Phi = \varrho_\Phi(f) = \overline{\varrho_\Phi}(f^*) = \inf\left\{\lambda > 0 : \int_E \Phi\left(\frac{|f|\,(t)}{\lambda}\right)\ dt \le 1\right\}.$$

Remark 1.16. We remark that if $\Phi_p(t) = t^p$, $1 \le p < \infty$, then

$$\|f\|_{\Phi_p} = \|f\|_p \, .$$

A more recent generalization of $\|.\|_p$ is Lorentz norm defined in the following definition.

Definition 1.17 (Lorentz norm)**.** Let $E \subset \mathbb{R}^n$, $p, q \in (0, \infty]$, then the Lorentz norm is given by

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^{|E|} \left(f^*(t) \cdot t^{\frac{1}{p}}\right)^q \cdot \frac{1}{t}\, dt\right)^{\frac{1}{q}}, & \text{if } 0 < q < \infty, \\ \sup_{0<t<\infty} t^{\frac{1}{p}} f^*(t), & \text{if } q = \infty. \end{cases}$$

Similar to an associated Lebesgue spaces L_p and $L_{p'}$ there exists associated pairs of r. i. spaces.

Definition 1.18 (Associated norm)**.** If ϱ is a function norm, its associate norm ϱ' is defined on $\mathfrak{M}(E)$ by

$$\varrho'(f) = \sup_{g \in \mathfrak{M}(E): \varrho(g) \le 1} \left(\int_R fg \, d\mu\right), \ f \in \mathfrak{M}(E).$$

If ϱ is a function norm, then the associated function norm ϱ' is itself a function norm. Let us note that $\varrho'' = \varrho$. The Lebesgue function norm $\|.\|_p$ has the associated function norm $\|.\|_{p'}$ if $p \in (1, \infty)$, where p' is such number that

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The associated norm to $\|.\|_1$ is $\|.\|_\infty$ and vice versa.

Let $1 < p < \infty$ and $1 \le q \le \infty$, then the associated space to the Lorentz space $L_{p,q}$, is the Lorentz space $L_{p',q'}$ where p' and q' satisfy

$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ and } \frac{1}{q} + \frac{1}{q'} = 1.$$

Let us note that in r. i. spaces we have the Hölder inequality of the following form.

Theorem 1.19. *Let X be a r. i. space with associated space X'. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int |fg| \le \|f\|_X \|g\|_{X'} \, .$$

Proof. This is basically Theorem 2.4 in [2, Chapter 1]. □

The basic tool for working with r. i. norms is the so-called Hardy-Littlewood-Pólya Principle.

Theorem 1.20. *Let $E \subset \mathbb{R}^n$ be a measurable set and suppose that f and g are locally-integrable on E. Then, for any r. i. norm ϱ on $\mathfrak{M}(E)$, one has*

$$\int_0^t f^*(s)\,ds \leq \int_0^t g^*(s)\,ds,\ 0 < t < |E|\,,$$

implies

$$\varrho(f) \leq \varrho(g).$$

Proof. This is Theorem 4.2 from [2, Chapter 2]. □

We will apply the Hardy-Littlewood-Pólya Principle to the following inequality.

Theorem 1.21. *As before, let $E \subset \mathbb{R}^n$ be measurable and suppose f and g are integrable functions on E. Then,*

$$\int_0^t (f^* - g^*)^*(s)\,ds \leq \int_0^t (f - g)^*(s)\,ds, \qquad 0 < t < |E|\,.$$

Proof. The proof can be found in [10]. □

Remark 1.22. The previous two theorems ensure that

$$\|f^* - s^*\|_\infty \leq \|(f - s)^*\|_\infty = \|f - s\|_\infty\,.$$

Definition 1.23. Let (R, μ) be a totally σ-finite measure space.

- The space $L_1 + L_\infty$ consists of all functions f in $\mathfrak{M}(R, \mu)$ that are representable as a sum $f = g + h$ of functions $g \in L_1$ and $h \in L_\infty$. For each $f \in L_1 + L_\infty$, let

  $$\|f\|_{L_1+L_\infty} = \inf\left\{\|g\|_{L_1} + \|h\|_{L_\infty}\right\},$$

  where the infimum is taken over all representations $f = g + h$ of the kind described above.

- For each f in the intersection $L_1 \cap L_\infty$ of L_1 and L_∞, let

  $$\|f\|_{L_1 \cap L_\infty} = \max\left\{\|f\|_{L_1}, \|f\|_{L_\infty}\right\}.$$

Because it will be useful later, let us recall the following interesting fact. If the (R, μ) is a non-atomic, σ-finite measurable space and X is an arbitrary rearrangement-invariant Banach function space over (R, μ), then

$$L_1 \cap L_\infty \hookrightarrow X \hookrightarrow L_1 + L_\infty.$$

This is proved in Theorem 6.6 in [2, Chapter 2].

It is possible to create a similar construction to that of Sobolev spaces over Lebesgues spaces. This generalization is showed in the following definition.

Definition 1.24. Let $\Omega \subset \mathbb{R}^n$ be an open domain and let $X(\Omega)$ be a r. i. space, we define the following Sobolev type spaces:

$$V^1X(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is weakly differentiable on } \Omega \text{ and } |\nabla u| \in X(\Omega)\},$$

$$W^1X(\Omega) = X(\Omega) \cap V^1X(\Omega)$$

and

$$V_0^1X(\Omega) = \{u\colon \Omega \to \mathbb{R} : u_0 \text{ is weakly differentiable on } \mathbb{R}^n \text{ and } |\nabla u| \in X(\Omega)\},$$

where

$$u_0(x) = \begin{cases} u(x), & \text{for } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover,

$$W_0^1X(\Omega) = X(\Omega) \cap V_0^1X(\Omega).$$

These spaces are equipped with the following norm

$$\|u\| = \|u\|_{X(\Omega)} + \big\||\nabla u|\big\|_{X(\Omega)}.$$

1.4 The finite element method

Before we can present the discussed algorithm, we need to build basics of theory of the finite element methods. Doing that, we will follow the approach of the book [3]. First, we define a finite element.

Definition 1.25 (finite element)**.** Let

1. $\mathcal{K} \subset \mathbb{R}^n$ be a domain with piecewise smooth boundary,
2. $\mathcal{P}$ be a finite-dimensional space of functions on $\mathcal{K}$ and
3. $\mathcal{N} = \{N_1, N_2, \ldots, N_k\}$ be a basis for a dual space $\mathcal{P}^*$.

Then $(\mathcal{K}, \mathcal{P}, \mathcal{N})$ is called a *finite element.*

Remark 1.26. It is implicitly assumed that the elements of $\mathcal{N}$ belong to the dual space of some larger function space.

Definition 1.27. Let $(\mathcal{K}, \mathcal{P}, \mathcal{N})$ be a finite element, and let $\{\phi_1, \phi_2, \dots, \phi_k\}$ be the basis for $\mathcal{P}$ such that $N_i(\phi_j) = \delta_{ij}$, $N_i \in \mathcal{N}$, then it is called the *nodal basis* for $\mathcal{P}$ or a dual base to $\mathcal{N}$.

Definition 1.28 (Local interpolant)**.** Given a finite element $(\mathcal{K}, \mathcal{P}, \mathcal{N})$, let the set $\{\phi_i : 1 \leq i \leq k\} \subset \mathcal{P}$ be the basis dual to $\mathcal{N}$. If v is a function for which all $N_i(v)$, $i = 1, \dots, d$, are defined, then we define the *local interpolant* by

$$\Pi_{\mathcal{K}}\, v := \sum_{i=1}^{k} N_i(v)\phi_i.$$

The most frequent finite elements are the triples of a simplex, some subspace of polynomials and a functional which appoints to a function it's value or a derivative at a given point. This is also the case of algorithms discussed in this thesis.

Our method of approximation will be based on linear Lagrange elements. The finite element method is usually based on a collection of finite elements which locally determines approximation of a problem and it's solution is taken to be approximation of the original solution. The following definitions specify the relationship of domains of finite elements used in our algorithm.

Definition 1.29 (linear Lagrange element)**.** Let T be a simplex in $\mathbb{R}^n$ with vertices $a_1, a_2, \dots, a_{n+1}$ and let us denote $N_1 = \{F_1, F_2, \dots, F_{n+1}\}$, where $F_i(f) = f(a_i)$ for any (continuous) $f\colon T \to \mathbb{R}$. Then the finite element (T, P_1, N_1) is called a *linear Lagrange element.*

Usually all the finite elements used to solve a problem are similar and differ only in the domains. Therefore it is useful to introduce a referencing simplex and define on it a similar finite element, so algorithms, properties and estimates can be described on an example of this particular finite element.

Definition 1.30. Let $T \subset \mathbb{R}^n$ be a simplex with diameter d, let assign $V_0 = [0, 0, \dots, 0]$ and $V_i = d \cdot e_i$, where $e_i = [0, \dots, 0, 1, 0, \dots, 0]$ has a non-zero value in the i-th coordinate. Then the *referencing simplex*, T_d, of the simplex T is the simplex with vertices $V_0, V_1, \cdots, V_n$. There are $n+1$ affine transformations which map simplex T_d on T. These affine transformations will be called *referencing tranformations.*

Definition 1.31 (triangulation)**.** Let $\Omega \subset \mathbb{R}^n$. Then a finite collection of subsets of Ω, $\mathcal{T}$, is called a *triangulation* if it meets the following conditions:

($\mathcal{T}1$) Each set $T \in \mathcal{T}$ is closed, connected and has non-empty interior.

($\mathcal{T}2$) The boundary, ∂T, of each $T \in \mathcal{T}$ is Lipschitz-continuous.

($\mathcal{T}3$) It holds that
$$\overline{\Omega} = \cup_{T \in \mathcal{T}} \overline{T}.$$

($\mathcal{T}4$) Intersection of interiors of any two sets in $\mathcal{T}$ is empty.

If all sets in $\mathcal{T}$ are simplices, then $\mathcal{T}$ can be called a simplex triangulation.

Definition 1.32 (conforming triangulation)**.** Let $\Omega \subset \mathbb{R}^n$ be a polygonal domain. Let $\mathcal{T}$ be a simplex triangulation. Then $\mathcal{T}$ is called a *conforming triangulation* if the following condition holds.

($\mathcal{T}5$) If $I = T_i \cap T_j$, then I is an m-facet of T_i and also of T_j for some $m \in \{0, \ldots, n-1\}$.

Now we can state the main definition of this section.

Definition 1.33 (finite element space)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary and let $\mathcal{T}$ be a triangulation of domain Ω which fulfil conditions ($\mathcal{T}1$) - ($\mathcal{T}4$). To all sets $T \in \mathcal{T}$ let there be assigned a finite element $(T, \mathcal{P}_T, \mathcal{N}_T)$. Let $T, T' \in \mathcal{T}$ a $\phi \in \mathcal{N}_T, \phi' \in \mathcal{N}_{T'}$. Then ϕ and ϕ' are equivalent if
$$\phi\left(v|_T\right) = \phi'\left(v|_{T'}\right)$$
holds for all $v \in \mathcal{C}^\infty\left(\mathbb{R}^n\right)$. Suppose that for all $T \in \mathcal{T}$, the functionals in $\mathcal{N}_T$ are pairwise non-equivalent. Therefore the set $\cup_{T \in \mathcal{T}} \mathcal{N}_T$ can be divided into subsets $\mathcal{N}_{h,1}, \mathcal{N}_{h,2}, \ldots, \mathcal{N}_{h,N}$, where $\mathcal{N}_{h,i}$ contains equivalent functionals and any two equivalent functionals from $\cup_{T \in \mathcal{T}} \mathcal{N}_T$ are in the same set $\mathcal{N}_{h,i}$. Let us introduce for $i = 1, 2, \ldots, N$ the symbol
$$\mathcal{T}_{h,i} = \left\{T \in \mathcal{T} : \mathcal{N}_{h,i} \cap \mathcal{N}_T \neq \emptyset\right\}.$$
Under these assumptions the set $\mathcal{N}_{h,i} \cap \mathcal{N}_T$ contains exactly one functional for a chosen $T \in \mathcal{T}_{h,i}$, we will denote this functional by $\phi_{T,i}$. Let us define the finite element space corresponding to the finite element domain $(\Omega, \mathcal{T}, \{(T, \mathcal{P}_T, \mathcal{N}_T); T \in \mathcal{T}\}$ as follows:
$$X = \left\{v_h \in L_2\left(\Omega\right) : v_k|_T \in P_T;\ \phi_{K,i}\left(v_h|_K\right) = \phi_{K',i}\left(v_h|_{K'}\right)\right\},$$
where $T \in \mathcal{T}$ and $K, K' \in \mathcal{T}_{h,i}$ for $i = 1, 2, \ldots, N$.

Definition 1.34. Let $P \subset \mathbb{R}^n$ be a polyhedron, let $\mathcal{T}$ be a conforming simplex triangulation of P. Let us assign to each $T \in \mathcal{T}$ a linear Lagrangian finite element and let X_h be the finite element space which corresponds to $\mathcal{T}$ and assigned finite elements. Then X_h is called a *Lagrangian finite element space* which corresponds to the triangulation $\mathcal{T}$.

Definition 1.35 (Finite element domain)**.** Let $P \subset \mathbb{R}^n$ be a polygon domain and let $\mathcal{T}$ be a triangulation of P which fulfills conditions $(\mathcal{T}1)$ – $(\mathcal{T}5)$. To all sets $T \in \mathcal{T}$ let there be assigned a finite element $(T, \mathcal{P}_T, \mathcal{N}_T)$. Then for sake of brevity let us call the domain P equipped by triangulation $\mathcal{T}$ and assigned finite elements a *finite element domain*. By simplex linear Lagrange finite element domain we mean a finite element domain such that the triangulation $\mathcal{T}$ contains only simplices and all assigned finite elements are linear Lagrange finite elements.

The following definition describes how an approximation from the finite elements space is assigned to a given function.

Definition 1.36 (interpolation operator)**.** Let $(\Omega, \mathcal{T}, \{(T, \mathcal{P}_T, \mathcal{N}_T)\,; T \in \mathcal{T}\})$ be a finite element domain and let $Q(\Omega)$ be a space of all real functions defined on $\overline{\Omega}$, such that for all $T \in \mathcal{T}$ all functionals from $\mathcal{N}_T$ are defined on $Q(\Omega)|_T$. The *interpolation operator* $\Pi_\Omega \colon Q(\Omega) \to X$ assigns to $v \in Q(\Omega)$ a function $\Pi_\Omega\, v \in X$ such that $\phi_{T,i}(v|_T) = \phi_{T,i}\left((\Pi_\Omega\, v)|_T\right)$ for all $T \in \mathcal{T}$ and $i \in \{1, \ldots, N\}$.

Remark 1.37. Let Ω be a finite element domain with triangulation $\mathcal{T}$ and finite elements $\{(T, \mathcal{P}_T, \mathcal{N}_T); T \in \mathcal{T}\}$. Let X be the corresponding finite element space. Because $\Pi_\Omega\, f$ is in X and therefore in $L_2(\Omega)$, formally speaking, $\Pi_\Omega\, f$ is a class of functions such that they are equal to each other almost everywhere. This is useful because of non-emptiness of intersection of some simplices in triangulation, where a problem could occur with the ambiguity of value of $\Pi_\Omega\, f$. However, to simplify our argumentation, let us assume that for all $x \in \Omega$ holds

$$\Pi_\Omega\, f(x) \in \cup_{T \in \mathcal{T} : x \in T} \left\{P(x) : P \in \mathcal{P}_T \text{ such that } \phi_j(P) = \phi_j(f),\, \forall \phi_j \in \mathcal{N}_T\right\}.$$

In other words, the values of $\Pi_\Omega\, f$ are spanned only from “suggestions” of involved finite elements. If we say that $\Pi_\Omega\, f$ is continuous, we mean that there exists a continuous representative in the class $\Pi_\Omega\, f$.

Lemma 1.38. *Let P be a linear Lagrange finite element domain with triangulation $\mathcal{T} = \{T_i\}_{i \in \mathbb{I}}$. Let $f \in \mathbf{C}(P)$. Then $s = \Pi_P\, f$ is continuous, too.*

Proof. Let be $x \in P$, we prove that s is continuous in x. Our proof splits in the following two cases.

- If x is in the interior of some $T \in \mathcal{T}$, then s is continuous in x because it is continuous on some neighborhood of x because it is linear on T.

- Let x be in the boundary of some $T_i \in \mathcal{T}$. Then, thanks to ($\mathcal{T}5$), it must lie in an s-facet , $0 \leq s \leq n-1$, common to simplices $T_j, j \in J \subset \mathbb{I}$. Let us denote that s-facet as F. Any linear function on s-simplex is defined by prescribed values on vertices of that simplex. Because all simplices T_j, $j \in J$, contain vertices of F and the linear function $s \restriction_{T_j}$ has values on F determined by values of f in vertices of F, it holds that $s \restriction_{T_j} (x) = s \restriction_{T_k} (x)$, for all $j, k \in J$. Using the classical Heine's theorem we can prove that for any sequence $\{x_k\}_{k=1}^{\infty}, x_k \in P$, for all $k \geq 1$ such that $\lim_{k\to\infty} x_k = x$ it hodls that $\lim_{k\to\infty} s(x_k) = x$. Let us just consider maximal subsequences of x_k denoted $\{x_{t_j}\}_{t_j=1}^{\infty}$ such that $x_{t_j} \in T_j$, for all $t_j \geq 1$, for $j \in J$. Then to a given $\varepsilon > 0$ we can found a $m_j \in \mathbb{N}$ such that $s(x_{t_j}) - s(x) < \varepsilon$ for all $t_j > m_j$. We set $m = \max_{j\in J} m_j$. Then $s(x_k) - s(x) < \varepsilon$ for all $k > m$ because $x_k \subset \cup_{j\in J} \cup_{t_j \geq 1} \{x_{t_j}\}$. Therefore s is continuous in x. □

We can describe our algorithm for approximation of the non-increasing rearrangement of a function in the following way. Let $\Omega \subset \mathbb{R}^n$ be a polyhedron. Let $f \colon \overline{\Omega} \to \mathbb{R}$. We approximate the rearrangement of the original function f by the rearrangement of the function $\Pi_\Omega f$ from a finite element space corresponding to some simplex-triangulation of domain Ω, where each simplex got assigned a linear Lagrange finite element.

1.5 Classical estimates on interpolation error

In this section we present some known results on bound of error of interpolation. The following two theorems are borrowed from [3].

Definition 1.39. Let $(\mathcal{K}, \mathcal{P}, \mathcal{N})$ be a finite element and let F be an affine map. The finite element $(\hat{\mathcal{K}}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ is *affine equivalent* to $(\mathcal{K}, \mathcal{P}, \mathcal{N})$ if

- $F(\mathcal{K}) = \hat{\mathcal{K}}$,

- $\left\{\hat{P} \circ F : \hat{P} \in \hat{\mathcal{P}}\right\} = \mathcal{P}$ and

- $\left\{\Phi_N : \Phi_N(f) = N(\hat{f} \circ F); f \text{ from domain space of } \hat{\mathcal{N}}; N \in \mathcal{N}\right\} = \hat{\mathcal{N}}$.

Definition 1.40. $\Omega \subset \mathbb{R}^n$ is *star-shaped with respect to* B if, for all $x \in \Omega$, the closed convex hull of $\{x\} \cup B$ is a subset of Ω.

Remark 1.41. Let us note that a simplex is star-shaped with respect to any ball it contains.

Definition 1.42. Suppose Ω has diameter d and is star-shaped with respect to a ball B, let $\mathcal{B}$ denote a set of all balls such that Ω is star-shaped with respect to them. We denote

$$\varrho_\Omega = \sup_{B \in \mathcal{B}} \operatorname{diam} B \quad \text{and} \quad h_\Omega = \operatorname{diam} \Omega,$$

then the *chunkiness parameter* of Ω is defined by

$$\gamma = \frac{h_\Omega}{\varrho_\Omega}.$$

Let Ω be a given domain and let $\{\mathcal{T}^h\}$, $0 < h \leq 1$, be a family of triangulations such that

$$\max\left(\operatorname{diam} T : T \in \mathcal{T}^h\right) \leq h \operatorname{diam} \Omega,$$

for all $h \in (0,1]$. If, for such a family, there exists a $\sigma > 0$ such that for all $T \in \mathcal{T}^h$ and for $h \in (0,1]$ it holds

$$\varrho_T \geq \sigma h_T, \tag{1.2}$$

then this family is called a *non-degenerate.*

Remark 1.43. The family $\mathcal{T}^h$ is non-degenerate if and only if the chunkiness parameter is uniformly bounded for all $T \in \mathcal{T}^h$ and for all $h \in (0,1]$.

To a given family of triangulations we can create a family of finite element domains by assigning to all sets in each triangulation a finite element. Such a family we will call a family of finite element domains.

Theorem 1.44. *Let* $(\mathcal{K}, \mathcal{P}, \mathcal{N})$ *be a finite element satisfying*

1. $\mathcal{K}$ *is a star-shaped with respect to some ball,*
2. $P_{m-1} \subset \mathcal{P} \subset W_{m,\infty}(\mathcal{K})$ *and*
3. $\mathcal{N} \subset \left(\mathbf{C}^l\left(\overline{K}\right)\right)^*$.

Suppose $1 \le p \le \infty$ *and either* $m - l - \frac{n}{p} > 0$ *when* $p > 1$ *or* $m - l - n \ge 0$ *when* $p = 1$*. Then for* $0 \le i \le m$ *nad* $v \in W_{m,p}(\mathcal{K})$ *we have*

$$|v - \Pi_{\mathcal{K}} v|_{i,p,\mathcal{K}} \le C_{m,n,\gamma,\sigma(\hat{\mathcal{K}})} (\operatorname{diam} \mathcal{K})^{m-i} |v|_{m,p,\mathcal{K}},$$

where $\hat{\mathcal{K}} = \left\{ \frac{1}{\operatorname{diam} \mathcal{K}} x : x \in \mathcal{K} \right\}$*,* γ *is the chunkiness parameter for* $\mathcal{K}$ *and* $\sigma K = \|\Pi_{\mathcal{K}} v\|_{\mathcal{L}(\mathbf{C}^l(\overline{K}), W_{m,p}(K))}$*.*

Proof. Proof can be found in [3, Theorem 4.4.4]. □

Theorem 1.45. *Let* $\{\mathcal{T}^h\}$*,* $0 < h \le 1$*, be a non-degenerate family of subdivisions of a polyhedral domain* $\Omega \in \mathbb{R}^n$*. Let* $(\mathcal{K}, \mathcal{P}, \mathcal{N})$ *be a reference element, satysfying the conditions of Theorem 1.44 for some* l*,* m *and* p*. For all* $T \in \mathcal{T}^h$*,* $0 < h \le 1$*, let* $(T, \mathcal{P}_T, \mathcal{N}_T)$ *be the affine-equivalent element. Then there exists a positive constant* C *depending on the reference element,* n*,* m*,* p *and the number* σ *in (1.2) such that for* $0 \le s \le m$*,*

$$\left(\sum_{T \in \mathcal{T}^h} \|v - \Pi_{\mathcal{T}^h} v\|_{s,p,T}^p \right)^{\frac{1}{p}} \le C h^{m-s} |v|_{m,p,\Omega}$$

for all $v \in W_{m,p}(\Omega)$*, where the left hand side should be interpreted, in the case* $p = \infty$*, as* $\max_{T \in \mathcal{T}^h} \|v - \Pi_{\mathcal{T}^h} v\|_{s,\infty,T}$*. For* $0 \le s \le l$*,*

$$\max_{T \in \mathcal{T}^h} \|v - \Pi_{\mathcal{T}^h} v\|_{s,\infty,T} \le C h^{m-s-\frac{n}{p}} |v|_{m,p,\Omega} \ \forall v \in W_{m,p}(\Omega).$$

Proof. Proof can be found in [3], Theorem 4.4.20. □

Remark 1.46. Let us examine the validity of assumptions of the preceding theorem in the case of a family which contains only simplex linear Lagrangian finite element domains.

In that case all assigned finite elements are affine equivalent to one linear Lagrange element (T, P_1, N_1), where T is a simplex. It is clear that (T, P_1, N_1) satisfies the first assumption from Theorem 1.44, the second condition is fulfilled only for $m \le 2$ and the third condition is fulfilled for any $l > 0$. This leaves us with $p \in (n, \infty]$ for $n > 1$. Working with the simple linear Lagrange finite elements, we can apply Theorem 1.45 with $s = 0, 1$, $m = 2$ and $p \in (n, \infty]$ provided that our family of triangulations is non-degenerated.

2. Description of the algorithm

To shortly introduce the algorithm presented in this thesis, we say that it uses the Lagrange finite element method with conforming simplex triangulation to approximate the rearrangement of the original function by the rearrangement of the approximation. In this chapter, we will go through this procedure in detail. Later, we will also quickly present some results in one dimension with splines of higher order. We conclude this chapter by presentation of an older algorithm.

We will summarize the situation in which our algorithm could be used. Let $P \subset \mathbb{R}^n$, be a polyhedron, $f : P \to \mathbb{R}$ a real function. Assume also that $\mathcal{T}$ is a conforming triangulation of P. If $T \in \mathcal{T}$ is a simplex, then we denote the linear function on T which agrees with f in all vertices of T as l_T. We assign to each $T \in \mathcal{T}$ a linear Lagrangian finite element and create the finite element space X_h of functions which are linear on each $T \in \mathcal{T}$. We denote $s = \Pi_P(f)$. It is worth noting that $s|_T = l_T$ for all $T \in \mathcal{T}$. Finally we can proceed to the description of our algorithm.

In order to determine s^* it suffices to compute $\mu_s = \sum_T \mu_{l_T}$. To do this we need to measure level sets

$$E_\lambda^{l_T,T} := \{x \in T : |l_T(x)| > \lambda\}$$

for each simplex T in the decomposition of P.

Suppose the vertices $V_1, V_2, \ldots, V_{n+1}$ of T have been so ordered that $z_1 \leq z_2 \leq \cdots \leq z_{n+1}$, where $z_k = |f(V_k)|$, $k = 1, 2, \ldots, n+1$. If $\lambda \leq z_1$, then $\mu_{l_T}(\lambda) = |T|$, while if $\lambda \geq z_{n+1}$, $\mu_{l_T}(\lambda) = 0$. Suppose, then, $z_{k_0-1} \leq \lambda \leq z_{k_0}$ for some $k_0 \in \{2, \ldots, n+1\}$. In that case the hyperplane $l_T(x) = \lambda$ divides T into two polyhedra P_1 and P_2.

The next step is to decide on which part, P_1 or P_2 , is $l_T > \lambda$ almost everywhere. The following notation will be useful in more complicated cases.

Notation 2.1. Let be $a, b \in \mathbb{R}^n$ and let l be a linear function on $\mathbb{R}^n$. Let be $l(a) = C$, $l(b) = D$, $C < D$ and let $\lambda \in [C, D]$. Then there exists exactly one point $\delta_{a,b,l,\lambda}$ on the line segment between a and b such that $l(\delta_{a,b,l,\lambda}) = \lambda$. Usually it will be obvious to which linear function and λ our symbol belongs and we will write only $\delta_{a,b}$ instead of $\delta_{a,b,l,\lambda}$. It holds that

$$\delta_{a,b,l,\lambda} = a\frac{D-\lambda}{D-C} + b\frac{\lambda - C}{D-C}.$$

Motivation of the previous notation is in denoting a point on edge of simplex with given value $l_T = \lambda$.

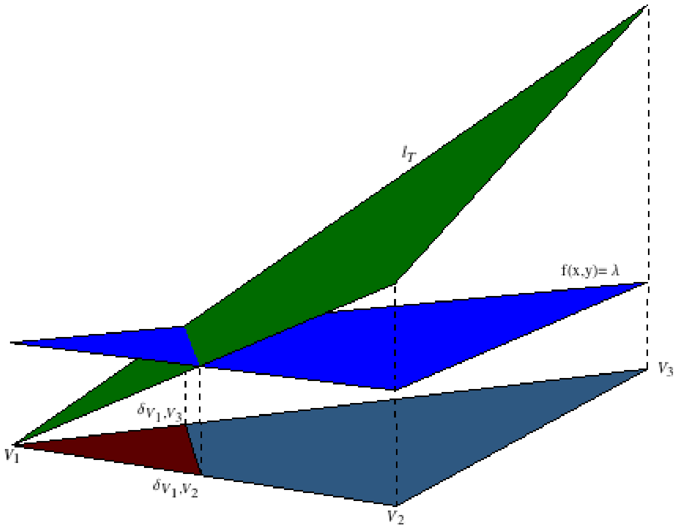

Figure 2.1: Example of a linear function on triangular domain, with $z_1 < \lambda < z_2$. The level set a gray polygon in base, the red triangle is the complement.

At this point our description of algorithm will split into cases according to dimension of domain of function. We restrict ourselves to cases $n = 1, 2, 3$. The following sections will deal with each case separately.

2.1 The case $n = 1$

However in the case $n = 1$ the situation is simpler and splines of higher order can be used (this matter is a subject of Section 2.4), let us present the algorithm with linear splines for the sake of completeness. Simplices in $\mathbb{R}$ are intervals. Let us have the interval I with endpoints x and y. We denote $a = l_I(x)$ and $b = l_I(y)$. Then the formula for distribution function of l_I is

$$\mu_{l_I}(\lambda) = \begin{cases} |x - y|, & \lambda \leq \min\{a, b\}, \\ |x - y| \frac{\max\{a,b\} - \lambda}{|b-a|}, & \min\{a, b\} < \lambda \leq \max\{a, b\}, \\ 0, & \max\{a, b\} < \lambda. \end{cases}$$

2.2 The case $n = 2$

Now, let $n = 2$. Let us leave aside degenerate cases and let us assume that $z_1 < z_2 < z_3$. When $z_1 < \lambda \leq z_2$, we are in the situation shown on Figure 2.1, in which

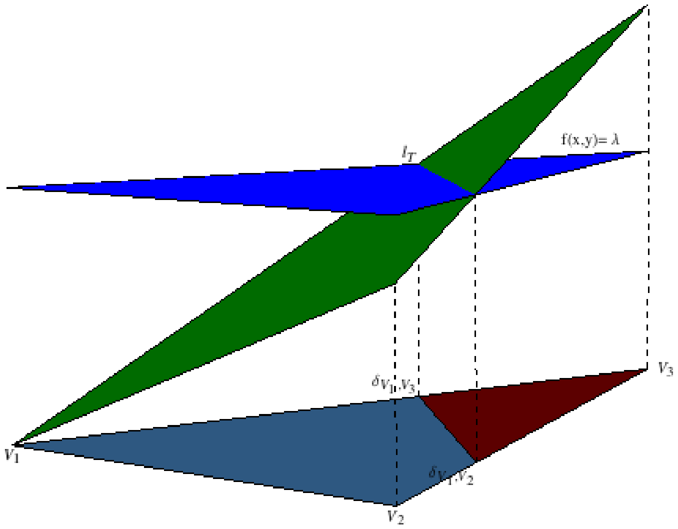

Figure 2.2: The second case, where $z_1 < \lambda < z_2$. The level set is the red triangle in base.

$$\left| E_\lambda^{l_T,T} \right| = |T| - |T_1|,$$

T_1 being the red triangle in the plane $v_1 v_2 v_3$. The case $z_2 < \lambda \leq z_3$ is pictured on Figure 2.2; here $\left| E_\lambda^{l_T,T} \right| = |T_2|$, where T_2 is the red triangle in plane $v_1 v_2 v_3$.

As is well-known, the area of a triangle can be expressed in terms of a determinant involving the coordinates of its vertices. Using this fact one can show that, on $[z_1, z_2]$, $\mu_{l_T}(\lambda)$ is a quadratic spline equal to $|T|$ at z_1 and equal to $\frac{z_3 - z_2}{z_3 - z_1} |T|$ at z_2 and to 0 at z_3, namely,

$$\mu_{l_T}(\lambda) = \begin{cases} |T|, & \lambda \leq z_1, \\ \left(1 - \frac{(\lambda - z_1)^2}{(z_2 - z_1)(z_3 - z_1)}\right) |T|, & z_1 < \lambda \leq z_2, \\ \frac{(z_3 - \lambda)^2}{(z_3 - z_2)(z_3 - z_1)} |T|, & z_2 < \lambda \leq z_3, \\ 0, & z_3 < \lambda. \end{cases}$$

2.3 The case $n = 3$

If $n = 3$ then the formula for $\mu_{l_T}(\lambda)$ will be split into five cases.

1. $\lambda \leq z_1$: This case is trivial, $\mu_{l_T}(\lambda) = |T|$.

2. $z_1 < \lambda < z_2$: The level set in this case is a polyhedron M, whose complement in T, let us call it T_M, is a tetrahedron with vertices V_1, δ_{V_1,V_2},

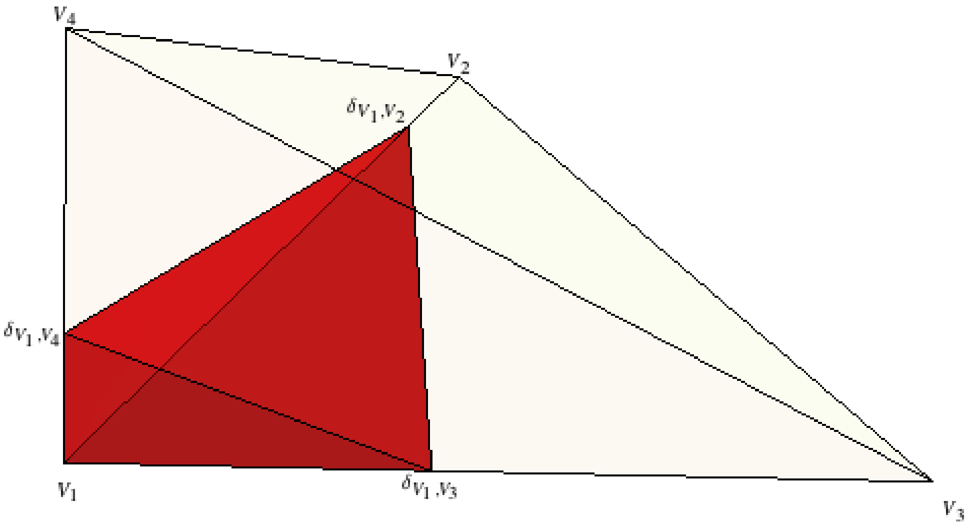

Figure 2.3: The situation in $\mathbb{R}^3$ if $z_1 < \lambda < z_2$. The red tetrahedron is the complement of the level set.

δ_{V_1,V_3} and δ_{V_1,V_4}. The situation is depicted in Figure 2.3. Therefore, we have

$$\left|E_\lambda^{l_T,T}\right| = |T| - |T_M|\,.$$

3. $z_2 < \lambda < z_3$: When λ is between z_2 and z_3 the situation is more complicated. Both the level set and its complement in T are not tetrahedra. We choose to compute the measure of a level set by dividing it into tetrahedra with intersection of measure zero. This can be done in several ways. Here we will divide this area into three tetrahedra, but in [13] could be find a decomposition into four tetrahedra with the advantage of a better proportion of tetrahedra in division. It might be worth noting that this problem is equivalent to that of dividing a triangle-based prismoid into tetrahedra. The situation and division of the level set is shown in Figure 2.4.

   The level set in this case is the polyhedron with vertices δ_{V_1,V_3}, δ_{V_1,V_4}, δ_{V_2,V_3}, δ_{V_2,V_4}, V_3 and V_4. Such a polyhedron can be divided into three tetrahedra T_R, T_G and T_B, where T_R has vertices δ_{V_2,V_3}, δ_{V_1,V_4},δ_{V_2,V_4} and V_4, the tetrahedron T_B has vertices δ_{V_2,V_3}, δ_{V_1,V_3}, δ_{V_1,V_4}, V_4 and the last tetrahedron T_G is defined by vertices δ_{V_2,V_3}, V_3, δ_{V_1,V_3} and V_4.

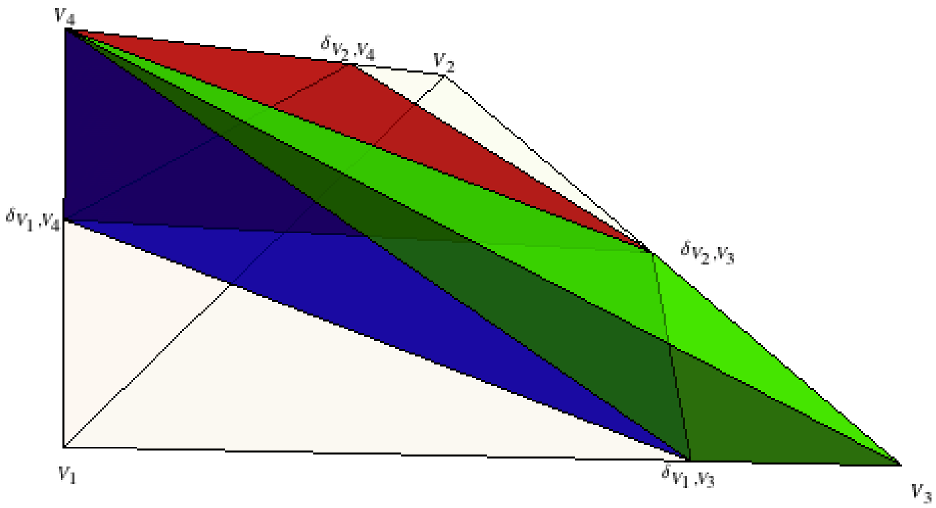

Figure 2.4: The situation if $z_2 < \lambda < z_3$, the tetrahedra T_R, T_G and T_B are the red, green and blue tetrahedra.

Hence, we get a formula

$$\left| E_\lambda^{l_T,T} \right| = |T_R| + |T_G| + |T_B| \,.$$

4. $z_3 < \lambda < z_4$: In this case the only vertex which lies in $E_\lambda^{l_T,T}$ is V_4. The plane $l_T(x) = \lambda$ cuts tetrahedron T into polyhedron P_1 with vertices $V_1, V_2,$ V_3, O_1, O_2 and O_3 and tetrahedron T_O with vertices O_1, O_2, O_3 and V_4, where points O_i, $i = 1, 2, 3$, lies on edge between vertices V_i and V_k and it holds $l_T(O_i) = \lambda$. Situation is illustrated in Figure 2.5. It holds that $l_T(x) \geq \lambda$ on T_O and therefore we get

$$\left| E_\lambda^{l_T,T}(\lambda) \right| = |T_O| \,.$$

To be specific we add vertices of T_O: δ_{V_1,V_4}, δ_{V_2,V_4}, δ_{V_3,V_4}, V_4.

5. $z_4 \leq \lambda$: Finally, volume of the level set in this case is zero, hence $\mu_{l_T}(\lambda) = 0$.

The previous paragraph shows how to convert the problem of computing a distribution function of a linear function defined on a tetrahedron into a sum of volumes of smaller tetrahedra. The previous facts, together with Remark 1.8,

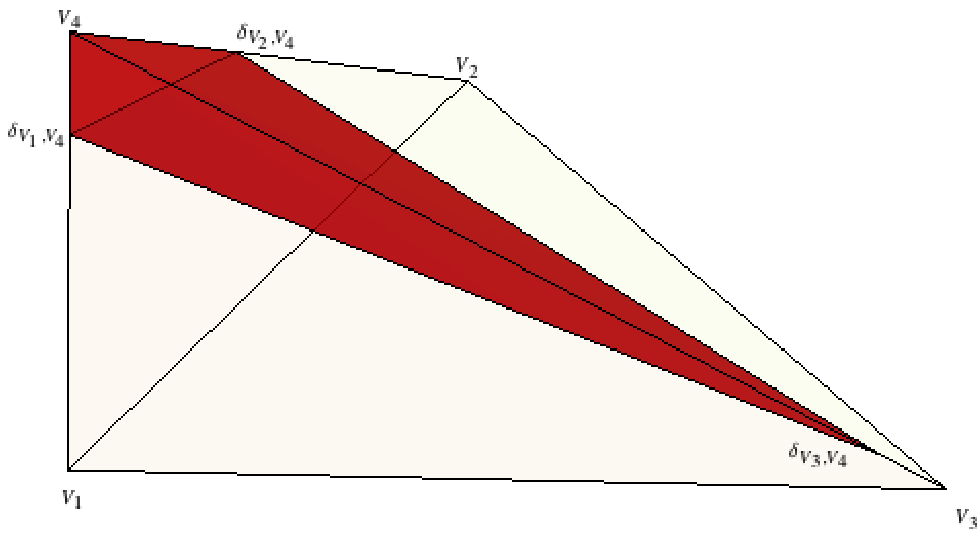

Figure 2.5: If $z_3 < \lambda < z_4$, then the level set is a tetrahedron similar to the marked red tetrahedron.

ensures that there is a similar formula as in the two-dimensional case but it is rather complicated and gives no further information, therefore we shall not present it here. This concludes the description of algorithm in three dimensional space.

Let us note that our method was implemented in the case $n = 2$ with linear splines and in the case $n = 1$ with clamped cubic splines. More information can be found in Appendix A. We will focus on the approximation by cubic splines in the following section.

2.4 Approximation by splines of higher order in one dimension

Focused on domain as finite interval $I \subset \mathbb{R}$ we can use the approximation by splines of higher order. Let us present the error estimates based on approximation by the so-called clamped cubic splines and natural cubic splines.

Definition 2.2 (Partition of interval)**.** Let $I = [a, b] \subset \mathbb{R}$ be an arbitrary interval. Let

$$\pi : a = x_0 < x_1 < \cdots < x_n = b$$

be a collection of knots partitioning interval I. We define

$$\Delta x_i = x_{i+1} - x_i, \; i = 0, 1, \dots, n-1.$$

The norm of the partition is defined as follows:

$$\|\pi\| = \max_{i\in\{0,\dots,n-1\}} \Delta x_i,$$

and the mesh ratio is

$$M_\pi = \frac{\|\pi\|}{\min_{i\in\{0,\dots,n-1\}} \Delta x_i}.$$

Definition 2.3 (cubic spline approximation)**.** Let π be a partition of interval $I = [a, b]$, $a, b \in \mathbb{R}$. Then the function s such that

$$s \restriction_{[x_i,x_{i+1}]} \in P\left([x_i, x_{i+1}]\right),$$

for $i \in \{0, 1, \dots, n-1\}$, and $s \in \mathbf{C}^2\left(I\right)$, is called a *cubic spline*. Let $f \colon I \to \mathbb{R}$ be such that there exist limits $l = \lim_{x\to a+} f'(x)$ and $r = \lim_{x\to b-} f'(x)$, then the *clamped cubic spline approximation* of f is the only spline such that

$$s(x_i) = f(x_i)$$

for all $i \in \{0, 1, \dots, n\}$, in addition it holds

$$\lim_{x\to a+} s'(x) = l \quad \text{and} \quad \lim_{x\to b-} s'(x) = r.$$

If there exist limits $L = \lim_{x\to a+} f^{(2)}(x)$ and $R = \lim_{x\to b-} f^{(2)}(x)$, then the *natural cubic spline approximation* of function f is a cubic spline which satisfies

$$s(x_i) = f(x_i)$$

for all $i \in \{0, 1, \dots, n\}$ while also

$$\lim_{x\to a+} s^{(2)}(x) = L \quad \text{and} \quad \lim_{x\to b-} s^{(2)}(x) = R.$$

Let us shortly discuss the algorithm of computing values of distribution function in case of approximation by a cubic spline. The definition of a cubic spline and error bounds are mentioned in Section 2.4. We will split the interval I into subintervals such that s in not only a cubic polynomial on each of them, but that I is monotone on them in addition. This will lead to a new partition of I, say π'. Computing of the value of the distribution function of a monotone cubic polynomial then turns into the question of finding its roots.

We present a theorem about error bounds for spline approximation which originated in [9, Theorem 5].

Theorem 2.4. *Let s be a clamped cubic or natural spline approximation of $f \in \mathbf{C}^4(I)$, $I = [a, b]$, $a, b \in \mathbb{R}$ corresponding to a partition π. Then*

$$\left\|(f-s)^{(r)}\right\|_\infty \leq C_r \left\|f^{(4)}\right\|_\infty \|\pi\|^{4-r},$$

for $r = 0, 1, 2, 3$, with constant

$$C_0 = \frac{5}{384},\ C_1 = \frac{1}{24},\ C_2 = \frac{3}{8},\ C_3 = \frac{\left(M_\pi + \frac{1}{M_\pi}\right)}{2}.$$

Proof. This is the main result of the paper [9]. □

This leads us to estimate the error involved in the approximation of a non-increasing rearrangement in one dimension. Advantage of approximation by splines of higher order is that we can use an easily calculated derivative of approximation and its rearrangement to approximate the derivative of the non-increasing rearrangement. The error bounds for this kind of approximation are included in the following theorem, too.

Theorem 2.5. *Let $I = [a, b]$, $a, b \in \mathbb{R}$ and consider $f \in \mathbf{C}(I)$. Suppose that π is a partition of I. Let s_1 be a linear spline interpolating f in the nodes of π. Then*

$$\|f^* - s_1^*\|_\infty \leq \max_{i\in\{0,1,\ldots,n-1\}} \int_{x_i}^{x_{i+1}} \left| f'(x) - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \right| dx.$$

when $f \in \mathbf{C}^1(I)$. Moreover, if $f \in \mathbf{C}^4(I)$ and s_3 is a clamped cubic spline approximation of f with respect to the partition π, then

$$\begin{aligned}
\|f^* - s_3^*\|_\infty &\leq \frac{5}{384} \left\|f^{(4)}\right\|_\infty \|\pi\|^4, \\
\left\|\left(f^{(1)}\right)^* - \left(s_3^{(1)}\right)^*\right\|_\infty &\leq \frac{1}{24} \left\|f^{(4)}\right\|_\infty \|\pi\|^3, \\
\left\|\left(f^{(2)}\right)^* - \left(s_3^{(2)}\right)^*\right\|_\infty &\leq \frac{3}{8} \left\|f^{(4)}\right\|_\infty \|\pi\|^2, \\
\left\|\left(f^{(3)}\right)^* - \left(s_3^{(3)}\right)^*\right\|_\infty &\leq \frac{1}{2}\left(M_\pi + \frac{1}{M_\pi}\right) \left\|f^{(4)}\right\|_\infty \|\pi\|.
\end{aligned}$$

Proof. First, we use the claim of Remark 1.22 to get

$$\|f^* - s^*\|_\infty \leq \|f - s\|_\infty,$$

for both s_1 and s_3. On $[x_i, x_{i+1}]$ we have

$$\begin{aligned} f(x) - s_1(x) &= \int_{x_i}^{x} f'(t) - s_1'(t)\, dt = \int_{x_i}^{x} f'(t) - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\, dt \\ &\leq \int_{x_i}^{x} \left| f'(t) - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \right| dt \\ &\leq \int_{x_i}^{x_{i+1}} \left| f'(t) - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \right| dt. \end{aligned}$$

Together then

$$\begin{aligned} \|f^* - s_1^*\|_\infty &\leq \|f - s_1\|_{\infty, I} \\ &\leq \max_{i \in \{0,1,\ldots,n-1\}} \|f - s_1\|_{\infty, [x_i, x_{i+1}]} \\ &\leq \max_{i \in \{0,1,\ldots,n-1\}} \int_{x_i}^{x_{i+1}} \left| f'(t) - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \right| dt. \end{aligned}$$

For the cubic spline approximation we use Theorem 2.4 and get

$$\|f^* - s_3^*\|_\infty \leq \|f - s_3\|_\infty \leq \frac{5}{384} \left\| f^{(4)} \right\|_\infty \|\pi\|^4 .$$

Estimates for derivative can be achieved analogously. □

2.5 Older algorithm

The only algorithm dealing with an approximation of the non-increasing rearrangement which is known to the author is that presented in the paper [12], due to Talenti. This algorithm approximates the rearrangement of the input function by that of a step function approximation. Considered are only real-valued functions defined on a finite interval in $\mathbb{R}$.

Let us shortly present the results of the mentioned paper, for proofs of stated theorems and more details please consult [12].

First, we need to define a step function approximation.

Notation 2.6. Let $f\colon I \to \mathbb{R}$, $I = [a,b], a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Then we define a step function approximation to f as follows:

$$S_{f,n}(t) = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right) \chi_{[x_{i-1}, x_i)}(t),$$

where

$$x_i = a + \frac{i(b-a)}{n},\ i = 0, 1, 2 \ldots, n.$$

Such an approximation of a function yields the following estimates.

Theorem 2.7. *Let be $a, b \in \mathbb{R}$, $I = [a, b]$, $f \colon I \to \mathbb{R}$, f is absolutely continuous and it's derivative u' is in $L_q(I)$ for some $q \geq 1$ and let there be $p \geq 1$ such that $q \leq p$. Then it holds*

$$\|u - S_{f,n}\|_p \leq C \left(\frac{h}{2}\right)^{1+\frac{1}{p}-\frac{1}{q}} \|u'\|_q ,$$

where

$$C = \left(1 + \frac{q}{p(q-1)}\right)^{\frac{1}{q}} \left(1 + p\left(1 - \frac{1}{q}\right)\right)^{-\frac{1}{p}} \frac{p}{\mathrm{B}\left(\frac{1}{p}, 1 - \frac{1}{q}\right)}$$

and B *stands for Euler's Beta function, and*

$$h = \frac{b-a}{n}.$$

Moreover, we have

$$\left\|f^* - S^*_{f,n}\right\|_p \leq \|f - S_{f,n}\|_p \leq C \left(\frac{h}{2}\right)^{1+\frac{1}{p}-\frac{1}{q}} \|u'\|_q .$$

We will now describe how to compute a non-increasing rearrangement of $S_{f,n}$. Let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite sequence such that

$$v_i = f\left(\frac{x_{i-1} + x_i}{2}\right),\ i = 1, \ldots, n.$$

We denote by $S = \{s_1, s_2, \ldots, s_n\}$ the sequence V sorted in a decreasing order. Then the non-increasing rearrangement of the step function approximation $S_{f,n}$ is defined as follows:

$$\left(S_{f,n}\right)^* (t) = \sum_{i=1}^{n} s_i \chi_{[x_{i-1}, x_i)}.$$

This algorithm could be easily generalized to higher dimensions. If we use an equidistant partition into cubes in $\mathbb{R}^n$ or a simplex partitions we get the similar algorithm for $\mathbb{R}^2$ or $\mathbb{R}^3$. Splines of higher order can be used, too. The relation of the algorithm which is the subject to this thesis to the algorithm described in this section is that it uses the linear interpolation instead of an approximation by a step function. The linear approximation then yields a sharper estimates in exchange for more complicated computation of rearrangement of approximation, which is no surprise.

3. Error estimates

The subject of this chapter is the presentation of error estimates of our method of approximation. We start with the result which follows from the known results on the error bound of interpolation from finite element spaces.

3.1 Error estimates based on the interpolation error

Taking advantage of general finite element theory we can readily state a general result.

Theorem 3.1. *Let $P \subset \mathbb{R}^n$ be a polyhedron, let be $f \in W_{2,\infty}(P)$. Let Ω^h, $h \in (0,1]$ be a non-degenerate family of linear Lagrangian finite element domains with conforming simplex triangulations. We denote $s_h = \Pi_{\Omega^h} f$*

$$\|f^* - s_h^*\|_\infty \leq Ch^2 \, |f|_{2,\infty} \, .$$

Proof. Using Remark 1.22 we get that

$$\|f^* - s_h^*\|_\infty \leq \|f - s_h\|_\infty \, .$$

Now, we are going to use Theorem 1.45 with reference to Remark 1.46. This gives us

$$\|f - s_h\|_\infty = |f - s_h|_{0,\infty,P} \leq Ch^2 \, |f - s_h|_{2,\infty,P} \, .$$

□

3.2 Supporting theorems

Before we proceed to the main theorems themselves we will state some results used in proofs of main theorems.

Lemma 3.2. *Let X be an r. i. space and let $\Omega \subset \mathbb{R}^n$ be an open set. Let $u \in V^1X(\Omega)$ be such that $|\{x \in \Omega : |u(x)| > t\}| < \infty$ for $t > 0$. Then u^* is locally absolutely continuous and*

$$n{\gamma_n}^{\frac{1}{n}} \left\| -\frac{du^*}{ds} s^{1-\frac{1}{n}} \right\|_{\overline{X},0,|\Omega|} \leq \big\| |\nabla u| \big\|_{X(\Omega)}. \tag{3.1}$$

Our proof will follow the direction of proof of Lemma 4.1. in [5] because these lemmas are almost the same. The only difference is that our lemma has slightly weaker conditions for u.

Proof. Let us set

$$\phi(s) = n\gamma_n^{\frac{1}{n}} \left(-\frac{du^*}{ds}\right) s^{1-\frac{1}{n}},$$

for $0 < s < |\Omega|$. We will prove that

$$\int_0^s \phi^*(r)\, dr \leq \int_0^s |\nabla u|^* (r)\, dr, \tag{3.2}$$

for $t \in (0, |\Omega|)$. The inequality (3.1) is then a consequence of Theorem 1.20 and (3.2). Let $0 \leq a < b$, $a, b \in \mathbb{R}$, and let v be the function defined by

$$v(x) = \begin{cases} 0, & |u(x)| \leq u^*(b), \\ u(x) - u^*(b), & u^*(b) < |u(x)| < u^*(a), \\ u^*(a) - u^*(b), & u^*(a) \leq |u(x)|. \end{cases}$$

Since $|\nabla u| \in X(\Omega)$ we have that $|\nabla u| \in X(\Omega) \subset L_1(\Omega) + L_\infty(\Omega)$ by Theorem 6.6 in the second chapter of [2]. Then, with aid of Theorem 6.2 from the same source, we have for $s > 0$

$$\int_0^s |\nabla u|^* (r)\, dr = \inf_{|\nabla u|^* = g+h} \left\{ \|g\|_{L_1} + s\, \|h\|_{L_\infty} \right\} \leq \begin{cases} \| |\nabla u|^* \|_{L_1 + L_\infty}, & s \leq 1, \\ s\, \| |\nabla u|^* \|_{L_1 + L_\infty}, & 1 < s, \end{cases}$$

hence

$$\int_0^s |\nabla u|^* (r)\, dr < \infty$$

from which follows $|\nabla u| \in L_1(G)$ for every $G \subset \Omega$ having finite measure. We will denote

$$E_{a,b} = \{x \in \Omega : u^*(a) > |u(x)| > u^*(b)\}.$$

Because $|\nabla u| \in L_1(G)$ for every $G \subset \Omega$ having finite measure and

$$|E_{a,b}| \leq b - a, \tag{3.3}$$

we have that $v \in V^1 L_1(\mathbb{R}^n)$. Because the v is also a bounded function with support of a finite measure, we get also that $v \in W^1 L_1(\mathbb{R}^n)$. The coarea formula for functions of bounded variation applied to v yields

$$\int_{E_{a,b}} |\nabla u|\, dx = \int_{u^*(b)}^{u^*(a)} P\left(\{|u| > t\}, \mathbb{R}^n\right)\, dt, \tag{3.4}$$

where $P(E, \mathbb{R}^n)$ denotes a perimeter of set P. The coarea formula for BV function can be found for example in [7], p. 185. The standard isoperimetric theorem tells us that

$$P\left(\{|u| > t\}, \mathbb{R}^n\right) \geq n\gamma_n^{\frac{1}{n}} \left|\{|u| > t\}\right|^{1-\frac{1}{n}}. \tag{3.5}$$

The last two inequalities imply that

$$\int_{E_{a,b}} |\nabla u| \; dx \geq n\gamma_n^{\frac{1}{n}} a^{\frac{1}{n'}} \left[u^*(a) - u^*(b)\right]. \tag{3.6}$$

The estimates (3.3) and (3.6) together ensure that u^* is locally absolutely continuous. Moreover, the inequalities (3.4) and (3.5) yield

$$\int_{E_{a,b}} |\nabla u| \; dx \;\; \geq \;\; n\gamma_n^{\frac{1}{n}} \int_{u^*(b)}^{u^*(a)} \left|\{|u| > t\}\right|^{1-\frac{1}{n}} \; dt.$$

We have

$$\int_a^b \phi(r) \; dr = n\gamma_n^{\frac{1}{n}} \int_a^b r^{1-\frac{1}{n}} \left(-\frac{du^*(r)}{dr}\right) \; dr.$$

If we acknowledge that for $r > 0$ it holds that

$$\text{if } \frac{du^*(r)}{dr} \neq 0 \text{ then } r = \left|\{|u| > u^*(r)\}\right|,$$

then the change of variables $u^*(r) = t$ together with the fact that $\frac{du^*(r)}{dr} \leq 0$ yields

$$\int_a^b \phi(r) \; dr = n\gamma_n^{\frac{1}{n}} \int_{u^*(a)}^{u^*(b)} \left|\{|u| > t\}\right|^{1-\frac{1}{n}} \; dt \leq \int_{E_{a,b}} |\nabla u| \; dx.$$

Let $\{(a_i, b_i)\}_{i=1}^{\infty}$ be any countable family of disjoint intervals in $(0, |\Omega|)$. Then Theorem 1.1 yields

$$\begin{aligned}
\int_{\cup_i (a_i,b_i)} \phi(r) \; dr &= \sum_i \int_{a_i}^{b_i} \phi(r) \; dr \\
&\leq \sum_i \int_{E_{a_i,b_i}} |\nabla u| (x) \; dx \\
&\leq \int_\Omega |\nabla u| (x) \sum_i \chi_{E_{a_i,b_i}}(x) \; dx \\
&\leq \int_0^{\sum_i |E_{a_i,b_i}|} |\nabla u|^* (r) \; dr.
\end{aligned}$$

Thus, by (3.3), we obtain

$$\int_{\cup_i (a_i,b_i)} \phi(r)\, dr \;\; \leq \;\; \int_0^{\sum_i (b_i - a_i)} |\nabla u|^* (r)\, dr.$$

The last estimate yields

$$\sup_{|E|=s} \int_E \phi(r)\, dr \leq \int_0^s |\nabla u|^* (r)\, dr,$$

since every measurable set $E \subset (0, |\Omega|)$ can be approximated from outside by sets of the form $\cup_i (a_i, b_i)$. Hence (3.2) follows, as its left-hand side coincides with the left-hand side of the previous inequality. □

The following theorem is a paraphrase of Theorem 6.3 in [6]. The presented proof is a adjustment of the original proof to our conditions.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\|.\|_R$ and $\|.\|_D$ be r. i. norms on $\mathfrak{M}_+(0, |\Omega|)$. If there exists $K > 0$ for which*

$$\left\| \int_t^{|\Omega|} f(s) s^{\frac{1}{n}-1}\, ds \right\|_R \leq K \|f\|_D, \; f \in \mathfrak{M}_+(0, |\Omega|), \tag{3.7}$$

then it holds

$$\|u^*\|_R \leq C \, \| |\nabla u|^* \|_D,$$

for all continuous $u \in V^1 L_D(\Omega)$ such that

$$\lim_{t \to |\Omega|-} u^*(t) = 0. \tag{3.8}$$

Moreover, the constant C reads as

$$C = \frac{K}{n \gamma_n^{\frac{1}{n}}}.$$

Proof. Let $u \in V^1 L_D(\Omega)$ satisfy the condition (3.8). According to Lemma 3.2, u^* is absolutely continuous on $[t, |\Omega| - \varepsilon]$ for eacht $t, \varepsilon > 0$ such that $0 < t < |\Omega| - \varepsilon$. Therefore

$$u^*(t) - u^*(|\Omega| - \varepsilon) = \int_t^{|\Omega| - \varepsilon} -\frac{du^*}{ds}\, ds.$$

Because of (3.8), we have, for $t \in (0, |\Omega|)$,

$$u^*(t) = \int_t^{|\Omega|} -\left(\frac{du^*}{ds}\right)(s)\; ds = \int_t^{|\Omega|} \left(-s^{1-\frac{1}{n}} \frac{du^*}{ds}\right) s^{\frac{1}{n}-1}\, ds.$$

Thus, given (3.7) and the rearrangement-invariance of $\|.\|_D$,

$$\|u^*\|_R \leq \left\| \left(\int_t^{|\Omega|} \left(-s^{1-\frac{1}{n}} \frac{du^*}{ds} \right) s^{\frac{1}{n}-1} \, ds \right) \right\|_R \leq K \left\| \left(-s^{1-\frac{1}{n}} \frac{du^*}{ds} \right)^* \right\|_D .$$

From Lemma 3.2 and Theorem 1.20,

$$K \left\| \left(-s^{1-\frac{1}{n}} \frac{du^*}{ds} \right)^* \right\|_D \leq \frac{K}{n\gamma_n^{\frac{1}{n}}} \left\| |\nabla u| \right\|_D$$

we conclude

$$\|u^*\|_R \leq \frac{K}{n\gamma_n^{\frac{1}{n}}} \, \| |\nabla u|^* \|_D .$$

□

Corollary 3.4. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n > 1$, and let f be a continuous function on $\overline{\Omega}$ having weak first order derivatives on Ω with*

$$\lim_{t \to |\Omega|-} f^*(t) = 0.$$

If $f \in W^1 L_{n,1}(\Omega)$ then

$$\|f\|_{\infty,\Omega} \leq \frac{1}{n\gamma_n^{\frac{1}{n}}} \left\| |\nabla f| \right\|_{n,1,\Omega} \tag{3.9}$$

and if $f \in W^1 L_{\frac{n}{2},1}(\Omega)$ then

$$\|f\|_{n,1,\Omega} \leq \frac{1}{\gamma_n^{\frac{1}{n}}} \left\| |\nabla f| \right\|_{\frac{n}{2},1,\Omega}. \tag{3.10}$$

Moreover,

$$\|f\|_{n',1,\Omega} \leq \frac{n}{n-1} \frac{1}{\gamma_n^{\frac{1}{n}}} \left\| |\nabla f| \right\|_1, \tag{3.11}$$

provided that $f \in W^1 L_1(\Omega)$.

Proof. To prove the inequality (3.9) we will use Theorem 3.3 with the following set-up: $\|.\|_R = \|.\|_{\infty,(0,|\Omega|)}$ and $\|.\|_D = \|.\|_{n,1,(0,|\Omega|)}$. We have

$$\left\| \int_t^1 h(s) s^{\frac{1}{n}-1} \, ds \right\|_{\infty,(0,|\Omega|)} \leq \int_0^{|\Omega|} h(s) s^{\frac{1}{n}-1} \, ds = \|h\|_{n,1,(0,|\Omega|)} ,$$

for $h \in \mathfrak{M}(0,|\Omega|)$, therefore the assertion of Theorem 3.3 holds with constant $K = 1$.

Proof of the second inequality is similar. We set $\|.\|_R = \|.\|_{n,1,(0,|\Omega|)}$ and $\|.\|_D = \|.\|_{\frac{n}{2},1,(0,|\Omega|)}$ which gives us

$$\left\| \int_t^{|\Omega|} h^*(s) s^{\frac{1}{n}-1}\, ds \right\|_{n,1,(0,|\Omega|)} = \int_0^{|\Omega|} \int_t^{|\Omega|} h^*(s) s^{\frac{1}{n}-1}\, ds\, t^{\frac{1}{n}-1}\, dt.$$

Using Fubini's theorem we obtain

$$\begin{aligned}
\left\| \int_t^{|\Omega|} h^*(s) s^{\frac{1}{n}-1}\, ds \right\|_{n,1,|\Omega|} &= \int_0^{|\Omega|} h^*(s) s^{\frac{1}{n}-1} \int_0^s t^{\frac{1}{n}-1}\, dt\, ds \\
&= n \int_0^{|\Omega|} h^*(s) s^{\frac{1}{n}-1}\, s^{\frac{1}{n}}\, ds \\
&= n \|h\|_{\frac{n}{2},1,(0,|\Omega|)},
\end{aligned}$$

for $h \in \mathfrak{M}(0,|\Omega|)$. We have verified the assumptions of Theorem 3.3 with the corresponding choice of $\|.\|_R$ and $\|.\|_D$ to prove

$$\|f\|_{n,1,(0,1)} \leq \frac{1}{\gamma_n^{\frac{1}{n}}} \big\| |\nabla f| \big\|_{\frac{n}{2},1,(0,1)}. \tag{3.12}$$

The inequality (3.11) is proved in the same way. We invoke Theorem 3.3 and check its assertions. We have $\varrho_R = \|.\|_{n',1}$ and $\varrho_D = \|.\|_1$, thus

$$\begin{aligned}
\left\| \int_t^{|\Omega|} f(s) s^{\frac{1}{n}-1}\, ds \right\|_{n',1} &= \int_0^{|\Omega|} \int_t^{|\Omega|} f(s) s^{\frac{1}{n}-1}\, ds\, t^{1-\frac{1}{n}-1}\, dt \\
&= \int_0^{|\Omega|} \int_t^{|\Omega|} f(s) s^{\frac{1}{n}-1} t^{-\frac{1}{n}}\, ds\, dt.
\end{aligned}$$

Fubini's theorem yields

$$\begin{aligned}
\left\| \int_t^{|\Omega|} f(s) s^{\frac{1}{n}-1}\, ds \right\|_{n',1} &= \int_0^{|\Omega|} \int_0^s f(s) s^{\frac{1}{n}-1} t^{-\frac{1}{n}}\, dt\, ds \\
&= \frac{n}{n-1} \int_0^{|\Omega|} f(s)\, ds = \frac{n}{n-1} \|f\|_1.
\end{aligned}$$

Thus, in this case, $K = \frac{n}{n-1}$. □

Lemma 3.5. *Let T be a simplex in $\mathbb{R}^n$ and $f \in C(\overline{T})$ such that there exists a point $x \in \overline{T}$ where f vanishes. Then the function f satisfies*

$$\lim_{t \to |T|-} f^*(t) = 0.$$

Proof. Let there be $\varepsilon > 0$. Provided that f is continuous there is neighborhood B (in T) of x such that $|f| < \varepsilon$ on B. Then we set $\delta := |B|$. Then for any $t \in (|T| - \delta, |T|]$ it holds

$$f^*(t) := \inf\{\lambda \in \mathbb{R} : \lambda \geq 0, \mu_f(\lambda) \leq t\} \leq \varepsilon,$$

because

$$\mu_f(\varepsilon) = |\{|f| > \varepsilon\}| < |T \setminus B| = |T| - \delta < t.$$

□

The rest of this section consists of few technical results.

Lemma 3.6. *Consider a simplex $T \subset \mathbb{R}^n$ and a function $f \in \mathbf{C}(T)$ having weak derivatives up to order two on T. Suppose that $\mathbf{A}$ is a referencing affine transformation onto T from the referencing simplex T'. Then for the following function*

$$g(x) := (f \circ \mathbf{A})(x), \; x \in T',$$

and a linear function

$$l_d(x) = g(V_0) + \sum_{r=1}^{n} (g(V_r) - g(V_0)) \frac{x_r}{d}, \; x = (x_1, \ldots, x_n) \in T',$$

it holds

$$|g_{ij}| \leq \alpha \left[\sum_{k,l=1}^{n} (f_{kl} \circ \mathbf{A})^2 \right]^{\frac{1}{2}}, \tag{3.13}$$

where

$$\alpha = \max_{j,k \in \{1,\ldots,n\}} \left[\sum_{l=1}^{n} (a_{lj})^2 \right] \left[\sum_{m=1}^{n} (a_{mk})^2 \right].$$

Proof. We have

$$\begin{aligned} g_i &= \frac{\partial g}{\partial x_i} = \frac{\partial}{\partial x_i} g = \frac{\partial}{\partial x_i} f \circ \mathbf{A} = \sum_{k=1}^{n} (f_k \circ A)\, a_{ki}, \\ g_{ij} &= \frac{\partial}{\partial x_j} \left(\sum_{k=1}^{n} (f_k \circ \mathbf{A})\, a_{ki} \right) = \sum_{k=1}^{n} \sum_{l=1}^{n} (f_{kl} \circ \mathbf{A})\, a_{ki}\, a_{lj}. \end{aligned}$$

Applying the Hölder inequality to sum over index l we obtain

$$\begin{aligned} |g_{ij}(x)| &= \left| \sum_{k=1}^{n} \sum_{l=1}^{n} (f_{kl} \circ \mathbf{A})(x) a_{ki}\, a_{lj} \right| \\ &= \left| \sum_{k=1}^{n} \left[\sum_{l=1}^{n} (f_{kl} \circ \mathbf{A})(x) a_{lj} \right] a_{ki} \right| \\ &\leq \left[\sum_{k=1}^{n} \left[\sum_{l=1}^{n} (f_{kl} \circ \mathbf{A})(x) a_{lj} \right]^2 \right]^{\frac{1}{2}} \left[\sum_{k=1}^{n} (a_{ki})^2 \right]^{\frac{1}{2}} . \end{aligned}$$

Using the Hölder inequality again one gets the rest of the proof:

$$\begin{aligned} &\leq \left[\sum_{k=1}^{n} \left[\sum_{l=1}^{n} (f_{kl} \circ \mathbf{A})^2 (x) \right] \left[\sum_{l=1}^{n} (a_{lj})^2 \right] \right]^{\frac{1}{2}} \left[\sum_{k=1}^{n} (a_{ki})^2 \right]^{\frac{1}{2}} \\ &= \left[\left[\sum_{l=1}^{n} (a_{lj})^2 \right] \sum_{k=1}^{n} \left[\sum_{l=1}^{n} (f_{kl} \circ \mathbf{A})^2 (x) \right] \right]^{\frac{1}{2}} \left[\sum_{k=1}^{n} (a_{ki})^2 \right]^{\frac{1}{2}} \\ &= \left[\sum_{k=1}^{n} (a_{ki})^2 \right]^{\frac{1}{2}} \left[\sum_{l=1}^{n} (a_{lj})^2 \right]^{\frac{1}{2}} \left[\sum_{k=1}^{n} \sum_{l=1}^{n} (f_{kl} \circ \mathbf{A})^2 (x) \right]^{\frac{1}{2}} \\ &\leq \alpha \left[\sum_{k,l=1}^{n} (f_{kl} \circ \mathbf{A})^2 (x) \right]^{\frac{1}{2}} . \end{aligned}$$

□

Lemma 3.7. *Consider a simplex $T \subset \mathbb{R}^n$ and a function $f \in \mathbf{C}(T)$ having weak derivatives up to order two on T. Suppose that $\mathbf{A}$ is a referencing affine transformation onto T from the referencing simplex T'. Then for the following function*

$$g(x) := (f \circ \mathbf{A})(x),\ x \in T',$$

and a linear function

$$l_d(x) = g(V_0) + \sum_{r=1}^{n} (g(V_r) - g(V_0)) \frac{x_r}{d},\ x = (x_1, \ldots, x_n) \in T',$$

we have

$$|\nabla(|\nabla(g - l_d)|)| \leq n^2 \alpha \max_{i,j \in \{1,\ldots,n\}} |f_{ij} \circ A|_{0,\infty,T'} ,$$

where

$$\alpha = \max_{k,l\in\{1,\ldots,n\}} \left[\sum_{r=1}^{n} a_{rk}^2\right]^{\frac{1}{2}} \left[\sum_{s=1}^{n} a_{sl}^2\right]^{\frac{1}{2}},$$

and a_{ij} *denotes the* ij*-th element of the transformation matrix of* $\mathbf{A}$*. In addition,*

$$\sum_{k=1}^{n} \left|\nabla\left(\frac{\partial}{\partial x_k}(g - l_d)\right)\right| \leq n^{2+\frac{1}{2}}\alpha \max_{i,j\in\{1,\ldots,n\}} \|f_{ij}\circ A\|_{\infty,T'}.$$

Proof. We have

$$\frac{\partial}{\partial x_i}(g - l_d) = g_i - \frac{g(V_i) - g(V_0)}{d}. \tag{3.14}$$

Let us denote $\frac{g(V_i)-g(V_0)}{d}$ by $\Delta_i g$, then we have

$$\begin{aligned} \frac{\partial}{\partial x_j}(|\nabla (g - l_d)|) &= \frac{\partial}{\partial x_j}\left(\left(\sum_{i=1}^{n}(g_i - \Delta_i g)^2\right)^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{n}(g_i - \Delta_i g)^2\right)^{-\frac{1}{2}} 2\sum_{i=1}^{n}(g_i - \Delta_i g)\, g_{ij}. \end{aligned}$$

If we apply the Hölder inequality to the second sum in the last expression, we get

$$\frac{\partial}{\partial x_j}(|\nabla (g - l_d)|) \leq \left(\sum_{i=1}^{n}(g_i - \Delta_i g)^2\right)^{-\frac{1}{2}} \left(\sum_{i=1}^{n}(g_i - \Delta_i g)^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n}(g_{ij})^2\right)^{\frac{1}{2}}.$$

Together we have

$$\begin{aligned} |\nabla(|\nabla (g - l_d)|)|^2 &= \sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}(|\nabla (g - l_d)|)\right)^2 \\ &\leq \sum_{j=1}^{n}\left(\sum_{i=1}^{n}(g_{ij})^2\right). \end{aligned}$$

The estimate for g_{ij} from Lemma 3.6 yields

$$|\nabla(|\nabla (g - l_d)|)|^2 \leq \sum_{i,j=1}^{n} \alpha \left[\sum_{k,l=1}^{n}(f_{kl}\circ \mathbf{A})^2\right]^{\frac{1}{2}}.$$

If we want our estimate to be independent of the indices i and j, we get

$$|g_{ij}| \leq \alpha n \left(\max_{k,l\in\{1,\ldots,n\}} |f_{kl} \circ \mathbf{A}|_{0,\infty,T'} \right). \tag{3.15}$$

Combining the previous results, we have

$$\begin{aligned}
|\nabla\left(|\nabla\left(g-l_d\right)|\right)| &\leq \left(\sum_{i,j=1}^{n} g_{ij}^2\right)^{\frac{1}{2}} \\
&\leq n\alpha \max_{k,l\in\{1,\ldots,n\}} |f_{kl}\circ\mathbf{A}|_{0,\infty,T'} \left(\sum_{i,j=1}^{n} 1^2\right)^{\frac{1}{2}} \\
&\leq n^2\alpha \max_{k,l\in\{1,\ldots,n\}} |f_{kl}\circ\mathbf{A}|_{0,\infty,T'}\,,
\end{aligned}$$

which completes the proof of first part of claim. From the (3.14) it follows that

$$\left|\nabla\left(\frac{\partial}{\partial x_k}\left(g-l_d\right)\right)\right|^2 = \sum_{i=1}^{n}\left(\frac{\partial^2}{\partial x_i\partial x_k}\left(g-l_d\right)\right)^2 = \sum_{i=1}^{n}\left(g_{ik}\right)^2,$$

hence

$$\begin{aligned}
\sum_{k=1}^{n}\left|\nabla\left(\frac{\partial}{\partial x_k}\left(g-l_d\right)\right)\right| &= \sum_{k=1}^{n}\left(\sum_{i=1}^{n}\left(g_{ik}\right)^2\right)^{\frac{1}{2}} \leq \sqrt{n}\sum_{k=1}^{n}\max_{i\in\{1,\ldots,n\}}|g_{ik}| \\
&\leq n^{1+\frac{1}{2}}\max_{i,j\in\{1,\ldots,n\}}|g_ij| \\
&\leq n^{2+\frac{1}{2}}\alpha\left(\max_{k,l\in\{1,\ldots,n\}}|f_{kl}\circ\mathbf{A}|_{0,\infty,T'}\right),
\end{aligned}$$

where the last inequality is a consequence of the inequality (3.15). □

Remark 3.8. Let us note that if $\mathbf{A}$ is a referencing affine transformation which maps a referencing simplex $T' \subset \mathbb{R}^n$ onto a simplex $T \subset \mathbb{R}^n$, $f\colon T \to \mathbb{R}^n$, $g = f \circ \mathbf{A}$ and

$$l_d(x) = g(V_0) + \sum_{r=1}^{n}\left(g(V_r) - g(V_0)\right)\frac{x_r}{d},\; x = (x_1,\ldots,x_n) \in T',$$

then the linear function $l_d\circ\mathbf{A}$ agrees with g on the vertices of T'. Consequently, having defined a linear Lagrange finite element on T', it holds

$$l_d = \Pi_{T'}(f).$$

Lemma 3.9. *Let $\Omega \subset \mathbb{R}^n$ be measurable, let $f \in \mathbf{C}(\Omega)$ have weak derivatives of the first order. Then*

$$\int_0^{|\Omega|} |\nabla f|^*(t)\, t^{\frac{1}{n}-1}\, dt \le n^{\frac{1}{n}+\frac{1}{2}} \sum_{k=1}^{n} \int_0^{|\Omega|} \left(\frac{\partial}{\partial x_k} f\right)^* (t)\, t^{\frac{1}{n}-1}\, dt. \tag{3.16}$$

Proof. Using basic properties of the rearrangement, we get

$$\begin{aligned}
\int_0^{|\Omega|} |\nabla f|^*(t)\, t^{\frac{1}{n}-1}\, dt &= \int_0^{|\Omega|} \left(\sqrt{\sum_{k=1}^{n} \left(\frac{\partial}{\partial x_k} f(t)\right)^2} \right)^* t^{\frac{1}{n}-1}\, dt \\
&= \int_0^{|\Omega|} \sqrt{\left(\sum_{k=1}^{n} \left(\frac{\partial}{\partial x_k} f(t)\right)^2 \right)^*}\, t^{\frac{1}{n}-1}\, dt \\
&\le \int_0^{|\Omega|} \sqrt{\sum_{k=1}^{n} \left(\left(\frac{\partial}{\partial x_k} f\left(\frac{t}{n}\right)\right)^2 \right)^*}\, t^{\frac{1}{n}-1}\, dt.
\end{aligned}$$

Simple adjustments yield:

$$\begin{aligned}
\int_0^{|\Omega|} |\nabla f|^*(t)\, t^{\frac{1}{n}-1}\, dt &\le \int_0^{|\Omega|} \sqrt{n \max_{k\in\{1,\dots,n\}} \left(\left(\frac{\partial}{\partial x_k} f\left(\frac{t}{n}\right)\right)^2 \right)^*}\, t^{\frac{1}{n}-1}\, dt \\
&\le \sqrt{n} \int_0^{|\Omega|} t^{\frac{1}{n}-1} \max_{k\in\{1,\dots,n\}} \sqrt{\left(\left(\frac{\partial}{\partial x_k} f\left(\frac{t}{n}\right)\right)^2 \right)^*}\, dt \\
&\le \sqrt{n} \int_0^{|\Omega|} t^{\frac{1}{n}-1} \max_{k\in\{1,\dots,n\}} \left(\sqrt{\left(\frac{\partial}{\partial x_k} f\left(\frac{t}{n}\right)\right)^2} \right)^* dt \\
&\le \sqrt{n} \int_0^{|\Omega|} t^{\frac{1}{n}-1} \max_{k\in\{1,\dots,n\}} \left(\frac{\partial}{\partial x_k} f\left(\frac{t}{n}\right)\right)^* dt.
\end{aligned}$$

We use the change of variables with $y = \frac{t}{n}$, consequently we have $dt = n\, dy$, $t^{\frac{1}{n}-1} = n^{\frac{1}{n}-1} y^{\frac{1}{n}-1}$, and we obtain

$$\begin{aligned}
\int_0^{|\Omega|} |\nabla f|^*(t)\, t^{\frac{1}{n}-1}\, dt &\le \sqrt{n} \int_0^{\frac{|\Omega|}{n}} n^{\frac{1}{n}-1}\, y^{\frac{1}{n}-1} \max_{k\in\{1,\dots,n\}} \left(\frac{\partial}{\partial x_k} f(y)\right)^* n\, dy \\
&\le n^{\frac{1}{n}} \sqrt{n} \int_0^{\frac{|\Omega|}{n}} y^{\frac{1}{n}-1} \sum_{k=1}^{n} \left(\frac{\partial}{\partial x_k} f(y)\right)^* dy \\
&\le n^{\frac{1}{n}} \sqrt{n} \sum_{k=1}^{n} \int_0^{|\Omega|} \left(\frac{\partial}{\partial x_k} f(y)\right)^* y^{\frac{1}{n}-1}\, dy.
\end{aligned}$$

□

We shall also need the following classical imbedding theorem by Gagliardno-Nirenberg-Sobolev. It can be found in [1].

Theorem 3.10 (Gagliardo-Nirenberg-Sobolev)**.** *Let us assume that u is a continuously differentiable real-valued function on $\mathbb{R}^n$ with compact support. Then for $1 \le p < n$ there exists a constant C depending only on n and p such that*

$$\|u\|_{p^*} \le C\big\||\nabla u|\big\|_p, \tag{3.17}$$

where

$$p^* = \frac{pn}{n-p} > p$$

is the Sobolev conjugate to p. Such a constant is

$$C = \frac{p\,(n-1)}{n-p},$$

but this may not be the best constant.

3.3 Error estimates

Theorem 3.11. *Consider a polyhedron, P, in $\mathbb{R}^n$. Assume that $f \in \mathcal{C}^1(\overline{P})$ with*

$$\int_0^{|P|} |\nabla f|^*(t) t^{\frac{1}{n}-1}\, dt < \infty.$$

Let the P be a linear Lagrangian finite element domain with triangulation $\mathcal{T} = \{T_i\}_{i\in\mathbb{I}}$. Let $A_i = \big(a_{jk}(i)\big)$ be a matrix representing a referencing affine trasformation onto T_i from the referencing simplex T_{d_i}. Then, given the linear spline $s = \Pi_{\mathcal{T}}(f)$, one has

$$\|f^* - s^*\|_{\infty,P} \;\le\; \frac{1}{n!^{\frac{1}{n}}\gamma_n^{\frac{1}{n}}} \max_{i\in\mathbb{I}} d_i \,|\nabla((f\circ \mathbf{A}_i) - l_{d_i})|_{0,\infty,T_{d_i}} \tag{3.18}$$

and if f has a weak derivatives up to second order and in addition it holds that

$$\int_0^{|P|} |Df|^*(t) t^{\frac{2}{n}-1}\, dt < \infty,$$

then

$$\|f^* - s^*\|_{\infty,P} \le \frac{n^{\frac{1}{n}+3}}{2\,n!^{\frac{2}{n}}\,\gamma_n^{\frac{2}{n}}}\, |f|_{2,\infty,P} \max_{i\in\mathbb{I}} d_i^2\, \alpha_i, \tag{3.19}$$

in which α_i denotes

$$\alpha_i = \max_{k,l\in\{1,\dots,n\}} \left[\sum_{r=1}^{n}\left(a_{rk}^i\right)^2\right]^{\frac{1}{2}} \left[\sum_{s=1}^{n}\left(a_{sl}^i\right)^2\right]^{\frac{1}{2}}$$

and d_i is the diameter of simplex T_i and a_{sr}^i denotes sr-th element of transformation matrix of the affine transformation $\mathbf{A}_i$. Moreover, it holds

$$\|f^* - s^*\|_{1,P} \;\le\; \frac{n^{\frac{1}{n}+3}}{n!^{1+\frac{2}{n}}2\,\gamma_n^{\frac{2}{n}}}\,|f|_{2,\infty,P}\sum_{i\in\mathbb{I}}\alpha_i\,d_i^{n+2}.$$

Proof. We fix an index $i \in \mathbb{I}$ and denote $T = T_i$, $T' = T_{d_i}$ and $\mathbf{A} = \mathbf{A}_i$. Function s restricted on T will be denoted as l and $l_d = l\circ\mathbf{A}$. First, we derive an error estimate on the simplex T. From the fact that $g - l_d$ vanishes in vertices of T' and Corollary 3.4 it follows, with $g = f\circ\mathbf{A}$,

$$|f-l|_{0,\infty,T} = |g-l_d|_{0,\infty,T'} \le \frac{1}{n\gamma_n^{\frac{1}{n}}}\int_0^{|T'|}|\nabla(g-l_d)|^*(t)\,t^{\frac{1}{n}-1}\,dt. \tag{3.20}$$

Hölder's inequality now yields

$$\begin{aligned}|f-l|_{0,\infty,T} &\le \frac{1}{n\gamma_n^{\frac{1}{n}}}\,|\nabla(g-l_d)|_{0,\infty,T}\int_0^{|T'|}t^{\frac{1}{n}-1}\,dt\\ &\le \frac{|T'|^{\frac{1}{n}}}{\gamma_n^{\frac{1}{n}}}\,|\nabla(g-l_d)|_{0,\infty,T}\,.\end{aligned}$$

Using the formula for volume of a simplex one obtains

$$|f-l|_{0,\infty,T} \;\le\; \frac{1}{n!^{\frac{1}{n}}\gamma_n^{\frac{1}{n}}}d\,|\nabla(g-l_d)|_{0,\infty,T}\,.$$

Hence, we have

$$|f-l|_{0,\infty,T} \le \frac{1}{n!^{\frac{1}{n}}\gamma_n^{\frac{1}{n}}}\max_{i\in\mathbb{I}} d_i\,|\nabla\left((f\circ\mathbf{A}_i)-l_{d_i}\right)|_{0,\infty,T}\,,$$

which completes the proof of (3.18). Now, let us go back to inequality (3.20). With aid of Lemma 3.9 we get

$$\begin{aligned}|f-l|_{0,\infty,T} &\le \frac{1}{n\gamma_n^{\frac{1}{n}}}\int_0^{|T'|}|\nabla(g-l_d)|^*(t)\,t^{\frac{1}{n}-1}\,dt\\ &\le \frac{n^{\frac{1}{n}-\frac{1}{2}}}{\gamma_n^{\frac{1}{n}}}\sum_{k=1}^{n}\int_0^{|T'|}\left(\frac{\partial}{\partial x_k}f\right)^*(t)t^{\frac{1}{n}-1}\,dt.\end{aligned}$$

Using Rolle's theorem we get that for $i = 1, \ldots, n$ there exist a point z_i on edge of T' between vertices V_0 and V_i, such that $\frac{\partial}{\partial x_i}(g - l_d)(z_i) = 0$, because $g - l_d$ vanishes at vertices of T'. Using the previous fact and the second inequality from Corollary 3.4 on all elements in sum, we obtain

$$|f - l|_{0,\infty,T} \leq \frac{n^{\frac{1}{n}-\frac{1}{2}}}{\gamma_n^{\frac{2}{n}}} \sum_{k=1}^{n} \int_0^{|T'|} \left| \nabla \left(\frac{\partial}{\partial x_k} (g - l_d) \right) \right|^* (t)\, t^{\frac{2}{n}-1}\, dt.$$

Invoking the estimate of $\sum_{k=1}^{n} \left(\left| \nabla \left(\frac{\partial}{\partial x_k} g - l_d \right) \right| \right)$ from Lemma 3.7, we have

$$\begin{aligned} |f - l|_{0,\infty,T} &\leq \frac{n^{\frac{1}{n}-\frac{1}{2}}}{\gamma_n^{\frac{2}{n}}} n^{2+\frac{1}{2}}\, \alpha \max_{i,j\in\{1,\ldots,n\}} |f_{ij} \circ \mathbf{A}|_{0,\infty,T'} \int_0^{|T'|} t^{\frac{2}{n}-1}\, dt \\ &\leq \frac{n^{\frac{1}{n}+3}}{2\gamma_n^{\frac{2}{n}}}\, |T'|^{\frac{2}{n}}\, \alpha \max_{i,j\in\{1,\ldots,n\}} |f_{ij} \circ \mathbf{A}|_{0,\infty,T'} . \end{aligned}$$

Formula for volume of a simplex yields:

$$\begin{aligned} |f - l|_{0,\infty,T} &\leq \frac{n^{\frac{1}{n}+3}}{2\gamma_n^{\frac{2}{n}}} \left(\frac{d^n}{n!} \right)^{\frac{2}{n}} \alpha \max_{i,j\in\{1,\ldots,n\}} |f_{ij} \circ \mathbf{A}|_{0,\infty,T'} \\ &\leq \frac{n^{\frac{1}{n}+3}}{2n!^{\frac{2}{n}} \gamma_n^{\frac{2}{n}}} d^2\, \alpha \max_{i,j\in\{1,\ldots,n\}} |f_{ij} \circ \mathbf{A}|_{0,\infty,T'} \\ &\leq \frac{n^{\frac{1}{n}+3}}{2\, n!^{\frac{2}{n}}\, \gamma_n^{\frac{2}{n}}} d^2\, \alpha \max_{i,j\in\{1,\ldots,n\}} |f_{ij}|_{0,\infty,T} , \end{aligned}$$

where d denotes $\operatorname{diam}(T)$.

Finally, we can combine the estimates of error on each simplex in decomposition and get

$$\begin{aligned} |(f^* - s^*)|_{0,\infty,P} &\leq \max_{i\in\mathbb{I}} |f - l_i|_{0,\infty,T_i} \\ &\leq \frac{n^{\frac{1}{n}+3}}{2\, n!^{\frac{2}{n}}\, \gamma_n^{\frac{2}{n}}} \max_{i\in\mathbb{I}} d_i^2\, \alpha_i \max_{k,j\in\{1,\ldots,n\}} |f_{kj}|_{0,\infty,T_i} \\ &\leq \frac{n^{\frac{1}{n}+3}}{2\, n!^{\frac{2}{n}}\, \gamma_n^{\frac{2}{n}}} |f|_{2,\infty,P} \max_{i\in\mathbb{I}} d_i^2\, \alpha_i, \end{aligned}$$

which is the inequality (3.19). It remains to prove the estimate of $\|f^*(t) - s^*\|_{1,P}$. We have

$$|f^* - s^*|_{0,1,P} = \sum_{i\in\mathbb{I}} \int_{T_i} |f^*(t) - s^*(t)| \leq \sum_{i\in\mathbb{I}} |T_i|\, |f^*(t) - s^*(t)|_{0,\infty,T_i} .$$

We can use the estimate for $|f^*(t) - s^*(t)|_{0,\infty,T_h}$ obtained above and we get

$$\begin{aligned} |f^* - s^*|_{0,1,P} &\le \sum_{i\in\mathbb{I}} |T_i|\; |f^* - s^*|_{0,\infty,T_i} \\ &\le \frac{n^{\frac{1}{n}+3}}{2\, n!^{\frac{2}{n}}\, \gamma_n^{\frac{2}{n}}} \sum_{i\in\mathbb{I}} |T_i|\, \alpha_i d_i^2 \; |f|_{2,\infty,T_i} \,. \end{aligned}$$

We use the formula for the volume of a simplex and our proof is completed:

$$\begin{aligned} |f^* - s^*|_{0,1,P} &\le \frac{n^{\frac{1}{n}+3}}{n!^{1+\frac{2}{n}}\, 2\, \gamma_n^{\frac{2}{n}}} \sum_{i\in\mathbb{I}} \alpha_i\, d_i^{n+2} \; |f|_{2,\infty,T_i} \\ &\le \frac{n^{\frac{1}{n}+3}}{n!^{1+\frac{2}{n}}\, 2\, \gamma_n^{\frac{2}{n}}} \; |f|_{2,\infty,P} \sum_{i\in\mathbb{I}} \alpha_i\, d_i^{n+2}. \end{aligned}$$

□

Theorem 3.12. *Consider a polytope $P \subset \mathbb{R}^n$, $n > 1$. Assume that $f \in \mathbf{C}(\overline{P})$ has weak first-order derivatives, with*

$$\int_P |\nabla f| < \infty.$$

Suppose P is a simplex linear Lagrange finite element domain with simplex triangulation $\mathcal{T} = \{T_i\}_{i\in\mathbb{I}}$. Let $\mathbf{A}_i = (a_{jk}(i))$ be a transformation matrix of an affine transformation on T_i from n-simplex, T_{d_i}, with vertices $V_0 = \{0, \ldots, 0\}$, $V_1 = \{d_i, 0, \ldots, 0\}$ and $V_n = \{0, \ldots, 0, d_i\}$, where d_i is the diameter of T_i, $i \in \mathbb{I}$.

Then, for the linear spline $s = \Pi_P(f)$, one has

$$\|f^* - s\|_1 \le \frac{n}{(n-1)\gamma_n^{\frac{1}{n}}} \sum_{i\in\mathbb{I}} |T_i|^{\frac{1}{n}} \int_{T_i} |\nabla(f - s)| \,. \tag{3.21}$$

If, further, $f \in \mathbf{C}^1(\overline{P})$ has weak second-order derivatives, with

$$\int_P \left|D^2 f\right| < \infty,$$

then,

$$\|f^* - s^*\|_1 \le (n-1)^2\, n^{\frac{1}{2}} \sum_{i\in\mathbb{I}} \alpha_i\, |T_i|^{\frac{2}{n}} \int_{T_i} \left|D^2 f\right| ; \tag{3.22}$$

here,

$$\alpha_i = \max_{j,k\in\{1,\ldots,n\}} \left[\sum_{l=1}^{n} (a_{lj}(i))^2\right] \left[\sum_{m=1}^{n} (a_{mk}(i))^2\right].$$

Proof. Fix $i_0 \in \mathbb{I}$ and define g on $T_{d_{i_0}}$ by

$$g(x) = f \circ A_{i_0}(x),\ x \in T_{d_{i_0}}.$$

Given the linear function, $l_{T_{i_0}}$, interpolating f at the vertices of T_{i_0}, one has that

$$l_{T_{d_{i_0}}} := l \circ A_{i_0}$$

is the linear function doing the same for g on $T_{d_{i_0}}$, in fact

$$l_{d_{i_0}}(x) = g(V_0) + \sum_{r=1}^{n} (g(V_r) - g(V_0)) \frac{x_r}{d_{i_0}},\ x = (x_1, \ldots, x_n).$$

Using Corollary 1.13 and the general Hölder inequality for associated Lorentz spaces $L_{n',1}$ and $L_{n,\infty}$, we get

$$\begin{aligned} \|f^* - s^*\|_1 \le \int_P |f - s| &= \sum_{i\in\mathbb{I}} \int_{T_i} |f - s| \\ &\le \sum_{i\in\mathbb{I}} \|1\|_{n,\infty,T_i} \|f - s\|_{n',1T_i} \\ &\le \sum_{i\in\mathbb{I}} |T_i|^{\frac{1}{n}} \|f - s\|_{n',1T_i}. \end{aligned}$$

The imbedding inequality (3.11) yields

$$\|f^* - s^*\|_1 \le \sum_{i\in\mathbb{I}} \frac{n}{(n-1)\gamma_n^{\frac{1}{n}}} \||\nabla (f - s)|\|_1 .$$

which is (3.21).

Next, from the Hölder's inequality and Theorem 3.10, again, we have

$$\begin{aligned} \int_{T_{i_0}} |f - s| &= \int_{T_{i_0}} \left|f - l_{T_{i_0}}\right| \\ &= \int_{T_{d_{i_0}}} \left|g - l_{T_{d_{i_0}}}\right| |\det A_{i_0}| \\ &\le \left|T_{d_{i_0}}\right|^{\frac{2}{n}} \left\|g - l_{T_{d_{i_0}}}\right\|_{(\frac{n}{2})',T_{d_{i_0}}} |\det A_{i_0}| \\ &\le (n-1)^2 \left|T_{d_{i_0}}\right|^{\frac{2}{n}} \left\|\nabla(g - l_{T_{d_{i_0}}})\right\|_{n',T_{d_{i_0}}} |\det A_{i_0}| \\ &\le (n-1)^2 n^{\frac{1}{2}} \left|T_{d_{i_0}}\right|^{\frac{2}{n}} \sum_{l=1}^{n} \left\|g_l - \frac{g(V_l) - g(V_0)}{d_{i_0}}\right\|_{n',T_{d_{i_0}}} |\det A_{i_0}| . \end{aligned}$$

Now, we may invoke the imbedding inequality Theorem 3.10. This yields:

$$\begin{aligned}\int_{T_{i_0}} |f-s| &\le (n-1)^2 n^{\frac{1}{2}} \left|T_{d_{i_0}}\right|^{\frac{2}{n}} \sum_{k=1}^{n} \int_{T_{d_{i_0}}} \left|\nabla\left(g_k - \frac{g(V_k)-g(V_0)}{d_{i_0}}\right)\right| |\det A_{i_0}| \\ &\le (n-1)^2 n^{\frac{1}{2}} \left|T_{d_{i_0}}\right|^{\frac{2}{n}} \int_{T_{d_{i_0}}} \sum_{l=1}^{n} \sum_{k=1}^{n} |g_{lk}| \left|\det A_{i_0}\right|.\end{aligned}$$

Altogether with aid of Lemma 3.6, then,

$$\begin{aligned}\|f-s\|_{1,P} &\le \sum_{i\in\mathbb{I}} \int_{T_i} |f-s| \\ &\le (n-1)^2 n^{\frac{1}{2}} \sum_{i\in\mathbb{I}} \alpha_i \left|T_{d_{i_0}}\right|^{\frac{2}{n}} \int_{T_{d_i}} \left|D^2 f\right| (A_i) \left|\det A_i\right| \\ &\le (n-1)^2 n^{\frac{1}{2}} \sum_{i\in\mathbb{I}} \alpha_i \left|T_{d_i}\right|^{\frac{2}{n}} \int_{T_i} \left|D^2 f\right|.\end{aligned}$$

□

Conclusion

Our main goal was to develop a new algorithm for approximation of the non-increasing rearrangement of a function. The algorithm can be applied to functions with a convex polyhedral domain in $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$. The new algorithm yields better estimates than the older one. Further, we present a couple of error estimates of our method. The first estimate depends mostly on results from the finite element theory, the other estimates use aid of embedding inequalities. The algorithm can be generalized to higher dimensions. This can be subject to further research.

A. Implementation notes

The described algorithm for approximation of a non-increasing rearrangement of a function was implemented for the case $n = 2$ and the case $n = 1$ with clamped cubic splines. The aim of this chapter is to give overview of some details of this implementation. Though the algorithm is rather straightforward, there are few issues which were left out in the description of algorithms in Chapter 2. But first we should note some general facts about implementation.

The program is written in language JAVA and it uses the arbitrary precision floating-point number format, therefore the only limitation of precision of calculation is the capability of computer which execute it. Number operations with the arbitrary precision numbers are supported by the library Apfloat.Plotting graphic output is done with the aid of the libraries JCommon and JFreeChart.

The first issue which needs more attention is how to create a triangulation which determines the used partially linear approximation. This is done by a two-step algorithm which is described in the following. The input data for our algorithm is a convex domain and a bound for area of triangles. The desired output is a triangulation of a given domain such that area of no triangle in it exceed the given bound.

The first step of the algorithm divides the domain into initial triangles. This is done by adding a central point as an arithmetical average of vertices of domain and then creating a triangulation which consists of triangles with one vertex being the central point and the other two being endpoints of an edge of the domain. To this initial triangulation is then applied the second step of the algorithm which consists of several iterations. In each iteration step each triangle in triangulation is checked if it's area is smaller than a given parameter. If the triangle is too large, then it is divided into four smaller triangles. This iteration is repeated until all triangles have the sufficiently small area.

Let us show the process of division on the case of a triangle T with vertices v_1, v_2 and v_3. Triangle T is divided into four smaller triangles T_1, T_2, T_3 and T_4. Vertices of T_1 are $v_1, v_{1\,2}$ and $v_{1\,3}$, vertices of T_2 are $v_2, v_{1\,2}$, $v_{2\,3}$, vertices of T_3 are $v_3, v_{2\,3}, v_{1\,3}$ and vertices of the last triangle are $v_{1\,2}, v_{2\,3}, v_{1\,3}$, where $v_{i\,j} = \frac{v_i + v_j}{2}$. Triangle T is then removed from triangulation and triangles T_1, T_2, T_3, T_4 are added. Newly added triangles have smaller area.

However, the triangulation described above does not satisfy condition ($\mathcal{T}5$). This condition could be satisfied if we slightly change the second step of the algorithm. We change the condition for division of a triangle in trian-

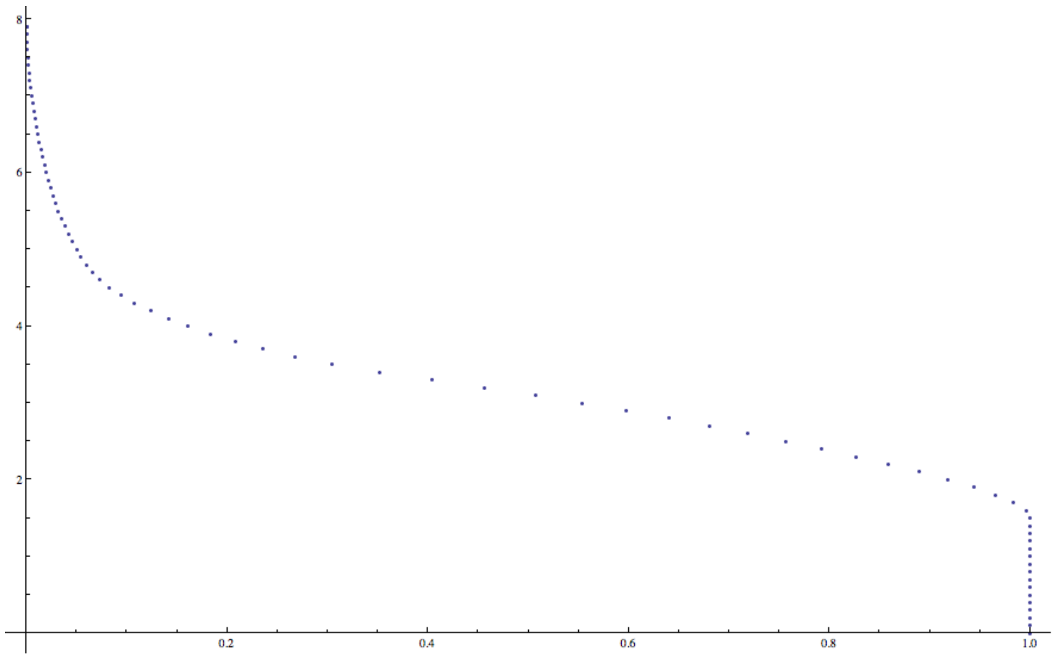

Figure A.1: Example of different density of nodes of a graph of a non-increasing rearrangement.

gulation. If there is any triangle in triangulation with too big area we have to divide all triangles in the triangulation from beginning of this iteration.

The second issue is how to plot the approximation of the rearrangement. The values of the non-increasing rearrangement are calculated through the values of the distribution function. We could divide the interval $[\min(|f|), \max(|f|)]$ with equidistant division and evaluate the distribution function in nodes of this division. Unfortunately, if we evaluate the distribution function in some equidistant division of its domain, then nodes for graph of the rearrangement usually accumulate at some part of the domain and in other areas the density of nodes could be too small as it is illustrated in Figure A.1. This effect is diminished by the following algorithm. The user specifies the maximal gap between two nodes of the final graph. The program then uses more precise division in parts of domain of distribution function where the distribution function changes more quick and less points where it is not necessary.

Bibliography

[1] Adams, R. A., and Fournier, J. J. F. *Sobolev Spaces*. Elsevier, Oxford, 2007.

[2] Bennett, C., and Sharpley, R. *Interpolation of Operators*. Academic Press, Princeton, 1988.

[3] Brenner, S. C., and Scott, L. R. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag New York, Inc., New York, 1994.

[4] Chiti, G. Rearrangements of functions and convergence in orlicz spaces. *Applicable Analysis 9* (1979), 23–27.

[5] Cianchi, A., and Pick, L. Sobolev embeddings into BMO, VMO, and L_∞. *Arkiv för Matematik 36* (1998), 433–446.

[6] Edmunds, D. E., Kerman, R., and Pick, L. Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms. *Journal of Functional Analysis 170* (2000), 307–355.

[7] Evans, L. C., and Gariepy, R. F. *Measure theory and fine properties of functions*. CRC Press Inc., Boca Raton, Florida, 1992.

[8] Franců, M., Kerman,R., Sayfy, A., and Phipps, C. Finite element approximations to rearrangements. *Preprint no. MATH-KMA 396* (2012), 1–20.
URL: "http://www.karlin.mff.cuni.cz/kma-preprints/"

[9] Hall, C. A., and Meyer, W. W. Optimal error bounds for cubic spline interpolation. *Journal of Approximation Theory 16* (1976), 105–122.

[10] Lorentz, G. G., and Shimogaki, T. Interpolation theorems for operators in function spaces. *Journal of Functional Analysis 2* (1968), 31–51.

[11] Steiner, J. *Gesammelte Werke*. Berlin, 1881–1882.

[12] Talenti, G. Rearrangements of functions and partial differential equations. *Lecture Notes in Mathematics 1224* (1986), 153–178.

[13] Zhang, S. On a nested refinement of anisotropic tetrahedral grids under hessian metrics. *Unpublished manuscript,* (2006).
URL: "http://www.math.udel.edu/~szhang/research/p/3dhessian.pdf"