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DIPLOMOVÁ PRÁCE

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Kvadraturní formule Clenshaw-Curtisova typu pro Gegenbauerovu váhovou funkci

Katedra numerické matematiky

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Prohlašuji, že jsem tuto diplomovou práci vypracoval samostatně a výhradně s použitím citovaných pramenů, literatury a dalších odborných zdrojů.

Beru na vědomí, že se na moji práci vztahují práva a povinnosti vyplývající ze zákona č. 121/2000 Sb., autorského zákona v platném znění, zejména skutečnost, že Univerzita Karlova v Praze má právo na uzavření licenční smlouvy o užití této práce jako školního díla podle §60 odst. 1 autorského zákona.

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Abstrakt: Táto práca sa venuje predovšetkým kvadratúrnym vzorcom založeným na Čebyševovom rozvoji, známym ako Clenshaw-Curtisove kvadratúry. V začiatkoch práce sa tak zaoberáme Čebyševovými polynómami, ich definíciami a vlastnosťami. Tieto vedomosti využijeme k odvodeniu Clenshaw-Curtisovej kvadratúry. Značná časť textu je venovaná porovnaniu tejto kvadratúry s obecne známou Gaussovou kvadratúrou ako teoreticky, tak aj na príkladoch. Clenshaw-Curtisovu kvadratúru následne rozšírime o Gegenbauerovu váhovú funkciu, čím získame nové metódy pre numerickú integráciu. Tieto metódy nám umožnia riešenie ďalších problémov, čo zdôrazníme na numerických experimentoch.

Kľúčové slová: Čebyševove polynómy, Clenshaw-Curtisova kvadratúra, Gaussova kvadratúra, Numerická integrácia

Title: : A quadrature formula of Clenshaw-Curtis type for the Gegenbauer weight-function
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Abstract: In this thesis we study especially quadrature formulae based on the Chebyshev expansion, known as the Clenshaw-Curtis quadrature. The first part is focused on the Chebyshev polynomials, their definitions and properties. This knowledge will be used to derivate the Clenshaw-Curtis quadrature. Considerable part of this work is dedicated to comparison of this and the well-known Gauss quadrature both theoretically and practicaly. In the further work we will extend the Clenshaw-Curtis quadrature by the Gegenbauer weight function which gives us new methods for numerical integration. These methods allow us to find a solution of some known problems what will be pointed out also on some numerical experimets.

Keywords: Chebyshev polynomials, Clenshaw-Curtis quadrature, Gauss quadrature, numerical integration

Contents

Introduction

Numerical integration is a very important part of the numerical analysis. We want to compute a definite integral by a quadrature using numerical techniques. Such a formula approximates the value of a definite integral by the use of known properties about the integrand at a set of discrete points. We will evaluate the quadrature formula based on the Chebyshev expansion and as we will derive, notable points of Chebyshev polynomials can be used as a set of discrete points. Such a quadrature formula is known as the Clenshaw-Curtis quadrature because C. W. Clenshaw and A. R. Curtis were the first to introduce this approach in 1960. As a bit curious fact can be considered that Hungarian mathematician L. Fejér presented two quadrature rules very similar to Clenshaw-Curtis quadrature in 1933.

Neverthless, we shall firstly introduce Chebyshev polynomials and their properties which we need for further work. In this part of work we use to refer to [14] and so we do not need to prove all the theorems. Some of the very important properties are the orthogonality and the even parity, which will be used through the whole work. Then we will take a short look at well known methods for numerical integration - Simpson's rule and Gauss quadrature. The observation and comparison of these two quadratures is motivating us to derive another method for numerical integration. Based on our knowledge of the Chebyshev expansion we will be able to derive Clenshaw-Curtis quadrature in the similar way as the founders made it in their work [2]. This quadrature will play the main role in the rest of the work.

Having a new quadrature we will firstly compare it with already existing and frequently used Gauss quadrature both theoreticaly and practically. Surprising results arise as the new quadrature is almost as accurate as Gauss quadrature which satisfies the factor-of-2 advantage in efficiency. Observations about these results were done already by C. W. Clenshaw and A. R. Curtis, but we will built this part of work upon the article of L. N. Trefethen [16] who proved the new error estimations of the Clenshaw-Curtis quadrature.

In the next part of work we briefly introduce Gegenbauer polynomials and the Gegenbauer weight function which is also known as "ultraspherical" weight function. With this function we will extend the previously derived Clenshaw-Curtis method for numerical integration. Within this part of work a lot of research was done by H. V. Smith and D. B. Hunter in their article [4]. We shall analyse this article and derive also method based on a new set of discrete points. After providing error estimations we will use these methods[1] in examples. For this purpose numerical software Matlab 7.5.0 is used.
H. V. Smith[9] has already pointed out that the "classic" numerical integration, where the quadrature is applied iteratively, is not the only option for obtaining a satisfying result. Based no his knowledge and Theorems proved in his previous articles [7], [8], [10] and [11], he developed with D. B. Hunter [12] a method where the quadrature rule is applied only once. After this is done, the exact error term is calculated.
In the end of this work we briefly mention use of Chebyshev expansion and therefore connection with Clenshaw-Curtis quadrature in the question of simple ordinary equation. Two basic methods are introduced shortly.
At the end I would like to point out a fact that Clenshaw-Curtis quadratures have been for a long time in a shadow of other methods for numerical integration. But recently, the team from the Oxford University under the supervision of L. N. Trefethen has developed "chebfun", what is a collection of algorithms which extends familiar powerful methods of numerical computation where a big impact is given on Cheyshev polynomials and also a Clenshaw-Curtis quadrature. This together with new theorems recently posted implies that methods based on Chebyshev polynomials are getting to the forefront.

[1]Corresponding Matlab source codes can be found on attached medium

Chapter 1

Chebyshev polynomials

Chebyshev polynomials are named after Russian mathematician Pafnuty Lvovich Chebyshev (May 16, 1821 - December 8, 1894) who is considered a founding father of Russian mathematics. As we will see, Chebyshev polynomials are a sequence of orthogonal polynomials which can be defined in many ways, for example recursively. We distinguish between four kinds of Chebyshev polynomials, but we shall focus on the Chebyshev polynomials of the first kind which will be used in the further work. The widely used notation T_n for these polynomials becomes from the alternative transliterations of the name Chebyshev used in French - Tchebycheff or Germany (Tschebyschow).

1.1 Chebyshev polynomials

Firstly, we shall introduce the Chebyshev differential equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + n^2y = 0, \quad |x| < 1, \quad n = 0, 1, 2, \dots, \tag{1.1}$$

where $y = y(x)$. If we substitute $x = \cos t$ we obtain

$$\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}x} &= -\frac{1}{\sin t}, \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{1}{\sin t}\frac{\mathrm{d}y}{\mathrm{d}t}, \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{1}{\sin t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sin t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{\sin t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right) = \\
&= \frac{1}{\sin^2 t}\left(\left(-\frac{\cos t}{\sin t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right).
\end{aligned}$$

Using these identities we can simplify the original equation to the form

$$\frac{\mathrm{d}^2 y}{dt^2} + n^2 y = 0.$$

General solution of this equation is $y(t) = A\cos nt + B\sin nt$. When we transform back to the variable x we get the form

$$y = A\cos(n\arccos x) + B\sin(n\arccos x), \quad |x| < 1, \tag{1.2}$$

or equivalently

$$y = AT_n(x) + BU_n(x), \quad |x| < 1,$$

where $T_n(x)$ and $U_n(x)$ respectively are Chebyshev polynomials of the first and second kind of degree n. The following definitions of Chebyshev polynomials can be found in [14] (Chapter 1).

Definition 1.1. *(Chebyshev polynomials of the first kind.)*
Denote $T_n(x)$ the Chebyshev polynomial of the first kind of the degree n above the interval $[-1, 1]$ defined by the recurrence relation

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x). \end{aligned} \tag{1.3}$$

They can be equivalently expressed by the following explicit formula

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k},$$

where $\lfloor n/2 \rfloor$ denotes the integer part of the real number $n/2$.
It leads to (see [14], Chapter 1)

$$T_n(x) = \cos\left(n \arccos x\right). \tag{1.4}$$

Moreover, when we substitute $x = \cos\theta$, where as the range for corresponding θ can be taken $[0, \pi]$, we get the equation

$$T_n(x) = \cos n\theta. \tag{1.5}$$

In the literature we can also find different expression for T_n known as the Rodrigues' formula:

$$T_n(x) = \frac{(-1)^n (1-x^2)^{1/2} \sqrt{\pi}}{2^{n+1}\Gamma(n+1/2)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left((1-x^2)^{n-1/2}\right),$$

where Γ is the well-known Gamma function [17] (Chapter 6)

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t, \quad \Re(z) > 0. \tag{1.6}$$

The Gamma function is very important for the rest of this work. Because its properties are well-known we will not write them again, they can be easily found in a numerous literature, for instance [17] (Chapter 6).

Definition 1.2. *(Chebyshev polynomials of the second kind.)*
Denote $U_n(x)$ the Chebyshev polynomial of the second kind of the degree n above the interval $[-1, 1]$ defined by the recurrence relation

$$\begin{aligned} U_0(x) &= 1, \\ U_1(x) &= 2x, \\ U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x). \end{aligned} \tag{1.7}$$

Also in this case we can use the explicit formula

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(n-k)!}{k!(n-2k)!} (2x)^{n-2k} = \frac{T'_{n+1}(x)}{n+1}.$$

If we use the same substitution as above, we can obtain the following formula (see [14], Chapter 1)

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \theta \in [0, \pi]. \tag{1.8}$$

Rodrigues' formula for the Chebyshev polynomial of the second kind $U_n(x)$ has the following form

$$U_n(x) = \frac{(-1)^n (n+1)\sqrt{\pi}}{(1-x^2)^{1/2} 2^{n+1} \Gamma(n+3/2)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left((1-x^2)^{n+1/2} \right).$$

There are also Chebyshev polynomials of the third and fourth kind. They are sometimes called "airfoil polynomials". We can find them in [14].

Definition 1.3. *(Chebyshev polynomials of the third kind.)*
Denote $V_n(x)$ the Chebyshev polynomial of the third kind of the degree n above the interval $[-1, 1]$ defined by the equation

$$V_n(x) = \frac{\cos\left(n + \frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}}, \tag{1.9}$$

where $x = \cos\theta$ as above.
The recurrence definition is given by relations

$$\begin{aligned} V_0(x) &= 1, \\ V_1(x) &= 2x - 1, \\ V_n(x) &= 2xV_{n-1}(x) - V_{n-2}(x). \end{aligned} \tag{1.10}$$

Definition 1.4. *(Chebyshev polynomials of the fourth kind.)*
Denote $W_n(x)$ the Chebyshev polynomial of the fourth kind of the degree n above the interval $[-1, 1]$ defined by the equation

$$W_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}. \tag{1.11}$$

where $x = \cos\theta$ as above.
The recurrence definition is given by relations

$$\begin{aligned} W_0(x) &= 1, \\ W_1(x) &= 2x + 1, \\ W_n(x) &= 2xW_{n-1}(x) - W_{n-2}(x). \end{aligned} \tag{1.12}$$

There exist numerous identities between these polynomials which can be easily found in many books and articles, for instance see [14]. They are not stated here because we are using only the Chebyshev polynomials of the first kind in the rest of this work.

Remark. We shall use the simplified notation "Chebyshev polynomial" instead of "Chebyshev polynomial of the first kind".

1.2 Properties of the Chebyshev polynomials

There are many interesting and important known properties of the Chybeshev polynomials. We introduce some of them which will be used later, others can be easily found in literature, for example see [14].

Chebyshev polynomials of the p-th kind ($p = 1, 2, 3, 4$) are orthogonal with respect to the corresponding weight as it is shown in [14] (Section 4.2).

Theorem 1.5. *Chebyshev polynomials T_n are orthogonal with respect to the weight $1/\sqrt{1-x^2}$. Thus*

$$\begin{aligned} \int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} \mathrm{d}x &= 0, \quad i \neq j, \\ &= \pi, \quad i = j = 0, \\ &= \frac{\pi}{2}, \quad i = j \neq 0. \end{aligned} \tag{1.13}$$

It is essential to know where the zeros and extrema of the Chebyshev polynomial are. These important points can be obtained from the definition (1.5) and we can find them together with the basic derivative relation in [14] (Section 2.2).

Theorem 1.6. *Roots of the Chebyshev polynomial T_n ($n > 0$) are points*

$$x_k = \cos\left(\frac{\pi(2k+1)}{2n}\right), \quad k = 1, \dots, n. \tag{1.14}$$

Theorem 1.7. *The derivative of T_n can be expressed by the following equation*

$$\frac{\mathrm{d}}{\mathrm{d}x} T_n(x) = \frac{\frac{\mathrm{d}}{\mathrm{d}\theta} \cos n\theta}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{n \sin n\theta}{\sin\theta}, \quad x = \cos\theta. \tag{1.15}$$

By using the properties of $\sin$ *and reccurence definition (1.3) of the T_n we can also obtain an interesting reccurence relation*

$$(1-x^2)T_n'(x) = -nxT_n(x) + nT_{n-1}(x), \quad n > 1. \tag{1.16}$$

Theorem 1.8. *Extrema of the Chebyshev polynomial T_n ($n > 1$) are attained if*

$$x_k = \cos\left(\frac{\pi k}{n}\right), \quad k = 1, \dots, n. \tag{1.17}$$

Chebyshev polynomials also hold the discrete orthogonality as is shown in [14] (Section 4.6), over the discrete point set $\{x_k\}$ consisting of the zeros of $T_{n+1}(x)$ and over the set consisting of the extrema of $T_n(x)$. We shall introduce both these conclusions.

Theorem 1.9. *If we choose $\{x_k\}$ to be the set of extrema of $T_n(x)$, Chebyshev polynomials satisfy the following condition for the discrete orthogonality*

$$\begin{aligned} \sum_{k=0}^{n}{}'' T_i(x_k)T_j(x_k) &= 0, \quad i \neq j \leq n, \\ &= \frac{n}{2}, \quad 0 < i = j < n, \\ &= n, \quad i = j \in \{0, n\}, \end{aligned} \tag{1.18}$$

where x_k are the extrema of $T_n(x)$ and the double dash $''$ indicates that the first and the last term in the sum are to be halved.

In the case when we choose $\{x_k\}$ to be the set of zeros of $T_{n+1}(x)$, the relation is modified

$$\begin{aligned}\sum_{k=1}^{n+1} T_i(x_k)T_j(x_k) &= 0, \quad i \neq j \leq n,\\ &= \frac{n+1}{2}, \quad 0 < i = j \leq n,\\ &= n+1, \quad i = j = 0,\end{aligned} \tag{1.19}$$

where x_k are the zeros of $T_{n+1}(x)$.

Theorem 1.10. *Let $f(x)$ be a continuous function over the interval $[-1, 1]$. Then it can be expanded as a series of the Chebyshev polynomials ([14], Section 5.2)*

$$f(x) = \frac{1}{2}A_0T_0(x) + A_1T_1(x) + \cdots = \sum_{n=0}^{\infty}{}^{*} A_nT_n(x), \tag{1.20}$$

where the asterisk means that the first term is to be halved. The coefficients A_n are given by the following formula

$$A_n = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}}\,\mathrm{d}x, \quad n = 0, 1, 2, 3, \ldots. \tag{1.21}$$

The property which will be widely used later is the one describing the even parity. It can be easily obtained from the relation (1.3).

Theorem 1.11. *Whether Chebyshev polynomial is an even or odd function depends on its degree $n \in \mathbb{N}_0$. Chebyshev polynomials satisfy the following relationship*

$$T_n(-x) = (-1)^nT_n(x). \tag{1.22}$$

Thus if n is even $T_n(x)$ is even as well; otherwise $T_n(x)$ is odd.

Proof. This property deduces directly from the relation (1.3) as $g(x) = x$ is odd function and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ (starting with $T_0(x) = 1$ and $T_1(x) = x$) is
- equal to odd function multiplied by odd function with subtracted even function, thus it is even function,
- equal to odd function multiplied by even function with subtracted odd function, thus it is odd function. □

From the reccurence definition (1.3) of T_n we can also easily see the values of Chebyshev polynomials in their boundaries.

Theorem 1.12. *The values of Chebyshev polynomials in their boundaries* ± 1 *are*

$$T_n(1) = 1 \quad n = 0, 1, 2, \dots, \tag{1.23}$$

and

$$\begin{aligned} T_n(-1) &= 1 \qquad n = 0, 2, 4, 6, \dots, \\ T_n(-1) &= -1 \quad n = 1, 3, 5, 7, \dots. \end{aligned} \tag{1.24}$$

The following property may be obtained by applying the formula for derivative (1.15) on Chebyshev polynomials T_{n+1} and T_{n-1}.

Theorem 1.13. *Chebyshev polynomials* T_n *satisfy the following property*

$$T_n(x) = \frac{1}{2}\left(\frac{T'_{n+1}(x)}{n+1} - \frac{T'_{n-1}(x)}{n-1}\right), \quad n > 1. \tag{1.25}$$

Based on this formula we can prove ([14], Chapter 2.4.4) the following property for the integration of the Chebyshev polynomial which will be used later in the Section 2.3.

Theorem 1.14. *The indefinite integral of Chebyshev polynomials can be expressed in terms of Chebyshev polynomials as follows ([14], Chapter 2.4.4)*

$$\begin{aligned} &\int T_0(x)\mathrm{d}x = x + const, \\ &\int T_1(x)\mathrm{d}x = \frac{x^2}{2} + const, \\ &\int T_n(x)\mathrm{d}x = \frac{1}{2}\left(\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right) + const, \quad n = 2, 3, 4, \dots. \end{aligned} \tag{1.26}$$

The following two relations can be found in the article [3], (p. 126, eq. (3.1) and (3.2)).

Theorem 1.15. *Suppose* $\lambda > -\frac{1}{2}$ *and* $r \in \mathbb{Z}$*. Then the Chebyshev polynomials and Gamma function satisfy the following identity ([3], Eq.(3.1))*

$$\int_{-1}^{1} \left(1 - x^2\right)^{\lambda - 1/2} T_{2r}(x)\mathrm{d}x = \frac{(-1)^r \Gamma\left(\lambda + \frac{1}{2}\right)\Gamma(\lambda + 1)\sqrt{\pi}}{\Gamma\left(\lambda + r + 1\right)\Gamma\left(\lambda - r + 1\right)}. \tag{1.27}$$

Gamma function holds also the following property ([3], Eq.(3.2))

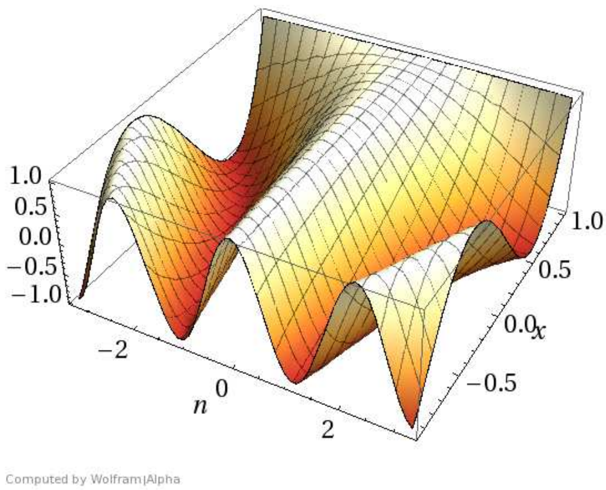
$$\frac{(-1)^r\,(\Gamma(\lambda+1))^2}{\Gamma(\lambda+r+1)\Gamma(\lambda-r+1)} = \prod_{j=1}^{r}\frac{j-1+\lambda}{j+\lambda}. \tag{1.28}$$

Based on this equation denote

$$\begin{aligned} G_r(\lambda) &= \prod_{j=1}^{r}\frac{j-1-\lambda}{j+\lambda}, & r>0,\\ &= 1, & r=0,\\ &= G_{-r}(\lambda), & r<0. \end{aligned} \tag{1.29}$$

The previous Theorem 1.15 plays an important role in the process of derivation the desired method in Chapter 5 and its error estimation. We will be using the same notation G_r as we have just defined.

The visualisation of Chebyshev polynomials T_n (for $n < 0$ expanded symetrically) is very interesting even for small values of n as we can see on the following graph.



We have introduced Chebyshev polynomials and some properties of Chebyshev polynomials of the first kind. With this knowledge we can derive the Clenshaw-Curtis quadrature. The theory of the Chebyshev polynomials is very large and the polynomials of second, third and fourth kind which have huge impact in various theories. There are also the shifted Chebyshev polynomials of all four kinds. It is available in various literature, for example [14].

Chapter 2

Derivation of the Clenshaw-Curtis quadrature

Within this chapter we derive the Clenshaw-Curtis quadrature which was firstly presented by C. W. Clenshaw and A. R. Curtis in [2]. This kind of numerical integration is based on an expansion of the integrand in terms of Chebyshev polynomials. We will find out that there is also a very close connection between the Chebyshev expansion, which is used as an essential of Clenshaw-Curtis quadrature, and Fourier transformation.

2.1 Motivation

At the begining of this section we shall introduce two well known methods of numerical integration. One of them is (composite) Simpson's rule which is important tool in the theory of numerical integration and can be found in every basic handbook of numerical analysis, for example [15] (p. 365-375).

Method 2.1. *(Simpson's rule.)*
Let $f \in C^4[a,b]$. The three-point Newton-Cotes formula, known as Simpson's rule is given by

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{b+a}{2}\right) + f(b)\right). \tag{2.1}$$

If we split the interval $[a,b]$ into even number of equal subintervals ($x_0 = a$, $x_n = b$) we get the following formula known as the composite Simpson's rule *with equidistant nodes*

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{3}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots \\ \cdots + 4f(x_{n-1}) + f(x_n)\big), \tag{2.2}$$

where $h = (b-a)/n$ *and* $x_j = a + jh$ *for* $j = 0, 1, \ldots n-1$.
The error arising by using this method is in absolute value bounded by the value

$$\frac{h^4}{180}(b-a)\max_{x\in[a,b]}\left|f^{(4)}(x)\right|. \tag{2.3}$$

Method 2.2. *(Guass quadrature.)*
Gauss quadrature (also know as Gauss-Legendre quadrature) is given by the formula ([17], p.887)

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} w_i f(x_i), \tag{2.4}$$

where

$$w_i = \frac{2}{(1-x_i^2)\left(P_n'(x_i)\right)^2}. \tag{2.5}$$

and P_n *denotes the Legendre polynomial of degree* n *and* x_i *is the* i*-th zero of* $P_n(x)$.

$$\frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3}f^{(2n)}(\xi), \quad -1 \le \xi \le 1. \tag{2.6}$$

Using the transition from $[-1,1]$ *to* $[a,b]$ *we get the formula*

$$\int_a^b f(y)\mathrm{d}y \approx \frac{b-a}{2}\sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right),$$

Remark. The zeros of $P_n(x)$ are not equidistant and neither are the nodes of this method.

The error arising using this method is limited by
Knowing the Newton-Cottes and Gauss formulae for numerical integration the next question can be asked. Why do we need another one?
Every method has its own pros and cons. For example, Gauss formula does not have problem with rounding errors and converge for any continuous function. It even has a factor-of-2 advantage in efficiency. But it is rather inappropriate to

indefinite integration. Also iteration with growing n requires a new set of x_i and weights w_i without using previously computed values. Thus a lot of computation has to be done what raises the prize of this method.
On the other hand, Simpson's rule is relatively easy to implement. The iteration also uses previously computed values. But taking into account for example highly-oscilated function, much more function values will be needed to estimate satisfactory the integral, comparing to the previous Gauss quadrature scheme. And still we can encounter a problem with check failure. This problem occurs when we are trying to establish the correctness of our result (with using n nodes) by comparing to double amount of nodes. We can get both results wrong and neverthless one could think that this result is correct. The higher order Newton-Cotes formulae can also have negative coefficients what can lead to significant rounding errors.
We will derive the Clenshaw-Curtis formula which is based on the term by term integration of the function expressed by a series of the Chebyshev polynomials. As we will see, the unique advantage of such a method is that its accuracy may be checked before the integration is completed. There are also some other advantages such as increasing the number of the ordinates without previous work being wasted (similarly to Simpson's rule) or accuraccy which is surprisingly comparable to the Gauss formula. This phenomenom will be discussed more precisely in the Chapter 3.

2.2 Relation with the Fourier transformation

From the first look we can suppose that there is some relation between the Chebyshev expansion and the Fourier cosine transformation. Such relations can be found in several publications, for example [14] (Section 5.3).
Suppose $f \in \mathcal{L}_2[-1,1]$ with respect to the weight function $(1-x^2)^{-\frac{1}{2}}$. Thus via the usual change of variable we can define a new function (also as we used to do it earlier with $x = \cos\theta$)

$$g(\theta) = f(\cos\theta), \quad \theta \in [0,\pi]. \tag{2.7}$$

We can extend this definition to $\theta \in \mathbb{R}$ by introducing

$$\begin{aligned} g(\theta + 2\pi) &= g(\theta), \\ g(-\theta) &= g(\theta). \end{aligned}$$

Thus g becomes an $\mathcal{L}_2$-integrable, even, 2π-periodic function what is preferable for developing into a Fourier series. Since g is even we get the Fourier series with only the cosine terms

$$g(\theta) = \sum_{k=0}^{\infty}{}^{*} a_k \cos k\theta, \tag{2.8}$$

where the asterisk means that the first term is to be halved and

$$a_k = \frac{2}{\pi} \int_0^{\pi} g(\theta) \cos k\theta \, \mathrm{d}\theta. \tag{2.9}$$

If we transform back to the variable $x = \cos\theta$ we get the Chebyshev expansion given by the equation (1.20) with coefficients given by (1.21). Thus, apart from the change of variables, the Chebyshev series expansion is identical to the Fourier cosine series and the coefficients a_k occurring in these two expansions have identical values.
Thus if we choose the extrema of the Chebyshev polynomials (1.17) to be the set of points we can evaluate a_k by the discrete cosine transformation thanks to orthogonality (1.18)

$$a_k \approx \frac{2}{n} \left(\sum_{j=0}^{n}{}'' f\left(\cos \frac{\pi j}{n}\right) \cos \frac{\pi k j}{n} \right).$$

2.3 Derivation of the Clenshaw-Curtis quadrature

We shall use the same approach as it was done in [2]. Assume the integration of a non-singular function in a finite range.

Remark. We may observe that an infinite range may be transformed to a finite range, or approximated by a large finite range.

Every function $f(x)$ which is continuous and of bounded variation (real-valued function whose total variation is bounded) in (a, b) can be expanded as follows ([2], Chapter 5)

$$f(x) = F(t) = \frac{1}{2} a_0 T_0(t) + a_1 T_1(t) + a_2 T_2(t) + \cdots, \quad a \le x \le b, \tag{2.10}$$

where

$$T_r(t) = \cos(r \arccos t), \quad t = \frac{2x - (b+a)}{b-a}. \tag{2.11}$$

Integrating over an interval $[a, x]$ we obtain

$$\begin{aligned}\frac{2}{b-a}\int_a^x f(x)\mathrm{d}x &= \int_{-1}^t F(t)\mathrm{d}t,\\ &= \int_{-1}^t \left(\frac{1}{2}a_0T_0(t) + a_1T_1(t) + a_2T_2(t) + \cdots\right)\mathrm{d}t.\end{aligned}$$

If we use the propeties for integration (1.26) and the known boundaries of Chebyshev polynomials (1.24) we obtain a formula

$$\int_{-1}^t F(t)\mathrm{d}t = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{a_{r-1}-a_{r+1}}{2r} + \sum_{r=1}^{\infty}\frac{a_{r-1}-a_{r+1}}{2r}T_r(t).$$

If we now denote

$$\begin{aligned}b_r &= \frac{a_{r-1}-a_{r+1}}{2r}, \quad r = 1, 2, \ldots,\\ b_0 &= 2b_1 - 2b_2 + 2b_3 - \cdots,\end{aligned} \tag{2.12}$$

we can write

$$\int_{-1}^t F(t)\mathrm{d}t = \frac{1}{2}b_0 + b_1T_1(t) + b_2T_2(t) + \cdots. \tag{2.13}$$

The definite integral is then given by

$$\frac{2}{b-a}\int_a^b f(x)\mathrm{d}x = \int_{-1}^1 F(t)\mathrm{d}t = \frac{1}{2}b_0 + b_1 + b_2 + \cdots = 2(b_1 + b_3 + \cdots), \tag{2.14}$$

and the indefinite integral is given by the sum of series (2.13). Since Chebyshev polynomials are orthogonal, any polynomial $f(x)$ of degree N can be written in the form

$$\begin{aligned}f(x) = F(t) &= \frac{1}{2}a_0 + a_1T_1(t) + \cdots + a_{n-1}T_{N-1}(t) + \frac{1}{2}a_NT_N(t),\\ &= \sum_{r=0}^{N}{}'' a_rT_r(t),\end{aligned} \tag{2.15}$$

where $-1 \leq t \leq 1$ and $\sum''$ denotes a finite sum whose first and last terms are to be halved.

We can choose the extrema of the Chebyshev polynomial as the points t_s (see orthogonality (1.18) and Method 5.1) to get a Method based on the practical abscissae.
The coefficients a_r are then given by (see also Section 2.2)

$$a_r = \frac{2}{N} \sum_{s=0}^{N}{}'' F_s \cos \frac{\pi r s}{N}, \quad F_s = F\left(\cos \frac{\pi s}{N}\right), \tag{2.16}$$

which can be rewritten in the form

$$a_r = \frac{2}{N} \sum_{s=0}^{N}{}'' F_s T_s(t_r), \quad t_r = \cos \frac{\pi r}{N}. \tag{2.17}$$

Any function which satisfies the conditions necessary for convergence of its Chebyshev expansion can be approximated to any required accuracy by a finite series of the form (2.15) with coefficients given by the above formula (2.17).

We have pointed out the relation with the Fourier transformation. We already know that this kind of transformation can be achieved using the well-known Fast Fourier Transformation. This connection will be mentioned in the next chapter where we will also point out that the Clenshaw-Curtis quadrature is comparatebly accurate to Gauss quadrature.

Chapter 3

Comparison with the Gauss quadrature

One of the goals of this work is to compare general Clenshaw-Curtis quadrature with Gauss quadrature. As we know, the Guass quadrature has the advantage of the factor-of-2 in efficiency. On the other hand, Clenshaw-Curtis is much easier to implement but from the first look is half as efficient. However the numerical results are very surprising as was reported already by C. W. Clenshaw and A. R. Curtis ([2], 1960). Observations made by H. O'Hara and F. J. Smith ([6], 1968) indicates that both formulas are about equally accurate. Since then also other mathematicians made the same observation but L. N. Trefethen ([16], 2008) is the one who took a step further. He pointed out that for most of the integrands both quadratures reach very similar accuracy. And by using the FFT, Clenshaw-Curtis can be implemented in $O(n \log n)$ operations which is much better than solve the tridiagonal eigenvalue problem which arises in the implementation of Gauss quadrature.

Assume that we have given a continuous function f on $[-1, 1]$ and we wish to approximate the integral $I = I(f) = \int_{-1}^{1} f(x)\mathrm{d}x$ by sum

$$I_n = I_n(f) = \sum_{k=0}^{n} w_k f(x_k),$$

where the nodes x_k depend on n but not on f, for various n. Due to the fact that I_n is desired to be an interpolary quadrature, the weights w_k are given uniquely. Thus I_n integrates exactly polynomials of degree at most n. As we already know from (1.20), the Chebyshev series for $f \in \mathcal{C}[-1,1]$ is defined by

$$f(x) = \sum_{k=0}^{\infty}{}^{*} a_k T_k(x), \quad \text{with} \quad a_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}} \mathrm{d}x,$$

where the asterisk indicates that the first term ($k = 0$) is to be halved.

The Chebyshev polynomials T_n are orthogonal (1.13) with respect to the weight function $1/\sqrt{1-x^2}$. Thus we can define the Chebyshev weighted norm as it was done in [16] (p. 75, Eq. (4.5)).

Definition 3.1. *The Chebyshev-weighted 1-norm $\|.\|_T$ is defined by*

$$\|u\|_T = \left\| \frac{u'(x)}{\sqrt{1-x^2}} \right\|_1,$$

where u is of bounded variation (real-valued function whose total variation is finite).

Several inequalities concerning this norm are proved in [16] (Chapter 4). They set bounds for the coefficients $|a_k|$ and lead to the following theorem which holds for the Gauss quadrature as well as for the Clenshaw-Curtis quadrature.

Theorem 3.2. *Let Gauss or Clenshaw-Curtis quadrature be applied to a function $f \in \mathcal{C}[-1,1]$. If $f, f', \ldots, f^{(k-1)}$ are absolutely continuous on the interval $[-1,1]$ and $\|f^{(k)}\|_T = V < \infty$ for some $k \geq 1$, then*

$$|I - I_n| \leq \frac{32V}{15\pi k(2n+1-k)^k}, \tag{3.1}$$

- *for $n \geq \frac{k}{2}$ for the Gauss quadrature.*
- *for $n \geq n_k$ where n_k depends on k for the Clenshaw-Curtis quadrature.*

Proof. We can find the proof in the [16] (p. 79-80). The main fact is that of "aliasing". On the grid in $[0, 2\pi]$ of $2n$ equally spaced points θ_k

$$\theta_k = \frac{\pi k}{n}, \quad 0 \leq k \leq 2n-1,$$

the following functions

$$\cos{(n+p)\pi\theta_k}, \quad \cos{(n-p)\pi\theta_k},$$

are indistinguishable for any $p \in \mathbb{Z}$. We use to say that the numbers $(n-p)\pi\theta_k$ and $(n+p)\pi\theta_k$ are "aliases" of one another on this grid. Based on this property one can see (transplanting $x = \cos\theta$) that

$$T_{n+p}(x_k) = T_{n-p}(x_k), \quad 0 \leq k \leq n, \tag{3.2}$$

on the grid of extreme points of Chebyshev polynomials $x_k = \cos\frac{\pi k}{n}$ (method based on the practical abscissae, see Method 5.1)[1]. Thus

$$\begin{aligned} I_n(T_{n+p}) = I_n(T_{n-p}) = I(T_{n-p}) &= 0, && n \pm p \quad \text{is odd}, \\ &= \frac{2}{1-(n-p)^2}, && n \pm p \quad \text{is even}, \end{aligned} \tag{3.3}$$

and the error in integrating is given by

$$\begin{aligned} I(T_{n+p}) - I_n(T_{n+p}) &= 0, \quad n \pm p \quad \text{is odd}, \\ &= \frac{8pn}{n^4 - 2(p^2+1)n^2 + (p^2-1)^2}, \quad n \pm p \quad \text{is even}. \end{aligned} \tag{3.4}$$

Remark. Here we can already see why is the Clenshaw-Curtis quadrature so accurate. If n is even then the first few terms in the Chebyshev expansion of f that contribute to the error $I(f) - I_n(f)$ are $a_{n+2}T_{n+2},\ a_{n+4}T_{n+4},\ \ldots$.

Now we can estimate

$$I(f) - I_n(f) = \sum_{k=0}^{\infty}{}^{*} a_k \left(I(T_k) - I_n(T_k)\right) \leq S_1 + S_2 + S_3 + S_4,$$

[1] Because of aliasing we have to evaluate only the coefficients a_k up to $k = n/2$. If $x_i = \cos\frac{\pi i}{n}$ then $T_n(x_i) = T_i(x_n)$.

where asterisk means that the first term of sum is to be halved and

$$S_1 = \sum_{k=0}^{n}{}^{*} |a_k|\,|I(T_k) - I_n(T_k)|,$$

$$S_2 = \sum_{k=n+1}^{2n-\lfloor n^{\frac{1}{3}} \rfloor} |a_k|\,|I(T_k) - I_n(T_k)|,$$

$$S_3 = \sum_{k=2n+1-\lfloor n^{\frac{1}{3}} \rfloor}^{2n+1} |a_k|\,|I(T_k) - I_n(T_k)|,$$

$$S_4 = \sum_{k=2n+2}^{\infty} |a_k|\,|I(T_k) - I_n(T_k)|.$$

The term $S_1 = 0$ because the quadrature formula is interpolatory. The other terms are estimated via inequalities proved in [16] and thanks to aliasing. In the case of Gauss quadrature the terms S_2 and S_3 are equal to 0 as well. Consider Clenshaw-Curtis quadrature.

The estimation of the term S_2 is based on a relation (3.4). Because of that relation we can write that the factors $|I(T_k) - I_n(T_k)|$ are of order at worst $n^{-\frac{2}{3}}$ and by inequality ([16], p. 75)

$$|a_n| \leq \frac{2V}{\pi n(n-1)\cdots(n-k)}, \text{ for each } n \geq k+1,$$

the coefficients a_j are of order Vn^{-k-1}. Thus S_2 consists of $O(n)$ terms of size $O(Vn^{-k-\frac{5}{3}})$ which means a total magnitude $O\left(Vn^{-k-\frac{2}{3}}\right)$.

In the same way the term S_3 consists of $O\left(n^{\frac{1}{3}}\right)$ terms of size $O\left(Vn^{-k-1}\right)$ which means a total magnitude $O\left(Vn^{-k-\frac{2}{3}}\right)$.

The term S_4 is still remaining for both quadratures. We know about the even parity of T_n (1.22) which means that T_j is odd whenever j is odd. Thus from the relation (3.3) for $j \geq 4$ (j is from $2n+2$)

$$\begin{aligned} |I(T_k) - I_n(T_k)| &\leq \frac{32}{15}, \text{ if } k \text{ is even}, \\ &\leq 0, \quad \text{if } k \text{ is odd}. \end{aligned}$$

In particular, the statement in the Theorem for the Clenshaw-Curtis method allows us to increase n a little further. Thus we can ensure $k \geq 6$ ($n > 2$ is enough for such case) and the constant $\frac{32}{15}$ can be improved to $2 + \frac{2}{6^2-1} = \frac{72}{35}$ which gives the result

$$S_1 + S_2 + S_3 + S_4 \leq O(Vn^{-k-\frac{2}{3}}) + \frac{72V}{35\pi k(2n+1-k)^k} < \frac{32V}{35\pi k(2n+1-k)^k}.$$

□

As we can see, the factor 2^{-k} in the error bound from the previous relation (3.1) is common for both quadratures. Yet the Clenshaw-Curtis formula has essentially the same performance for most integrands as Gauss formula. L. N. Trefethen shown also another explanation of this, mostly unexpected, phenomenom based on the rational approximation as can be found in [16] (Section 6).
We have already pointed out the relation with the Fourier cosine transformation in the Section 2.2. Thus we know that the Chebyshev coefficients a_k can be expressed as the *cosine* transformation. And such a formula is mostly calculated by the Fast Fourier Transformation algorithm (**FFT**). C. W. Clenshaw and A. R. Curtis published their work in 1960 while the FFT was introduced in 1965. The connection with the FFT was pointed out by W. M. Gentleman (*Implementing Clenshaw-Curtis quadrature*, 1972). The Clenshaw-Curtis method based on the extrema of Chebyshev polynomial can be implemented in Matlab environment as it was done in [16].

```
function I = clenshaw_curtis(f,n) //(n+1)-pt C.-C. quad. of f
x = cos(pi*(0:n)'/n); //extrema of Chebyshev polynomials
fx = feval(f,x); //f evaluated at these points
g = real(fft(fx([1:n+1 n:-1:2])/(2*n))); //Fast Fourier Transform
a = [g(1); g(2:n)+g(2*n:-1:n+2); g(n+1)]; //Chebyshev coeffs
w = 0*a'; w(1:2:end) = 2./(1-(0:2:n).^2); //weight factor
I = w*a; //the integral
```

The following function shows one of the possible ways how to implement the Gauss formula in the Matlab enviroment as it was done in [16].

```
function I = gauss(f,n) //(n+1)-pt Gauss quadrature of f
beta = .5./sqrt(1-(2*(1:n)).^(-2)); //3-term recurrence coeffs
T = diag(beta,1)+diag(beta,-1); //Jacobi matrix
[V,D] = eig(T); //eigenvalue decomposition
x = diag(D); [x,i] = sort(x); //nodes (= Legendre points)
w=2*V(1,i).^2; //weights
I=w*feval(f,x); //the integral
```

The main difference between these two codes is that the Clenshaw-Curtis representation is more efficient than the Gauss representation where we have to face the eigenvalue problem. In the case of Clenshaw-Curtis formula we can also store all previously calculated values of the function and use them again whenever we decide to use more points ($n_2 > n$). Therefore Clenshaw-Curtis does not require many more evaluations of function to converge to a desired accuracy.

For ilustration we can provide the distribution of Chebyshev, Gauss and Newton-Cotes (equidistant) points for $n = 42$.

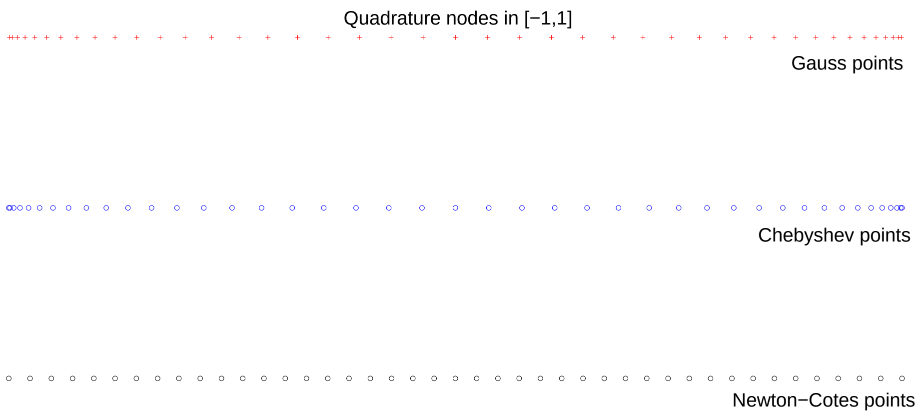


Figure 3.1: Distribution of Chebyshev, Gauss and Newton-Cotes points.

Remark. This figure was obtained by Matlab function *display_points.m* which can be found on attached DVD.

We shall provide some simple examples to compare these two methods practically. Within these examples we provide rounded results of both methods as we are more interested in absolute error $|I(f)-I_n(f)|$ which we display also graphically. The absolute error is on the axis y; on the axis x are the numbers n as we use $(n+1)$-pt. quadrature.

Example 3.3. *We can integrate exactly the polynomial*

$$I = \int_{-1}^{1} x^{12} \mathrm{d}x = 2/13 \doteq 0.153846153846154.$$

The table of values provided by method of Gauss and Clenshaw-Curtis for chosen n *follows.*

	Clenshaw-Curtis	abs. error	Gauss	abs. error
$n = 4$	0.15000	3.846×10^{-3}	0.14585	7.994×10^{-3}
$n = 6$	0.14777	6.078×10^{-3}	0.15385	1.110×10^{-16}
$n = 12$	0.15385	5.551×10^{-17}	0.15385	2.776×10^{-15}

Remark. The absolute error is calculated by the difference $|I - I_n|$ where I_n is the result obtained by chosen $(n + 1)$-pt. quadrature.

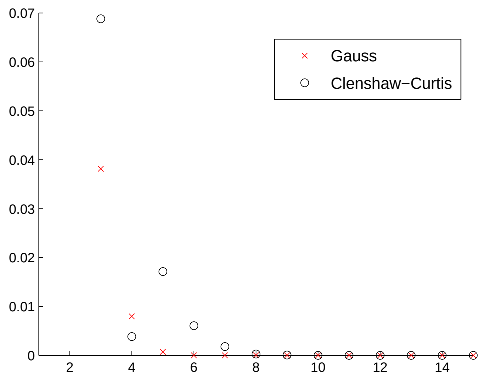


Figure 3.2: The absolute error.

Remark. It is no surprise that we have achieved these results for a polynomial of degree 12. Gauss quadrature integrates exactly polynomials of degree $2n+1$ and thus we have the error comparable to the *machine epsilon*[2] since $n = 6$. On the other hand, Clenshaw-Curtis quadrature integrates exactly polynomials of degree n.

[2] We are using double precision by default, which means that *machine epsilon* $\doteq 1.11 \times 10^{-16}$

Example 3.4. *The exact value of the integral of the exponential function is*

$$\int_{-1}^{1} e^x \mathrm{d}x \doteq 2.350402387287603.$$

We shall provide similar table as in the example above.

	Clenshaw-Curtis	abs. error	Gauss	abs. error
$n = 2$	2.3621	1.165×10^{-2}	2.3503	6.546×10^{-5}
$n = 6$	2.3505	2.059×10^{-8}	2.3504	3.109×10^{-15}
$n = 12$	2.3504	0	2.3504	1.332×10^{-15}

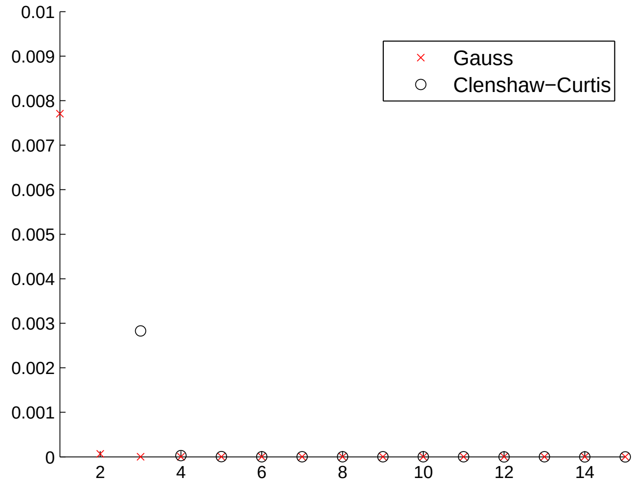


Figure 3.3: The absolute error.

The exponential function is entire and therefore it is analytic through the complex plane. As we can see, Gauss quadrature significantly outperforms Clenshaw-Curtis quadrature for small n as both quadratures converge very fast.

Remark. Within the next examples we shall take as the exact solution of the following integrals the result obtained by the function *quad($f(x)$,a,b, tol)* which is built in the environment Matlab 7.5.0. This method approximates the integral of function over the integral $[a, b]$ via recursive adaptive Simpson quadrature. Its basic tolerance is set to 10^{-6}, but we shall use 10^{-10} instead.

Example 3.5. *We can write the exact (see* Remark *above) result*

$$\int_{-1}^{1} \frac{1}{1+8x^4} \mathrm{d}x \doteq 0.870419751312476.$$

We shall provide table with rounded values of the selected methods.

	Clenshaw-Curtis	abs. error	Gauss	abs. error
$n = 10$	0.8725	2.098×10^{-3}	0.8719	7.617×10^{-4}
$n = 20$	0.8704	2.407×10^{-6}	0.8704	7.462×10^{-7}
$n = 30$	0.8704	5.537×10^{-10}	0.8704	7.842×10^{-10}

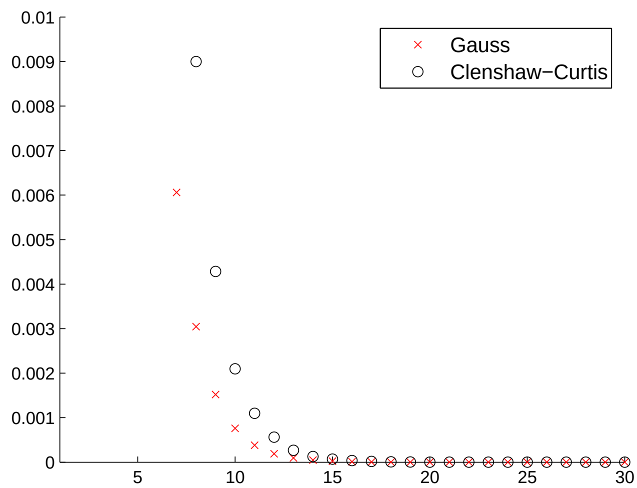


Figure 3.4: The absolute error.

This function is analytic in a neighborhood of $[-1, 1]$ *but not through the complex plane. We can see that Gauss quadrature again outperforms Clenshaw-Curtis quadrature but this time not as significantly as in the previous examples.*

Example 3.6. *We shall investigate the frequently presented example*

$$\int_{-1}^{1} e^{-x^{-2}} \mathrm{d}x \doteq 0.178147711893461.$$

We shall provide table with rounded values of the selected methods.

	Clenshaw-Curtis	abs. error	Gauss	abs. error
$n = 10$	0.1787	5.441×10^{-4}	0.1781	1.866×10^{-5}
$n = 18$	0.1782	1.373×10^{-5}	0.1782	4.302×10^{-6}
$n = 35$	0.1781	2.465×10^{-8}	0.1781	4.221×10^{-9}

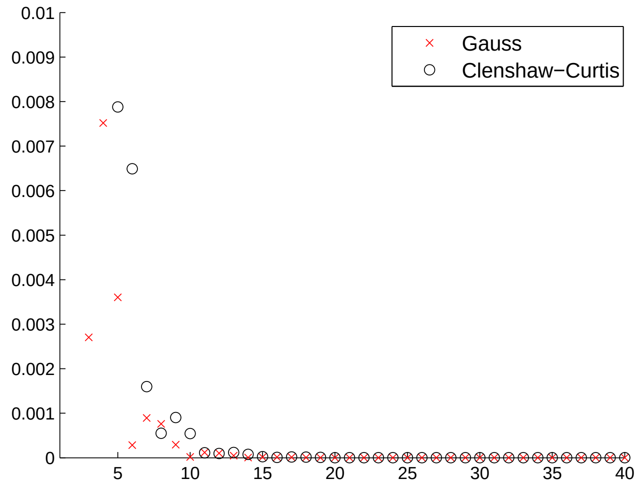


Figure 3.5: The absolute error.

This integrated function is from $C^\infty(\mathbb{R})$ and we can see that there is not such a big difference in the results provided by these two quadratures as in the previous example.

Example 3.7. *In this example we integrate a function which is not smooth*

$$\int_{-1}^{1} \sqrt{|2x+1|}\mathrm{d}x \doteq 2.065384140890834.$$

We shall provide table with rounded values of the selected methods.

	Clenshaw-Curtis	abs. error	Gauss	abs. error
$n = 151$	2.0635	1.866×10^{-3}	2.0656	1.875×10^{-4}
$n = 576$	2.0654	4.306×10^{-5}	2.0654	5.546×10^{-5}
$n = 1001$	2.0654	1.759×10^{-5}	2.0654	2.269×10^{-5}

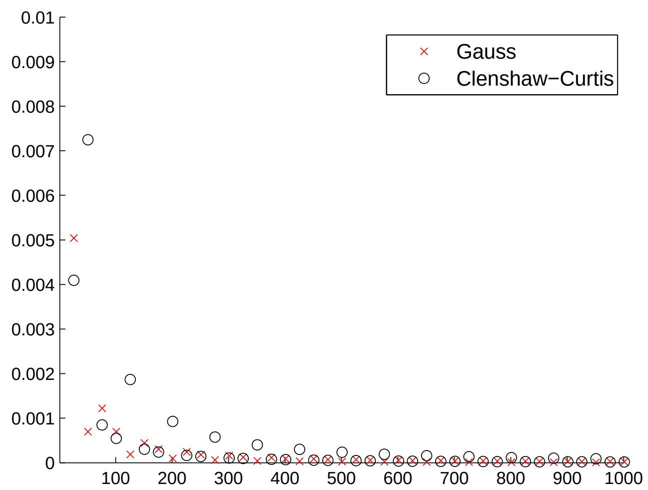


Figure 3.6: The absolute error.

In this case both methods converge very slowly but both achieve very similar accuracy. The visible difference is again only for n *small enough where are both quadratures very inaccurate.*

From the previous examples we can see that Gauss quadrature significantly outperforms Clenshaw-Curtis quadrature for functions analytic in a sizable neighborhood of $[-1, 1]$. But as we have also found out, for such functions both quadratures converge very fast and therefore there is not required many more function evaluations for Clenshaw-Curtis quadrature. Moreover, for functions that are not analytic in a sizable neighborhood of $[-1, 1]$, this quadrature is achieving comparable accuracy as Gauss quadrature. This can be seen from the condition in the Theorem 3.2 where the error estimation for Clenshaw-Curtis formula holds for $n > n_k$.

Chapter 4

Gegenbauer weight function

Within this chapter we introduce the Gegenbauer weight function. It is a weight function of the Gegenbauer polynomials which are a type of orthogonal polynomials. We shall shortly desribe these polynomials as they generalize Chebyshev polynomials. Firstly, define this weight function.

Definition 4.1. *(Gegenbauer weight function.)*
The Gegenbauer weight function is defined by the following formula

$$w(x) = \left(1 - x^2\right)^{\lambda - \frac{1}{2}}, \quad \lambda > -\frac{1}{2}. \tag{4.1}$$

Gegenbauer weight function is used in the theory of Gegenbauer (or ultraspherical) polynomials $C_n^{(\lambda)}(x)$. Those are orthogonal on the interval $[-1, 1]$ with respect to this weight function ([17], p. 773-785).

Definition 4.2. *(Gegenbauer polynomials.)*
Gegenbauer polynomials C_n^λ of degree n are defined by the recurrence relation

$$\begin{aligned} C_0^\lambda(x) &= 1, \\ C_1^\lambda(x) &= 2\lambda x, \\ C_n^\lambda(x) &= \frac{1}{n}\left(2x(n+\lambda-1)C_{n-1}^\lambda(x) - (n+2\lambda-2)C_{n-2}^\lambda(x)\right). \end{aligned} \tag{4.2}$$

They are orthogonal on the interval $[-1, 1]$ with respect to the weight function given by (4.1).
The equivalent Rodrigues' formula follows ([17], p. 785)

$$C_n^\lambda(x) = \frac{(-2)^n\,\Gamma(n+\lambda)\,\Gamma(n+2\lambda)}{n!\,\Gamma(\lambda)\,\Gamma(2n+2\lambda)}\left(1-x^2\right)^{\frac{1}{2}-\lambda}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\left(1-x^2\right)^{n+\lambda-\frac{1}{2}}\right). \tag{4.3}$$

Gegenbauer polynomials are particular solutions of the Gegenbauer differential equation

$$(1-x^2)y^{''} - (2\lambda+1)xy' + n(n+2\lambda)y = 0, \text{ where } y = y(x), \tag{4.4}$$

and they generalize other well known polynomials such as the Legendre polynomials or the Chebyshev polynomials. If $\lambda = \frac{1}{2}$ the equation (4.4) reduces to the following equation which is called the Legendre equation

$$(1-x^2)y^{''} - 2xy' + n(n+1)y = 0,$$

and the corresponding Gegenbauer polynomials are known as the Legendre polynomials P_n, see [17] (p. 779).
If $\lambda = 0$ the equation (4.4) reduces to the Chebyshev equation presented also in this work in the Chapter 1 (1.1).

Authors D. B. Hunter and H. V. Smith used Gegenbauer weight function as some "generalization" of the Clenshaw-Curtis quadrature in their work [4], [12] and we are looking for the integral of function g which can be written as $g(x) = (1-x^2)^{\lambda-1/2} f(x)$. That means we are looking for the approximation of the integral

$$I^{(\lambda)}(f) = \int_{-1}^{1} \left(1-x^2\right)^{\lambda-1/2} f(x)\mathrm{d}x, \quad \lambda > -\frac{1}{2}, \tag{4.5}$$

by using the Chebyshev expansion of the function $f(x)$. In the next Chapter we will introduce methods based on the term-by-term integration of the approximate Chebyshev series to achieve this objective.

Chapter 5

A quadrature formula of the Clenshaw-Curtis type for the Gegenbauer weight function

In this chapter we finally investigate a desired method which can be regarded as a generalization of the Clenshaw-Curtis quadrature by involving the Gegenbauer weight function. We will derive a method where we have to face the question - How shall we obtain the coefficients $a_{n,k}$ from the Chebyshev expansion? We have already suggested a choice of the extrema of the Chebyshev polynomials which is also one of the methods we will present more in details. Another method will be based on the zeros of these polynomials. We will also evaluate the method based on a different choice of nodes. The description is completed with the error estimates caused by these methods. At the end of this we will look at a method based on different approach, which H. V. Smith introduced [9].

5.1 Description of the method

Suppose a function f is analytic over some region of the complex plane containing the interval $[-1, 1]$ in its interior. Then we can denote by $I^{(\lambda)}(f)$ the integral

$$I^{(\lambda)}(f) = \int_{-1}^{1} \left(1 - x^2\right)^{\lambda - 1/2} f(x) \mathrm{d}x, \quad \lambda > -\frac{1}{2}. \tag{5.1}$$

We can approximate the function $f(x)$ by a finite Chebyshev expansion

$$f(x) \approx \sum_{r=0}^{n}{}'' a_{n,r} T_r(x), \tag{5.2}$$

where the double dash indicates sum whose first and last term are to be halved. There are several details depending on the parity of n. We shall set

$$n = 2s + \sigma, \tag{5.3}$$

where s is an integer and σ is equal to 0 or 1.
Now we would like to approximate $I^{(\lambda)}(f)$. Because of the symmetry of Chebyshev polynomial (1.22) we can see that

$$\begin{aligned} \int_{-1}^{1} \left(1 - x^2\right)^{(\lambda - \frac{1}{2})} \sum_{r=0}^{n}{}'' a_{n,r} T_r(x) \mathrm{d}x &= \int_{1}^{-1} \left(1 - x^2\right)^{(\lambda - \frac{1}{2})} \sum_{r=0}^{s}{}^{**} a_{n,2r} T_{2r}(x) \mathrm{d}x, \\ &= \sum_{r=0}^{s}{}^{**} a_{n,2r} \int_{-1}^{1} \left(1 - x^2\right)^{(\lambda - \frac{1}{2})} T_{2r}(x) \mathrm{d}x, \end{aligned}$$

where the double asterisk ** indicates that the first term is to be halved and also last term for n even. We receive this expression

$$\Psi_n^{(\lambda)}(f) = \sum_{r=0}^{s}{}^{**} a_{n,2r} I^{(\lambda)} \left(T_{2r}\right). \tag{5.4}$$

This can be simplified using equations previously introduced in Theorem 1.15 to the form

$$\Psi_n^{(\lambda)}(f) = \frac{\sqrt{\pi}\, \Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda + 1)} \sum_{r=0}^{s}{}^{**} a_{n,2r} G_r(\lambda). \tag{5.5}$$

Remark.

$$\begin{aligned} G_r(\lambda) &= \prod_{j=1}^{r} \frac{j - 1 - \lambda}{j + \lambda}, & r > 0, \\ &= 1, & r = 0, \\ &= G_{-r}(\lambda), & r < 0. \end{aligned}$$

If we write in details terms of this sum

$$\sum_{r=0}^{s}{}^{**} a_{n,2r}G_r(\lambda) = \frac{1}{2}a_{n,0} + a_{n,2}\frac{-\lambda}{1+\lambda} + a_{n,4}\frac{-\lambda}{1+\lambda}\frac{1-\lambda}{2+\lambda} + \\ + a_{n,6}\frac{-\lambda}{1+\lambda}\frac{1-\lambda}{2+\lambda}\frac{2-\lambda}{3-\lambda} + \\ + \dots a_{n,2s}\frac{-\lambda}{1+\lambda}\cdots\frac{s-1-\lambda}{s+\lambda}\frac{\sigma+1}{2},$$

we can see that this sum can be calculated by following reccurence

$$\begin{aligned} u_s &= \frac{\sigma+1}{2}a_{n,2s}, \\ u_r &= \frac{r-\lambda}{r+\lambda+1}u_{r+1} + a_{n,2r}, \quad r = s-1, s-2, \dots, 1, \\ u_0 &= -\frac{\lambda}{\lambda+1}u_1 + \frac{1}{2}a_{n,0}. \end{aligned} \tag{5.6}$$

Then using all the previous identities, we obtain

$$\Psi_n^{(\lambda)}(f) = \frac{\sqrt{\pi}\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda+1)}u_0. \tag{5.7}$$

Here appears the question - how shall we obtain the coefficients $a_{n,r}$? There are two common methods, one is based on evaluating on the coefficients in the extrema of T_n ("practical abscissae") and the other one based on the zeros of the T_{n+1} ("classical abscissae"). We can find these methods in [4]. We will also try another way by using points

$$x_k = \cos\frac{(6k-2)\pi}{3n+1}, \quad k = 1, 2, \dots n+1. \tag{5.8}$$

Method 5.1. *(Practical abscissae.)*
According to (1.17) the extreme points of T_n can be easily found. As we have already shown ((2.17) and Section 2.2) with such a choice we have

$$a_{n,r} = \frac{2}{n}\sum_{i=0}^{n}{}'' f(x_i)T_i(x_r), \text{ where } x_i = \cos\frac{\pi i}{n},$$

which can be evaluated as

$$a_{n,r} = \frac{2}{n}\left(\frac{f(1)}{2} + \sum_{i=1}^{n-1} f\left(\cos\frac{i\pi}{n}\right)\cos\frac{ri\pi}{n} + \frac{(-1)^r}{2}f(-1)\right). \tag{5.9}$$

If we denote

$$w_{n,i} = \frac{2}{n} \sum_{r=0}^{s}{}^{*} I^{(\lambda)}(T_{2r}) \cos \frac{2\pi r i}{n},$$

and because of the symetry shown by (1.22) we can rearrange the equation (5.4) to the form

$$\Psi_n^{(\lambda)}(f) = \sum_{i=0}^{n}{}'' w_{n,i} f\left(\cos \frac{i\pi}{n}\right). \tag{5.10}$$

Method 5.2. *(Classical abscissae.)*
The zeros of T_n are given by (1.14). If we choose the zeros of T_{n+1} for the points x_j from the previous method, then the approximation (5.2) of the function $f(x)$ can be replaced by following sum ([5], p.236-237)

$$f(x) \approx \sum_{r=0}^{n}{}^{*} b_{n,r} T_r(x), \tag{5.11}$$

where the asterisk indicates that the first term is to be halved and

$$b_{n,r} = \frac{2}{n+1} \sum_{j=0}^{n} f(x_j) T_r(x_j), \quad r = 0, 1, \ldots, n. \tag{5.12}$$

Zeros of T_{n+1} occur if $x_j = \cos \frac{(2j+1)\pi}{2n+2}$. Thus

$$b_{n,r} = \frac{2}{n+1} \sum_{j=0}^{n} f\left(\cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)\right) \cos\left(\frac{(2j+1)r\pi}{2(n+1)}\right), \quad r = 0, 1, \ldots, n. \tag{5.13}$$

Now denote the approximation of $I^{(\lambda)}(f)$ by $\Phi^{(\lambda)}(f)$ (instead of $\Psi_n^{(\lambda)}$ which we used in previous case). Then

$$\Phi_n^{(\lambda)}(f) = \sum_{r=0}^{s}{}^{*} b_{n,2r} I^{(\lambda)}(T_{2r}) = \frac{\sqrt{\pi}\,\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda+1)} \sum_{r=0}^{s}{}^{*} b_{n,2r} G_r(\lambda). \tag{5.14}$$

Method 5.3. *(The new ascissae.)*
We derive a new method based on the points given by equation (5.8).
Firstly, we shall deduce the orthogonality of the Chebyshev polynomials on this set of points.
Consider a sum

$$s_n(\theta) = \sum_{k=1}^{n+1} \cos\left(k - \frac{1}{3}\right)\theta = \cos\frac{2}{3}\theta + \cos\frac{5}{3}\theta + \cdots + \cos\left(n + \frac{1}{2}\theta\right). \tag{5.15}$$

We can immediately see that

$$s_n(0) = n+1, \quad s_n(2\pi) = -\frac{1}{2}(n+1). \tag{5.16}$$

Now consider a sum of the geometric progression

$$z^{\frac{2}{3}}\left(1+z+z^2+\cdots+z^n\right) = z^{\frac{1}{3}}\frac{z^n-1}{z-1}. \tag{5.17}$$

If we now substitute $z = e^{i\theta}$ we can use the Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

to rewrite the sum (5.16) in the form

$$\left(\cos\frac{2}{3}\theta + i\sin\frac{2}{3}\theta\right)\left(1+\cos\theta+i\sin\theta+\cdots+\cos n\theta+i\sin n\theta\right).$$

The real part of the sum above consists of the following terms

$$\cos\frac{2}{3}\theta\cos k\theta = \frac{1}{2}\left(\cos\left(\left(k+2/3\right)\theta\right)+\cos\left(\left(k-2/3\right)\theta\right)\right),$$
$$i^2\sin\frac{2}{3}\theta\sin k\theta = -\frac{1}{2}\left(\cos((k-2/3)\,\theta)-\cos((k+2/3)\,\theta)\right),$$

which means that the real part of this progression is the same as the series presented in (5.15).
To evaluate the series (5.17), we shall begin with the denominator which we have to multiply by the complex conjugate term. Thus we get the denominator

$$\left(\left(e^{ix}-1\right)\left(e^{-ix}-1\right)\right) = 2-2\cos\theta = 4\sin^2\frac{\theta}{2}.$$

The nominator is equal to

$$e^{2i\theta/3}\left(e^{in\theta}-1\right)\left(e^{-i\theta}-1\right) = e^{i\theta(n-1/3)} - e^{i\theta(n+2/3)} - e^{i\theta(-1/3)} + e^{2i\theta/3},$$

whose real part can be evaluated in the form

$$\underbrace{\cos\left(n-1/3\right)\theta-\cos\left(n+2/3\right)\theta}_{2\sin\frac{\theta}{2}\sin\left(n+\frac{1}{6}\right)\theta}\underbrace{-\cos(\theta/3)+\cos(2\theta/3)}_{2\sin\frac{\theta}{2}\sin\frac{\theta}{6}},$$

which means that s_n given by formula (5.15) can be calculated in the form

$$s_n = \frac{\sin\left(\frac{3n+1}{6}\right)\theta\ \cos\frac{n\theta}{2}}{\sin\frac{\theta}{2}}. \tag{5.18}$$

We can see that if (r is integer)

$$\theta = \frac{6}{3n+1}r\pi, \quad 0 < r \le 3n, \tag{5.19}$$

then $s_n(\theta) = 0$.
Now consider a set of points given by (5.8), namely

$$x_k = \cos\theta_k, \quad \theta_k = \frac{6(k-\frac{1}{3})\pi}{3n+1}.$$

Then we can write for integers p, q, such as $0 < p, q < n$ that

$$\begin{aligned} c_{pq} \sum_{k=1}^{n+1} \cos p\theta_k \cos q\theta_k &= \frac{1}{2}\sum_{k=1}^{n+1}\left(\cos(p+q)\theta_k + \cos(p-q)\theta_k\right), \\ &= \frac{1}{2}\left(s_n\left(\frac{6(p+q)\pi}{3n+1}\right) + s_n\left(\frac{6(p-q)\pi}{3n+1}\right)\right), \end{aligned}$$

and from evaluated values of s_n above we can see that

$$\begin{aligned} c_{pq} &= 0, && p \ne q \le n, \\ &= \frac{1}{2}(n+1), && 0 < p = q \le n, \\ &= n+1, && p = q = 0. \end{aligned}$$

This means that we have just proved the orthogonality of Chebyshev polynomials on the set of $\{x_k\}$ given by (5.8)

$$\begin{aligned} \sum_{k=1}^{n+1} T_i(x_k)T_j(x_k) &= 0, && p \ne q \le n, \\ &= \frac{1}{2}(n+1), && 0 < p = q \le n, \\ &= n+1, && p = q = 0. \end{aligned}$$

We can write the n-th degree polynomial $p_n(x)$ interpolating $f(x)$ in the points given by (5.8) as a sum of Chebyshev polynomials

$$p_n(x) = \sum_{j=0}^{n}{}^{*} a_j T_j(x),$$

where the asterisk means that the first term is to be halved.
Setting $f(x_k) = p_n(x_k)$ follows that

$$f(x_k) = \sum_{j=0}^{n}{}^{*} a_j T_j(x_k).$$

We can multiply this equation by $\frac{2}{n+1} T_i(x_k)$ $(i \leq n)$ and sum for $k = 1$ to $n + 1$. Then, because of orthogonality, we receive the following relation

$$\frac{2}{n+1} \sum_{k=1}^{n+1} f(x_k) T_i(x_k) = \sum_{j=0}^{n}{}^{*} a_j \left(\frac{2}{n+1} \sum_{k=1}^{n+1} T_j(x_k) T_i(x_k) \right) = a_i.$$

To use the same notation as we used before we shall shift the index k so that $x_{r+1} = x_k$ what means that the set of points is given by

$$x_r = \cos \frac{(6r+4)\pi}{3n+1}, \quad r = 0, 1, 2, \ldots n. \tag{5.20}$$

Then the approximating formula is

$$f(x) \approx \sum_{r=0}^{n}{}^{*} a_{n,r} T_r(x),$$

where

$$a_{n,r} = \frac{2}{n+1} \sum_{k=0}^{n} f(x_k) T_r(x_k).$$

The approximation $\Psi^{(\lambda)}(f)$ of $I^{(\lambda)}(f)$ is then given by

$$\Psi_n^{(\lambda)}(f) = \frac{\sqrt{\pi}\, \Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda + 1)} \sum_{r=0}^{s}{}^{*} a_{n,2r} G_r(\lambda).$$

5.2 Error estimation

Within this section we will provide error analysis of methods presented above. This analysis is based on the fact which can be found in [20] (Section 3.6) for functions which are analytic within and on some contour containing points $\{x_k\}$. The error $f(x) - p_n(x)$ (where p_n is the interpolating polynomial of degree n) can be written as a contour integral given by the equation (5.22).

Let us denote the error in approximation (5.4) by

$$E_n^{(\lambda)}(f) = I^{(\lambda)}(f) - \Psi_n^{(\lambda)}(f). \tag{5.21}$$

Now we will distinguish between the cases of chosen abscissae presented above.

5.2.1 Practical abscissae

Suppose that f is analytic within and on some contour C in the complex plane containing the interval $[-1, 1]$ in its interior. As we know from aliasing (3.2), zeros of $T_{n+1}(x) - T_{n-1}(x)$ are points $x_{n,i} = \cos\frac{i\pi}{n}$, where $i = 0, 1, \ldots, n$, which is equal to the points where $T_n(x)$ reaches its extrema. Then based on the relation presented in [20] (Theorem 3.6.1) the error of the interpolation for $f(x)$ is given by the contour integral

$$e_n(x) = \frac{1}{2\pi i}\int_C \frac{(T_{n+1}(x) - T_{n-1}(x))\, f(z)}{(z-x)\,(T_{n+1}(z) - T_{n-1}(z))}\mathrm{d}z, \quad x \in [-1, 1]. \tag{5.22}$$

The error of the Clenshaw-Curtis quadrature is then given by formula

$$E_n^{(\lambda)}(f) = \int_{-1}^{1} e_n(x)\mathrm{d}x = \frac{1}{\pi i}\int_C \frac{\left(Q_{n+1}^{(\lambda)}(z) - Q_{n-1}^{(\lambda)}(z)\right) f(z)}{T_{n+1}(z) - T_{n-1}(z)}\mathrm{d}z, \tag{5.23}$$

where

$$Q_n^{(\lambda)}(z) = \frac{1}{2}\int_{-1}^{1} \frac{\left(1-x^2\right)^{\lambda-\frac{1}{2}} T_n(x)}{z-x}\mathrm{d}x, \quad z \notin [-1, 1]. \tag{5.24}$$

Authors ([4],[12],[18], [19]) choose the contour C to be an ellipse. We shall do so as well because of its simplicity.
Let us define the ellipse E_ρ with foci at ± 1 by the following equation

$$E_\rho = \frac{1}{2}\left(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}\right) :\; 0 \le \theta < 2\pi, \quad \rho > 1. \tag{5.25}$$

In practise we will work in terms of variable $\xi = \rho e^{i\theta}$, which is related to z so that $z = \frac{1}{2}(\xi + \xi^{-1})$ and $|\xi| > 1$.
We can develop the function $Q_n^{(\lambda)}(z)$ by an expansion. To do so, we can use the following theorem found in [4].

Theorem 5.4. *Let n be given by (5.3). Setting $z = \frac{1}{2}(\xi + \xi^{-1})$ where $|\xi| > 1$, $Q_n^{(\lambda)}(z)$ can be expanded as follows*

$$Q_n^{(\lambda)} = \frac{\Gamma(\lambda + \frac{1}{2})\sqrt{\pi}}{2\Gamma(\lambda + 1)} \sum_{k=1}^{\infty} \xi^{1-\sigma-2k} \left(Z_{k+s+\sigma}(\lambda) + Z_{k-s}(\lambda)\right). \tag{5.26}$$

Here

$$\begin{aligned} Z_r(\lambda) &= \prod_{j=1}^{r-1} \frac{j-\lambda}{j+\lambda}, \quad r \geq 1, \\ &= -Z_{1-r}(\lambda), \quad r \leq 1. \end{aligned} \tag{5.27}$$

Proof. Proof can be found, [4] (p. 392). It is based on the following idea. Setting $z = \frac{1}{2}\left(\xi + \xi^{-1}\right)$ and $x = \cos\theta$ in the equation (5.24) gives

$$Q_n^{(\lambda)}(z) = \int_0^{\pi} \frac{sin^{2\lambda}\theta \cos n\theta}{\xi - 2\cos\theta + \xi^{-1}} \mathrm{d}\theta.$$

Since we know, that ([18], p. 653)

$$\frac{\sin\theta}{\xi - 2\cos\theta + \xi^{-1}} = \sum_{r=1}^{\infty} \frac{\sin r\theta}{\xi^r},$$

we have

$$\begin{aligned} Q_n^{(\lambda)}(z) &= \int_0^{\pi} \sin^{2\lambda}\theta \cos n\theta \sum_{r=1}^{\infty} \frac{\sin r\theta}{\xi^r \sin\theta} \mathrm{d}\theta, \\ &= \frac{1}{2} \sum_{r=1}^{\infty} \xi^{-r} \int_0^{\pi} \sin^{2\lambda-1}\theta \left(\sin{(r+n)\theta} + \sin{(r-n)\theta}\right) \mathrm{d}\theta. \end{aligned} \tag{5.28}$$

From the equations in [13] (p. 397) we know that for $k \in \mathbb{N}$

$$\int_0^{\pi} \sin^{2\lambda-1}\theta \sin 2k\theta \mathrm{d}\theta = 0,$$

$$\int_0^{\pi} \sin^{2\lambda-1}\theta \sin{(2k-1)\theta} \mathrm{d}\theta = \frac{(-1)^{k+1}\pi\Gamma(2\lambda)}{2^{2\lambda-1}\Gamma(\lambda+k)\Gamma(\lambda-k+1)},$$

and by using the Duplication formula [17] (p. 256)

$$\Gamma(2\lambda) = (2\pi)^{-\frac{1}{2}} 2^{2\lambda-\frac{1}{2}} \Gamma(\lambda)\Gamma\left(\lambda + \frac{1}{2}\right),$$

with some further manipulations (required properties of Gamma function can be found in [17](Section 6.1)) we obtain

$$\int_0^{\pi} \sin^{2\lambda-1}\theta \sin(2k-1)\theta \mathrm{d}\theta = \frac{\Gamma(\lambda+\frac{1}{2})\sqrt{\pi}}{\Gamma(\lambda+1)} Z_k(\lambda).$$

By setting $r = 2k - 1 + \sigma$ in (5.28) we get the desired result. □

The following theorem is shown in [4] (p. 393).

Theorem 5.5. *The error $E_n^{(\lambda)}(f)$ satisfies the following estimation*

$$\left|E_n^{(\lambda)}(f)\right| \leq \frac{4\Gamma\left(\lambda+\frac{1}{2}\right)\sqrt{\pi}\rho^{\sigma} M(\rho)}{\Gamma(\lambda+1)(\rho^2-1)(\rho^n-\rho^{-n})}, \tag{5.29}$$

where

$$M(\rho) = \max_{z \in E_\rho} |f(z)|, \tag{5.30}$$

and the ellipse E_ρ is defined by equation (5.25) and $n = 2s + \sigma$.

Proof. Proof can be found in [4] (p.393). We shall present its idea. Choosing the ellipse E_ρ as the contour C, so that $\xi = \rho e^i\theta$ in the identity (5.23), we obtain the following estimation

$$\left|E_n^{(\lambda)}\right| \leq \frac{1}{\pi} \int_{E_\rho} \frac{\left|Q_{n+1}^{(\lambda)}(z) - Q_{n-1}^{(\lambda)}(z)\right|}{|T_{n+1}(z) - T_{n-1}(z)|} |f(z)| |\mathrm{d}z|.$$

We can find the connection between Z_r given by (5.27) and G_r given by (1.29)

$$G_r(\lambda) = \frac{1}{2}\left(Z_{r+1}(\lambda) - Z_r(\lambda)\right).$$

Thus, based on the relation (5.26) we can write the difference in the form

$$Q_{n+1}^{(\lambda)}(z) - Q_{n+1}^{(\lambda)}(z) = \frac{\Gamma(\lambda+1/2)\sqrt{\pi}}{\Gamma(\lambda+1)} \sum_{k=1}^{\infty} \xi^{\sigma-2k}\left(G_{k+s}(\lambda) - G_{k-s-\sigma}(\lambda)\right),$$

and denoting $U_n(\lambda)$ as an upper bound for $|G_{k+s}(\lambda) - G_{k-s-\sigma}(\lambda)|$ we receive an estimation

$$\left|Q_{n+1}^{(\lambda)}(z) - Q_{n+1}^{(\lambda)}(z)\right| \leq \frac{\Gamma(\lambda+1/2)\sqrt{\pi}}{\Gamma(\lambda+1)} \sum_{k=1}^{\infty} U_n(\lambda)\rho^{\sigma-2k} = \\ = \frac{\Gamma(\lambda+1/2)\sqrt{\pi}U_n(\lambda)\rho^{\sigma}}{\Gamma(\lambda+1)(\rho^2-1)}.$$

We can set $U_n(\lambda) = 2$ as $G_0(\lambda) = 1$ and for $r \neq 0$ and $\lambda > -\frac{1}{2}$ is $|G_r(\lambda)| < 1$. The remaining term to estimate is $\frac{|\mathrm{d}z|}{|T_{n+1}(z)-T_{n-1}(z)||f(z)|}$ which we can find in [18] (p. 655)

$$\frac{|\mathrm{d}z|}{|T_{n+1}(z) - T_{n-1}(z)|} \leq \frac{|\mathrm{d}\xi|}{\rho(\rho^n - \rho^{-n})} = \frac{\mathrm{d}\theta}{\rho^n - \rho^{-n}}.$$

If we take all these relations into account we have the estimation

$$\left|E_n^{(\lambda)}\right| \leq \frac{1}{\pi}\int_0^{2\pi} \frac{\Gamma(\lambda+1/2)\sqrt{\pi}U_n(\lambda)M(\rho)}{\Gamma(\lambda+1)(\rho^2-1)(\rho^n-\rho^{-n})}\mathrm{d}\theta,$$

which is by integrating and setting $U_n(\lambda) = 2$, as is shown above, equal to

$$\left|E_n^{(\lambda)}\right| \leq \frac{4\Gamma(\lambda+1/2)\sqrt{\pi}M(\rho)}{\Gamma(\lambda+1)(\rho^2-1)(\rho^n-\rho^{-n})}.$$

□

The Theorem 5.5 is very rough and can be improved in some ways depending on λ. We can from the proof that for different λ the behaviour of the terms in this estimation vary. Again, we can find the following theories in [4] (p. 394-396).

Theorem 5.6. *The error $E_n^{(\lambda)}(f)$ satisfies the following estimation*

$$\left|E_n^{(\lambda)}(f)\right| \leq \frac{2\Gamma\left(\lambda+\frac{1}{2}\right)\sqrt{\pi}\rho^{\sigma}M(\rho)(1-G_n(\lambda))}{\Gamma(\lambda+1)(\rho^2-1)(\rho^n-\rho^{-n})}, \quad -\frac{1}{2} < \lambda < 1, \tag{5.31}$$

where all variables are defined as above.

Proof. Proof is shown in [4] (p. 394). The idea is to set more accurate estimation for the term $U_n(\lambda)$ from previous proof. It is shown in the reffered article that by choosing $\lambda \in (-\frac{1}{2}, 1)$ we obtain a relation for the upper bound of the term $|G_{k+s}(\lambda) - G_{k-s-\sigma}(\lambda)|$

$$U_n(\lambda) = (1 - G_n(\lambda)),$$

and hence based on the previous proof we get the desired estimation. □

As a corollary of this theorem we obtain (see [18], p. 655) that for $\lambda = \frac{1}{2}$

$$\left|E_n^{1/2}(f)\right| \leq \frac{16n^2\rho^\sigma M(\rho)}{(4n^2-1)(\rho^2-1)(\rho^n-\rho^{-n})}.$$

From the definition of the term G_r given by (1.29) one can see that more simplifications can occur if λ is an integer and $\lambda < s+\sigma$.

Remark. $n = 2s+\sigma$, where σ is equal to 0 or 1.

Then we can simplify the expansion of $Q_n^{(\lambda)}(z)$ to the form

$$Q_n^{(\lambda)}(z) = (-1)^\lambda 2^{-2\lambda}\pi\xi^{-n}(\xi-\xi^{-1})^{2k-1}, \tag{5.32}$$

as is shown in [4] (p. 396 - 397). Based on this relation we can write the following theorem, which can be again found in [4] (p. 397).

Theorem 5.7. *If λ is an integer and $\lambda < s+\sigma$ then*

$$\left|E_n^{(\lambda)}(f)\right| \leq \frac{\pi(\rho+\rho^{-1})^{2\lambda}M(\rho)}{2^{2\lambda-1}(\rho^{2n}-1)}, \tag{5.33}$$

where all variables are defined as above.

Proof. Proof can be found in [4]. It is similar to the one of the Theorem 5.5. Thus we are again looking for estimation of the difference

$$\left|Q_{n+1}^{(\lambda)} - Q_{n-1}^{(\lambda)}\right|,$$

using the previous relation (5.32) for $z \in E_\rho$. We have

$$Q_{n+1}^{(\lambda)} - Q_{n-1}^{(\lambda)} = \frac{(-1)^{\lambda+1}\pi(\xi - xi^{-1})^{2\lambda}}{2^{2\lambda}\xi^n},$$

which means that we have the following estimation

$$\left|Q_{n+1}^{(\lambda)} - Q_{n-1}^{(\lambda)}\right| \leq \frac{\pi(\rho+\rho^{-1})^{2\lambda}}{2^{2\lambda}\rho^n}.$$

The rest of this proof is the same as the proof of the Theorem 5.5. □

5.2.2 Classical abscissae

Suppose that f is analytic within and on some contour C in the complex plane containing interval $[-1,1]$ in its interior. The error caused by the method based on the classical abscissae is then given by the equation ([18], p.653)

$$E_n^{(\lambda)}(f) = I^{(\lambda)}(f) - \Phi_n^{(\lambda)}(f),$$

$$E_n^{(\lambda)}(f) = \frac{1}{\pi i}\int_C \frac{Q_{n+1}^{(\lambda)}(z)f(z)}{T_{n+1}(z)}\mathrm{d}z, \tag{5.34}$$

where $Q_n^{(\lambda)}(z)$ is defined as before (5.24).
If we choose the contour C to be an ellipse E_ρ denoted by (5.25) and taking $z = \frac{1}{2}(\xi + \xi^{-1})$ where $|\xi| > 1$ as above, then

$$|\mathrm{d}z| = \left|\frac{1}{2}\left(1-\xi^{-2}\right)\mathrm{d}\xi\right| \le \frac{1}{2}\left(\rho+\rho^{-1}\right)\mathrm{d}\theta,$$

and

$$|T_{n+1}(z)| = \left|\frac{1}{2}\left(\xi^{n+1}-\xi^{n-1}\right)\right| \ge \frac{1}{2}\left(\rho^{n+1}-\rho^{-n-1}\right).$$

Similarly, like in the Theorem 5.4 we have

$$Q_{n+1}^{(\lambda)}(z) = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)\sqrt{\pi}}{2\Gamma(\lambda+1)}\sum_{k=1}^{\infty}\xi^{\sigma-2k}\big(Z_{k+s+1}(\lambda) - Z_{k-s-\sigma}(\lambda)\big),$$

where $Z_r(\lambda)$ is defined by equation (5.27). If $\lambda \ge 0$ then $|Z_r(\lambda)|$ has its maximum value equal to 1 (when $r=0$ or $r=1$). Thus we can set the following inequality

$$|Z_{k+s+1}(\lambda) - Z_{k-s-\sigma}(\lambda)| \le 2, \quad \lambda \ge 0,$$

which leads to

$$\left|Q_{n+1}^{(\lambda)}(z)\right| \le \frac{\Gamma\left(\lambda+\frac{1}{2}\right)\sqrt{\pi}}{2\Gamma(\lambda+1)}2\sum_{k=1}^{\infty}\rho^{\sigma-2k} = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)\sqrt{\pi}\rho^{\sigma}}{\Gamma(\lambda+1)(\rho^2-1)}.$$

Based on these inequations we can write the following theorem which is analogue of Theorem 5.5.

Theorem 5.8. *If $\lambda \ge 0$, the error $E_n^{(\lambda)}(f)$ satisfies the bound*

$$\left|E_n^{(\lambda)}(f)\right| \le \frac{2\Gamma\left(\lambda+\frac{1}{2}\right)\sqrt{\pi}\rho^{\sigma}M(\rho)\left(\rho+\rho^{-1}\right)}{\Gamma(\lambda+1)(\rho^2-1)(\rho^{n+1}-\rho^{-n-1})}, \tag{5.35}$$

where all variables are defined as before.

The theorem above can be proved in the same way as the Theorem 5.5. The knowledge from the previous section, where we were looking for the error estimation for method based on the practical abscissae, motivate us to look for the improvement in the case that $0 \le \lambda \le 1$. Then we can estimate

$$|Z_{k+s+1}(\lambda) - Z_{k-s-\sigma}(\lambda)| \le 1 + Z_{n+2}(\lambda), \quad 0 \le \lambda \le 1,$$

as $Z_r(\lambda)$ is nonnegative for $r > 0$ and nonpositive for $r \le 0$. The maximum value (equality) occurs when $k = s + \sigma + 1$. This leads us to the theorem which is analogue of Theorem 5.6.

Theorem 5.9. *The error $E_n^{(\lambda)}(f)$ satisfies the following equation*

$$\left|E_n^{(\lambda)}(f)\right| \le \frac{(1 + Z_{n+2}(\lambda))\Gamma(\lambda + 1/2)\sqrt{\pi}M(\rho)(\rho + \rho^{-1})}{\Gamma(\lambda + 1)(\rho^2 - 1)(\rho^{n+1} - \rho^{-n-1})}, \quad 0 \le \lambda \le 1, \tag{5.36}$$

where variables are defined as above.

As a corollary of this theorem we receive the estimation for $\lambda = 1/2$ which is also introduced in [18] (p. 655)

$$\left|E_n^{(1/2)}(f)\right| \le \frac{4(n + 2)\rho^\sigma M(\rho)(\rho + \rho^{-1})}{(2n + 3)(\rho^2 - 1)(\rho^{n+1} - \rho^{-n-1})}.$$

There still remains the case when $-1/2 < \lambda < 0$. From the definition of $Z_r(\lambda)$ we can find out that for each integer r as λ decreases from 0 to $-1/2$, $Z_r(\lambda)$ varies monotonically from $Z_r(0) = \pm 1$ to $Z_r(-1/2) = 2r - 1$. This leads us to estimation

$$|Z_r(\lambda)| \le |2r - 1|, \quad -1/2 < \lambda < 0.$$

From this relation and equation (5.26) it follows that

$$\begin{aligned}\left|Q_{n+1}^{(\lambda)}(z)\right| &\le \frac{\Gamma(\lambda + 1/2)\sqrt{\pi}}{2\Gamma(\lambda + 1)} \sum_{k=1}^{\infty} \rho^{\sigma - 2k} \left(|Z_{k+\sigma+1(\lambda)}| + |Z_{k-s-\sigma}(\lambda)|\right) = \\ &= \frac{\Gamma(\lambda + 1/2)\sqrt{(\pi)}\rho^\sigma \left((2s + 3)\rho^2 - (2s + 1)\right)}{\Gamma(\lambda + 1)(\rho^2 - 1)^2}.\end{aligned}$$

Thus we can write the following theorem.

Theorem 5.10. *The error* $E_n^{(\lambda)}(f)$ *satisfies the following estimation*

$$\left|E_n^{(\lambda)}(f)\right| \le \frac{2\Gamma(\lambda+1/2)\sqrt{\pi}\rho^{\sigma}(\rho+\rho^{-1})M(\rho)\left((2s+3)\rho^2-(2s+1)\right)}{\Gamma(\lambda+1)(\rho^2-1)^2(\rho^{n+1}-\rho^{-n-1})}, \quad -\frac{1}{2}<\lambda<0, \tag{5.37}$$

where variables are defined as above.

We have also analogue of the Theorem 5.7 which we mention at this place.

Theorem 5.11. *The error* $E_n^{(\lambda)}(f)$ *satisfies the following estimation for* λ *integer such that* $\lambda \le s$

$$\left|E_n^{(\lambda)}(f)\right| \le \frac{\pi\left(\rho+\rho^{-1}\right)^{2\lambda}M(\rho)}{2^{2\lambda-1}\left(\rho^{2n+2}-1\right)}, \tag{5.38}$$

where variables are defined as before.

5.2.3 The new abscissae

Suppose that f is analytic within and on some contour C in the complex plane containing interval $[-1,1]$ in its interior. If we choose points as we did in (5.20), namely

$$x_k = \cos\frac{(6k+4)\pi}{3n+1}, \quad k=0,1,2,\dots n,$$

we can show that they are the zeros of

$$T_{3n+2}(x) - T_{3n}(x),$$

because we know that

$$\cos{(3n+2)x_k} - \cos{3nx_k} = -2\sin{(3n+1)x_k}\sin x_k.$$

Hence we can write similarly as in the Section 5.2.1 that the error is given by the equation

$$E_n^{(\lambda)}(f) = \frac{1}{\pi i}\int_C \frac{Q_{3n+2}^{(\lambda)}(z)-Q_{3n}^{(\lambda)}(z)}{T_{3n+2}(z)-T_{3n}(z)}f(z)\mathrm{d}z,$$

where $Q_n^{(\lambda)}(z)$ is defined as before by equation (5.24)

$$Q_n^{(\lambda)}(z) = \frac{1}{2}\int_{-1}^{1}\frac{\left(1-x^2\right)^{\lambda-\frac{1}{2}}T_n(x)}{z-x}\mathrm{d}x, \quad z\notin[-1,1].$$

If we choose the ellipse E_ρ given by equation (5.25) as contour C, the main difference to Section 5.2.1 is that we are looking for the estimations of the differences

$$\left|Q_{3n+2}^{(\lambda)}(z) - Q_{3n}^{(\lambda)}\right|,$$

and

$$\left|T_{3n+2}(z) - T_{3n}(z)\right|.$$

As we already know, choosing ellipse E_ρ as the contour C allow us to write

$$\left|T_{3n+2}(z) - T_{3n}(z)\right| = \frac{1}{2}\left|(\xi^{3n+2} + \xi^{-3n})(1 - \xi^{-2})\right|,$$

and we obtain the estimation

$$\begin{aligned} \frac{|\mathrm{d}z|}{|T_{3n+2}(z) - T_{3n}(z)|} &= \frac{|(1-\xi^{-2})\mathrm{d}\xi|}{|(\xi^{3n+2} + \xi^{-3n})(1-\xi^{-2})|}, \\ &\leq \frac{|\mathrm{d}\xi|}{\rho^{3n+2} + \rho^{-3n}} = \frac{\mathrm{d}\theta}{\rho^{3n+1} + \rho^{-3n-1}}. \end{aligned} \tag{5.39}$$

Now denote

$$m = 3n = 6s + 3\sigma,$$

and put $m = 2p + q$, where $q = 1$ or 0. Thus if n is even $p = 3s$ and if n is odd $p = 3s + 1$ which allow us to write that $p = 3s + \sigma$ and $q = \sigma$. Then based on the Theorem 5.4 we can estimate the difference

$$Q_{3n+2}^{(\lambda)}(z) - Q_{3n}^{(\lambda)}(z) = \frac{\Gamma(\lambda + 1/2)\sqrt{\pi}}{\Gamma(\lambda+1)} \sum_{k=1}^{\infty} \xi^{1-\sigma-2k}(G_{k+p+\sigma}(\lambda) - G_{k-p-1}(\lambda)). \tag{5.40}$$

So, if $U_n\lambda$ is an upper bound for $|G_{k+p+\sigma}(\lambda) - G_{k-p-1}(\lambda)|$ we obtain (similarly like in the proof of Theorem 5.5) that

$$\left|Q_{3n+2}^{(\lambda)}(z) - Q_{3n}^{(\lambda)}(z)\right| \leq \frac{\Gamma(\lambda + 1/2)\sqrt{\pi}U_n(\lambda)\rho^{1-\sigma}}{\Gamma(\lambda+1)(\rho^2 - 1)}.$$

Again, we set $U_n(\lambda) = 2$ and we can write a theorem similar to Theorem 5.5.

Theorem 5.12. *The error $E_n^{(\lambda)}(f)$ satisfies the following estimation*

$$\left|E_n^{(\lambda)}(f)\right| \leq \frac{4\Gamma\left(\lambda + \frac{1}{2}\right)\sqrt{\pi}\rho^{1-\sigma}M(\rho)}{\Gamma(\lambda+1)(\rho^2-1)(\rho^{3n+1} - \rho^{-3n-1})}, \tag{5.41}$$

where

$$M(\rho) = \max_{z \in E_\rho} |f(z)|, \lambda > -\frac{1}{2}, \tag{5.42}$$

and the ellipse E_ρ is defined by equation (5.25) and $n = 2s + \sigma$.

5.3 Different approach

There are also two different views at numerical integration. One is "classic" where the quadrature rule is applied repeatedly and the approximations obtained converge to the correct value. At the same time we choose the error bounds. We have presented examples of such a method in the Section 5.1.
On the other hand, H. V. Smith introduced a different approach to the numerical integration in his works [9], [7], [8], [10] and [11] which he used with D. B. Hunter in deriving the method presented in [12]. Within this approach the quadrature rule is applied only once and the error term is numerically evaluated assuming the degree of the rule to be fixed. Consider a method based on the practical abscissae described earlier in the Method 5.1 and Section 5.2.1. We use the same notation and require the same conditions as in the previous sections.
If we choose the ellipse (5.25) E_ρ as the contour C in the equation (5.23) and assume f to be analytic in its interior. Then f can be developed into an infinite Chebyshev series (1.20) with coefficients A_r given by the formula (1.21). Substituting this series to the equation (5.23) we get the following expression

$$E_n^\lambda = \frac{1}{\pi i}\int_{E_\rho} \frac{Q_{n+1}^{(\lambda)}(z) - Q_{n-1}^{(\lambda)}(z)}{T_{n+1}(z) - T_{n-1}(z)} \sum_{r=0}^{\infty}{}^{*} A_r T_r(z)\,\mathrm{d}z,$$

which can be simplified to the form

$$E_n^{(\lambda)}(f) = \sum_{r=0}^{\infty}{}^{*} A_r e_{n,r}^{(\lambda)}, \tag{5.43}$$

where the asterisk means that the first term is to be halved and

$$e_{n,r}^{(k)} = \frac{1}{\pi i}\int_{E_\rho} \frac{\left(Q_{n+1}^{(\lambda)}(z) - Q_{n-1}^{(\lambda)}(z)\right)}{T_{n+1}(z) - T_{n-1}(z)} T_r(z)\,\mathrm{d}z. \tag{5.44}$$

We shall require that n is even

$$n = 2s,$$

as this choice will make some further relations more simple.
Following three lemmas are required for $z \in E_\rho$ to gain estimation for $e_{n,r}^{(k)}$. They can be found in [19] (Section 3) which authors used in [12]. It is good to remind the definition (5.25) of the ellipse E_ρ at this place

$$E_\rho = \frac{1}{2}\left(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}\right) : 0 \le \theta < 2\pi, \quad \rho > 1,$$

and for $z \in E_\rho$ we write $z = \frac{1}{2}\left(\xi + \xi^{-1}\right)$. Then the following lemma can be easily proved using the Euler's formula of the cosine

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right).$$

Lemma 5.13. *For $z \in E_\rho$*

$$T_r(z) = \frac{1}{2}\left(\xi^r + \xi^{-r}\right). \tag{5.45}$$

Lemma 5.14. *For $z \in E_\rho$*

$$T_{n+1}(z) - T_{n-1}(z) = \frac{1}{2}\xi^{n+1}\left(1 - \xi^{-2}\right)\left(1 - \xi^{-2n}\right). \tag{5.46}$$

We already know from the Theorem 1.15 and identities (5.4), (5.5) that

$$I^{(\lambda)}(T_{2r}) = \frac{\sqrt{\pi}\,\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda+1)} G_r(\lambda),$$

where G_r is defined by the equation (1.29). If we denote

$$B = \frac{\sqrt{\pi}\,\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda+1)}, \tag{5.47}$$

then it becomes

$$I^{(\lambda)}(T_{2r}) = BG_r(\lambda).$$

The following lemma (see [19] p. 304 or [12]) is based on the Theorem 5.4.

Lemma 5.15. *For $z \in E_\rho$ and $n = 2s$ is*

$$Q^{(\lambda)}_{n+1}(z) - Q^{(\lambda)}_{n-1}(z) = B\sum_{k=1}^{\infty} \frac{H^{(\lambda)}_{s,k}}{\xi^{2k}}, \tag{5.48}$$

where

$$H^{(\lambda)}_{s,k} = G_{k+s}(\lambda) - G_{k-s}(\lambda), \tag{5.49}$$

and $G_r(\lambda)$ is defined as before (1.29).

Proof. Proof is based on the Theorem 5.4. We have the following equation

$$\frac{1}{2}\left(Z_{r+1}(\lambda) - Z_r(\lambda)\right) = G_r(\lambda).$$

The desired equation is then obtained directly by expressing $Q^{(\lambda)}_{n+1}$ and $Q^{(\lambda)}_{n-1}$ via their expansions (5.26). □

From the last two lemmas we can get the following equation

$$\begin{aligned}\frac{Q_{n+1}^{(\lambda)}(z)-Q_{n-1}^{(\lambda)}(z)}{T_{n+1}(z)-T_{n-1}(z)} &= \frac{2B}{\xi^{n+1}(1-\xi^{-2})}\left(1-\frac{1}{\xi^{2n}}\right)^{-1}\left(\sum_{k=1}^{\infty}\frac{H_{s,k}^{(\lambda)}}{\xi^{2k}}\right),\\ &= \frac{2B}{\xi^{2s+3}(1-\xi^{-2})}\left(\sum_{k=1}^{\infty}\frac{H_{s,k}^{(\lambda)}}{\xi^{2k-2}}\right)\left(\sum_{h=0}^{\infty}\frac{1}{\xi^{4hs}}\right).\end{aligned}$$

If we multiply these two series we obtain

$$\frac{Q_{n+1}^{(\lambda)}(z)-Q_{n-1}^{(\lambda)}(z)}{T_{n+1}(z)-T_{n-1}(z)} = \frac{2B}{\xi^{2s+3}(1-\xi^{-2})}\sum_{j=0}^{\infty}\sum_{k=0}^{n-1}\frac{1}{\xi^{4js+2k}}\sum_{i=0}^{j}H_{s,2si+k+1}^{(\lambda)}. \quad (5.50)$$

Denote

$$J^{(\lambda)}(j,k,s) = \sum_{i=0}^{j}H_{s,2si+k+1}^{(\lambda)}. \quad (5.51)$$

Substituting (5.50) into (5.44) we get

$$e_{n,r}^{(\lambda)} = \frac{2B}{\pi i}\sum_{j=0}^{\infty}\sum_{k=0}^{n-1}J^{(\lambda)}(j,k,s)\int_{E_\rho}\frac{1}{\xi^{4sj+2k+2s+3}(1-\xi^{-2})}T_r(z)\,\mathrm{d}z.$$

Because of the Lemma 5.13 this becomes

$$\begin{aligned}e_{n,r}^{(\lambda)} &= \frac{B}{\pi i}\sum_{j=0}^{\infty}\sum_{k=0}^{n-1}J^{(\lambda)}(j,k,s)\int_{E_\rho}\xi^{r-2k-4js-2s-3}\left(1+\xi^{-2r}\right)\mathrm{d}\xi,\\ &= \frac{B}{2\pi i}\sum_{j=0}^{\infty}\sum_{k=0}^{n-1}J^{(\lambda)}(j,k,s)\int_{E_\rho}\xi^{r-2k-4js-2s-3}\mathrm{d}\xi,\end{aligned} \quad (5.52)$$

since the term ξ^{-2r} makes no contribution to the integral. We shall remind the definition of the ellipse E_ρ as $\xi = \rho e^{i\theta}$ and

$$E_\rho = \frac{1}{2}\left(\rho e^{i\theta}+\rho^{-1}e^{-i\theta}\right):\ 0\le\theta<2\pi,\quad \rho>1.$$

If we apply the residue theorem to the integral in (5.52) (or using the direct method for evaluating this contour integral) we obtain the following relation (use the direct method).

$$\frac{\mathrm{d}\xi}{\mathrm{d}\theta} = \rho i e^{i\theta},$$

and setting $c = r - 2k - 4js - 2s - 3$we solve the integral

$$\int_0^{2\pi} \rho^{c+1} i e^{i\theta(c+1)} \mathrm{d}\theta = \frac{\rho^{c+1}}{c+1} \left(e^{2\pi i(c+1)} - 1 \right).$$

As $e^{2\pi i(c+1)} - 1 = 0$, the non-zero terms can occur if $c + 1 = 0$ or equivalently if

$$r = 2k + 4js + 2s + 2, \quad j = 0, 1, \dots, \infty, \quad k = 0, 1, \dots, 2s - 1.$$

Thus we can see that whenever r is odd or equal to $0, 2, 4, \dots 2s$ then $e_{n,r}^{(\lambda)} = 0$, as the first non-zero term occur for $k = 0$, $j = 0$ and $r = 2s + 2$ which allow us to rewrite the error term (5.43) in the form

$$E_n^{(\lambda)} = \sum_{p=1}^{\infty} A_{n+2p} e_{n,n+2p}^{(\lambda)}. \tag{5.53}$$

Taking these relations into account we can write the following theorem ([12], p.1036).

Theorem 5.16. *Assume p and $n = 2s$ to be fixed then*

$$e_{n,n+2p}^{(\lambda)} = B \sum_{i=0}^{j} H_{s,ni+k+1}^{(\lambda)}, \quad k = 0, 1, \dots, n-1, \quad j = 0, 1, \dots, \tag{5.54}$$

where j is the quotient and k the reminder when $p - 1$ is divided by n.

Proof. Proof can be found in [12] (p. 1036). The idea is that in the equation (5.52) we set $r = 2s + 2p$ which gives the integral term in the form

$$\int_{E_\rho} \xi^{4s\left(\frac{p-1}{2s} - j - \frac{k}{2s}\right) - 1} \mathrm{d}\xi.$$

As it is pointed out in [12] (p. 1036), this integral is non-zero for those values of j and k such that j is the quotient and k the remainder when $p - 1$ is divided by $2s$. The value of this integral is then $2\pi i$. Because the quotient j and remainder k are unique, the result follows from (5.51) and (5.52). □

This means that when exact Chebyshev coefficients are known, the identity (5.52) and Theorem 5.16 allow us to evaluate the error term $E_{2s}^{(\lambda)}(f)$. Unfortunatelly, we usually do not know the exact Chebyshev coefficients. But as H. V. Smith and D. B. Hunter [12] (p. 1036-1037) suggested, approximate coefficients can be used instead of the exact ones as well. The final method is then based on evaluating $\Psi_{2s}^{(\lambda)}$ given by the equation (5.7) for fixed $n = 2s$ and adding r non-zero terms of the relation (5.53).

Chapter 6

Examples

We use previously described methods in this chapter. All of the experiments, which we introduce, were evaluated using Matlab 7.5.0. We are interested in a comparison of presented methods with the exact result. Such a result is obtained via the in-built function *quad(fun,a,b)* which evaluates integral of function *fun* over the interval $[a, b]$ using recursive adaptive Simpson quadrature. The default tolerance is set to 10^{-6}, unless it is stated differently. We use the Gegenbauer weight function, thus we shall evaluate exactly the function $g^{(\lambda)}(x)$ given by

$$g^{(\lambda)}(x) = \left(1 - x^2\right)^{\lambda - 1/2} f(x).$$

Remark. We will face a problem of obtaining exact result for $\lambda < 1/2$ which will be discussed already in the following example.

Source codes of used functions can be found in appendix and also on the attached medium. The algorithms are based exactly on the results we have presented in this work.

The absolute error is calculated as $|I - I_n|$ where I is the exact value of the integral and I_n is the approximated value obtained by $(n + 1)$-pt. quadrature formula.

As a relative error we denote a percentual ratio of the error with respect to the exact solution

$$\frac{|I - I_n|}{|\text{exact}|} \times 100.$$

We will use the following notation:

PRAE - absolute error caused by method based on practical abscissae presented in Method 5.9,

CLAE - absolute error caused by method based on classical abscissae presented in Method 5.2,

NEWAE - absolute error of invented method presented in Method 5.3,
PRRE - relative error of method based on practical abscissae, given in percentage,
CLRE - relative error of method based on classical abscissae, given in percentage,
NEWRE - relative error of invented method, given in percentage.

Example 6.1.

$$I^{(\lambda)} = \int_{-1}^{1} \left(1 - x^2\right)^{\lambda - 1/2} e^x \mathrm{d}x.$$

We have already evaluated this integral for $\lambda = 1/2$ in Example 3.4 and thus we shall start with this choice of variable λ. The exact value is

$$I^{(1/2)} = 2.350402,$$

and the errors of the methods are presented by table

$\lambda = 1/2$	PRAE	NEWAE	CLAE
$n = 2$	1.1651361e-002	2.3623730e-001	9.2965772e-002
$n = 8$	8.1557667e-009	1.0274792e-002	4.9532631e-009
$n = 32$	8.1353027e-009	6.0246067e-004	8.1353022e-009

$\lambda = 1/2$	PRRE	NEWRE	CLRE
$n = 2$	4.9571772e-001	1.0050930e+001	3.9553130e+000
$n = 8$	3.4699449e-007	4.3715033e-001	2.1074107e-007
$n = 32$	3.4612382e-007	2.5632235e-002	3.4612381e-007

We will provide the same excersise for two different choices for λ, from the interval$(-\frac{1}{2}, \frac{1}{2})$ and $\lambda > \frac{1}{2}$. If we choose $\lambda < 1/2$ we face the problem with singularities in the endpoints of the integral. We will discuss this problem after some observation.
Usin the function quadgk *with received warning about singularities we obtain a value*

$$I^{(-1/4)} \doteq 7.120594.$$

Neverthless, we shall take it as exact value and offer the table of errors.

$\lambda = -1/4$	PRAE	NEWAE	CLAE
$n = 2$	2.2172501e-002	8.5686525e-001	2.4892748e-001
$n = 8$	2.6654097e-004	2.7290964e-001	2.6645197e-004
$n = 32$	2.6654084e-004	1.3521458e-001	2.6654084e-004

$\lambda = -1/4$	PRRE	NEWRE	CLRE
$n = 2$	3.1138554e-001	1.2033620e+001	3.4958807e+000
$n = 8$	3.7432405e-003	3.8326807e+000	3.7419907e-003
$n = 32$	3.7432387e-003	1.8989227e+000	3.7432387e-003

We can observe rather big errors in this case. But we have to keep in mind that there are singularities in the endpoints and the "exact" value obtained by quadgk *is probably inaccurate. If we make an attempts with basic Matlab function* quad(fun,a,b) *and compare the results obtained one the intervals* $[-99, 99]$ *and* $[0.999999999, 0.999999999]$ *with tolerancy set to* 10^{-12} *we can observer increasement fo the value of the integral. It is easy to see that as* $x \to 1_-$ *(*$x \to -1_+$ *) the function* $g^{(\lambda)}(x)$ *is growing which agrees with these observations. Therefore we shall look at values obtained by our methods for* $n = 32$ *within the interval* $[-1, 1]$ *which are slightly larger than that obtained by* quadgk.

$\lambda = -1/4$	Pract. result	Classic. result	New meth. result
$n = 32$	7.120860723662652	7.120860723662654	6.985379606027591

Because we do not have to face the problem with singularities within these methods, we can take into account as the exact solution the result evaluated by method based on the practical abscissae with $n = 1000$.

The table of errors can be then modified as follows:

$$I^{(-1/4)} \doteq 7.120860723662654.$$

$\lambda = -1/4$	PRAE	NEWAE	CLAE
$n = 2$	2.1905960e-002	8.5713179e-001	2.4919402e-001
$n = 8$	1.2685675e-010	2.7317618e-001	8.8870103e-008
$n = 32$	1.7763568e-015	1.3548112e-001	0

$\lambda = -1/4$	PRRE	NEWRE	CLRE
$n = 2$	3.0763078e-001	1.2036913e+001	3.4994930e+000
$n = 8$	1.7814805e-009	3.8362804e+000	1.2480247e-006
$n = 32$	2.4945816e-014	1.9025947e+000	0

From this example we can see that these methods are suitable for such a problems and they elegantly avoid the problem with singularities in the endpoints (with respect to the conditions presented in Method 5.9).

Remark. In the next examples for $-1/2 < \lambda < 1/2$ we will take the value obtained by the method based on the practical abscissae 5.9 with $n = 1000$ (see also Theorem 5.6) as the exact value of the integral.

Now we can investigate in the usual way the integral

$$I^{(4)} = 0.9028868.$$

$\lambda = 4$	PRAE	NEWAE	CLAE
$n = 2$	2.7946923e-003	2.6757724e-001	9.3254317e-002
$n = 8$	5.0175190e-007	1.3840794e-003	5.0292383e-007
$n = 32$	5.0169820e-007	5.0169820e-007	5.0169820e-007

$\lambda = 4$	PRRE	NEWRE	CLRE
$n = 2$	3.0952855e-001	2.9635747e+001	1.0328462e+001
$n = 8$	5.5571963e-005	1.5329490e-001	5.5701761e-005
$n = 32$	5.5566016e-005	5.5566016e-005	5.5566016e-005

With this choice of λ we get a function which is well integrable via selected methods as well and the error gets on the tolleration level very fast.

Example 6.2.

$$I^{(\lambda)} = \int_{-1}^{1} \left(1 - x^2\right)^{\lambda - 1/2} x^{12} \mathrm{d}x.$$

We have already evaluated this integral for $\lambda = 1/2$ in Example 3.3 and thus we shall start with this choice of variable λ. The exact value is

$$I^{(1/2)} = 0.153852.$$

The errors of methods are given by the following table.

$\lambda = 1/2$	PRAE	NEWAE	CLAE
$n = 2$	5.1281449e-001	1.0713415e-001	4.3901732e-002
$n = 8$	2.8074548e-004	6.9116673e-004	7.5162060e-004
$n = 32$	6.0202006e-006	6.0202006e-006	6.0202006e-006

$\lambda = 1/2$	PRRE	NEWRE	CLRE
$n = 2$	3.3331638e+002	6.9634475e+001	2.8535009e+001
$n = 8$	1.8247742e-001	4.4924079e-001	4.8853427e-001
$n = 32$	3.9129773e-003	3.9129773e-003	3.9129773e-003

Now we choose the $\lambda = -1/4$ which indicates the same problem with singularities in the endpoints as in the previous example.

$$I^{(-1/4)} = 2.304050574023278.$$

$\lambda = -1/4$	PRAE	NEWAE	CLAE
$n = 2$	1.1920262e+000	1.1882678e+000	1.5781196e+000
$n = 8$	1.6006845e-003	2.0807364e-002	2.2353702e-002
$n = 32$	1.7763568e-015	2.2204460e-015	1.7763568e-015

$\lambda = -1/4$	PRRE	NEWRE	CLRE
$n = 2$	5.1736111e+001	5.1572990e+001	6.8493271e+001
$n = 8$	6.9472628e-002	9.0307756e-001	9.7019146e-001
$n = 32$	7.7097129e-014	9.6371411e-014	7.7097129e-014

We keep in mind that in this case the exact solution was obtained by the method based on practical abscissae with significant amout of nodes ($n = 1000$).

Now investigate this problem with $\lambda = 5/2$

$$I^{(5/2)} = 0.0048293.$$

$\lambda = 5/2$	PRAE	NEWAE	CLAE
$n = 2$	1.4755156e-001	8.4214510e-002	7.6532219e-002
$n = 8$	1.1983703e-005	3.6145251e-005	4.4383541e-005
$n = 32$	2.8425176e-006	2.8425176e-006	2.8425176e-006

$\lambda = 5/2$	PRRE	NEWRE	CLRE
$n = 2$	3.0552846e+003	1.7437924e+003	1.5847186e+003
$n = 8$	2.4814121e-001	7.4844364e-001	5.8858748e-002
$n = 32$	5.8858748e-002	5.8858748e-002	5.5566016e-005

With this choice of λ we have a function which is well integrable via selected methods as well as by Gauss quadrature and the error gets on the tolerance level very fast.

Example 6.3.

$$I^{(\lambda)} = \int_{-1}^{1} \left(1 - x^2\right)^{\lambda - 1/2} e^{-x^{-2}} \mathrm{d}x.$$

We have already evaluated this integral for $\lambda = 1/2$ in Example 3.6 and thus we can start with this choice of variable λ. The exact value is

$$I^{(1/2)} = 0.178147.$$

The errors of methods are given in the following table.

$\lambda = 1/2$	PRAE	NEWAE	CLAE
$n = 2$	6.7105233e-002	9.6092258e-002	1.1473798e-001
$n = 8$	5.5027449e-004	5.1640241e-004	9.9186341e-004
$n = 32$	5.1875247e-008	4.2114768e-008	6.5777744e-008

$\lambda = 1/2$	PRRE	NEWRE	CLRE
$n = 2$	3.7668307e+001	5.3939649e+001	6.4406087e+001
$n = 8$	3.0888662e-001	2.8987314e-001	5.5676456e-001
$n = 32$	2.9119230e-005	2.3640362e-005	3.6923145e-005

Now we set $\lambda = -1/3$ which indicates problem with singularities in the endpoints as described above. Based on this knowledge we obtain

$$I^{(-1/3)} = 1.911361361051442.$$

$\lambda = -1/3$	PRAE	NEWAE	CLAE
$n = 2$	9.8902586e-002	3.4878472e-001	3.1089796e-001
$n = 8$	1.1437968e-003	5.0462403e-003	4.3932375e-003
$n = 32$	1.1609431e-007	1.9617014e-006	1.9123102e-006

$\lambda = -1/3$	PRRE	NEWRE	CLRE
$n = 2$	5.1744577e+000	1.8247974e+001	1.6265787e+001
$n = 8$	5.9841994e-002	2.6401289e-001	2.2984861e-001
$n = 32$	6.0739071e-006	1.0263373e-004	1.0004964e-004

If we set $\lambda = 5$ then

$$I^{(5)} = 0.004561.$$

The table of errors can be found on next page.

$\lambda = 5$	PRAE	NEWAE	CLAE
$n = 2$	1.9140700e-002	6.5572180e-002	7.4692437e-002
$n = 8$	9.5406117e-004	6.6494888e-006	2.6428544e-005
$n = 32$	1.1829866e-006	1.1737899e-006	1.1606985e-006

$\lambda = 5$	PRRE	NEWRE	CLRE
$n = 2$	4.1968406e+002	1.4377530e+003	1.6377262e+003
$n = 8$	2.0918998e+001	1.4579846e-001	5.7947927e-001
$n = 32$	2.5938477e-002	2.5736830e-002	2.5449783e-002

In this case we did face any new problems.

We have just presented advantages of methods we have desribed in the earlier parts of this thesis. As we observed, these methods are well suited also for integrals with singularities in the endpoints, if the problem can be transformed to

$$\int_{-1}^{1} \left(1 - x^2\right)^{(\lambda - 1/2)} f(x) \mathrm{d}x, \quad \lambda > -\frac{1}{2}.$$

All three methods elegantly avoid the problem with singularities and provide accurate results for $-1/2 < \lambda < 1/2$. As if $\lambda = 1/2$ we receive a well-solvable problem with analytic function, we can compare these results with previously calculated in Section 3. We can see in the examples, that even when all three methods usually provide very accurate results, the method based on the extrema of Chebyshev polynomials use to slightly outperform other methods. This fact registered already D. Elliott [5] (p. 243) who described this observation and suggested use of the method based on "practical ascissae". That is the reason of the name of this method.

Chapter 7

Simple Ordinary Differential Equation and Further Extension

In this chapter we shall briefly introduce two methods based on the Chebyshev polynomials for solving a simple linear ordinary differential equation. This is one of the fields where this work can be extended in the future as there can arise some interesting connections. For example, the trio of authors from the Republic of Korea [1] is already studying application of the generalized Clenshaw-Curtis quadrature rule to a collocation least-squares method.
For the next sections consider the simple, one dimensional, linear, two-point boundary-value problem on the range $[-1, 1]$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) = f(x), \quad u(-1) = a, \quad u(1) = b, \tag{7.1}$$

where the function f and the boundary values a, b are given.

7.1 Collocation method

Suppose that we approximate $u(x)$ in the following way (see Eq. (5.11)) by $n+1$ terms of its Chebyshev expansion

$$u(x) \approx \sum_{k=0}^{n}{}^{*} c_k T_k(x), \tag{7.2}$$

where the asterisk indicates that the first term of this sum is to be halved. Now we need the property of Chebyshev polynomial which can be found in [14] (Eq.

(10.5))

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}T_k(x) = \sum_{r=0 \quad (k-r)\ \text{even}}^{k-2}{}^{*} (k-r)k(k+r)T_r(x), \quad k \geq 2. \tag{7.3}$$

Thus if select $n-1$ points $\{x_i\}_{i=1}^{n-1}$ in the range of integration and require $u_n(x)$ to satisfy the differential equation (7.1) at these points, called *collocation points*, we obtain the following system of $n+1$ (incl. boundaries) linear equations

$$\sum_{k=2}^{n} \sum_{r=0 \quad (k-r)\ \text{even}}^{k-2}{}^{*} (k-r)k(k+r)c_k T_r(x_i) = f(x_i), \quad i = 1, \dots, n-1, \tag{7.4}$$

with boundary values (thanks to Theorem 1.12) given by equations

$$\sum_{k=0}^{n}{}^{*} (-1)^k c_k = a, \quad \sum_{k=0}^{n}{}^{*} c_k = b. \tag{7.5}$$

Now arises the question about choosing these $n-1$ points. We can choose zeros of $T_{n-1}(x)$

$$x_r = \cos\frac{(r-1/2)\pi}{n-1},$$

as is show in the relation (1.14), which means that we can use the discrete orthogonality of Chebyshev polynomials on this set of points (see (1.19)). We will use this property after multiplying (7.4) by $2T_j(x_i)$ (j is integer, $0 \leq j \leq n-2$) and summing $\sum_{i=1}^{n-1}$. Thus from the relations

$$\begin{aligned} \sum_{i=1}^{n-1} T_r(x_i)T_j(x_i) &= 0, && 0 \leq r \neq j \leq n-2, \\ &= n-1, && r = j = 0, \\ &= \frac{1}{2}(n-1), && 0 < r = j \leq n-2, \end{aligned}$$

we can deduce that

$$\sum_{k=j+2 \ (k-j)\ \text{even}}^{n} (k-j)k(k+j)c_k = \frac{2}{n-1}\sum_{i=1}^{n-1} T_j(x_i)f(x_i), \quad j = 0, \dots, n-2. \tag{7.6}$$

If we use this equation in reverse order (start with $j = n-2$)) we can determine coefficients $c_n, \dots, c_3, c_2$. Using the boundary conditions we can determine coefficients c_1, c_2.

7.2 Projection method

We shall describe the method presented in [14] (Section 10.2.3). Approximate the $u_n(x)$ in the same way as above

$$u_n(x) \approx \sum_{k=0}^{n}{}^{*} c_k T_k(x),$$

where the asterisk indicates that the first term of the sum is to be halved.
Suppose that we select $n-1$ independent test functions $\{\psi_j(x)\}_{j=1}^{n-1}$ and a positive weight function $w(x)$ and solve the system of $n+1$ linear equations (including boundary conditions)

$$\begin{aligned}
&\int_{-1}^{1} w(x)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_n(x) - f(x)\right)\psi_j(x)\mathrm{d}x = \\
&= \int_{-1}^{1} w(x)\left(\sum_{k=2}^{n}\sum_{\substack{r=0 \\ (k-r)\text{ even}}}^{k-2}{}^{*} (k-r)k(k+r)c_k T_r(x) - f(x)\right)\psi_j(x)\mathrm{d}x, \\
&= 0, \quad j = 1,\dots,n-1,
\end{aligned} \tag{7.7}$$

with boundary conditions

$$\sum_{k=0}^{n}{}^{*}(-1)^k c_k = a, \qquad \sum_{k=0}^{n}{}^{*} k c_k = b. \tag{7.8}$$

This means that the residual

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_n(x) - f(x),$$

is orthogonal to each of the $n-1$ test functions with respect to the weight $w(x)$. Let us choose $\psi_j(x) = T_{j-1}(x)$ and $w(x) = \frac{2}{\pi}\frac{1}{\sqrt{1-x^2}}$ then because of the orthogonality of Chebyshev polynomials presented in (1.13) we can represent the residual in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_n(x) - f(x) = \sum_{k=0}^{\infty} \tau_{k-1} T_{k-1}(x),$$

for some sequence of undetermined coefficients (τ_k). [1] Because of the orthogonality we can reduce the first $n-1$ equations from the system (7.7) to

$$\sum_{k=j+2 (k-j)\ (even)}^{n} (k-j)k(k+j)c_k = \frac{2}{\pi}\int_{-1}^{1} \frac{T_j(x)f(x)}{\sqrt(1-x^2)}\mathrm{d}x, \quad j = 0,\ldots,n-2, \tag{7.9}$$

which means that we can obtain coefficients c_k in a similar way as in the previous section.

Remark. This is no coincidence. If we apply basic Clenshaw-Curtis quadrature (based on the zeros of T_{n-1}) on the right-hand side of (7.9) we receive the right-hand side of (7.6).

The difference between these two methods ((7.6) and (7.9)) is that in some context we may have a better option of evaluating the integrals more accurately.

[1] The method is often refered to as the tau method although it slightly differs from the original Lanczos' tau method which is based on the representation of $u_n(x)$ as the sum of powers of x, $u_n(x) = \sum_{k=0}^{n} a_k x^k$.

7.3 Further extension

There are many posibilities to extend this work. The previous two sections can be extended for nonlinear equations and based on the collocation method we can also study the Eigenvalue problem as is shown in [14] (Chapter 10).
There is also a method of least-square collocation which is used, for example, in the geophysics and geodesy what can be found in a literature. Authors C. Kim, S. D. Kim and J. Yoon in their work [1] extend Clenshaw-Curtis quadrature to multidimensional convex domain and apply to a collocation least-squares method to solve a first-order system of linear equations with an elliptic boundary value problem.

The Clenshaw-Curtis quadrature had not been mentioned very often for a long time, but recently the team under the supervision of L. N. Trefethen developed Chebfun - collection of algorithms, and a software system in object-oriented MATLAB, which extends familiar powerful methods of numerical computation. A big emphasis is put on the use of Chebyshev polynomials. This software is still under developement and can be found on http://www2.maths.ox.ac.uk/chebfun/. This together with the articles which are published in recent years gives a good presumption that methods based on the Clenshaw-Curtis quadrature will become more casual in the near future and we can also expect new theorems and maybe also surprising results.

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Appendix

Source code of functions used for the numerical experiments.

```
function I=simpson(f,n) %composite Simpson’s rule;
                        %n denotes the number of
                        %subintervals (must be even)
a=-1; b=1;              %over the interval [a,b]=[-1,1];
h = (b-a)/n;            %step h

S=feval(f,a);           %f(a)

for i=1:(n/2)
    x=a+h*(2*i-1);      %odd indexes (x_1, x_3 ... x_{n-1})
    S=S+4*feval(f,x);
end

for i=1:(n/2-1)
    x=a+h*2*i;          %even indexes (x_2, x_4 ... x_{n-2})
    S=S+2*feval(f,x);
end

S=S+feval(f,b);         %f(b)

I=h*S/3;                %the integral
}

======================================================
```

```
function [c,g,s]=points(n)  %function returns n+1 points for
                            %Chebyshev, Gauss and Newton-Cotes
                            %in [-1,1]
c = cos(pi*(0:n)'/n);       %extrema of Chebyshev

beta = .5./sqrt(1-(2*(1:n)).^(-2));
                        %3-term recurrence coeffs
T = diag(beta,1)+diag(beta,-1);
                        %Jacobi matrix
[V,D] = eig(T);         %eigenvalue decomposition
g = diag(D); [g,i] = sort(g);
                        %Gauss nodes (= Legendre points)

h = 2/n;                %step h, interval [-1,1]
for i=0:n
    s(i+1)=-1+h*i;      %Newton-Cotes points (equidistant)
end
s=s';                   %function returns [c,g,s];
```

==

```
function display_points(n) %function displays n+1 points
                        for Chebyshev, Gauss and Simpson
                           %in [-1,1]
figure(1)
[c,g,s]=points(n);

hold on
whitebg('white');
title('Quadrature nodes in [-1,1]')
plot(c(1:n+1),0.1,'bo');
text(1.05, 0.1, 'Chebyshev points')
plot(g(1:n+1),0.2,'r+');
text(1.05, 0.2, 'Gauss points')
```

```
plot(s(1:n+1),0,'blacko');
text(1.05, 0, 'Newton-Cotes points')
axis tight
axis off
hold off
return
```

==

```
function I = gauss(f,n) %(n+1)-pt Gauss quadrature of f

beta = .5./sqrt(1-(2*(1:n)).^(-2));
                       %3-term recurrence coeffs
T = diag(beta,1)+diag(beta,-1);   %Jacobi matrix
[V,D] = eig(T);        %eigenvalue decomposition
x = diag(D); [x,i] = sort(x);%nodes (= Legendre points)
w=2*V(1,i).^2;           %weights

I=w*feval(f,x);          %the integral
```

==

```
function I = clenshaw_curtis(f,n)
%(n+1)-pt Clenshaw-Curtis quadrature of f based on
%the practical abscissae (extrema of Cheb.polyn.)

x = cos(pi*(0:n)'/n); %extrema of Chebyshev polynomials
fx = feval(f,x);      %f evaluated at these points
g = real(fft(fx([1:n+1 n:-1:2])/(2*n)));
                              %Fast Fourier Transform
```

```
a = [g(1); g(2:n)+g(2*n:-1:n+2); g(n+1)];
                             %Chebyshev coeffs
w = 0*a'; w(1:2:end) = 2./(1-(0:2:n).^2);
                                   %weight factor

I = w*a;       %the integral
```

==

```
function I = practical(f,n,lambda)
%(n+1)-pt Clenshaw-Curtis quadrature of f based on
%the practical abscissae (extrema of Cheb.polyn.)
%with Gegenbauer w.f.

format long;
x = cos(pi*(0:n)'/n);
                %extrema of Chebyshev polynomials
fx = feval(f,x);    %f evaluated at these points
s=floor(n / 2);
sigma=n-2*s;

for r=0:2:n           %we are evaluating only 2r
    k=r/2;
    a(k+1)=0;
    for j=1:(n-1)
        a(k+1)=a(k+1)+fx(j+1)*cos(r*j*pi/n);
    end
    a(k+1)=2/n*(a(k+1)+1/2*fx(1)+fx(n+1)/2*(-1)^r);
end

B=gamma(lambda+1/2)*sqrt(pi)/gamma(lambda+1);

u=(0:s);
u(s+1)=(sigma+1)/2*a(s+1);             %reccurence
```

```
if s>1
  for r=(s-1):-1:1
     u(r+1)=(r-lambda)/(r+lambda+1)*u(r+2)+a(r+1);
  end
end
u(1)=-lambda/(lambda+1)*u(2)+a(1)/2;
I=B*u(1);
```

==

```
function I = classical(f,n,lambda)
%(n+1)-pt Clenshaw-Curtis quadrature of f based on
%the classical abscissae (zeros of Cheb.polyn.)
%with Gegenbauer w.f.

format long;
x = cos(pi*(1:2:(2*n+1))'/(2*(n+1))); %zeros of Tn
fx=feval(f,x);        %f evaluated at this points

s=floor(n/2);
sigma = n-2*s;

for r=0:2:n              %evaluate just terms b_2r
    k=r/2;
    b(k+1)=0;
    for i=0:n
b(k+1)=b(k+1)+fx(i+1)*cos(r*(2*i+1)*pi/(2*(n+1)));
    end
end
b=b*2/(n+1);

B=gamma(lambda+1/2)*sqrt(pi)/gamma(lambda+1);

u=(0:s);
```

```
u(s+1)=(sigma+1)/2*b(s+1);   %reccurence relation
if s>1
  for r=(s-1):-1:1
     u(r+1)=(r-lambda)/(r+lambda+1)*u(r+2)+b(r+1);
  end
end
u(1)=-lambda/(lambda+1)*u(2)+b(1)/2;

I=B*u(1);
```

==

```
function I = new(f,n,lambda)
%(n+1)-pt Clenshaw-Curtis quadrature of f based
%on the new abscissae
%with Gegenbauer w.f.

format long;
x = cos(pi*(4:6:(6*n+4))'/(3*(n+1)));%points x_k
fx=feval(f,x);       %f evaluated at this points

s=floor(n/2);
sigma = n-2*s;

for r=0:2:n           %evaluate just terms a_2r
    k=r/2;
    a(k+1)=0;
    for i=0:n
a(k+1)=a(k+1)+fx(i+1)*cos(r*(6*i+4)*pi/(3*(n+1)));
    end
end
a=a*2/(n+1);
B=gamma(lambda+1/2)*sqrt(pi)/gamma(lambda+1);
```

```
u=(0:s);

u(s+1)=(sigma+1)/2*a(s+1);   %reccurence relation
if s>1
  for r=(s-1):-1:1
     u(r+1)=(r-lambda)/(r+lambda+1)*u(r+2)+a(r+1);
  end
end

u(1)=-lambda/(lambda+1)*u(2)+a(1)/2;

I=B*u(1);
```