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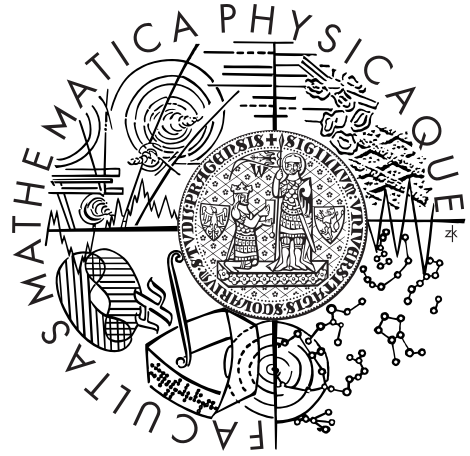

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Noise-induced transitions in nonlinear dynamics of stochastic systems

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I would like to thank doc. Petr Chvosta for introducing me to the wonderful world of stochastic modeling and for his guidance during this work. I am also grateful to everyone from the theoretical group at the Department of Macromolecular Physics for providing me with a friendly and encouraging work environment.

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Šumem indukované přechody v nelineární dynamice stochastických systémů.

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Abstrakt: V této bakalářské práci se soustředíme na studium jednodimenzionální difúze v náhodném potenciálu, který je dán dichotomickým šumem s obecnými parametry. V [5] bylo ukázáno, že tento problém má velmi blízko studiu stochastické Riccatiho rovnice. Ve stejném článku bylo nalezeno řešení pro difúzi na polopřímce za pomoci Chapman-Kolmogorovy rovnice. Abychom jsme se přiblížili k řešení i pro konečný interval, přistoupíme k tomuto problému za pomoci metody Carlemanovy linearizace. Odvodíme vztah pro momenty řešení Riccatiho rovnice v Laplacovském obrazu, který má tvar maticového elementu matice nekonečné dimenze. Tento maticový element se pokusíme vypočíst v limitě difúze na polopřímce a v limitě nekonečného času, ale zjistíme, že výsledek se neshoduje s předpovědí numerické simulace. Dále se zabýváme numerickou simulací Riccatiho rovnice za pomoci metody Monte Carlo. Správnost simulací je ověřena srovnáním s analytickými výsledky získaných pomocí Chapman-Kolmogorovy rovnice.

Klíčová slova: difůze v náhodném potenciálu, Carlemanova linearizace, Riccatiho rovnice

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Abstract: In this thesis we focuse on one-dimensional diffusion in a random potential given by the general Markov dichotomous process. It was shown in [5] that this problem is closely related to the study of the stochastic Riccati equation. Using Kolmogorov forward equation we have a solution in the case of a semi-infinite interval. In order to overcome the restriction of a semi-infinite interval we present an approach to solution based on the method of Carleman embedding. We give an expression for the moments in the Laplace domain in terms of an infinite-dimensional matrix element and we try to evaluate it in the limit of infinite time and semi-infinite interval. However we find a discrepancy between our result, numerical simulation and different theoretical approach to the same problem. We also develop Monte Carlo simulations of the Riccati equations and we compare them to analytical results.

Keywords: diffusion in random potential, Carleman embedding, Riccati equation

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Introduction

The ultimate goal of physics has always been to write down a set of fundamental laws out of which everything we observe could be built. While tempting it is well known that such an approach breaks down even if we try to describe systems that are relatively simple compared to the complex structures that surround us. By breaking down we mean that even numerical simulations of these systems are not possible since current technology does not provide us with enough computing power. To overcome this difficulty physicists have developed variety of approximate descriptions of Nature. This thesis focuses on one such phenomenological description that revolves around ideas from probability theory.

Noise in dynamical systems

There are two possible ways to introduce randomness into a system. Imagine an isolated system that moves through its phase space according to certain deterministic law (*e.g.* a box filled with gas). We are unable to solve for the trajectory of this system so we make the following simplification. We assign probabilities that in the next moment the system will be in the state A given that during previous times it visited states A_1, A_2, ..., A_n. For example if $n = 1$, this transforms the problem to a continuous-time Markov process that is governed by a system of linear differential equations whose solution gives the probabilities of occupying given state at arbitrary time. This kind of randomness is referred to as *intrinsic noise*. Obviously this description is valid only if we make a smart choice of the transition probabilities which must be motivated by either the original deterministic law and/or by experiment. Study of such probabilistic description of systems is the central idea of both equilibrium and non-equilibrium statistical physics.

To illustrate the second possibility of introducing randomness into a system, consider the DNA molecules in our body that are being exposed to ultraviolet radiation that can damage them or cause mutations in our genome (see figure 1). Since there is no hope for predicting when a high-energy photon hits a given DNA molecule, we instead give the probability that at time t a given DNA molecule is hit by a photon of energy E. We think of this situation as a system (the DNA molecule) interacting with a *random environment* (the ultraviolet radiation). This random environment gives rise to a randomness in the system which is referred to as *extrinsic noise*.

Statistical physics is successful due to its ability to neglect physically uninteresting degrees of freedom. This extraction of only interesting information from a system enabled us to explain the large spectrum of different phases of matter we know today. It is therefore natural to ask if the replacement of a complex environment by an approximation using a statistical description offers us a new

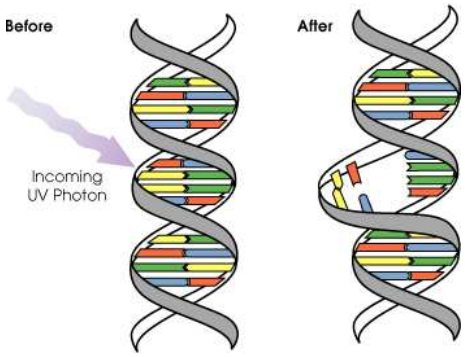

Figure 1: Illustration of the damaging effect UV radiation can have on DNA molecules. Since we do not know when and where the DNA molecule is going to be hit, we treat it as a random process. Picture adapted from the Wikipedia file "Image:DNA UV mutation.gif", *http://schools-wikipedia.org/images/788/78870.gif.htm.*

insight into the behavior of the system as well. Intuitively we would expect the following recipe to hold. Replace the random environment by its average state, then solve for the evolution of the system and in the end add some fluctuations around this solution to account for the environmental randomness. It turns out however that this approach holds only in a few special cases.

To be more specific we know that it holds in the situation when we deal with white environmental fluctuations (*i.e.* fluctuations uncorrelated in time) that are coupled additively to the system. Let us concentrate mainly on the stationary behavior. In this case the probability distribution function corresponding to the system will have peaks centered exactly at the stable points of the system when the random environment is replaced by its average. If however we look at the situation where environmental noise is coupled multiplicatively to the system it is possible to find that not only the peaks of the distribution are shifted away from the deterministic stable points but also that the distribution has more peaks than the number of these stable points. Since stable points of the deterministic system correspond to different phases, we see that a random environment can essentially create a new phase. Simple one-dimensional model that exhibits this behavior is the so-called genetic model described in [1]. It corresponds to the equation

$$\frac{dX(t)}{dt} = \alpha - X(t) + \lambda(t)X(t)(1 - X(t)),\ X(t) \in [0, 1],$$

where $\lambda(t) = \lambda + \sigma^2 \frac{dW(t)}{dt}$ describes the fluctuating environment (modeled as Gaussian white noise). In the deterministic case ($\sigma = 0$) the system has only one stable point however if the fluctuations are large enough a new phase appears. This model can be mapped for example onto a problem in population genetics or it can be interpreted as a chemical reaction. Remarkably all the theoretical predictions have been verified experimentally.

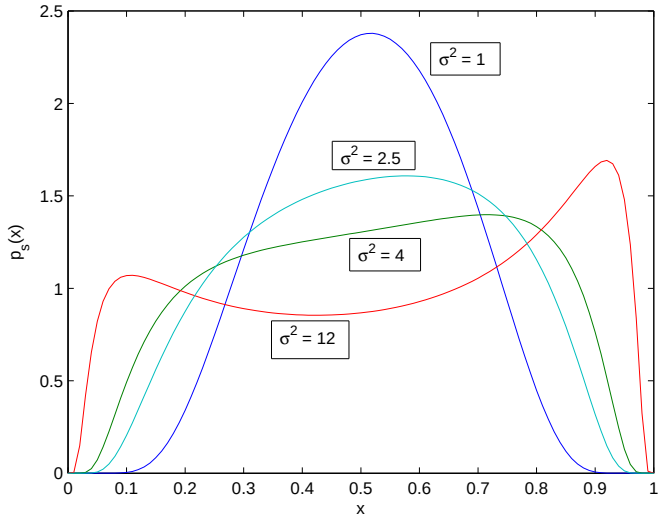

Figure 2: The stationary probability distribution in the genetic model for different intensities of the environmental noise. We used $\alpha = 0.5$, $\lambda = 0.1$. As the noise intensity σ^2 increases the form of this probability distribution changes from an unimodal to a bimodal one, *i.e.* it undergoes a noise-induced phase transition. The emergence of the bimodal distribution has no deterministic analog, it is a purely stochastic effect. Picture is a modified version of the one found in [1].

Effects that a fluctuating environment has on the stationary state of a system are encoded in the probability distribution that describes it. It is therefore important to study the dependence of this distribution on parameters of the environmental noise. The qualitative change of this probability distribution (*i.e.* number of maxima or behavior at boundary) with a change in the parameters of the noise is called a *noise-induced transition*.

Random environment however does not alter only the stationary state and it is important to study the transient behavior as well. While restricting to the study of moments of the random variable describing the state of a system is not always enough to discover a noise-induced transition in the stationary state, it might be enough to see non-trivial transient behavior. As will be seen in the problem of this thesis the asymptotic behavior of moments is sometimes characterized by a power law with non-trivial *dynamical exponents*. This is an effect not present in the deterministic system.

Diffusion in a random potential

Before we move to the problem of a random potential let us review diffusion under the influence of a deterministic force field. Imagine a particle submerged

in a liquid that consist of much smaller particles. It is well know that the particle will follow an unpredictable trajectory. This so-called Brownian motion is caused by the interactions of the particle with other particles in the liquid. Since it is impossible to predict these interactions we make an approximation. We consider the liquid to be a random environment that at each moment of time pushes the particle with a random force. We assume that the magnitude of this force is uncorrelated with the magnitude of a push at any other time and we also assume that it is normally distributed. The first assumption is physically intuitive since the correlation time of these pushes is much smaller than any other time scale in the motion of the diffusing particle. The second assumption is a consequence of the central limit theorem since one push actually consists of hits from many particles that can be assumed to be independent. Last assumption is that inertial effects are negligible, *i.e.* that we can treat the motion of diffusing particle as overdamped. This leads to the conclusion that we can describe Brownian motion by the following diffusion process (for formal derivation see [11]).

$$\frac{d\boldsymbol{X}(t)}{dt} = \frac{F(\boldsymbol{X}(t), t)}{\eta} + \sqrt{2D}\frac{d\boldsymbol{W}(t)}{dt},$$

where $\frac{d\boldsymbol{W}(t)}{dt}$ is the Gaussian white noise, $F(x,t)$ an arbitrary deterministic force field that acts on the diffusing particle, η is the viscosity of the liquid and D is the diffusion constant. For transparency we assume only one-dimensional motion. This stochastic differential equation is equivalent to the following partial differential equation called Fokker-Planck equation for the probability density $\rho(x,t)$ of $\boldsymbol{X}(t)$

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[-D\frac{\partial \rho(x,t)}{\partial x} + \frac{F(x)}{\eta}\rho(x,t)\right]. \tag{1}$$

This sort of description of Brownian motion is very phenomenological and is based on our intuition about liquids. We are however also interested for example in calculating the conductivity of a crystalline material. In that case it is much more intuitive to imagine diffusion as a motion of an impurity in a periodic potential. Since the impurity is in thermal equilibrium with the lattice, we know from statistical physics that the probability of jumping from one site to another will be proportional to the height of the potential barrier between adjacent sites. It can be shown (see [2]) that in the limit of small separation of lattice sites this description leads to the Fokker-Planck equation as well.

Usually when we talk about a crystalline material we imagine its structure as a precisely periodic lattice. However we know that there are no perfect lattices in Nature so a question arises how to treat diffusion in such an imperfect media. Natural approach is to consider the lattice spacing (and therefore also the jump probabilities from one site onto another) to be random variables. That brings us to the problem of diffusion in random potential.

Review of the problem of one-dimensional random walk with random jump probabilities can be found in Alexander *et al.* ([9]). To mention few applications

it has been applied to the study of superionic conductors, Anderson tight-binding model or to magnetic models with random ferromagnetic interactions. The model considered in this thesis deals with the continuous limit and therefore with the Fokker-Planck equation where the force field $F(x,t) = F(x)$ is considered to be random. Since the Fokker-Planck equation is closely related to the Schrödinger equation, this problem is similar to the study of the dynamics of a quantum particle in a random potential ([15]).

As a final point of this introduction we would like to elaborate on the connection between diffusion in a random potential and effects of noise on non-linear systems. In the nomenclature of the preceding section we are dealing with a system (the diffusing particle) coupled to a random environment that consists of two parts. Thermal part responsible for the standard Brownian motion and a disorder in the form of random potential. These two parts are completely independent however what is the form of the coupling to the system is a non-trivial question. We will see that the problem reduces to the study of the stochastic Riccati equation

$$\frac{dQ(x)}{dx} = -Q^2(x) - F(x)Q(x) + z,$$

from which it is obvious that the coupling is multiplicative.

In this thesis we will be interested in the case where $F(x)$ is given by the dichotomous random force. This problem has been already approached in [5] however the presented solution works only if the diffusion takes place on a semi-infinite line. We will present a completely different approach to this problem that we believe might be more easily adaptable to the finite interval case.

1. Diffusion in a piecewise linear random potential

1.1 From Fokker-Planck to the stochastic Riccati equation

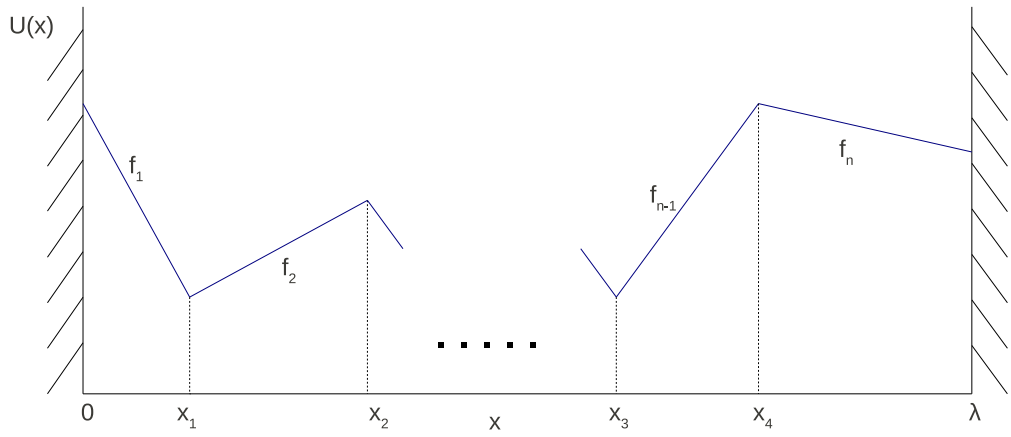

Figure 1.1: Illustration of the piecewise linear potential. The parameters f_i represent the slopes of the potential (*i.e.* forces) within the corresponding segments.

The problem at hand is pictured in figure 1.1. We assume an interval of length λ and a deterministic piecewise linear potential. The probability density describing the position of a Brownian particle moving in the potential $U(x) = \int_0^x F(x')\,\mathrm{d}x'$ is given by the Fokker-Planck equation (1). We assume the particle started at point y and we rewrite this equation in terms of dimensionless parameters as

$$\frac{\partial\rho(x,t;y)}{\partial t} = -\frac{\partial}{\partial x}\left[-\frac{\partial\rho(x,t;y)}{\partial x} + f(x)\rho(x,t;y)\right],\ \rho(x,0;y) = \delta(x-y). \quad (1.1)$$

If the variables in the original Fokker-Planck equation are marked with an apostrophe, the new dimensionless variables are

$$x = \frac{F_0}{D\eta}x',\ \rho(x,t;y) = \frac{D\eta}{F_0}\rho'(x',t';y'),\ t = \frac{F_0^2}{D\eta^2}t',$$

where we defined $F(x) \equiv F_0 f(x)$, with $f(x)$ dimensionless.

Since there is only linear dependence on the time derivative, it is natural to perform a Laplace transform in the variable t. Let us call the Laplace variable z. We will distinguish the Laplace transform of $\rho(x,t;y)$ from the function itself only by replacing the variable t with z. The Fokker-Planck equation then reads

$$\left(\frac{d^2}{dx^2} - f(x)\frac{d}{dx} - z\right)\rho(x,z;y) = -\delta(x-y). \tag{1.2}$$

From now on we will assume that there are reflecting boundaries at $x = 0$ and at $x = \lambda$. Defining the probability current as

$$j(x,z;y) \equiv -\frac{\partial\rho(x,z;y)}{\partial x} + f(x)\rho(x,z;y), \tag{1.3}$$

this condition says that $j(0,z;y) = j(\lambda,z;y) = 0$.

The form of the Fokker-Planck equation implies that both the density $\rho(x,t;y)$ and the current $j(x,t;y)$ must be continuous functions. We can therefore solve the equation (1.2) on each interval of constant force and then glue these solutions together. It is convenient to combine the density and current on kth interval (let us call it I_k) into a vector. Solution on this interval then reads

$$\begin{aligned}|G_k(x,z;y)\rangle \equiv \begin{pmatrix}\rho_k(x,z;y)\\ j_k(x,z;y)\end{pmatrix} = \begin{pmatrix} e^{\alpha_k^+(z)x} & e^{-\alpha_k^-(z)x}\\ -\alpha_k^-(z)e^{\alpha_k^+(z)x} & \alpha_k^+(z)e^{-\alpha_k^-(z)x}\end{pmatrix}\begin{pmatrix}c_k^+(z)\\ c_k^-(z)\end{pmatrix}\\ + \begin{cases}0 & \text{if } y\notin I_k\\ |\sigma_k(x,y;z)\rangle & \text{if } y\in I_k\end{cases},\end{aligned} \tag{1.4}$$

where the constants $c_k^\pm$ are to be determined from the boundary or gluing conditions and

$$\alpha_k^\pm(z) \equiv \sqrt{z + \left(\frac{f_k}{2}\right)^2} \pm \frac{f_k}{2}, \tag{1.5}$$

$$|\sigma_k(x,z;y)\rangle \equiv -\frac{\Theta(y-x)}{\alpha_k^+(z)+\alpha_k^-(z)}\begin{pmatrix} e^{-\alpha_k^-(z)(x-y)} - e^{\alpha_k^+(z)(x-y)}\\ \alpha_k^-(z)e^{\alpha_k^+(z)(x-y)} + \alpha_k^+(z)e^{-\alpha_k^-(z)(x-y)}\end{pmatrix}. \tag{1.6}$$

Now all that is left is to impose the continuity condition at each x_k. In order to do so it is useful to define the following matrix

$$W_k(l;z) \equiv \frac{1}{\alpha_k^+(z)+\alpha_k^-(z)}\begin{pmatrix}\alpha_k^-(z)e^{\alpha_k^-(z)l} + \alpha_k^+(z)e^{-\alpha_k^-(z)l} & e^{\alpha_k^-(z)l} - e^{-\alpha_k^+(z)l}\\ z\left[e^{\alpha_k^-(z)l} - e^{-\alpha_k^+(z)l}\right] & \alpha_k^+(z)e^{\alpha_k^-(z)l} + \alpha_k^-(z)e^{-\alpha_k^+(z)l}\end{pmatrix}. \tag{1.7}$$

It is then easy to show that the solutions in two consecutive intervals are related by

$$|G_{k-1}(x,z;y)\rangle = W_{k-1}(x_{k-1}-x;z)\,|G_k(x_{k-1},z;y)\rangle\,. \tag{1.8}$$

By iterating this relation and employing the reflecting boundary condition it is easy to write down a solution of (1.2) at any given point in terms of products of W matrices. The result is rather tedious and since we will not need it in this thesis, we omit it. The central quantity that we will work with is the value of the probability density $\rho(x, z; y)$ at the origin. This can be expressed using a simple formula provided we set $y = 0$. This is not a very restricting assumption since we will be mostly interested in the limit $t \to \infty$ where the initial condition plays no role. Therefore from now on we will assume $y = 0$, omitting all the appearances of y in our equations. The formula for density at the origin reads

$$\rho(0, z) = \frac{\langle 1, 0| \, W_1(x_1, z) W_2(x_2 - x_1, z) \dots W_n(\lambda - x_{n-1}, z) \, |1, 0\rangle}{\langle 0, 1| \, W_1(x_1, z) W_2(x_2 - x_1, z) \dots W_n(\lambda - x_{n-1}, z) \, |1, 0\rangle}. \tag{1.9}$$

Power of this approach is apparent when we realize that the matrix $W_k(l, z)$ can be written as an exponential

$$W_k(l, z) = \exp\left[\begin{pmatrix} -f_k & 1 \\ z & 0 \end{pmatrix} l\right]. \tag{1.10}$$

Let us define functions $S(l; z)$ and $R(l; z)$ as a solution to the differential equation

$$\frac{d}{dl}\begin{pmatrix} S(l;z) \\ R(l;z) \end{pmatrix} = \begin{pmatrix} -f(\lambda - l) & 1 \\ z & 0 \end{pmatrix}\begin{pmatrix} S(l;z) \\ R(l;z) \end{pmatrix}, \begin{pmatrix} S(0;z) \\ R(0;z) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1.11}$$

Thanks to the exponential nature of the W matrix, one can easily become convinced using equation (1.9) that a function defined as $P(\lambda; z) \equiv S(\lambda; z)/R(\lambda; z)$ corresponds to the value $\rho(0, z)$ of the probability density given that the interval on which diffusion occurs has length λ.

Now we are really interested in the case when the force field is random. Throughout this thesis, random variables will be typeset as capital letters in a sans-serif format. We will denote the stochastic process describing the force field by $\mathsf{\Phi}(x)$. We make the physically sensible assumption that it is *stationary*. Therefore it follows that $\mathsf{\Phi}(\lambda - x) = \mathsf{\Phi}(x)$. Making this substitution in (now stochastic) equation (1.11), relabeling the variable l by λ to emphasize its meaning as the interval length and taking the derivative of $\mathsf{P}(\lambda; z) = \mathsf{S}(\lambda; z)/\mathsf{R}(\lambda; z)$, we obtain the following non-linear stochastic differential equation

$$\frac{d\mathsf{P}(\lambda; z)}{d\lambda} = -z\mathsf{P}^2(\lambda; z) - \mathsf{\Phi}(\lambda)\mathsf{P}(\lambda; z) + 1, \;\; \mathsf{P}(0; z) = \infty. \tag{1.12}$$

Solution of this equation is a random variable describing the value of the density $\rho(x, z)$ at the origin.

1.2 Stochastic Riccati equation and the dichotomous random force

If we look at the derivation of equation (1.12) we can conclude that it is valid for any random potential whose realizations can be obtained as a limit of piecewise linear functions. Therefore this approach to the problem of diffusion in random potential is very general. Furthermore it turns out that the variable $\mathsf{P}(\lambda; z)$ is enough to calculate properties like the random variable that corresponds to the mean position or mean velocity of the Brownian particle. For derivation and exact statements of these results see [5].

From now on we will assume that $\mathsf{\Phi}(\lambda)$ is a stationary dichotomous Markov process (for an overview see Appendix A). We will denote the two possible forces ϕ_+ and ϕ_- and the corresponding average times between jumps as $1/n_+$ and $1/n_-$. Since we will be mostly interested in the behavior when $t \to \infty$ which, in the z-domain, corresponds to the limit $\lim_{z\to 0} z\mathsf{P}(\lambda; z)$, we make the substitution $\mathsf{Q}(\lambda; z) = z\mathsf{P}(\lambda; z)$. This transforms the equation (1.12) to

$$\frac{d\mathsf{Q}(\lambda; z)}{d\lambda} = -\mathsf{Q}^2(\lambda; z) - \mathsf{\Phi}(\lambda)\mathsf{Q}(\lambda; z) + z, \;\; \mathsf{Q}(0, z) = \infty, \tag{1.13}$$

which we will refer to as a stochastic Riccati equation.

To qualitatively study the behavior of this equation we imagine that it describes an overdamped motion of a particle under the influence of a force field corresponding to the right hand side. This force field randomly jumps between two possible forms that we describe in terms of a potential function as

$$V_\pm(q; z) = -\int_0^q (-q'^2 - \phi_\pm q' + z)\,\mathrm{dq}' = \frac{1}{3}q^3 + \frac{\phi_\pm}{2}q^2 - zq + V(0; z). \tag{1.14}$$

The minima resp. maxima lie at the points

$$q_\pm^{\min}(z) = \sqrt{z + \left(\frac{\phi_\pm}{2}\right)^2} - \frac{\phi_\pm}{2}, \;\; q_\pm^{\max}(z) = -\sqrt{z + \left(\frac{\phi_\pm}{2}\right)^2} - \frac{\phi_\pm}{2}. \tag{1.15}$$

For sample values of the noise parameters these functions together with a sample realization of the Riccati equation are plotted in figure 1.2. The main advantage of this geometrical interpretation is that it allows us to easily determine the support of the random variable $\mathsf{Q}(\lambda; z)$. Suppose we are interested in the behavior $\lambda \to \infty$. The fictional particle starts to slide down from infinity towards one of the minima. Then the potentials switch and it moves towards the other minimum. Sooner or later it has to cross the minimum of $V_-(q; z)$ and when it does it can never move back to the right of $q_-^{\min}(z)$. However it also cannot move beyond the minimum of $V_+(q; z)$ although it can reach it in the limit $\lambda \to \infty$. Therefore we see that the support of $\mathsf{Q}(\lambda; z)$ in the limit $\lambda \to \infty$ is $\left[q_+^{\min}(z), q_-^{\min}(z)\right]$.

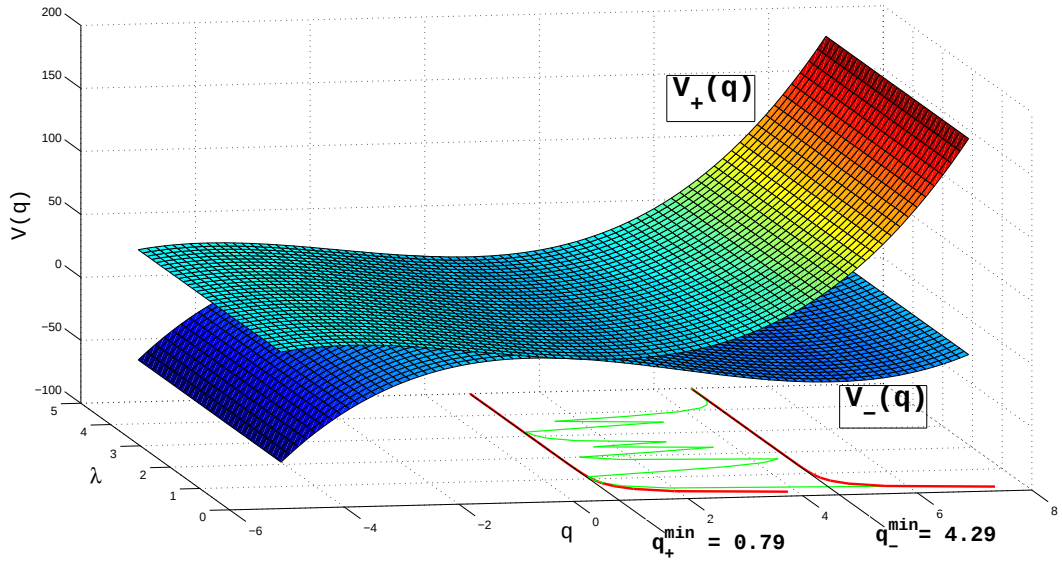

Figure 1.2: Visualization of the dynamics based on the stochastic Riccati equation (1.13). The red lines correspond to the support of the variable $Q(\lambda; z)$. The green line is a sample realization that was simulated using the parameters $n_+ = 2$, $n_- = 1$, $\phi_+ = 3$, $\phi_- = -4$, and $z = 3$. The surfaces represent the potentials corresponding to the two possible forces. Their minima are located at $q_+^{\min} = 0.79$ and $q_-^{\min} = 4.29$.

Similarly it is easy to see that for general λ the boundary of the support will correspond to the positions of two fictional particles each of which is moving under the influence of only one of the two potentials. Since a solution to the Riccati equation with a constant force $\Phi(\lambda) = \phi$ is (we defined $q^{\min}(z) = q_-^{\min}(z) = q_+^{\min}(z)$ and similarly for maximum)

$$Q(\lambda; z) = \frac{q^{\min}(z) - q^{\max}(z)e^{-(q^{\min}(z)-q^{\max}(z))\lambda}}{1 - e^{-(q^{\min}(z)-q^{\max}(z))\lambda}}, \tag{1.16}$$

we see that the support will be

$$\left[\frac{q_+^{\min}(z) - q_+^{\max}(z)e^{-(q_+^{\min}(z)-q_+^{\max}(z))\lambda}}{1 - e^{-(q_+^{\min}(z)-q_+^{\max}(z))\lambda}}, \frac{q_-^{\min}(z) - q_-^{\max}(z)e^{-(q_-^{\min}(z)-q_-^{\max}(z))\lambda}}{1 - e^{-(q_-^{\min}(z)-q_-^{\max}(z))\lambda}}\right]. \tag{1.17}$$

Because we have the ordering $q_+^{\max}(z) < q_-^{\max}(z) < 0 < q_+^{\min}(z) < q_-^{\min}(z)$, the conclusions we have just made are completely general.

Another property we can see from this visualization is that for finite λ the probability distribution corresponding to $Q(\lambda; z)$ will always contain two delta functions. This is caused by the dichotomous process since no matter in what

state it starts, the evolution of the system will at first follow a deterministic trajectory until a jump occurs. We can also immediately predict that these delta functions will be multiplied by $e^{-n_\pm \lambda}$ (*i.e.* by a factor proportional to the probability of a jump). This is an important point since these delta functions do not always follow easily from the equation describing the evolution of the probability density.

1.3 Limit of a semi-infinite interval

Corresponding to the equation (1.13) we can write down a Kolmogorov forward equation, *i.e.* a partial differential equation for the joint probability density of the random variables $\mathcal{Q}(\lambda; z)$ and $\mathcal{\Phi}(\lambda)$. This probability density has two components that we will call $\kappa_\pm(q, \lambda; z)$ which are defined as

$$\kappa_\pm(q, \lambda; z)\mathrm{dq} = \text{probability that } \mathcal{Q}(\lambda; z) \in [q, q + \mathrm{dq}] \text{ and } \mathcal{\Phi}(\lambda) = \phi_\pm. \quad (1.18)$$

The Kolmogorov forward equation then reads

$$\frac{\partial}{\partial \lambda}\begin{pmatrix}\kappa_-(q,\lambda;z)\\ \kappa_+(q,\lambda;z)\end{pmatrix} = -\frac{\partial}{\partial q}\begin{pmatrix}(-q^2-\phi_- q+z)\kappa_-(q,\lambda;z)\\ (-q^2-\phi_+ q+z)\kappa_+(q,\lambda;z)\end{pmatrix} - \begin{pmatrix} n_- & -n_+\\ -n_- & n_+\end{pmatrix}\begin{pmatrix}\kappa_-(q,\lambda;z)\\ \kappa_+(q,\lambda;z)\end{pmatrix},$$

$$\begin{pmatrix}\kappa_-(q,0;z)\\ \kappa_+(q,0;z)\end{pmatrix} = \delta(q)\frac{1}{n_- + n_+}\begin{pmatrix}n_+\\ n_-\end{pmatrix}. \quad (1.19)$$

If interested only in the behavior for $\lambda \to \infty$ we can set the left hand side of this equation to zero. This case has been solved in [5]. The physically important result is that if the forces satisfy $\phi_- < 0 < \phi_+$, non-trivial properties emerge. In this case non-trivial means that they cannot be obtained by simply replacing the random potential with its average. More precisely due to the existence of traps in the potential (see figure 1.3) the particle can be slowed down resulting in an asymptotic behavior of the mean velocity that scales as a power law with exponent controlled by the parameter $\Theta = n_-/|\phi_-| - n_+/\phi_+$. As seen from figure 1.3, this parameter is related to the average depth of traps in the potential. Except of these dynamical exponents this system also exhibits noise-induced transitions in the stationary probability density.

While effective for the regime $\lambda \to \infty$, the equation (1.19) is useless if we are interested in the behavior for general λ. To this end we approach the stochastic Riccati equation from a completely different point of view.

1.4 Method of Carleman embedding

The method of Carleman embedding also called Carleman linearization is a systematic way to transform a finite-dimensional problem with a polynomial nonlinearity to an infinite-dimensional linear system. Usually it is applied to the

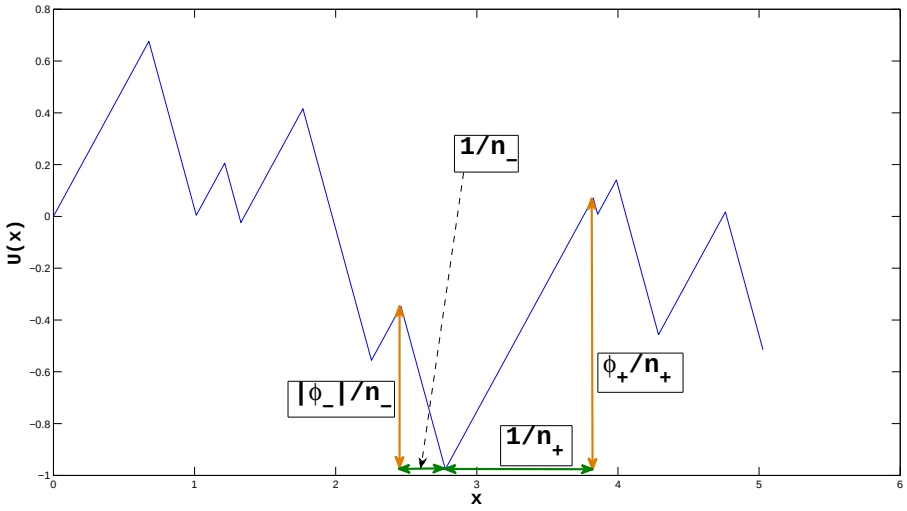

Figure 1.3: Illustration of a possible series of traps in the potential that cause the slowing down of the Brownian particle. As it moves it can get stuck in a specific trap and it stays there until it gets enough energy from the thermal reservoir to penetrate the potential barrier and to jump to a next trap. The parameter controlling the asymptotic behavior of the Brownian particle is proportional to the average depth of a trap.

study of deterministic dynamical systems however we will add one more ingredient and we will use it to find an expression for moments of all orders of the random variable $Q(\lambda; z)$. We will omit presenting this method in its full generality and we will just illustrate it on the problem of the stochastic Riccati equation (1.13). For a general overview see [6].

Let us start by defining the infinite-dimensional vector-valued stochastic process

$$|Y'(\lambda; z)\rangle \equiv \begin{pmatrix} Q(\lambda; z) \\ Q^2(\lambda; z) \\ Q^3(\lambda; z) \\ \vdots \end{pmatrix}. \tag{1.20}$$

All vectors and matrices will be assumed to be written in the basis denoted $\{|1\rangle, |2\rangle, \ldots\}$. By using the Riccati equation (1.13) we can write

$$\frac{dQ^k(\lambda; z)}{d\lambda} = -kQ^{k+1}(\lambda; z) - \Phi(\lambda)kQ^k(\lambda; z) + zkQ^{k-1}(\lambda; z). \tag{1.21}$$

Using this identity we formulate a stochastic differential equation for $|Y'(\lambda; z)\rangle$

as

$$\frac{d\,|Y'(\lambda;z)\rangle}{d\lambda} = -\left[B_- - zB_+ + \Phi(\lambda)N\right]|Y'(\lambda;z)\rangle + z\,|1\rangle\,,\ |Y'(0;z)\rangle = \begin{pmatrix} q_0 \\ q_0^2 \\ q_0^3 \\ \vdots \end{pmatrix} \equiv |q_0\rangle\,, \quad (1.22)$$

where

$$N \equiv \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 3 & \\ \vdots & \vdots & & \ddots \end{pmatrix},\ B_- \equiv NE_- = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},\ B_+ \equiv NE_+ = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 3 & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}, \quad (1.23)$$

$$E_- \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},\ E_+ \equiv \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}.$$

Now we see that we transformed the original problem (1.13) to an infinite-dimensional linear problem (1.22), therefore completing the step of Carleman embedding. However before we move forward we define the random evolution matrix $\mathsf{U}(\lambda;z)$ as a solution to the equation

$$\frac{d\mathsf{U}(\lambda;z)}{d\lambda} = -\left[B_- - zB_+ + \Phi(\lambda)N\right]\mathsf{U}(\lambda;z),\ \ \mathsf{U}(0;z) = 1. \quad (1.24)$$

In terms of this matrix the solution to (1.22) can be written as

$$|Y'(\lambda;z)\rangle = \mathsf{U}(\lambda;z)\,|q_0\rangle + z\int_0^\lambda \mathrm{d}\lambda'\mathsf{U}(\lambda';z)\,|1\rangle\,. \quad (1.25)$$

This allows us to concentrate only on the equation

$$\frac{d\,|\mathsf{Y}(\lambda;z)\rangle}{d\lambda} = -\left[B_- - zB_+ + \Phi(\lambda)N\right]|\mathsf{Y}(\lambda;z)\rangle\,, \quad (1.26)$$

with initial value either $|\mathsf{Y}(0;z)\rangle = |q_0\rangle$ or $|1\rangle$.

1.5 Moments as infinite-dimensional matrix elements

As mentioned earlier we will be interested in the moments of the random variable $\mathsf{Q}(\lambda;z)$. By means of equation (1.25), these can be obtained if we know the average of the vector $|\mathsf{Y}(\lambda;z)\rangle$ which we will denote as

$$|m(\lambda;z)\rangle \equiv \overline{|\mathsf{Y}(\lambda;z)\rangle}. \quad (1.27)$$

Next step is to decompose the dichotomous process $\boldsymbol{\Phi}(\lambda)$ into a deterministic and a random part as

$$\boldsymbol{\Phi}(\lambda) = \phi_0 + \tilde{\boldsymbol{\Phi}}(\lambda), \tag{1.28}$$

where ϕ_0 is arbitrary real number and $\tilde{\boldsymbol{\Phi}}(\lambda)$ is a dichotomous process with the same transition rates as $\boldsymbol{\Phi}(\lambda)$ but with amplitudes shifted by ϕ_0. This may seem rather unintuitive at the moment but it will considerably simplify our results.

Plugging this decomposition into equation (1.26) and taking the average, we obtain

$$\frac{d\,|m(\lambda;z)\rangle}{d\lambda} = -(\phi_0 N + B_- - zB_+)\,|m(\lambda;z)\rangle - N\,|l(\lambda;z)\rangle\,, \tag{1.29}$$

where

$$|l(\lambda;z)\rangle = \overline{\tilde{\boldsymbol{\Phi}}(\lambda)\,|\boldsymbol{Y}(\lambda;z)\rangle}. \tag{1.30}$$

In order to solve for $|m(\lambda;z)\rangle$, we need an equation for $|l(\lambda;z)\rangle$. In Appendix B it is derived that

$$\frac{d\,|l(\lambda;z)\rangle}{d\lambda} = -(\phi_0' N + \gamma + B_- - zB_+)\,|l(\lambda;z)\rangle + (\tilde{\phi_-}\tilde{\phi_+} N + \gamma\mu')\,|m(\lambda;z)\rangle\,, \tag{1.31}$$

where

$$\begin{aligned}
\tilde{\phi_\pm} &\equiv \phi_\pm - \phi_0, \\
\phi_0' &\equiv \phi_+ + \phi_- - \phi_0, \\
\gamma &\equiv n_- + n_+, \\
\mu' \equiv \overline{\tilde{\boldsymbol{\Phi}}(\lambda)} &= \frac{n_-\tilde{\phi_+} + n_+\tilde{\phi_-}}{n_- + n_+}.
\end{aligned} \tag{1.32}$$

Equations (1.30) and (1.31) form a system of two coupled infinite-dimensional linear equations. Natural approach to solution is taking the Laplace transform with respect to the variable λ which we know is positive. Let us call the corresponding Laplace variable u. Using the fact that $|m(0;z)\rangle = |\boldsymbol{Y}(0;z)\rangle$ and $|l(0;z)\rangle = \mu'\,|\boldsymbol{Y}(0;z)\rangle$, we can express the vector $|m(u;z)\rangle$ as

$$\begin{aligned}
|m(u;z)\rangle =& \frac{1}{u + \phi_0 N + B_- - zB_+ + N\frac{1}{u+\gamma+\phi_0' N + B_- - zB_+}\left[\tilde{\phi_-}\tilde{\phi_+} N + \gamma\tilde{\mu}\right]} \cdot \\
&\cdot \left(1 - N\frac{1}{u + \gamma + \phi_0' N + B_- - zB_+}\tilde{\mu}\right)|\boldsymbol{Y}(0)\rangle\,.
\end{aligned} \tag{1.33}$$

Therefore we see that calculating the moments amounts to inverting an infinite-dimensional matrix. This equation is the starting point of any future calculations.

1.6 Evaluating the matrix element for $t \to \infty$

1.6.1 Preliminary assumptions

In this thesis we will concentrate only on the case $z \to 0$ which corresponds to the limit $t \to \infty$. We will also assume that

$$\phi_- < \phi_+ < 0. \tag{1.34}$$

For clarity we make the following definitions.

$$P(\lambda) \equiv \lim_{z\to 0} Q(\lambda; z) = \lim_{z\to 0} zP(\lambda; z), \tag{1.35}$$

$$|m(\lambda)\rangle \equiv \lim_{z\to 0} |m(\lambda; z)\rangle\,, \tag{1.36}$$

$$p_0 = q_0. \tag{1.37}$$

Since in this limit the non-homogeneous term in equation (1.22) drops out, we can immediately identify $\langle k|m(\lambda)\rangle$ with the kth moment of $P(\lambda)$.

Now we turn our attention to evaluating (1.33). We set $\phi_0 = \phi_-$ which implies

$$\tilde{\phi_-} = 0,\ \tilde{\phi_+} = |\phi_-| - |\phi_+|,\ \phi_0 = -|\phi_+|,\ \mu' = \frac{n_-}{\gamma}(|\phi_-| - |\phi_+|),$$

and we rewrite the expression in terms of L matrices which we define as

$$L(a, b) \equiv \frac{1}{a - bN + B_-},\ b > 0. \tag{1.38}$$

This leads to

$$\begin{aligned} |m(u)\rangle =& \frac{1}{L^{-1}(u, |\phi_-|) + n_-(|\phi_-| - |\phi_+|)NL(u + \gamma, |\phi_+|)} \cdot \\ & \cdot \left[1 - \frac{n_-}{\gamma}(|\phi_-| - |\phi_+|)NL(u + \gamma, |\phi_+|)\right] |p_0\rangle\,. \end{aligned} \tag{1.39}$$

1.6.2 Properties of the L matrix

Evaluation of the expression (1.39) for general u will be based on the following identity which we prove in Appendix C.

$$L(a, b)\,|p_0\rangle = \frac{1}{a} {}_2F_1(N, \frac{a}{b}, \frac{a}{b} + 1; 1 - \frac{b}{p_0})\,|b\rangle\,, \tag{1.40}$$

$$\langle k|b\rangle = b^k. \tag{1.41}$$

In this thesis we are however concerned only with the case $\lambda \to \infty$ and all we will need is that

$$\lim_{u\to 0} uL(u, b)\,|p_0\rangle = |b\rangle\,, \tag{1.42}$$

which is a simple consequence of the preceding identity.

Another useful identity is

$$E_- \left|b\right\rangle = b \left|b\right\rangle . \tag{1.43}$$

Now suppose $f(x)$ is an analytic function (*i.e.* it has a Taylor expansion). Then it follows immediately from this identity and the fact that $B_- = NE_-$ that

$$f(-bN + B_-) \left|b\right\rangle = f(0) \left|b\right\rangle , \text{in particular } L(a,b) \left|b\right\rangle = \frac{1}{a} \left|b\right\rangle . \tag{1.44}$$

On the lowest level, the only two matrices that we are dealing with are N and E_-. Fortunately these satisfy the commutation relation

$$NE_- = E_-(N-1), \tag{1.45}$$

which we can generalize to

$$NB_- = B_-(N-1),\ f(-bN + B_-)B_- = B_- f(b - bN + B_-). \tag{1.46}$$

Last ingredient we need is the representation of $\left|b\right\rangle$ using a geometric series as

$$\left|b\right\rangle = \frac{b}{1 - bE_+} \left|1\right\rangle . \tag{1.47}$$

This can be used to employ the following trick. The matrix in (1.39) consists only of the N and E_- matrices that we know satisfy the relation (1.45). Therefore there must be a way to write it as a power series in E_- with matrix coefficients that depend only on N. This allows us to write

$$\begin{aligned}
\left\langle k|m(u)\right\rangle &= \left\langle k\right| \sum_{n=0}^{\infty} a_n(N) E_-^n \left|p_0\right\rangle \\
&= \left\langle 1\right| E_-^{k-1} \sum_{n=0}^{\infty} a_n(N) E_-^n \frac{p_0}{1 - p_0 E_+} \left|1\right\rangle \\
&= p_0 \left\langle 1\right| \sum_{n=0}^{\infty} a_n(N + k - 1) E_-^{n+k-1} \sum_{l=0}^{\infty} p_0^l E_+^l \\
&= p_0 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} a_n(k) p_0^l \left\langle 1\right| E_-^{n+k-1} E_+^l \left|1\right\rangle \\
&= p_0 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} a_n(k) \delta_{n+k-1,l} \left\langle 1\right| E_-^{n+k-1} E_+^l \left|1\right\rangle p_0^{n+k-1} \\
&= p_0^k \left\langle k\right| \sum_{n=0}^{\infty} a_n(N) (p_0 E_-)^n \left|e\right\rangle ,
\end{aligned} \tag{1.48}$$

where the components of $\left|e\right\rangle$ are $\left\langle k|e\right\rangle = 1$ for all k. In other words every occurrence of E_- (B_-) can be replaced by $p_0 E_-$ ($p_0 B_-$) provided we change the vector $\left|p_0\right\rangle$ (or any other vector with the same structure) to $\left|e\right\rangle$.

1.6.3 The case $\lambda \to \infty$

We will denote the kth moment by $m_k(u) \equiv \langle k|m(u)\rangle$. In the limit $\lambda \to \infty$ we just drop the argument so

$$m_k \equiv \lim_{\lambda\to\infty} m_k(\lambda) = \lim_{u\to 0} u m_k(u). \tag{1.49}$$

Our goal is to express m_k in the form ($f(x)$ is some analytic function)

$$\begin{aligned} m_k &= \langle k| \frac{1}{1 - f(-bN + B_-)B_-} |b\rangle \\ &= \langle k| \sum_{l=0}^{\infty} [f(-bN + B_-)N]^l |b\rangle \\ &= \langle k| \sum_{l=0}^{\infty} N(N+1)\cdots(N+l-1)E_-^l f(-bN + B_- + b)\cdots f(-bN + B_- + lb) |b\rangle \\ &= \sum_{l=0}^{\infty} (k)_l b^l \prod_{n=1}^{l} f(nb). \end{aligned} \tag{1.50}$$

To this end we divide the formula for $m_k(u)$ (1.39) into two parts

$$m_k^{(1)} = \left(1 - \frac{n_-}{\gamma}\right) \langle k| \frac{1}{L^{-1}(u, |\phi_-|) + n_-(|\phi_-| - |\phi_+|)NL(u+\gamma, |\phi_+|)} |p_0\rangle , \tag{1.51}$$

$$\begin{aligned} m_k^{(2)} =& \frac{n_-}{\gamma} \langle k| \frac{1}{L^{-1}(u, |\phi_-|) + n_-(|\phi_-| - |\phi_+|)NL(u+\gamma, |\phi_+|)} \cdot \\ & \cdot [1 - (|\phi_-| - |\phi_+|)NL(u+\gamma, |\phi_+|)] |p_0\rangle . \end{aligned} \tag{1.52}$$

Using the matrix identities

$$\frac{1}{A+B} = \frac{1}{1 + A^{-1}B} A = A \frac{1}{1 + BA^{-1}}, \tag{1.53}$$

and the definition of $L(a, b)$ we can rearrange these expressions to

$$m_k^{(1)} = \left(1 - \frac{n_-}{\gamma}\right) \langle k| \frac{1}{1 + n_-(|\phi_-| - |\phi_+|)L(u, |\phi_-|)NL(u+\gamma, |\phi_+|)} L(u, |\phi_-|) |p_0\rangle , \tag{1.54}$$

$$m_k^{(2)} = \frac{n_-}{\gamma} \langle k| \frac{1}{1 + (|\phi_-| - |\phi_+|)L(u, |\phi_-|)N \frac{L(u+\gamma, |\phi_-|)}{L(u+n_-, |\phi_-|)}} L(u, |\phi_-|) |p_0\rangle . \tag{1.55}$$

Now we can finally take the limit $\lim_{u\to 0} u m_k(u)$. The $1/u$ pole necessary for this limit to be non-zero comes from the rightmost matrix $L(u, |\phi_-|)$ according to the identity (1.40). Therefore we can use equation (1.42) on this part of the

equations and set $u = 0$ everywhere else. After employing the trick (1.48) and doing some tedious algebra with the definition of $L(a, b)$, we arrive at the form

$$m_k^{(1)} = \frac{1}{\nu}\left(1 - \frac{n_-}{\gamma}\right)|\phi_-|^k \langle k| [L^{-1}(\nu, 1) - \chi(\nu_- + B_-)] \frac{1}{1 - \chi B_- \frac{L(1,1)L(\nu+1,1)}{L(\nu_-+1,1)}} |e\rangle , \tag{1.56}$$

$$m_k^{(2)} = \frac{n_-}{\gamma}|\phi_-|^k \langle k| [1 - xL(\nu, 1)L^{-1}(\nu_-, 1)] \frac{1}{1 - xB_- \frac{L(1,1)L(\nu,1)}{L(\nu_-,1)}} |e\rangle , \tag{1.57}$$

where

$$\nu_- = \frac{n_-}{|\phi_-|},\ \nu = \frac{n_-}{|\phi_-|} + \frac{n_+}{|\phi_+|},\ x = 1 - \frac{|\phi_-|}{|\phi_+|}. \tag{1.58}$$

This result resembles the form in equation (1.50). In the same spirit we write down a series for these functions. This series can be summed giving the result

$$\begin{aligned} m_k^{(1)} &= \left(1 - \frac{n_-}{\gamma}\right)|\phi_-|^k \Big[{}_2F_1(k, \nu_- + 1, \nu; x) - x\frac{\nu_-}{\nu}{}_2F_1(k, \nu_- + 1, \nu + 1; x)] \\ &\quad -x\frac{k}{\nu}{}_2F_1(k + 1, \nu_- + 1, \nu + 1; x)\Big] \\ &= \left(1 - \frac{n_-}{\gamma}\right)|\phi_-|^k {}_2F_1(k - 1, \nu_-, \nu; x), \end{aligned} \tag{1.59}$$

$$\begin{aligned} m_k^{(2)} &= \frac{n_-}{\gamma}|\phi_-|^k \Big[{}_2F_1(k, \nu_-, \nu; x) - x\frac{\nu_-}{\nu}{}_2F_1(k, \nu_- + 1, \nu + 1; x)\Big] \\ &= \frac{n_-}{\gamma}|\phi_-|^k {}_2F_1(k - 1, \nu_-, \nu; x), \end{aligned} \tag{1.60}$$

where we used the Gauss' relations for contiguous functions (see [7], page 558). Therefore we obtain the result

$$m_k = m_k^{(1)} + m_k^{(2)} = |\phi_-|^k {}_2F_1(k - 1, \nu_-, \nu; x). \tag{1.61}$$

This however does not agree with the same result obtained by direct integration of the probability density given in [5]. The correct formula that we checked using numerical simulations is

$$m_k = |\mu||\phi_-|^{k-1} {}_2F_1(k - 1, \nu_-, \nu; x), \tag{1.62}$$

where μ is the mean force given by

$$\mu = \frac{n_-\phi_+ + n_+\phi_-}{n_- + n_+}. \tag{1.63}$$

Unfortunately mainly due to time limitation we were not able to find a flaw in our calculations. This discrepancy is the first thing that needs to be solved before we move on to calculate the formula for general u.

2. Numerical simulations of the Riccati equation

In order to check theoretical results about the solution of the stochastic Riccati equation with dichotomous noise (1.13), we developed a numerical simulation of this equation. All simulations were carried out using *Matlab 7.12.0*.

2.1 Description of the Monte Carlo algorithm

Let us overview all the parameters that we have available and that will serve as input for the program that executes the simulation. We have the two possible forces $\phi_\pm$ that act on the Brownian particle and we also have the transition rates $n_\pm$ that determine the average times the particle is under the influence of the force $\phi_\pm$. The least intuitive parameter is z which is the Laplace variable corresponding to time. Although in general it is a complex number, in simulations we will always assume it is real and positive. Its best interpretation is that small z represents large times according to the Tauber's theorem ([3])

$$\lim_{z\to 0} zf(z) = \lim_{t\to\infty} f(t). \tag{2.1}$$

Last parameter is λ which represents the length of the interval on which diffusion occurs.

Simulation of the riccati equation is considerably simplified by using the dichotomous process as an input noise because it allows us to generate its *exact* realizations. This is achieved in two steps (although they can be combined into one loop).

Step 1:

We make a realization of the dichotomous noise. First we have to decide in what state it starts. The stationary state is described by the probabilities

$$\pi_+ = \frac{n_-}{n_- + n_+} \text{ and } \pi_- = \frac{n_+}{n_- + n_+}. \tag{2.2}$$

Therefore we draw a uniformly distributed random number and if it is less than π_+ we start in the state ϕ_+, otherwise we start with ϕ_-.

Next we generate an exponentially distributed random variable with mean $1/n_\pm$ dependent on the initial state of $\boldsymbol{\Phi}(x)$. This number λ_1 marks the length of the first interval of constant force. If $\lambda_1 > \lambda$, we change λ_1 to λ and the generation of a realization is complete. Otherwise we switch the state and generate another exponentially distributed random variable with mean corresponding to the new state. This number λ_2 is the length of the second interval on which the force is constant. If $\lambda_1 + \lambda_2 > \lambda$, we change λ_2 to $\lambda - \lambda_1$, obtaining the sought for

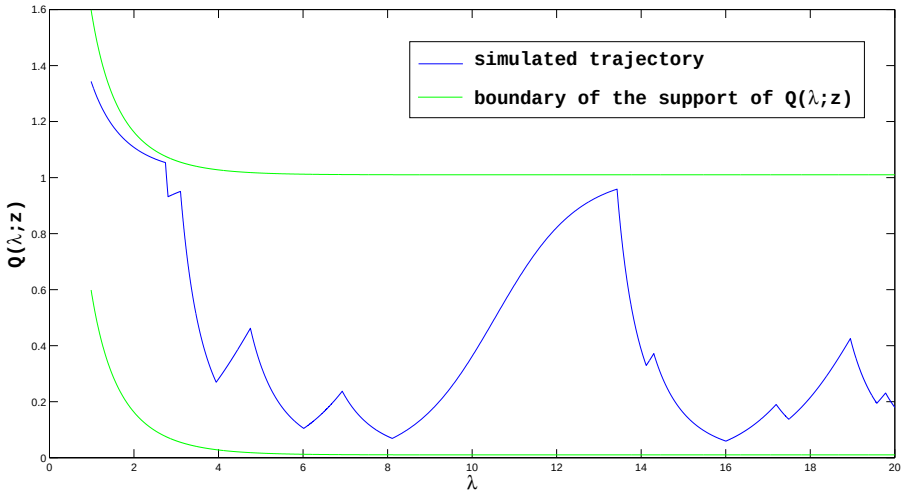

Figure 2.1: A sample realization of the solution to the stochastic Riccati equation. We used parameters $n_+ = 2$, $n_- = 1$, $\phi_+ = 1$, $\phi_- = -1$, and $z = 0.1$. The green lines show the boundaries of the support of $\boldsymbol{Q}(\lambda; z)$.

realization. Otherwise we switch the state again and iterate this process until we reach the total length λ. The result of this step is a sequence of numbers $\lambda_1, \ldots, \lambda_n$ that describe the partition of our interval into segments on which the realization $\phi(x)$ of $\boldsymbol{\Phi}(x)$ is constant.

Step 2:

Corresponding to the realization $\phi(x)$ of $\boldsymbol{\Phi}(x)$ we find a solution of the Riccati equation (1.13). To that end we need a solution to the Riccati equation with constant force ϕ and *arbitrary* initial value $Q(\lambda_0) = Q_0$. Since in this case the Riccati equation is separable, it is easy to find that the solution we are looking for is

$$Q(\lambda; z) = \frac{q^{\min}(z) - q^{\max}(z)\frac{Q_0 - q^{\min}(z)}{Q_0 - q^{\max}(z)} e^{-\left[q^{\min}(z) - q^{\max}(z)\right](\lambda - \lambda_0)}}{1 - \frac{Q_0 - q^{\min}(z)}{Q_0 - q^{\max}(z)} e^{-[q^{\min}(z) - q^{\max}(z)](\lambda - \lambda_0)}}, \tag{2.3}$$

where similarly as in section 1.2

$$q^{\min}(z) = \sqrt{z + \left(\frac{\phi}{2}\right)^2} - \frac{\phi}{2}, \; q^{\max}(z) = -\sqrt{z + \left(\frac{\phi}{2}\right)^2} - \frac{\phi}{2}. \tag{2.4}$$

Now it is easy to construct the solution for $\phi(x)$. We start with $Q_0 = \infty$ and using equation (2.3), we construct a solution $Q_1(\lambda; z)$ on the interval $[0, \lambda_1]$ where we know $\phi(x)$ is constant. Then we set $Q_0 = Q_1(\lambda_1; z)$ and using this initial value we find a solution on the interval $[\lambda_1, \lambda_1 + \lambda_2]$. We iterate this procedure constructing a realization of the process $\boldsymbol{Q}(\lambda; z)$ for the interval $[0, \lambda]$. It should

be noted that we can formulate the solution (2.3) in terms of $1/Q_0$ instead of Q_0 which enables us to treat the case $Q_0 = \infty$ exactly.

A sample realization of the process $\mathcal{Q}(\lambda; z)$ is in figure 2.1. We can iterate this algorithm large number of times therefore obtaining a collection of realizations of the random variable $Q(\lambda; z)$. In order to simulate the corresponding probability density, we divide the support of $Q(\lambda; z)$ into small bins and count the number of realizations that ended up in each one of them. Plot of this histogram is the result of the simulation. If we are only interested in the moments of $Q(\lambda; z)$, we simply calculate it by averaging over the collected realizations without the necessity of making a histogram first.

2.2 Comparison of analytical results with simulations

In [5] the probability distribution for $Q(\lambda; z)$ in the case that $\lambda \to \infty$ was derived to be

$$\kappa(q; z) = \frac{1}{C(z)} \left\{ \frac{1}{[q_-(z) - q][q - q'_-(z)]} + \frac{1}{[q - q'_+(z)][q - q_+(z)]} \right\} \cdot \\ \cdot \left[\frac{q_-(z) - q}{q - q'_-(z)} \right]^{\alpha_-(z)} \left[\frac{q - q_+(z)}{q - q'_+(z)} \right]^{\alpha_+(z)} \Theta(q - q_+(z)) \Theta(q_-(z) - q), \quad (2.5)$$

where the normalization constant is

$$C(z) = \frac{1}{\gamma} \left(\frac{n_-}{\sqrt{(4z + \phi_+^2)}} + \frac{n_+}{\sqrt{(4z + \phi_-^2)}} \right) \frac{[q_-(z) - q_+(z)]^{\alpha_-(z) + \alpha_+(z)}}{[q_+(z) - q'_-(z)]^{\alpha_-(z)} [q_-(z) - q'_+(z)]^{\alpha_+(z)}} \cdot \\ \cdot B[\alpha_-(z), \alpha_+(z)]_2 F_1(\alpha_-(z), \alpha_+(z), \alpha_-(z) + \alpha_+(z) + 1; -u(z)), \quad (2.6)$$

$$u(z) = \frac{[q_-(z) - q_+(z)][q'_-(z) - q'_+(z)]}{[q_-(z) - q'_+(z)][q_+(z) - q'_-(z)]}. \quad (2.7)$$

B(x,y) denotes the beta function. And the other variables are defined as

$$q_\pm(z) = q_\pm^{\min}(z), \; q'_\pm(z) = q_\pm^{\max}(z), \; \alpha_\pm = \frac{n_\pm}{\sqrt{(4z + \phi_\pm^2)}}. \quad (2.8)$$

We compared this exact result with our simulations for the parameters $n_+ = 3$, $n_- = 2$, $\phi_- = -2$, $\phi_+ = -1$ and three different values of z. As is clear from figure 2.2 we found a good match. We also tested the formula (1.62) for $k = 2$ *i.e.* for second moment of $\mathcal{Q}(\lambda; z)$. Used parameters and results are displayed in figure 2.3. We see again that the formula is consistent with simulations therefore confirming the prediction that our result is flawed.

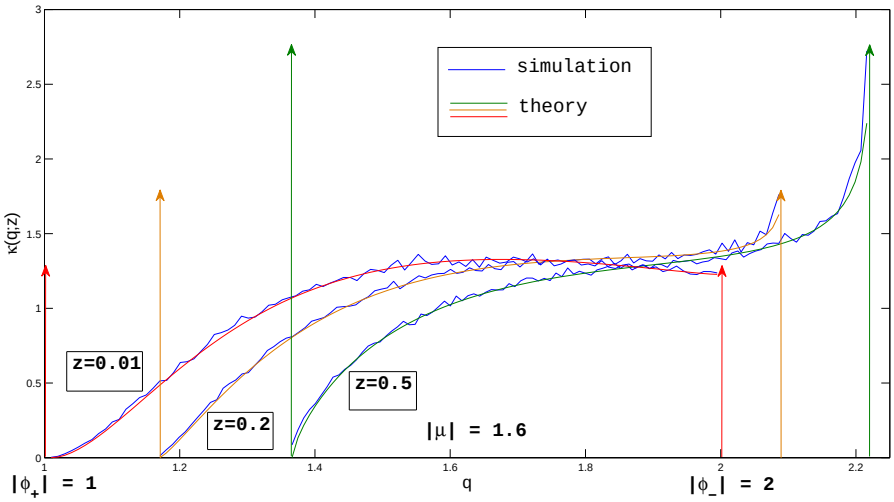

Figure 2.2: Comparison of the results of numerical simulations of the stochastic Riccati equation with the probability density obtained in [5]. Used parameters of the dichotomous process were $n_+ = 3$, $n_- = 2$, $\phi_- = -2$, and $\phi_+ = -1$. Each blue curve is based on a simulation using 300000 realizations of the process $Q(\lambda; z)$ and the supports of these functions were divided into 100 bins. For the interval length we used $\lambda = 10$. The arrows mark the boundaries of supports.

2.3 Simulations for the case of finite interval

The value of these simulations is that it confirms the results predicted theoretically however it does not reveal much about physics. Physically interesting variables are for example dynamical exponents which are however much harder to obtain numerically. Therefore it does not make much sense to simulate all possible scenarios for finite λ before we actually have some analytical results to compare with. Nevertheless since our algorithm works for any finite λ we show in figure 2.4 the λ-dependence of first moment of $Q(\lambda; z)$ for set of parameters specified in the figure. If we looked at the first moment at even longer times we would find that the red and green curve in the figure would converge close to zero while the green curve would have a non-zero limit as expected since it corresponds to a situation with negative mean force.

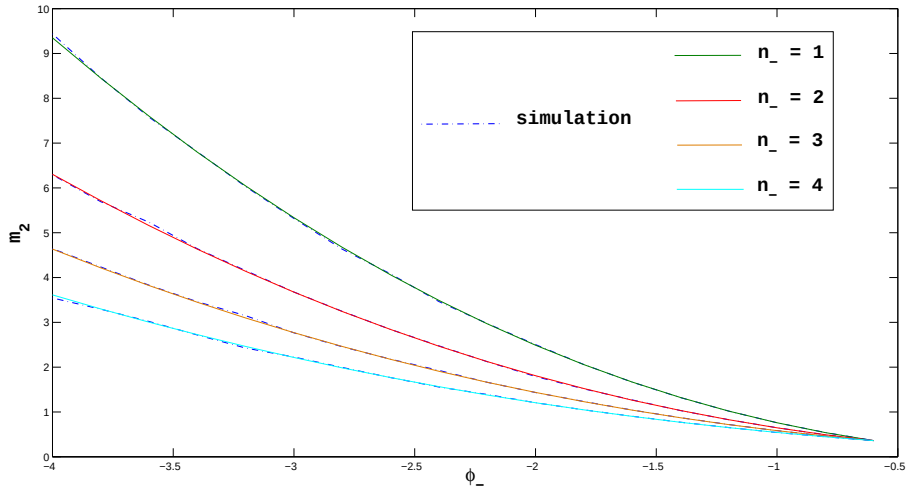

Figure 2.3: Comparison of the analytical results based on equation (1.62) with numerical simulations. The figure shows the dependence of the second moment of $Q(\lambda; z)$ in the limit $z \to 0$, $\lambda \to \infty$ on the force ϕ_- while keeping the values of $\phi_+ = -0.6$, $n_+ = 2$ and n_- constant. The simulation is carried out for four different values of $n_- = 1,\ 2,\ 3,\ 4$. For each ϕ_-, the second moment was calculated using 10000 realizations.

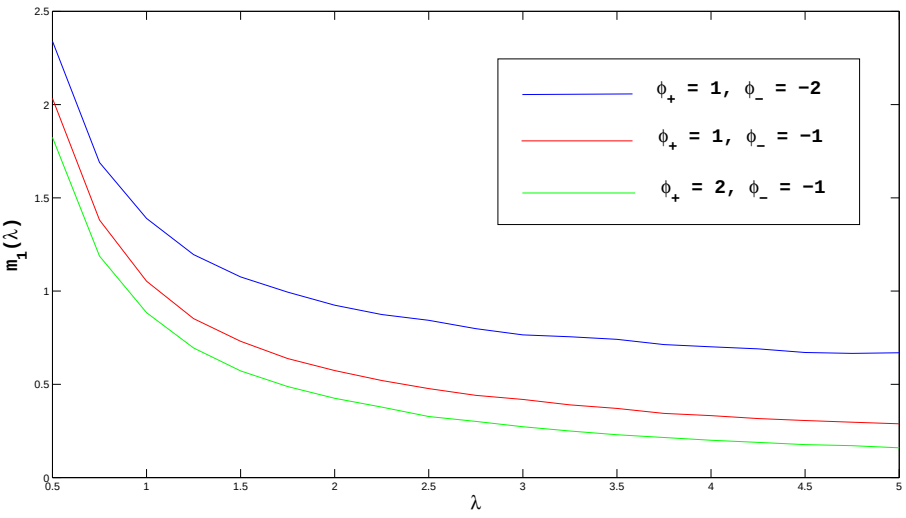

Figure 2.4: Simulation of the λ-dependence of the first moment of $Q(\lambda; z)$. We used parameters $n_- = 1 = n_+$, $z = 0.01$. Other parameters are specified in the figure. For each λ, moments were calculated from 10000 realizations.

Conclusion

In this thesis we have presented a previously unexplored method of solution of the stochastic Riccati equation with multiplicative dichotomous noise. The method is based on the idea of Carleman linearization and it allowed us to express the Laplace transformed moments of the solution in terms of the matrix elements of an infinite-dimensional matrix. We have developed a purely algebraic approach which enables an analytic evaluation of these matrix elements. Unfortunately we were unable to obtain a result consistent with numerical simulations and analytical results based on a different theoretical approach. However the results differ only by a multiplicative factor and we believe that the discrepancy is caused by our mistake and not by a flaw in the method which has been tested on simpler systems. Unfortunately time restrictions did not allow us to invest appropriate amount of time to the resolution of this discrepancy therefore we decided to postpone it to further work beyond this thesis.

Once we overcome this obstacle we hope to find an exact formula for the moments even for a finite λ. In fact using the identities and methods we presented we should be able to write down this answer in terms of a power series. However the hardest part of this problem is to find an appropriate expansion parameter and/or a form of the equation (1.33) that will actually allow us to sum this series.

As an important element of this work, we have created a *Matlab* simulation that generates realizations of the solution to the stochastic Riccati equation. We tested its correctness on known theoretical results. This simulation is easily generalized even to other input noises whose realizations can be approximated by a piecewise constant functions and it certainly will be useful for future work on this problem.

Appendix A - Dichotomous process

Dichotomous process is a Markov process $\boldsymbol{\Phi}(x)$ with state space ϕ_-, ϕ_+. Let us define the vector of occupation probabilities of these two states as

$$|\pi(x)\rangle = \begin{pmatrix} \pi_-(x) \\ \pi_+(x) \end{pmatrix}. \tag{2.9}$$

The dichotomous process then satisfies the Master equation

$$\frac{d\,|\pi(x)\rangle}{dx} = -\Gamma\,|\pi(x)\rangle\,, \tag{2.10}$$

where

$$\Gamma = \begin{pmatrix} n_- & -n_+ \\ -n_- & n_+ \end{pmatrix}. \tag{2.11}$$

The parameters $n_\pm$ will be referred to as the *transition rates*. It is easy to find an exact solution of this equation however it is not essential for the present thesis. We just note that the intervals between jumps from one state to another are exponentially distributed with mean lengths $1/n_\pm$.

In our calculations we will however deal with the correlation functions

$$C_n(x_1, x_2, \ldots, x_n) \equiv \overline{\boldsymbol{\Phi}(x_1)\boldsymbol{\Phi}(x_2)\cdots\boldsymbol{\Phi}(x_n)}, \tag{2.12}$$

where it is assumed that $x_1 < x_2 < \ldots < x_n$.

The evolution operator corresponding to equation (2.10) is

$$R(x - x') = \exp\left[-\Gamma(x - x')\right]. \tag{2.13}$$

Using this matrix we can write down the following expression for the correlation functions

$$C_n(x_1, x_2, \ldots, x_n) = \langle e|\, DR(x_n - x_{n-1})DR(x_{n-1} - x_{n-2})\cdots DR(x_1)\,|\pi(0)\rangle\,, \tag{2.14}$$

where

$$D = \begin{pmatrix} \phi_- & 0 \\ 0 & \phi_+ \end{pmatrix} \text{ and } |e\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2.15}$$

Proof of this identity follows immediately if we interpret $R_{ij}(x - x')$ as the probability that $\boldsymbol{\Phi}(x) = i$ given that $\boldsymbol{\Phi}(x') = j$ and if we rewrite the equation (2.14) using component notation.

The following result is essential for our calculation.

Lemma (Derivative of correlation functions). *For any n and any x, x_1, x_2, ..., x_n such that $x_1 < x_2 < \ldots < x_n < x$, we have*

$$\frac{d}{dx}C_{n+1}(x_1,\ldots,x_n,x) = -(n_-+n_+)C_{n+1}(x_1,\ldots,x_n,x)+(n_+\phi_-+n_-\phi_+)C_n(x_1,\ldots,x_n).$$

Proof. From equation (2.14) and (2.13) we have

$$\frac{d}{dx}C_{n+1}(x_1,\ldots,x_n,x) = -\left\langle e\right| D\Gamma R(x-x_n)DR(x_n-x_{n-1})\cdots DR(x_1)\left|\pi(0)\right\rangle.$$

Simple algebra shows that

$$\left\langle e\right| D\Gamma = (n_- + n_+)\left\langle e\right| D - (n_+\phi_- + n_-\phi_+)\left\langle e\right|.$$

Combining this result with the preceding equation and using the fact that

$$\left\langle e\right| R(y) = \left\langle e\right| \text{ for all } y,$$

which is true since $R(y)$ conserves total probability, we arrive at the statement of this lemma. □

Appendix B[1]

The purpose of this appendix is to derive a differential equation for

$$|l(\lambda;z)\rangle = \overline{\tilde{\boldsymbol{\Phi}}(\lambda)\,|\boldsymbol{Y}(\lambda;z)\rangle},$$

where $|\boldsymbol{Y}(\lambda;z)\rangle$ satisfies the equation

$$\frac{d\,|\boldsymbol{Y}(\lambda;z)\rangle}{d\lambda} = -\Big[B_- - zB_+ + \phi_0 N + \tilde{\boldsymbol{\Phi}}(\lambda)N\Big]\,|\boldsymbol{Y}(\lambda;z)\rangle\,. \tag{2.16}$$

We assume a general initial value $|\boldsymbol{Y}(0;z)\rangle$. First step is to transform this equation into an "interaction" picture by defining

$$|\boldsymbol{Y}_{\rm I}(\lambda;z)\rangle = e^{(\phi_0 N + B_- - zB_+)\lambda}\,|\boldsymbol{Y}(\lambda;z)\rangle\,. \tag{2.17}$$

After this change of variables, equation (2.16) reads

$$\frac{d\,|\boldsymbol{Y}_{\rm I}(\lambda;z)\rangle}{d\lambda} = -\tilde{\boldsymbol{\Phi}}(\lambda)N'(\lambda;z)\,|\boldsymbol{Y}_{\rm I}(\lambda;z)\rangle\,,\ |\boldsymbol{Y}_{\rm I}(0;z)\rangle = |\boldsymbol{Y}(0;z)\rangle\,, \tag{2.18}$$

where

$$N'(\lambda;z) = e^{(\phi_0 N + B_- - zB_+)\lambda} N e^{-(\phi_0 N + B_- - zB_+)\lambda}. \tag{2.19}$$

We can immediately write down a solution to this equation in a form of a Dyson series as

$$\begin{aligned}|\boldsymbol{Y}_{\rm I}(\lambda;z)\rangle = \Bigg(1 - \int_0^\lambda \mathrm{d}\lambda'\,\tilde{\boldsymbol{\Phi}}(\lambda')N'(\lambda';z) + \\ + \int_0^\lambda \mathrm{d}\lambda' \int_0^{\lambda'} \mathrm{d}\lambda''\,\tilde{\boldsymbol{\Phi}}(\lambda')\tilde{\boldsymbol{\Phi}}(\lambda'')N'(\lambda';z)N'(\lambda'';z) + \ldots\Bigg)\,|\boldsymbol{Y}(0;z)\rangle\,.\end{aligned} \tag{2.20}$$

Using this formal solution we find

$$\begin{aligned}\frac{d}{d\lambda}\overline{\tilde{\boldsymbol{\Phi}}(\lambda)\,|\boldsymbol{Y}_{\rm I}(\lambda;z)\rangle} = -\,N'(\lambda;z)\overline{\tilde{\phi}^2(\lambda)\,|\boldsymbol{Y}_{\rm I}(\lambda;z)\rangle} + \\ + \left[\frac{d\overline{\tilde{\boldsymbol{\Phi}}(\lambda)}}{d\lambda} - \int_0^\lambda \mathrm{d}\lambda'\,\frac{d}{d\lambda}\overline{\tilde{\boldsymbol{\Phi}}(\lambda)\tilde{\boldsymbol{\Phi}}(\lambda')}N'(\lambda';z) + \ldots\right]\,|\boldsymbol{Y};\boldsymbol{z}(0)\rangle\,.\end{aligned} \tag{2.21}$$

Now we can use the lemma from Appendix A about the derivatives of correlation functions of $\tilde{\boldsymbol{\Phi}}(\lambda)$. Inserting these relations into the last equation gives

$$\frac{d\overline{\tilde{\boldsymbol{\Phi}}(\lambda)\,|\boldsymbol{Y}_{\rm I}(\lambda)\rangle}}{d\lambda} = -N'(\lambda;z)\overline{\tilde{\phi^2}(\lambda)\,|\boldsymbol{Y}_{\rm I}(\lambda;z)\rangle} - \gamma\overline{\tilde{\boldsymbol{\Phi}}(\lambda)\,|\boldsymbol{Y}_{\rm I}(\lambda;z)\rangle} + \gamma\mu'\overline{|\boldsymbol{Y}_{\rm I}(\lambda;z)\rangle}. \tag{2.22}$$

Employing the definition (2.17) of $|\boldsymbol{Y}_{\rm I}(\lambda;z)\rangle$ and the simple identity

$$\tilde{\phi^2}(\lambda) = (\tilde{\phi_-} + \tilde{\phi_+})\tilde{\boldsymbol{\Phi}}(\lambda) - \tilde{\phi_-}\tilde{\phi_+}, \tag{2.23}$$

it is now just a matter of algebra to arrive at the sought for result

$$\frac{d\,|l(\lambda;z)\rangle}{d\lambda} = -(\phi_0' N + \gamma + B_- - zB_+)\,|l(\lambda;z)\rangle + (\tilde{\phi_-}\tilde{\phi_+}N + \gamma\mu')\,|m(\lambda;z)\rangle\,. \tag{2.24}$$

[1]For definitions of variables used in this appendix see section 1.5.

Appendix C

In this appendix we prove the identity

$$L(a,b)\,|c\rangle = \frac{1}{a}{}_2F_1(N, \frac{a}{b}, \frac{a}{b}+1; 1-\frac{b}{c})\,|b\rangle\,, \tag{2.25}$$

where the components of vectors $|b\rangle$ and $|c\rangle$ are given by

$$\langle k|b\rangle = b^k,\ \langle k|c\rangle = c^k. \tag{2.26}$$

The proof is based on the same methods that were used in section 1.6.3. First we employ the trick (1.48) to write

$$\langle k|\,L(a,b)\,|c\rangle = c^k\,\langle k|\,\frac{1}{a-bN+cB_-}\,|e\rangle\,. \tag{2.27}$$

Second step is to expand this expression

$$\begin{aligned}
\langle k|\,L(a,b)\,|c\rangle &= c^k\frac{1}{b}\,\langle k|\,\frac{1}{L^{-1}\left(\frac{a}{b},1\right)-\left(1-\frac{c}{b}\right)B_-} \\
&= c^k\frac{1}{a}\,\langle k|\,\frac{1}{1-\left(1-\frac{c}{b}\right)L\left(\frac{a}{b},1\right)B_-} \\
&= c^k\frac{1}{a}\,\langle k|\sum_{n=0}^{\infty}\left(1-\frac{c}{b}\right)^n B_-^n\frac{1}{\left(1+\frac{a}{b}\right)_n}\,|e\rangle \\
&= c^k\frac{1}{a}{}_2F_1\left(1,k,1+\frac{a}{b};1-\frac{c}{b}\right) \\
&= b^k\frac{1}{a}{}_2F_1\left(k,\frac{a}{b},1+\frac{a}{b};1-\frac{b}{c}\right),
\end{aligned} \tag{2.28}$$

where in the last equality we used a linear transformation formula for Gauss hypergeometric functions ([7], page 559).

Looking at the last formula we can immediately rewrite it as

$$\langle k|\,L(a,b)\,|c\rangle = \frac{1}{a}\,\langle k|\,{}_2F_1\left(N,\frac{a}{b},\frac{a}{b}+1;1-\frac{b}{c}\right)|b\rangle\,, \tag{2.29}$$

therefore proving the identity (2.25).

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List of Symbols

General notation

$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{N}$	The set of non-negative integers.
$f(s)$	$f(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$ is the Laplace transform of the real-valued function $f(t)$, $t \in \mathbb{R}^+$, $s \in \mathbb{C}$.
$(a)_n$	Pochhammer symbol, $(a)_n \equiv a(a+1)\dots(a+n-1)$ if $n > 0$, $(a)_0 \equiv 1$, $n \in \mathbb{N}$.
${}_2F_1(a,b,c;z)$	Gauss hypergeometric function, ${}_2F_1(a,b,c;z)$ is the analytical continuation of the series $\sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$, $\lvert z\rvert < 1$.
$\lvert k\rangle$	Orthonormal basis $\lvert k\rangle = (0\dots1\dots0)$ where the 1 is at position k, $k \in \{1,2,\dots\}$.
X	A random variable, x denotes a member of its state space.
$\mathsf{X}(t)$	A stochastic process indexed by t, t belongs to some interval, $x(t)$ denotes a member of the space of all possible realizations of $\mathsf{X}(t)$.

Problem-specific notation

λ	Length of the interval on which diffusion occurs.
z	Laplace variable corresponding to time in the Fokker-Planck equation.
$\mathsf{\Phi}(x)$	Stochastic process describing the force field in which diffusion takes place. Usually assumed to be dichotomous noise.
$\mathsf{P}(\lambda;z)$	For fixed z and λ, $\mathsf{P}(\lambda;z)$ is a random variable corresponding to the value of a solution to the Fokker-Planck equation at the origin when random force field is assumed.
$\mathsf{Q}(\lambda;z)$	$\mathsf{Q}(\lambda;z) \equiv z\mathsf{P}(\lambda;z)$. Solution to the stochastic Riccati equation.
$\mathsf{P}(\lambda)$	$\mathsf{P}(\lambda) \equiv \lim_{z\to 0} \mathsf{Q}(\lambda;z)$.
$\lvert\mathsf{Y}(\lambda)\rangle$,$\lvert\mathsf{Y}'(\lambda;z)\rangle$	Vectors with components $\langle k\rvert\mathsf{Y}(\lambda)\rangle = \mathsf{P}^k(\lambda)$, $\langle k\rvert\mathsf{Y}'(\lambda;z)\rangle = \mathsf{Q}^k(\lambda;z)$.
$\lvert\mathsf{Y}(\lambda;z)\rangle$	Solution to equation (1.26).
$\lvert m(\lambda)\rangle$,$\lvert m(\lambda;z)\rangle$	$\lvert m(\lambda\rangle = \overline{\lvert\mathsf{Y}(\lambda)\rangle}$, $\lvert m(\lambda;z)\rangle = \overline{\lvert\mathsf{Y}(\lambda;z)\rangle}$.

$\|l(\lambda;z)\rangle$	$\|l(\lambda;z)\rangle = \overline{\tilde{\boldsymbol{\Phi}}(\lambda)\,\|\boldsymbol{Y}(\lambda;z)\rangle}$.
$m_k(\lambda)$, m_k	$m_k(\lambda) = \langle k\|m(\lambda)\rangle$, $m_k = \lim_{\lambda\to\infty} m_k(\lambda)$.
$q_\pm^{\min}(z)$, $q_\pm^{\max}(z)$	$q_\pm^{\min}(z) = \sqrt{z + \left(\frac{\phi_\pm}{2}\right)^2} - \frac{\phi_\pm}{2}$, $q_\pm^{\max}(z) = -\sqrt{z + \left(\frac{\phi_\pm}{2}\right)^2} - \frac{\phi_\pm}{2}$.

Notation related to the dichotomous noise

$\boldsymbol{\Phi}(\lambda)$	Stationary dichotomous process with state space $\{\phi_-, \phi_+\}$, $\phi_- < \phi_+$.
n_-, n_+, γ	n_-, n_+ are transition rates between the two states of the process $\boldsymbol{\Phi}(\lambda)$. $\gamma = n_- + n_+$
ν_-, ν_+, ν	$\nu_- \equiv \frac{n_-}{\|\phi_-\|}$, $\nu_+ \equiv \frac{n_+}{\|\phi_+\|}$, $\nu = \nu_- + \nu_+$.
μ	Mean value of $\boldsymbol{\Phi}(\lambda)$, *i.e.* mean force acting on a diffusing particle, $\mu = \frac{\phi_+ n_- + \phi_- n_+}{n_- + n_+}$.
ϕ_0, $\tilde{\boldsymbol{\Phi}}(\lambda)$	Decomposition of the process $\boldsymbol{\Phi}(\lambda) = \phi_0 + \tilde{\boldsymbol{\Phi}}(\lambda)$, $\tilde{\boldsymbol{\Phi}}(\lambda)$ is dichotomous process with transition rates n_-, n_+.
$\tilde{\phi_\pm}$	$\tilde{\phi_\pm} = \phi_\pm - \phi_0$ are the two states of the process $\tilde{\boldsymbol{\Phi}}(\lambda)$.
μ'	$\mu' = \overline{\tilde{\boldsymbol{\Phi}}(\lambda)} = \frac{n_-\tilde{\phi_+} + n_+\tilde{\phi_-}}{n_- + n_+}$.
ϕ_0'	$\phi_0' = \phi_- + \phi_+ - \phi_0$.

Matrices that occur in the main text

$$N = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 3 & \\ \vdots & \vdots & & \ddots \end{pmatrix},\ B_- = NE_- = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},\ B_+ = NE_+ = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 3 & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix},$$

$$E_- = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},\ E_+ = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}.$$

$$L(a,b) = \frac{1}{a - bN + B_-},\ b > 0.$$