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BACHELOR THESIS

Financial Risk Measures: Review and Empirical Applications

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Declaration of Authorship

The author hereby declares that he compiled this thesis independently, using only the listed resources and literature.

I declare that this thesis was not used to receive degree from any other institution.

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Prague, July 31, 2012

Signature

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Abstract

This thesis focuses on several classes of risk measures, related axioms and properties. We have introduced and compared monetary, coherent, convex and deviation classes of risk measures and subsequently their properties have been discussed and in selected cases demonstrated on data. Furthermore the relatively promising and advanced class of risk measures, the spectral risk measures, has been introduced. In addition to that we have outlined selected topics from portfolio theory that are relevant for applications of selected risk measures and then derived theoretical solution of portfolio selection using chosen risk measures. In the end we have highlighted the potential consequences of improper employment of certain risk measures in portfolio optimization.

Keywords: Risk Measures, Risk, Coherentness, Portfolio Theory
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Abstrakt

Tato bakalářská práce je zaměřena na rozličné třídy rizikových měr, související axiomy a vlastnosti. Představujeme a porovnáváme monetární, koherentní, konvexní a deviační třídy rizikových měr, přičemž následně byly diskutovány jejich vlastnosti a ve vybraných případech demonstrovány na datech. Dále jsou uvedeny perspektivní a pokročilé spektrální míry rizika. V další byly popsány související a vybrané partie z teorie portfolia, které jsou relevantní pro aplikaci vybraných rizikových měr, a rovněž bylo odvozeno teoretické řešení optimalizace portfolia za užití vybraných měr rizika. Na konec bylo poukázáno na potenciální dopady nevhodného užití určitých rizikových měr při výběru optimálního portfolia.

Klíčová slova: Míry rizika, Riziko, Koherentnost, Teorie portfolia
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Contents

Acronyms

CDF	Cumulative Distribution Function
CVaR	Conditional Value at Risk
ES	Expected Shortfall
FTSE	Financial Times and the London Stock Exchange
IID	Independent Identically Distributed
NASDAQ	National Association of Securities Dealers Automated Quotations system
PSG	Portfolio Safeguard
SR	Sharpe Ratio
TCE	Tail Conditional Expectation
TM	Tail Mean
VaR	Value at Risk
WCE	Worst Case Expectation

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1 Introduction

The concept of risk has been cardinal to the theory and practice of finance since the Markowitz's influential work exactly 60 years ago. Risk management has developed as a field of an independent study in the past 20 years and significant advances in the discipline of risk measurement have been main contributor to this noteworthy development as new risk measures have been introduced and their properties scrutinized. These risk measures, which in the past have been applied only to the market risk, are nowadays being applied to liquidity, credit ant operational risk as well to portfolio optimization. Nevertheless the risk management is much more than mindless application of quantitative techniques. A very long list of financial catastrophes demonstrates that a profound understating of the nature of risk, risk measures and their properties and limitations is vitally important.

Quantifying the risk of the uncertainty of the future value of an asset or a portfolio is one of the crucial tasks of risk management. This quantification is prevalently achieved by modeling the uncertain return or loss as a random variable, to which then a specific function is applied. Such functions are called risk measures. The financial industry standard, Value at Risk, is very often criticized for arguable statistical properties. These deficiencies have lead to a search for more appropriate possibilities. As a first tangible results of this search a specification of certain desirable axioms for risk measures were introduced. In a subsequent step the researchers and practitioners try to characterize those measures that satisfy defined axioms.

In this thesis, we intent to provide a broader overview of financial risk measures, where the development in the recent decades will be stressed and assessed. Afterwards we would like to supply the reader with desired and real statistical properties of these measures and highlight their advantages and disadvantages. Moreover brief introduction to the probably most important field of finance where these risk measures are applied will be rendered. With a deeper detail will be presented parts of portfolio theory dealing with the application of defined risk measures and subsequently demonstrated possible consequences of improper application of risk measures without knowing their properties using simulated and real market data.

In the first section of this thesis are introduced coherent, convex, deviation and spectral classes of risk measures, presented related axioms and outlined advantages and disadvantages of individual measures. In the second section a theory concerning multi-period portfolio optimization will be provided and applied jointly with selected risk measures introduced in the first section in order to illustrate the impact of theoretical properties in practice.

2 Theory of Risk Measures

The portfolios of financial institutions consist of several classes of financial assets and thus these institutions need to manage their risk expositions and are required to operate in conformity with regulatory constraints. For the measurement of risk associated with these expositions financial firms have to use appropriate measures of risk. In the view of the fact, that the theory of static risk measures is already well developed, we provide in following subsections an adequate overview of risk measures and related issues.

2.1 Monetary Measures of Risk

The uncertainty in the value of a portfolio in the future is usually described by function $X: \Omega \rightarrow R$, where Ω stands for a fixed sets of scenarios (Artzner, Delbaen, Eber, Heath, 1997). By way of illustration X can represent the (discounted) value of the portfolio. The main task is to quantify a number $\rho(\mathbf{X})$ that measures the risk and can serve as a capital requirement. For instance, it might be the minimal amount of cash which, if added to the portfolio and invested in a risk-free asset, makes the new position acceptable. Let *X* denote a certain linear space of functions $X: \Omega \rightarrow R$. Then a mapping

$$\rho: X \rightarrow R \cup \{+\infty\}$$

is called *monetary risk measure* if $\rho(0)$ is finite and the risk measure ρ satisfies for all X,Y ∈ X the following conditions (Artzner, Delbaen, Eber, Heath, 1999):

- *Monotonicity:* If the returns of X are always lower than returns of Y, then the risk of X must be higher. Formally:

$$If\ X \leq Y, then\ \rho(\mathrm{X}) \geq \rho(\mathrm{Y}).$$

- *Translation invariance:* Adding certain amount of cash *w* to some existing position reflects on risk reduction of the position and by an equivalent amount.

$$\rho(\mathrm{X} + w) = \rho(\mathrm{X}) - \ w$$

In general, monetary risk measures represent a class of risk measures equating the risk of certain asset allocation with the minimum quantity of funds that is needed to supplement to the specific investment in order to make the corresponding risk acceptable to the regulator or investor (Dhaene, Goovaerts, Kaas, 2001). In brevity monetary risk measure **ρ** could be defined as follows:

$$\rho(\boldsymbol{X}) \coloneqq min_{w \geq 0}[investment\ in\ a\ postition\ (X + w)\ is\ acceptable],$$

given $\boldsymbol{w}$ stands for an amount of funds and $\boldsymbol{X}$ represents the monetary profit and loss of evaluated portfolio or investment during some specified time horizon. Nevertheless the definition of "acceptable investment" is subject to individual assessment of investor or regulator. One of the most remarkable advantages of this approach to risk measurement is its relative simplicity and the fact that it gives direct information about the requested amount of financial reserves, which should be kept by bank or insurance company in order to face given risk. The monetary risk measure can be in short characterized by key attributes (Basel Committee on Banking Supervision, 2011):

- The exposure to risk is expressed as a monetary amount in an arbitrary accounting unit.
- The risk measure **ρ(.)** represents the distance between potential loss of given investment and an acceptable quantity of loss.

For more detailed information see (Artzner, Delbaen, Eber, Heath, 1999)

2.1.1 Value at Risk

Value at Risk is quite surely one of the leading risk measures and constitutes a central role in international insurance and banking regulatory directives, such as Solvency and Basel Accord (Hull, 2002). The VaR of a position or a portfolio quantifies the maximal loss that could be incurred within given time horizon within a given confidence level of "1-α" (under condition that α stands for values between 0 and 1). Before we establish VaR, definitions of upper and lower quantile are needed (Jorion, 1997).

$$x_\alpha = q_\alpha(X) = inf\{x \in R: P[X \leq x] \geq \alpha\} \; is \; the \; lower \; \alpha - quantile \; of \; X,$$

$$x^\alpha = q^\alpha(X) = inf\{x \in R: P[X \leq x] > \alpha\} \; is \; the \; upper \; \alpha - quantile \; of \; X.$$

The second equality can be re-writed as follows:

$$x^\alpha = sup\{x \in R: P[X \leq x] < \alpha\}\,.$$

From relation

$$\{x \in R: P[X \leq x] > \alpha\} \subset \{x \in R: P[X \leq x] \geq \alpha\}$$

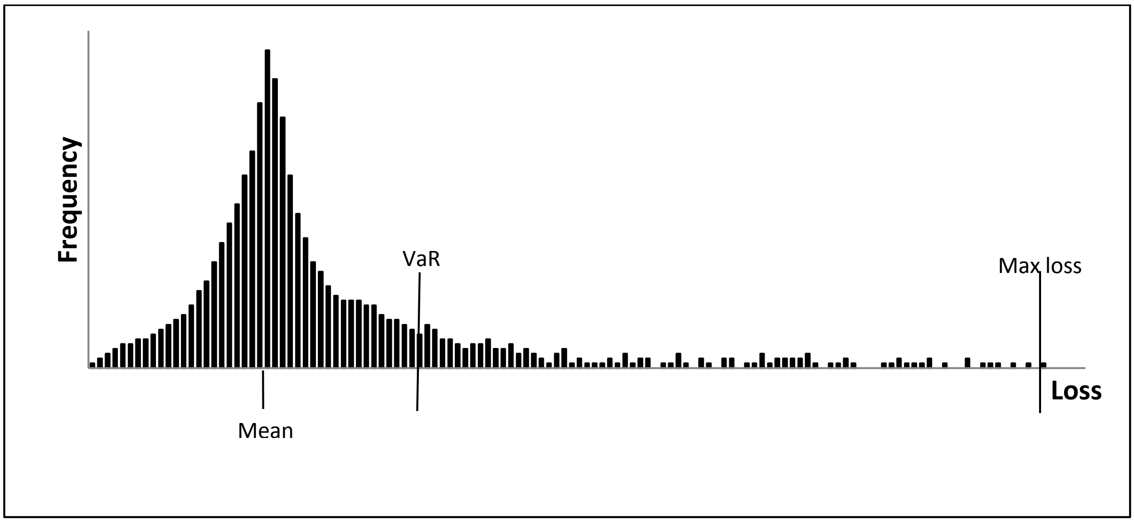
is clear that $x_\alpha \leq x^\alpha$. Moreover, the strict equality $x_\alpha = x^\alpha$ is valid if and only if $P[X \leq x] = \alpha$ for at most one x.

The Value at Risk of a portfolio is then defined in a following way:

$$VaR^{\alpha} = VaR^{\alpha}(X) = -x_{\alpha}$$

There is a minus in the formula because the random variable X stands for returns and VaR is intended to be a positive number. There are some other formal definitions of VaR, nevertheless all of them embody the same think. The VaR plainly expresses that the portfolio suffers a loss greater than X in no more that α % of cases.

Figure 2.1: Value at Risk



2.1.1.1 Estimation Methods

Here we turn to the estimation of risk measures. This process requires estimation all or part of the profit and loss distribution function. Generally speaking, there are three main classes of approach that practitioners can take – parametric methods, nonparametric methods and stochastic simulation methods (Dowd, 2006).

2.1.1.1.1 Parametric Methods

Parametric methods are based on the assumption that underlying loss distribution posses a particular parametric form, thus the first task is to determine which one it might be. The distribution selection would be driven by informal diagnostics (e.g. mean excess function plots, q-q plots, etc.) in which we generally check the goodness-of-fit of a wide range of potential distributions. The weak point of this approach is possible choice of inappropriate distribution. Depending on the application, the distribution might be any of large list, including: normal, lognormal, t-distribution, log-t-distribution, elliptical, hyperbolic, jump-diffusion, skew-t, and others (Bartoszynski, Niewiadomska-Bugaj, 2008).

Having selected the distribution, the involved parameters of distribution function have to be estimated using a method suitable to the identified distribution: maximum likelihood, method of moments, least squares, etc. When we subsequently plug the parameter estimates into our function we obtain desired certain distribution.

Nevertheless in the field of finance we are frequently dealing with multiple distributions and so we would want to model a corresponding multivariate distribution. These distributions require us to either specify a certain multivariate distribution and dependence structure among the random variables through the correlation matrix, or to specify a copula[1] function. In the latter approach is more flexible since it requires only marginal distributions for each random variable and a copula function describing the dependence structure. On the other hand it is harder to work with copulas and in practice require stochastic simulation. The first approach is easier to work with, but is usually applicable only to elliptical distributions.

Parametric methods are designed for risk management problems under the condition that the concerned distribution are known or can be reliably estimated. However, this condition had been proved to be too much stringent in the practice, especially when we are dealing with small sample sizes.

2.1.1.1.2 Nonparametric Methods

This approach seeks to estimate risk without any strong assumptions about the distribution of profit an losses. Instead of applying some parametric distribution on the considered data, the nonparametric methods estimate risk form the observed empirical distribution and this is the major attraction of this approach – they avoid the unpleasant possibility of misspecifying the distribution, which has to lead to cardinal errors in risk estimation.

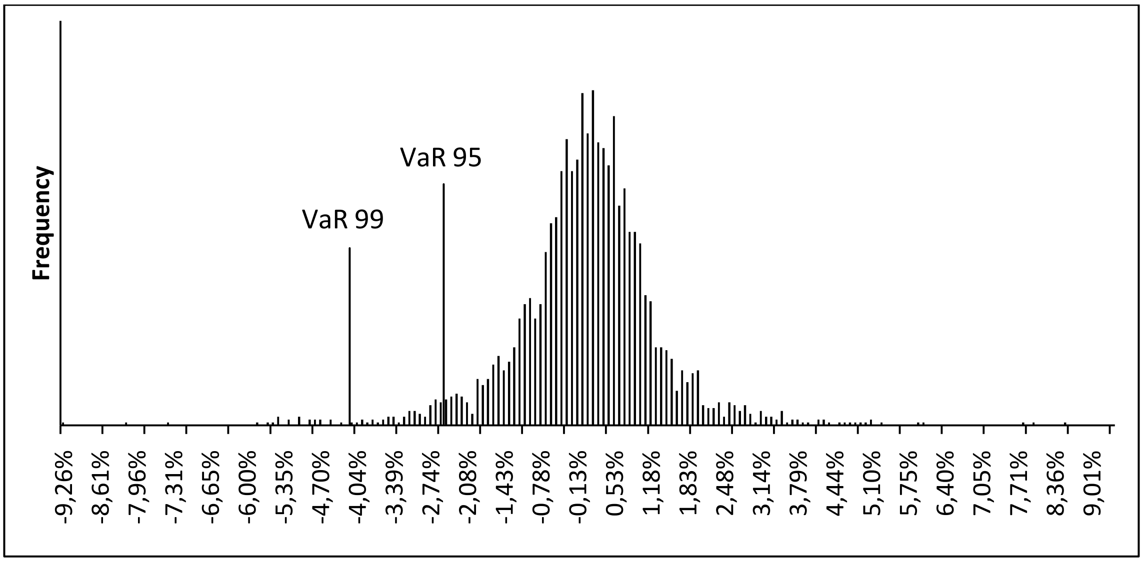
The core idea of nonparametric methods is that the future will be sufficiently similar to the recent past and thus we can use recent historical data to forecast the following future. The usefulness in practice depends whether this assumption is fulfilled or not. The weak point of this approach are tail regions of empirical distributions, because there

[1] For sufficiently summarizing overview of copulas and related information we would like to recommend McNeil, Frey, Embrechts (2005).

are the data sparse and as a result, nonparametric methods often have difficulty handling extremes.

Undoubtedly the most common method belonging into this class is historical simulation, in which we get the quantiles from a histogram of historical gains and losses. The practitioner can refine the historical simulation using weights based on data age or volatility to take into account of changing market conditions. The general idea in this approach is quite simple. We simply reorganize actual historical returns and put them in order from the worst to the best. Then, for a given confidence level α it is easy to determine a cut-off point γ_α such that the amount of occurrences exceeding this value is α%, or analogously, the amount of occurrences strictly below this level is $(1-\alpha)$%. The following graph depicts daily FTSE return from 2002 till 2012. The VaR 99 and VaR 95 represent cut-off points γ_{99} and γ_{95}, respectively.

Figure 2.2: FTSE Log-Returns 2002-2012



2.1.1.1.3 Stochastic Simulation Approach

The last approach is stochastic simulation, known as a Monte Carlo simulation. This method simulates the profit and loss distribution using a pseudo-random number generation engine, and is noticeable more flexible and powerful than the two preceding methods. This is because the underlying distribution of losses is derived from a calculation engine that accounts virtually for any level of complexity. The basic idea is to specify the model generating the loss distribution, where all factors influencing the loss are involved. Subsequently a large amount of trials is performed; each of them produces a loss based on a set of simulated values of each random variable in the specified model. If we execute a large amount of such trials, then obtained distribution provide quite good approximation to the true and unknown distribution that we are looking for. In the last step we apply nonparametric methods to these simulated losses.

Stochastic methods are useful for wide range of risk management issues, and will usually provide the most reliable way of dealing with the tasks we are likely to encounter.

2.1.1.2 Shortcomings of Value at Risk

Notwithstanding due to its easy interpretation, the VaR has become quickly a standard measure of risk in finance industry and has been several times criticized by academics and professionals for lacking subadditivity. That is a property implying compartmentalized risk measurement based on Value at Risk methodology is not necessary appropriate due to possibility of omitting diversification effects. Value at Risk is subadditive only if the value of an investment follows an elliptic distribution, for example normal or t-distribution and thus the value of portfolio is a linear function of corresponding asset prices. Nonetheless in this case the VaR becomes equivalent to mean-variance methodology. However this assumption about distribution is not relevant, because majority of asset distribution evince more or less signs of skeweness and thus the subadditivity of VaR is not relevant in these cases any more.

In this place we provide often uses exemplary demonstration of lacking subadditivity (Danıelsson, 2011). Consider two assets returns A and B that are customarily normally distributed, but subject to some occasional independent shocks:

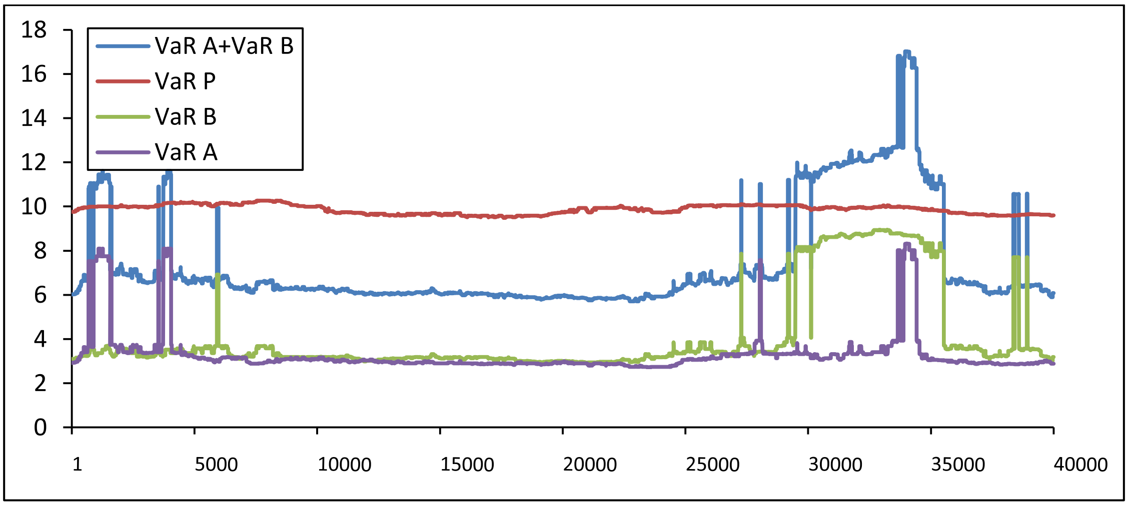
$$A = \epsilon + \eta, \quad \epsilon \sim IID\ \mathrm{N}(0,1);$$

$$\eta = \begin{cases} 0\ with\ probability\ 0{,}991 \\ 1\ with\ probability\ 0{,}009 \end{cases}$$

Here the 1% VaR is in the majority of time around 3.4, since the probability that the η equals 10 is less than 1%. Suppose that B has the same distribution independently from A and that we comprise an equally weighted portfolio of A and B. In that case, the 1% VaR of this portfolio is around 9,9. The reason is simple – for (A + B) the probability of getting the shock of 10 for either A or B is much higher. These conclusions are based on two simulated paths, each with 50 000 observations, where we established a rolling window with size of 10 000 observations and moved it forwards 40 000 times by a size of one observation. Within each window were corresponding values of VaR calculated via historical simulation and then plotted. The following graph provides sufficient evidence of lacking subadditivity – the VaR of equally weighted portfolio consisting of A and B is in the central part of graph remarkably higher than a sum of VaR(A) and VaR(B). Clearly the identity excluding subadditivity is valid on amply long interval on axis x:

$$VaR(A+B) = 9{,}9 > VaR(A) + VaR(B) = 6{,}8\,.$$

Figure 2.3: Value at Risk: Violation of Subadditivity Axiom



Another relevant issue connected not only to VaR is time-varying volatility. It is clear that some underlying risk factors, such as equities or interest rates, exhibit remarkably strong time-varying volatility. Many firms in financial industry advocate the implementation of fast reacting risk measures with incorporated time-weighted volatility factors. The justification for these measures is that such models provide faster caution against changes of market conditions. It had been several time showed that relying on historical simulation of VaR without incorporated time-varying volatility factors may significantly underestimate the risk, when that volatility is exhibited by corresponding underlying risk factors (Pritsker, 1997). Thus incorporating this volatility in VaR models seems to be necessary because it is prevalent in many risk factors. In contrast, there are some voices emphasizing that time-varying volatility in VaR models might not be imperative, or could be in some applications, especially in the long horizons, even inappropriate (Acerbi, Tasche, 2002). This is because of the concerns about pro-cyclicality and following instabilities in economy associated with regulatory

VaR models capturing the time-varying volatility (Basel Committee on Banking Supervision, 2011).

2.1.1.2.1 Time-scaling of Risk

And another issue is often the time-scaling using square root of time (Butler, 1999). The unlucky practice is transforming one-day VaR into "n-day VaR" using square root of time:

$$\text{n-day VaR} = \sqrt{n} * \text{"}one - day\ VaR\text{"}\ .$$

But this identity is valid if and only if the loss or gain distribution is normal and independent and identically distributed, which is a very restrictive condition. And almost never fulfilled one (Danielsson, Zigrand, 2003).

2.2 Coherent Measures of Risk

In the field of risk management there are number of quantitative approaches to risk measurement and in order to clarify this issue there have been described some properties that appropriate risk measure should or should not have. Artzner et al (1999) defined coherent measures of risk as a class of monetary risk measures. These measures satisfy *monotonicity, translation invariance* and following two properties of coherence (Artzner, Delbaen, Eber, Heath, 1997):

- *Subadditivity:* The risk of two portfolios can't be any worse than the summation of the risks concerning the individual positions:

  $$\rho(X + Y) \leq \rho(X) + \rho(Y).$$

  This crucial property of suitable risk measure enables for diversification effects and hence this property could be viewed as an extension of the diversification concept presented by Harry H. Markowitz in 1952 (Markowitz, 1952).

- *Homogenity:* Given the position in X is increased by certain proportion *k*, then the risk of this position increases by the same proportion *k.* Speaking mathematically:

  $$\rho(kX) = k\rho(X).$$

  This property implies that the risk growths according to the quantity of positions taken. But it is important to mention, that homogenity property does not take into account for liquidity risk increase that might emerge when the position increases.

The inherent question is whether the coherence is necessary. The answer is, as it usually is, not clear. It depends on whether one is portfolio manager, regulator or banker, some of the above mentioned properties will be more relevant, and some less for each of them. The typical primordial attribute required in the field of portfolio management is the subadditivity or when speaking about regulatory purposes, the main concern is about translation invariance. Earlier mentioned risk measure VaR is coherent only under considerably strict conditions, but in general is not considered to be coherent because it fails the subadditivity property.

Standard deviation determined using distribution function estimate of asset returns is not a monetary measure and thus cannot be coherent. Standard deviation determined using profit and loss distribution is monetary measure but it clearly does not satisfy the required subadditivity property (Rockafellar, Uryasev, Zabarankin, 2008).

2.2.1.1 Tail Conditional Expectation

The main shortcoming of VaR is being indifferent of severity of losses beyond given threshold. If the probability distribution function of the portfolio losses is continuous, then the statistics removing the previous deficiency of VaR is simply determined by a conditional expected value below the certain quantile. The commonly name used for this statistics is a Tail Conditional Expectation (Acerbi, Tasche, 2002). Assume $E[X^-] < \infty$, then

$$TCE_\alpha = -E\{X|X \le x_\alpha\}$$

is the lower tail conditional expectation at level α. The upper tail conditional expectation at level α is defined

$$TCE^\alpha(X) = -E\{X|X \le x^\alpha\}$$

For general distributions this statistics is not appropriate since the event $\{X \le x^\alpha\}$ may have the probability much larger than the set of selected worst cases which we want to quantify. Thus the TCE is coherent risk measure merely when restricted to continuous distribution functions.

2.2.1.2 Expected Shortfall

The Expected Shortfall is probably one of the most well-known measures of risk following the Value at Risk. The ES is frequently employed due its conceptual intuitiveness, firm theoretical background and correcting the three main shortcomings of the VaR approach. The ES takes into consideration the size of losses beyond the confidence threshold. This attribute is important especially for the regulatory bodies which are concerned in particular about these losses. Second, this risk measure is always subadditive and thus coherent.

First of all we need to define a Tail Mean (Acerbi, Tasche, 2002). The α-tail mean at level α is

$$\bar{x}_\alpha = TM_\alpha = \alpha^{-1}\left(E\left[X\,1_{\{X \le x_\alpha\}}\right] + x_\alpha(\alpha - P[X \le x_\alpha])\right).$$

The definition of the Tail Mean can be rewrited as follows:

$$\bar{x}_\alpha = \alpha^{-1}\int_0^\alpha x_u du.$$

The meaning is absolutely the same as before, but notwithstanding the latter definition could be easier to understand and apply.

Now we can define the Expected Shortfall:

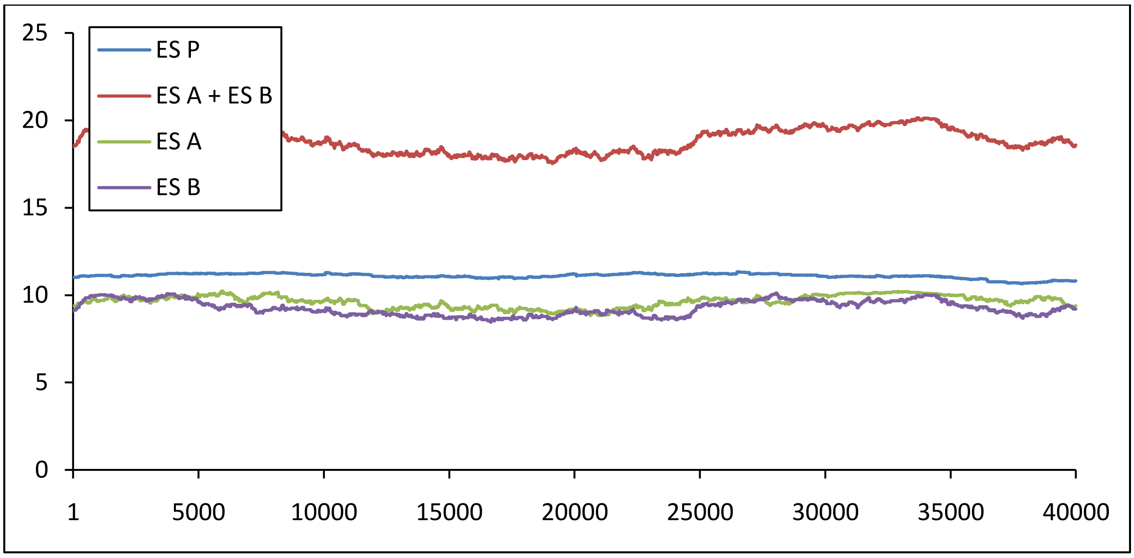
$$ES_\alpha = -\bar{x}_\alpha$$

Employing the data used for the demonstration of subadditivity violation by the Value at Risk, we show that the Expected Shortfall is subadditive. The used methodology and data in this demonstration is exactly the same as before when we were providing example of subadditivity lacking of VaR. We simulated two paths with 50 000 observations and used 10 000 of observation as a rolling window that has been shifted 40 000 times and each time was corresponding value of ES determined through the historical simulation approach. The following graph provides evidence of subadditivity of ES – the ES of equally weighted portfolio consisting of A and B is remarkably higher than a sum of ES(A) and ES(B).

As we see the inequality endorsing the subadditivity is valid:

$$ES(A+B) = 11 \le ES(A) + ES(B) \cong 19\,.$$

Figure 2.4: Subadditivity of Expected Shortfall

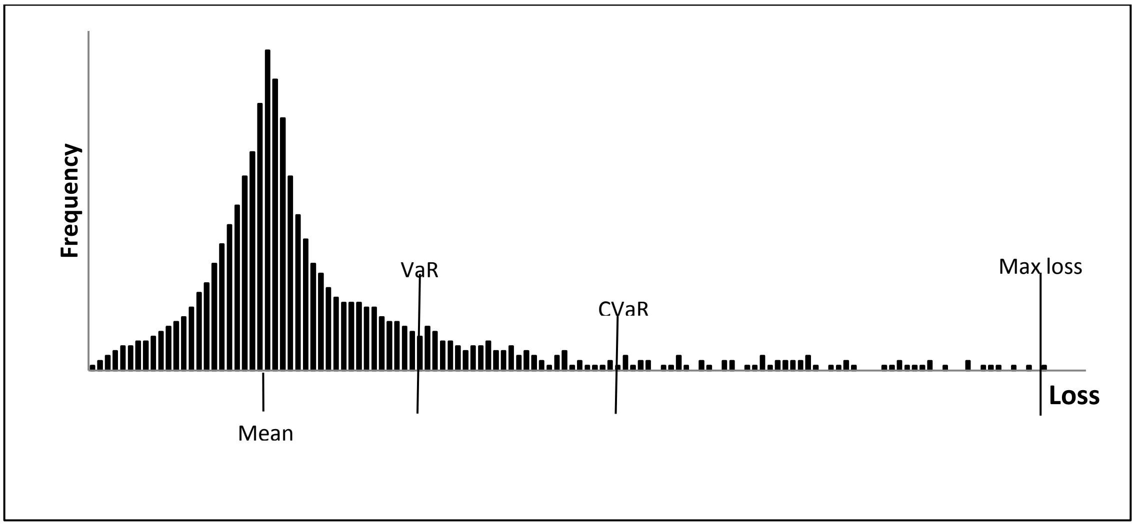


And last but not least, ES is continuous with respect to the confidence level, which means the impact of particular choice of single confidence level is mitigated.

2.2.1.3 Conditional Value at Risk

The Conditional Value at Risk is very closely related to Value at Risk (Rockafellar, Uryasev, 2001). For continuous distributions, the CVaR is defined as a conditional expected loss given that it exceeds VaR. For this kind of distribution is also known as Mean Shortfall or Tail Value at Risk. Although the CVaR is very similar to the VaR it has more attractive properties. It can be proven that it is subadditive and convex. Moreover it is a coherent risk measure in the sense of previously defined coherentness. It is worth to mention, when the return-loss distribution function is normal, CVaR and VaR are coherent and equivalent. However for skewed distribution functions these measures provide in general different information about the risk exposition.

Figure 2.5: Value at Risk and Conditional Value at Risk



Let the function $f(\mathbf{x}, \mathbf{y})$ represent the loss associated with the vector $\mathbf{x}$[2] representing portfolio components chosen from certain subset $\mathbf{X} \in \mathrm{R}^{\mathrm{n}}$. The vector $\mathbf{y}$ stands for uncertainties (market prices, interest rates and others), that might influence the loss of portfolio. For each $\mathbf{x}$, the loss function $f(\mathbf{x}, \mathbf{y})$ is a clearly random variable and has a distribution in R evoked by the distribution of $\mathbf{y}$. For convenience we allow for reasonable assumption that underlying probability distribution of $\mathbf{y} \in \mathrm{R}^{\mathrm{m}}$ has its density $\boldsymbol{p}$(y). Then the probability of loss function does not exceed the threshold ζ is given by

$$\psi(\mathbf{x}, \zeta) = \int_{f(\mathbf{x},\mathbf{y}) \leq \zeta} p(\mathbf{y})\, d\mathbf{y} \,.$$

For fixed $\mathbf{x}$, the $\psi(\mathbf{x}, \zeta)$ is the *cdf* of the loss associated with $\mathbf{x}$. The function $\psi(\mathbf{x}, \zeta)$ is nondecreasing in ζ and we state an assumption of continuousness of the function with respect to ζ. The CVaR and VaR values for the random variable representing loss associated with vector $\mathbf{x}$ and some probability level α are denoted by $\phi_\alpha(\mathbf{x})$ and $\varsigma_\alpha(\mathbf{x})$. With respect to previous definition, the VaR and CVaR can be defined as follows:

$$\varsigma_\alpha(\mathbf{x}) = min\{\varsigma \in R \colon \psi(\mathbf{x}, \zeta) \geq \alpha\}$$

$$\phi_\alpha(\mathbf{x}) = (1-\alpha)^{-1} \int_{f(\mathbf{x},\mathbf{y}) \leq \varsigma_\alpha(\mathbf{x})} f(\mathbf{x}, \mathbf{y})\, p(\mathbf{y})\, d\mathbf{y}.$$

[2] The boldface fonts are used in the interest of distinguishing them from scalars.

2.2.1.4 *Worst Case Expectation*

This quite novel risk measure has been proposed by Artzner et al. in 1999 as a theoretical example of coherent measure of risk. Nevertheless the Zhu und Fukushima (2005) provided new characterization of WCE based on CVaR. The implementation of WCE is suitable, when we do not know the exact probability distribution *p*, but we know that *p(.)* belongs to some family P of the probability distributions characterized by certain properties. Then

$$WCE_{\alpha}(X) = \sup_{p(.)\in \mathrm{P}} \{\phi_{\alpha}(\mathbf{x})\}\,.$$

However the proper application of WCE requires the precise knowledge of the whole underlying probability space. This claim is obviously in general impossible to fulfill and thus the true WCE is useful practically only in a theoretical settings.

2.3 Convex Measures of Risk

The convex risk measures are a novel and quite emerging area of risk management research. Follmer and Schied (2002a) presented the concept of convex risk measure. The convexity rise from the reality that price of some instruments might not change linearly with the size of their position. This property is in direct contradiction to the *homogenity* assumption of the coherent risk measures. Thus the *homogenity* and the *subadditivity* properties were relaxed in favor of the *convexity.* The monetary risk measure is *convex* if it satisfies

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \qquad \text{given } 0 \leq \lambda \leq 1.$$

The typical representative of convex risk measure is CVaR. Let us to provide slightly different definition of Conditional Value at Risk:

$$CVaR = \lambda VaR + (1-\lambda)CVaR^{+}; \quad 0 \leq \lambda \leq 1,$$

where $CVaR^{+}$is "upper CVaR". Basically it is expected loss strictly exceeding VaR, in other words the Expected Shortfall in the sense of the former definition. For the proper determination of parameter λ subsequent definition is needed:

$$\lambda = \frac{\Psi(\mathrm{VaR}) - \alpha}{1-\alpha},$$

$$\Psi(VaR) = probability\ that\ losses\ do\ not\ exceed\ VaR\ or\ equal\ to\ VaR.$$

The convexity of CVaR is crucial for the portfolio optimization, because as the preceding graph depicts – the VaR is not convex thus in the common optimization procedures could be selected as an "optimum" local optimum not global one.

2.4 Deviation Measures

The minimization of standard deviation, or equivalently variance, is a indivisible part of classical portfolio theory. It has been many times subject to criticism, because standard deviation, or variance, does not correspondingly take into account the phenomenon of fat tails in profit and loss distributions and thus penalizes downs and ups equally. In the course of time, more convenient tool than standard deviation has been developed – the Value at Risk. But the VaR has been controversial because of some mathematical shortcomings (lack of convexity, subadditivity, as well as continuity in parameter alpha) and its incapacity to the magnitude of the threatening losses below the threshold it does identify (Mausser, Rosen, 1999). A similar concept, the Conditional Value at Risk has been proved to be superior in these respects and thus more suitable for the optimization purposes in portfolio theory. Although these and another quantile based measures are more or less contributive to risk management, another class of "risk measures", deviation measures, had been developed. These measures satisfy slightly different axioms and have their own interpretation and importance, with standard deviation just being one of them. The following text concerning deviation measures is based on Rockafellar, Uryasev and Zabarankin (2002). We start with axiomatizing what is meant to be a deviation measure, and then explain how these measures are related to other risk measures.

We consider a state space Ω with elements ω representing possible future states, and suppose it to be supplied with some probability measure P. This probability measure stands for the true distribution of the future states. The random variables are treated as functions $X: \Omega \to \mathrm{R}$ that belong to linear space$\mathcal{L}^2 = \mathcal{L}^2(\Omega, M, P)$. Id est the measurable functions where the corresponding mean and variance exist (i.e. the following integrals are well defined) and can be expressed in this manner:

$$\mu(x) = EX = \int_\Omega X(\omega) dP(\omega)$$

$$\sigma^2(X) = E[X - EX]^2 = \int_\Omega [X(\omega) - \mu(x)]^2 dP(\omega)$$

By *deviation measure* on space $\mathcal{L}^2$ will be meant any functional

$\mathcal{D}: \mathcal{L}^2 \to [0; infinity)$ satisfying:

$$(D1)\quad \mathcal{D}(X+C) = \mathcal{D}(X)\ for\ all\ X\ and\ constants\ C,$$
$$(D2)\quad \mathcal{D}(0) = 0, and\ \mathcal{D}(\lambda X) = \lambda\mathcal{D}(X)\ for\ all\ X\ and\ all\ \lambda > 0$$
$$(D3)\quad \mathcal{D}(X+X') \leq \mathcal{D}(X) + \mathcal{D}(X')\ for\ all\ X\ and\ X'$$
$$(D4)\quad \mathcal{D}(X) > 0\ for\ all\ nonconstatnt\ X, whereas\ \mathcal{D}(X) = 0\ for\ constant\ X$$

Under previous axioms, measure $\mathcal{D}(X)$ depends solely on $X - EX$ (results from the case D(1) where $C = -EX$). If $\mathcal{D}$ is the deviation measure then its $reflection\,\widetilde{\mathcal{D}}$ and its $symmetrization\,\widetilde{\widetilde{\mathcal{D}}}$ is that kind of measure as well. The reflection and the symmetrization are given by

$$\widetilde{\mathcal{D}}(X) = \mathcal{D}(-X),\ \ \widetilde{\widetilde{\mathcal{D}}}(X) = 1/2\big[\mathcal{D}(X) + \widetilde{\mathcal{D}}(X)\big].$$

Axiom D2 guarantees *positive homogenity*. With combination with axiom D3 we get property well known as *sublinearity*. *Positive homogenity* and *sublinearity* imply that $\mathcal{D}$ is *convex* function on given space $\mathcal{L}^2$.

For the standard deviation $\mathcal{D}(X) = \sigma(X)$ all properties D1 – D4 hold. This measure is symmetric which is not always desired property of appropriate risk measure. Therefore we define standard *semi*deviations $\sigma_+(X)$and $\sigma_-(X)$ as

$$\sigma_+(X) = (E\big[max\{X - \mu(x); 0\}^{1/2}\big])^2$$

$$\sigma_-(X) = (E\big[max\{\mu(x) - X; 0\}^{1/2}\big])^2$$

Standard *semi*deviations are deviation measures in the sense of previous definition, but are not symmetric.

Albeit deviation risk measures are intended to serve for purposes in risk analysis, they are not "risk measures" in the sense defined by Artzner et al. (1997) in his land-breaking paper. The relation between risk measures and deviation measures is quite close, but a crucial distinction has to be clearly distinguished. The deviation measure inform about the uncertainty of X, but a risk measure evaluate the "severity of possible losses" associated with random variable X. In order to bridge the difference between these classes of measures (Rockafellar, Uryasev, Zabarankin, 2008) define *expectation-bounded risk measures.*

By an *expectation-bounded risk measure* on the space $\mathcal{L}^2$ will be meant any functional $\mathcal{R}: \mathcal{L}^2 \to (-\infty; +\infty)$ satisfying

$$(R1)\quad \mathcal{R}(X + C) = \mathcal{R}(X) - C \; for \; all \; X \; and \; constants \; C,$$
$$(R2)\quad \mathcal{R}(0) = 0, and \; \mathcal{R}(\lambda X) = \lambda\mathcal{R}(X) \; for \; all \; X \; and \; all \; \lambda > 0$$
$$(R3)\quad \mathcal{R}(X + X') \leq \mathcal{R}(X) + \mathcal{R}(X') \; for \; all \; X \; and \; X'$$
$$(R4)\quad \mathcal{R}(X) > E[-X] \; for \; all \; nonconstatnt \; X,$$
$$whereas \; \mathcal{R}(X) = E[-X] \; for \; constant \; X$$

An *expectation-bounded risk measure* is said to be coherent if it satisfies the additional axiom

$$(R5)\quad \mathcal{R}(X) \leq \mathcal{R}(X') \; for \; all \; X \geq X'$$

When the inequality $\mathcal{R}(X) < infinity$ is valid, we speak about *finite* expectation-bounded risk measure on given space.

A certain functional $\mathcal{R}$, given that R1, R2, R3 and R hold, is coherent risk measure in the sense of definition of Artzner et al. (1997). The property of expectation-boundedness has not been identified as sufficiently important for a risk measure. But here we need that property in order to create connection between risk and deviation measures. Deviation measure is in one-to-one accordance with expectation-bounded risk measure under following restrictions

$$i)\; \mathcal{D}(X) = \mathcal{R}(X - EX)$$

$$ii)\; \mathcal{R}(X) = \mathcal{D}(X) - EX$$

Under condition that $\mathcal{R}$ is an expectation-bounded risk measure and $\mathcal{D}$ is in line with definition in *i)*, then $\mathcal{D}$ is deviation measure that generates $\mathcal{R}$ throught *ii)*. If relations *i)* and *ii)* are fulfilled, then $\mathcal{R}$ is coherent if and only if $\mathcal{D}$ has following property

$$(D5)\quad \mathcal{D}(X) \leq EX - \inf X \; for \; all \; X.$$

Employing D5 we can handle with $\mathcal{D}$ as with a *coherent* deviation measure. Naturally, only if axioms D1-D4 hold.

2.4.1 Value at risk

For any an arbitrary level $\alpha \in (0; 1)$, the quantile function

$$\mathcal{R}(X) = VaR_\alpha(X) = -inf\{z; P\{X \leq x\} > \alpha\}$$

is finite and apparently satisfies R1, R2, R5 but fails to satisfy the expectation-boundedness condition R4 and the subadditivity property in R3. We define the corresponding function

$$\mathcal{D}(X) = VaR_\alpha^\Delta(X) = VaR_\alpha(X - EX),$$

where D1 and D2 hold and D3 and D5 not. The $\mathcal{D}(X)$ can in general not fulfill D4 because the equality $\mathcal{D}(X) = 0$ sometimes hold for nonconstant X. This shortcoming corresponds to the failure of function $\mathcal{R}(X) = VaR_\alpha(X)$ to satisfy R4. Although $VaR_\alpha(X)$ is definitely nondecreasing with respect to α, there can be intervals of constancy. For example it is imaginable having $VaR_\alpha(X) = VaR_{0,5}(X)$ for some range of α values around 0,5 and thus one will get $VaR_\alpha(X) = -EX$ though X is not constant.

2.4.2 Conditional Value at Risk

Based on Uryasev et al. (2002) we would like to start with following definition:

$$\mathcal{R}(X) = CVaR_\alpha(X) = -E[lower\ \alpha - tail\ of\ distribution\ of\ X]$$

This function is again finite, expectation-bounded, coherent and

$$VaR_\alpha(X) \leq CVaR_\alpha(X).$$

The corresponding coherent deviation measure is

$$\mathcal{D}(X) = CVaR_\alpha^\Delta(X) = CVaR_\alpha(X - EX).$$

$\mathcal{D}$ is not symmetric. For arbitrary, fixed X, both $CVaR_\alpha^\Delta(X)$ and $VaR_\alpha^\Delta(X)$ are continuous and nondecreasing with respect to α (Rockafellar, Uryasev, Zabarankin, 2002). Additionally, the following limit relations are valid:

$$\lim_{\alpha \to 0} CVaR_\alpha(X) = -\inf X, \quad \lim_{\alpha \to 1} CVaR_\alpha(X) = -EX,$$

$$\lim_{\alpha \to 0} CVaR_\alpha^\Delta(X) = EX - \inf X, \quad \lim_{\alpha \to 1} CVaR_\alpha^\Delta(X) = 0.$$

It is worth emphasizing that the second expression of these above is not an expectation-bounded risk measure because it apparently fails the strict inequalities in R4.

The clear distinction between $CVaR_{\alpha}^{\Delta}$ and $CVaR_{\alpha}$, or VaR_{α}^{Δ} and VaR_{α} has to be made. At times in finance, the "CVaR" or "VaR" of X refer to $CVaR_{\alpha}^{\Delta}$ or VaR_{α}^{Δ}, instead to $CVaR_{\alpha}$ or VaR_{α}. But that is not consistent with definitions of functions $CVaR_{\alpha}$ or VaR_{α} as risk measures (Grechuk, Molyboha, Zabarankin, 2008). The appropriate title for $CVaR_{\alpha}^{\Delta}$ and VaR_{α}^{Δ} should be "CVaR-deviation" and "VaR-deviation" of X.

It is easy to prove, that CVaR is possible to express as an average of VaR (Acerbi, Nordio, Sirtori, 2001):

$$CVaR_{\alpha}(\mathrm{X}) = \frac{1}{\alpha}\int_{0}^{\alpha} \mathrm{VaR_z}(\mathrm{X})dz.$$

Since VaR is left-continuous and nonincreasing function with respect to α, this formula implies, that $\alpha CVaR_{\alpha}(\mathrm{X})$ is concave as a function of α. Therefore it possess left (and right as well) derivatives with respect to α, the left one being (Acerbi, Nordio, Sirtori, 2001):

$$\frac{d^{-}}{d\alpha}[\alpha CVaR_{\alpha}(\mathrm{X})] = VaR_{\alpha}(\mathrm{X}).$$

2.4.3 Mixed Conditional Value at Risk and Risk Profile

For any arbitrary weighting measure $d\lambda$ on (0,1), provided $\int_0^1 p^{-1}d\lambda(\mathrm{p}) < infinity$, the mixed CVaR function defied below

$$\mathcal{R}(X) = \int_{0}^{1} CVaR_{\alpha}(X)\,d\lambda(\alpha)$$

is finite, expectation-bounded coherent risk measure (Uryasev, Rockafellar, Zabarankin, 2004). There is equal expression using spectral formula

$$\mathcal{R}(X) = \int_{0}^{1} VaR_{\alpha}(X)\varphi(\alpha)\,d(\alpha).$$

The functional $\varphi(\alpha)$on (0,1) provides the corresponding *risk profile* and is defined as

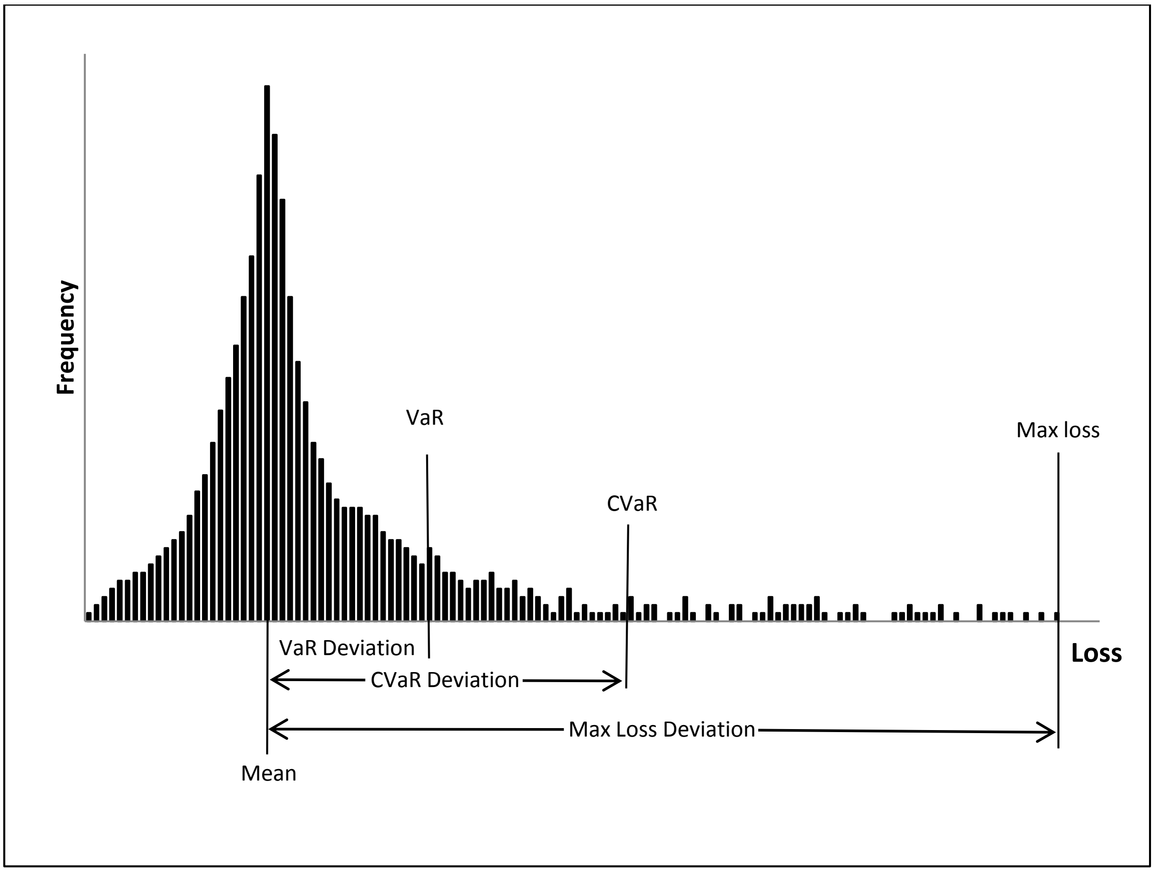
$$\varphi(\alpha) = \int_{\alpha}^{1} p^{-1} d\lambda(\mathrm{p}).$$

The associated deviation measures fulfilling coherency properties is defined by this formula:

$$\mathcal{D}(X) = \int_{0}^{1} CVaR_{\alpha}^{\Delta}(X)\, d\lambda(\alpha) = \int_{0}^{1} VaR_{\alpha}^{\Delta}\varphi(\alpha)\, d(\alpha).$$

For more information about this topic see Sarykalin, Serraino, Uryasev (2008).

Figure 2.6: Value at Risk, Conditional Value at Risk and Corresponding Deviations



2.5 Spectral Risk Measures

Within the last decade, the CVaR and, as its natural further extension, spectral risk measures have become popular tools in risk management (Follmer, Schied, 2002). Initially, these risk measures were proposed as an alternative to criticized Value at Risk. During the last decade the purpose of spectral risk measures has changed, away from the regulatory evaluation of solvency capital, towards the implementation in portfolio theory, where it is used for an investor's preferences modeling (Adam, Houkari, Laurent, 2008), (Acerbi, Simonetti, 2008). Nevertheless, these two approaches must be distinctly distinguished from each other, since they are based on different approach to diversification. The regulatory employment of spectral risk measures is appropriate for the assessment of regulatory capital but there are some deficiencies when applied to portfolio selection.

In the context of banking industry regulation, the regulatory requirements add to a utility function a constraint (Brandtner, 2011):

$$max_{X \in \chi}\, \pi(X), \qquad s.t.\, \rho(X) \leq \bar{\rho}$$

The logic of these requirements is straightforward. The utility function of assessed bank is allowed to be applied to those alternatives X out of the set of all alternatives $\boldsymbol{\chi}$, whose requirements for capital $\rho(X)$ don't go beyond its solvency capital $\bar{\rho}$. In the framework of Basel II, the ρ usually corresponds to Value at Risk. More suitable alternative at least in term of theory is spectral risk measure, especially Conditional Value at Risk. Under certain conditions, these progressive measures can be applied instead of Value at Risk in Basel-based regulation within advanced internal ratings-based approach.

As mentioned above, the spectral risk measures are used in different context as well. Within a modern portfolio theory, they are employed to derive $(\mu, \rho_{\emptyset}) -$ efficient frontiers by minimization of a spectral risk measure $\rho_{\emptyset}$ for given expected return $\overline{\mu}$:

$$min_{X \in \chi}\, \rho_{\emptyset}(X), \qquad s.t.\, EX = \overline{\mu}$$

The proper properties of spectral risk measure, based on definition of Acerbi (2004), are following:

A functional $\rho_{\emptyset}\colon \boldsymbol{\chi} \to R$ *is spectral risk measure if it satisfies these properties*

- *Monotonicity with respect to the first order stochastic dominance: For X, Y* $\in \chi$ *with* $F_X(t) \geq F_Y(t), if\ t \in R,\ \rho_{\emptyset}(X) \geq \rho_{\emptyset}(Y)$.
  The substance of monotonicity is obvious – financial position X having probability of falling under a threshold *t* higher than a position Y requires more solvency capital than position Y.
- *Translation invariance: For X* $\in \chi$ *and* $c \in R,\ \rho_{\emptyset}(X + c) = \rho_{\emptyset}(X) - c$
- *Subadditivity: For X, Y* $\in \chi,\ \rho_{\emptyset}(X + Y) \leq \rho_{\emptyset}(X) + \rho_{\emptyset}(Y)$

Subadditivity guarantees that spectral risk measures reward an investor for diversification. In general, the diversification results from the dependence structure among the financial positions in the portfolio.

- *Comonotonic Additivity: For comonotonic X, Y* $\in \chi$, $\rho_{\emptyset}(X+Y) = \rho_{\emptyset}(X) + \rho_{\emptyset}(Y)$

  The random variables *X, Y* $\in \chi$ are said to be comonotonic if

  $\left(X(\omega_i) - X(\omega_j)\right) * \left(Y(\omega_i) - Y(\omega_j)\right) \geq 0, for\ all\ \omega_i, \omega_j \in \Omega, P(\Omega) = 1.$

  In other words, the X and Y are comonotonic if and only if they decrease and increase simultaneously in their state-dependent realizations. In fact, the comonotonicity is a sort of generalization of perfect positive correlation. Thus perfect positive correlation has to imply comonotonicity, but vice versa does not always hold.

- *Positive homogenity:* $For\ an\ arbitrary\ X \in \chi, \lambda \geq 0\ and\ \lambda \in$ R $is\ valid\ following\ \ \rho_{\emptyset}(\lambda X) = \lambda \rho_{\emptyset}(X).$

  The determination of this property is not essential, since comonotonic additivity and monotonicity imply positive homogenity.

Now we can state a proper definition of spectral risk measure based on Brandtner (2011):

Any spectral risk measure $\rho_{\emptyset}$ of a random variable *X* is of the form

$$\rho_{\emptyset}(X) = -\int_0^1 F_X^*(p)\,\varphi(p)dp,$$

where $F_X^*(p) = sup\{x \in R; F_x < p\}, p \in (0,1]$ are the $\mathrm{p}-$quantiles of the

cumulative distribution function $F_{x,}$and the risk spectrum φ: $[0,1] \rightarrow R$ satisfies

- *positivity:* $\varphi(p) \geq 0\ for\ all\ p \in (0,1]$,
- *normalization:* $\int_0^1 \varphi(p)d\mathrm{p} = 1$,
- *monotonicity:* $\varphi(p) \geq \varphi(q)\ for\ all\ 0 \leq p \leq q \leq 1.$

The spectral risk measures are especially characterized by a risk spectrum φ which assigns different weights to the p-quantiles.

Nowadays, the most discussed measure of that kind is the Conditional Value at Risk and its corresponding risk spectrum is given by

$$\varphi(p) = \begin{cases} \alpha^{-1}\ for\ all\ 0 < p \leq \alpha \\ 0 \quad for\ all\ \alpha < p \leq 1 \end{cases}$$

Furthermore, the spectral risk measures according to the previous definitions are extension of CVaR, as any convex combination of particular CVaRs' yields spectral risk measure (Frittelli, Gianin, 2004). Even the negative mean $\rho_{\emptyset}(X) = -EX$ can be considered to be spectral risk measure where $\varphi(p) = 1,\ p \in (0,1]$. On the contrary, the variance of position X is not spectral risk measure, since it does not satisfy any of the required properties.

2.6 Comparative analysis of VaR and CVaR

2.6.1 VaR: Pros and Cons

The Value at Risk is quite uncomplicated risk management notion. The intuition behind the $\alpha -$ percentile of a distribution is easily understood and VaR has straightforward interpretation: how much you might loose with specified confidence level (Duffie, Pan, 1997). Specifying the VaR for all confidence levels completely defines the distribution. In this sense, the Value at Risk is superior to the widely used standard deviation. But unlike standard deviation, the VaR focuses especially on a specific part of the distribution. One of the most important properties of this measure is stability of estimations. Since this measure disregards the tail, it is unaffected by very high tail losses, which are usually difficult to measure. But this feature can be considered as a substantial weakness because it does not account for properties of the distribution beyond the chosen confidence level. As a consequence VaR may increase dramatically with small change in confidence level (Pritsker, 1997). In order to adequately estimate the risk in the tail, we need to calculate several VaRs with different confidence level. The worst consequences can have using VaR for risk control, when the underlying distribution is skewed then the VaR-based risk management may lead to undesirable results (Crouhy, Galai, Mark, 2006).

2.6.2 CVaR: Pros and Cons

Defining CVaR for all confidence levels completely specifies the return distribution, just as VaR. But CVaR has several prominent mathematical properties: coherentness, convexity, continuousness with respect to α. In contrast to VaR, these properties result in quite easy application of CVaR in portfolio theory (Krokhmal, Palmquist, Uryasev, 2001). Since VaR is generally non-smooth and non-convex function, the finding of global extrema is challenging topic. Nonetheless some progress has been made (Uryasev, 2000).

On the other hand, the CVaR is more sensitive than VaR to estimation errors. If there is not available suitable model for the tail of distribution, this measure can be misleading (Rockafellar, Uryasev, 2001).

2.7 Omega Risk Measure

Due to criticism concerning to the mean-variance approach in portfolio theory[3] which is based on the mostly impracticable assumption of normal distribution of gains, Keating and Shadwick (2002) proposed a new performance measure called Omega reflecting all statistic properties of profit and loss distribution. In practice it incorporates all its moments, not only the mean and variance.

Majority of widely used performance indicators are based on two simplifications:

- Mean and variance completely describe the distributions of returns;
- Risk-return characteristics of an asset or a portfolio can be fully described only with mean on returns.

These two simplifications are in general valid if it is assumed a normal distribution of profit and loss. But it is widely accepted empirical fact that this assumption is unrealistic. Thus, besides variance and mean, higher order moments would be required to improve the distribution description.

Omega incorporates actually all moments of the distribution and thus it provides a complete description of the distribution. Instead of estimating some moments, the Omega measures total impact, which is surely of interest of decision-makers.

This measure is defined as a ratio of the weighted conditional expectation of losses over the weighted conditional expectation of gains:

$$\Omega = \frac{P(\ell > 0)E(\ell|\ell > 0)}{-P(\ell < 0)E(\ell|\ell < 0)}.$$

[3] This approach is discussed in subsequent section and here is mentioned only as a motivation for Omega.

The following figure depicts some risk measures for a continuous distribution *f* of losses and the corresponding cdf *F*.

The integral I_1 represents the denominator of previous fraction (the weighted conditional expectation of losses) and I_2 represents the weighted conditional expectation of gains (Keating, Shadwick, 2002). Using mathematical notation these two integral are:

$$I_1 = \int_{-\infty}^{0} F(z)dz \ ; \ I_2 = \int_{0}^{\infty} \left(1 - \mathrm{F}(\mathrm{z})\right)\mathrm{dz}\,.$$

Provided that the integral exist, the Omega measure can be computed as

$$\Omega = \frac{\mathrm{I}_2}{\mathrm{I}_1}.$$

The expected loss can be then estimated by

$$E(\ell) = \mathrm{I}_2 - \mathrm{I}_1.$$

Keating (202) noticed that the Expected Shortfall can be computed as

$$ES = VaR + \frac{1}{\beta} I_3\,.$$

Figure 2.7: Omega Risk Measure

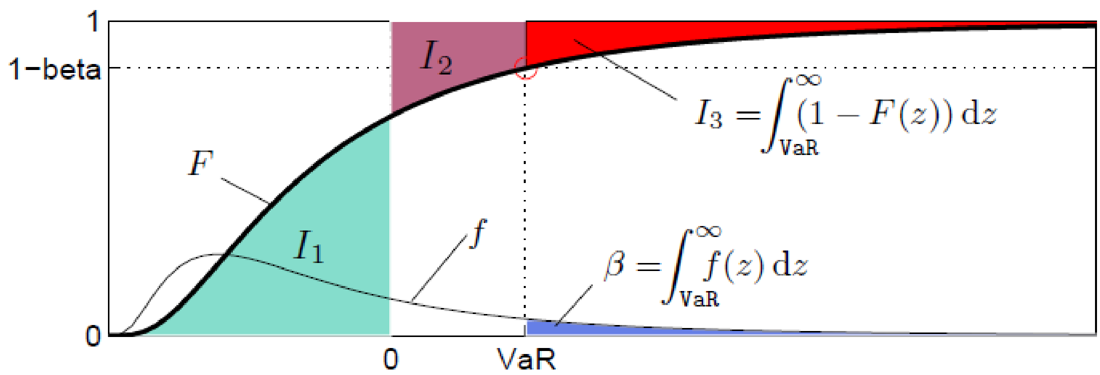


Figure adopted from M. Gilli, E. Kellezi, H. Hysi (2006)

3 Portfolio Optimization

Modern portfolio theory is part of finance theory with an attempt to maximize expected return of a portfolio for a given level of portfolio risk. Or equivalently, the aim of this theory can be the minimization of risk for a chosen level of expected return. These results are supposed to be achieved especially by choosing the optimal properties of various assets. The Modern Portfolio Theory is mathematical embodiment of the diversification concept in investing, with a goal of determining a collection of certain assets that has collectively lower risk than any individual asset.

Foundations of this theory have been laid by Henry Markowitz in 1952 by his groundbreaking paper "Portfolio Selection" (Markowitz, 1952), (Steinbach, 2000). He, Merton Miller and William F. Sharpe were appraised by Nobel Prize for their contribution to field of finance in 1990.

The principal concept underlying this theory is that the investment asset in a portfolio should not be picked out individually, each on its own merit. Rather, it is fundamental to consider how each asset develops in price relative to developments in prices of other assets in portfolio. Since an investing is a trade-off between expected risk and expected return, we can say, that assets with higher expected return bear in general more risk. Universally, the objective of Modern Portfolio Theory is finding the optimal trade-off and thus the best diversification strategy.

3.1 Mean-Variance Optimization

The Mean-Variance approach is based on Markowitz's work from 1952, where the main thought is that the distribution of returns can be fully described by two moments – by mean and variance regardless whether we are considering univariate or multivariate case.

We are considering possible investment in *n* risky assets $x_1, \dots, x_n$. The historical data of returns for T periods are at our disposal as well and the data for *i*-th period are represented by vector $r_i = [r_{i1}, \dots, r_{in}]'$.

The expected return of the portfolio is

$$r_p(w) = w'E(r) \ ,$$

where w represents vector of portfolio weights $[w_1, \ldots, w_n]'$ and *E(r)* typifies vector of expected returns of portfolio components $[E(r_1), \ldots, E(r_n)]'$.

Another crucial part of the portfolio selection is the risk. Within this framework the risk is defined as a variance of portfolio return. The term Σ_{ia} stands for the covariance between returns on asset *a* and *i*:

$$\Sigma_{ia} = E\left((r_{ij} - \bar{r}_i)(r_{aj} - \bar{r}_a)\right), \quad i, a \in \{1, .., n\}, j \in \{1, .., t\},$$

$$E(r_i) = \bar{r}_i = \frac{\sum_{j=1}^{k} r_{ij}}{K}, \quad i \in \{1, .., n\}, j \in \{1, .., t\}, K \in N.$$

The semi-positive definite covariance matrix Σ is defined as:

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \cdots & \Sigma_{1n} \\ \vdots & \ddots & \vdots \\ \Sigma_{n1} & \cdots & \Sigma_{nn} \end{bmatrix}.$$

Thus the variance of portfolio returns is

$$variance_p(w) = w'\Sigma\mathrm{w}\,.$$

Nonetheless the widely spread rule is using a standard deviation as a risk measure:

$$\rho_p(w) = \sqrt{w'\Sigma\mathrm{w}}\,.$$

The entire Mean-Variance framework is based on assumption of normally distributed returns. If this assumption is violated, the results implying this concept are not correct.

Generally, there are three different ways how to find optimal portfolio.

3.1.1 Optimum Portfolio for Selected Rate of Return

As the heading suggest, the portfolio selection procedure is grounded in choosing a particular rate of return and subsequently in minimization of risk embodied by variance:

$$\min \boldsymbol{w'\Sigma\mathrm{w}}$$

$$s.t.\, r_p(w) \geq \mu \;\wedge w_i \geq 0 \wedge w'e = 1,$$

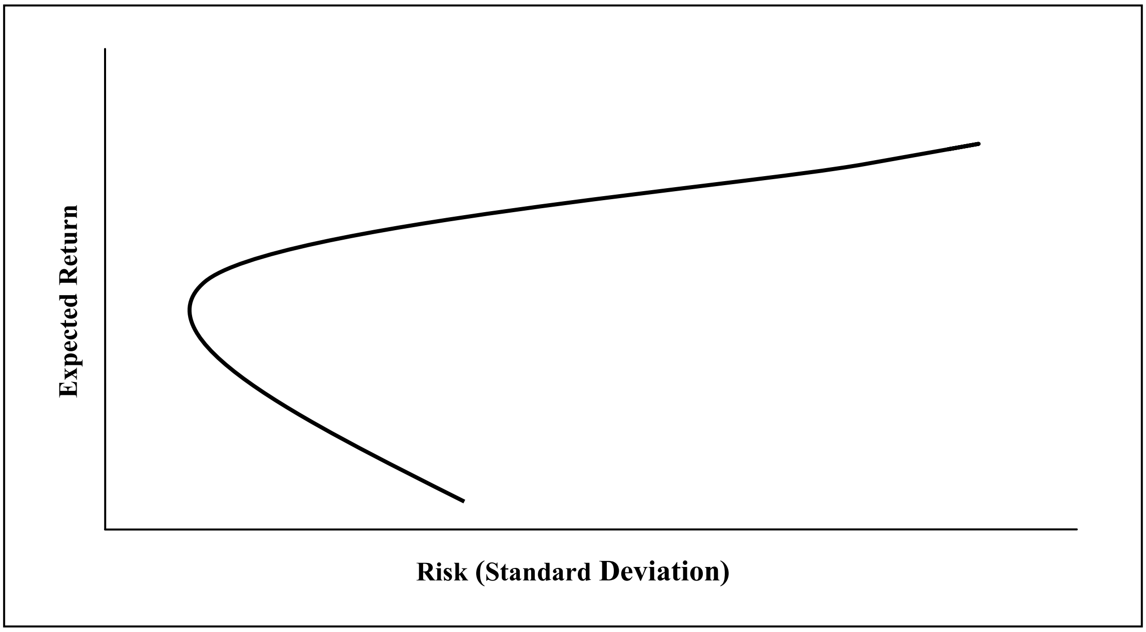
where the term μ represents desired rate of return and *e* is column vector of ones.

Although here is minimized the variance, on the grounds of the definition of standard deviation this term is minimized as well.

3.1.1.1 *Efficient Frontier*

By varying in the μ we obtain different values of minimized variance (Campbell, Huisman, Koedijk, 2001). When plotting these pairs of values in risk-return space, we will get the efficient frontier, the line constituting optimal portfolios in the sense, that there is no other portfolio generating higher expected return with the same variance. Thus no rational investor should select portfolio below the frontier.

Figure 3.1: Efficient Frontier



3.1.2 Optimum Portfolio for Selected Rate of Risk

Another approach, similar to the previous one, is available (Krokhmal, Palmquist, Uryasev, 2001).This selection procedure consists in stating certain level of risk and maximization of rate of return:

$$\max\ r_p(w)$$

$$s.t.\, \boldsymbol{w'\Sigma w} \leq \sigma\ \wedge w_i \geq 0 \wedge w'e = 1,$$

given that the σ deputize variance. By varying the value of σ, the efficient frontier will be generated. Regardless we minimize risk or maximize return, the result of this procedures are identical (Uryasev, 2000).

3.1.3 Optimum Portfolio with Highest Performance Ratio

Probably the most employed method in portfolio selection field is maximization (or minimization) of some ratio measuring the performance. Sharpe (Sharpe, 1994) introduced reward-to-variability ratio designated for Markowitz's mean-variance framework:

$$SR(w) := \frac{\bar{d}}{\sigma_d} = \frac{r_p(w) - r_f}{\sigma_d},$$

where $\bar{d}$ represents the "expected differential return" (i.e. difference between expected return of selected portfolio and expected return of risk free asset). The term r_f stands for risk free asset generating guaranteed rate of future return, predominantly government bonds or cash. Given that the risk free rate is uncorrelated with risky assets the SR can be adjusted in following way:

$$SR(w) := \frac{r_p(w) - r_f}{w_p \sigma_p}.$$

The variable w_p denotes the risky part of portfolio. However as we recently have seen in Greece the "risk-free" rate is sometimes not always absolutely risk free, thus instead of Sharpe ratio several others performance measures have been proposed. Using matrix algebra we propose following ratio:

$$performance\ ratio := \frac{\mu(\boldsymbol{w}'E(r))}{\rho(\boldsymbol{w}'\boldsymbol{\Sigma}\mathbf{w})}.$$

3.1.4 Data

For our purposes we have generated in MATLAB two datasets. The first one consists of four paths, each with 660 observations. Each of them has been generated 600 times, thus we have a matrix with 660 columns and 2400 rows. These paths representing asset prices are normally distributed. The second dataset is nearly identical to the previous, only with two exceptions: the data posses fat tailed distribution with positive mean and randomly evince positive and negative shocks. Furthermore we have prepared one extra dataset from years 2005 – 2012 consisting of nine components of NASDAQ index. Basic statistical description of the market data is in table below and additional information about these are enclosed in appendix.

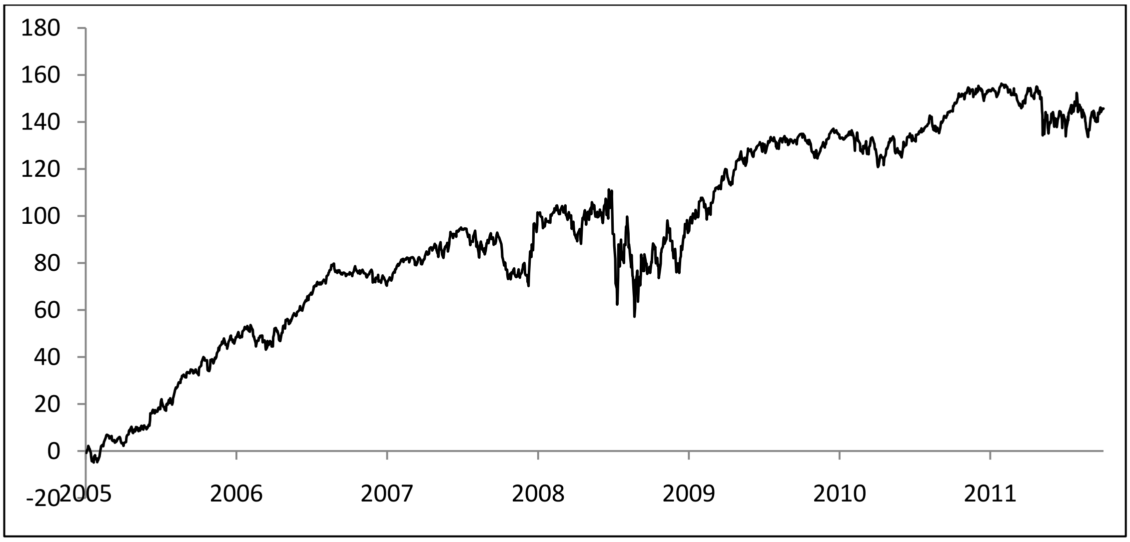
Table 3.1: Basic Statistical Properties of Daily Log-Returns from Years 2005 - 2012

	Mean	Standard Deviation	Skewness	Kurtosis
Microsoft Corporation	0,000056	0,018296	0,181311	10,402480
Deutsche Bank AG	-0,000471	0,033314	-0,050669	9,164695
Apple Inc.	0,001447	0,024307	-0,250435	4,594080
Cisco Systems, Inc.	-0,000038	0,021027	-0,320740	9,772347
Vertex Pharmaceuticals Inc.	0,000668	0,031071	0,461198	5,998642
Texas Instruments Inc.	0,000105	0,020349	-0,428003	3,898109
Starbucks Corporation	-0,000161	0,028436	-7,531710	185,873753
Wells Fargo & Company	-0,000459	0,038297	-2,819319	71,932301
Bank of America Corporation	-0,001205	0,041636	-0,242428	16,463302

3.1.4.1 Application on the Selected Market Data

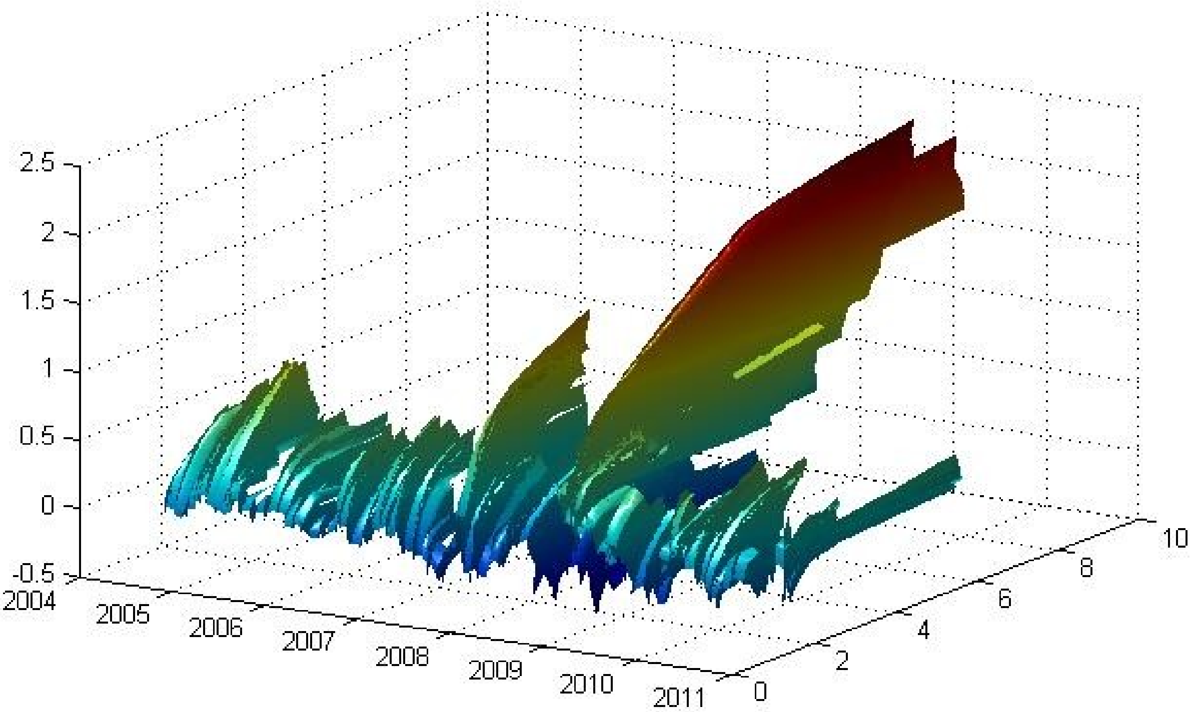
Under the assumption of no available risk free asset, we have conducted multi-period portfolio selection procedure. The window size was set to the value of 60 observations and within the each step the shift of the window was 20 observations. The following graph describes the development of this portfolio in time. The slump in the value of portfolio in 2008 is evident.

Figure 3.2: Value of Optimized Market Portfolio



Here follows the three-dimensional efficient frontier. The X-axis represents returns, Y-axis time and Z-axis the risk measured by standard deviation. The "crisis years" 2007 and 2008 are characterized by sharply higher volatility, thus we can see in these years major changes in structure of the multi-period efficient frontier depicted below.

Figure 3.3: Development of Market Efficient Frontier in Time



3.2 Single-period Mean-VaR Optimization

In this section we would like to introduce a portfolio model that allocates assets by maximizing the expected return subject to a certain risk constraint, where the amount of risk exposure is measures by a Value at Risk (Campbell, Huisman, Koedijk, 2001).The maximal expected loss of the optimal portfolio should not exceed the VaR for a given investment horizon at chosen confidence level α. We suppose the possibility of lending and borrowing at the exogenous market interest rate.

We define *W(0)* as the wealth of an investor to be invested for a horizon *T*, such that the chosen portfolio meets given VaR limit. This limit may be set in order to meet the Basel capital adequacy requirement, or in order to be in line with private investor's individual aversion to risk. The amount *W(0)* can therefore be invested along with figure *B* representing the amount money that can be lent ($B < 0$) or borrowed ($B > 0$) at risk free rate r_f. There are accessible *n* financial assets and term w_i denotes the fraction of resources invested in the *i-th* risky asset. Therefore $\sum_{i=1}^{n} w_i = 1$. Given that the term $p_{i,t}$ represents the price of *i-th* asset at time *t*, the initial value of portfolio value is subject to budget constraint:

$$W(0) + B = \sum_{i=1}^{n} w_i p_{i,0} \, . \qquad (1)$$

Allocating the assets and determining the amount to lend or borrow such that the maximum expected level o final wealth is achieved leads to the definition of portfolio allocation problem. Establishing the desired level of VaR as VaR* we subsequently formulate the downside risk constraint:

$$P\{W(0) - W(T,p) \geq VaR^*\} \leq 1 - \alpha \, , \qquad (2)$$

where "P" denotes the expected probability conditioned on all accessible information at time zero, for some portfolio *p*. The α represents the chosen confidence level. This equation can be rewrited in following form:

$$P\{W(T,p) \leq W(0) - VaR^*\} \leq 1 - \alpha \, . \qquad (3)$$

Since the VaR should be the worst suffered loss over the time horizon *T*, which can be expected with confidence α, the level of risk aversion of certain investor is mirrored in both the level of the VaR, and the confidence level α associated with it. The optimal portfolio derived according to the previous equations will therefore necessarily reflect it.

We assume that a rational investor is interested in wealth maximization at the end of the investment horizon *T*. The expression *r(p)* represents the total expected return on portfolio *p* in time *T*. The expected wealth from investing in portfolio *p* at the end of period is

$$E_0\big(W(T,p)\big) = \big(W(0) + B\big)\big(1 + R(p)\big) - B\big(1 + r_f\big). \qquad (4)$$

Substituting for B and the downside risk constraint (3), the final expected return is maximized for an investor concerned about the downside risk by the portfolio p' maximizing *S(p)* in ensuing equation:

$$p' : max_p S(p) = \frac{r(p) - r_f}{W(0)(r_f - q(\alpha,p))} \, , \qquad (5)$$

given the *q(α,p)* defines the quantile that corresponds to the probability $1 - \alpha$ of occurrence. It is important to notify, that although the term *W(0)* is in the denominator of *S(p),* it has no influence on the choice of optimal portfolio since it is only constant in the maximization. In other words, the asset allocation process is simply independent of initial wealth of an investor.

Since the negative quantile of the distribution multiplied by *W(0)* is the VaR associated with the selected portfolio for a confidence level, we are able to derive an expression for the risk faced by an owner of the portfolio as φ. Given that $VaR(\alpha, p)$ denote the portfolio p's VaR, the denominator of equation (5) may be rewritten as

$$\varphi(\alpha, p) = W(0)r_f - VaR(\alpha, p). \qquad (6)$$

Thus the risk-return ration *S(p)* can be therefore written as follows:

$$p': max_p S(p) = \frac{r(p) - r_f}{\varphi(\alpha, p)}. \qquad (7)$$

It is obvious, that the *S(p)* is performance measure similar to the Sharpe index used for an evaluation of the efficiency of portfolios. Forsooth under the assumptions that the risk-free rate is zero and the expected portfolio return posses normal distribution, the ratio *S(p)* collapses to a multiple of the Sharpe index. The explanation is apparent. The VaR is expressed as a multiple of the standard deviation of the expected returns and thus the point at which both performance indices (Sharpe index and *S(p)*) are maximized will requisitely lead to the same best portfolio being selected. Only a slight difference in the chosen portfolio weights occurs for positive risk free rates, for adequately small time horizon. As we have mentioned above, the optimal portfolio is independent of the initial wealth *W(0)*. The portfolio selection procedure in not dependent on the desired level of VaR, since the measure $\varphi(\alpha, p)$ depends on the VaR of the estimated portfolio, not on the determined level of VaR*. However, since the level of risk aversion is defined by the selected VaR*, the level of lending or borrowing needed to meet the VaR* constraint has to be determined. This property is significant benefit of the model, because it allows for easy and quite accurate determination of the desired risk-return trade-off with the specific amount of lending or borrowing. The amount to be borrowed is defined by succeeding expression:

$$B = \frac{W(0)(VaR^* - VaR(\alpha, p'))}{\varphi'(\alpha, p')}. \qquad (8)$$

The optimal portfolio is independent of any assumptions about distribution.

3.3 Mean - VaR Optimization

In this section we would like to propose a multi-period portfolio model. The following text is based on Rengifo, Rombouts (2004). Let us to define $W_t$as the wealth of an investor at time *t*, and let be b_t the amount of money that can be lent ($b_t < 0$) or borrowed ($b_t > 0$) at risk free rate r_f at time *t.* There are again *n* financial assets available with prices $p_{i,t}$, where *i* = 1, …,*n*. The set of portfolios weights at time *t* is defined as follows

$$X_t \equiv \left\{ x_t \in R^n : \sum_{i=1}^{n} x_{i,t} = 1 \right\}.$$

The $w_{i,t}$ represents the number of shares of *i*-th asset at time *t*. Then the investor's budget constrain is

$$W_t + b_t = \sum_{i=1}^{n} w_{i,t} p_{i,t} = w_t' p_t \ . \qquad (1)$$

Then the value of chosen portfolio at time *t+1* is given by

$$W_{t+1}(w_t) = (W_t + b_t)\big(1 + R_{t+1}(w_t)\big) - b_t\big(1 + r_f\big)\,, \qquad (2)$$

where $R_{t+1}(w_t)$ is the return of portfolio at maturity. As mentioned above, the VaR is the maximal expected loss over a given investment horizon at chosen confidence level α:

$$P_t\{W_{t+1}(w_t) \leq W_t - VaR^*\} \leq 1 - \alpha\,. \qquad (3)$$

The P_t stands for probability conditioned on the all known information at time *t* and VaR* is a VaR level desired by investor. We can express the optimization problem in terms of the maximization f the expected returns $E_t\{W_{t+1}(w_t)\}$, subject to the VaR constraint and budget constraint:

$$w_t^* = arg\ max_{w_t}(W_t + b_t)(1 + E_t\{R_{t+1}(w_t)\}) - b_t\big(1 + r_f\big)\,, \qquad (4)$$

s.t. (1) and (3). The term $E_t\{R_{t+1}(w_t)\}$ stands for the return of whole portfolio given the known information at time *t.* Nevertheless this optimization problem can be restated in unconstrained way. Thus we replace (1) in (2) and take expectations:

$$E_t\{W_{t+1}(w_t)\} = w_t'p_t\big(E_t\{R_{t+1}(w_t)\} - r_f\big) + W_t\big(1 + r_f\big)\,. \qquad (5)$$

This equation shows that if th expected return of a given portfolio is higher than the risk free rate, then an rational investor with an aversion towards risk wants to invest a fraction of his wealth in risky assets. By substitution of identity (5) in (3) we get:

$$P_t\{w_t'p_t\big(R_{t+1}(w_t) - r_f\big) + W_t\big(1 + r_f\big) \ \le W_t - VaR^*\} \le 1 - \alpha\,. \qquad (6)$$

We can that rewrite in more suitable form:

$$P_t\left\{R_{t+1}(w_t) \le r_f - \frac{VaR^* + W_t r_f}{w_t'p_t}\right\} \le 1 - \alpha\,. \qquad (7)$$

The identity (7) defines the quantile $\mathrm{q}(w_t, \alpha)$ of the return distribution of the chosen portfolio at a given confidence level α. We can now express the portfolio in following way:

$$w_t'p_t = \frac{VaR^* + W_t r_f}{r_f - \mathrm{q}(w_t, \alpha)}\,. \qquad (8)$$

In the end substituting (8) in (5) and dividing by initial wealth W_t we acquire:

$$\frac{E_t\{W_{t+1}(w_t)\}}{W_t} = \frac{VaR^* + W_t r_f}{W_t\left(r_f - \mathrm{q}(w_t, \alpha)\right)}\big(E_t\{R_{t+1}(w_t)\} - r_f\big) + W_t\big(1 + r_f\big)\,, \qquad (9)$$

and so

$$w_t^* = arg\ max_{w_t} \frac{E_t\{R_{t+1}(w_t)\} - r_f}{W_t\left(r_f - \mathrm{q}(w_t, \alpha)\right)}\,. \qquad (10)$$

It's clear, that the two funds separation theorem[4] applies (id est the initial wealth of investor and desired $VaR = W_t \mathrm{q}(w_t, \alpha)$ easily do not affect in any way the maximization procedure). As in traditional theory, the investor first determines the allocation of the risky assets and then decides about the amount of lending and borrowing. The latter mirrors by how much the VaR of specific portfolio differs from the selected VaR* level due to the degree of risk aversion of an investor.

[4] for information see for example Wenzelburger (2008)

The amount of money the rational investor borrows or lends is established by replacing (1) in (8):

$$b_t = \frac{VaR^* + W_t \mathrm{q}(\mathrm{w}_\mathrm{t}^*, \alpha)}{r_f - \mathrm{q}(\mathrm{w}_\mathrm{t}^*, \alpha)}.$$

3.4 Mean - CVaR Optimization

For this section, we refer to Uryasev and Rockafellar (2000). Nowadays, the Value at Risk has achieved remarkable high status of being written into banking industry regulations. Nevertheless it is weird that it had happened, because for a long time it was quite difficult to optimize VaR numerically when losses were not normally distributed. Fortunately recently great progress has been made and efficient VaR optimization routines were included in commercial packages. We decided to use Portfolio Safeguard[5] in combination with MATLAB in preceding VaR optimization and succeeding VaR-deviation optimization. As a tool for modeling, the CVaR has superior properties. CVaR-based optimization is consistent with VaR optimization and generates the same results for normal or elliptical distributions (elliptical distributions and their properties are understandably explained in Ebrechts et al.(2002)). For models, with such distribution, working with CVaR, VaR or even Markowitz's minimum variance is equivalent (for deeper cognition see Rockafellar and Uryasev (2002)). In the same paper was proposed an approach extremely useful for CVaR based portfolio selection. Most importantly, the CVaR can be superseded by function that can be used in minimization routine instead of CVaR:

$$F_\alpha(x, \zeta) = \zeta + \frac{1}{1-\alpha} E\{[f(x, y) - \zeta]^+\}.$$

Rockafellar and Uryasev proved that:

- $F_\alpha(x, \zeta)$ is convex with respect to α;
- $VaR_\alpha(x)$ is a minimum point of function $F_\alpha(x, \zeta)$ w.r.t. ζ;
- minimizing $F_\alpha(x, \zeta)$ w.r.t. ζ gives $CVaR_\alpha(x)$:

$$CVaR_\alpha(x) = min_\alpha\, F_\alpha(x, \zeta)\ .$$

In optimization tasks, CVaR can enter into constraints or objective or both. A cardinal advantage of CVaR over VaR is the preservation of convexity (if $f(x, y)$ is convex in x,

[5] Portfolio Safeguard is tool for complex optimizations problems. For wider description see please Appendix.

then $CVaR_\alpha(x)$ is convex in *x*). This convexity is crucial because minimizing $F_\alpha(x,\zeta)$ over $(x,\zeta) \in X \times \Re$ results necessarily in minimizing $CVaR_\alpha(x)$:

$$min_{x\in X} CVaR_\alpha(x) = min_{(x,\zeta)\in X\times\Re} F_\alpha(x,\zeta) .$$

Given that the pair (x^*,ζ^*) minimize F_α over $X \times \Re$, then not only does x^* optimize the $CVaR_\alpha(x)$ over X, but also

$$CVaR_\alpha(x^*) = F_\alpha(x^*,\zeta^*)$$

Rockafellar and Uryasev additionally showed that for an arbitrary choice of confidence levels α_i and loss tolerances ω_i, *i*=1,…,*l,* the problem

$$min_{x\in X}\ g(x)$$

$$s.t. CVaR_{\alpha_i}(x) \le \omega_i$$

is tantamount to the problem

$$\min_{x,\zeta_1,\ldots,\zeta_l\in X\times\Re\times\ldots\times\Re} g(x)$$

$$s.t. F_{\alpha_i}(x,\zeta) \le \omega_i .$$

Under the assumption that X and *g* are convex and $f(x,y)$ is convex in *x*, the above stated optimization problems ones of quadratic programming, thus convenient for computation. When Y is some discrete probability space where are elements y_k, $k = 1,\ldots,N$, with probabilities p_k, we obtain

$$F_{\alpha_i}(x,\zeta_i) = \zeta_i + \frac{1}{1-\alpha_i}\sum_{k=1}^{N} p_k[f(x,y_k) - \zeta_i]^+ .$$

The restrictive constraint $F_\alpha(x,\zeta) \le \omega$ is usually replaced by a system of inequalities by introducing additional variables π_k:

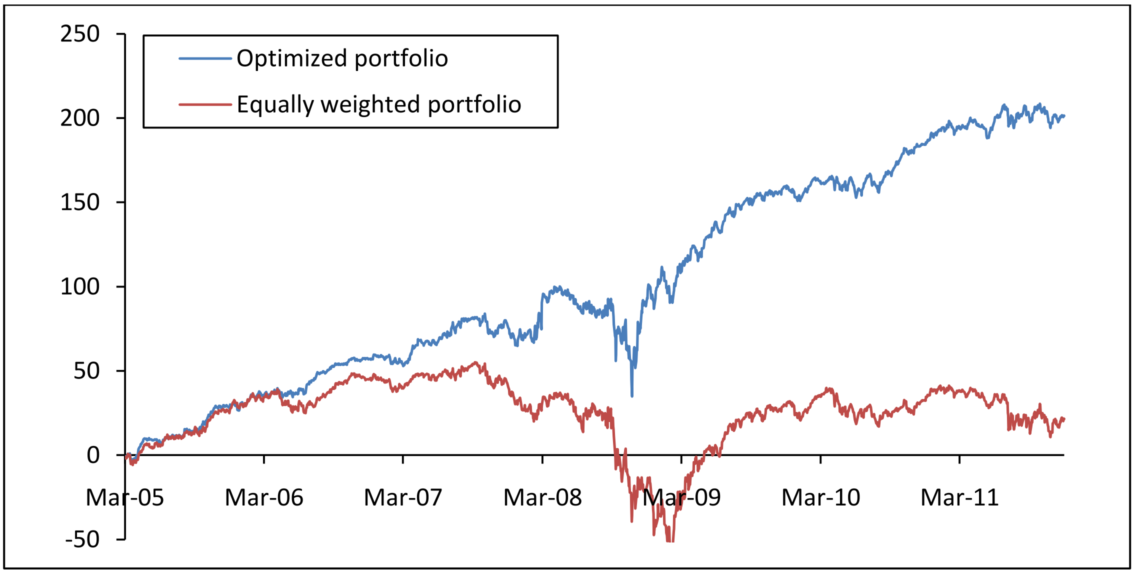
$$\pi_k \ge 0\ ;\ f(x,y_k) - \zeta - \pi_k \le 0\ ; k = 1,\ldots,N;$$

$$\zeta + \frac{1}{1-\alpha}\sum_{k=1}^{N} p_k\,\pi_k \le \omega .$$

The earlier stated minimization problem $min_{x \in X}\ g(x)$ with its constraints can be transformed into a minimization of *g(x)* with the constraints $F_{\alpha_i}(x,\zeta) \leq \omega_i$ being replaced with above presented system of inequalities. Moreover if *f* is linear in *x*, the constraints in the system of inequalities are linear as well.

We have employed the Uryasev's method on our market and simulated data. Concerning the market data – we have considered two portfolios, the first one was optimized, the second one was equally weighted. The values of our portfolios have developed in following ways:

Figure 3.4: Value of Optimized and Equally Weighted Market Portfolio



These results testify unequivocally in favor of CVaR portfolio optimization, at least when we are speaking about absolute return. However the risk we have to undergo in order to achieve such returns is crucial. The risk measured by volatility of returns is ensuing:

Figure 3.5: Volatility of Optimized Market Portfolio

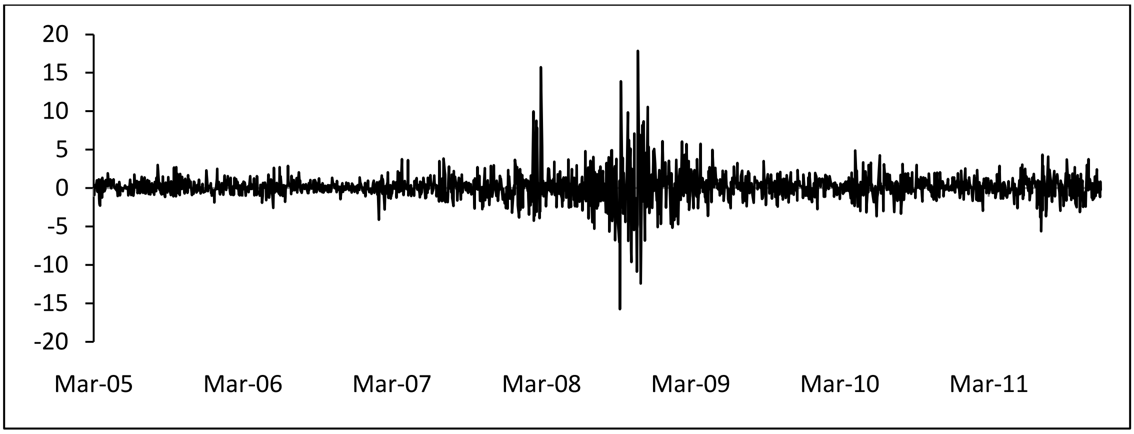
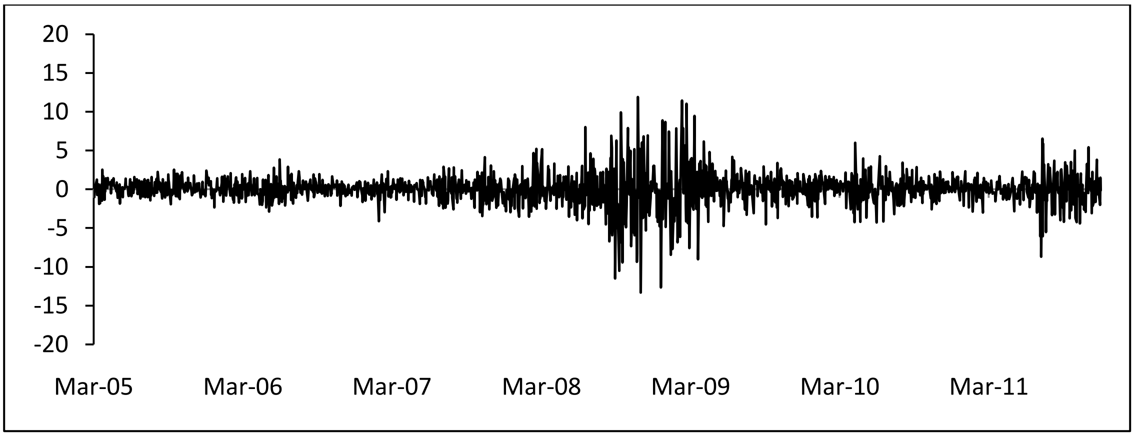
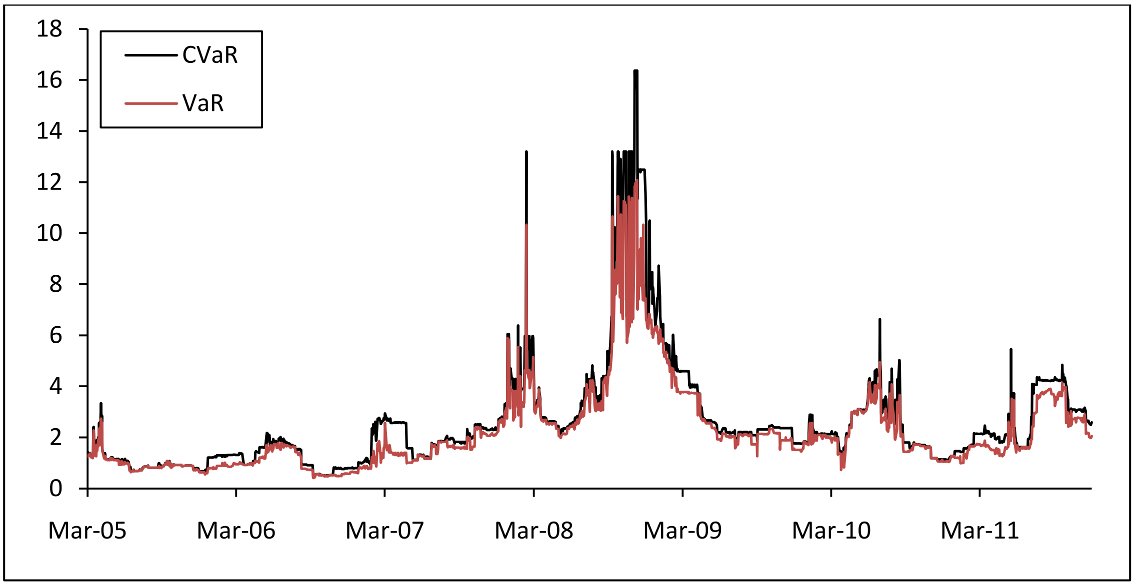


Figure 3.6: Volatility of Equally Weighted Market Portfolio



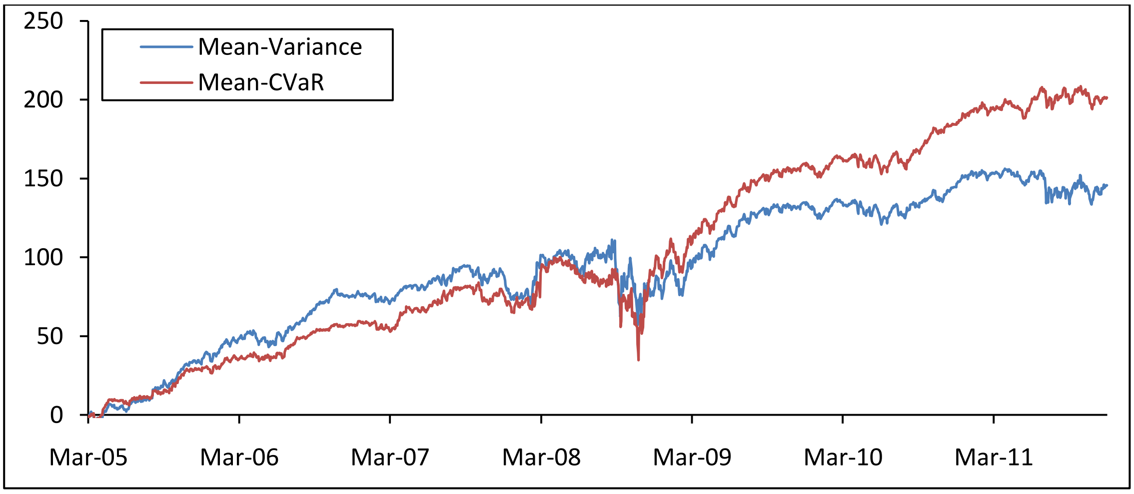
As we can see, the volatilites are quite similar, the optimized portfolio is slightly less risky during the whole period with a exception of years 2008-2009 when the econonomic crisis in USA started. The development of optimized portfolio risk measured by VaR and CVaR has been plotted as well. The confidence levels were selected at a value of .95.

Figure 3.7: VaR and CVaR of Optimized Market Portfolio



The CVaR values are similar to VaR values, although is mostly moderately higher. Sure, the CVaR has to be always higher or equal (for unskewed elliptical distributions) then VaR, but here we observe some dramatic differences in these values. These differences have been noted in crisis years and were caused by extreme losses that are beyond the confidence level of the VaR. Here follows graph comparing Mean-Variance and Mean-VaR approach. As we can see, the latter approach outperforms the first one. We assume that the reason for this is non-normality of underlying returns distribution. In favor of this opinion testifies the fact that the CVaR is often slightly than VaR, which is usually result of non-normality.

Figure 3.8: Portfolio Optimization: Mean-Variance vs. Mean-CVaR Approach



3.5 Mean - (C)VaR Deviation and Mean - (C)VaR Optimization

In this section we would like to show, that the (C)VaR-deviation measures and (C)VaR measure perform equally when the underlying distribution of returns is elliptical. As outlined before, we have simulated four assets paths with 660 observations. The distribution of returns was normal, the means of return were selected randomly and the positive-definite variance-covariance matrix was stable over time. Subsequently the portfolio consisting of four assets has been optimized on monthly basis (every 20 observations) and then determined the cumulative return of the portfolio over the whole period. This process, including simulation of new data, was conducted 1000 times (thus we have got 1 000 values of cumulative gains). Absolutely the same procedure was performed once again, but the underlying distribution possessed fat tailed t-distribution. After a simply adjustment the mean of this distribution was not zero, but was positive. We have done this in order to make look this path more similar to real asset path. The confidence levels have been set up to 0.95 in both cases. Ensuing table presents values we have obtained for normal distribution:

Table 3.2: Median and Variance of Cumulative Gains: VaR and CVaR Based Portfolio Selection Under Normality

	$\mathrm{VaR}_{0,95}$	$\mathrm{CVaR}_{0,95}$	$\mathrm{VaR}^{\Delta}_{0,95}$	$\mathrm{CVaR}^{\Delta}_{0,95}$
Median of cumulative gains	0,0985	0,0993	0,0994	0,0996
Variance of cumulative gains	0,0016	0,0016	0,0015	0,0016

Now we can easily advocate the opinion that under the assumption of normality is not crucial which of the four measures will be applied in portfolio selection. In all cases, the median and variance of the 1 000 cumulative gains was very similar. Since the distribution of gains can be fully described by mean and variance, provided it is normal, we can conclude, that the result of portfolio selection is independent of the choice of one of the four previous functionals as a risk measure.

Following histograms of cumulative gains indicate the distribution. Regardless whether VaR or CVaR was used, or (C)VaR-deviation, the histogram of cumulative gains is in both cases very similar.

Figure 3.9: Histogram of Cumulative Gains: VaR and CVaR Based Portfolio Selection Under Normality

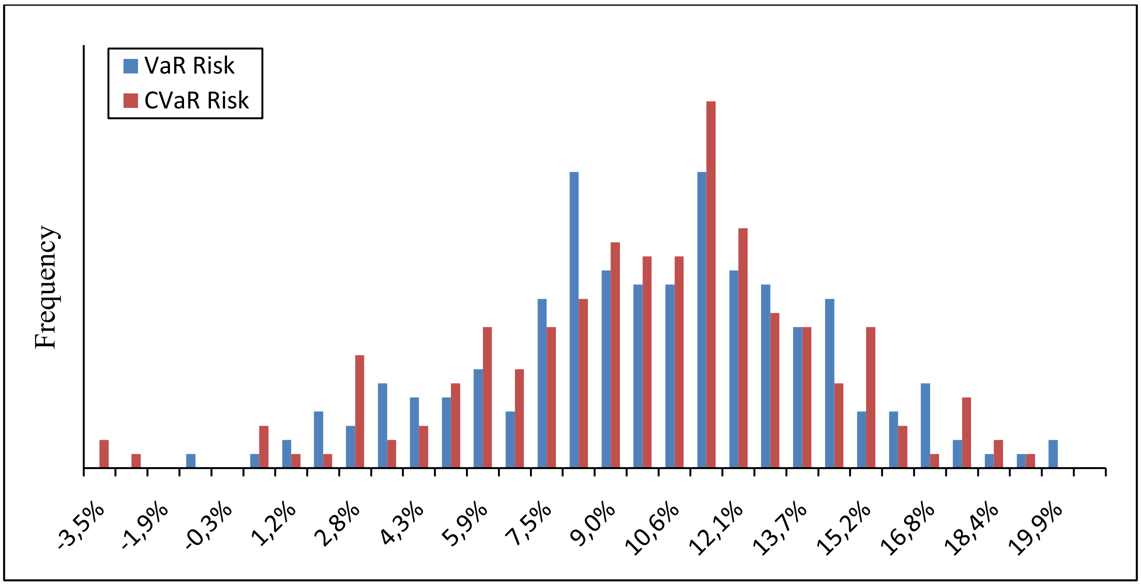
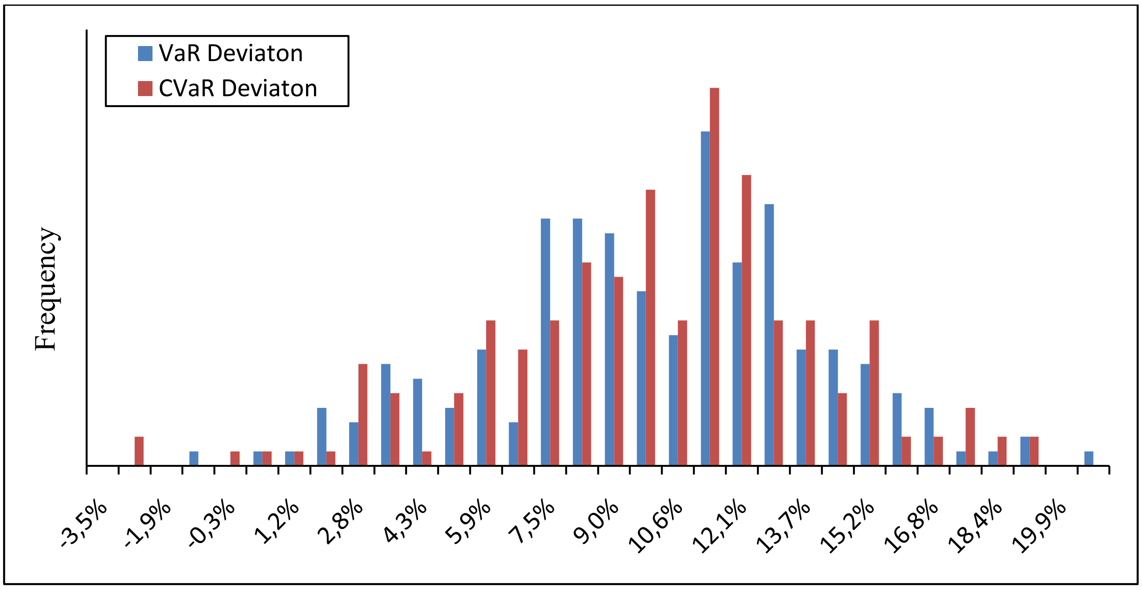


Figure 3.10: Histogram of Cumulative Gains: VaR Deviation and CVaR Deviation Based Portfolio Selection Under Normality



Now we can see that the results of optimizations based on these different measures are in this case practically same. Here follow the results for symmetrical fat-tailed distribution.

Table 3.3: Median and Variance of Cumulative Gains: VaR and CVaR Based Portfolio Selection, t-distribution

	$\mathrm{VaR}_{0,95}$	$\mathrm{CVaR}_{0,95}$	$\mathrm{VaR}^{\Delta}_{0,95}$	$\mathrm{CVaR}^{\Delta}_{0,95}$
Median of cumulative gains	0,1171	0,1157	0,1164	0,1157
Variance of cumulative gains	0,0014	0,0014	0,0014	0,0014

The conclusion based on previous figures is the same as before. The median of cumulative gains demonstrates only very little changes and the variance of cumulative gains are even without any perceptible change. The plots are available as well:

Figure 3.11: Histogram of Cumulative Gains: VaR and CVaR Based Portfolio Selection Under Fat Tailed Distribution

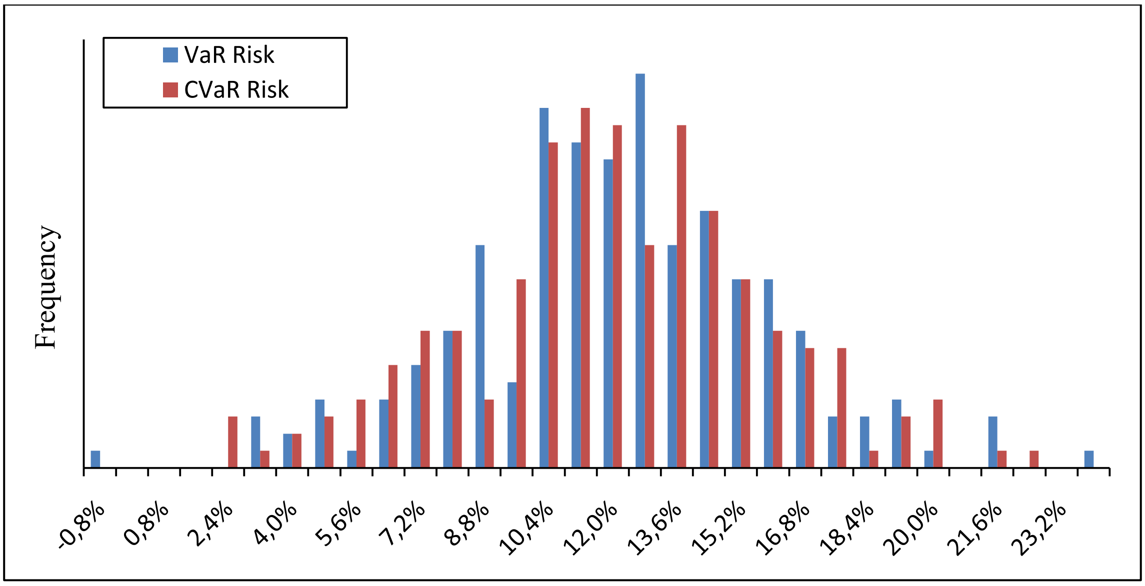
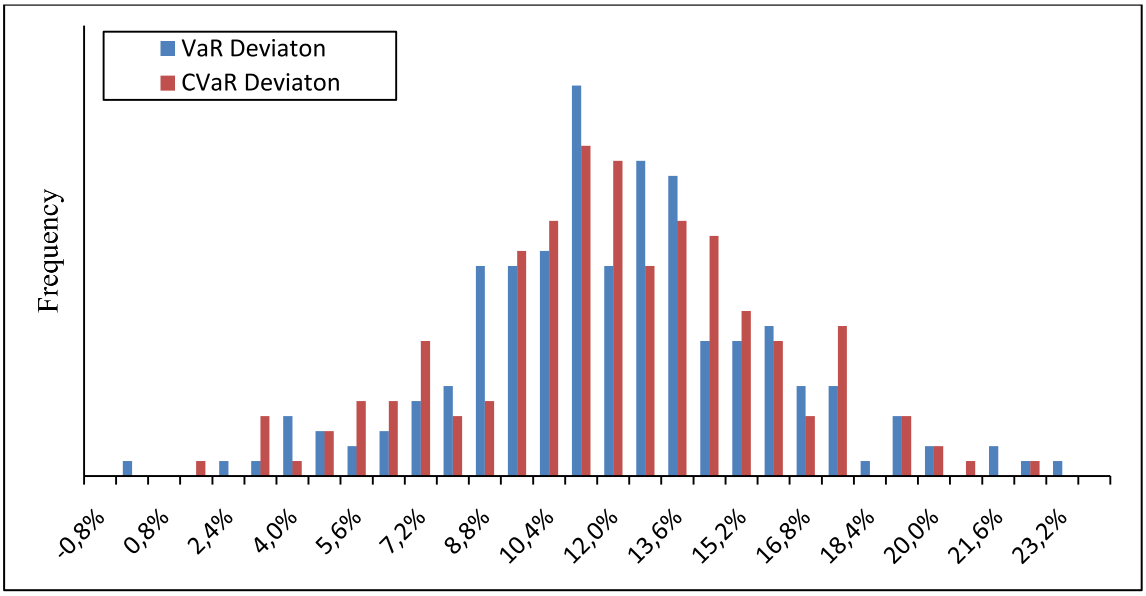


Figure 3.12: Histogram of Cumulative Gains: VaR Deviation and CVaR Deviation Based Portfolio Selection Under Fat Tailed Distribution



Now is clear, that for these distribution is not important which of the four above mentioned measures is selected, the results of portfolio selection are the same. The similar results are reachable for other elliptical distributions. For example for logistic and Laplace distribution the results are the same.

3.6 Risk Control Based on VaR

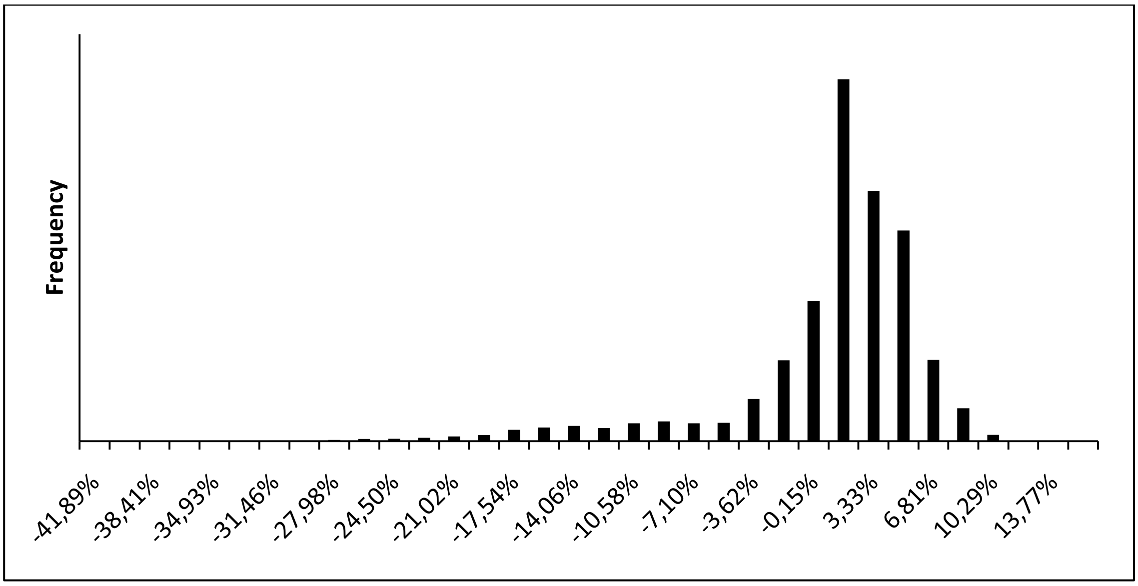
As indicated before, the risk control using VaR may lead to very paradoxical results for distinctively skewed distributions. In these cases, the minimization of VaR often may lead to a stretch of the tail of the distribution exceeding the VaR level. The primary purpose of VaR minimization is to restrict extreme losses. However, the VaR minimization under certain circumstances may lead to a growth in the extreme losses that we were supposed to control (Brandtner, 2011). This is the unwanted feature of VaR minimization that can cause huge losses. We have simulated in MATLAB dataset consisting of 10 paths (each with 1000 steps) where the underlying profit and loss distribution is skewed. Then we have solved two optimization problems. In the first task we have minimized $\mathrm{CVaR}_{99\%}$-deviation of losses subject to constraints on required returns. In the second problem was minimized $\mathrm{VaR}_{99\%}$-deviation of losses subjected to the same constraints. Afterwards we got two optimal portfolios for our problems and have evaluated different risk functions at the optimal points. The results are summarized in subsequent table.

Table 3.4: Risk Control Based on VaR

	$\min \mathrm{CVaR}^{\Delta}_{0.99}$	$\min \mathrm{VaR}^{\Delta}_{0.99}$	Ratio
$\mathrm{CVaR}_{0,99}$	0,00341	0,00590	1,729723
$\mathrm{CVaR}^{\Delta}_{0.99}$	0,04507	0,04746	1,053049
$\mathrm{VaR}_{0,99}$	0,00150	0,00359	2,393506
$\mathrm{VaR}^{\Delta}_{0.99}$	0,04016	0,03797	0,945421
$\text{Max loss} = \mathrm{CVaR}_1$	0,01271	0,02007	1,579592
$\text{Max loss deviation} = \mathrm{CVaR}^{\Delta}_1$	0,05437	0,06163	1,133643

Let say, we have started with the selected portfolio having minimal $\mathrm{CVaR}^{\Delta}_{0.99}$. But minimization of $\mathrm{VaR}^{\Delta}_{0.99}$ results in 72,9% increase in $\mathrm{CVaR}_{0,99}$, compared with $\mathrm{CVaR}_{0,99}$ of the optimal $\mathrm{CVaR}^{\Delta}_{0.99}$ portfolio. In other words, if the distribution is skewed the minimization of VaR deviation will probably lead to a stretch of the tail of "optimal" portfolio compared to CVaR deviation optimal portfolio. This finding is crucial when we look at the financial risk management regulations like Solvency or Basel that are based on minimization of VaR deviation (Sarykalin, Serraino, Uryasev, 2008). Here follows the distribution of returns:

Figure 3.13: Histogram of Returns, Skewed Underlying Distribution



Albeit we have showed that sometimes the VaR may be inferior to CVaR in portfolio optimization procedures, it can be still useful. The graph above depicts the distribution an unoptimized portfolio, where the maximal loss is 41,89%. While the maximal loss of a portfolio based on $\mathrm{VaR}^{\Delta}_{0.99}$ minimization is "only" 6,16%. Thus this way of portfolio optimization has surely some shortcomings, but if the practitioners are fully aware of them, it can be still very powerful method for risk management and portfolio selection.

4 Conclusion

The main aim of this thesis was providing an overview of several classes of risk measures and their application in finance. First of all, we have started with an introduction of monetary risk measures which are characterized by two crucial properties – by monotonicity and translation invariance. The irreplaceable representative of this class is popular Value at Risk, but the subject of risk measurement has covered a long way since the Value at Risk was presented in the early 1990s. The risk management has evolved and now we are endowed by many respectable risk measures that can be used for theoretical and practical purposes in the field of finance.

In spite of indisputable merits of Value at Risk, this measure is failing to satisfy probably the most significant property of suitable risk measure – the subadditivity. This property ensures that the measure takes into consideration the vital concept in finance – the diversification. Because the lacking of subadditivity can have serious consequences, the new class of subadditive risk measures has been proposed by Artzner et al. (1999). This new class of coherent risk measures satisfies the two preceding properties, and the desired subadditivity and homogenity. Within this class were described several measures, some of them had convenient properties, some of them not. The respectable representatives of this class are Expected Shortfall and its successor Conditional Value at Risk. Nevertheless the risk measurement and management is highly progressive field and thus the coherent measures of risk have developed into convex risk measures. This class came into existence in 2002 when it has been introduced by Follmer and Schied. The dominant characteristic is convexity, a property replacing subadditivity and homogenity. The convex risk measures are most likely the most promising kind of static risk measures.

Contemporaneously with monetary, coherent and convex risk measures have been developing somewhat less known class of deviation measures. The essential member of deviation measures is popular standard deviation, but it is only one constituent of this wide class. In the thesis have been shown, that at the first sight, the deviation measures could be perceived as quite distant from coherent measures, but the very opposite is true. In the thesis have been proved that these two classes could be easily closely connected together.

Subsequently a considerably advanced measure of risk have been introduced, the spectral risk measure. These measures are characterized by risk spectrum which assigns different weights to different quantiles of quantile based risk measures.

In the second part of this thesis some selected parts from portfolio theory were introduced. After brief digression to history three prevailing ways of portfolio selection were described and later used in single- or multi-period portfolio selection.

Based on Campbell et al. (2001) the theoretical solution of single- and multi-period portfolio selection with Value at Risk has been derived. One of the most important findings is that the selection procedure is not dependent on the desired confidence level of risk measure. Albeit the Value at Risk and its application in portfolio theory is frequent, due to its improper attributes, more suitable portfolio selection method has been described. Inspired by Uryasev and Rockafellar (2000) a Condtional-Value-at-Risk-Based portfolio selection have been presented and subsequently used on market data. Using CVaR-based optimization, we have demonstrated the undisputable power of portfolio optimization. When compared with not optimized and equally weighted portfolio, the optimized one produced substantially higher cumulative gains and exhibited lower volatility.

The unquestionable advantage of CVaR over standard deviance or VaR is its ability to deal better with assets where the underlying loss distribution is non-normal. Thus in these cases the CVaR usually outperforms the VaR and deviation. But in the case of normality, the CVaR- and VaR- based portfolio selection produces the same results. This has been demonstrated using simulated data. The empirical part of this thesis was concluded by demonstration of possible inappropriate application of VaR. There have been shown, that under certain circumstances the VaR-based portfolio selection may lead to substantial losses.

In summary, this thesis provided brief introduction into several classes of risk measures and outlined the relations among them. Substantial space has been devoted to the desired statistical properties, because the incomprehension of their meaning and importance can potentially result in severe difficulties of financial institution. The prospective extension of this thesis can be done in the field of spectral and dynamic risk measures and their application in advanced portfolio theory, but this matter is very sophisticated and there are reasonable doubts whether their potential can be fully employed in Czech environment.

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6 Appendix

6.1 Computations

6.1.1 MATLAB

MATLAB is a sophisticated computing environment equipped with own programming language. This software allows for matrix manipulation, advanced multidimensional plotting, implementation and development of algorithms and user interfaces, and interfacing with programs written in other languages. All codes are available upon request.

6.1.2 Portfolio Safeguard

Portfolio Safeguard (PSG) is decision-support tool intended for solving complex optimization problems. The reason for application within this thesis is its ability of optimizing non-convex functions such as VaR and the possibility of calling it via MATLAB.

6.2 Market Data

All prices are in USD and all ranges and market capitalizations are related to July 2012.

Table 6.1: Additional Characteristics Selected MASDAQ Components

Name	Ticker	Beta	52 Week Range	Market Capitalization
Microsoft Corporation	MSFT	1,00	23,65 - 32,95	257,74 B
Deutsche Bank AG	DB	2,08	28,57 - 61,23	35,19 B
Apple Inc.	AAPL	0,94	310,50 - 644,00	521,97 B
Cisco Systems, Inc.	CSCO	1,31	13,30 - 21,30	90,01 B
Vertex Pharmaceuticals Inc.	VRTX	0,84	26,50 - 64,50	13,64 B
Texas Instruments Inc.	TXN	1,23	24,34 - 35,53	35,05 B
Starbucks Corporation	SBUX	1,05	33,72 - 62,00	40,71 B
Wells Fargo & Company	WFC	1,25	22,58 - 34,59	172,22 B
Bank of America Corporation	BAC	1,96	4,92 - 12,11	79,21 B

6.2.1 Daily Log-Returns

Figure 6.1: Daily Log-Returns: Microsoft Corporation

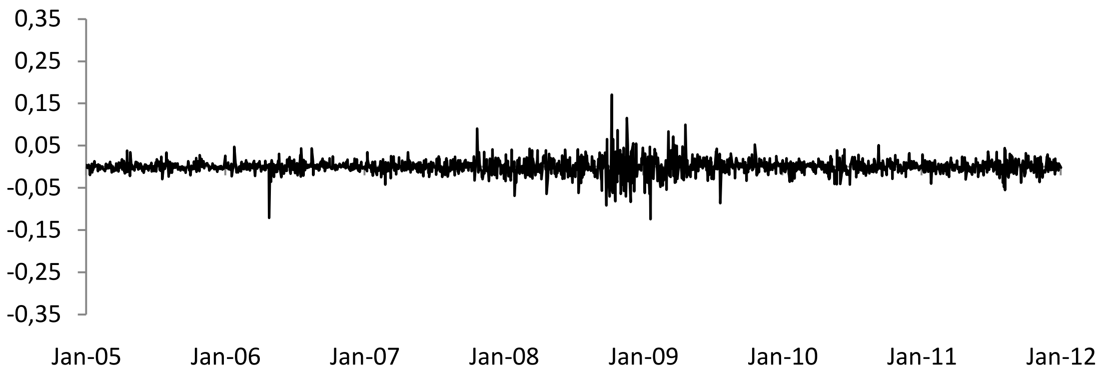


Figure 6.2: Daily Log-Returns: Deutsche Bank AG

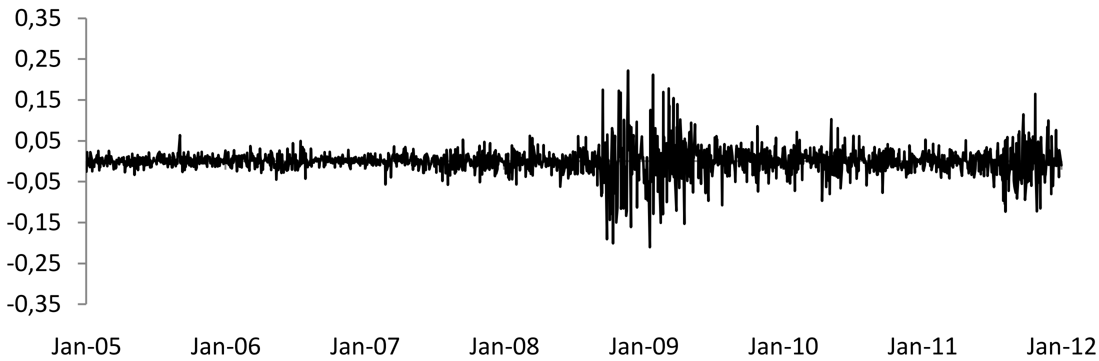


Figure 6.3: Daily Log-Returns: Apple Inc.

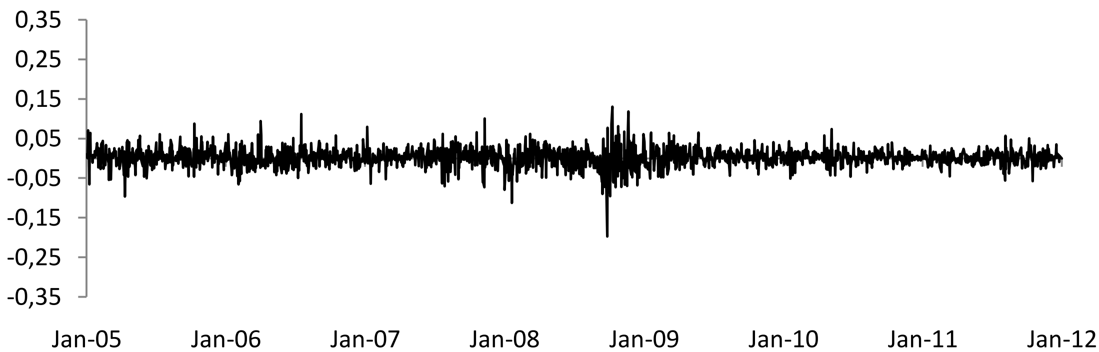


Figure 6.4: Daily Log-Returns: Cisco Systems, Inc.

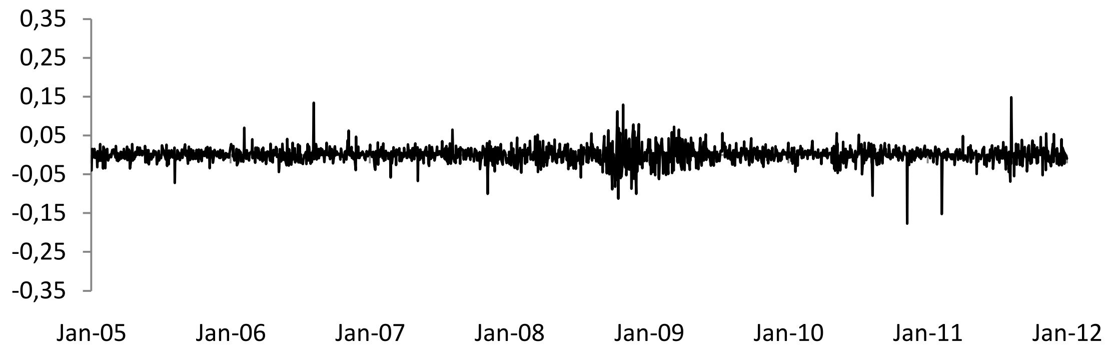


Figure 6.5: Daily Log-Returns: Vertex Pharmaceuticals Inc.

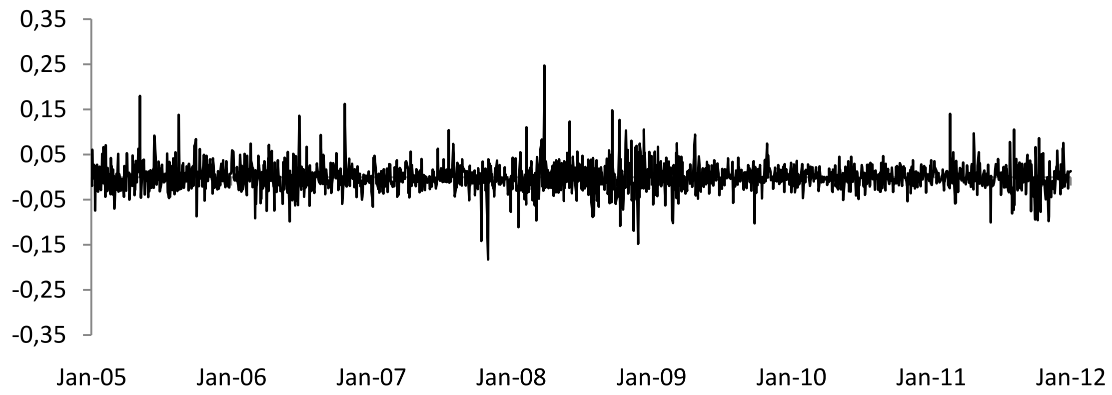


Figure 6.6: Daily Log-Returns: Texas Instruments Inc.

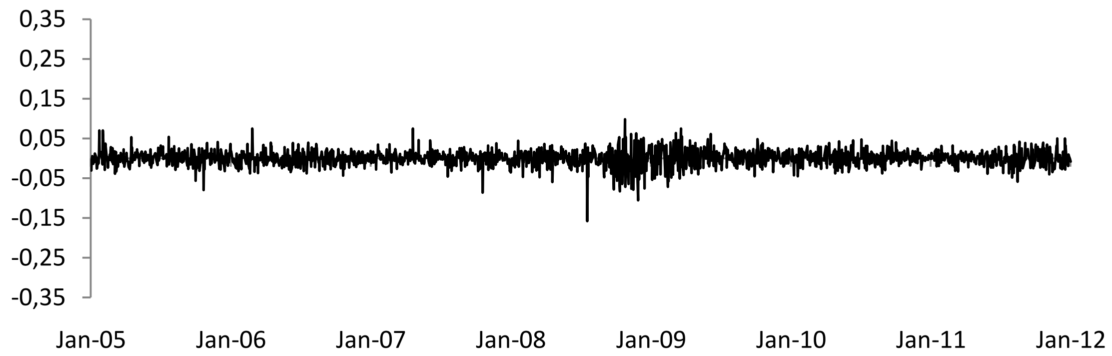


Figure 6.7: Daily Log-Returns: Starbucks Corporation

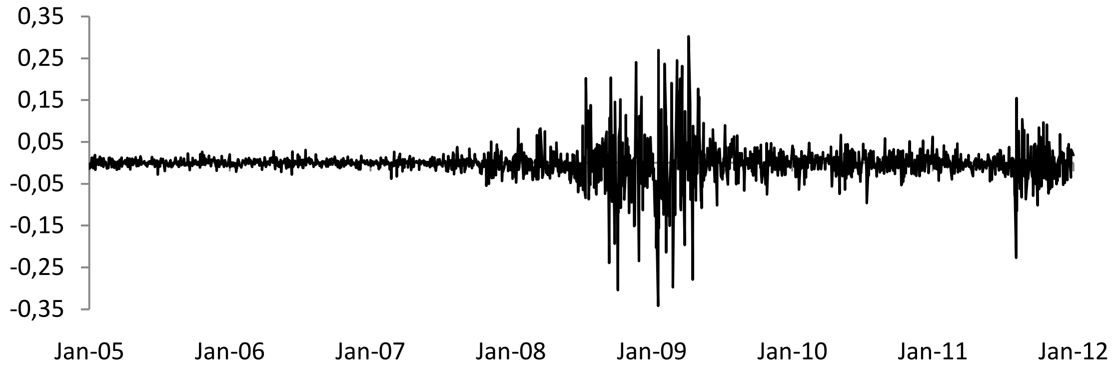


Figure 6.8: Daily Log-Returns: Wells Fargo & Company

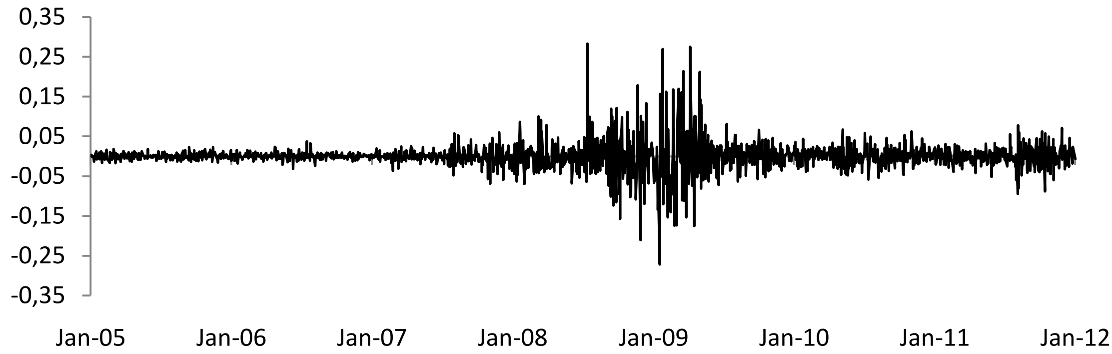
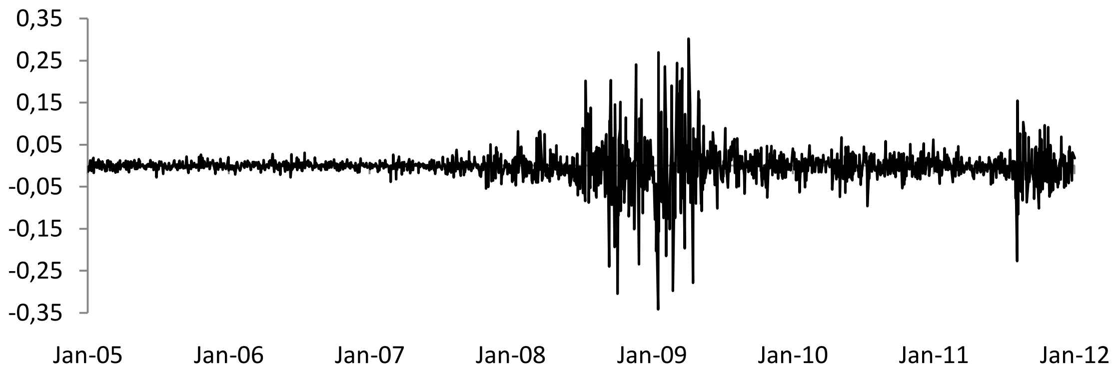


Figure 6.9: Daily Log-Returns: Bank of America Corporation



6.2.2 Q-Q Plots of Log-Returns Against Normal Distribution

Figure 6.10: Q-Q Plot: Microsoft Corporation

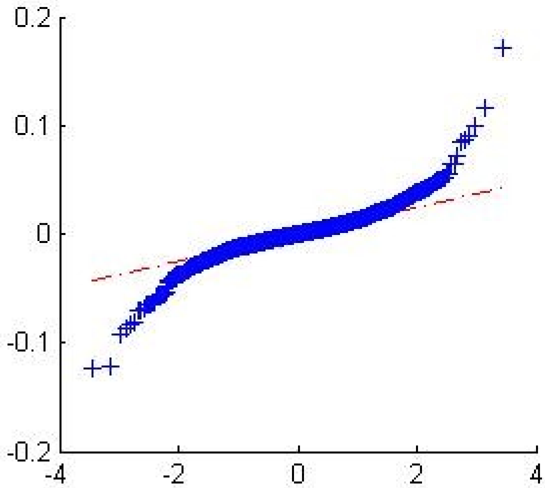


Figure 6.11: Q-Q Plot: Deutsche Bank AG

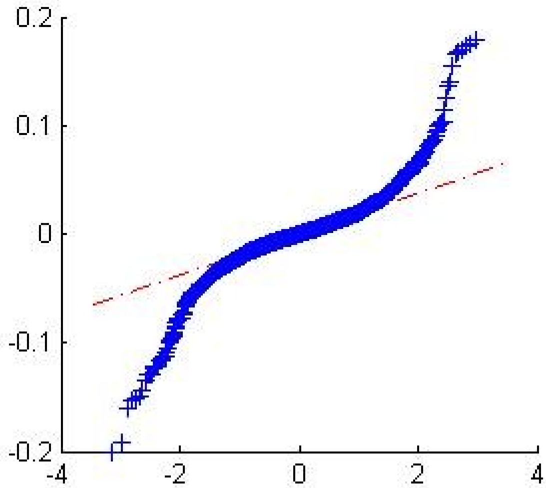


Figure 6.12: Q-Q Plot: Apple Inc.

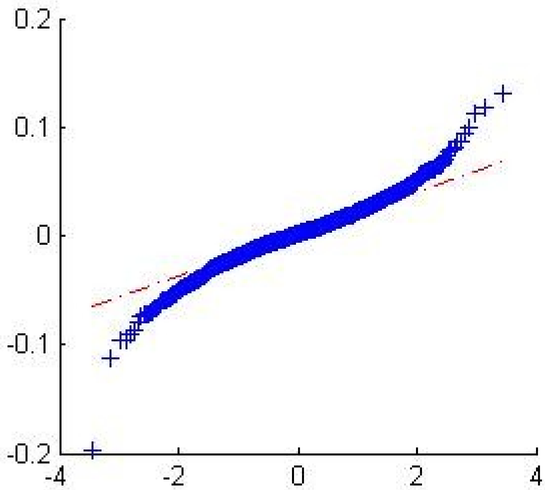


Figure 6.13: Q-Q Plot: Cisco Systems, Inc.

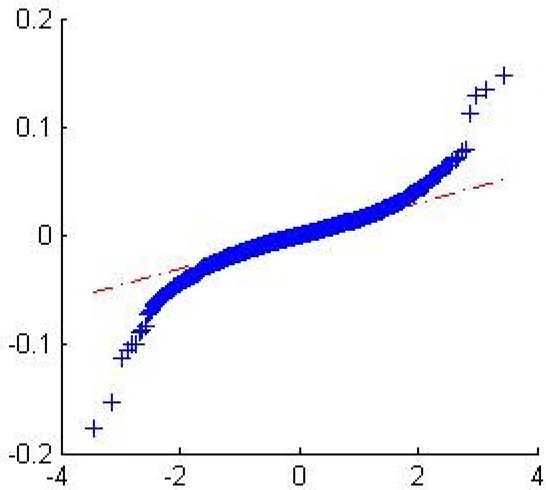


Figure 6.14: Q-Q Plot: Vertex Pharmaceuticals Inc.

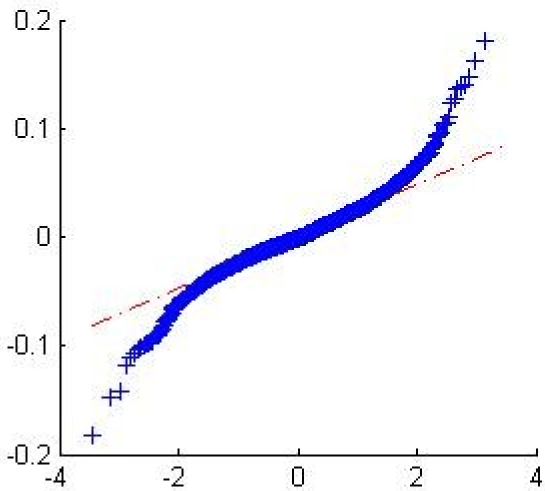


Figure 6.15: Q-Q Plot: Texas Instruments Inc.

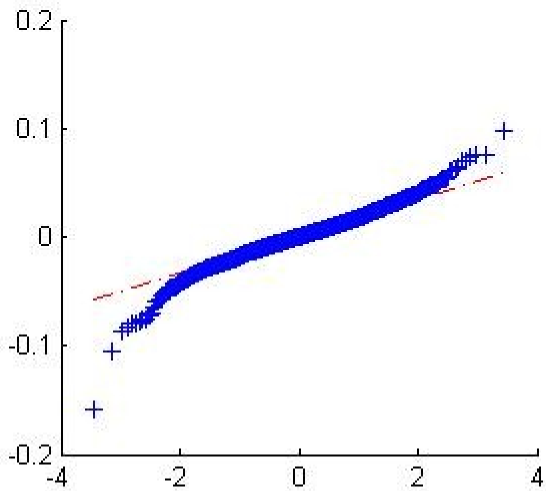


Figure 6.16: Q-Q Plot: Starbucks Corporation

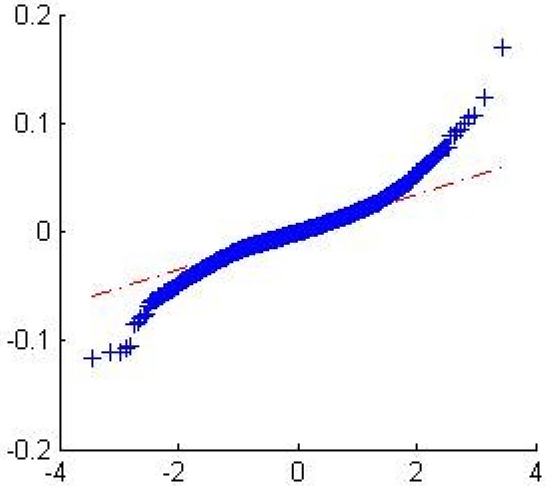


Figure 6.17: Q-Q Plot: Wells Fargo & Company

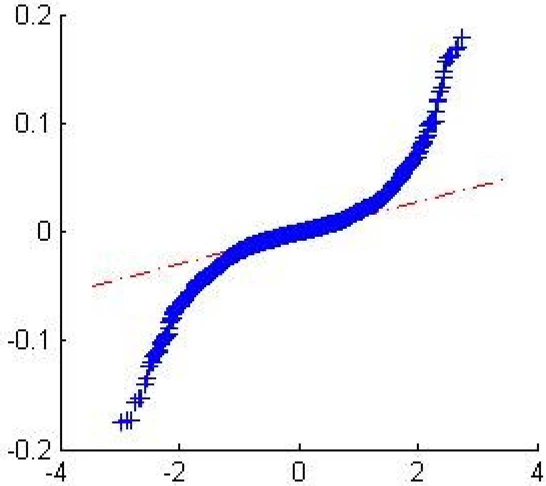
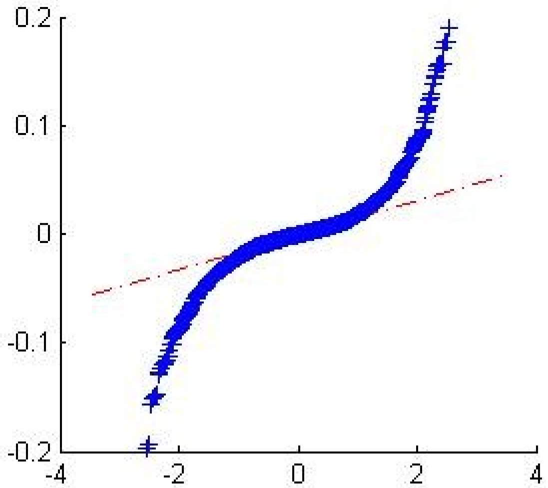


Figure 6.18: Q-Q Plot: Bank of America Corporation



6.3 Thesis Proposal

Bachelor Thesis Proposal

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Specialization	*Economics*	Defense Planned:	June 2012

Expected Title

Financial Risk Measures: Review and Empirical Applications

Topic Characteristics:

The aim of this thesis is to provide comprehensive review of currently used risk measures such as Value at Risk or Expected Shortfall; compare them with more advanced methods such Tail-Value at Risk or Conditional Tail Expectation; and analyzes their shortcomings. Selected risk measures will be compared on a referential highly liquid market data.

After the analysis of main advantages and disadvantages of particular risk measures, their combinations will be analyzed and compared. On the grounds of these analyses, we can further identify appropriate combinations of particular methods that mitigate their individual limitations and weaknesses. Than we are able to attain synergic effect and get more complex information about the risk.

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