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MASTER THESIS

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Kompaktní objekty v kategoriích modulů

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Kompaktní objekty v kategoriích modulů

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Abstrakt: V práci uvedeme základní přehled vlastností kompaktních objektů ve vhodných kategoriích ako např. kategorie modulů, stabilní faktor kategorie nad perfektním okruhem a Grothendieckovy kategorie. Najdeme okruh nad kterým je třída malých modulů za dodatečného množinově-teoretického předpokladu uzavřená na direktní součiny. Na závěr zkoumáme podmínky, kdy jsou spočetně generované projektivní moduly konečné, vyjádřené tvarem ich Grothendieckova monoidu.

Klíčová slova: kompaktní, malý modul, stabilní kategorie modulů, projektivní, samomalý

Title: Compact objects in categories of modules

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Abstract: In the thesis we state baic properties of compact objects in various appropriate categories like categories of modules, stable factor category over a perfect ring and Grothendieck categories. We find a ring R such that the class of dually slender R-modules is closed under direct products under some set-theoretic assumption. Finally, we characterize the conditions, when countably generated projective modules are finitely generated, expressed by their Grothendieck monoid.

Keywords: compact, dually slender module, stable module category, projective module, self-small

Contents

Introduction

Compactness conditions appear in various parts of topology and algebra and the general idea behind them is the possibility to deal with properties involving infinitely many objects using only some finite subset of them.

In the first chapter we focus on the category of right modules **Mod**-R over an associative ring R with identity 1_R. Here we introduce a dually slender right module as the compact object. From the categorical point it is one whose induced covariant Hom functor preserves all direct sums. It turns out that this is equivalent to not being a countably infinite sum of its own submodules. Consequently, a finitely generated R-module is a natural representative of a compactness. We will also observe the amount of dually slender modules present in a particular category of modules. If they only coincide with finitely generated ones, the category of modules seems not to be a rich supply. The situation is completely different if they are closed under arbitrary direct products (moreover, dually slender modules are closed under factormodules). The main result states that for a particular non-artinian Von Neumann regular ring this happens, taking some set-theoretic assumption consistent with Zermelo-Fraenkel set theory with the Axiom of Choice.

In the second chapter we make another generalization for right R-modules. For a module M we weaken the categorical condition for $Hom_R(M, -)$ to preserve only direct sums of copies of some fixed module N. In case $M = N$ we say that M is self-dually slender. Again we take a look on direct products of these modules.

The third chapter is a purely categorical point of view on compact objects. We state the characterization for Grothendieck categories. Then we step out of these and we describe compact objects in a stable module category over a right perfect rings. Recall that stable module category is a factor category where projective morphisms are killed.

Finally we study when countably generated projective modules are finitely generated (and therefore compact), assuming the finite generation of some factor over a submodule generated by an ideal contained in the Jacobson radical of a ring. We use the notion of Grothendieck monoid of countably generated projective modules which is a set of their isomorphism classes endowed with a commutative binary operation + imposed by taking the isomorphism class of direct sum of its arguments and zero module as a zero constant.

The very basic notation and results could be found in [AndFul92] in the first place or in [Lam99] in the second.

Chapter 1

Dually Slender Modules

The are two fundamental isomorphisms of (abelian) homomorphism groups relating the direct sum and the direct product over a family $(A_i \mid i \in I)$ of R-modules of an arbitrary cardinality I. Let M be a right R-module. The functor $Hom_R(M, -) : \mathbf{Mod}\text{-}R \to \mathcal{AB}$ preserves direct products via the canonical isomorphism:

$$\tau_1 : \prod_{i\in I} \mathrm{Hom}_R(M, A_i) \to \mathrm{Hom}_R(M, \prod_{i\in I} A_i) \tag{1.1}$$

and the contravariant functor $Hom_R(-, M) : \mathbf{Mod}\text{-}R \to \mathcal{AB}$ converts coproducts into products via the isomorphism:

$$\tau_2 : \prod_{i\in I} \mathrm{Hom}_R(A_i, M) \to \mathrm{Hom}_R(\bigoplus_{i\in I} A_i, M) \tag{1.2}$$

Let us exchange the direct product with the direct sum in the equation (1.1) and we consider the canonical mapping:

$$\rho : \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_R(M, A_\lambda) \to \mathrm{Hom}_R(M, \bigoplus_{\lambda\in\Lambda} A_\lambda) \tag{1.3}$$

defined by $\rho((\cdots, f_\lambda, \cdots))(m) = (\cdots, f_\lambda(m), \cdots)$ for $f_\lambda \in \mathrm{Hom}_R(M, A_\lambda)$, $\lambda \in \Lambda$.

The mapping ρ is always injective but we will see that generally it is not an isomorphism and therefore it will make a sense to establish the following definition:

Definition 1. *We call a right R-module M_R* dually slender *if the mapping ρ is an isomorphism, i.e. the covariant functor $Hom_R(M_R, -)$ commutes with arbitrary direct sums of modules.*

Dually slender modules are known under various names (module of type Σ, Σ-compact, $\cup$-compact and small). The notion *small module* is quite common but we will rather not use it because of the similarity with small (superfluous) submodules.

First we observe that for a dually slender R-module it is enough to consider preservations of direct sums of families of countable cardinality. Now we provide the basic characterization.

Theorem 1. *Let R be a ring. For a right R-module M the following is equivalent:*

(D1) M is dually slender,

(D2) for every countable family of R-submodules $(M_i \mid i \in \omega)$ of M such that $\sum\limits_{i\in\omega} M_i = M$ there is an $n \in \omega$ with $\sum\limits_{i=0}^{n} M_i = M$,

(D2') for every countable increasing chain of R-submodules $(M_i \mid i \in \omega)$ such that $\bigcup\limits_{i\in\omega} M_i = M$ there is an $n \in \omega$ with $M_n = M$,

(D3') for every countable family of R-modules $(A_i \mid i \in I)$ and every R-homomorphism $\varphi : M \to \bigoplus\limits_{i\in\omega} A_i$ there is an $n \in \omega$ with im $(\varphi) \subseteq \bigoplus\limits_{i=0}^{n} A_i$.

Proof. $(D1) \to (D2)$: Let $N_n := \sum_{i=0}^n M_i$. Denote by $\pi_n : M \to M/N_n$ the canonical projection and define $\varphi := \bigoplus\limits_{n<\omega} \pi_n$. Then $\varphi \in Hom_R(M, \bigoplus\limits_{n<\omega} M/N_n)$, so by (D1) φ has an inverse by ρ in $\bigoplus\limits_{n<\omega} \mathrm{Hom}_R(M, N_n)$. Because the sum is direct, there exists some $m \in \omega$ such that $M = N_m$.

$(D2) \to (D2')$: Let $M = \bigcup\limits_{n<\omega} M_n$ for an increasing chain $(M_n \mid n < \omega)$ of submodules of M. Then $M_n = \sum_{0\leq i\leq n} M_i$. By (D2) there exists $m \in \omega$ such that $M = M_m$.

$(D2') \to (D3')$: The inverse image of a submodule under R-homomorphism is a submodule so if (D3') is not true than there exists a countably infinite strictly increasing chain of $M_n := \varphi^{-1}[\bigoplus_{j\leq n}]$, $n < \omega$ such that the union of $M_n, n < \omega$ is M and (D2') is not true.

$(D3') \to (D1)$: Let $\varphi \in \mathrm{Hom}_R(M, \bigoplus_{i<\omega})$ By (D3') it follows that there is some $n < \omega$ such that $\pi_k \circ \varphi = 0$ for all $k \geq n$. Denote $\psi := \bigoplus_{j=0}^n \pi_j \circ \varphi$. Then $\psi \in \bigoplus_{i<\omega} \mathrm{Hom}_R(M, A_i)$ and $\rho(\psi) = \varphi$. ∎

As a corollary we get that the class of dually slender modules are closed under factormodules. The usual characterization of finitely generated module M is following: for every set of submodules $(M_\lambda \mid \lambda \in \Lambda)$ of M such that $\sum_\lambda M_\lambda = M$ there is a finite subset Λ_0 such that $\sum_{\lambda\in\Lambda_0} M_\lambda = M$. Now it is obvious that every finitely generated module is dually slender. Let us provide an example that the converse is not true:

We call a right R-module M *uniserial* if the submodules of M are linerly ordered by inclusion. These provide a trivial source of dually slender modules that are not finitely generated.

Example 1. Every uncountably generated uniserial module is dually slender.

Some classes of rings represent the least possible source of dually slender modules, i.e. only the finitely generated ones. We call these rings *right resp. left steady* depending on the category of modules involved. Examples of right steady rings are classes satisfying some finitness conditions. Recall that a ring R is *right perfect* if every right R-module M has a projective cover and that the following characterize them.

Fact 2. *Let R be a ring and $\mathcal{J}(R)$ be its Jacobson radical. Then R is left perfect if and only if $R/\mathcal{J}(R)$ is semisimple and $\mathcal{J}(R)$ is left T-nilpotent if and only if $M\mathcal{J}(R)$ is superfluous for every nonzero R-module M. In particular for every* *right R-module M, if the radical factor $M/RadM$ is finitely generated, then M is finitely generated.*

Proof. Proved in [[AndFul92], Lemma 28.3, Theorem 28.4(Bass)]. The additional statement follows from [[AndFul92], Corolary 15.18] which states that $M\mathcal{J}(R) = Rad(M)$. ∎

Proposition 3. *If R is a right perfect ring. Then R is right steady.*

Proof. Let M be a dually slender module and let $0 \to K \to P \to M \to 0$ be its projective cover. Assume that M is not finitely generated. Thus, P is not finitely generated and P is a countable union of strictly increasing chain of its submodules, $P = \bigcup_{i<\omega} P_i$. Because π is surjective, $M = \bigcup_{i<\omega} \pi[P_i]$ so there is some $n < \omega$ such that $M = \pi[P_n]$. Then $P = \ker \pi + P_n$. Because $\ker \pi$ is superfluous, $P = P_n$, a contradiction. ∎

Let $\mathcal{S}$ be a representative set of all simple R-modules. We say that M has a $\mathcal{S}$-*socle filtration* $(S_\alpha \mid \alpha \leq \sigma)$, if it is an increasing continuous chain of submodules of M starting with $S_0 = 0$, ending in $S(\sigma) = M$ satisfying $S_{\alpha+1}/S_\alpha$ is isomorphic to a direct sum of modules from $\mathcal{S}$ for all $\alpha < \sigma$. The ordinal σ we call *the $\mathcal{S}$-socle length* of M.

A ring is R called *right semiartinian* if R_R has a $\mathcal{S}$-socle filtration, where $\mathcal{S}$ is a representative set of simple modules. It is proved in [EGT97], Proposition 2.1, that every right semiartinian ring with the countable $\mathcal{S}$-socle length is right steady.

Proposition 4. *If R is right noetherian, then R is right steady.*

Proof. The proof is based on the fact that any category of modules has a cogenerator $K := \bigoplus_{i\in I} E(S_i)$ which is injective in this case, because injective modules are closed under direct sums in right noetherian rings and it is a characterization of them [AndFul92], Proposition 18.13. It is located in [[Ren67], 7^0]. ∎

Let us provide some definitions of cardinals that will be useful in the sequel.

Definition 2. *We say that a cardinal κ is*

- Ulam-measurable *if there is a countably complete (i.e. σ-complete) nonprincipal ultrafilter on κ*
- measurable *if there is a κ-complete nonprincipal ultrafilter on κ*
- strongly inaccesible *if $\kappa = cf(\kappa)$ and for all $\lambda < \kappa$, $2^\lambda < \kappa$*

We say that a R-module M is *ω_1-reducing* (also *countably finite* or *(ω, ω)-reducing*)if every countably generated submodule N of M is contained in a finitely generated submodule. The class of ω_1-reducing modules lies strictly between finitely generated ones and dually slender. Let us provide an example of dually slender module that is not ω_1-reducing. Recall that for an infinite set X with a discrete topology the *Čech-Stone compactification* $\beta(X)$ is a set of ultrafilters on X with the basis $\mathcal{B}$ given $\{O(Y) \mid Y \subseteq X\}$ where $O(Y) = \{p \in \beta(X) \mid Y \in p\}$. Then $\beta(X)$ with the topology generated by $\mathcal{B}$ is a compact Hausdorff topological space satisfying that the closure of any open set in $\beta(X)$ is open (so called *extremely disconnected* space) [Eng89], 6.2.28.

Lemma 5. *Let X be an infinite set with the discrete topology and let U be an ultrafilter on X. If $p = \bigcup_{i<\omega} F_i$ for an increasing chain of subfilters of p, then there is some $n < \omega$ such that $p = F_n$.*

Proof. For a contradiction assume that there is a strictly increasing chain $(F_i \mid i < \omega)$ of filters such that $p = \bigcup_{n<\omega} F_i$. For every $i < \omega$ define a set $W'_i := \{X \setminus A\} \cup F_i$ for a choise of a set A such that $A \in F_{i+1} \setminus F_i$. Then all W'_i have an finite intersection property so for all $i < \omega$ there exist an ultrafilter W_i extending W'_i such that $F_{i+1} \nsubseteq W_i$. Define:

$$\mathcal{U} := p \cup \{W_i \mid i < \omega\}$$

Then $\mathcal{U}$ is a countably inifinite subset of Čech-Stone compactification $\beta(X)$. Indeed, we show that $\mathcal{U}$ is closed in $\beta(X)$. Let $q \in \beta(X) \setminus \mathcal{U}$ and we find an open neighborough of q that is disjoint with $\mathcal{U}$. Because $\beta(X)$ is Hausdorff, there is an open subset $O(Y) \subseteq \beta(X)$ for some Y containing q but not containing p. Hence $X \setminus Y \in p$ and there is $k_0 < \omega$ such that $X \setminus Y \in F_k$ for all $k > k_0$. Then $W_i \notin O(Y)$ for every $k > k_0$. Set $\mathcal{B} := O(Y) \cap \bigcap_{i=0}^{k} \beta(X) \setminus W_i$, then $\mathcal{B}$ is an open neighborough containing q and it is disjoint with $\mathcal{U}$.

We have found a countably infinite closed subset of $\beta(X)$. Recall that every infinite closed subset of an infinite Hausdorff space contains a copy of the set of natural numbers with the discrete topology. Because it is closed it contains also a copy of $\beta(\mathbb{N})$ and by [[Eng89], Proposition 3.6.12], it has the cardinality $2^{\aleph_0}$, which leads to a contradiction. ∎

Example 2. Let K be a field. Let $\kappa > \aleph_0$ be not a Ulam-measurable cardinal.

Then there is a dually slender right K^κ-module that is ω_1-reducing.

Proof. The proof is in [Trl95]. The idea is following:

First, denote by $\mathcal{F}$ be the lattice of all filters on κ and by $\mathcal{I}$ the lattice of all two-sided ideals of K^κ. Define $\varphi : \mathcal{F} \to \mathcal{I}$ by

$$F \mapsto \{\mathbf{k} := (k_\alpha \mid \alpha < \kappa) \in K^\kappa \mid \exists\ X \in F : \pi_\alpha(\mathbf{k}) = 0\ \forall \alpha \in X\}$$

for all $F \in \mathcal{F}$. Then φ is an injective lattice homomorphism.

Let $p \in \mathcal{F}$ be a non-principal ultrafilter on κ. By Lemma 5 p is not a strictly increasing countably infinite chain of its subfilters. By [Trl95], Lemma 2.4(i), $\varphi(p)$ is dually slender. Because κ is not Ulam-measurable, the non-principal ultrafilter p is not countably complete and by [Trl95], Lemma 2.2(ii) $\varphi(p)$ is not ω_1-reducing. ∎

Now we study rings with larger classes of dually slender modules. Let us start with a question.

Question 1. *Does there exist a ring R such that dually slender right R-modules are closed under direct products? (definitely not right steady). Denote it as the condition (DS-P).*

The question has also an another reason, because an analogical statement holds in the categorically dual situation for so called *slender* modules (if the coproduct is exchanged with the product and vice versa in equation 1.2) - slender modules are closed under arbitrary direct sums.

For a ring Q we observe that every dually slender Q-module keeps this property in the module category over any subring of Q in which Q is dually slender.

Lemma 6. *Let R be a unital subring of a ring Q, M be a right Q-module and suppose that Q_R is dually slender as an R-module. Then M is a dually slender Q-module if and only if it is dually slender as an R-module.*

Proof. Assume that M is a dually slender Q-module. Let $M = \bigcup_{i<\omega} M_i$ for a countable chain of R-submodules $M_0 \subseteq M_1 \subseteq \dots$. For each $i < \omega$ define $N_i = \{m \in M \mid mQ \subseteq M_i\}$. Obviously, $N_0 \subseteq N_1 \subseteq \dots$ forms a chain of Q-submodules of M and $N_i \subseteq M_i$ for every $i < \omega$. For every $m \in M$, $(mQ)_R$ as the homomorphic image of dually slender module Q_R is dually slender so there exists $k < \omega$ such that $mQ \subseteq M_k$, hence $M = \bigcup_{i<\omega} N_i$. Now, by the assumption there exists $n < \omega$ such that $N_n = M$, hence $M_n = M$.

The converse is clear, because every Q-module is also an R-module. ∎

Proposition 7. *Let R be a subring of simple Von Neumann regular non-artinian ring Q such that Q_R is finitely generated as a right R-module.*

Then every injective Q-module is dually slender as an R-module.

Proof. By Lemma 6 it is enough to prove that every injective Q- module is dually slender (or ω_1-reducing). Let E_Q be any injective right Q-module and $E_Q = \bigcup\limits_{n<\omega} N_n$. For a contradiction assume that $(N_n \mid n \in \omega)$ is strictly increasing chain of submodules of E, i.e. $N_n \subsetneq N_{n+1}$ for all $n \in \omega$. The ring Q contains an infinite set $(e_i \mid i < \omega)$ of orthogonal idempotents and because Q is simple $Qe_nQ = Q$ for all $n < \omega$. Then $Ee_nQ = E$ and for all $n \in \omega$ there exists $x_n \in E$ such that $x_ne_nQ \nsubseteq N_n$. Define $\varphi : \bigoplus_{n<\omega} e_nQ \to E$ by $e_nq \mapsto x_ne_nq$. By Baer's Criterion applied for the injective module E_Q the following diagram commutes for some $\widetilde{\varphi}$:

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus\limits_{n\in\omega} e_nQ \arrow[r, "\iota"] \arrow[d, "\varphi"'] & Q_Q \arrow[dl, squiggly, "\widetilde{\varphi}"'] \\ & E_Q & \end{tikzcd}$$

and there is some $m \in E$ such that $\widetilde{\varphi}(q) = mq$ for all $q \in Q$. Hence $\sum_{n<\omega} x_ne_nQ$ is contained in a cyclic Q-module $mQ \subseteq E$ and therefore in N_n for some $n < \omega$, a contradiction. ∎

The class of projective modules sis not a rich source of dually slender modules. By Kaplansky's Theorem every projective module is a direct sum of countably generated projective modules [AndFul92], Corollary 26.2. and those are not dually slender unless finitely generated.

For a right R-module define:

$$Z(M_R) := \{m \in M \mid \ \mathrm{rann}_R(m) \text{ essential in } R\ \}$$

where submodule U of V is *essential* in V if $U \cap W = 0$ implies $W = 0$ for a submodule W of V.

We say that ring R is *right non-singular*, if $Z(R_R) = 0$. We observe that simple rings are an example of a class of non-singular rings and as a fact we state a deep statement about their maximal right rings of quotients.

Proposition 8. *(i) Every simple ring is right and left non-singular.*

(ii) if R is a right non-singular ring, then $Q_{\max}(R)$ the maximal right ring of quotients of R is Von Neumann regular and right self-injective.

Proof. (i) First we prove that $Z(R_R)$ is two-sided ideal. It is an abelian group because $\operatorname{rann}_R(u) \cap \operatorname{rann}_R(v) \subseteq \operatorname{rann}_R(u-v)$ for all $u, v \in R_R$ and essential right ideals are closed under finite intersections and oversets.

Let $u \in Z(R_R)$ and $r \in R$ be arbitrary. We can assume $ur \neq 0$. Let a be arbitrary and we want to prove $\operatorname{rann}_R(ur) \cap aR \neq 0$. If $ura = 0$, then there is nothing to prove so assume $ura \neq 0$. Therefore $ra \notin \operatorname{rann}_R(u)$, but $raR \cap \operatorname{rann}_R(u) \neq 0$ from the essentiality of $\operatorname{rann}_R(u)$. It follows $uras = 0$ for some $ras \neq 0$. Then $0 \neq as \in aR \cap \operatorname{rann}_R(ur)$. On the other hand, $\operatorname{rann}_R(u) \subseteq \operatorname{rann}_R(ru)$, so we are done.

Assume R is not right singular. Then $Z(R) \neq 0$ and $Z(R) = R$ by simplicity of R. But $\operatorname{rann}_R(1_R) = 0$ is not an essential right ideal, a contradiction.

(ii) Proved in [[Ste75], Proposition XII.2.1]. ∎

Lemma 9. *Let R satisfy (DS - P). Denote $Q = Q_{\max}(R)$ the maximal right ring of quotients of R.*

(i) Every injective right R-module is dually slender.

(ii) If R is a non-singular ring, then Q satisfies (DS - P).

(iii) Every factorring of R satisfies (DS - P)

Proof. (i) Let E_R be an injective R-module and let $\pi : R^{(\kappa)} \to E$ be an epimorphism. Since the canonical injection $R^{(\kappa)} \to R^\kappa$ is a monomorphism, by the injectivity of E, π can be extended to an epimorphism $R^\kappa \to E$. Because $(R_R)^\kappa$ is dually slender by the hypothesis, the module E is a homomorphic image of a dually slender module and therefore also dually slender.

(ii) By Proposition 8(ii) Q_R is injective, so by (i) it is dually slender as an R-module. Thus every product of dually slender Q-modules is dually slender as an R-module by the hypothesis and Lemma 6, hence it is a dually slender Q-module.

(iii) Modules over any factor ring have a natural structure of R-modules. ∎

Corollary 10. *If a ring R satisfies (DS - P) and I is a maximal two-sided ideal, then R/I is (right) non-singular and $Q_{\max}(R/I)$ is a non-artinian self-injective simple ring satisfying (DS - P).*

Definition 3. *A ring S is said to be* Dedekind finite *if for all $r, s \in S$, $rs = 1$ implies $sr = 1$. We say that a ring R is* right purely infinite *if R has no nonzero idempotent $e = e^2$ such that eRe is Dedekind finite.*

Lemma 11. *Let κ be an infinite cardinal, R be a non-artinian self-injective purely infinite ring and $(M_\alpha \mid \alpha < \kappa)$ be a system of R-modules.*

(i) if every M_α in the system is ω_1-reducing, then $\prod_{\alpha<\kappa} M_\alpha$ is ω_1-reducing as well.

(ii) the product of any system of finitely generated modules is ω_1-reducing.

(iii) if $\kappa = \omega$, then $\prod_{\alpha<\omega} M_\alpha / \bigoplus_{\alpha<\omega} M_\alpha$ is ω_1-reducing.

Proof. Put $M = \prod\limits_{\alpha<\kappa} M_\alpha$. For any product $\prod_\alpha M_\alpha$ denote by $\nu_\alpha : M_\alpha \to \prod_\alpha M_\alpha$ the natural embedding and $\pi_\alpha : \prod_\alpha M_\alpha \to M_\alpha$ the natural projection.

Similarly we define ν_J and π_J for any subset of $\{\alpha\}$.

(i) First we show that for every injective module M is ω_1-reducing. Because R is right purely infinite we can form a sequence $0 \to K(\simeq R^{(\omega)}) \to R$ with $K \leq R$ a right ideal of R. Choose $C = \{m_i \mid i < \omega\} \subseteq M$ arbitrary. Let $(x_i \mid i < \omega)$ be a free basis of K and define $\varphi : K \to M, x_i \mapsto m_i$. So φ is an R-homomorphism and by injectivity of M we can extend it to $\widetilde{\varphi} : R \to M$ such that $m_i = \widetilde{\varphi}(x_i) = \widetilde{\varphi}(1_R x_i) = \widetilde{\varphi}(1_R)x_i$ for all $i < \omega$, so $C \subseteq \widetilde{\varphi}(1_R)R$ is ω_1-reducing. [[Trl95], Example 2.8].

Note that $\prod\limits_{\alpha<\kappa} R^{(n_\alpha)} \cong R^\kappa$ is injective for all finite n_α, hence ω_1-reducing. Fix a countable set $D := \{m_n \mid n < \omega\} \subseteq M$. By hypothesis on M_α, for each $\alpha < \kappa$ there is some finitely generated submodule F_α of M_α such that $\{\pi_\alpha(m_n) \mid n < \omega\} \subseteq F_\alpha$ and there is some n_α such that we can write F_α as a factormodule of a finitely generated free R-module $R^{(n_\alpha)}$. Hence $D \subseteq \prod_{\alpha<\kappa} F_\alpha$ and the exact sequence $\prod_{\alpha<\kappa} R^{(n_\alpha)} \to \prod_{\alpha<\kappa} F_\alpha \to 0$ shows that the middle term is a factor-module of an ω_1-reducing R-module and it is itself ω_1-reducing. Then there exists a finitely generated R-module F of $\prod_{\alpha<\kappa} F_\alpha$ such that $D \subseteq F(\subseteq M)$.

(ii) As finitely generated R-modules are ω_1-reducing, (ii) is a consequence of (i).

(iii) Put $S = \bigoplus_{\alpha<\omega} M_\alpha$. Fix a countable set $D' := \{m_n \mid n < \omega\} \subseteq M$ and for each $\alpha < \omega$ define (a finitely generated) R-module $G_\alpha = \sum_{j\leq\alpha} \pi_\alpha(m_j)R$. Observe that $D' \subseteq \prod_{\alpha<\omega} G_\alpha$. By (ii) $\prod_{\alpha<\omega} G_\alpha$ is ω_1-reducing, hence a factor-module $\prod_{\alpha<\omega} G_\alpha + S/S$ is also ω_1-reducing. Then there exists a finitely generated module $F \subseteq \prod_{\alpha<\omega} G_\alpha(\subseteq M)$ such that $m_n + S \in F + S/S$ for all $n < \omega$. ∎

Definition 4. *For a set X, we call a system $\mathcal{I}$ of subsets of X* an (set-theoretic) ideal *if it is*

- *closed under subsets, i.e. if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.*
- *closed under finite unions, i.e. if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.*

Moreover, we call a system $\mathcal{I}$ a prime ideal *if it is an ideal and for all subsets A, B of X, $A \cap B \in \mathcal{I}$ implies $A \in \mathcal{I}$ or $B \in \mathcal{I}$. It is easy to seet that this is equivalent to statement that for every $A \subseteq X$, $A \in \mathcal{I}$ or $X \backslash A \in \mathcal{I}$. If $Y \subseteq X$ we say that the set $\mathcal{I} \mid Y := \{Y \cap A \mid A \in \mathcal{I}\}$ is* a trace *of $\mathcal{I}$ on Y. Observe that the trace of an ideal is also an ideal.*

Remark 1. Let X be a set. Then there is a dual corresponcence between ultrafilters and prime ideals on X defined by $I \mapsto \mathcal{P}(X) \backslash I$ for an ideal I.

Lemma 12. *Let R be a non-artinian self-injective purely infinite ring and let $(M_\alpha \mid \alpha \in I)$ be a family of dually slender modules. Let $M = \prod_{\alpha\in I} M_\alpha$ be the direct product of the family and let $M = \bigcup_{n<\omega} N_n$ be a union of a countable increasing chain of submodules $(N_n \mid n < \omega)$. Denote $\mathcal{A}_n = \{J \subseteq I \mid \prod\limits_{\alpha\in J} M_\alpha \subseteq N_n\}$ and $\mathcal{A} = \bigcup\limits_{n<\omega} \mathcal{A}_n$. Assume M is not dually slender. Then the following holds:*

(i) $\mathcal{A}_n$ is an (set-theoretic) ideal

(ii) $\mathcal{A}$ is closed under countable unions of sets

(iii) There exists $n<\omega$ for which $\mathcal{A}=\mathcal{A}_n$.

(iv) there exists a subset $I_0 \subseteq I$ such that the trace of $\mathcal{A}$ on I_0 is prime

Proof. (i) Obviously $\emptyset \in \mathcal{A}$ and because M is not dually slender, $I \notin \mathcal{A}$. The closure of $\mathcal{A}_n$ under subsets is obvious by the definition. The closure of $\mathcal{A}_n$ under finite unions follows from the decomposition $\prod_{\alpha\in J\cup K} M_\alpha = \prod_{\alpha\in J} M_\alpha \oplus \prod_{\alpha\in K\setminus J} M_\alpha \subseteq N_n$.

(ii) First we show that $\mathcal{A}$ is closed under countable unions of pairwise disjoint unions. Let $K_j \in \mathcal{A}$ be pairwisely disjoint subsets of I for all $j<\omega$. We show that there exists $k<\omega$ such that $K_j \in \mathcal{A}_k$ for each $j<\omega$. Assume by contradiction that for every $n<\omega$ there exists (possibly distinct) $i(n)$ such that $K_{i(n)} \notin \mathcal{A}_n$. Hence there is $f_n \in \prod_{\alpha\in K_{i(n)}} M_\alpha$ for which $\nu_{K_{i(n)}}(f_n) \notin N_n$. Since $\prod_{j<\omega} f_j R = \bigcup_{n<\omega}(f_j R\cap N_j)$ is dually slender by Lemma 11(ii) there is $k<\omega$ such that $\nu_{K_{i(k)}}(f_k) \in \prod_{j<\omega} f_j R \subseteq N_k$, a contradiction.

Put $P_j = \prod_{\alpha\in K_j} M_\alpha$ for $j<\omega$. Observe that there is some $k<\omega$ such that $P_j \subseteq N_k$ and it follows that $\bigoplus_{j<\omega} P_j \subseteq N_k$. Let $P=\prod_{j<\omega} P_j = \prod\{M_\alpha \mid \alpha \in \bigcup_{j<\omega} K_j\}$ be a countably generated module. As $P/\bigoplus_{j<\omega} P_j$ is dually slender by Lemma 11(iii) there exists some $l\geq k$ such that $P=\bigcup_{j<\omega}(P\cap N_j)\subseteq N_l$.

Now let J_j, $j<\omega$ be any subsets of I and put $J_0=K_0$ and $J_i = K_i \setminus \bigcup_{j<i} K_j$ for $i>0$. So $\bigcup_{j<\omega} J_j = \bigcup_{j<\omega} K_j$ and by the preceding we get the result.

(iii) Assume that $\mathcal{A} \neq \mathcal{A}_n$. Then there exists a sequence $(J_n \in \mathcal{A}\setminus\mathcal{A}_n \mid n\in\omega)$. Since $\bigcup_{j<\omega} J_j \in \mathcal{A}$, we obtain a contradiction with (ii).

(vi) There exists $I_0 \subseteq I$ such that for every $K \subseteq I_0, K \in \mathcal{A} or I_0\setminus K \in \mathcal{A}$. Assume that such I_0 does not exist. Then we may construct a countably infinite sequence of disjoint sets $(K_i \mid i<\omega)$ (K_i non-empty for $i>0$) in the following way: Put $K_0=\emptyset$ and $J_0=I_0$. There exist disjoint sets $J_{i+1}, K_{i+1} \subset J_i$ such that $J_i = J_{i+1}\cup K_{i+1}$ where $J_{i+1}, K_{i+1} \notin \mathcal{A}$. Now, for each $n\geq 1$ there exists $g_n \in \prod_{\alpha\in K_n} M_\alpha$ such that $\nu_{K_n}(g_n) \notin N_n$ which contradicts to the fact that $\prod_{j\geq 1} g_j R \subseteq N_m$ for some $m<\omega$ (cf. the proof of (iii)). ∎

Proposition 13. *Let R be a non-artinian self-injective purely infinite ring. Then the following holds:*

(i) A countable product of dually slender R-modules is dually slender.

(ii) If there exists a system $(M_\alpha \mid \alpha<\kappa)$ of dually slender R-modules such that the product $\prod_{\alpha<\kappa} M_\alpha$ is not dually slender, then there exists an uncountable cardinal $\lambda<\kappa$ and an σ-complete ultrafilter on λ.

Proof. (i) Follows immediatelly from Lemma 12(iii).

(ii) If we take $M=\prod\limits_{\alpha\in I} M_\alpha$ which is not dually slender and a set-theoretic ideal $\mathcal{A}$ and $I_0\subseteq I$ from Lemma 12 and if we define $\mathcal{F}=\{I_0\setminus A \mid A\in\mathcal{A}\}$ then from Lemma 12(i),(iii) it follows that $\mathcal{F}$ is an ultrafilter and by Lemma 12(ii) it is σ-complete. ∎

The relation to the set theory is established in the following proposition. Recall that Goedels Second Incompleteness Theorem states that a consistent axiomatizable theory containing a fragment of arithmetic (like ZFC set theory) does not prove its own consistency.

Proposition 14. *The following is true:*

(i) every Ulam-measurable cardinal is greater or equal to the first measurable cardinal

(ii) every measurable cardinal is strongly inaccesible.

(iii) it is consistent with ZFC that there is no strongly inaccesible cardinal

Proof. (i) because every cardinal greater then Ulam-measurable is also Ulam-measurable, it is enough to show that the first Ulam-measurable cardinal is measurable. Denote it κ and let $p \in \mathcal{U}(\kappa)$ be a nonprincipal countably complete ultrafilter on κ.

Assume that p is not κ-complete. Then there is $\mu_0 < \kappa$ and a partition $(A_\alpha \mid \alpha < \mu_0)$ such that $A_\alpha \notin p$ for all A_α, $\alpha < \mu_0$. Indeed, let

$$\mathcal{A} := \{\mu < \kappa \mid \exists \, \{C_\alpha \in p \mid \alpha < \mu\} \text{ such that } \bigcap_{\alpha<\mu} C_\alpha \notin p\}$$

Then by the assuption $\mathcal{A}$ is nonempty and therefore it has the smallest element, denote it μ_0. Because $C_{\alpha'} \setminus \bigcap_{\alpha<\mu_0, \alpha \neq \alpha'} C_\alpha \in p$ for all $\alpha' < \mu_0$ we can assume $\bigcap_{\alpha<\mu_0} C_\alpha = \emptyset$. Define $B_0 := \kappa$ and $B_\alpha := \bigcap_{\beta<\alpha} C_\beta$ for all $0 < \alpha < \mu_0$ and set $A_\alpha := B_\alpha \setminus B_{\alpha+1}$ for all $\alpha < \mu_0$. Then the system $(A_\alpha \mid \alpha < \mu_0)$ forms a partition of κ (because sets B_α form a chain) and $A_\alpha \notin F$ for all $\alpha < \mu_0$, because $B_{\alpha+1} \in F$ by minimality of μ_0 and so $\kappa \setminus B_{\alpha+1}$ (an overset of A_α) is not in F by the ultrafilter property.

Define a function $f : \kappa \to \mu_0$ that maps very element of κ to the index of a member of the partition it belong to, i.e. $\beta \mapsto \alpha$ if $\beta \in A_\alpha$ for all $\beta < \kappa$. Observe that $f[F]$ is a nonprincipal countably complete ultrafilter on μ_0, a contradiction with minimality of κ.

(ii) let κ be measurable and let F be a nonprincipal κ-complete ultrafilter on κ.

First assume that $cf(\kappa) < \kappa$ and let $(\kappa_\alpha \mid \alpha < cf(\kappa))$ be a cofinal sequence of cardinals in κ. If $\kappa \setminus \kappa_\alpha \in F$ for every $\alpha < cf(\kappa)$, then $\emptyset = \bigcap_{\alpha<cf(\kappa)} \kappa \setminus \kappa_\alpha \in F$ because F is κ-complete. So there is some $\alpha < cf(\kappa)$ such that $\kappa_\alpha \in F$. Obviously $\bigcap_{\beta<\kappa_\alpha} \kappa_\alpha \setminus \{\beta\} = \emptyset$. Since F is nonprincipal, $\kappa_\alpha \setminus \{\beta\} \in F$ for all $\beta < \kappa_\alpha$ (otherwise $\{\beta\} = k_\alpha \cap (\kappa \setminus \kappa_\alpha \cup \{\beta\}) \in F$) and since F is κ-complete, $\bigcap_{\beta<\kappa_\alpha} \kappa_\alpha \setminus \{\beta\} \in F$.

Let $\lambda < \kappa$ be a cardinal such that $\kappa < 2^\lambda$. Then there exists an injective function $F : \kappa \to 2^\lambda$. For every $i < 2$ and every $\gamma < \lambda$ define:

$$A^i_\gamma := \{\alpha < \kappa \mid F(\alpha)(\gamma) = i\}$$

Observe that $(A^0_\gamma \in p$ or $(A^1_\gamma \in p$ for every $\gamma < \lambda$. Pick the function $g \in^\lambda 2$ such that $A^{g(\gamma)}_\gamma \in p$ for all $\gamma < \lambda$. Because κ is κ-complete $\bigcap_{\gamma<\lambda} A^{g(\gamma)}_\gamma \in p$ and

$$\bigcap_{\gamma<\lambda} A^{g(\gamma)}_\gamma = \{\alpha < \kappa \mid F(\alpha)(\gamma) = g(\gamma) \forall \gamma < \lambda\} =$$

$$= \{\alpha < \kappa \mid F(\alpha) = g\} = F^{-1}[g]$$

By injectivity of F it follows that the intersection of all $A_\gamma, \gamma < \lambda$ is a singleton and p is not a principal ultrafilter, a contradiction.

(iii) Define $V_0 := \emptyset$ and by induction $V_{S(\alpha)} := \mathcal{P}(V_\alpha)$ for a successor ordinal $S(\alpha)$ and $V_\alpha := \bigcup_{\beta<\alpha} V_\beta$ for a limit ordinal α. The proof will follow from the fact ([Jec97], Lemma 12.13) that V_κ is a model of ZFC for a strongly inaccesible cardinal κ. Let κ be a strongly inaccesible cardinal.

Let *Inac* denote a statement "there is a strongly inaccesible cardinal". Assume the consistency of ZFC implies the consistency of ZFC + Inac. By ([Jec97], Lemma 12.13) the theory ZFC + Inac proves the existence of a model V_κ of ZFC and therefore a consistency of ZFC. So we have that ZFC + Inac is consistent. But we got that ZFC + Inac proves its own consistency, a contradiction with Goedel's Second Incompleteness Theorem. ∎

Using Proposition 14(iii) and Corolary 13 we can state our main result:

Corollary 15. *Let R be a non-artinian self-injective, purely infinite ring.*

If we assume that there is no inaccesible cardinal, then the class of dually slender R-modules is closed under direct products.

Chapter 2

Self-Dually Slender Modules

It this chapter we will step even farther from finitely generated modules. Recall that the $\mathrm{Hom}_R(M, -)$ functor with a dually slender module M commutes with all direct sums. If we weaken the property such that we want the canonical isomorphism only with all direct sums of some fixed module N we get a generalization of dually slender modules and we will call the module M *a N-dually slender module.* In the case $M = N$ we speak about self-dually slender (or *self-small*) modules and they are precisely the compact objects in the category of $Add(M)$.

Let M be a R-module. For a subset $X \subseteq M$ define $X^* := \{f \in End_R(M) \mid f[X] = 0\}$ and for a subset $D \subseteq E$ the set $D^* := \{m \in M \mid f(m) = 0 \text{ for all } f \in D\}$. Obviously, X^* is a left ideal of E, D^* is a submodule of M and $X \subseteq X^{**}$ and $D \subseteq D^{**}$. We call a left ideal $L \leqslant End_R(M)$ *an annihilator left ideal* if $L = L^{**}$ and a submodule N of M *a kernel submodule* if $N = N^{**}$. We say that a system of morphism $(f_\lambda \in Hom_R(M, M_\lambda) \mid \lambda \in \Lambda)$ is *summable* if for every $m \in M$ there is only finitely many $\lambda \in \Lambda$ such that $f_\lambda(m) \neq 0$.

Theorem 16. *Let R be a ring, M a right R-module. The following is equivalent:*

(1) M is self-dually slender

(2) every summable family $S \subseteq End_R(M)$ of endomorphisms of M is finite

(3) for every decreasing chain of left annihilator ideals $(L_i \mid i \in \omega)$ of $End(M)$ such that $M = \bigcup_{i<\omega} L_i^*$ there is an $n \in \omega$ such that $L_n = 0$

(4) for every countable increasing chain of kernel submodules $(K_i \mid i \in \omega)$ of M such that $\bigcup_{i\in\omega} K_i = M$ there is an $n \in \omega$ with $K_n = M$

Proof. (1) $\to$ (2) : assume there is an infinite summable system $S \subseteq End_R(M)$ and define a mapping $\varphi : M \to \bigoplus_{\alpha\in S} M_\alpha$ by $m \mapsto \sum_{\alpha\in S} \alpha(m)$. The system S is summable so φ is a correctly defined homomorphism. But $\operatorname{im}\varphi \not\subseteq \bigoplus_{\alpha\in S_0} M_\alpha$ for any finite subset $S_0 \subseteq S$ and M is not self-dually slender.

(2) $\to$ (3): for every $i \in \omega$ choose some $f_i \in L_i \setminus L_{i+1}$ or $f_i = 0$ if no such exists. This forms a system $S := (f_i \mid i \in \omega)$. Choose $m \in M$. Then there exists some $n \in \omega$ such that $m \in L_n^*$ and so $f(m) = 0$ for all $f \in L_{n'}$, for all $n' \geq n$. By (2) S is finite, so only finitely many nonzero f_i were posible to choose and $L_l = L_{l+1}$ for some l. But this should be zero, because every element of M could

be otherwise annihilated by a nonzero function from the least annihilator ideal, a contradiction with the assumption that M is union of the sets L_i^*.

(3) → (4): Let $(K_i \mid i \in \omega)$ is an increasing chain of kernel submodules such that $A = \bigcup_{i\in\omega} K_i$; Then $(K_i^* \mid i \in \omega)$ is a decreasing chain of left annihilator ideals. By (3) there is some $n \in \omega$ such that $K_n^* = 0$. But K_n is a kernel submodule, so there is some $L \subseteq End_R(A)$ such that $L^* = K_n$. Then $L^{**} = K_n^* = 0$, so $L = 0$ and it follows that $L^* = M$.

(4) → (1): Assume M is not self-dually slender. Then there is some $\varphi : M \to \bigoplus_{i\in\omega} A$ such that $\pi_i \circ \varphi \neq 0$ for all $i \in \omega$. Define $A_i := \{x \in M \mid \pi_i \circ \varphi \neq 0 \text{ for all } j > i\ \}$. Then $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots A_n \subseteq \cdots$ is an increasing chain of submodules of M such that $M = \bigcap_{i\in\omega} A_i$. Because $A_n \subseteq A_n^{**}$, we get an increasing chain of kernel submodules such that $M = \bigcap_{i\in\omega} A_i^{**}$. By (4) then there is some $n \in \omega$ with $A_n^{**} = M$. Then $A_n^* = 0$, a contradiction with the strictly increasing chain of A_i. ∎

It is known that a countable generated self-dually slender projective R-module is necessary finitely generated.

Here we have a sufficient condition (which is relatively strong) when the product of a system of self-dually slender modules stays self-dually slender.

Proposition 17 ([Zem08], Proposition 1.6). *Let $(M_i \mid i \in I)$ be a system of self-dually slender modules satisfying the condition $Hom_R(\prod_{j\in I\setminus\{i\}} M_j, M_i) = 0$ for every $i \in I$. Then $\prod_{i\in I} M_i$ is self-dually slender.*

Proof. Denote the product of the system by M and suppose $M = \bigcup_{n\in\omega} N_n$ for an increasing chain of submodules of M.

For each $i \in I$ consider the set $S_i := \{n \in \omega \mid f[\pi_i[N_n]] = 0 \text{ implies } f = 0\}$. Observe that $M_i = \bigcup_{n\in\omega} \pi_i[N_n]$. Then $M_i = \bigcup_{n\in\omega} (\pi_i[N_n])^{**}$ and $(\pi_i[N_n]^* \mid n \in \omega)$ is a decreasing chain of left annihilator ideals. By assumption then there is $n \in \omega$ such that $\pi_i[N_n]^* = 0$, S_i is nonempty and has the least element. Form a sequence of integers $(n_i \mid i \in I)$.

If it is unbounded, by minimality of its elements, for every $n \in \omega$ there is some $i_n \in I$, $i_n < n_i$ with $(\pi_{i_n}[N_n])^* \neq 0$. But then $\pi_{i_n}[N_n] \neq M_{i_n}$ so we can choose $m \in M$ such that $m \notin N_n$ for every $n \in \omega$, a contradiction.

So the sequence is bounded with an upper bound say n. We consider $\prod_{i\in I\setminus\{j\}} M_i$ as a submodule of $\prod_{i\in I} M_i$. Let $\varphi \in End_R(M)$ be such that $\varphi[N_n] = 0$. We want to show that $\varphi = 0$. By assumption, for every $i \in I$, $\pi_i \circ \varphi[\prod_{i\in I\setminus\{j\}} M_i] = 0$. Let us observe that for every $m \in M$, $m - (\iota_j \circ \pi_j)(m) \in \prod_{i\in I\setminus\{j\}} M_i$, so $(\pi_i \circ \varphi) \circ (\iota_i \circ \pi_i(m) - m) = 0$ for every $m \in M$, i.e. $\pi_i \circ \varphi = \pi_i \circ \varphi \circ \iota_i \circ \pi_i$. Since for every $i \in I$, $(\pi_i[N_n])^*$ and $\pi_i \circ \varphi \circ \iota_i \in End_R(M_i)$, it follows $\pi_i \circ \varphi \circ \iota_i = 0$ for every $i \in I$, hence $\pi_i \circ \varphi = 0$ for every $i \in I$ and finally it implies $\varphi = 0$. ∎

Remark 2. Let $\mathcal{S}$ be a *co-abstract* set of simple R-modules, i.e. all members of $\mathcal{S}$ are pairwisely non-isomorphic. Let $T \in \mathcal{S}$ and denote $P := \prod_{S\in\mathcal{S}\setminus\{T\}} S$. If R is a principle ideal domain then $Hom_R(P,T) = 0$. Indeed, let $f \in Hom_R(P,T)$. It is known that annihilator of a module is an ideal hence $ann_R(T) = Ra$ for some $a \in R$. Fix an arbitrary simple R-module $S \in \mathcal{S}$, $S \not\simeq T$. Then $ann_R(S) \nsubseteq ann_R(T)$ and $ann_R(T) \nsubseteq ann_R(S)$ so $Sa \neq 0$. Because S is simple, $S = Sa = SaR$. Finally $P = PaR = Pann_R(T) \subseteq \ker f$, $f = 0$.

Now let us provide an example of a self-dually slender module that is not dually slender. Denote by $\mathbb{P}$ the set of all positive primes and let $(\mathbb{Z}_p \mid p \in \mathbb{P})$ be a system of abelian groups. Then $Hom_{\mathbb{Z}}(\prod_{p\in\mathbb{P}\setminus q} \mathbb{Z}_p, \mathbb{Z}_q) = 0$ by the previous part. By previous theorem $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is self-dually slender $\mathbb{Z}$-module that is not dually slender, because $\mathbb{Z}$ is noetherian and hence (right) steady.

Chapter 3

Compact objects in Grothendieck Categories

Let $\mathcal{C}$ be a category. A category is *locally small* if for every object $C \in \mathcal{C}$ the class of subobjects is a set. We say that $\mathcal{C}$ is an *additive* category if it is locally small with an abelian group operation on $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for all pairs if objects A,B of $\mathcal{C}$ such that composition is biadditive:

$$(\alpha + \beta) \circ \gamma = \alpha \circ \gamma + \beta \circ \gamma$$
$$\alpha \circ (\gamma + \delta) = \alpha \circ \gamma + \alpha \circ \delta$$

whenever the sums and compositions are defined.

The category $\mathcal{C}$ is an *abelian* if it is an additive category with a zero object, biproducts, kernels, and cokernels, in which every monomorphism is a kernel and every epimorphism is a cokernel.

We call a cocomplete abelian category $\mathcal{C}$ *Grothendieck category* if direct limits are exact in $\mathcal{C}$ and $\mathcal{C}$ has a generator. A Grothendieck category is an example of a locally small category, because it contains a generator and by [Ste75], Proposition IV.6.6 every such a category is locally small. We say that a category is *pseudo-complemented* if for every object $C \in \mathcal{C}$ the lattice of subobjects of C is pseudocomplemented, i.e. for every subobject A of C there is a subobject M of C such that $A \cap M = 0$ and $A \cap B = 0$ implies $B \leq M$ for every subobject B of C. A Grothendieck category is an example of a pseudo-complemented category by [Ste75], Proposition III.6.3.

Let $\mathcal{C}$ be an abelian category with arbitrary coproducts. We call an object M of $\mathcal{C}$ *dually slender* if the functor $Hom_{\mathcal{C}}(M, -)$ preserves arbitrary direct sums. By [Ste75], Exercise V.3.13 it is enough to consider only countable direct sums. Let N be an object of $\mathcal{C}$. It is said that an object M in $\mathcal{C}$ is N-*dually slender* if the functor $Hom_{\mathcal{C}}(M, -)$ preserves direct sums of N. For $M, N \in \mathcal{C}$ and any X a subobject of M denote $V_{M,N}(X) = \{f \in \mathrm{Hom}_{\mathcal{C}}(M, N) \mid X \subseteq \ker f\}$. A subobject X_0 of X is *essential* if $X_0 \cap Y = 0$ implies $Y = 0$ for every Y subobject of X. We say that a monomorphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is *essential* if $\mathrm{im}\, \alpha$ is an essential subobject in Y. Recall that an essential monomorphism $\mu : 0 \to M \to Q$ with Q injective is called *injective envelope* of M. The following two facts imply the existence of injective envelopes in Grothendieck categories.

Fact 18 ([Ste75], Proposition V.2.5)**.** *Let $\mathcal{C}$ be a locally small, pseudocomplemented abelian category. Then if C is a subobject of an injective object then C*

has an injective envelope.

Fact 19 ([Ste75], Proposition X.4.3)**.** *Let $\mathcal{C}$ be a Grothendieck category. Then every object is a subobject of an injective object.*

Now we are able to formulate a generalization of compact objects for Grothendieck categories.

Proposition 20. *Let $\mathcal{C}$ be a Grothendieck category and let M be an object of $\mathcal{C}$ Then M is dually slender if and only if M is Q-dually slender for every injective objects Q in $\mathcal{C}$.*

Proof. Let $(A_i \mid i < \omega)$ be a countable family of objects of $\mathcal{C}$ and denote $A := \prod_{i<\omega} A_i$. Let $f \in \mathrm{Hom}_{\mathcal{C}}(M, A)$ be a morphism. Because $\mathcal{C}$ is a Grothendieck category, for every $i < \omega$ there exists an injective envelope $\nu_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, E(A_i))$. Let $Q := E(A)$ and let $\mu_i \in \mathrm{Hom}_{\mathcal{C}}(E(A_i), Q)$ be the canonical inclusion for every $i < \omega$. By the assuption the morphism $\bigoplus_{i<\omega} \nu_i \circ \bigoplus_{i<\omega} \mu_i \circ f \in \mathrm{Hom}_{\mathcal{C}}(M, Q^{(\omega)})$ factors through a finite coproduct Q^n for some $n < \omega$ and so $\mathrm{im}\ (\bigoplus_{i<\omega} \mu_i \circ f) \subseteq \bigoplus_{i<n} E(A_i)$. By essentiality of the morphism ν_i, $i < \omega$ it follows that f factors through a finite coproduct, which concludes the proof. ∎

Proposition 21. *Let $\mathcal{C}$ be a Grothendieck category and let M, N be objects of $\mathcal{C}$. Then*

(1) M is N-dually slender

(2) for every increasing chain $M_0 \subseteq M_1 \subseteq M_2 \cdots \subseteq M$ of proper subobjects of M, either $\sum\limits_{i\in\omega} M_i$ is proper subobject of M or there is some $n < \omega$ such that $V_{M,N}(M_n) = 0$.

Proof. (1) $\to$ (2): let $M_0 \subseteq M_1 \subseteq M_2 \cdots \subseteq M$ be an increasing chain of proper subobjects of M such that $\sum\limits_{i<\omega} M_i = M$ and $V(M_i) \neq 0$ for all $i < \omega$. Then for every $i \in \omega$ exists a nonzero homomorphism $f_i \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ such that $f_i[M_i] = 0$. Let $\varphi \in \mathrm{Hom}_{\mathcal{C}}(M, N^{\omega})$ be a morphism such that $\pi_i \circ \varphi = f_i, i < \omega$ where $\pi_k \in \mathrm{Hom}_{\mathcal{C}}(N^{(\omega)}, N)$ is a natural projection. Since $f_k[M_n] = 0$ for all $k \geq n$, it follows $(\pi_k \circ \varphi)[M_n] = 0$ for all $k \geq n$ and $\mathrm{im}\ \varphi \subseteq N^{(\omega)}$. Then there is a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{C}}(M, N^{(\omega)})$ such that $\pi_i \circ \varphi = f_i, i < \omega$. By (1) f factors through a finite coproduct and so there exists some $n < \omega$ such that $\pi_k \circ \varphi = 0$ for all $k \geq n$, a contradiction.

(2) $\to$ (1): assume there exists $f \in \mathrm{Hom}_{\mathcal{C}}(M, N^{(\omega)})$ such that $\pi_i \circ f \neq 0$ for all $i \in \omega$, where $\pi_i \in \mathrm{Hom}_{\mathcal{C}}(N^{(\omega)}, N)$ is a natural projection. Define $M_n := \bigcap\limits_{i<\omega} \ker(\pi_i \circ f)$. Then $M_0 \subseteq M_1 \subseteq M_2 \cdots$ is an increasing chain of subobjects of M. Observe that if X is a finitely generated subobject of M, then there exists $n < \omega$ with $(\pi_k \circ f)[X] = 0$ for every $k \geq n$ and so $X \subseteq \bigcap\limits_{i>n} \ker(\pi_i \circ f)$. Hence $M = \sum\limits_{i<\omega} M_i$. ∎

An easy example of a Grothendieck category is the category of R-modules over a ring R. But the previous theorem provides a characterization of dually slender objects also for another Grothendieck categories like the category of all abelian p-groups (with group homomorphisms between its objects) or the functor category between an additive category and abelian groups ([Ste75], V.1. Examples).

Let $\mathcal{C}$ be an additive category and let $\mathcal{T}$ be a full additive subcategory. The stable category $\underline{\mathcal{C}}$ obtained from $\mathcal{C}$ is the category whose objects are the same as $\mathcal{C}$ and whose morphisms $\mathrm{Hom}_{\underline{\mathcal{C}}}(X,Y)$ are the equivalence classes with equivalence defined on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ such that two morphisms are equivalent if their difference factors through an object of $\mathcal{T}$.

Recall that a pushout (in a category theory) (P,u,v) of (B,f) and (C,g) is a colimit of the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, rightsquigarrow, "u"] \\
C \arrow[r, rightsquigarrow, "v"] & P
\end{tikzcd}$$

Later we will need to know how epimorphisms look in a stable category $\underline{\mathcal{C}}$. The next lemma shows that they can be represented by epimorphisms of $\mathcal{C}$. For a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ we denote f^k the kernel of f.

Lemma 22. *Let $f \in Hom_{\mathcal{C}}(X,Y)$ be an epimorphism in an abelian category $\mathcal{C}$ and assume that also $\overline{f} \in Hom_{\underline{\mathcal{C}}}(X,Y)$ is also an epimorphism. Then for every $h \in Hom_{\mathcal{C}}(X,T)$ with $T \in \mathcal{T}$ there is a morphism $\widetilde{h} \in Hom_{\mathcal{C}}(X,\ im\ (h \circ f^k))$ such that h and $\widetilde{h}$ coincide on* $\ker(f)$

Proof. Let (P,u,v) be a pushout of (h,T) and (f,Y) by X. Then $\overline{0} = \overline{u} \circ \overline{h} = \overline{v} \circ \overline{f}$ in $\underline{\mathcal{C}}$. Because $\overline{f}$ is an epimorphism it follows that $\overline{v} = 0$ and v factors through some $T' \in \mathcal{T}$, say via v_2 and v_1. By projectivity of T' we get a homomorphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(T',T)$ such that $u \circ \varphi = v_2$.From the equality $v = v_2 \circ v_1 = (u \circ \varphi) \circ v_1$ we get $0 = u \circ h - v \circ f = u(h - \varphi \circ v_1 \circ f)$. Denote $h' := h - \varphi \circ v_1 \circ f$. From the universal property of the kernel of u we get a morphism $\widetilde{h} \in \mathcal{C}(X, \ker(u))$ such that $u^k \circ \widetilde{h} = h'$:

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r, "f^k"] & X \arrow[rr, "f"] \arrow[dd, Rightarrow, bend right=40, "\widetilde{h}"'] \arrow[dd, bend right, "h'"'] \arrow[dd, bend left, "h"] & & Y \arrow[r] \arrow[dl, rightsquigarrow, "v_1"] \arrow[dd, "v"] & 0 \\
& & & T' \arrow[dl, dashed, "\varphi"] \arrow[dr, rightsquigarrow, "v_2"] & & \\
0 \arrow[r] & \ker(u) \arrow[r, "u^k"'] & T \arrow[rr, "u"'] & & P \arrow[r] & 0
\end{tikzcd}$$

It follows that $u^k \circ \widetilde{h} \circ f^k = h \circ f^k$. ∎

Recall that a *split* epimorphism (monomorphism) $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a morphism such that there is $g \in \mathrm{Hom}_{\mathcal{C}}(B,A)$ such that $f \circ g = 1_X$ ($g \circ f = 1_X$), i.e f has a left (right) inverse.

Theorem 23. *Let $\mathcal{C}$ be an abelian category and let $f \in Hom_{\mathcal{C}}(X,Y)$ be an epimorphism. Then the following is equivalent:*

(1) $\overline{f}$ is an isomorphism in $\underline{\mathcal{C}}$

(2) f is a split epimorphism in $\mathcal{C}$ with $\ker(f) \in \mathcal{T}$

(3) f is a split epimorphism in $\mathcal{C}$ whose kernel map f^k factors through an object of $\mathcal{T}$

Proof. (1) $\to$ (3): $\overline{f}$ is an isomorphism, so we can choose some $g \in \mathcal{C}(Y, X)$ such that $\overline{g} = \overline{f^{-1}}$. Then $\overline{1_X - g \circ f} = \overline{1} - \overline{1} = \overline{0}$, so the morphism $1_X - g \circ f$ factors through some object $T' \in \mathcal{T}$ via some h and v:

$$\begin{tikzcd}
X \arrow[rr, "1_X - g\circ f"] \arrow[dr, squiggly, "h"'] & & X \\
& T' \arrow[ur, squiggly, "v"'] &
\end{tikzcd}$$

We have that $(v \circ h) \circ f^k = (1_X - g \circ f) \circ f^k = f^k - g \circ f \circ f^k = f^k - g \circ 0 = f^k$, so f^k factors through an object $T' \in \mathcal{T}$ with morphisms $h \circ f^k$ and v.

There exists $\widehat{v} \in \mathrm{Hom}_{\mathcal{C}}(\text{ im } h \circ f^k, \ker f)$ such that and $f^k \circ \widehat{v} = v \circ m$. Indeed, by the universal property of the kernel f^k we need to show $f \circ (v \circ m) = 0$. We have

$$f \circ (v \circ m) \circ e = f \circ v \circ (h \circ f^k) = f \circ (1 - g \circ f) \circ f^k = 0$$

and e is an epimorphism so the result follows.

Let $h \circ f^k = m \circ e$ be the epi-mono factorization of $h \circ f^k$ through im $h \circ f^k$. Because $\overline{f}$ is an epimorphism in $\underline{\mathcal{C}}$, so applying Lemma 22 for h there is some $\widetilde{h}$ such that $m \circ \widetilde{h} \circ f^k = h \circ f^k$ in the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[rrr, "f^k"] \arrow[dd, equal] \arrow[dr, "e"] & & & X \arrow[r, "f"] \arrow[dl, "h"] \arrow[dll, squiggly, "\widetilde{h}"'] \arrow[dd, "1-g\circ f"] & Y \arrow[r] & 0 \\
& & \text{im } (h \circ f^k) \arrow[r, "m"'] \arrow[dl, "\widehat{v}"] & T' \arrow[dr, "v"] & & & \\
0 \arrow[r] & \text{im } (v \circ h \circ f^k) \arrow[rrr, "f^k"] & & & X & &
\end{tikzcd}$$

Finally, $f^k \circ (\widehat{v} \circ \widetilde{h} \circ f^k) = (f^k \circ \widehat{v}) \circ \widetilde{h} \circ f^k = (v \circ m) \circ \widetilde{h} \circ f^k = v \circ (m \circ \widetilde{h} \circ f^k) = v \circ (h \circ f^k) = f^k$. But f^k is a monomorphism, so $(\widehat{v} \circ \widetilde{h}) \circ f^k = 1_{\ker(f)}$ and f^k is a split monomorphism.

(3) $\to$ (2) : By (3) f^k is a split monomorphism so there exists a morphism t such that $t \circ f^k = 1_{\ker(g)}$. But f^k also factors through an object $T \in \mathcal{T}$, lets say $f^k = b \circ a$. Then $1_{\ker(f)} = t \circ (b \circ a) = (t \circ b) \circ a$, so a is a split monomorphism and $\ker(f)$ is a direct summand of T. The class $\mathcal{T}$ is closed under direct summands, so $\ker(f) \in \mathcal{T}$.

(2) $\to$ (1): We have that $X \simeq Y \oplus \ker(f)$ via $\overline{g} = [\overline{g_1}, \overline{g_2}]^T$ with $\ker(f) \in \mathcal{T}$ so $\overline{f} = [\overline{1}_Y, \overline{0}] \circ [\overline{g_1}, \overline{g_2}]^T = \overline{1} \circ \overline{g_1} + \overline{0} \circ \overline{g_2}$ is an isomorphism. ∎

Let R be a ring and let $\mathcal{C} = \mathbf{Mod}\text{-}R$ the corresponding category of right R-modules. Consider $\mathcal{T}$ the subcategory of all projective right R-modules. We will call the category $\underline{\mathcal{C}}$ *the stable module category* of R and denote it $\underline{\mathbf{Mod}}\text{-}R$.

The following theorem describes a method how compact objects transfer from the category of modules over a left perfect ring R to the stable module category of R. The assumption on the ring is relatively strong and the proof is not so obvious as in the case of the particular category of modules.

Theorem 24 ([Miy07], Theorem 3)**.** *Let R be a perfect ring, $M \in \underline{\mathbf{Mod}}\text{-}R$ a compact object. Then there is a finitely generated module $M' \in \mathbf{Mod}\text{-}R$ such that $M \simeq M'$ in $\underline{\mathbf{Mod}}\text{-}R$.*

Proof. Because R is a perfect ring $\overline{M} := M/M\mathcal{J}(R)$ is semisimple i.e. it is isomorphic to $\bigoplus_{i\in I} S_i$ via ϕ with S_i simple for every $i \in I$. Let p be a composition of ϕ with the canonical projection $\pi : M \to M/M\mathcal{J}(R)$. Because M is a compact object in $\underline{\mathbf{Mod}}\text{-}R$, there is an R-homomorphism $f : M \to T_1$, where $T_1 \simeq \bigoplus_{i\in I_0} S_i$ for a finite subset $I_0 \subseteq I$ and T_2 is a complement of T_1 in $\overline{M}$ such that $p - (f, 0)$ factors through a projective R-module, say Q. Write $(M \xrightarrow{p} \bigoplus_{i\in I} S_i) = (M \xrightarrow{\binom{p_1}{p_2}} T_1 \oplus T_2)$. So we have a commutative diagram:

$$\begin{tikzcd}
 & Q \arrow[r, dashed, "\binom{g_1}{g_2}"] \arrow[dr, squiggly, "b"'] & P_1 \oplus P_2 \arrow[d, "\begin{pmatrix}\pi_1 & 0\\ 0 & \pi_2\end{pmatrix}"] \\
M \arrow[ur, squiggly, "a"] \arrow[rr, "\binom{p_1-f}{p_2-0}"'] & & T_1 \oplus T_2 \arrow[d] \\
 & & 0
\end{tikzcd}$$

because $b \circ a = p - f$ and $b = \binom{\pi_1\circ g_1}{\pi_2\circ g_2}$ for a R-homomorphism g (exists by the projectivity of Q).

Epimorphism π_2 is superfluous and $\pi_2 \circ (g_2 \circ a)$ $(= p_2)$ is an epimorphism, so by [AndFul92][Corollary 5.15] $g_2 \circ a$ is also an epimorphism. By the projectivity of P_2 it follows that $M \xrightarrow{g_2\circ a} P_2 \to 0$ splits (i.e. there is a R-homomorphism z such that $(g_2 \circ a) \circ z = 1_{P_2}$. We have $M \simeq \ker(g_2 \circ a) \oplus P_2$.

We show that $M' := \ker(g_2 \circ a)$ is finitely generated. Write $(M \xrightarrow{\binom{p_1}{p_2}} T_1 \oplus T_2) = (M' \oplus P_2 \xrightarrow{\binom{p_{11} p_{12}}{p_{21} p_{22}}} T_1 \oplus T_2)$. We get commutative diagrams:

$$\begin{tikzcd}
M' \oplus P_2 \arrow[r, "\binom{p_{11}p_{12}}{p_{21}p_{22}}"] \arrow[d, "(0\ 1)"'] & T_1 \oplus T_2 \arrow[d, "(0\ 1)"] \\
P_2 \arrow[r, "\pi_2"'] & T_2
\end{tikzcd}
\qquad
\begin{tikzcd}
M' \oplus P_2 \arrow[r, "\binom{p_{11}p_{12}}{p_{21}p_{22}}"] \arrow[d, "(1\ 0)"'] & T_1 \oplus T_2 \arrow[d, "(1\ 0)"] \\
M' \arrow[r, "p_{11}"'] & T_1
\end{tikzcd}$$

i.e. equations $(\pi_2 \circ 0, \pi_2 \circ 1) = (0 \circ p_{11} + p_{21}, 0 \circ p_{12} + p_{22})$ imply $p_{21} = 0$ and $p_{22} = \pi_2$. So $0 \to \ker(p_{11}) \to M' \to T_1 \to 0$ is a short exact sequence with T_1 finitely generated and $\ker(p_{11}) \subseteq M'\mathcal{J}(R)$ which si superfluous in M' by Fact 2 and we conclude the proof. ∎

Chapter 4

Grothendieck monoids of projective modules

Let R be a ring with identity. We consider a commutative monoid $(V^*(R_R), \oplus, 0)$ where $V^*(R_R)$ is a set of isomorphism classes of countably generated projectives and $\oplus$ is a binary operation of taking direct sums and 0 is the zero module and we call it *a Grothendieck monoid* of countably generated projective right R-modules. Denote $V(R_R)$ the analogical monoid of finitely generated projective right R-modules.

Let I be a two-sided ideal of R. Then we define

$$V_I(R_R) := \{\langle P \rangle \mid P \text{ projective, } P/PI \text{ is finitely generated}\}$$

Observe that $V(R_R) = V_0(R_R)$, $V^*(R_R) = V_R(R_R)$.

Let $\varphi : R \to S$ be a ring homomorphism. Then $V(\varphi) : V(R) \to V(S)$ induced via the functor $- \otimes_R S$ by $\langle P \rangle \mapsto \langle P \otimes_R S \rangle$ for every finitely generated projective module P is a monoid homomorphism of corresponding Grothendieck monoids. In case $S = R/I$ we have one more expression for elements of $V(S)$, because R/I is an R-R/I-bimodule and $P_R \otimes_R R/I$ is isomorphic to P/PI in $\mathbf{Mod}$-R/I via $p \otimes (r + I) \mapsto pr + AI$. Let $\pi : R \to R/I$ be a canonical ring isomorphism, then $V(\pi)$ is an injective monoid homomorphism.

We are interested in a situation when $V_I(R_R) = V(R)$, i.e. if a projective module is finitely generated modulo some ideal factor then it is itself finitely generated. First observe that $V_I(R_R) \subseteq V^*(R_R)$ if I is contained in $\mathcal{J}(R)$. Indeed, by Kaplansky's Theorem $P \simeq \bigoplus_{\lambda \in \Lambda} P_\lambda$ with P_λ countably generated for all $\lambda \in \Lambda$. Then

$$P/PI \simeq P \otimes_R R/I = \left(\bigoplus_{\lambda \in \Lambda} P_\lambda \right) \otimes_R R/I \simeq \bigoplus_{\lambda \in \Lambda} (P_\lambda \otimes_R R/I) \simeq \bigoplus_{\lambda \in \Lambda} P_\lambda / P_\lambda I$$

By the assuption there is a finite subset $\Lambda_0 \subseteq \Lambda$ such that $P_\lambda / P_\lambda I = 0$ for all $\lambda \in \Lambda \backslash \Lambda_0$. By an analogical statement of [[Pri07], Theorem 2.2] for $I \subseteq \mathcal{J}(R)$ it follows $P_\lambda \simeq 0$ for such all $\lambda \in \Lambda \backslash \Lambda_0$. So $P \simeq \bigoplus_{\lambda \in \Lambda_0} P_\lambda$ is countably generated.

We say that a submodule N of an R-module M is *pure* in M if $N \cap MP = NP$ for every right ideal K of R. It is known that M/N is flat if and only if N is pure in M. There is a useful criterion how to test if a module is flat.

Fact 25. *Let R be a ring and let $0 \to K \xrightarrow{\alpha} F \to C \to 0$ be a short exact sequence of R-modules with F free with the basis $(e_i \mid i \in I)$. Then C is flat if and only if for every finite set $u_1, u_2, \cdots, u_n \in K$ there is a homomorphism $\varphi : F \to K$ identical on the given elements.*

Proof. Proved in [Lam99], Proposition 4.23. ∎

Definition 5 (Ideal-supplement, Ideal projectivity)**.** *Let R be a ring and let I be a two-sided ideal of R. Then an R-module P is I-*projective *if for all right R-modules X and Y with $YI = 0$, every R-epimorphism $f : X \to Y$ and every homomorphism $\varphi : P \to Y$ there exists a homomorphism $g : P \to X$ such that $f \circ g = \varphi$, i.e. the diagram*

$$\begin{tikzcd} & P \arrow[dl, squiggly, "\exists\, g"'] \arrow[d, "\varphi"] & \\ X \arrow[r, "f"'] & Y \arrow[r] & 0 \end{tikzcd}$$

commutes.

*We call a submodule N of an R-module M an I-*supplement *if there is a submodule G in M such that $N + G = M$ and $N \cap G \subseteq NI$. (Note that direct summands are exactly 0-supplements.)*

Denote

$$T_I(M) = \{f \in End_R(M) \mid \text{ im } f \subseteq MI\}$$
$$fEnd_R(M) := \{f \in End_R(M) \mid \exists a : M \to R^n, b : R^n \to M \text{ with } f = b \circ a\}$$

A property of being I-projective can be for a special case of finitely generated modules characterized like this:

Lemma 26. *Let $I \trianglelefteq R$ be a two-sided ideal and let M be a finitely generated right R-module. Then the following is equivalent:*

(1) M is I-projective

(2) for every epimorphism $A \xrightarrow{f} B$ and every morphism $\varphi : M \to B$ there is a morphism $g : M \to A$ such that im $(\varphi - f \circ g) \subseteq BI$

(3) for the canonical projection $\pi : M \to M/MI$ there are homomorphisms φ, ψ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[d, squiggly, "\exists\varphi"'] & M/MI \\ F \arrow[r, squiggly, "\exists\psi"'] & M \arrow[u, "\pi"'] \end{tikzcd}$$

commutes for some finitely generated free R-module F.

(4) $fEnd_R(M) + T_I(M) = End_R(M)$

Proof. (1) → (2): We get the following commutative diagram:

$$\begin{tikzcd}
& M \arrow[r, squiggly, "k"] \arrow[dl, squiggly, "h"'] \arrow[dr, "g"] & BI \arrow[d, "\pi^k"] & \\
A \arrow[rr, "f"'] \arrow[drr, dashed, "\pi\circ f"'] & & B \arrow[r] \arrow[d, "\pi"] & 0 \\
& & B/BI &
\end{tikzcd}$$

Indeed, by I-projectivity of M there is a homomorphism $h : M \to A$ such that $\pi\circ g = (\pi\circ f)\circ h$, so $\pi\circ(g - f\circ h) = 0$. Obviously $\pi\circ\pi^k = 0$ where π^k is a kernel map of π and so by the universal property of the cokernel there is a homomorphism $k : M \to BI$ such that $\pi^k \circ k = g - f \circ h$. Then $\text{im}\,(g - f \circ h) \subseteq \text{im}\,(\pi^k) = BI$ and (2) follows.

(2) → (1) : Let $A \xrightarrow{f} B \to 0$ be an exact sequence with $BI = 0$ and let $g : M \to B$ be arbitrary. By (2) there is a homomorphism $h : M \to A$ such that $\text{im}\,(f \circ h - g) \subseteq BI = 0$, so h is also a witness of I-projectivity of M.

(1) → (4): It is enough to prove that $1_M \in fEnd_R(M) + T_I(M)$. Write M as the homomorphic image of some finitely generated free right module F, i.e. $F \xrightarrow{g} M \to 0$. Because $(M/MI)I = 0$, by (1) there is some homomorphism $f : M \to F$ such that $\pi = (\pi \circ g) \circ f$, i.e. $\pi \circ (1_M - g \circ f) = 0$. It follows that $\text{im}\,(1_M - g \circ f) \subseteq MI$ and of course $g \circ f \in fEnd_R(M)$.

(3) → (1) Let $g : A \to B \to 0$ be an epimorphism with $BI = 0$ and $\varphi : M \to B$ some homomorphism. Let $\pi : M \to M/MI$ be a canonical projection. Because $\varphi(MI) \subseteq \varphi(M)I = 0$ by the Homomorphism Theorem $\varphi = \widetilde{\varphi} \circ \pi$ for some $\widetilde{\varphi} : M/MI \to B$. By (3) there is a free module F and some homomorphisms $a : M \to F, b : F \to M$ such that $\pi = \pi \circ b \circ a$. By projectivity of F we get a homomorphism $c : F \to A$ such that $g \circ c = \varphi \circ b$. Then $g \circ (c \circ a) = (g \circ c) \circ a = (\varphi \circ b) \circ a = ((\widetilde{\varphi} \circ \pi) \circ b) \circ a = \widetilde{\varphi} \circ (\pi \circ b \circ a) = \widetilde{\varphi} \circ \pi = \varphi$.

(4) → (3): Let $1_M = x + y$ with $x \in fEnd_R(M)$ and $y \in T_I(M)$. So x factors through a finitely generated free R-module F via homomorphisms $a : M \to F$ and $b : F \to M$. Then $0 = \pi \circ y = \pi \circ (1_M - x) = \pi \circ (1_M - a \circ b)$ and (3) follows. ∎

A finitely generated R-module P is projective if and only if $fEnd_R(P) = End_R(P)$. Indeed, $fEnd_R(P)$ is an ideal, hence a finitely generated module P is projective if and only if $1_P \in fEnd_R(P)$. Similarly like in the case of modules, we say that an ideal I of a ring R is *superfluous* in R if $I + J = R$ implies $J = 0$ for every two-sided ideal J of R. It would be interesting to know whether the ideal $T_I(R)$ is superfluous in $End_R(M)$.

Remark 3. Let R be a ring and $K \trianglelefteq R$ a nonzero ideal. Denote by $\mathcal{G}(R)$ the Brown-McCoy radical of R. Recall that a Brown-McCoy radical is the intersection of all maximal two-sided ideals of R or the intersection of all two-sided ideals K such that R/K is a simple ring. Then K is superfluous in R if and only if $K \subseteq \mathcal{G}$.

Proof. Let $K \subseteq \mathcal{G}(R)$. Assume that I is not superfluous in R, i.e. $K + L = R$ for some proper nonzero ideal L of R. Then L is contained in some maximal ideal M of R, hence $K + M = R$. Then R/M is simple and $\mathcal{G}(R) \subseteq M$, a contradiction with $K + M = R$.

Let K be superfluous in R and assume $K \not\subseteq \mathcal{G}(R)$. Then there is some ideal L of R such that $K \not\subseteq L$ with R/L a simple ring, so L is a maximal ideal and $K + L = R$. But by the assumption $L = R$, a contradiction. ∎

Fact 27 ([AndFul92], Theorem 10.4)**.** *A right R-module M is finitely generated if and only if $Rad(M)$ is superfluous in M and $M/Rad(M)$ is finitely generated.*

If P,Q are projective modules and $f : P \to Q$ is a R-homomorphism, then by *the induced homomorphism* $\overline{f} : P/PI \to Q/QI$ it is meant the natural map defined by $p + PI \mapsto f(p) + QI$ for all $p \in P$. For any R/I-homomorphism $\alpha : P/PI \to Q/QI$ there exists an R-homomorphism $f : P \to Q$ such that the induced homomorphism $\overline{f}$ equals α and we say that f is *a lift of* α.

Proposition 28 ([FacHerSak05], Proposition 6.1)**.** *Let R be a ring and $I \trianglelefteq R$ be an ideal contained in $\mathcal{J}(R)$. Let P, Q be projective right R-modules and let $\alpha : P/PI \to Q/QI$ be an R/I-homomorphism. Let f be a lift of α. If α is a pure monomorphism, then f is a pure monomorphism.*

Proof. First choose an R-module P' such that $P \oplus P'$ is free, then an R-module Q' such that $(Q \oplus P') \oplus Q'$ is free. Let $\epsilon : P' \to P' \oplus Q'$ denote the inclusion onto the direct summand. Without lack of generality we now suppose that P and Q are free, because if $f \oplus \epsilon$ is a pure monomorphism then f is a pure monomorphism.

Fix a finitely generated free direct summand M of P with a complement M'. We show that the image $f[M]$ is a direct summand of Q. We find a free submodule N of Q such that $f[M] \subseteq N$ and N is a direct summand with a complement N'. Denote $j : N \to Q$ and $i : M \to P$ be the inclusions, f_M the restriction of f on M and let $q : Q \to N$ and $p : P \to M$ be homomorphisms such that $p \circ i = 1_M$ and $q \circ j = 1_N$. Then we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"'] \arrow[d, squiggly, "f_M"] & P \arrow[r] \arrow[d, "f"] \arrow[l, bend right, squiggly, "p"'] & M' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r, "j"] & Q \arrow[r] \arrow[l, bend left, dashed, "q"] & N' \arrow[r] & 0
\end{tikzcd}$$

Inducing with $- \otimes_R R/I$ we get a commutative diagram in **Mod**-R/I with split exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M/MI \arrow[r, "\overline{i}"] \arrow[d, "\overline{f_M}"] & P/PI \arrow[r] \arrow[d, "\alpha"] & M'/M'I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N/NI \arrow[r, "\overline{j}"'] \arrow[d] & Q/QI \arrow[r] & N'/N'I \arrow[r] & 0 \\
& Coker\ \overline{f_M} & & &
\end{tikzcd}$$

Both $\alpha, \overline{i}$ are pure monomorphisms so $\alpha \circ \overline{i}$ is also a pure monomorphism. Hence $\overline{j} \circ \overline{f_M}$ is a pure monomorphism. It follows that $\overline{f_M}$ is a pure monomorphism and $Coker\ \overline{f_M}$ is a finitely presented flat R/I-module, so there is a homomorphism $\beta : N/NI \to M/MI$ such that $\beta \circ \overline{f_M} = 1_{M/MI}$. Let g be any lift of β.

We get that $g \circ f_M$ is an automorphism of M. In particular, f_M is a split monomorphism. This concludes the observation that $f[M] = f_M[M]$ is a direct summand in Q.

We show that f injective. Let $x \in \ker f$ be arbitrary. Then there is a finitely generated free summand $f_M[M]$ of Q containing $f(x)$ for some M, hence $x \in \ker(f_M) = 0$.

Now f is a direct limit of split short exact sequences arising from f_M, M finitely generated so by [Lam99], Examples 4.84(c), f is pure exact. ∎

Recall that a projective dimension of a module M equals one if there exists a projective presentation $0 \to K \to P \to M \to 0$ of M with K projective.

Lemma 29. *Let R be a ring and let M be a right R-module. Then the following holds*

(i) (Schanuel Lemma) Assume we have two presentations $0 \to K \xrightarrow{i} P \xrightarrow{\pi} M \to 0$ and $0 \to L \xrightarrow{j} Q \xrightarrow{\rho} M \to 0$ of M with P projective. Then we have a short exact sequence $0 \to K \to L \oplus P \to Q \to 0$.

(ii) every countably presented flat R-module has the projective dimension ≤ 1.

In particular, if a countably presented flat R-module has a presentation $0 \to K \to P \to M \to 0$ with P projective, then K is also projective.

Proof. (i) Proved in [Lam99], Lemma 5.1.
(ii) Proved in [Laz69], Théoreme 3.2. ∎

Lemma 30. *For a commutative diagram in* **Mod**-*R with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "p"] \arrow[d, "\beta"] \arrow[dl, dashed, "\varphi"'] & C \arrow[r] \arrow[d, "\gamma"] \arrow[dl, dashed, "\psi"] & 0 \\
0 \arrow[r] & A' \arrow[r, "j"'] & B' \arrow[r, "q"'] & C' \arrow[r] & 0
\end{tikzcd}$$

holds that there exists $\varphi : B \to A'$ such that $\varphi \circ i = \alpha$ if and only if there exists $\psi : C \to B'$ such that $\psi \circ q = \gamma$. In particular, if γ is an isomorphism and the first commutative square admits a diagonal fill-in from B to A' then the lower row splits.

Proof. We show only one direction, the second is categorically dual. Assume that there exists $\varphi : B \to A'$ such that $\varphi \circ i = \alpha$. Then

$$0 = \beta \circ i - j \circ \alpha = \beta \circ i - j \circ (\varphi \circ i) = (\beta - j \circ \varphi) \circ i$$

By the universal property of cokernel C there exists a homomorphism $\psi : C \to B'$ such that $\psi \circ p = \beta - j \circ \varphi$. We show that ψ is required homomorphism. Because p is an epimorphisms it is enough to show that $\gamma \circ p = q \circ \psi \circ p$. Indeed,

$$q \circ (\psi \circ p) = q \circ (\beta - j \circ \varphi) = q \circ \beta - (q \circ j) \circ \varphi = q \circ \beta = \gamma \circ p$$ ∎

Now we give an equivalent conditions connecting ideal supplements and ideal projectivity with the equality $V_I(R_R) = V(R_R)$ for an ideal I contained in the Jacobson radical. Recall that a homomorphism $\pi : P \to M$ is a projective cover of M if and only if π is a $\mathcal{P}$-cover where $\mathcal{P}$ is a class of all projective modules, i.e. $P \in \mathcal{P}$ two conditions holds: the first, for every $Q \in \mathcal{P}$ and every $\varphi : Q \to M$ there is $\psi : G \to P$ such that $\pi \circ \psi = \varphi$ and the second, if $Q = P$ and $\varphi = \pi$ then every endomorphism ψ is an automorphism.

Theorem 31. *Let R be a ring and $I \trianglelefteq R$ be an ideal contained in $\mathcal{J}(R)$. Then the following is equivalent:*

(A) for every finitely generated projective R-module P, every finitely generated I-supplement in P is 0-supplement

(B) every finitely generated (presented) I-projective R-module is projective

(C) every finitely generated flat R-module M with projective R/I-module M/MI is projective

(D) for every projective R-module Q, if the factor-module Q/QI is finitely generated then Q is finitely generated

Proof. $(A) \to (B)$: Let K be a finitely generated I-projective right R-module and let $\pi : K \to K/KI$ be the canonical projection. Then by Proposition 26(1) $\to$ (3) there are homomorphisms $p : K \to P$ and $q : P \to K$ such that $\pi \circ (q \circ p) = \pi$ for some finitely generated projective (free) R-module P. Then we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "p"] \arrow[d, "\pi"] & P \arrow[r] \arrow[d, "\pi"] & Coker(p) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & K/KI \arrow[r, "\widetilde{p}"'] \arrow[d] & P/KI \arrow[r] \arrow[d] & (P/KI)))/(K/KI) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Choosing $\varphi := \pi \circ q$, by Lemma 30 we get that the lower row is split exact and so there is some submodule C of P such that $p(K) + C = P$ and $p(K) \cap C \subseteq KI$. This just means that $p(K)$ is an I-supplement of C in P. By the condition (A) it means that $p[K]$ is a direct summand of a projective module, hence it is projective.

$(B) \to (C)$: Let M be a finitely generated flat right R-module. Then there is a short exact sequence $0 \to \ker(p) \to F \xrightarrow{p} M \to 0$ with F finitely generated free. Denote $K := \ker(p)$. By assumption on M the induced sequence $0 \to K/KI \to F/FI \to M/MI \to 0$ splits in **Mod**-R/I. Then K/KI is finitely generated and so $K = K_0 + KI$ for a finitely generated submodule K_0 of K. By Lemma 25 applied on M there is a homomorphism $f : F \to K$ that is identical on generators of K_0 and therefore on the whole K_0. Define a (finitely presented) module $P := F/K_0$. Because $K_0 \subseteq \ker(1_F - f)$ we have some homomorphism g

that makes the diagram commuting:

$$\begin{tikzcd} F \arrow[r, "1-f"] \arrow[d, "\pi_{K_0}"'] & F \\ F/K_0 \arrow[ur, squiggly, "g"] & \end{tikzcd}$$

We want to show that P is I-projective. By characterization of I-projectivity it is enough to show that the square commutes:

$$\begin{tikzcd} F/K_0 \arrow[r, "\pi_I"] \arrow[d, squiggly, "g"'] & \frac{F/K_0}{(K_0+FI)/K_0} \\ F \arrow[r, squiggly, "\pi_{K_0}"] & F/K_0 \arrow[u, "\pi_I"'] \end{tikzcd}$$

Observe that

$$\pi_I \circ \pi_{K_0} \circ (g \circ \pi_{K_0}) = (\pi_I \circ \pi_{K_0}) \circ (1 - f) = \pi_I \circ \pi_{K_0},$$

because $\operatorname{im} f \subseteq K \subseteq K_0 + FI$. Because π_{K_0} is an epimorphism the square commutes.

By (B) P is projective and so K_0 is a direct summand of F. Denote by G a complement of K_0 in F, which is obviously projective, because it is also a direct summand of F. Then the factormodule $G/(G \cap K) \simeq (K + G)/K = F/K$ is isomorphic to M and therefore it is flat. Denote by $q : G \to G/(G \cap K)$ the canonical projection and let $x \in G \cap K$ be arbitrary. Because G is finitely generated $Rad(G)$ is superfluous in G. We have the inclusion

$$G \cap K \subseteq G \cap (K_0 + (K_0 + G)I) = G \cap (K_0 + HG) = (G \cap K_0) + GI = GI$$

and $GI \subseteq G\mathcal{J}(R) \subseteq Rad(G)$ so by Fact 27 it follows that $G \cap K$ is superfluous in G, i.e. q is a projective cover of $G/(G \cap K)$. Now by Lemma 25 applied on $G/G \cap K$ there is a homomorphism $f_x : G \to G \cap K$ identical on x and it follows $q \circ (1 - f_x) = q$. Because q is a projective cover, $(1 - f_x)$ is an automorphism so $\ker(1 - f_x) = 0$ and $x = 0$. This is true for all $x \in G \cap K$ and $G \cap K = 0$. Then $G \simeq M$ and M is projective.

$(C) \to (D)$: Let Q be projective and the ideal factor Q/QI be finitely generated. Then there is an embedding $0 \to Q/QI \xrightarrow{\alpha} (R/I)^{(n)}$ such that α is a split monomorphism. By Proposition 28 there exists $f : Q \to R^{(n)}$ such that $\overline{f} = \alpha$ and it is a pure monomorphism. Denote $M := Coker\ f$. Then M is a finitely generated flat R-module such that $M \otimes_R R/I$ is isomorphic to a direct summand of $(R/I)^{(n)}$ so by the condition (C) M is projective. We have that f is a split monomorphism and Q is finitely generated.

$(D) \to (A)$: Let P be a finitely generated projective R-module and N be a finitely generated submodule of P such that it is an I-supplement, denote by ι the embedding N into P. That means there exists a submodule G of P such that $N + G = P$ and $N \cap G \subseteq PI$ and the short exact sequence $0 \to N/(N \cap G) \to P/(N \cap G) \to G/(N \cap G) \to 0$ is split exact. Let $M := P/N$. Because P is projective, there is some $\gamma \in End_R(P)$ such that $\operatorname{im} \gamma \subseteq N$ and $\operatorname{im} (1_P - \gamma) \subseteq G$.

Observe that $\text{im } \gamma^2 = \text{im } \gamma = N$. Indeed, $\text{im } \gamma^2 + \text{im } (\gamma \circ (1-\gamma)) + G = P$. But $\text{im } (\gamma \circ (1-\gamma)) = \text{im } (1-\gamma) \circ \gamma \subseteq N \cap G \subseteq NI$ which is superfluous in P, hence $\text{im } \gamma^2 + G = P$. Then $N = (\text{im } \gamma^2 + G) \cap N = \text{im } \gamma^2 + (G \cap N) = \text{im } \gamma^2$ by modularity of modules and because $N \cap G$ is superfluous in N. By induction we get that $N = \text{im } \gamma^n$ for all $n \in \mathbb{N}$.

By projectivity of P there exists a homomorphism $\tau \in End_R(P)$ such that the diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, squiggly, "\exists\, \tau"'] \arrow[d, "\gamma"] & \\ P \arrow[r, "\gamma^2"'] & N \arrow[r] & 0 \end{tikzcd}$$

By induction we get that $\gamma = \gamma^{n+1} \circ \tau^n$ for every $n \in \mathbb{N}$.

Define $G_1 := \sum_{i=1}^{\infty} \ker \gamma^i$. Then $G_1 \subseteq G$ and $N + G_1 = P$. We claim that G_1 is pure in P. We show that P/G_1 is flat using Lemma 25. Let $x \in G_1$, so there is some $m \in \mathbb{N}$ depending on x such that $m \in \ker \gamma^m$. Define $\alpha_n := 1_P - \tau^n \circ \gamma^n$ for all $n \in \mathbb{N}$. Then $\text{im } \alpha_m \subseteq G_1$. Indeed, $\gamma^{m+1} \circ (1_P - \tau^m \circ \gamma^m) = \gamma^{m+1} - \gamma \circ \gamma^m = 0$ and so $\text{im } \alpha_m \subseteq \ker \gamma^{m+1}$ and $G_1 = \sum_{n=1}^{\infty} \text{ im } \alpha_n$. Finally, $\alpha(x) = 1_P - \tau^m \circ \gamma^m(x) = x$.

Observe that G_1/G_1I is finitely generated. Indeed, from $N \cap G_1 \subseteq PI$ we get

$$\frac{N + PI}{PI} \oplus \frac{G_1 + PI}{PI} = P/PI$$

Because G_1 is pure in P we have $G_1 \cap PI = G_1 I$. Then

$$G_1 + PI/PI \simeq G_1/G_1 \cap PI = G_1/G_1 I$$

and G_1/G_1I is isomorphic to a direct summand of a finitely generated R/I-module P/PI.

The factormodule P/G_1 is countably presented, finitely generated and flat, so by Lemma 29 G_1 is projective. By the condition (D) G_1 is finitely generated, hence $G_1 = \ker \gamma^m$ for some $m \in \mathbb{N}$. Let $y \in G_1 \cap N$. Then $y = \gamma^m(z)$ for some $z \in P$ and $\gamma^m(y) = 0$. It follows that $z \in \ker \gamma^{2m} = \ker \gamma^m$ and so $y = 0$. We have showed that $N = \text{ im } \gamma^m$ is a direct summand of P. ∎

We say that a subset X of R is *locally nilpotent* if for every finite subset X_0 of X there exists $k = k(X_0) \in \mathbb{N}$ depending on X_0 such that every product of k elements from X_0 is zero. Now we show that $V_{\mathcal{L}(R)}(R_R) = V(R_R)$ where $\mathcal{L}(R)$ is the *Levitzki radical* of the ring R which is the set of all $x \in R$ such that xR is locally nilpotent subset of R. Observe that the sum of two locally nilpotent right ideals are locally nilpotent. Let $\mathcal{A} \in M_n(\mathcal{L}(R))$ and $X_0 = \{x_1, x_2, \cdots, x_{n^2}\}$ be the set of entries of $\mathbb{A}$. Then X_0 is a finite subset of a locally nilpotent $\sum_{i=1}^{n^2} x_i R$ and $\mathbb{A}^{k(X_0)} = 0$.

Proposition 32 (based on [MohSan89], Corollary 3.5)**.** *Let R be a ring and $I \trianglelefteq R$ be an ideal contained in $\mathcal{J}(R)$. If the ideal $M_n(I)$ contains only nilpotent elements, then $V_I(R) = V(R)$.*

Proof. We check the condition (B) from the previous theorem. Let $P = \sum\limits_{i=1}^{n} Ry_i$ be n-generated left R-module and denote $E := End_R(P)$. Since P is I-projective, it is enough to show that $T_I(P)$ contains only nilpotent elements because then it is trivially contained in the Brown-McCoy radical of $End_R(P)$ and so $T_I(P)$ is superfluous in E which concludes that P is projective.

Let $t \in T_I(P)$ and write $t(y_i) = \sum_{j=1}^{n} a_{ij} y_j$ for $i = 1, 2, \ldots, n$ and $a_{ij} \in I$, write them in a matrix $\mathbb{A}$. Let F be a finitely generated free R-module with a basis $\{e_1, e_2, \ldots, e_n\}$ and define $g : F \to P$ by $e_i \mapsto y_i$. Then we have a commutative diagram:

$$\begin{array}{ccccccc}
F & \xrightarrow{\mathbb{A}} & F & \xrightarrow{\mathbb{A}} & F & \xrightarrow{\mathbb{A}} & \ldots \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \\
P & \xrightarrow{t} & P & \xrightarrow{t} & P & \xrightarrow{t} & \ldots
\end{array}$$

because $g(\mathbb{A}e_i) = g(\sum_{j=1}^{n} a_{ij} e_i) = \sum_{j=1}^{n} a_{ij} g(e_i) = \sum_{j=1}^{n} a_{ij} y_i = t(y_i) = (t \circ g)(e_i)$ for $i = 1, 2, \ldots, n$ and by induction we get $(t^m \circ g) = g \circ \mathbb{A}^m$ for all $m \in \mathbb{N}$. But $\mathbb{A}$ is contained in $M_n(I)$ that contains only nilpotent elements by the assumption, so there is an index $m_0 \in \mathbb{N}$ such that $t^{m_0} \circ g = 0$. Because g is an epimorphism, it follows $t^{m_0} = 0$. ∎

There is a construction of a ring R such that $V_{\mathcal{J}(R)}(R_R) \neq V(R_R)$ [GerSak84]. Together with the last proposition it makes a sense to state the following question.

Question 2. *Let I be a nil ideal. Does it hold that $V_I(R) = V(R)$?*

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