Charles University in Prague

Faculty of Mathematics and Physics

MASTER THESIS

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Compactness of higher-order Sobolev embeddings

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Study programme: Mathematics

Specialization: Mathematical Analysis

Prague 2012

I would like to express my thanks to my supervisor, prof. Luboš Pick, for lots of consultations and for many valuable comments concerning the thesis. I also thank my parents for their support.

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague on April 12, 2012 Lenka Slavíková

Název práce: Kompaktnost Sobolevových vnoření vyššího řádu

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Abstrakt: V předložené práci studujeme kompaktnost Sobolevových vnoření m-tého řádu na oblasti $\Omega \subseteq \mathbb{R}^n$ vybavené pravděpodobnostní mírou ν a splňující jistou izoperimetrickou nerovnost. Odvodíme podmínku na dvojici prostorů $X(\Omega,\nu)$ a $Y(\Omega,\nu)$ invariantních vůči nerostoucímu přerovnání, která zaručuje kompaktnost vnoření Sobolevova prostoru $V^m X(\Omega,\nu)$ do $Y(\Omega,\nu)$. Tato podmínka je vyjádřena pomocí kompaktnosti jistého operátoru na reprezentačních prostorech. Získaný výsledek poté využijeme k charakterizaci kompaktních Sobolevových vnoření na konkrétních prostorech s mírou, kterými jsou Johnovy oblasti, Maz'yovy třídy oblastí v eukleidovském prostoru a součinové pravděpodobnostní prostory, jejichž standardním příkladem je Gaussův prostor.

Klíčová slova: kompaktnost, prostor invariantní vůči nerostoucímu přerovnání, Sobolevův prostor, izoperimetrická funkce

Title: Compactness of higher-order Sobolev embeddings

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Abstract: The present work deals with m-th order compact Sobolev embeddings on a domain $\Omega \subseteq \mathbb{R}^n$ endowed with a probability measure ν and satisfying certain isoperimetric inequality. We derive a condition on a pair of rearrangement-invariant spaces $X(\Omega,\nu)$ and $Y(\Omega,\nu)$ which suffices to guarantee a compact embedding of the Sobolev space $V^m X(\Omega,\nu)$ into $Y(\Omega,\nu)$. The condition is given in terms of compactness of certain operator on representation spaces. This result is then applied to characterize higher-order compact Sobolev embeddings on concrete measure spaces, including John domains, Maz'ya classes of Euclidean domains and product probability spaces, among them the Gauss space is the most standard example.

Keywords: compactness, rearrangement-invariant space, Sobolev space, isoperimetric function

Contents

Introduction

Compact embeddings of Sobolev spaces into other function spaces play a very important role in modern functional analysis and, especially, in its applications to finding solutions of partial differential equations. Although Sobolev spaces on Euclidean domains having a Lipschitz boundary are discussed the most frequently, it turns out that Sobolev spaces on different measure spaces would be of interest as well.

Suppose that Ω is an open connected subset of $\mathbb{R}^n$ endowed with a probability measure ν fulfilling $d\nu(x) = \omega(x)\,dx$, where ω is a strictly positive continuous density. Let $X(\Omega,\nu)$ and $Y(\Omega,\nu)$ be rearrangement-invariant spaces, in the sense described in the following chapter. Given $m \in \mathbb{N}$, the Sobolev space $V^mX(\Omega,\nu)$ consists of all m-times weakly differentiable functions in Ω whose m-th order weak derivatives belongs to $X(\Omega,\nu)$. Our main aim is to prove a sufficient condition for

$$V^mX(\Omega,\nu) \hookrightarrow\hookrightarrow Y(\Omega,\nu)$$

given in terms of compactness of the one-dimensional operator

$$H_I^m f(t) = \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds, \quad f \in L^1(0,1),\ t \in (0,1),$$

on representation spaces. This operator corresponds to a function I which should be chosen in such a way that it is dominated by the isoperimetric function of (Ω,ν) and satisfies some regularity assumptions. Our method is based on a result which yields that boundedness of such an operator implies validity of the corresponding Sobolev embedding. This result was proved in [10] for first-order Sobolev embeddings and then it was generalized in [3] for Sobolev embeddings of an arbitrary order m by iterating of first-order embeddings. In contrast with this, in a big part of our proof we do not need to distinguish whether $m = 1$ or $m > 1$, although there is an exception in the case when $Y(\Omega,\nu) = L^\infty(\Omega,\nu)$.

Our method strongly depends on the use of so called almost-compact embeddings, called also absolutely continuous embeddings in some literature, which were studied, e.g., in [6] and [13]. It is well known that such embeddings have a great significance for deriving compact Sobolev embeddings.

Our key one-dimensional result which, in fact, provides a connection between the one-dimensional and n-dimensional case, says that compactness of the operator H_I^m from a rearrangement-invariant space $X(0,1)$ into a rearrangement-invariant space $Y(0,1) \neq L^\infty(0,1)$ is equivalent to an almost-compact embedding of certain rearrangement-invariant space $X_{m,I}(0,1)$ into $Y(0,1)$. The space $X_{m,I}(0,1)$ is related to $X(0,1)$ by the fact that it is the smallest rearrangement-invariant space fulfilling that the operator H_I^m is bounded from $X(0,1)$ into this space.

The case when $Y(0,1) = L^\infty(0,1)$ is slightly different since here we first derive the first-order result based on a certain almost-compact embedding, while the higher-order result is transformed by iteration to the case when $Y(0,1) \neq L^\infty(0,1)$. However, in contrast with [3], this iteration is quite straightforward, using only the well known Hardy-Littlewood-Pólya inequality instead of a non-standard inequality needed in [3].

Moreover, in most cases of possible interest the function I can be chosen in such a way that compactness of the operator H_I^m is not only sufficient but also necessary for compactness of the corresponding Sobolev embedding. This is the case, e.g., when Ω is a John domain, that is, a bounded Euclidean domain whose isoperimetric function is equivalent to $s^{1/n'}$ near 0. Here, $n' = n/(n-1)$. Note that for the particular family of bounded domains having a Lipschitz boundary, such a result is already known, see [7]. Also compact Sobolev embeddings on more general Euclidean domains belonging to so called Maz'ya classes can be characterized by compactness of certain one-dimensional operator, in the sense that there is always one domain in each class for which we have the necessity. Finally, the product probability spaces belong into this framework. Among them, the Gauss space, i.e., $\mathbb{R}^n$ endowed with the probability measure

$$d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{\frac{-|x|^2}{2}}\, dx,$$

is the most standard example.

The structure of the thesis is as follows. We start by recalling some basic facts from the theory of rearrangement-invariant spaces and by describing measure spaces on which we will study compact Sobolev embeddings, see Chapter 1, Section 1.1 and Section 1.2, respectively. In Chapter 2 we derive several characterizations of compactness of the one-dimensional operator H_I^m, which will be needed in Chapter 3 where we state and prove our main results. The last chapter of the thesis, devoted to the study of compact Sobolev embeddings on concrete measure spaces, starts with some technical results contained in Section 4.1. In Section 4.2 we characterize compact Sobolev embeddings on John and Maz'ya domains and in the final Section 4.3 we deal with compact Sobolev embeddings on product probability spaces.

1. Preliminaries

1.1 Rearrangement-invariant spaces

In this section we recall some basic facts from the theory of rearrangement-invariant spaces.

Let (R,μ) be a nonatomic measure space satisfying $\mu(R)=1$. Denote by $\mathcal{M}(R,\mu)$ the set of all real valued μ-measurable functions in R.

Suppose that $f\in\mathcal{M}(R,\mu)$. The *nonincreasing rearrangement* of f is the function f^*_μ defined by

$$f^*_\mu(t)=\inf\{\lambda\geq 0:\mu\left(\{x\in R:|f(x)|>\lambda\}\right)\leq t\},\quad t\in(0,\infty).$$

Observe that $f^*_\mu(t)=0$ whenever $t\geq 1$. For this reason, we sometimes consider the function f^*_μ to be defined on $(0,1)$ instead of on $(0,\infty)$.

Assume that a functional $\|\cdot\|_{X(R,\mu)}:\mathcal{M}(R,\mu)\to[0,\infty]$ is such that for all functions f, $g\in\mathcal{M}(R,\mu)$, for all sequences $(f_k)_{k=1}^\infty$ in $\mathcal{M}(R,\mu)$ and for all constants $a\geq 0$, the following conditions are satisfied:

(P1) $\|f\|_{X(R,\mu)}=0\Leftrightarrow f=0$ μ-a.e., $\|af\|_{X(R,\mu)}=a\|f\|_{X(R,\mu)}$,
$\|f+g\|_{X(R,\mu)}\leq\|f\|_{X(R,\mu)}+\|g\|_{X(R,\mu)}$;

(P2) $0\leq f\leq g$ μ-a.e. $\Rightarrow\|f\|_{X(R,\mu)}\leq\|g\|_{X(R,\mu)}$;

(P3) $0\leq f_k\uparrow f$ μ-a.e. $\Rightarrow\|f_k\|_{X(R,\mu)}\uparrow\|f\|_{X(R,\mu)}$;

(P4) $\|1\|_{X(R,\mu)}<\infty$;

(P5) $\int_R|f|\,d\mu\leq C\|f\|_{X(R,\mu)}$ for some constant $C>0$ independent of f;

(P6) $f^*_\mu=g^*_\mu\Rightarrow\|f\|_{X(R,\mu)}=\|g\|_{X(R,\mu)}$.

The collection of all functions $f\in\mathcal{M}(R,\mu)$ for which $\|f\|_{X(R,\mu)}<\infty$ is then called the *rearrangement-invariant space* $X(R,\mu)$ and the functional $\|\cdot\|_{X(R,\mu)}$ is called the *rearrangement-invariant norm* of $X(R,\mu)$.

If $X(R,\mu)$ and $Y(R,\mu)$ are rearrangement-invariant spaces, the continuous embedding $X(R,\mu)\hookrightarrow Y(R,\mu)$ holds if and only if $X(R,\mu)\subseteq Y(R,\mu)$, see [2, Chapter 1, Theorem 1.8]. We shall write $X(R,\mu)=Y(R,\mu)$ if the set of functions belonging to $X(R,\mu)$ coincides with the set of functions belonging to $Y(R,\mu)$. In this case, the norms on $X(R,\mu)$ and $Y(R,\mu)$ are equivalent, in the sense that there are positive constants C_1, C_2 such that

$$C_1\|f\|_{X(R,\mu)}\leq\|f\|_{Y(R,\mu)}\leq C_2\|f\|_{X(R,\mu)},\quad f\in\mathcal{M}(R,\mu).$$

The *Fatou lemma* [2, Chapter 1, Lemma 1.5 (iii)] tells us that whenever $(f_k)_{k=1}^\infty$ is a sequence in $X(R,\mu)$ converging to some function f μ-a.e. and fulfilling that $\liminf_{k\to\infty}\|f_k\|_{X(R,\mu)}<\infty$, then $f\in X(R,\mu)$ and

$$\|f\|_{X(R,\mu)}\leq\liminf_{k\to\infty}\|f_k\|_{X(R,\mu)}.$$

Furthermore, the *Hardy-Littlewood inequality* [2, Chapter 2, Theorem 2.2] yields that

$$\int_R |fg|\,d\mu \le \int_0^1 f^*_\mu(s) g^*_\mu(s)\,ds \tag{1.1}$$

is satisfied for all functions $f,\, g \in \mathcal{M}(R,\mu)$.

Given a rearrangement-invariant space $X(R,\mu)$, the *associate space* $X'(R,\mu)$ is the rearrangement-invariant space consisting of all functions $g \in \mathcal{M}(R,\mu)$ for which

$$\|g\|_{X'(R,\mu)} = \sup_{\|f\|_{X(R,\mu)}\le 1} \int_R |fg|\,d\mu < \infty.$$

For every $f \in X(R,\mu)$ and $g \in X'(R,\mu)$, we have the *Hölder inequality*

$$\int_R |fg|\,d\mu \le \|f\|_{X(R,\mu)} \|g\|_{X'(R,\mu)},$$

see [2, Chapter 1, Theorem 2.4].

For each rearrangement-invariant space $X(R,\mu)$ there exists a rearrangement-invariant space $X((0,1),\lambda_1)$ such that

$$\|f\|_{X(R,\mu)} = \|f^*_\mu\|_{X((0,1),\lambda_1)}, \quad f \in X(R,\mu), \tag{1.2}$$

see [2, Chapter 2, Theorem 4.10]. Here, λ_1 denotes the one-dimensional Lebesgue measure. The space $X((0,1),\lambda_1)$ is called the *representation space* of the space $X(R,\mu)$.

In other words, each rearrangement-invariant space $X(R,\mu)$ can be defined in terms of a rearrangement-invariant space $X((0,1),\lambda_1)$ by (1.2). We shall therefore always start with a rearrangement-invariant space $X((0,1),\lambda_1)$ and then denote by $X(R,\mu)$ the rearrangement-invariant space whose norm is given by (1.2). For simplicity, we shall write $(0,1)$ instead of $((0,1),\lambda_1)$ and, analogously, we shall omit the lower index λ_1 when dealing with nonincreasing rearrangements.

Let $X(0,1)$ be a rearrangement-invariant space. The function φ_X defined by

$$\varphi_X(s) = \|\chi_{(0,s)}\|_{X(0,1)}, \quad s \in (0,1),$$

is called the *fundamental function* of $X(0,1)$ (or $X(R,\mu)$). It is *quasiconcave*, i.e., it is nondecreasing in $(0,1)$ and $\varphi_X(s)/s$ is nonincreasing in $(0,1)$. Furthermore, each rearrangement-invariant space can be equivalently renormed such that its fundamental function is concave.

We say that a function $f \in \mathcal{M}(R,\mu)$ has an *absolutely continuous norm* in $X(R,\mu)$ if for every sequence $(E_k)_{k=1}^\infty$ of μ-measurable subsets of R fulfilling $\chi_{E_k} \to 0$ μ-a.e. we have

$$\lim_{k\to\infty} \|\chi_{E_k} f\|_{X(R,\mu)} = 0.$$

An easy observation yields that this can be equivalently reformulated by

$$\lim_{a\to 0_+} \|\chi_{(0,a)} f^*_\mu\|_{X(0,1)} = 0.$$

Suppose that $X(0,1)$ and $Y(0,1)$ are rearrangement-invariant spaces. We say that $X(R,\mu)$ is *almost-compactly embedded* into $Y(R,\mu)$ and write $X(R,\mu) \overset{*}{\hookrightarrow} Y(R,\mu)$ if

$$\lim_{k\to\infty} \sup_{\|f\|_{X(R,\mu)}\le 1} \|\chi_{E_k} f\|_{Y(R,\mu)} = 0$$

is satisfied for every sequence $(E_k)_{k=1}^{\infty}$ of μ-measurable subsets of R fulfilling $\chi_{E_k} \to 0$ μ-a.e. It can be deduced that this is the same as if

$$\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,a)} f^*\|_{Y(0,1)}.$$

Note that the relation $X(R,\mu) \overset{*}{\hookrightarrow} Y(R,\mu)$ always implies $X(R,\mu) \hookrightarrow Y(R,\mu)$.

We shall now give some examples of rearrangement-invariant spaces over $(0,1)$. A basic example are the *Lebesgue spaces* $L^p(0,1)$, $p \in [1,\infty]$, consisting of all $f \in \mathcal{M}(0,1)$ for which the functional

$$\|f\|_{L^p(0,1)} = \begin{cases} \left(\int_0^1 |f|^p(s)\,ds\right)^{1/p} & p < \infty; \\ \operatorname{ess\,sup}_{s\in(0,1)} |f|(s) & p = \infty \end{cases}$$

is finite. Recall that for each rearrangement-invariant space $X(0,1)$, we have

$$L^\infty(0,1) \hookrightarrow X(0,1) \hookrightarrow L^1(0,1). \tag{1.3}$$

Since none of the conditions $L^\infty(0,1) \overset{*}{\hookrightarrow} L^\infty(0,1)$ and $L^1(0,1) \overset{*}{\hookrightarrow} L^1(0,1)$ is satisfied (see [13, Remark 4.2]), it follows from (1.3) that there is no rearrangement-invariant space $X(0,1)$ for which $X(0,1) \overset{*}{\hookrightarrow} L^\infty(0,1)$ or $L^1(0,1) \overset{*}{\hookrightarrow} X(0,1)$. Furthemore, it is well known that the fact that a rearrangement-invariant space $X(0,1)$ is different from $L^\infty(0,1)$ can be characterized by $\lim_{s\to 0_+} \varphi_X(s) = 0$.

One can consider also more general sets of functions $L^{p,q}(0,1)$ and $L^{p,q;\alpha}(0,1)$ which were studied, e.g., in [5] and [12]. They consists of all $f \in \mathcal{M}(0,1)$ for which

$$\|f\|_{L^{p,q}(0,1)} = \left\| f^*(s) s^{\frac{1}{p}-\frac{1}{q}} \right\|_{L^q(0,1)} < \infty$$

and

$$\|f\|_{L^{p,q;\alpha}(0,1)} = \left\| f^*(s) s^{\frac{1}{p}-\frac{1}{q}} \left(\log(e/s)\right)^\alpha \right\|_{L^q(0,1)} < \infty,$$

respectively. Here, we assume that $p \in [1,\infty]$, $q \in [1,\infty]$, $\alpha \in \mathbb{R}$, and use the convention that $1/\infty = 0$. Note that $L^p(0,1) = L^{p,p}(0,1)$ and $L^{p,q}(0,1) = L^{p,q;0}(0,1)$ for every such p and q. However, it turns out that under these assumptions on p, q and α, $L^{p,q}(0,1)$ and $L^{p,q;\alpha}(0,1)$ do not have to be rearrangement-invariant spaces. To ensure that $L^{p,q;\alpha}(0,1)$ is a rearrangement-invariant space (up to equivalent norms), we need to assume that one of the following conditions is satisfied:

$$p = q = 1, \quad \alpha \ge 0; \tag{1.4}$$
$$1 < p < \infty; \tag{1.5}$$
$$p = \infty, \quad q < \infty, \quad \alpha + \frac{1}{q} < 0; \tag{1.6}$$
$$p = q = \infty, \quad \alpha \le 0. \tag{1.7}$$

In this case, $L^{p,q}(0,1)$ is called a *Lorentz space* and $L^{p,q;\alpha}(0,1)$ is called a *Lorentz-Zygmund space*.

Suppose that $L^{p_1,q_1;\alpha_1}(0,1)$ and $L^{p_2,q_2;\alpha_2}(0,1)$ are rearrangement-invariant spaces (up to equivalent norms). Then

$$L^{p_1,q_1;\alpha_1}(0,1) \hookrightarrow L^{p_2,q_2;\alpha_2}(0,1)$$

holds if and only if $p_1 > p_2$, or $p_1 = p_2$ and one of the following conditions is satisfied:

$$\begin{aligned} &p_1 < \infty, \quad q_1 \le q_2, \quad \alpha_1 \ge \alpha_2; \\ &p_1 = \infty, \quad q_1 \le q_2, \quad \alpha_1 + \frac{1}{q_1} \ge \alpha_2 + \frac{1}{q_2}; \\ &q_2 < q_1, \quad \alpha_1 + \frac{1}{q_1} > \alpha_2 + \frac{1}{q_2}. \end{aligned} \tag{1.8}$$

Let φ be a nonnegative nondecreasing concave function in $(0,1)$. The *Lorentz endpoint space* $\Lambda_\varphi(0,1)$ is the rearrangement-invariant space consisting of all functions $f \in \mathcal{M}(0,1)$ for which

$$\|f\|_{\Lambda_\varphi(0,1)} = \varphi(0_+)\|f\|_{L^\infty(0,1)} + \int_0^1 f^*(s)\varphi'(s)\,ds < \infty.$$

Here, we use the convention $0 \cdot \infty = 0$. Recall that $\Lambda_\varphi(0,1)$ has fundamental function φ and, moreover, it is the smallest rearrangement-invariant space having this fundamental function.

Throughout the thesis we shall adopt the following convention. When a functional ρ is defined on $\mathcal{M}(R,\mu)$ and $f : R \to [0,\infty]$ is a μ-measurable function on R such that $f = \infty$ on a subset E of R satisfying $\mu(E) > 0$, then we set $\rho(f) = \infty$.

1.2 Measure spaces

In this section we describe measure spaces on which we will later study compact Sobolev embeddings. Our most general results, namely those appering in Chapter 3, correspond to the following situation.

Let $n \in \mathbb{N}$ and let Ω denote an open connected subset of $\mathbb{R}^n$ endowed with a measure ν satisfying $\nu(\Omega) = 1$. Moreover, we assume that there exists a strictly positive continuous function ω on Ω such that

$$\nu(E) = \int_E \omega(x)\,dx \tag{1.9}$$

for every Lebesgue measurable subset $E \subseteq \Omega$.

For every E as above we define its *perimeter* in (Ω,ν) by

$$P_\nu(E,\Omega) = \int_{\Omega\cap\partial^M E} \omega(x)\,d\mathcal{H}^{n-1}(x),$$

where $\partial^M E$ stands for the essential boundary of E, in the sense of geometric measure theory (see [11]), and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. The *isoperimetric function* $I_{(\Omega,\nu)} : [0,1] \to [0,\infty)$ is then defined by

$$I_{(\Omega,\nu)}(s) = \inf\left\{P_\nu(E,\Omega) : E \subseteq \Omega,\ s \le \nu(E) \le \frac{1}{2}\right\}$$

if $s \in [0,1/2]$ and by $I_{(\Omega,\nu)}(s) = I_{(\Omega,\nu)}(1-s)$ if $s \in (1/2,1]$. Throughout the thesis we shall assume that (Ω,ν) is such that there exists a constant $C_1 > 0$ for which

$$I_{(\Omega,\nu)}(s) \geq C_1 s, \quad s \in [0,1/2]. \tag{1.10}$$

Moreover, if we denote $n' = n/(n-1)$ when $n > 1$ and $n' = \infty$ when $n = 1$, we have

$$C_2 s^{\frac{1}{n'}} \geq I_{(\Omega,\nu)}(s), \quad s \in [0,1/2], \tag{1.11}$$

for some constant $C_2 > 0$ independent of $s \in [0,1/2]$, see [3, Proposition 5.1].

Throughout the thesis, λ_n denotes the n-dimensional Lebesgue measure. For simplicity of notation we shall write Ω instead of (Ω, λ_n).

Let $X(0,1)$ be a rearrangement-invariant space. We denote

$$V^m X(\Omega,\nu) = \{u : \ u \text{ is an } m\text{-times weakly differentiable function in } \Omega \text{ such that } |\nabla^m u| \in X(\Omega,\nu)\},$$

where $\nabla^m u$ is the vector of all m-th order weak derivatives of the function u. According to [3, Proposition 5.2], the inclusion $V^m X(\Omega,\nu) \subseteq L^1(\Omega,\nu)$ is satisfied. Hence, the expression

$$\|u\|_{V^m X(\Omega,\nu)} = \|u\|_{L^1(\Omega,\nu)} + \||\nabla^m u|\|_{X(\Omega,\nu)} \tag{1.12}$$

defines a norm on $V^m X(\Omega,\nu)$.

Proposition 1.1. *The Sobolev space $V^m X(\Omega,\nu)$ equipped with the norm* (1.12) *is a Banach space.*

Proof. Let $(u_k)_{k=1}^\infty$ be a Cauchy sequence in $V^m X(\Omega,\nu)$. If $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multiindex, denote by $(D^\alpha u_k)_{k=1}^\infty$ the sequence consisting of weak derivatives with respect to the multiindex α of elements of the sequence $(u_k)_{k=1}^\infty$. Furthermore, set $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Owing to the completeness of $L^1(\Omega,\nu)$ and $X(\Omega,\nu)$, there is a function u on Ω such that $u_k \to u$ in $L^1(\Omega,\nu)$, and for each multiindex β satisfying $|\beta| = m$ there is a function v_β on Ω such that $D^\beta u_k \to v_\beta$ in $X(\Omega,\nu)$.

Suppose that Ω_0 is an open subset of Ω such that $\overline{\Omega_0}$ is compact in Ω. Since ω is continuous and strictly positive in Ω, there is a constant $c > 0$ such that $\omega \geq c$ in Ω_0. Hence, for every function $w \in L^1(\Omega,\nu)$ we have

$$\|w\|_{L^1(\Omega_0)} = \int_{\Omega_0} |w(x)|\,dx \leq \frac{1}{c}\int_{\Omega_0} |w(x)|\omega(x)\,dx = \frac{1}{c}\|w\|_{L^1(\Omega_0,\nu)} \leq \frac{1}{c}\|w\|_{L^1(\Omega,\nu)}. \tag{1.13}$$

Thus, $u_k \to u$ in $L^1(\Omega_0)$ and $D^\beta u_k \to v_\beta$ in $L^1(\Omega_0)$ for each multiindex β fulfilling $|\beta| = m$ (in the latter case we are using the fact that $X(\Omega,\nu) \hookrightarrow L^1(\Omega,\nu)$).

Let $k, l \in \mathbb{N}$ and let α be a multiindex satisfying $1 \leq |\alpha| < m$. By [9, Remark 5.11.4], by inequality (1.13) and by embedding $X(\Omega,\nu) \hookrightarrow L^1(\Omega,\nu)$, we have

$$\begin{aligned}
\|D^\alpha u_k - D^\alpha u_l\|_{L^1(\Omega_0)} &\leq C_1 \Big(\|u_k - u_l\|_{L^1(\Omega_0)} + \||\nabla^m u_k - \nabla^m u_l|\|_{L^1(\Omega_0)}\Big) \\
&\leq \frac{C_1}{c}\Big(\|u_k - u_l\|_{L^1(\Omega,\nu)} + \||\nabla^m u_k - \nabla^m u_l|\|_{L^1(\Omega,\nu)}\Big) \\
&\leq C_2\Big(\|u_k - u_l\|_{L^1(\Omega,\nu)} + \||\nabla^m u_k - \nabla^m u_l|\|_{X(\Omega,\nu)}\Big) \\
&= C_2\|u_k - u_l\|_{V^m X(\Omega,\nu)},
\end{aligned}$$

where $C_1 > 0$, $C_2 > 0$ are positive constants independent of k and l. Hence, $(D^\alpha u_k)_{k=1}^\infty$ is a Cauchy sequence in $L^1(\Omega_0)$. Owing to the completeness of $L^1(\Omega_0)$, for every such α we can find a function v_α such that $D^\alpha u_k \to v_\alpha$ in $L^1(\Omega_0)$. Passing, if necessary, to a subsequence, we may assume that $D^\alpha u_k \to v_\alpha$ a.e. in Ω_0. Hence, by different choices of Ω_0, it is correct to consider v_α to be defined a.e. in the entire Ω. Now, a standard argument (see, e.g., [11, Theorem 1.1.12]) yields that u is m-times weakly differentiable in Ω and $D^\alpha u = v_\alpha$ for every α fulfilling that $1 \le |\alpha| \le m$. Hence, $u_k \to u$ in $V^m X(\Omega, \nu)$, as required. □

Note that it follows from [3, Proposition 5.2] that for every $k = 1, 2, \dots, m-1$, we have the inclusion $V^m X(\Omega, \nu) \subseteq V^k L^1(\Omega, \nu)$. Hence, since the graph of the indentity map from $V^m X(\Omega, \nu)$ into $V^k L^1(\Omega, \nu)$ is closed, the closed graph theorem yields that the inclusion is continuous, that is,

$$V^m X(\Omega, \nu) \hookrightarrow V^k L^1(\Omega, \nu), \quad k = 1, 2, \dots, m-1. \tag{1.14}$$

Thanks to the embedding $X(\Omega, \nu) \hookrightarrow L^1(\Omega, \nu)$, (1.14) holds also for $k = m$.

We now describe concrete measure spaces we will deal with in Chapter 4, namely, John domains, Maz'ya classes of Euclidean domains and product probability spaces.

Let $n \in \mathbb{N}$, $n \ge 2$. A bounded domain $\Omega \subseteq \mathbb{R}^n$ endowed with the n-dimensional Lebesgue measure λ_n and fulfilling that $\lambda_n(\Omega) = 1$ is called a *John domain* if the reverse inequality to (1.11) is satisfied, i.e., if there is a constant $C_3 > 0$ such that

$$I_\Omega(s) \ge C_3 s^{\frac{1}{n'}}, \quad s \in [0, 1/2].$$

Let $\alpha \in [1/n', 1]$. We denote by $\mathcal{J}_\alpha$ the *Maz'ya class* of all bounded Euclidean domains $\Omega \subseteq \mathbb{R}^n$ with $\lambda_n(\Omega) = 1$ fulfilling that there is a positive constant C_4 such that

$$I_\Omega(s) \ge C_4 s^\alpha, \quad s \in [0, 1/2].$$

Assume that $\Phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function such that it is twice continuously differentiable and convex in $(0, \infty)$, $\sqrt{\Phi}$ is concave in $(0, \infty)$ and $\Phi(0) = 0$. Define the one-dimensional probability measure $\mu_\Phi = \mu_{\Phi,1}$ by

$$d\mu_\Phi(x) = c_\Phi e^{-\Phi(|x|)}\, dx, \tag{1.15}$$

where the constant $c_\Phi > 0$ is chosen in such a way that $\mu_\Phi(\mathbb{R}) = 1$. We also define the product measure $\mu_{\Phi,n}$ on $\mathbb{R}^n$, $n \ge 2$, by

$$\mu_{\Phi,n} = \underbrace{\mu_\Phi \times \cdots \times \mu_\Phi}_{n-times}. \tag{1.16}$$

Then $(\mathbb{R}^n, \mu_{\Phi,n})$ is a probability space for every $n \in \mathbb{N}$ and we have

$$d\mu_{\Phi,n}(x) = (c_\Phi)^n e^{-(\Phi(|x_1|) + \Phi(|x_2|) + \cdots + \Phi(|x_n|))}\, dx.$$

Define the function $F_\Phi : \mathbb{R} \to (0, 1)$ by

$$F_\Phi(t) = \int_t^\infty c_\Phi e^{-\Phi(|r|)}\, dr, \ t \in \mathbb{R},$$

the function $I_\Phi : [0,1] \to [0,\infty)$ by

$$I_\Phi(t) = c_\Phi e^{-\Phi(|F_\Phi^{-1}(t)|)}, \quad t \in (0,1),$$

and $I_\Phi(0) = I_\Phi(1) = 0$, and the function $L_\Phi : [0,1] \to [0,\infty)$ by

$$L_\Phi(t) = t\Phi'\left(\Phi^{-1}\left(\log\frac{2}{t}\right)\right), \quad t \in (0,1],$$

and $L_\Phi(0) = 0$. Then the isoperimetric function of $(\mathbb{R}^n, \mu_{\Phi,n})$ satisfies

$$I_{(\mathbb{R}^n,\mu_{\Phi,n})}(t) \approx I_\Phi(t) \approx L_\Phi(t), \quad t \in [0,1/2], \tag{1.17}$$

see [1, Proposition 13 and Theorem 15].

The main example of product probability measures we have just defined is the n-dimensional *Gauss measure*

$$d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{\frac{-|x|^2}{2}}\, dx,$$

which can be obtained by setting

$$\Phi(t) = \frac{1}{2}t^2, \quad t \in [0,\infty),$$

into (1.15) (if n=1) or (1.16) (if $n > 1$).

More generally, measures associated with

$$\Phi(t) = \frac{1}{\beta}t^\beta, \quad t \in [0,\infty),$$

for some $\beta \in [1,2]$ are also examples of product probability measures. For each $\beta \in [1,2]$, such n-dimensional measure is denoted by $\gamma_{n,\beta}$ and satisfies

$$d\gamma_{n,\beta}(x) = c_\beta^n e^{\frac{-|x|^\beta}{\beta}}\, dx,$$

where $c_\beta > 0$ is chosen in such a way that $\gamma_{1,\beta}(\mathbb{R}) = 1$. We of course have $\gamma_{n,2} = \gamma_n$.

2. Compact operators

Let $J : [0,1] \to [0,\infty)$ be a measurable function satisfying

$$\inf_{t\in(0,1)} \frac{J(t)}{t} > 0. \tag{2.1}$$

We set

$$J_a = \inf_{t\in[a,1]} J(t), \quad a \in (0,1), \tag{2.2}$$

and observe that for every $a \in (0,1)$,

$$J_a \geq Ca > 0,$$

where $C = \inf_{t\in(0,1)} J(t)/t$.

Let $m \in \mathbb{N}$. We define the operator H_J^m by

$$H_J^m f(t) = \int_t^1 \frac{f(s)}{J(s)} \left(\int_t^s \frac{dr}{J(r)} \right)^{m-1} ds, \quad f \in L^1(0,1),\ t \in (0,1). \tag{2.3}$$

Consider also the operator H_J defined by

$$H_J f(t) = \int_t^1 \frac{f(s)}{J(s)}\, ds, \quad f \in L^1(0,1),\ t \in (0,1).$$

Then

$$H_J^m = (m-1)!\, \underbrace{H_J \circ H_J \circ \cdots \circ H_J}_{m-times}, \tag{2.4}$$

see [3, Remarks 10.1]. Furthermore, observe that whenever $f \in L^1(0,1)$ is nonnegative in $(0,1)$ then $H_J^m f$ is nonincreasing in $(0,1)$.

Let $X(0,1)$ be a rearrangement-invariant space. For every $f \in \mathcal{M}(0,1)$ define the functional $\|\cdot\|_{(X_{m,J})'(0,1)}$ by

$$\|f\|_{(X_{m,J})'(0,1)} = \left\| \frac{1}{J(s)} \int_0^s \left(\int_t^s \frac{dr}{J(r)} \right)^{m-1} f^*(t)\, dt \right\|_{X'(0,1)} \tag{2.5}$$

and let $(X_{m,J})'(0,1)$ be the collection of all $f \in \mathcal{M}(0,1)$ for which $\|f\|_{(X_{m,J})'(0,1)} < \infty$. Then, according to [3, Proposition 8.1], $(X_{m,J})'(0,1)$ is a rearrangement-invariant space and

$$H_J^m : X(0,1) \to X_{m,J}(0,1), \tag{2.6}$$

where $X_{m,J}(0,1)$ denotes the associate space to the space $(X_{m,J})'(0,1)$. Moreover, $X_{m,J}(0,1)$ is the optimal (i.e., the smallest) rearrangement-invariant space for which (2.6) is satisfied.

Remark 2.1. (i) The function J defined above does not denote the class of equivalence of all functions which coincide a.e. Instead of this, we suppose that J is one particular representative defined everywhere in [0,1].

(ii) In [3], the function J is supposed to be nondecreasing in $[0,1]$. However, this additional assumption has no significance for the proof of (2.4) and (2.6).

We start by proving that the image by H_J^m of the unit ball of each rearrangement-invariant space is compact in measure.

Lemma 2.2. *Let $J : [0,1] \to [0,\infty)$ be a measurable function satisfying* (2.1) *and let $m \in \mathbb{N}$. Suppose that $X(0,1)$ is a rearrangement-invariant space and $(f_k)_{k=1}^\infty$ is a bounded sequence in $X(0,1)$. Then there is a subsequence $(f_{k_\ell})_{\ell=1}^\infty$ of $(f_k)_{k=1}^\infty$ such that $(H_J^m f_{k_\ell})_{\ell=1}^\infty$ converges pointwise a.e. in $(0,1)$.*

Proof. Because every rearrangement-invariant space $X(0,1)$ is embedded into $L^1(0,1)$, the sequence $(f_k)_{k=1}^\infty$ is bounded in $L^1(0,1)$. First, suppose that $m=1$. Let $j>1$ be an integer. Thanks to (2.6) and (1.3), we have

$$H_J : L^1(0,1) \to (L^1)_{1,J}(0,1) \hookrightarrow L^1(0,1),$$

so the sequence $(H_J f_k)_{k=1}^\infty$ is bounded in $L^1(0,1)$. Therefore, in particular, $(\chi_{(1/j,1)} H_J f_k)_{k=1}^\infty$ is bounded in $L^1(1/j,1)$. For every $k \in \mathbb{N}$, we have that the function $\chi_{(1/j,1)} H_J f_k$ is absolutely continuous in $(1/j,1)$ and

$$\left|(\chi_{(1/j,1)} H_J f_k)'(t)\right| = \frac{|f_k(t)|}{J(t)} \le \frac{|f_k(t)|}{J_{1/j}}$$

for a.e. $t \in (1/j,1)$. Since $(f_k)_{k=1}^\infty$ is bounded in $L^1(0,1)$, $((\chi_{(1/j,1)} H_J f_k)')_{k=1}^\infty$ is bounded in $L^1(1/j,1)$ and $(\chi_{(1/j,1)} H_J f_k)_{k=1}^\infty$ is therefore bounded in $V^1L^1(1/j,1)$.

Denote $f_k^1 = f_k$, $k \in \mathbb{N}$. By induction, for every integer $j>1$ we will construct a subsequence $(f_k^j)_{k=1}^\infty$ of the sequence $(f_k^{j-1})_{k=1}^\infty$ such that $(H_J f_k^j)_{k=1}^\infty$ converges a.e. in $(1/j,1)$. Suppose that, for some $j>1$, we have already found the sequence $(f_k^{j-1})_{k=1}^\infty$. Due to the results of the previous paragraph, $(\chi_{(1/j,1)} H_J f_k^{j-1})_{k=1}^\infty$ is bounded in $V^1L^1(1/j,1)$. Then, thanks to the compact embedding $V^1L^1(1/j,1) \hookrightarrow\hookrightarrow L^1(1/j,1)$, we can find a subsequence $(f_k^j)_{k=1}^\infty$ of the sequence $(f_k^{j-1})_{k=1}^\infty$ such that $(H_J f_k^j)_{k=1}^\infty$ converges a.e. in $(1/j,1)$, as required. Now, the diagonal sequence $(H_J f_k^k)_{k=1}^\infty$ converges a.e. in $(0,1)$. This completes the proof in the case that $m=1$.

Finally, suppose that $m>1$. Then, due to (2.6) and (1.3),

$$H_J^{m-1} : L^1(0,1) \to (L^1)_{m-1,J}(0,1) \hookrightarrow L^1(0,1),$$

so $(H_J^{m-1} f_k)_{k=1}^\infty$ is bounded in $L^1(0,1)$ and the first part of the proof implies that there is a subsequence $(f_{k_\ell})_{\ell=1}^\infty$ of $(f_k)_{k=1}^\infty$ such that $(H_J(H_J^{m-1} f_{k_\ell}))_{\ell=1}^\infty$ converges a.e. in $(0,1)$. But (2.4) gives

$$\begin{aligned} H_J^m &= (m-1)!\, \underbrace{H_J \circ H_J \circ \cdots \circ H_J}_{m-times} \\ &= (m-1) H_J \circ ((m-2)!\, \underbrace{H_j \circ H_J \circ \cdots \circ H_J}_{(m-1)-times}) = (m-1) H_J \circ H_J^{m-1}, \end{aligned} \tag{2.7}$$

i.e., $(H_J^m f_{k_\ell})_{\ell=1}^\infty$ converges a.e. in $(0,1)$. □

The main result of this chapter is the following

Theorem 2.3. *Let $J : [0,1] \to [0,\infty)$ be a measurable function satisfying* (2.1) *and let $m \in \mathbb{N}$. Suppose that $X(0,1)$ and $Y(0,1)$ are rearrangement-invariant spaces. Then, except of the case that $X(0,1) = L^1(0,1)$, $Y(0,1) = L^\infty(0,1)$, $\int_0^1 1/J(t)\,dt < \infty$ and $m = 1$, the following two conditions are equivalent:*

(i) $H_J^m : X(0,1) \to\to Y(0,1)$;

(ii) $\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|H_J^m(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0$.

In the case that $X(0,1) = L^1(0,1)$, $Y(0,1) = L^\infty(0,1)$, $\int_0^1 1/J(t)\,dt < \infty$ and $m = 1$, the implication (i) $\Rightarrow$ (ii) *is still true.*

Moreover, provided that

$$Y(0,1) \neq L^\infty(0,1) \text{ or } \int_0^1 \frac{dt}{J(t)} = \infty, \tag{2.8}$$

conditions (i) *and* (ii) *are equivalent to*

(iii) $\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,a)} H_J^m|f|\|_{Y(0,1)} = 0$

and

(iv) $X_{m,J}(0,1) \overset{*}{\hookrightarrow} Y(0,1)$.

Proof. (i) $\Rightarrow$ (ii) For every $k \in \mathbb{N}$ we can find a nonnegative measurable function f_k in $(0,1)$ such that $\|f_k\|_{X(0,1)} \le 1$ and

$$\sup_{\|f\|_{X(0,1)}\le 1} \|H_J^m(\chi_{(0,1/k)}|f|)\|_{Y(0,1)} < \|H_J^m(\chi_{(0,1/k)}f_k)\|_{Y(0,1)} + \frac{1}{k}. \tag{2.9}$$

Because the sequence $(\chi_{(0,1/k)}f_k)_{k=1}^\infty$ is bounded in $X(0,1)$, the assumption (i) yields that there is a subsequence $(f_{k_\ell})_{\ell=1}^\infty$ of $(f_k)_{k=1}^\infty$ such that $(H_J^m(\chi_{(0,1/k_\ell)}f_{k_\ell}))_{\ell=1}^\infty$ converges to some function f in $Y(0,1)$. Moreover, the subsequence can be found in such a way that $(H_J^m(\chi_{(0,1/k_\ell)}f_{k_\ell}))_{\ell=1}^\infty$ converges to f a.e. in $(0,1)$. But $H_J^m(\chi_{(0,1/k_\ell)}f_{k_\ell}) = 0$ in $(1/k_\ell, 1)$, which implies that $H_J^m(\chi_{(0,1/k_\ell)}f_{k_\ell}) \to 0$ pointwise. Thus, $f = 0$ a.e. in $(0,1)$. This yields

$$\lim_{\ell\to\infty} \|H_J^m(\chi_{(0,1/k_\ell)}f_{k_\ell})\|_{Y(0,1)} = 0.$$

Now, the inequality (2.9) gives

$$\lim_{\ell\to\infty} \sup_{\|f\|_{X(0,1)}\le 1} \|H_J^m(\chi_{(0,1/k_\ell)}|f|)\|_{Y(0,1)} = 0.$$

Because the function

$$a \mapsto \sup_{\|f\|_{X(0,1)}\le 1} \|H_J^m(\chi_{(0,a)}|f|)\|_{Y(0,1)}$$

is nondecreasing in $(0,1)$, we obtain (ii), as required.

(ii) $\Rightarrow$ (iii) We assume that (2.8) is satisfied. Let $\varepsilon > 0$. Due to (ii), we can find $a \in (0,1)$ such that

$$\sup_{\|f\|_{X(0,1)}\le 1} \|H_J^m(\chi_{(0,a)}|f|)\|_{Y(0,1)} < \frac{\varepsilon}{2}.$$

Once we show that there is $b \in (0,1)$ such that

$$\sup_{\|f\|_{X(0,1)} \le 1} \|\chi_{(0,b)} H_J^m(\chi_{(a,1)}|f|)\|_{Y(0,1)} < \frac{\varepsilon}{2}, \tag{2.10}$$

it will be easy to complete the proof. Indeed, we have

$$\begin{aligned}
&\sup_{\|f\|_{X(0,1)} \le 1} \|\chi_{(0,b)} H_J^m |f|\|_{Y(0,1)} \\
&\le \sup_{\|f\|_{X(0,1)} \le 1} \|\chi_{(0,b)} H_J^m(\chi_{(0,a)}|f|)\|_{Y(0,1)} + \sup_{\|f\|_{X(0,1)} \le 1} \|\chi_{(0,b)} H_J^m(\chi_{(a,1)}|f|)\|_{Y(0,1)} < \varepsilon.
\end{aligned}$$

Then, using that the function

$$b \mapsto \sup_{\|f\|_{X(0,1)} \le 1} \|\chi_{(0,b)} H_J^m |f|\|_{Y(0,1)}$$

is nondecreasing in $(0,1)$, we obtain (iii), as required.

It remains to find $b \in (0,1)$ such that (2.10) is satisfied. For every function $f \in \mathcal{M}(0,1)$ fulfilling $\|f\|_{X(0,1)} \le 1$ and for every $t \in (0,a)$ we have

$$\begin{aligned}
H_J^m(\chi_{(a,1)}|f|)(t) &= \int_a^1 \frac{|f(s)|}{J(s)} \left(\int_t^s \frac{dr}{J(r)} \right)^{m-1} ds \\
&\le \frac{1}{J_a} \left(\int_t^1 \frac{dr}{J(r)} \right)^{m-1} \int_a^1 |f(s)|\, ds \\
&\le \frac{1}{J_a} \left(\int_t^1 \frac{dr}{J(r)} \right)^{m-1} \|f\|_{L^1(0,1)} \\
&\le \frac{C}{J_a} \left(\int_t^1 \frac{dr}{J(r)} \right)^{m-1} \|f\|_{X(0,1)} \\
&\le \frac{C}{J_a} \left(\int_t^1 \frac{dr}{J(r)} \right)^{m-1},
\end{aligned} \tag{2.11}$$

where $C > 0$ is the constant from the embedding $X(0,1) \hookrightarrow L^1(0,1)$.

Now, suppose that

$$\int_0^1 \frac{dt}{J(t)} < \infty. \tag{2.12}$$

Then for every f and t as above we have

$$H_J^m(\chi_{(a,1)}|f|)(t) \le \frac{C}{J_a} \left(\int_t^1 \frac{dr}{J(r)} \right)^{m-1} \le \frac{C}{J_a} \left(\int_0^1 \frac{dr}{J(r)} \right)^{m-1} = D.$$

Since (2.12) is in progress, we necessarily have $Y(0,1) \neq L^\infty(0,1)$. So, as it was pointed out in Section 1.1, there is $b \in (0,a)$ such that $\|\chi_{(0,b)}\|_{Y(0,1)} < \varepsilon/2D$. This implies

$$\sup_{\|f\|_{X(0,1)} \le 1} \|\chi_{(0,b)} H_J^m(\chi_{(a,1)}|f|)\|_{Y(0,1)} \le D \|\chi_{(0,b)}\|_{Y(0,1)} < \frac{\varepsilon}{2}.$$

Finally, we will discuss the case when

$$\int_0^1 \frac{dt}{J(t)} = \infty. \tag{2.13}$$

We can find $a_1 \in (0,a)$ such that

$$\int_t^1 \frac{dr}{J(r)} \geq \frac{C}{J_a}$$

for every $t \in (0,a_1)$. Then, owing to (2.11), for any such t and for every f from the unit ball of $X(0,1)$, we have

$$H_J^m(\chi_{(a,1)}|f|)(t) \leq \frac{C}{J_a}\left(\int_t^1 \frac{dr}{J(r)}\right)^{m-1} \leq \left(\int_t^1 \frac{dr}{J(r)}\right)^m. \tag{2.14}$$

Choose $c>0$ such that $\|\chi_{(0,1)}c\|_{X(0,1)}=1$. Then

$$\lim_{d\to 0_+} \|H_J^m(\chi_{(0,d)}c)\|_{Y(0,1)} \leq \lim_{d\to 0_+} \sup_{\|f\|_{X(0,1)}\leq 1} \|H_J^m(\chi_{(0,d)}|f|)\|_{Y(0,1)} = 0,$$

so

$$\lim_{d\to 0_+} \|H_J^m(\chi_{(0,d)})\|_{Y(0,1)} = 0. \tag{2.15}$$

Fix $d\in(0,1)$ and $t\in(0,d)$. Denote

$$\varphi(s) = \int_t^s \frac{dy}{J(y)}, \quad s\in(t,d).$$

Then φ is an increasing Lipschitz function in (t,d), because $|\varphi'(s)| = 1/J(s) \leq 1/J_t$ for a.e. $s\in(t,d)$. So, due to the change of variables theorem,

$$\begin{aligned}
H_J^m(\chi_{(0,d)})(t) &= \chi_{(0,d)}(t)\int_t^d \frac{1}{J(s)}\left(\int_t^s \frac{dy}{J(y)}\right)^{m-1} ds \\
&= \chi_{(0,d)}(t)\int_t^d \varphi'(s)\,(\varphi(s))^{m-1}\,ds \\
&= \chi_{(0,d)}(t)\int_{\varphi(t)}^{\varphi(d)} r^{m-1}\,dr \\
&= \frac{1}{m}\chi_{(0,d)}(t)(\varphi(d))^m = \frac{1}{m}\chi_{(0,d)}(t)\left(\int_t^d \frac{dy}{J(y)}\right)^m.
\end{aligned} \tag{2.16}$$

Combining this with (2.15), we obtain

$$\lim_{d\to 0_+}\left\|\chi_{(0,d)}(t)\left(\int_t^d \frac{dy}{J(y)}\right)^m\right\|_{Y(0,1)} = 0.$$

Thus, we can find $a_2 \in (0,a_1)$ such that

$$\left\|\chi_{(0,a_2)}(t)\left(\int_t^{a_2} \frac{dy}{J(y)}\right)^m\right\|_{Y(0,1)} < \frac{\varepsilon}{2^{m+1}}. \tag{2.17}$$

Thanks to (2.13), there is $b\in(0,a_2)$ such that for every $t\in(0,b)$ we have

$$\left(\int_t^1 \frac{dr}{J(r)}\right)^m = \left(\int_t^{a_2} \frac{dr}{J(r)} + \int_{a_2}^1 \frac{dr}{J(r)}\right)^m \leq \left(2\int_t^{a_2} \frac{dr}{J(r)}\right)^m.$$

This inequality together with (2.14) and (2.17) gives

$$\sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,b)} H_J^m(\chi_{(a,1)}|f|)\|_{Y(0,1)} \le 2^m \left\| \chi_{(0,a_2)}(t) \left(\int_t^{a_2} \frac{dr}{J(r)} \right)^m \right\|_{Y(0,1)} < \frac{\varepsilon}{2}.$$

(iii) $\Rightarrow$ (iv) Using the definition of the associate norm and the Fubini theorem, we get

$$\begin{aligned}
&\lim_{a\to 0_+} \sup_{\|f\|_{Y'(0,1)}\le 1} \|\chi_{(0,a)} f^*\|_{(X_{m,J})'(0,1)} \\
&= \lim_{a\to 0_+} \sup_{\|f\|_{Y'(0,1)}\le 1} \left\| \frac{1}{J(s)} \int_0^s \chi_{(0,a)}(t) f^*(t) \left(\int_t^s \frac{dy}{J(y)} \right)^{m-1} dt \right\|_{X'(0,1)} \\
&= \lim_{a\to 0_+} \sup_{\|f\|_{Y'(0,1)}\le 1} \sup_{\|g\|_{X(0,1)}\le 1} \int_0^1 \frac{|g(s)|}{J(s)} \int_0^s \chi_{(0,a)}(t) f^*(t) \left(\int_t^s \frac{dy}{J(y)} \right)^{m-1} dt\, ds \\
&= \lim_{a\to 0_+} \sup_{\|f\|_{Y'(0,1)}\le 1} \sup_{\|g\|_{X(0,1)}\le 1} \int_0^1 \chi_{(0,a)}(t) f^*(t) \int_t^1 \frac{|g(s)|}{J(s)} \left(\int_t^s \frac{dy}{J(y)} \right)^{m-1} ds\, dt \\
&= \lim_{a\to 0_+} \sup_{\|g\|_{X(0,1)}\le 1} \sup_{\|f\|_{Y'(0,1)}\le 1} \int_0^1 \chi_{(0,a)}(t) f^*(t) H_J^m|g|(t)\, dt \\
&= \lim_{a\to 0_+} \sup_{\|g\|_{X(0,1)}\le 1} \sup_{\|f\|_{Y'(0,1)}\le 1} \int_0^1 f^*(t) \left(\chi_{(0,a)} H_J^m|g| \right)^*(t)\, dt \\
&= \lim_{a\to 0_+} \sup_{\|g\|_{X(0,1)}\le 1} \|\chi_{(0,a)} H_J^m|g|\|_{Y(0,1)} = 0.
\end{aligned}$$

Note that the second last equality holds because $\chi_{(0,a)} H_J^m|g|$ is nonincreasing in $(0,1)$ for every $a \in (0,1)$ and $g \in X(0,1)$. Thus, we have proved that $Y'(0,1) \overset{*}{\hookrightarrow} (X_{m,J})'(0,1)$, which is a condition equivalent to (iv), see [6, Section 4, property 5].

(iv) $\Rightarrow$ (i) Suppose that $(f_k)_{k=1}^\infty$ is a sequence bounded in $X(0,1)$. According to Lemma 2.2, there is a subsequence $(f_{k_\ell})_{\ell=1}^\infty$ of the sequence $(f_k)_{k=1}^\infty$ such that $(H_J^m f_{k_\ell})_{\ell=1}^\infty$ converges to some function f a.e. in $(0,1)$. Moreover, (2.6) implies that $(H_J^m f_{k_\ell})_{\ell=1}^\infty$ is bounded in $X_{m,J}(0,1)$. Hence, by the Fatou lemma,

$$\|f\|_{X_{m,J}(0,1)} \le \liminf_{\ell\to\infty} \|H_J^m f_{k_\ell}\|_{X_{m,J}(0,1)} < \infty,$$

so, $f \in X_{m,J}(0,1)$. Thus, $(H_J^m f_{k_\ell} - f)_{\ell=1}^\infty$ is bounded in $X_{m,J}(0,1)$ and, by using (iv) and [13, Theorem 3.1], we get that $(H_J^m f_{k_\ell} - f) \to 0$ in $Y(0,1)$, i.e., $H_J^m f_{k_\ell} \to f$ in $Y(0,1)$. Therefore, $H_J^m : X(0,1) \to\to Y(0,1)$.

(ii) $\Rightarrow$ (i) This implication has already been proved in the case that (2.8) is satisfied. Thus, we can suppose that $Y(0,1) = L^\infty(0,1)$ and $\int_0^1 1/J(t)\, dt < \infty$.

We first observe that it is enough to show that for every $a \in (0,1)$, the operator $H_{J,a}^m : f \mapsto H_J^m(\chi_{(a,1)} f)$ is compact from $X(0,1)$ into $L^\infty(0,1)$. Indeed, thanks to (ii) and to the fact that $|H_J^m(\chi_{(0,a)} f)| \le H_J^m(\chi_{(0,a)}|f|)$ for every $f \in X(0,1)$ and $a \in (0,1)$, we have

$$\begin{aligned}
\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|H_J^m f - H_{J,a}^m f\|_{L^\infty(0,1)} &= \lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|H_J^m(\chi_{(0,a)} f)\|_{L^\infty(0,1)} \\
&\le \lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|H_J^m(\chi_{(0,a)}|f|)\|_{L^\infty(0,1)} = 0,
\end{aligned}$$

so H_J^m will be a norm limit of compact operators, and thus itself a compact operator.

Let $a \in (0,1)$ and let f be a measurable function on $(0,1)$ satisfying $\|f\|_{X(0,1)} \leq 1$. Consider $H_{J,a}^m f = H_J^m(\chi_{(a,1)} f)$ to be defined by (2.3) in the entire $[0,1]$. Such a definition is correct since the integral in (2.3) is convergent for every $t \in [0,1]$. Indeed,

$$\begin{aligned} |H_{J,a}^m f(t)| &\leq \int_t^1 \frac{\chi_{(a,1)}(s)|f(s)|}{J(s)} \left(\int_t^s \frac{dr}{J(r)}\right)^{m-1} ds \\ &\leq \int_a^1 \frac{|f(s)|}{J(s)} \left(\int_0^1 \frac{dr}{J(r)}\right)^{m-1} ds \\ &\leq \frac{1}{J_a} \left(\int_0^1 \frac{dr}{I(r)}\right)^{m-1} \|f\|_{L^1(0,1)} \\ &\leq \frac{C}{J_a} \left(\int_0^1 \frac{dr}{I(r)}\right)^{m-1} \|f\|_{X(0,1)} \leq D \end{aligned} \tag{2.18}$$

for every $t \in [0,1]$, where $C > 0$, $D > 0$ are as in the proof of implication (ii) $\Rightarrow$ (iii). Inequality (2.18) also implies that the image by $H_{J,a}^m$ of the unit ball of $X(0,1)$ is equibounded in $[0,1]$.

Suppose that $0 \leq t_1 < t_2 \leq 1$. We have

$$\begin{aligned} &|H_{J,a}^m f(t_1) - H_{J,a}^m f(t_2)| \\ &= \left| \int_{t_1}^1 \frac{\chi_{(a,1)}(s)f(s)}{J(s)} \left(\int_{t_1}^s \frac{dr}{J(r)}\right)^{m-1} ds - \int_{t_2}^1 \frac{\chi_{(a,1)}(s)f(s)}{J(s)} \left(\int_{t_2}^s \frac{dr}{J(r)}\right)^{m-1} ds \right| \\ &\leq \int_{t_1}^{t_2} \frac{\chi_{(a,1)}(s)|f(s)|}{J(s)} \left(\int_{t_1}^s \frac{dr}{J(r)}\right)^{m-1} ds \\ &+ \int_{t_2}^1 \frac{\chi_{(a,1)}(s)|f(s)|}{J(s)} \left(\left(\int_{t_1}^s \frac{dr}{J(r)}\right)^{m-1} - \left(\int_{t_2}^s \frac{dr}{J(r)}\right)^{m-1} \right) ds. \end{aligned} \tag{2.19}$$

Assume that $m > 1$. Then

$$\begin{aligned} &|H_{J,a}^m f(t_1) - H_{J,a}^m f(t_2)| \leq \frac{1}{J_a} \left(\int_{t_1}^{t_2} \frac{dr}{J(r)}\right)^{m-1} \|f\|_{L^1(0,1)} \\ &+ \frac{1}{J_a} \int_{t_2}^1 |f(s)| \left(\int_{t_1}^s \frac{dr}{J(r)} - \int_{t_2}^s \frac{dr}{J(r)}\right) \left(\sum_{i=0}^{m-2} \left(\int_{t_1}^s \frac{dr}{J(r)}\right)^i \left(\int_{t_2}^s \frac{dr}{J(r)}\right)^{m-2-i} \right) ds \\ &\leq \frac{C}{J_a} \left(\int_{t_1}^{t_2} \frac{dr}{J(r)}\right)^{m-1} \|f\|_{X(0,1)} + \frac{m-1}{J_a} \left(\int_0^1 \frac{dr}{J(r)}\right)^{m-2} \int_{t_1}^{t_2} \frac{dr}{J(r)} \|f\|_{L^1(0,1)} \\ &\leq C' \int_{t_1}^{t_2} \frac{dr}{J(r)}, \end{aligned}$$

where $C > 0$, $C' > 0$ are constants independent of f. Thanks to the absolute continuity of the Lebesgue integral, the last expression goes to 0 when $t_2 - t_1$ tends to 0.

Let $m = 1$. We need the additional assumption $X(0,1) \neq L^1(0,1)$ which implies that $X'(0,1) \neq L^\infty(0,1)$. Then, using (2.19) and the Hölder inequality, we deduce that

$$\sup_{\|f\|_{X(0,1)}\le 1} |H^m_{J,a}f(t_1) - H^m_{J,a}f(t_2)| \le \frac{1}{J_a} \sup_{\|f\|_{X(0,1)}\le 1} \int_{t_1}^{t_2} |f(s)|\,ds$$
$$\le \frac{1}{J_a} \sup_{\|f\|_{X(0,1)}\le 1} \|f\|_{X(0,1)} \|\chi_{(t_1,t_2)}\|_{X'(0,1)}$$
$$= \frac{1}{J_a}\|\chi_{(t_1,t_2)}\|_{X'(0,1)} = \frac{1}{J_a}\|\chi_{(0,t_2-t_1)}\|_{X'(0,1)},$$

which goes to 0 when $t_2 - t_1$ tends to 0. This proves the equicontinuity. Arzela-Ascoli theorem now yields that $H^m_{J,a}$ maps the unit ball of $X(0,1)$ into a relatively compact set in $C([0,1])$. Because for every $f \in X(0,1)$, the norm of $H^m_{J,a}f$ in $C([0,1])$ coincides with its norm in $L^\infty(0,1)$, the operator $H^m_{J,a}$ is compact from $X(0,1)$ into $L^\infty(0,1)$. The proof is complete. □

Remark 2.4. It is easy to observe that the only function having an absolutely continuous norm in $L^\infty(0,1)$ is the constant function 0. Thus, none of the conditions (iii) and (iv) from the statement of Theorem 2.3 can hold with $Y(0,1) = L^\infty(0,1)$. In particular, if $\int_0^1 1/J(t)\,dt = \infty$ and the condition (i) (or, equivalently, (ii)) is satisfied, we have $Y(0,1) \ne L^\infty(0,1)$. In this sense, it would be equivalent to replace the assumption (2.8) just with $Y(0,1) \ne L^\infty(0,1)$.

Furthermore, one can observe that in the proof of (iii) ⇒ (iv) and (iv) ⇒ (i), we make no use of (2.8). However, these two implications have no significance in the case that $Y(0,1) = L^\infty(0,1)$, because none of the assumptions (iii) and (iv) can be satisfied.

Example 2.5. Suppose that $J : [0,1] \to [0,\infty)$ is a measurable function satisfying (2.1). We will show that, in general, the condition

$$\lim_{a\to 0_+} \sup_{\|f\|_{L^1(0,1)}\le 1} \|H_J(\chi_{(0,a)}|f|)\|_{L^\infty(0,1)} = 0 \tag{2.20}$$

does not imply

$$H_J : L^1(0,1) \to\to L^\infty(0,1). \tag{2.21}$$

Indeed, for an arbitrary $\beta > 0$, set $J(t) = t^{-\beta}$, $t \in (0,1]$, and $J(0) = 0$. Then J is measurable in $[0,1]$ and satisfies (2.1) as well as (2.20), because

$$\lim_{a\to 0_+} \sup_{\|f\|_{L^1(0,1)}\le 1} \|H_J(\chi_{(0,a)}|f|)\|_{L^\infty(0,1)} = \lim_{a\to 0_+} \sup_{\|f\|_{L^1(0,1)}\le 1} \int_0^a |f(s)| s^\beta\,ds$$
$$= \lim_{a\to 0_+} \left\|\chi_{(0,a)}(s)s^\beta\right\|_{L^\infty(0,1)} = \lim_{a\to 0_+} a^\beta = 0.$$

Let $t \in (0,1/2)$. Define the function f_t by $f_t(s) = \chi_{(1-t,1)}(s)1/t$, $s \in (0,1)$. Then $\|f_t\|_{L^1(0,1)} = 1$. We can consider $H_J f_t$ to be a continuous function defined in the entire $[0,1]$. Then we have

$$H_J f_t(1-t) - H_J f_t(1) = \int_{1-t}^1 \chi_{(1-t,1)}(s)\frac{s^\beta}{t}\,ds \ge \frac{1}{2^\beta}\int_{1-t}^1 \chi_{(1-t,1)}(s)\frac{1}{t}\,ds = \frac{1}{2^\beta},$$

so the image by H_J of the unit ball of $X(0,1)$ is not equicontinuous in $[0,1]$, hence, due to the Arzela-Ascoli theorem, it is not relatively compact in $C([0,1])$. Because the norm of $H_J f$ in $C([0,1])$ coincides with its norm in $L^\infty(0,1)$ for every $f \in L^1(0,1)$, the condition (2.21) cannot be satisfied.

Suppose that a function $J : [0,1] \to [0,\infty)$ is nondecreasing and satisfies (2.1) and

$$\int_0^1 \frac{dt}{J(t)} < \infty. \tag{2.22}$$

Define the function

$$\Psi(t) = \int_0^t \frac{ds}{J(s)}, \quad t \in (0,1).$$

Since Ψ is nonnegative, nondecreasing and concave in $(0,1)$, we can consider the rearrangement-invariant space $A_J(0,1) = \Lambda_\Psi(0,1)$. The main properties of this space are described in the following

Lemma 2.6. *Let $J : [0,1] \to [0,\infty)$ be a nondecreasing function satisfying* (2.1) *and* (2.22)*. Then for every $f \in \mathcal{M}(0,1)$ we have*

$$\|f\|_{A_J(0,1)} = \int_0^1 \frac{f^*(t)}{J(t)}\,dt.$$

Furthermore, the rearrangement-invariant space $A_J(0,1)$ is different from $L^\infty(0,1)$ and fulfills

$$H_J : A_J(0,1) \to L^\infty(0,1).$$

Proof. Due to the absolute continuity of the Lebesgue integral,

$$\lim_{t\to 0_+} \Psi(t) = \lim_{t\to 0_+} \int_0^t \frac{ds}{J(s)} = 0. \tag{2.23}$$

Hence, $A_J(0,1) = \Lambda_\Psi(0,1)$ is different from $L^\infty(0,1)$. Furthemore, for every $f \in \mathcal{M}(0,1)$ we have

$$\|f\|_{A_J(0,1)} = \|f\|_{\Lambda_\Psi(0,1)} = \|f\|_{L^\infty(0,1)} \lim_{t\to 0_+} \Psi(t) + \int_0^1 f^*(t)\Psi'(t)\,dt = \int_0^1 \frac{f^*(t)}{J(t)}\,dt.$$

Finally, using the Hardy-Littlewood inequality (1.1), we obtain

$$\begin{aligned} \|H_J f\|_{L^\infty(0,1)} &= \left\| \int_t^1 \frac{f(s)}{J(s)}\,ds \right\|_{L^\infty(0,1)} \leq \left\| \int_t^1 \frac{|f(s)|}{J(s)}\,ds \right\|_{L^\infty(0,1)} \\ &= \int_0^1 \frac{|f(s)|}{J(s)}\,ds \leq \int_0^1 \frac{f^*(s)}{J(s)}\,ds = \|f\|_{A_J(0,1)} \end{aligned} \tag{2.24}$$

for every $f \in A_J(0,1)$, so, $H_J : A_J(0,1) \to L^\infty(0,1)$. This completes the proof. □

The previous lemma shall be now applied to obtain further characterizations of compactness of the operator H_J^m from a rearrangement-invariant space $X(0,1)$ into $L^\infty(0,1)$.

Theorem 2.7. *Let $J : [0,1] \to [0,\infty)$ be a nondecreasing function satisfying* (2.1) *and* (2.22) *and let $m \in \mathbb{N}$. Suppose that $X(0,1)$ is a rearrangement-invariant space. Then the following two conditions are equivalent:*

(i) $H_J^m : X(0,1) \to\to L^\infty(0,1)$;
(ii) $\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)} \leq 1} \|H_J^m(\chi_{(0,a)}|f|)\|_{L^\infty(0,1)} = 0.$

Moreover, if $m = 1$, conditions (i) *and* (ii) *are equivalent to*

(iii) $X(0,1) \overset{*}{\hookrightarrow} A_J(0,1)$;

and if $m > 1$, they are equivalent to

(iv) $H_J^{m-1} : X(0,1) \to\to A_J(0,1)$.

Proof. To show that (i) $\Leftrightarrow$ (ii) it suffices to prove the implication (ii) $\Rightarrow$ (i) in the particular case $X(0,1) = L^1(0,1)$ and $m = 1$, because the rest of the proof follows from Theorem 2.3.

Suppose that (ii) holds with $X(0,1) = L^1(0,1)$ and $m = 1$. Then

$$\begin{aligned} 0 &= \lim_{a\to 0_+} \sup_{\|f\|_{L^1(0,1)}\le 1} \|H_J(\chi_{(0,a)}|f|)\|_{L^\infty(0,1)} = \lim_{a\to 0_+} \sup_{\|f\|_{L^1(0,1)}\le 1} \int_0^a \frac{|f(s)|}{J(s)}\,ds \\ &= \lim_{a\to 0_+} \left\| \frac{\chi_{(0,a)}}{J} \right\|_{L^\infty(0,1)} = \sup_{t\in(0,1]} \frac{1}{J(t)}, \end{aligned}$$

because $1/J$ is nonincreasing in $[0,1]$. But $1/J > 0$ in $(0,1]$, so the assumption (ii) cannot be satisfied in this situation. The proof of (i) $\Leftrightarrow$ (ii) is complete.

(ii) $\Leftrightarrow$ (iii) We suppose that $m = 1$. First, observe that for every $g \in \mathcal{M}(0,1)$ we have

$$\|g\|_{A_J(0,1)} = \int_0^1 \frac{g^*(s)}{J(s)}\,ds = \left\| \int_t^1 \frac{g^*(s)}{J(s)}\,ds \right\|_{L^\infty(0,1)} = \|H_J g^*\|_{L^\infty(0,1)}. \tag{2.25}$$

Let $a \in (0,1)$. Then

$$\begin{aligned} \sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,a)} f^*\|_{A_J(0,1)} &= \sup_{\|f\|_{X(0,1)}\le 1} \left\| H_J(\chi_{(0,a)} f^*) \right\|_{L^\infty(0,1)} \\ &\le \sup_{\|f\|_{X(0,1)}\le 1} \left\| H_J(\chi_{(0,a)} |f|) \right\|_{L^\infty(0,1)}, \end{aligned}$$

where the inequality holds thanks to the fact that whenever a measurable function f belongs to the unit ball of the space $X(0,1)$, then also $f^* = |f^*|$ has the same property. Conversely, using (2.24) we obtain that for every $a \in (0,1)$,

$$\begin{aligned} \sup_{\|f\|_{X(0,1)}\le 1} \left\| H_J(\chi_{(0,a)} |f|) \right\|_{L^\infty(0,1)} &\le \sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,a)} |f|\|_{A_J(0,1)} \\ &= \sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,a)} (\chi_{(0,a)}|f|)^*\|_{A_J(0,1)} \\ &\le \sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,a)} f^*\|_{A_J(0,1)}. \end{aligned}$$

Hence,

$$\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,a)} f^*\|_{A_J(0,1)} = \lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \left\| H_J(\chi_{(0,a)} |f|) \right\|_{L^\infty(0,1)}.$$

This establishes the equivalence of (ii) and (iii).

(ii) $\Leftrightarrow$ (iv) We suppose that $m > 1$. Then

$$\begin{aligned}
\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} &\|H_J^{m-1}(\chi_{(0,a)}|f|)\|_{A_J(0,1)} \\
&= \lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|H_J(H_J^{m-1}(\chi_{(0,a)}|f|))^*\|_{L^\infty(0,1)} \\
&= \lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|H_J(H_J^{m-1}(\chi_{(0,a)}|f|))\|_{L^\infty(0,1)} \\
&= \frac{1}{m-1} \lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|H_J^m(\chi_{(0,a)}|f|)\|_{L^\infty(0,1)}. \qquad (2.26)
\end{aligned}$$

Note that the first equality can be obtained by using (2.25) and the third equality holds thanks to (2.7).

Thus, (ii) is equivalent to

$$\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|H_J^{m-1}(\chi_{(0,a)}|f|)\|_{A_J(0,1)} = 0. \qquad (2.27)$$

Since $A_J(0,1) \neq L^\infty(0,1)$ (see Lemma 2.6), Theorem 2.3 implies that (2.27) is equivalent to (iv), as required. $\square$

Theorem 2.3 and Theorem 2.7 yield altogether the following

Corollary 2.8. *Let $J : [0,1] \to [0,\infty)$ be a nondecreasing function satisfying (2.1) and let $m \in \mathbb{N}$. Suppose that $X(0,1)$ and $Y(0,1)$ are rearrangement-invariant spaces. Then*

$$H_J^m : X(0,1) \to\to Y(0,1)$$

is satisfied if and only if

$$\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|H_J^m(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0.$$

3. Main results

Let (Ω, ν) be as in Section 1.2. The results from Chapter 2 shall be now applied to a function $I : [0,1] \to [0,\infty)$ satisfying

$$I_{(\Omega,\nu)}(t) \geq cI(ct), \quad t \in [0, 1/2], \tag{3.1}$$

for some constant $c \in (0,1)$ and fulfilling certain regularity assumptions, which are encoded in the notion of admissibility introduced in [3].

Definition 3.1. We say that a function $I : [0,1] \to [0,\infty)$ is *admissible* if it is nondecreasing in $[0,1]$, $I(t)/t$ is strictly positive and nonincreasing in $(0,1]$ and one of the following conditions is satisfied:

(a) There exists $k \in \mathbb{N}$ and a constant $A > 0$ such that for every $t \in (0,1)$,

$$\frac{1}{t}\int_0^t \left(\frac{I(s)}{s}\right)^k ds \leq A\left(\frac{I(t)}{t}\right)^k, \tag{3.2}$$

and, at the same time,

$$\inf_{t\in(0,1)} \frac{I(t)^{k+1}}{t^k} > 0. \tag{3.3}$$

(b) For every $m \in \mathbb{N}$ there exists a constant $A_m > 0$ such that for every $t \in (0,1)$,

$$\frac{1}{t}\int_0^t \left(\frac{I(s)}{s}\right)^m ds \leq A_m\left(\frac{I(t)}{t}\right)^m.$$

Remark 3.2. Each admissible function I is nondecreasing in $[0,1]$ and satisfies

$$\inf_{t\in(0,1)} \frac{I(t)}{t} \geq I(1) > 0.$$

Hence, all results from Chapter 2 can be indeed applied to the choice $J = I$.

Examples 3.3. (i) Let $\alpha \in (0,1]$. Then the function $I(s) = s^\alpha$, $s \in [0,1]$, is admissible. Indeed, I is obviously nondecreasing in $[0,1]$ and $I(s)/s$ is positive and nonincreasing in $(0,1]$. Moreover, if $\alpha \in (0,1)$ then (3.2) is satisfied if and only if $k < 1/(1-\alpha)$ and (3.3) is fulfilled if and only if $k \geq \alpha/(1-\alpha) = 1/(1-\alpha) - 1$. Since there is always a positive integer in the interval $[\frac{1}{1-\alpha} - 1, \frac{1}{1-\alpha})$, condition (a) is satisfied. On the other hand, if $\alpha = 1$ then, clearly, I fulfills (b).

(ii) The function I fulfilling $I(s) = s\sqrt{\log 2/s}$, $s \in (0,1]$, and $I(0) = 0$, is admissible. Indeed, it is not hard to observe that I is nondecreasing in $[0,1]$ and that $I(s)/s$ is nonincreasing in $(0,1]$. Furthermore, I satisfies (b) since for every $m \in \mathbb{N}$,

$$\frac{1}{t}\int_0^t \left(\log\frac{2}{s}\right)^{\frac{m}{2}} ds \approx \left(\log\frac{2}{t}\right)^{\frac{m}{2}}, \quad t \in (0,1),$$

up to multiplicative constants independent of $t \in (0,1)$.

(iii) Suppose that $m > 1$ is an integer and set $\alpha = 1 - 1/m$. Let $\beta < 0$ and let I be the function defined by $I(s) = s^\alpha(\log 2/s)^\beta$, $s \in (0,1]$, and by $I(0) = 0$. Then there is $\delta > 0$ such that I is nondecreasing in $[0,\delta]$ and $I(s)/s$ is nonincreasing in $(0,\delta]$. However, we will show that I is not equivalent to an admissible function.

Since $\beta < 0$, condition (3.3) is satisfied if and only if $\alpha(k+1) - k < 0$, or, equivalently, $k > \alpha/(1-\alpha) = m-1$. Furthermore, a necessary condition for (3.2) to be fulfilled is that

$$\int_0^t \left(\frac{I(s)}{s}\right)^k ds = \int_0^t s^{(\alpha-1)k} \left(\log \frac{2}{s}\right)^{\beta k} ds < \infty, \quad t \in (0,1),$$

which holds if and only if $k < 1/(1-\alpha) = m$, or $k = 1/(1-\alpha) = m$ and $\beta m < -1$. Hence, (3.2) and (3.3) cannot be satisfied simultaneously unless $k = m$ and $\beta m < -1$. However, in this case we have for every $t \in (0,1)$

$$\frac{1}{t}\int_0^t \left(\frac{I(s)}{s}\right)^m ds = \frac{1}{t}\int_0^t \frac{\left(\log \frac{2}{s}\right)^{\beta m}}{s} ds \approx \frac{1}{t}\left(\log \frac{2}{t}\right)^{\beta m+1} = \left(\frac{I(t)}{t}\right)^m \log \frac{2}{t},$$

so, (3.3) is not satisfied. Hence, (a) is not true. Moreover, (b) is also not satisfied since (3.2) does not hold for $k > m$.

The following result, which will be crucial for us, is a consequence of [3, Theorem 3.3 and Theorem 3.4].

Theorem 3.4. *Assume that* (Ω, ν) *is as in Section 1.2 and* $I : [0,1] \to [0,\infty)$ *is an admissible function satisfying* (3.1). *Let* $m \in \mathbb{N}$ *and let* $X(0,1)$ *and* $Y(0,1)$ *be rearrangement-invariant spaces. Then the condition*

$$H_I^m : X(0,1) \to Y(0,1)$$

implies

$$V^m X(\Omega,\nu) \hookrightarrow Y(\Omega,\nu).$$

In particular, we have

$$V^m X(\Omega,\nu) \hookrightarrow X_{m,I}(\Omega,\nu).$$

We shall now prove a result in the spirit of Theorem 3.4 concerning compact Sobolev embeddings.

Theorem 3.5. *Assume that* (Ω, ν) *is as in Section 1.2 and* $I : [0,1] \to [0,\infty)$ *is an admissible function satisfying* (3.1). *Let* $m \in \mathbb{N}$ *and let* $X(0,1)$ *and* $Y(0,1)$ *be rearrangement-invariant spaces. Then, provided that*

$$H_I^m : X(0,1) \to\to Y(0,1), \tag{3.4}$$

or, equivalently,

$$\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|H_I^m(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0, \tag{3.5}$$

we have

$$V^m X(\Omega,\nu) \hookrightarrow\hookrightarrow Y(\Omega,\nu).$$

Our proof starts with the following lemma which shows that the unit ball of each Sobolev space $V^m X(\Omega,\nu)$ is compact in measure.

Lemma 3.6. *Assume that (Ω, ν) is as in Section 1.2. Let $m \in \mathbb{N}$ and let $X(0,1)$ be a rearrangement-invariant space. Then every sequence $(u_k)_{k=1}^{\infty}$ bounded in $V^m X(\Omega, \nu)$ contains a subsequence $(u_{k_\ell})_{\ell=1}^{\infty}$ converging ν-a.e. in Ω. In particular, the subsequence $(u_{k_\ell})_{\ell=1}^{\infty}$ is convergent in measure.*

Proof. For every $x \in \Omega$ we can find a ball B_x centered in x such that $\overline{B_x} \subseteq \Omega$. Then $\Omega \subseteq \cup_{x \in \Omega} B_x$ and, due to the separability of Ω, there is a sequence $(x_j)_{j=1}^{\infty}$ of points in Ω such that $\Omega \subseteq \cup_{j=1}^{\infty} B_{x_j}$. As it was pointed out in (1.14) for $k = 1$, the sequence $(u_k)_{k=1}^{\infty}$ is bounded in $V^1 L^1(\Omega, \nu)$. Moreover, for every $j \in \mathbb{N}$, $(u_k)_{k=1}^{\infty}$ is bounded in $V^1 L^1(B_{x_j}, \nu)$. Since $\overline{B_{x_j}}$ is compact in Ω and ω is strictly positive and continuous in Ω, there is a constant $c_j > 0$ such that $\omega \geq c_j$ in B_{x_j}. Thus, for every $k \in \mathbb{N}$,

$$
\begin{aligned}
\|u_k\|_{V^1 L^1(B_{x_j}, \nu)} &= \int_{B_{x_j}} (|u_k(x)| + |\nabla u_k(x)|)\, \omega(x)\, dx \\
&\geq c_j \int_{B_{x_j}} (|u_k| + |\nabla u_k|)\, dx = c_j \|u_k\|_{V^1 L^1(B_{x_j})}.
\end{aligned}
$$

Hence, $(u_k)_{k=1}^{\infty}$ is bounded in $V^1 L^1(B_{x_j})$. Denote $u_k^0 = u_k$, $k \in \mathbb{N}$. By induction, for every $j \in \mathbb{N}$ we will construct a subsequence $(u_k^j)_{k=1}^{\infty}$ of the sequence $(u_k^{j-1})_{k=1}^{\infty}$ converging a.e. in B_{x_j}. Suppose that, for some $j \in \mathbb{N}$, we have already found the sequence $(u_k^{j-1})_{k=1}^{\infty}$. Since $(u_k^{j-1})_{k=1}^{\infty}$ is bounded in $V^1 L^1(B_{x_j})$ and the compact embedding $V^1 L^1(B_{x_j}) \hookrightarrow\hookrightarrow L^1(B_{x_j})$ holds, we can find a subsequence $(u_k^j)_{k=1}^{\infty}$ of $(u_k^{j-1})_{k=1}^{\infty}$ converging in $L^1(B_{x_j})$. Passing, if necessary, to another subsequence, $(u_k^j)_{k=1}^{\infty}$ can be found in such a way that it converges a.e. in B_{x_j}. Now, the diagonal sequence $(u_k^k)_{k=1}^{\infty}$ converges a.e. (or, what is the same, ν-a.e.) in $\cup_{j=1}^{\infty} B_{x_j} = \Omega$, as required. Furthermore, it is a well known fact that each sequence converging ν-a.e. is convergent in measure. □

Proof of Theorem 3.5. Conditions (3.4) and (3.5) are equivalent according to Corollary 2.8.

First, suppose that

$$
Y(0,1) \neq L^\infty(0,1) \text{ or } \int_0^1 \frac{ds}{I(s)} = \infty.
$$

Then, due to Theorem 2.3, (3.4) and (3.5) imply $X_{m,I}(0,1) \overset{*}{\hookrightarrow} Y(0,1)$, or, what is the same, $X_{m,I}(\Omega, \nu) \overset{*}{\hookrightarrow} Y(\Omega, \nu)$.

Assume that $(u_k)_{k=1}^{\infty}$ is a sequence bounded in $V^m X(\Omega, \nu)$. Due to Lemma 3.6, we can find its subsequence $(u_{k_\ell})_{\ell=1}^{\infty}$ which converges to some function u ν-a.e. in Ω. Because $V^m X(\Omega, \nu) \hookrightarrow X_{m,I}(\Omega, \nu)$ (Theorem 3.4), $(u_{k_\ell})_{\ell=1}^{\infty}$ is bounded in $X_{m,I}(\Omega, \nu)$. Hence, by the Fatou lemma,

$$
\|u\|_{X_{m,I}(\Omega,\nu)} \leq \liminf_{\ell \to \infty} \|u_{k_\ell}\|_{X_{m,I}(\Omega,\nu)} < \infty,
$$

so $u \in X_{m,I}(\Omega, \nu)$ and $(u_{k_\ell} - u)_{\ell=1}^{\infty}$ is therefore bounded in $X_{m,I}(\Omega, \nu)$ as well. We have $X_{m,I}(\Omega, \nu) \overset{*}{\hookrightarrow} Y(\Omega, \nu)$, so, according to [13, Theorem 3.1], $(u_{k_\ell} - u) \to 0$ in $Y(\Omega, \nu)$, i.e., $u_{k_\ell} \to u$ in $Y(\Omega, \nu)$. Thus, $V^m X(\Omega, \nu) \hookrightarrow\hookrightarrow Y(\Omega, \nu)$.

Now, assume that $Y(0,1) = L^\infty(0,1)$ and $\int_0^1 1/I(s)\,ds < \infty$. We start with the case that $m = 1$. Then, due to Theorem 2.7, conditions (3.4) and (3.5) imply

$$X(0,1) \overset{*}{\hookrightarrow} A_I(0,1). \tag{3.6}$$

Moreover, Lemma 2.6 together with Theorem 3.4 give that

$$V^1 A_I(\Omega,\nu) \hookrightarrow L^\infty(\Omega,\nu). \tag{3.7}$$

Let $u \in V^1 A_I(\Omega,\nu)$ be a nonnegative function fulfilling $\operatorname{med}(u) = 0$, that is,

$$\nu(\{x \in \Omega : u > 0\}) \le \frac{1}{2}.$$

Then, using (3.7), [3, Proposition 5.2] and the fact that $A_I(\Omega,\nu) \hookrightarrow L^1(\Omega,\nu)$, we get

$$\begin{aligned} \|u\|_{L^\infty(\Omega,\nu)} &\le C_1(\||\nabla u|\|_{A_I(\Omega,\nu)} + \|u\|_{L^1(\Omega,\nu)}) \\ &\le C_1(\||\nabla u|\|_{A_I(\Omega,\nu)} + C_2\||\nabla u|\|_{L^1(\Omega,\nu)}) \le C_3\||\nabla u|\|_{A_I(\Omega,\nu)}, \end{aligned} \tag{3.8}$$

where C_1, C_2, C_3 are positive constants independent of u.

Let $(u_k)_{k=1}^\infty$ be a sequence in $V^1 X(\Omega,\nu)$ such that

$$\|u_k\|_{V^1 X(\Omega,\nu)} \le 1, \quad k \in \mathbb{N}. \tag{3.9}$$

Due to Lemma 3.6, there is a subsequence $(v_k)_{k=1}^\infty$ of the sequence $(u_k)_{k=1}^\infty$ which converges in measure to some function v. Choose $\varepsilon > 0$ arbitrarily. Due to (3.6), we can find $\delta \in (0, 1/2)$ such that

$$\sup_{\|f\|_{X(0,1)} \le 1} \|\chi_{(0,\delta)} f^*\|_{A_I(0,1)} < \frac{\varepsilon}{4C_3}. \tag{3.10}$$

Since $(v_k)_{k=1}^\infty$ converges to v in measure, there exists $k_0 \in \mathbb{N}$ such that whenever $k \ge k_0$, we have $\nu(\{x \in \Omega : |v_k(x) - v(x)| > \varepsilon/4\}) < \delta/2$. Because for every $k, \ell \ge k_0$,

$$\begin{aligned} &\{x \in \Omega : |v_k(x) - v_\ell(x)| > \varepsilon/2\} \\ &\subseteq \{x \in \Omega : |v_k(x) - v(x)| > \varepsilon/4\} \cup \{x \in \Omega : |v_\ell(x) - v(x)| > \varepsilon/4\}, \end{aligned}$$

we deduce that

$$\begin{aligned} &\nu(\{x \in \Omega : |v_k(x) - v_\ell(x)| > \varepsilon/2\}) \\ &\le \nu(\{x \in \Omega : |v_k(x) - v(x)| > \varepsilon/4\}) + \nu(\{x \in \Omega : |v_\ell(x) - v(x)| > \varepsilon/4\}) < \delta. \end{aligned} \tag{3.11}$$

Observe that

$$|v_k - v_\ell| = \min\{|v_k - v_\ell|, \varepsilon/2\} + \max\{|v_k - v_\ell| - \varepsilon/2, 0\}. \tag{3.12}$$

Let Ω' be an open subset of Ω such that $\overline{\Omega'}$ is compact in Ω. Since v_k, v_l, $|\nabla v_k|$ and $|\nabla v_l|$ are locally integrable in Ω, we have $v_k \in V^{1,1}(\Omega')$ and $v_l \in V^{1,1}(\Omega')$. Obviously, the constant function $\varepsilon/2 \in V^{1,1}(\Omega')$. This implies that the function $|v_k - v_\ell| - \varepsilon/2$ belongs to $V^{1,1}(\Omega')$ as well and

$$\nabla(|v_k - v_\ell| - \varepsilon/2) = \nabla|v_k - v_\ell| = \operatorname{sgn}(v_k - v_\ell)\nabla(v_k - v_l) = \operatorname{sgn}(v_k - v_\ell)(\nabla v_k - \nabla v_\ell)$$

a.e. in Ω'. Furthermore, because the constant function 0 belongs to $V^{1,1}(\Omega')$, we have $\max\{|v_k - v_\ell| - \varepsilon/2, 0\} \in V^{1,1}(\Omega')$ and

$$\nabla \max\{|v_k - v_\ell| - \varepsilon/2, 0\} = \begin{cases} \operatorname{sgn}(v_k - v_\ell)(\nabla v_k - \nabla v_\ell) & \text{a.e. in } \Omega' \cap \{|v_k - v_\ell| > \varepsilon/2\}, \\ 0 & \text{a.e. in } \Omega' \cap \{|v_k - v_\ell| \le \varepsilon/2\}, \end{cases}$$

i.e.,

$$\nabla \max\{|v_k - v_\ell| - \varepsilon/2, 0\} = \chi_{\{|v_k - v_\ell| > \varepsilon/2\}} \operatorname{sgn}(v_k - v_\ell)(\nabla v_k - \nabla v_\ell) \tag{3.13}$$

a.e. in Ω'.

Suppose that $\varphi \in C^\infty(\Omega)$ has a compact support in Ω and denote this support by K. Because K is bounded in $\mathbb{R}^n$, there is an open ball B in $\mathbb{R}^n$ such that $K \subseteq B$. Moreover, since $\mathbb{R}^n \backslash \Omega$ is closed in $\mathbb{R}^n$, the Euclidean distance $d = d(K, \mathbb{R}^n \backslash \Omega) > 0$. Define

$$\Omega' = \left\{ x \in \mathbb{R}^n : d(x, K) < \frac{d}{2} \right\} \cap B.$$

Then Ω' is a bounded (i.e., relatively compact) open subset of $\mathbb{R}^n$ and

$$K \subseteq \Omega' \subseteq \overline{\Omega'} \subseteq \left\{ x \in \mathbb{R}^n : d(x, K) \le \frac{d}{2} \right\} \cap \overline{B} \subseteq \Omega.$$

Thus, $\overline{\Omega'}$ is compact in Ω and φ has a compact support in Ω'. Hence, owing to (3.13), for every $i \in 1, 2, \ldots, n$ we have

$$\begin{aligned} &\int_\Omega \max\{|v_k - v_\ell| - \varepsilon/2, 0\} \cdot \frac{\partial \varphi}{\partial x_i} \\ &\qquad = \int_{\Omega'} \max\{|v_k - v_\ell| - \varepsilon/2, 0\} \cdot \frac{\partial \varphi}{\partial x_i} \\ &\qquad = -\int_{\Omega'} \varphi \cdot \chi_{\{|v_k - v_\ell| > \varepsilon/2\}} \operatorname{sgn}(v_k - v_\ell) \left(\frac{\partial v_k}{\partial x_i} - \frac{\partial v_\ell}{\partial x_i} \right) \\ &\qquad = -\int_{\Omega} \varphi \cdot \chi_{\{|v_k - v_\ell| > \varepsilon/2\}} \operatorname{sgn}(v_k - v_\ell) \left(\frac{\partial v_k}{\partial x_i} - \frac{\partial v_\ell}{\partial x_i} \right). \end{aligned}$$

This yields that $\max\{|v_k - v_\ell| - \varepsilon/2, 0\}$ is weakly differentiable in the entire Ω and

$$\nabla \max\{|v_k - v_\ell| - \varepsilon/2, 0\} = \chi_{\{|v_k - v_\ell| > \varepsilon/2\}} \operatorname{sgn}(v_k - v_\ell)(\nabla v_k - \nabla v_\ell)$$

holds a.e. in Ω. Consequently,

$$|\nabla \max\{|v_k - v_\ell| - \varepsilon/2, 0\}| = \chi_{\{|v_k - v_\ell| > \varepsilon/2\}} |\nabla v_k - \nabla v_\ell| \tag{3.14}$$

a.e. in Ω. Moreover, $\max\{|v_k - v_l| - \varepsilon/2, 0\}$ is nonnegative in Ω and

$$\operatorname{med}(\max\{|v_k - v_l| - \varepsilon/2, 0\}) = 0. \tag{3.15}$$

Indeed, (3.11) implies that

$$\nu(\{x\in\Omega : \max\{|v_k-v_\ell|-\varepsilon/2,0\}>0\}) = \nu(\{x\in\Omega : |v_k-v_\ell|>\varepsilon/2\}) < \delta < \frac{1}{2}.$$

Altogether, we obtain

$$\begin{aligned}
&\|v_k-v_\ell\|_{L^\infty(\Omega,\nu)}\\
&\le \|\min\{|v_k-v_\ell|,\varepsilon/2\}\|_{L^\infty(\Omega,\nu)} + \|\max\{|v_k-v_\ell|-\varepsilon/2,0\}\|_{L^\infty(\Omega,\nu)} \quad \text{(by (3.12))}\\
&\le \frac{\varepsilon}{2} + C_3\||\nabla \max\{|v_k-v_\ell|-\varepsilon/2,0\}|\|_{A_I(\Omega,\nu)} \quad \text{(by (3.15) and (3.8))}\\
&= \frac{\varepsilon}{2} + C_3\|\chi_{\{|v_k-v_\ell|>\varepsilon/2\}}|\nabla v_k - \nabla v_\ell|\|_{A_I(\Omega,\nu)} \quad \text{(by (3.14))}\\
&\le \frac{\varepsilon}{2} + C_3\|\chi_{\{|v_k-v_\ell|>\varepsilon/2\}}|\nabla v_k|\|_{A_I(\Omega,\nu)} + C_3\|\chi_{\{|v_k-v_\ell|>\varepsilon/2\}}|\nabla v_\ell|\|_{A_I(\Omega,\nu)}\\
&= \frac{\varepsilon}{2} + C_3\|(\chi_{\{|v_k-v_\ell|>\varepsilon/2\}}|\nabla v_k|)^*_\nu\|_{A_I(0,1)} + C_3\|(\chi_{\{|v_k-v_\ell|>\varepsilon/2\}}|\nabla v_\ell|)^*_\nu\|_{A_I(0,1)}\\
&= \frac{\varepsilon}{2} + C_3\|\chi_{(0,\delta)}(\chi_{\{|v_k-v_\ell|>\varepsilon/2\}}|\nabla v_k|)^*_\nu\|_{A_I(0,1)}\\
&+ C_3\|\chi_{(0,\delta)}(\chi_{\{|v_k-v_\ell|>\varepsilon/2\}}|\nabla v_\ell|)^*_\nu\|_{A_I(0,1)} \quad \text{(by (3.11))}\\
&\le \frac{\varepsilon}{2} + C_3\|\chi_{(0,\delta)}(|\nabla v_k|)^*_\nu\|_{A_I(0,1)} + C_3\|\chi_{(0,\delta)}(|\nabla v_\ell|)^*_\nu\|_{A_I(0,1)}\\
&\le \frac{\varepsilon}{2} + 2C_3 \sup_{\|f\|_{X(0,1)}\le 1}\|\chi_{(0,\delta)}f^*\|_{A_I(0,1)} < \varepsilon. \quad \text{(by (3.9) and (3.10))}
\end{aligned}$$

It follows that $(v_k)_{k=1}^\infty$ is a Cauchy sequence in $L^\infty(\Omega,\nu)$. Thanks to the completeness of $L^\infty(\Omega,\nu)$, $(v_k)_{k=1}^\infty$ is convergent in $L^\infty(\Omega,\nu)$, as required.

Let $m>1$. According to Theorem 2.7, conditions (3.4) and (3.5) imply

$$H_I^{m-1} : X(0,1) \to\to A_I(0,1).$$

Because $A_I(0,1)\ne L^\infty(0,1)$ (Lemma 2.6), the first part of the proof gives that

$$V^{m-1}X(\Omega,\nu) \hookrightarrow\hookrightarrow A_I(\Omega,\nu). \tag{3.16}$$

Let $(u_k)_{k=1}^\infty$ be a bounded sequence in $V^mX(\Omega,\nu)$. Then $(u_k)_{k=1}^\infty$ is bounded in $L^1(\Omega,\nu)$, therefore $(\int_\Omega u_k\,d\nu)_{k=1}^\infty$ is a bounded sequence of real numbers and we can find a subsequence $(u_k^0)_{k=1}^\infty$ of $(u_k)_{k=1}^\infty$ such that the sequence $(\int_\Omega u_k^0\,d\nu)_{k=1}^\infty$ is convergent.

For $i=1,2,\dots,n$, consider the sequence $(D_iu_k^0)_{k=1}^\infty$ consisting of weak derivatives with respect to the i-th variable of elements of the sequence $(u_k^0)_{k=1}^\infty$. Thanks to the continuous embedding $V^mX(\Omega,\nu)\hookrightarrow V^1L^1(\Omega,\nu)$, all these sequences are bounded in $L^1(\Omega,\nu)$, and, owing to the boundedness of $(u_k)_{k=1}^\infty$ in $V^mX(\Omega,\nu)$, they are bounded also in $V^{m-1}X(\Omega,\nu)$. Now, the compact embedding (3.16) yields that we can inductively find sequences $(u_k^i)_{k=1}^\infty$, $i=1,2,\dots,n$, such that $(u_k^i)_{k=1}^\infty$ is a subsequence of $(u_k^{i-1})_{k=1}^\infty$ fulfilling that $(D_ju_k^i)_{k=1}^\infty$ is convergent in $A_I(\Omega,\nu)$ for $j=1,2,\dots,i$. Thus, in particular, $(D_ju_k^n)_{k=1}^\infty$ is a Cauchy sequence in $A_I(\Omega,\nu)$ for every $j\in\{1,2,\dots,n\}$.

Let $\varepsilon>0$. Thanks to the embedding $V^1A_I(\Omega,\nu)\hookrightarrow L^\infty(\Omega,\nu)$, there is a constant $C>0$ such that for every $u\in V^1A_I(\Omega,\nu)$,

$$\left\|u-\int_\Omega u\,d\nu\right\|_{L^\infty(\Omega,\nu)} \le C\,\||\nabla u|\|_{A_I(\Omega,\nu)} \le C\sum_{j=1}^n \|D_ju\|_{A_I(\Omega,\nu)}, \tag{3.17}$$

see [3, Proposition 5.3]. Since $(D_j u_k^n)_{k=1}^\infty$ is a Cauchy sequence in $A_I(\Omega,\nu)$ for every $j \in \{1,2,\dots,n\}$, we can find $k_0 \in \mathbb{N}$ such that $\|D_j u_k^n - D_j u_\ell^n\|_{A_I(\Omega,\nu)} < \varepsilon/Cn$ whenever $k,\ell \ge k_0$ and $j \in \{1,2,\dots,n\}$. Thus, inequality (3.17) implies that for every $k,\ell \ge k_0$,

$$\left\| u_k^n - u_\ell^n - \int_\Omega (u_k^n - u_\ell^n)\, d\nu \right\|_{L^\infty(\Omega,\nu)} \le C \sum_{j=1}^n \|D_j u_k^n - D_j u_\ell^n\|_{A_I(\Omega,\nu)} < \varepsilon,$$

so, $(u_k^n - \int_\Omega u_k^n\, d\nu)_{k=1}^\infty$ is a Cauchy sequence in $L^\infty(\Omega,\nu)$. Due to the completeness of $L^\infty(\Omega,\nu)$, $(u_k^n - \int_\Omega u_k^n\, d\nu)_{k=1}^\infty$ is convergent in $L^\infty(\Omega,\nu)$. Since the sequence $(\int_\Omega u_k^n\, d\nu)_{k=1}^\infty$ consisting of constant functions is convergent in $L^\infty(\Omega,\nu)$ as well, $(u_k^n)_{k=1}^\infty$ is convergent in $L^\infty(\Omega,\nu)$ and $V^m X(\Omega,\nu) \hookrightarrow\hookrightarrow L^\infty(\Omega,\nu)$, as required. □

We shall now prove that in the special case when there is a constant $C > 0$ such that

$$\int_0^s \frac{dr}{I(r)} \le C \frac{s}{I(s)}, \quad s \in (0,1), \tag{3.18}$$

the kernel operator H_I^m in (3.4) and (3.5) can be (almost) equivalently replaced by a much simpler weighted Hardy type operator K_I^m defined by

$$K_I^m f(t) = \int_t^1 f(s) \frac{s^{m-1}}{(I(s))^m}\, ds, \quad f \in L^1(0,1),\ t \in (0,1).$$

Moreover, the rearrangement-invariant space $X_{m,I}(0,1)$, useful to characterize when (3.4) and (3.5) hold, coincides with the rearrangement-invariant space $X_{m,I}^\sharp(0,1)$ whose associate space consists of all $f \in \mathcal{M}(0,1)$ for which

$$\|f\|_{(X_{m,I}^\sharp)'(0,1)} = \left\| \frac{t^{m-1}}{(I(t))^m} \int_0^t f^*(s)\, ds \right\|_{X'(0,1)} < \infty.$$

The fact that the functional $\|\cdot\|_{(X_{m,I}^\sharp)'(0,1)}$ is a rearrangement-invariant norm was proved in [3, Proposition 8.2].

Finally, notice that whenever inequality (3.18) is satisfied, we have

$$\int_0^1 \frac{dr}{I(r)} < \infty. \tag{3.19}$$

Let us now state the considerations above more precisely.

Theorem 3.7. *Assume that* (Ω,ν) *is as in Section 1.2 and* $I : [0,1] \to [0,\infty)$ *is an admissible function satisfying* (3.1) *and* (3.18). *Let* $m \in \mathbb{N}$ *and let* $X(0,1)$ *and* $Y(0,1)$ *be rearrangement-invariant spaces. Consider the following conditions:*

(i) $K_I^m : X(0,1) \to\to Y(0,1)$;
(ii) $\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|K_I^m(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0$;
(iii) $V^m X(\Omega,\nu) \hookrightarrow\hookrightarrow Y(\Omega,\nu)$.

Then (i) ⇒ (ii) ⇒ (iii).

Moreover, provided that $X(0,1) \ne L^1(0,1)$ *or* $Y(0,1) \ne L^\infty(0,1)$ *or* $(I(t))^m/t^{m-1}$ *is nondecreasing in* $(0,1]$, *conditions* (i) *and* (ii) *are equivalent. In the case when* $Y(0,1) \ne L^\infty(0,1)$, (i) *(and therefore also* (ii)*) is satisfied if and only if*

(iv) $X^{\sharp}_{m,I}(0,1) \overset{*}{\hookrightarrow} Y(0,1)$,

and in the case when $Y(0,1) = L^\infty(0,1)$ *and* $(I(t))^m/t^{m-1}$ *is nondecreasing in* $(0,1]$, *each of* (i), (ii) *is equivalent to*

(v) $X(0,1) \overset{*}{\hookrightarrow} A_{(I(t))^m/t^{m-1}}(0,1)$.

Proof. Consider the function J defined by

$$J(t) = \begin{cases} 0 & t = 0; \\ (I(t))^m/t^{m-1} & t \in (0,1]. \end{cases}$$

Then J is nonnegative and measurable in $[0,1]$ and

$$\inf_{t\in(0,1)} \frac{J(t)}{t} = \inf_{t\in(0,1)} \left(\frac{I(t)}{t}\right)^m \geq (I(1))^m > 0, \tag{3.20}$$

so J satisfies (2.1). Observe that $K_I^m = H_J$ and

$$\int_0^1 \frac{dt}{J(t)} = \int_0^1 \frac{t^{m-1}}{(I(t))^m}\, dt \leq \sup_{t\in(0,1)} \left(\frac{t}{I(t)}\right)^{m-1} \int_0^1 \frac{dt}{I(t)} < \infty,$$

since (3.18) implies (3.19). Hence, Theorem 2.3 yields that (i) implies (ii) and that (ii) implies (i) provided that $X(0,1) \neq L^1(0,1)$ or $Y(0,1) \neq L^\infty(0,1)$. Furthermore, due to Corollary 2.8, (ii) implies (i) also in the case that $(I(t))^m/t^{m-1}$ is nondecreasing in $(0,1]$.

According to [3, Proposition 10.15], there are positive constants C_1 and C_2 such that for every nonnegative measurable function f in $(0,1)$ we have

$$C_1 \|H_I^m f\|_{Y(0,1)} \leq \|K_I^m f\|_{Y(0,1)} \leq C_2 \|H_I^m f\|_{Y(0,1)}.$$

Thus, for every $a \in (0,1)$,

$$\begin{aligned} C_1 \sup_{\|f\|_{X(0,1)}\leq 1} \|H_I^m(\chi_{(0,a)}|f|)\|_{Y(0,1)} &\leq \sup_{\|f\|_{X(0,1)}\leq 1} \|K_I^m(\chi_{(0,a)}|f|)\|_{Y(0,1)} \\ &\leq C_2 \sup_{\|f\|_{X(0,1)}\leq 1} \|H_I^m(\chi_{(0,a)}|f|)\|_{Y(0,1)}, \end{aligned}$$

which proves the equivalence of (ii) and (3.5) (or, equivalently, (3.4)). The implication (ii) $\Rightarrow$ (iii) then follows from Theorem 3.5.

Suppose that $Y(0,1) \neq L^\infty(0,1)$. Then (i) is equivalent to (3.4) which is equivalent to

$$X_{m,I}(0,1) \overset{*}{\hookrightarrow} Y(0,1),$$

see Theorem 2.3. Owing to [3, Corollary 3.7], $X_{m,I}(0,1) = X^{\sharp}_{m,I}(0,1)$. This yields the equivalence of (i) and (iv).

Finally, let $Y(0,1) = L^\infty(0,1)$ and assume that $(I(t))^m/t^{m-1}$ is nondecreasing in $(0,1]$. Then, due to Theorem 2.7 applied to J as above, (i) is equivalent to

$$X(0,1) \overset{*}{\hookrightarrow} A_{(I(t))^m/t^{m-1}}(0,1).$$

This completes the proof. □

4. Compact Sobolev embeddings on concrete measure spaces

4.1 Two propositions

In this section we state and prove two propositions which will be needed in the rest of this chapter. The first proposition says that, under some assumptions, compactness of a Sobolev embedding implies compactness of a certain operator. This is the key step when proving the reverse implication to the one stated in Theorem 3.5 (we will do it in particular cases in the remaining two sections).

Proposition 4.1. *Assume that (Ω, ν) is as in Section 1.2. Let $m \in \mathbb{N}$ and let $X(0,1)$ and $Y(0,1)$ be rearrangement-invariant spaces satisfying*

$$V^m X(\Omega, \nu) \hookrightarrow\hookrightarrow Y(\Omega, \nu). \tag{4.1}$$

Let $\alpha \in (0,1]$. Denote

$$X_+^\alpha = \{f \in X(0,1) : f \geq 0 \text{ a.e. in } (0,\alpha) \text{ and } f = 0 \text{ a.e. in } (0,1)\backslash(0,\alpha)\}.$$

Suppose that H is an operator defined on X_+^α, with values in $\mathcal{M}(0,1)$. Assume that there exists an operator L defined on X_+^α, with values in $V^m X(\Omega, \nu)$, satisfying the following conditions.

(i) *There is $\beta > 0$ and a real valued function K defined in $(0,\beta) \times (0,\beta)$ such that $K(\cdot, t)$ is measurable in $(0,\beta)$ for every $t \in (0,\beta)$, and*

$$(Lf)_\nu^*(t) \approx \chi_{(0,\beta)}(t) \int_t^\beta f\left(\frac{s}{\beta}\right) K(s,t)\, ds, \quad t > 0, \tag{4.2}$$

holds for every $f \in X_+^\alpha$, up to multiplicative constants independent on f and t.

(ii) *The inequalities*

$$\|Lf\|_{V^m X(\Omega,\nu)} \leq C_1 \|f\|_{X(0,1)}, \tag{4.3}$$
$$\|Hf\|_{Y(0,1)} \leq C_2 \|Lf\|_{Y(\Omega,\nu)} \tag{4.4}$$

hold for some positive constants C_1 and C_2 and for all $f \in X_+^\alpha$.

Then

$$\lim_{a \to 0_+} \sup_{\|f\|_{X(0,1)} \leq 1} \|H(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0. \tag{4.5}$$

Proof. We first observe that whenever k is a positive integer satisfying $1/k \leq \alpha$ and $f \in X(0,1)$ fulfills $f \geq 0$ a.e. in $(0,1)$, then $\chi_{(0,1/k)} f \in X_+^\alpha$ and the functions $H(\chi_{(0,1/k)} f)$ and $L(\chi_{(0,1/k)} f)$ are thus well defined. Consequently, for every $k \in \mathbb{N}$ satisfying $1/k \leq \alpha$ we can find a nonnegative measurable function f_k in $(0,1)$ such that $\|f_k\|_{X(0,1)} \leq 1$ and

$$\sup_{\|f\|_{X(0,1)} \leq 1} \|H(\chi_{(0,1/k)}|f|)\|_{Y(0,1)} < \|H(\chi_{(0,1/k)} f_k)\|_{Y(0,1)} + \frac{1}{k}. \tag{4.6}$$

Because the sequence $(\chi_{(0,1/k)}f_k)_{k=\lceil 1/\alpha\rceil}^{\infty}$ is bounded in $X(0,1)$, it follows that $(L(\chi_{(0,1/k)}f_k))_{k=\lceil 1/\alpha\rceil}^{\infty}$ must be bounded in $V^mX(\Omega,\nu)$ due to (4.3). Thanks to (4.1), there is a subsequence $(f_{k_\ell})_{\ell=1}^{\infty}$ of $(f_k)_{k=\lceil 1/\alpha\rceil}^{\infty}$ such that $(L(\chi_{(0,1/k_\ell)}f_{k_\ell}))_{\ell=1}^{\infty}$ converges to some function g in $Y(\Omega,\nu)$. Moreover, passing, if necessary, to a subsequence, we can assume that $L(\chi_{(0,1/k_\ell)}f_{k_\ell}) \to g$ ν-a.e. in Ω.

Observe that for every $\ell \in \mathbb{N}$, we have $(L(\chi_{(0,1/k_\ell)}f_{k_\ell}))_\nu^*(t) = 0$ when $t > \beta/k_\ell$, thanks to (4.2). Because the functions $L(\chi_{(0,1/k_\ell)}f_{k_\ell})$ and $(L(\chi_{(0,1/k_\ell)}f_{k_\ell}))_\nu^*$ are equimeasurable, the distribution function of $L(\chi_{(0,1/k_\ell)}f_{k_\ell})$ according to ν coincides with that of $(L(\chi_{(0,1/k_\ell)}f_{k_\ell}))_\nu^*$ according to the one-dimensional Lebesgue measure λ_1. In particular,

$$\begin{aligned}&\lim_{\ell\to\infty} \nu\left(\left\{x\in\Omega : \left|L(\chi_{(0,1/k_\ell)}f_{k_\ell})(x)\right| > 0\right\}\right)\\ &= \lim_{\ell\to\infty} \lambda_1\left(\left\{s>0 : \left(L(\chi_{(0,1/k_\ell)}f_{k_\ell})\right)_\nu^*(s) > 0\right\}\right) \le \lim_{\ell\to\infty}\frac{\beta}{k_\ell} = 0. \qquad (4.7)\end{aligned}$$

Let S be the set of all points $x\in\Omega$ such that $\left|L(\chi_{(0,1/k_\ell)}f_{k_\ell})(x)\right| > 0$ for infinitely many $\ell\in\mathbb{N}$. Then, due to (4.7), $\nu(S)$=0. Thus, $L(\chi_{(0,1/k_\ell)}f_{k_\ell}) \to 0$ ν-a.e. in Ω. This implies that $g=0$ ν-a.e. in Ω. So, owing to (4.4),

$$\lim_{\ell\to\infty}\|H(\chi_{(0,1/k_\ell)}f_{k_\ell})\|_{Y(0,1)} \le C_2 \lim_{\ell\to\infty}\|L(\chi_{(0,1/k_\ell)}f_{k_\ell})\|_{Y(\Omega,\nu)} = 0.$$

Inequality (4.6) now yields

$$\lim_{\ell\to\infty}\sup_{\|f\|_{X(0,1)}\le 1}\|H(\chi_{(0,1/k_\ell)}|f|)\|_{Y(0,1)} = 0.$$

Using that the function

$$a \mapsto \sup_{\|f\|_{X(0,1)}\le 1}\|H(\chi_{(0,a)}|f|)\|_{Y(0,1)}$$

is nondecreasing in $(0,\alpha]$, we obtain (4.5). ☐

The second proposition provides a characterization of almost-compact embeddings between Lorentz-Zygmund spaces and gives us therefore a tool for studying compact embeddings of Sobolev spaces built upon Lorentz-Zygmund spaces. Let us note that almost-compact embeddings between even more general classical and weak Lorentz spaces have already been studied in [8]. Our proof is, however, independent on arguments from [8].

Proposition 4.2. *Let p_1, p_2, q_1, $q_2 \in [1,\infty]$, α_1, $\alpha_2 \in \mathbb{R}$ be such that both $L^{p_1,q_1;\alpha_1}(0,1)$ and $L^{p_2,q_2;\alpha_2}(0,1)$ are rearrangement-invariant spaces (up to equivalent norms). Then*

$$L^{p_1,q_1;\alpha_1}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2;\alpha_2}(0,1) \qquad (4.8)$$

holds if and only if $p_1 > p_2$, or $p_1 = p_2$ and the following conditions are satisfied:

$$\text{if } p_1 = p_2 < \infty \text{ and } q_1 \le q_2 \text{ then } \alpha_1 > \alpha_2; \qquad (4.9)$$

$$\text{if } p_1 = p_2 = \infty \text{ or } q_1 > q_2 \text{ then } \alpha_1 + \frac{1}{q_1} > \alpha_2 + \frac{1}{q_2}. \qquad (4.10)$$

In particular, if p_1, p_2, q_1, $q_2 \in [1,\infty]$ are such that both $L^{p_1,q_1}(0,1)$ and $L^{p_2,q_2}(0,1)$ are rearrangement-invariant spaces (up to equivalent norms) then

$$L^{p_1,q_1}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2}(0,1) \tag{4.11}$$

if and only if $p_1 > p_2$.

Proof. Suppose that $p_1 = p_2$ and denote $p = p_1 = p_2$. Assume that conditions (4.9) and (4.10) are satisfied. Then we can find $\varepsilon > 0$ such that $L^{p,q_2;\alpha_2+\varepsilon}(0,1)$ is a rearrangement-invariant space (up to equivalent norms). Indeed, except of the case $L^{p,q_2;\alpha_2}(0,1) = L^\infty(0,1)$, it easily follows from conditions (1.4) – (1.7). However, if $L^{p,q_2;\alpha_2}(0,1) = L^\infty(0,1)$ then, using that $L^{p,q_1;\alpha_1}(0,1) = L^{\infty,q_1;\alpha_1}(0,1)$ satisfies (1.6) or (1.7) and that (4.10) is in progress, we get $0 \geq \alpha_1 + 1/q_1 > 0$, which is a contradiction. Condition (4.10) is therefore never fulfilled with $L^{p,q_2;\alpha_2}(0,1) = L^\infty(0,1)$ and we are done.

The constant ε from the previous paragraph can be found in such a way that (4.9) and (4.10) are satisfied with $\alpha_2+\varepsilon$ in place of α_2. Then (1.8) is fulfilled with $\alpha_2+\varepsilon$ in place of α_2 as well and we have

$$L^{p,q_1;\alpha_1}(0,1) \hookrightarrow L^{p,q_2;\alpha_2+\varepsilon}(0,1).$$

Thus,

$$\begin{aligned}
&\lim_{a\to 0_+} \sup_{\|f\|_{L^{p,q_1;\alpha_1}(0,1)}\leq 1} \|\chi_{(0,a)} f^*\|_{L^{p,q_2;\alpha_2}(0,1)} \\
&= \lim_{a\to 0_+} \sup_{\|f\|_{L^{p,q_1;\alpha_1}(0,1)}\leq 1} \|\chi_{(0,a)}(s) f^*(s)\,(\log e/s)^{-\varepsilon}\, s^{\frac{1}{p}-\frac{1}{q_2}}\,(\log e/s)^{\alpha_2+\varepsilon}\,\|_{L^{q_2}(0,1)} \\
&\leq \lim_{a\to 0_+} \|\chi_{(0,a)}(s)\,(\log e/s)^{-\varepsilon}\,\|_{L^\infty(0,1)} \sup_{\|f\|_{L^{p,q_1;\alpha_1}(0,1)}\leq 1} \|f^*(s) s^{\frac{1}{p}-\frac{1}{q_2}}\,(\log e/s)^{\alpha_2+\varepsilon}\,\|_{L^{q_2}(0,1)} \\
&= \sup_{\|f\|_{L^{p,q_1;\alpha_1}(0,1)}\leq 1} \|f\|_{L^{p,q_2;\alpha_2+\varepsilon}(0,1)} \lim_{a\to 0_+} (\log e/a)^{-\varepsilon} = 0,
\end{aligned}$$

i.e., (4.8) is satisfied.

On the other hand, suppose that $p = p_1 = p_2$ and (4.8) is in progress. Then, in particular,

$$L^{p,q_1;\alpha_1}(0,1) \hookrightarrow L^{p,q_2;\alpha_2}(0,1),$$

so one of the conditions (1.8) must be satisfied. We shall distinguish between three cases when both $L^{p,q_1;\alpha_1}(0,1)$ and $L^{p,q_2;\alpha_2}(0,1)$ are rearrangement-invariant spaces (up to equivalent norms), some of the conditions (1.8) holds but (4.9) or (4.10) not. The first one is

$$p < \infty, \quad q_1 \leq q_2, \quad \alpha_1 = \alpha_2, \tag{4.12}$$

the second one is

$$p = \infty, \quad q_1 \leq q_2, \quad \alpha_1 + \frac{1}{q_1} = \alpha_2 + \frac{1}{q_2} < 0, \tag{4.13}$$

and the third one is

$$p = \infty, \quad q_1 = q_2 = \infty, \quad \alpha_1 = \alpha_2 = 0. \tag{4.14}$$

Using [12, proof of Theorem 6.3] we get that in all cases, fundamental functions of $L^{p,q_1;\alpha_1}(0,1)$ and $L^{p,q_2;\alpha_2}(0,1)$, denoted by $\varphi_{p,q_1;\alpha_1}$ and $\varphi_{p,q_2;\alpha_2}$, respectively, are equivalent up to multiplicative constants. Indeed, this trivially holds in the case (4.14), while in the case (4.12) we have

$$\varphi_{p,q_1;\alpha_1}(s) \approx s^{\frac{1}{p}}(\log e/s)^{\alpha_1} = s^{\frac{1}{p}}(\log e/s)^{\alpha_2} \approx \varphi_{p,q_2;\alpha_2}(s), \quad s \in (0,1),$$

and in the case (4.13) we have

$$\varphi_{p,q_1;\alpha_1}(s) \approx (\log e/s)^{\alpha_1+\frac{1}{q_1}} = (\log e/s)^{\alpha_2+\frac{1}{q_2}} \approx \varphi_{p,q_2;\alpha_2}(s), \quad s \in (0,1).$$

Therefore, a necessary condition for (4.8) to be true,

$$\lim_{s\to 0_+} \frac{\varphi_{p,q_2;\alpha_2}(s)}{\varphi_{p,q_1;\alpha_1}(s)} = 0, \tag{4.15}$$

is not satisfied (note that the fact that (4.8) yields (4.15) was shown in [6, Section 3]). Hence, (4.8) always implies (4.9) and (4.10).

Let us now discuss the case when $p_1 \neq p_2$. First, observe that if (4.8) is satisfied then, in particular,

$$L^{p_1,q_1;\alpha_1}(0,1) \hookrightarrow L^{p_2,q_2;\alpha_2}(0,1),$$

which implies that $p_1 \geq p_2$. On the other hand, suppose that $L^{p_1,q_1;\alpha_1}(0,1)$ and $L^{p_2,q_2;\alpha_2}(0,1)$ are rearrangement-invariant spaces (up to equivalent norms) fulfilling $p_1 > p_2$. Then $L^{p_2,q_2;\alpha_2}(0,1) \neq L^{\infty}(0,1)$, so, as it was explained in the first paragraph of this proof, we can find $\varepsilon > 0$ such that $L^{p_2,q_2;\alpha_2+\varepsilon}(0,1)$ is a rearrangement-invariant space (up to equivalent norms) as well. It follows from the first part of the proof that

$$L^{p_2,q_2;\alpha_2+\varepsilon}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2;\alpha_2}(0,1).$$

Moreover, we have

$$L^{p_1,q_1;\alpha_1}(0,1) \hookrightarrow L^{p_2,q_2;\alpha_2+\varepsilon}(0,1).$$

Altogether, we get (4.8), as required.

To complete the proof, observe that conditions (4.9) and (4.10) are satisfied with $\alpha_1 = \alpha_2 = 0$ only in the case that $p_1 = p_2 = \infty$ and $q_2 > q_1$. This, however, necessarily means that $q_1 < \infty$, i.e., the norm of $L^{\infty,q_1}(0,1)$ is not equivalent to a rearrangement-invariant norm. Therefore, (4.11) holds if and only if $p_1 > p_2$. $\square$

4.2 Compactness of Euclidean-Sobolev embeddings

In this section we characterize compact Sobolev embeddings on John domains and on Maz'ya classes of domains.

Let $n \in \mathbb{N}$, $n \geq 2$. For $\alpha \in [1/n', 1]$, consider the function $I_\alpha(s) = s^\alpha$, $s \in [0,1]$. Then I_α is admissible (see Examples 3.3) and if $\alpha \in [1/n', 1)$ then (3.18) is satisfied with $I = I_\alpha$.

Let $m \in \mathbb{N}$. Define the operator M_α^m by

$$M_\alpha^m f(t) = K_{I_\alpha}^m f(t) = \int_t^1 f(s) s^{-1+m(1-\alpha)}\, ds, \quad f \in L^1(0,1),\ t \in (0,1),$$

if $\alpha \in [1/n', 1)$ and by

$$M_1^m f(t) = H_{I_1}^m f(t) = \int_t^1 f(s) \frac{\left(\log \frac{s}{t}\right)^{m-1}}{s}\, ds, \quad f \in L^1(0,1),\ t \in (0,1).$$

We will also consider the rearrangement-invariant space $X_{m,\alpha}(0,1)$ defined by $X_{m,\alpha}(0,1) = X^\sharp_{m,I_\alpha}(0,1)$ if $\alpha \in [1/n', 1)$ and by $X_{m,1}(0,1) = X_{m,I_1}(0,1)$. The associate space of $X_{m,\alpha}(0,1)$ therefore fulfills for every $f \in \mathcal{M}(0,1)$

$$\|f\|_{(X_{m,\alpha})'(0,1)} = \left\| s^{-1+m(1-\alpha)} \int_0^s f^*(r)\, dr \right\|_{X'(0,1)}$$

if $\alpha \in [1/n', 1)$ and

$$\|f\|_{(X_{m,1})'(0,1)} = \left\| \frac{1}{s} \int_0^s \left(\log \frac{s}{r}\right)^{m-1} f^*(r)\, dr \right\|_{X'(0,1)}.$$

We first focus on compact Sobolev embeddings on John domains. In this situation, the operator $M^m_{1/n'}$ having the form

$$M^m_{1/n'} f(t) = \int_t^1 f(s) s^{-1+\frac{m}{n}}\, ds, \quad f \in L^1(0,1), \quad t \in (0,1),$$

and the rearrangement-invariant space $X_{m,1/n'}(0,1)$ fulfilling

$$\|f\|_{(X_{m,1/n'})'(0,1)} = \left\| s^{-1+\frac{m}{n}} \int_0^s f^*(r)\, dr \right\|_{X'(0,1)}, \quad f \in \mathcal{M}(0,1),$$

come into play.

Theorem 4.3. *Let $n \in \mathbb{N}$, $n \geq 2$, let $m \in \mathbb{N}$ and let Ω be a John domain in $\mathbb{R}^n$. Suppose that $X(0,1)$ and $Y(0,1)$ are rearrangement-invariant spaces. If $m \leq n$ then the following conditions are equivalent:*

(i) $V^m X(\Omega) \hookrightarrow\hookrightarrow Y(\Omega)$;
(ii) $M^m_{1/n'} : X(0,1) \to\to Y(0,1)$;
(iii) $\lim_{a \to 0_+} \sup_{\|f\|_{X(0,1)} \leq 1} \|M^m_{1/n'}(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0$.

If, in addition, $Y(0,1) \neq L^\infty(0,1)$, the previous conditions are equivalent to

(iv) $X_{m,1/n'}(0,1) \overset{*}{\hookrightarrow} Y(0,1)$,

and if $Y(0,1) = L^\infty(0,1)$, conditions (i) *–* (iii) *are equivalent to*

(v) $X(0,1) \overset{*}{\hookrightarrow} L^{\frac{n}{m},1}(0,1)$.

Furthermore, if $m > n$ then conditions (i) *and* (iii) *are satisfied independently of the choice of $X(0,1)$ and $Y(0,1)$ while condition* (ii) *is true for all pairs of rearrangement-invariant spaces $X(0,1)$ and $Y(0,1)$ except of $X(0,1) = L^1(0,1)$ and $Y(0,1) = L^\infty(0,1)$.*

The proof of the previous theorem will be moved to the end of this section since its major part is a consequence of more general results for Maz'ya classes of domains, which will be proved first.

Theorem 4.4. *Let $n \in \mathbb{N}$, $n \geq 2$, let $m \in \mathbb{N}$ and let $\alpha \in [1/n', 1]$. Suppose that $X(0,1)$ and $Y(0,1)$ are rearrangement-invariant spaces. If $m(1-\alpha) \leq 1$ (notice that this is true for every $m \in \mathbb{N}$ provided that $\alpha = 1$) then the following assertions are equivalent:*

(i) $V^m X(\Omega) \hookrightarrow\hookrightarrow Y(\Omega)$ *holds for every* $\Omega \in \mathcal{J}_\alpha$;
(ii) $M_\alpha^m : X(0,1) \to\to Y(0,1)$;
(iii) $\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\leq 1} \|M_\alpha^m(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0$.

If, in addition, $Y(0,1) \neq L^\infty(0,1)$ or $\alpha = 1$, conditions (i), (ii) *and* (iii) *are equivalent to*

(iv) $X_{m,\alpha}(0,1) \overset{*}{\hookrightarrow} Y(0,1)$,

and if $\alpha \in [1/n', 1)$ and $Y(0,1) = L^\infty(0,1)$, then (i), (ii) *and* (iii) *are equivalent to*

(v) $X(0,1) \overset{*}{\hookrightarrow} L^{\frac{1}{m(1-\alpha)},1}(0,1)$.

Furthermore, if $m(1-\alpha) > 1$ then conditions (i) *and* (iii) *are satisfied independently of the choice of $X(0,1)$ and $Y(0,1)$, while condition* (ii) *is true for all pairs of rearrangement-invariant spaces $X(0,1)$ and $Y(0,1)$ except of $X(0,1) = L^1(0,1)$ and $Y(0,1) = L^\infty(0,1)$.*

Proof. Suppose that (i) is satisfied. Then, in particular, $V^m X(\Omega) \hookrightarrow\hookrightarrow Y(\Omega)$ holds for Ω given by [3, Proposition 11.1]. Define the function N_α by

$$N_\alpha(s) = \begin{cases} (1-(1-\alpha)s)^{\frac{1}{1-\alpha}} \text{ for } s \in [0, \frac{1}{1-\alpha}] & \text{if } \alpha \in [1/n', 1); \\ e^{-s} \text{ for } s \in [0,\infty) & \text{if } \alpha = 1. \end{cases}$$

Let f be any nonnegative function in $X(0,1)$ (or, what is the same, let f be an arbitrary function belonging to the set X_+^1 defined in Proposition 4.1). For $x = (x_1, \dots, x_n) \in \Omega$, we set

$$Lf(x) = \int_{N_\alpha(x_n)}^1 \frac{1}{r_1^\alpha} \int_{r_1}^1 \frac{1}{r_2^\alpha} \dots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha}\, dr_m\, dr_{m-1} \dots dr_1.$$

Let us also define $H = H_{I_\alpha}^m$. According to [3, proof of Theorem 6.4], Lf is an m-times weakly differentiable function in Ω satisfying

$$(Lf)_{\lambda_n}^*(s) = \frac{(Hf)(s)}{(m-1)!} \text{ and } (|\nabla^m Lf|)_{\lambda_n}^*(s) = f^*(s)$$

for $s \in (0,1)$. Here, λ_n denotes the n-dimensional Lebesgue measure. The first equality means, in particular, that L satisfies (4.2) with $\beta = 1$ and

$$K(s,t) = \frac{1}{I_\alpha(s)} \left(\int_t^s \frac{dr}{I_\alpha(r)} \right)^{m-1}, \quad (s,t) \in (0,1) \times (0,1).$$

Moreover,

$$\|Lf\|_{Y(\Omega)} = \|(Lf)^*_{\lambda_n}\|_{Y(0,1)} = \frac{\|Hf\|_{Y(0,1)}}{(m-1)!}$$

and

$$\||\nabla^m Lf|\|_{X(\Omega)} = \|(|\nabla^m Lf|)^*_{\lambda_n}\|_{X(0,1)} = \|f\|_{X(0,1)}.$$

Furthermore, in [3, proof of Theorem 6.4] it is shown that there is a constant $C > 0$ independent of f such that

$$\|Lf\|_{L^1(\Omega)} \le C\|f\|_{X(0,1)}.$$

Altogether, we get (4.3) and (4.4). Using Proposition 4.1 with H and L as above, we obtain that

$$\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|H^m_{I_\alpha}(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0,$$

which is, as it was shown in the proof of Theorem 3.7 if $\alpha \in [1/n', 1)$, equivalent to (iii).

As it has already been pointed out at the beginning of this section, the function I_α is admissible and if $\alpha \in [1/n', 1)$ then (3.18) is fulfilled. Moreover, each domain $\Omega \in \mathcal{J}_\alpha$ satisfies (3.1). Hence, according to Theorem 3.7 if $\alpha \in [1/n', 1)$ and to Theorem 3.5 if $\alpha = 1$, (iii) implies (i).

Assume that $\alpha = 1$. Then condition (ii) is equivalent to (iii) due to Theorem 3.5 and condition (iv) is equivalent to (iii) thanks to Theorem 2.3 and to the fact that $\int_0^1 1/s\,ds = \infty$.

Now, let $\alpha \in [1/n', 1)$. If $X(0,1) \ne L^1(0,1)$ or $Y(0,1) \ne L^\infty(0,1)$ or $m(1-\alpha) \le 1$ (which is the same as that the function $(I(s))^m/s^{m-1} = s^{1-m(1-\alpha)}$ is nondecreasing in $(0,1)$) then Theorem 3.7 gives the equivalence of (ii) and (iii). Moreover, if $Y(0,1) \ne L^\infty(0,1)$ then (iii) is equivalent to (iv), and if $Y(0,1) = L^\infty(0,1)$ and $m(1-\alpha) \le 1$ then (iii) is equivalent to

$$X(0,1) \overset{*}{\hookrightarrow} A_{s^{1-m(1-\alpha)}}(0,1) = L^{\frac{1}{m(1-\alpha)},1}(0,1).$$

It remains to examine the case when $m(1-\alpha) > 1$. In this situation, it follows from Example 2.5 applied to $\beta = m(1-\alpha) - 1 > 0$ that (iii) holds with $X(0,1) = L^1(0,1)$ and $Y(0,1) = L^\infty(0,1)$. Because in general $X(0,1) \hookrightarrow L^1(0,1)$ and $L^\infty(0,1) \hookrightarrow Y(0,1)$, we obtain that (iii) is true for all pairs of rearrangement-invariant spaces $X(0,1)$ and $Y(0,1)$. We have already proved that (iii) implies (i), so, (i) is satisfied independently of the choice of $X(0,1)$ and $Y(0,1)$. Furthermore, (iii) implies (ii) provided that $X(0,1) \ne L^1(0,1)$ or $Y(0,1) \ne L^\infty(0,1)$. Thus, (ii) holds under this assumption. Conversely, it follows from Example 2.5 that (ii) is not true when $X(0,1) = L^1(0,1)$ and $Y(0,1) = L^\infty(0,1)$. □

The remaining part of this section is devoted to applications of the previous theorem to concrete pairs of rearrangement-invariant spaces $X(0,1)$ and $Y(0,1)$. We will always assume that $m(1-\alpha) \le 1$, which can be done with no loss of generality since the situation when $m(1-\alpha) > 1$ was sufficiently described in Theorem 4.4.

We first focus on compact embeddings of the Sobolev space $V^m L^1(\Omega)$ into rearrangement-invariant spaces, and on compact embeddings of Sobolev spaces built upon rearrangement-invariant spaces into $L^\infty(\Omega)$. Towards doing this, the following observation will be of use.

Remark 4.5. Suppose that ψ is a nonnegative nondecreasing concave function in $(0,1)$ and $X(0,1)$ is a rearrangement-invariant space. If φ_X denotes the fundamental function of $X(0,1)$ then

$$\Lambda_\psi(0,1) \overset{*}{\hookrightarrow} X(0,1) \tag{4.16}$$

if and only if

$$\lim_{s\to 0_+} \frac{\varphi_X(s)}{\psi(s)} = 0. \tag{4.17}$$

Indeed, (4.17) is a necessary condition for (4.16) to be true, see [6, Section 3]. On the other hand, if (4.17) is satisfied then, according to [6, Example 3.1], $\Lambda_\psi(0,1) \overset{*}{\hookrightarrow} \Lambda_{\varphi_X}(0,1)$ which, together with the fact that $\Lambda_{\varphi_X}(0,1) \hookrightarrow X(0,1)$, implies (4.16).

Theorem 4.6. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $m \in \mathbb{N}$. Suppose that $X(0,1)$ is a rearrangement-invariant space and denote by φ_X its fundamental function. If $\alpha \in [1/n', 1)$ satisfies $m(1-\alpha) \leq 1$ then the condition*

$$V^m L^1(\Omega) \hookrightarrow\hookrightarrow X(\Omega) \tag{4.18}$$

is satisfied for every $\Omega \in \mathcal{J}_\alpha$ if and only if

$$\lim_{s\to 0_+} \frac{\varphi_X(s)}{s^{1-m(1-\alpha)}} = 0, \tag{4.19}$$

and the condition

$$V^m X(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \tag{4.20}$$

is satisfied for every $\Omega \in \mathcal{J}_\alpha$ if and only if

$$\lim_{a\to 0_+} \|\chi_{(0,a)}(s) s^{m(1-\alpha)-1}\|_{X'(0,1)} = 0. \tag{4.21}$$

Furthermore, there is no rearrangement-invariant space $X(0,1)$ such that condition (4.18) or condition (4.20) is satisfied for every $\Omega \in \mathcal{J}_1$.

Proof. Let $\alpha \in [1/n', 1)$. It follows from Theorem 4.4, condition (v), that (4.18) is not satisfied for every $\Omega \in \mathcal{J}_\alpha$ when $X(0,1) = L^\infty(0,1)$ (recall that there is no rearrangement-invariant space into which $L^1(0,1)$ is almost-compactly embedded). Thus, Theorem 4.4 yields that (4.18) holds for every $\Omega \in \mathcal{J}_\alpha$ if and only if

$$(L^1)_{m,\alpha}(0,1) \overset{*}{\hookrightarrow} X(0,1). \tag{4.22}$$

Due to [3, Theorem 6.8], we have

$$\left(L^1\right)_{m,\alpha}(0,1) = \begin{cases} L^{\frac{1}{1-m(1-\alpha)},1}(0,1) & \text{if } m(1-\alpha) < 1; \\ L^\infty(0,1) & \text{if } m(1-\alpha) = 1. \end{cases}$$

In the latter case, condition (4.22) is fulfilled if and only if

$$\lim_{s\to 0_+} \varphi_X(s) = 0,$$

see [13, Theorem 5.2].

If $m(1-\alpha) < 1$ we set

$$\psi(s) = \frac{s^{1-m(1-\alpha)}}{1-m(1-\alpha)}, \quad s \in (0,1).$$

Then ψ is a nonnegative nondecreasing concave function in $(0,1)$ and $\Lambda_\psi(0,1) = L^{1/(1-m(1-\alpha)),1}(0,1)$. Owing to Remark 4.5, (4.22) holds if and only if (4.19) is satisfied, as required.

Furthermore, due to Theorem 4.4, condition (4.20) is satisfied for every $\Omega \in \mathcal{J}_\alpha$ if and only if

$$\begin{aligned} 0 = \lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|M_\alpha^m(\chi_{(0,a)}|f|)\|_{L^\infty(0,1)} &= \lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \int_0^a |f(s)| s^{m(1-\alpha)-1}\, ds \\ &= \lim_{a\to 0_+} \left\| \chi_{(0,a)}(s) s^{m(1-\alpha)-1} \right\|_{X'(0,1)}. \end{aligned}$$

We will finally examine the case when $\alpha = 1$. According to Theorem 4.4 and [3, Theorem 6.13], (4.18) holds for every $\Omega \in \mathcal{J}_1$ if and only if

$$(L^1)_{m,1}(0,1) = L^1(0,1) \overset{*}{\hookrightarrow} X(0,1),$$

which is never satisfied (see Section 1.1). Similarly, (4.20) is fulfilled for every $\Omega \in \mathcal{J}_1$ if and only if

$$X_{m,\alpha}(0,1) \overset{*}{\hookrightarrow} L^\infty(0,1).$$

As it was pointed out in Section 1.1, this cannot be satisfied since there is no rearrangement-invariant space almost-compactly embedded into $L^\infty(0,1)$. □

Remark 4.7. Theorem 4.6 enables us to describe all compact Sobolev embeddings on Maz'ya classes of domains in the case when $m(1-\alpha) = 1$. Indeed, suppose that $X(0,1)$ is a rearrangement-invariant space different from $L^\infty(0,1)$. Then $\lim_{s\to 0_+} \varphi_X(s) = 0$, so, (4.18) is satisfied for every $\Omega \in \mathcal{J}_\alpha$. Furthermore, if $X(0,1)$ is a rearrangement-invariant space different from $L^1(0,1)$ then $X'(0,1) \neq L^\infty(0,1)$ and we have

$$\lim_{a\to 0_+} \|\chi_{(0,a)}(s)\|_{X'(0,1)} = \lim_{a\to 0_+} \varphi_{X'}(a) = 0.$$

Hence, (4.20) is satisfied for every $\Omega \in \mathcal{J}_\alpha$. Altogether, thanks to continuous embeddings $X(0,1) \hookrightarrow L^1(0,1)$ and $L^\infty(0,1) \hookrightarrow Y(0,1)$ which hold for all rearrangement-invariant spaces $X(0,1)$ and $Y(0,1)$, we have that $V^m X(\Omega) \hookrightarrow\hookrightarrow Y(\Omega)$ is satisfied for every $\Omega \in \mathcal{J}_\alpha$ provided that $X(0,1) \neq L^1(0,1)$ or $Y(0,1) \neq L^\infty(0,1)$. On the other hand, it follows from Theorem 4.6 that there is always some domain $\Omega \in \mathcal{J}_\alpha$ for which $V^m L^1(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega)$ is not satisfied.

We now focus on applications of Theorem 4.4 to the case when both $X(0,1)$ and $Y(0,1)$ are Lorentz spaces. Owing to Remark 4.7, we shall assume that $m(1-\alpha) < 1$.

Theorem 4.8. *Let $n \in \mathbb{N}$, $n \ge 2$, let $m \in \mathbb{N}$ and let $\alpha \in [1/n', 1]$ satisfy $m(1-\alpha) < 1$. Suppose that p_1, p_2, q_1, $q_2 \in [1,\infty]$ are such that both $L^{p_1,q_1}(0,1)$ and $L^{p_2,q_2}(0,1)$ are rearrangement-invariant spaces (up to equivalent norms). Then the following assertions are equivalent.*

(i) *The compact embedding*

$$V^m L^{p_1,q_1}(\Omega) \hookrightarrow\hookrightarrow L^{p_2,q_2}(\Omega)$$

holds for every $\Omega \in \mathcal{J}_\alpha$.

(ii) *The compact embedding*

$$V^m L^{p_1}(\Omega) \hookrightarrow\hookrightarrow L^{p_2}(\Omega)$$

holds for every $\Omega \in \mathcal{J}_\alpha$.

(iii) *One of the following conditions is satisfied:*

$$\alpha \in [1/n', 1), \quad p_1 < \frac{1}{m(1-\alpha)}, \quad p_2 < \frac{p_1}{1 - mp_1(1-\alpha)}; \tag{4.23}$$

$$\alpha \in [1/n', 1), \quad p_1 = \frac{1}{m(1-\alpha)}, \quad p_2 < \infty; \tag{4.24}$$

$$\alpha \in [1/n', 1), \quad p_1 > \frac{1}{m(1-\alpha)}; \tag{4.25}$$

$$\alpha = 1, \quad p_1 > p_2. \tag{4.26}$$

Proof. Let $\alpha \in [1/n', 1)$. First, suppose that $L^{p_2,q_2}(0,1) \neq L^\infty(0,1)$. Due to Theorem 4.4, (i) is satisfied if and only if

$$(L^{p_1,q_1})_{m,\alpha}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2}(0,1). \tag{4.27}$$

It follows from [3, Theorem 6.8] that

$$(L^{p_1,q_1})_{m,\alpha}(0,1) = \begin{cases} L^{\frac{p_1}{1-mp_1(1-\alpha)},q_1}(0,1) & \text{if } p_1 < \frac{1}{m(1-\alpha)}; \\ L^{\infty,q_1,-1}(0,1) & \text{if } p_1 = \frac{1}{m(1-\alpha)} \text{ and } q_1 > 1; \\ L^\infty(0,1) & \text{otherwise.} \end{cases}$$

Thus, if $p_1 < 1/(m(1-\alpha))$ then (4.27) is fulfilled if and only if $p_2 < p_1/(1 - mp_1(1-\alpha))$, see Proposition 4.2. In the case when $p_1 \geq 1/(m(1-\alpha))$, (4.27) is characterized by $p_2 < \infty$. Indeed, observe that the only Lorentz space having the first index equal to ∞ and being a rearrangement-invariant space (up to equivalent norms) at the same time is $L^\infty(0,1)$. Since there is no rearrangement-invariant space almost-compactly embedded into $L^\infty(0,1)$ (see Section 1.1), condition (4.27) cannot be satisfied with $p_2 = \infty$. On the other hand, it is satisfied with $p_2 < \infty$, see Proposition 4.2.

Let us now discuss the case when $L^{p_2,q_2}(0,1) = L^\infty(0,1)$. Theorem 4.4 yields that in this situation, (i) is satisfied if and only if

$$L^{p_1,q_1}(0,1) \overset{*}{\hookrightarrow} L^{\frac{1}{m(1-\alpha)},1}(0,1),$$

which is, owing to Proposition 4.2, equivalent to $p_1 > 1/(m(1-\alpha))$.

Let $\alpha = 1$. First, suppose that $L^{p_1,q_1}(0,1) \neq L^\infty(0,1)$. Then, according to Theorem 4.4 and [3, Theorem 6.13], (i) is satisfied if and only if

$$(L^{p_1,q_1})_{m,1}(0,1) = L^{p_1,q_1}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2}(0,1),$$

which is equivalent to $p_2 < p_1$, see Proposition 4.2. Finally, (i) is satisfied with $L^{p_1,q_1}(0,1) = L^\infty(0,1)$ if and only if

$$(L^\infty)_{m,1}(0,1) = L^{\infty,\infty,-m}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2}(0,1).$$

As observed above, this is equivalent to $p_2 < \infty$.

By applying the equivalence of (i) and (iii) to the particular case when $p_1 = q_1$ and $p_2 = q_2$, we obtain that (ii) is equivalent to (iii) as well. The proof is complete. ☐

We shall finish this section by proving Theorem 4.3 which characterizes compact Sobolev embeddings on John domains. Note that variations of Theorem 4.6 and Theorem 4.8 for John domains can then be obtained by using the fact that the compact embedding

$$V^m X(\Omega) \hookrightarrow\hookrightarrow Y(\Omega)$$

holds for one particular John domain Ω if and only if it holds for all $\Omega \in \mathcal{J}_{1/n'}$.

Proof of Theorem 4.3. Suppose that $m < n$. Let B_R be a ball of radius $R \in (0,\infty)$ such that $\overline{B_R} \subseteq \Omega$. Without loss on generality, we may assume that B_R is centered at 0. Let f be any nonnegative function in $X(0,1)$ (or, what is the same, let f be an arbitrary function belonging to the set X^1_+ defined in Proposition 4.1). Then we set

$$Lf(x) = \begin{cases} \int_{\kappa_n|x|^n}^{\kappa_n R^n} \int_{r_1}^{\kappa_n R^n} \dots \int_{r_{m-1}}^{\kappa_n R^n} f\left(\frac{r_m}{\kappa_n R^n}\right) r_m^{-m+\frac{m}{n}}\, dr_m \dots dr_1 & x \in B_R \\ 0 & x \in \Omega/B_R, \end{cases}$$

where κ_n denotes the volume of the unit ball in $\mathbb{R}^n$. Define also $H = M^m_{1/n'}$. Then, according to [3, proof of Theorem 6.1], Lf is an m-times weakly differentiable function in Ω satisfying (4.3) and (4.4). Furthermore, for every $s > 0$ we have

$$(Lf)^*_{\lambda_n}(s) = \chi_{(0,\kappa_n R^n)}(s) \int_s^{\kappa_n R^n} f\left(\frac{r}{\kappa_n R^n}\right) r^{-m+\frac{m}{n}} \frac{(r-s)^{m-1}}{(m-1)!}\, dr,$$

so L satisfies (4.2) with $\beta = \kappa_n R^n$ and

$$K(r,s) = r^{-m+\frac{m}{n}} \frac{(r-s)^{m-1}}{(m-1)!}, \quad (r,s) \in (0,\kappa_n R^n) \times (0,\kappa_n R^n).$$

The implication (i) ⇒ (iii) in the particular case $m < n$ thus follows from Proposition 4.1 used with H and L as above.

Moreover, the above argument can also be applied to the case when $m = n$ and $X(0,1) = L^1(0,1)$. However, we need to explain why there is a constant $C > 0$ such that $\||\nabla^m Lf|\|_{L^1(\Omega)} \le C\|f\|_{L^1(0,1)}$ holds for every $f \in (L^1)^1_+$ since the proof given in [3, proof of Theorem 6.1] does not work in this situation.

It follows from [3, proof of Theorem 6.1] that there are positive constants C' and C'' such that for all $f \in (L^1)^1_+$,

$$\||\nabla^m Lf|\|_{L^1(\Omega)} \le C'\|f\|_{L^1(0,1)} + C'' \left\| \sum_{i=1}^{m-1} s^{i-1} \int_s^1 f(r) r^{-i}\, dr \right\|_{L^1(0,1)}.$$

But for every $i = 1, \dots, m-1$ we have

$$\left\| s^{i-1} \int_s^1 f(r) r^{-i} \right\|_{L^1(0,1)} = \int_0^1 s^{i-1} \int_s^1 f(r) r^{-i} \, dr \, ds = \int_0^1 f(r) r^{-i} \int_0^r s^{i-1} \, ds \, dr$$
$$= \frac{1}{i} \int_0^1 f(r) \, dr = \frac{1}{i} \|f\|_{L^1(0,1)},$$

which gives the result.

By using Remark 4.7 applied to the case $\alpha = 1/n'$ and by taking the equality $m(1-1/n') = m/n$ into account, it follows that condition (iii) is satisfied whenever $m = n$ and $X(0,1) \neq L^1(0,1)$. Altogether, we have proved that (i) implies (iii) provided that $m \leq n$. The rest of the proof follows from Theorem 4.4 by using the fact that Ω belongs to $\mathcal{J}_{1/n'}$. ☐

4.3 Compactness of Sobolev embeddings in product probability spaces

Let $m \in \mathbb{N}$ and let Φ be as in Section 1.2. Since the function $J = L_\Phi$ is measurable in $[0,1]$ and satisfies (2.1) we can consider the operator $P_\Phi^m = H_{L_\Phi}^m$ defined by (2.3). Observe that

$$P_\Phi^m f(t) = \int_t^1 \frac{f(s)}{s\Phi'\left(\Phi^{-1}\left(\log \frac{2}{s}\right)\right)} \left(\int_t^s \frac{dr}{r\Phi'\left(\Phi^{-1}\left(\log \frac{2}{r}\right)\right)} \right)^{m-1} ds$$
$$= \int_t^1 f(s) \frac{\left(\Phi^{-1}\left(\log \frac{2}{t}\right) - \Phi^{-1}\left(\log \frac{2}{s}\right)\right)^{m-1}}{s\Phi'\left(\Phi^{-1}\left(\log \frac{2}{s}\right)\right)} ds, \quad f \in L^1(0,1), \ t \in (0,1).$$

Furthermore, if $X(0,1)$ is a rearrangement-invariant space, we shall consider the rearrangement-invariant space $X_{m,L_\Phi}(0,1)$ whose norm is given by (2.5). Similarly to the previous case, it is not hard to observe that for every $f \in \mathcal{M}(0,1)$,

$$\|f\|_{(X_{m,L_\Phi})'(0,1)} = \left\| \int_0^t f^*(s) \frac{\left(\Phi^{-1}(\log \frac{2}{s}) - \Phi^{-1}(\log \frac{2}{t})\right)^{m-1}}{t\Phi'(\Phi^{-1}(\log \frac{2}{t}))} ds \right\|_{X'(0,1)}.$$

The following theorem characterizes compact Sobolev embeddings in $(\mathbb{R}^n, \mu_{\Phi,n})$. Notice that, in contrast to the Euclidean setting, such embeddings do not depend on the dimension n, in the sense that we have the equivalence of the following two assertions.
(i) There exists $n \in \mathbb{N}$ for which $V^m X(\mathbb{R}^n, \mu_{\Phi,n}) \hookrightarrow\hookrightarrow Y(\mathbb{R}^n, \mu_{\Phi,n})$ is satisfied.
(ii) The compact embedding $V^m X(\mathbb{R}^n, \mu_{\Phi,n}) \hookrightarrow\hookrightarrow Y(\mathbb{R}^n, \mu_{\Phi,n})$ is satisfied for every $n \in \mathbb{N}$.

Theorem 4.9. *Let $n, m \in \mathbb{N}$ and let $X(0,1)$ and $Y(0,1)$ be rearrangement-invariant spaces. Then the following conditions are equivalent:*

(i) $V^m X(\mathbb{R}^n, \mu_{\Phi,n}) \hookrightarrow\hookrightarrow Y(\mathbb{R}^n, \mu_{\Phi,n})$;
(ii) $P_\Phi^m : X(0,1) \to\to Y(0,1)$;
(iii) $\lim_{a \to 0_+} \sup_{\|f\|_{X(0,1)} \leq 1} \|P_\Phi^m(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0$;
(iv) $X_{m,L_\Phi}(0,1) \overset{*}{\hookrightarrow} Y(0,1)$.

Proof. Assume that (i) holds. Let f be an arbitrary function belonging to the set $X_+^{1/2}$ defined in Proposition 4.1. For every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we set

$$Lf(x) = \int_{F_\Phi(x_1)}^1 \frac{1}{I_\Phi(r_1)} \int_{r_1}^1 \frac{1}{I_\Phi(r_2)} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{I_\Phi(r_m)}\, dr_m\, dr_{m-1} \dots dr_1.$$

Let us also define $H = P_\Phi^m$. According to [3, proof of Theorem 7.4], Lf is an m-times weakly differentiable function in $\mathbb{R}^n$ satisfying

$$(|\nabla^m Lf|)^*_{\mu_{\Phi,n}}(s) = f^*(s), \quad s \in (0,1).$$

Thus,

$$\||\nabla^m Lf|\|_{X(\mathbb{R}^n,\mu_{\Phi,n})} = \|(|\nabla^m Lf|)^*_{\mu_{\Phi,n}}\|_{X(0,1)} = \|f\|_{X(0,1)}. \tag{4.28}$$

Furthermore, in [3, proof of Theorem 7.4] it is shown that there is a constant $C > 0$ such that for each $f \in X_+^{1/2}$,

$$\|Lf\|_{L^1(\mathbb{R}^n,\mu_{\Phi,n})} \le C\|f\|_{X(0,1)}. \tag{4.29}$$

By adding (4.28) and (4.29), we obtain (4.3).

According to [3, proof of Theorem 7.4] once again, we get

$$(Lf)^*_{\mu_{\Phi,n}}(s) \approx (Hf)(s), \quad s \in (0,1),$$

up to multiplicative constants independent of $f \in X_+^{1/2}$ and $s \in (0,1)$. This, in particular, means that L satisfies (4.2) with $\beta = 1$ and

$$K(s,t) = \frac{\left(\Phi^{-1}\left(\log\frac{2}{t}\right) - \Phi^{-1}\left(\log\frac{2}{s}\right)\right)^{m-1}}{s\Phi'\left(\Phi^{-1}\left(\log\frac{2}{s}\right)\right)}, \quad (s,t) \in (0,1) \times (0,1).$$

Moreover,

$$\|Lf\|_{Y(\mathbb{R}^n,\mu_{\Phi,n})} = \|(Lf)^*_{\mu_{\Phi,n}}\|_{Y(0,1)} \approx \|Hf\|_{Y(0,1)} \tag{4.30}$$

up to multiplicative constants independent of $f \in X_+^{1/2}$, hence, (4.4) is satisfied as well. Using Proposition 4.1 with H and L as above, we obtain (iii).

According to [3, Lemma 12.3], there is $t_0 \in (0, 1/2]$ such that the function I defined by

$$I(t) = \begin{cases} I_\Phi(t) & t \in [0, t_0]; \\ I_\Phi(t_0) & t \in (t_0, 1]; \end{cases}$$

is admissible. Moreover,

$$I(t) \approx I_\Phi(t) \approx I_{(\mathbb{R}^n,\mu_{\Phi,n})}(t) \approx L_\Phi(t), \quad t \in [0, 1/2], \tag{4.31}$$

up to multiplicative constants independent of $t \in [0, 1/2]$. Since I is a constant function in $[1/2, 1]$ and there are positive constants C_1, C_2 such that $C_1 \le L_\Phi(t) \le C_2$, $t \in [1/2, 1]$, the equivalence $I \approx L_\Phi$ holds in the entire $[0,1]$. Thus, we have

$$H_I^m|f|(t) \approx H_{L_\Phi}^m|f|(t) = P_\Phi^m|f|(t), \quad f \in L^1(0,1),\ t \in (0,1),$$

up to multiplicative constants independent of f and t. Hence, (iii) is fulfilled if and only if

$$\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)} \le 1} \|H_I^m(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0.$$

Furthermore, thanks to (4.31), there is a constant $C_3 > 0$ such that $I_{(\mathbb{R}^n,\mu_{\Phi,n})}(t) \geq C_3 I(t)$ for every $t \in [0, 1/2]$. Denote $C_4 = \min(C_3, 1)$. Then, owing to the fact that I is nondecreasing in $[0, 1]$,

$$C_3 I(t) \geq C_4 I(t) \geq C_4 I(C_4 t), \quad t \in [0, 1/2].$$

Hence, I satisfies (3.1) with $(\Omega, \nu) = (\mathbb{R}^n, \mu_{\Phi,n})$. The implication (iii) $\Rightarrow$ (i) now follows from Theorem 3.5.

Finally, recall that the function $J = L_\Phi$ fulfills (2.1) and observe that

$$\begin{aligned}\int_0^1 \frac{ds}{J(s)} &= \int_0^1 \frac{ds}{L_\Phi(s)} = \lim_{t\to 0_+} \int_t^1 \frac{ds}{L_\Phi(s)} \\ &= \lim_{t\to 0_+} \left(\Phi^{-1}\left(\log \frac{2}{t}\right) - \Phi^{-1}(\log 2) \right) = \infty. \end{aligned} \tag{4.32}$$

Using Theorem 2.3 and the fact that $P^m_\Phi = H^m_{L_\Phi}$, we obtain the equivalence of (ii), (iii) and (iv). ☐

We now focus on compact Sobolev embeddings in generalized Gauss spaces. Let $\beta \in [1, 2]$ and let $m \in \mathbb{N}$. Consider the operator G^m_β defined by

$$G^m_\beta f(t) = \frac{1}{\left(\log \frac{2}{t}\right)^{\frac{\beta-1}{\beta}(m-1)}} \int_t^1 f(s) \frac{\left(\log \frac{s}{t}\right)^{m-1}}{s\left(\log \frac{2}{s}\right)^{\frac{\beta-1}{\beta}}}\, ds, \quad f \in L^1(0,1),\ t \in (0,1).$$

Furthermore, whenever $X(0,1)$ is a rearrangement-invariant space, define the functional $\|\cdot\|_{\left(X^G_{m,\beta}\right)'(0,1)}$ for every $f \in \mathcal{M}(0,1)$ by

$$\|f\|_{(X^G_{m,\beta})'(0,1)} = \left\| \frac{1}{t(\log \frac{2}{t})^{\frac{\beta-1}{\beta}}} \int_0^t f^*(s) \frac{(\log \frac{t}{s})^{m-1}}{(\log \frac{2}{s})^{\frac{\beta-1}{\beta}(m-1)}}\, ds \right\|_{X'(0,1)}.$$

According to [3, Theorem 7.5] (where the case $\beta = 2$ is treated, the general case is analogous), the functional $\|\cdot\|_{\left(X^G_{m,\beta}\right)'(0,1)}$ is a rearrangement-invariant norm and, if we denote by $X^G_{m,\beta}(0,1)$ the associate space to the space $\left(X^G_{m,\beta}\right)'(0,1)$, we have that $X^G_{m,\beta}(0,1) = X_{m,s(\log 2/s)^{(\beta-1)/\beta}}(0,1)$.

In the special case when $\beta = 2$ and $\gamma_{n,\beta}$ is therefore the n-dimensional Gauss measure, the operator $G^m = G^m_2$ having the form

$$G^m f(t) = \frac{1}{\left(\log \frac{2}{t}\right)^{\frac{m-1}{2}}} \int_t^1 f(s) \frac{\left(\log \frac{s}{t}\right)^{m-1}}{s\left(\log \frac{2}{s}\right)^{1/2}}\, ds, \quad f \in L^1(0,1),\ t \in (0,1),$$

and the rearrangement-invariant space $X^G_m(0,1) = X^G_{m,2}(0,1)$ fulfilling

$$\|f\|_{(X^G_m)'(0,1)} = \left\| \frac{1}{t(\log \frac{2}{t})^{1/2}} \int_0^t f^*(s) \frac{(\log \frac{t}{s})^{m-1}}{(\log \frac{2}{s})^{\frac{m-1}{2}}}\, ds \right\|_{X'(0,1)}, \quad f \in \mathcal{M}(0,1),$$

come into play.

Characterization of compact Sobolev embeddings in generalized Gauss spaces takes the following form.

Theorem 4.10. *Let $n, m \in \mathbb{N}$, let $\beta \in [1,2]$ and let $X(0,1)$ and $Y(0,1)$ be rearrangement-invariant spaces. Then the following conditions are equivalent:*

(i) $V^m X(\mathbb{R}^n, \gamma_{n,\beta}) \hookrightarrow\hookrightarrow Y(\mathbb{R}^n, \gamma_{n,\beta})$;
(ii) $G^m_\beta : X(0,1) \to\to Y(0,1)$;
(iii) $\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|G^m_\beta(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0$;
(iv) $X^G_{m,\beta}(0,1) \overset{*}{\hookrightarrow} Y(0,1)$.

Proof. Set

$$\Phi(s) = \frac{1}{\beta} s^\beta, \quad s \in [0,\infty).$$

Then $\mu_{\Phi,n} = \gamma_{n,\beta}$ and it is easy to observe that

$$L_\Phi(s) = s\left(\log\frac{2}{s}\right)^{\frac{\beta-1}{\beta}}, \quad s \in (0,1].$$

According to [3, proof of Theorem 7.7], for each nonnegative function $f \in X(0,1)$ we have

$$G^m_\beta f \approx P^m_\Phi f \tag{4.33}$$

up to multiplicative constants depending on β and m. Thus, condition (iii) is equivalent to

$$\lim_{a\to 0_+} \sup_{\|f\|_{X(0,1)}\le 1} \|P^m_\Phi(\chi_{(0,a)}|f|)\|_{Y(0,1)} = 0, \tag{4.34}$$

which is equivalent to (i) due to Theorem 4.9. Moreover, the fact that $X^G_{m,\beta}(0,1) = X_{m,s(\log 2/s)^{(\beta-1)/\beta}}(0,1) = X_{m,L_\Phi}(0,1)$ yields the equivalence of (iii) and (iv).

Assume that (iv) is satisfied. Let $(f_k)_{k=1}^\infty$ be a bounded sequence in $X(0,1)$. Then for every $s \in (0,1)$,

$$\left|\frac{f_k(s)}{\left(\log\frac{2}{s}\right)^{\frac{\beta-1}{\beta}}}\right| \le \frac{|f_k(s)|}{(\log 2)^{\frac{\beta-1}{\beta}}},$$

so $(f_k(s)/(\log 2/s)^{(\beta-1)/\beta})_{k=1}^\infty$ is bounded in $X(0,1)$ as well. Consider the function $J(s) = s$, $s \in [0,1]$, and the operator G^m_1 satisfying

$$G^m_1 f(t) = H^m_J f(t) = \int_t^1 f(s) \frac{\left(\log\frac{s}{t}\right)^{m-1}}{s}\, ds, \quad f \in X(0,1),\ t \in (0,1).$$

Due to Lemma 2.2 applied to J and to the sequence $(f_k(s)/(\log 2/s)^{(\beta-1)/\beta})_{k=1}^\infty$, there is a subsequence $(f_{k_\ell})_{\ell=1}^\infty$ of $(f_k)_{k=1}^\infty$ such that

$$G^m_1\left(\frac{f_{k_\ell}(s)}{\left(\log\frac{2}{s}\right)^{\frac{\beta-1}{\beta}}}\right)(t) = \int_t^1 f_{k_\ell}(s) \frac{\left(\log\frac{s}{t}\right)^{m-1}}{s\left(\log\frac{2}{s}\right)^{\frac{\beta-1}{\beta}}}\, ds$$

is convergent for a.e. $t \in (0,1)$. Thus also

$$G^m_\beta f_{k_\ell}(t) = \frac{1}{\left(\log\frac{2}{t}\right)^{\frac{\beta-1}{\beta}(m-1)}} G^m_1\left(\frac{f_{k_\ell}(s)}{\left(\log\frac{2}{s}\right)^{\frac{\beta-1}{\beta}}}\right)(t)$$

converges for a.e. $t \in (0,1)$. By using that $G^m_\beta : X(0,1) \to X^G_{m,\beta}(0,1)$ (see [3, Theorem 7.5 and Theorem 7.7]), the same proof as in Theorem 2.3, implication (iv) $\Rightarrow$ (i), gives that $G^m_\beta f_{k_\ell}$ is convergent in $Y(0,1)$, which implies (ii).

The implication (ii) $\Rightarrow$ (iii) can be proved in the same way as the implication (i) $\Rightarrow$ (ii) in Theorem 2.3. □

In the remaining part of this section we focus on applications of results from Theorem 4.10 to concrete pairs of rearrangement-invariant spaces $X(0,1)$ and $Y(0,1)$. Similarly as in the case of Maz'ya domains, we start with the situation when $X(0,1) = L^1(0,1)$. On the other hand, we do not study compact Sobolev embeddings into $L^\infty(0,1)$ since such embeddings are never fulfilled, see Remark 2.4. Instead of this, we focus on compact embeddings of the Sobolev space $V^m L^\infty(\mathbb{R}^n, \gamma_{n,\beta})$.

Theorem 4.11. *Let n, $m \in \mathbb{N}$ and $\beta \in [1,2]$. Suppose that $X(0,1)$ is a rearrangement-invariant space and denote by φ_X its fundamental function. Then the condition*

$$V^m L^1(\mathbb{R}^n, \gamma_{n,\beta}) \hookrightarrow\hookrightarrow X(\mathbb{R}^n, \gamma_{n,\beta}) \tag{4.35}$$

is satisfied if and only if

$$\lim_{s\to 0_+} \frac{\varphi_X(s)}{s(\log \frac{2}{s})^{\frac{m(\beta-1)}{\beta}}} = 0, \tag{4.36}$$

and the condition

$$V^m L^\infty(\mathbb{R}^n, \gamma_{n,\beta}) \hookrightarrow\hookrightarrow X(\mathbb{R}^n, \gamma_{n,\beta}) \tag{4.37}$$

is satisfied if and only if

$$\lim_{a\to 0_+} \left\| \chi_{(0,a)}(s) \left(\log \frac{2}{s}\right)^{\frac{m}{\beta}} \right\|_{X(0,1)} = 0. \tag{4.38}$$

Proof. Due to Theorem 4.10 and [3, Theorem 7.8], (4.35) is equivalent to

$$(L^1)^G_{m,\beta}(0,1) = L^{1,1;\frac{m(\beta-1)}{\beta}}(0,1) \overset{*}{\hookrightarrow} X(0,1).$$

Set

$$\psi(s) = \int_0^s \left(\log \frac{e}{r}\right)^{\frac{m(\beta-1)}{\beta}} dr, \quad s \in (0,1).$$

Then ψ is a nonnegative nondecreasing concave function in $(0,1)$ and $\Lambda_\psi(0,1) = L^{1,1;\frac{m(\beta-1)}{\beta}}(0,1)$. Using Remark 4.5 and the fact that

$$\int_0^s \left(\log \frac{e}{r}\right)^{\frac{m(\beta-1)}{\beta}} dr \approx s\left(\log \frac{e}{s}\right)^{\frac{m(\beta-1)}{\beta}} \approx s\left(\log \frac{2}{s}\right)^{\frac{m(\beta-1)}{\beta}}, \quad s \in (0,1),$$

the equivalence of (4.35) and (4.36) follows.

Futhermore, due to Theorem 4.10 and [3, Theorem 7.8], (4.37) is satisfied if and only if

$$(L^\infty)^G_{m,\beta}(0,1) = L^{\infty,\infty;-\frac{m}{\beta}}(0,1) \overset{*}{\hookrightarrow} X(0,1).$$

Observe that a function f belongs to the unit ball of the space $L^{\infty,\infty,-\frac{m}{\beta}}(0,1)$ if and only if

$$f^*(s) \le \left(\log \frac{e}{s}\right)^{\frac{m}{\beta}}, \quad s \in (0,1).$$

Thus,

$$\begin{aligned}\lim_{a\to 0_+} \sup_{\|f\|_{L^{\infty,\infty,-m/\beta}(0,1)}\le 1} \|\chi_{(0,a)}(s) f^*(s)\|_{X(0,1)} &= \lim_{a\to 0_+} \left\|\chi_{(0,a)}(s)\left(\log\frac{e}{s}\right)^{\frac{m}{\beta}}\right\|_{X(0,1)} \\ &\approx \lim_{a\to 0_+} \left\|\chi_{(0,a)}(s)\left(\log\frac{2}{s}\right)^{\frac{m}{\beta}}\right\|_{X(0,1)},\end{aligned}$$

which proves the equivalence of (4.37) and (4.38). □

We finish with the case when both $X(0,1)$ and $Y(0,1)$ are Lebesgue spaces, and then, more generally, when both $X(0,1)$ and $Y(0,1)$ are Lorentz-Zygmund spaces.

Theorem 4.12. (i) *Let n, $m \in \mathbb{N}$, $\beta \in (1,2]$ and $p \in [1,\infty)$. Then*

$$V^m L^p(\mathbb{R}^n, \gamma_{n,\beta}) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^n, \gamma_{n,\beta}).$$

Moreover, $L^p(\mathbb{R}^n, \gamma_{n,\beta})$ is the optimal (i.e., the smallest) Lebesgue space into which $V^m L^p(\mathbb{R}^n, \gamma_{n,\beta})$ is compactly embedded.

(ii) *Let $\beta \in (1,2]$ and $p = \infty$, or $\beta = 1$ and $p \in [1,\infty]$. Suppose that $q \in [1,\infty]$. Then*

$$V^m L^p(\mathbb{R}^n, \gamma_{n,\beta}) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^n, \gamma_{n,\beta})$$

if and only if $q < p$.

Theorem 4.13. *Let n, $m \in \mathbb{N}$ and $\beta \in [1,2]$. Furthermore, let p_1, p_2, q_1, $q_2 \in [1,\infty]$, α_1, $\alpha_2 \in \mathbb{R}$ be such that both $L^{p_1,q_1;\alpha_1}(0,1)$ and $L^{p_2,q_2;\alpha_2}(0,1)$ are rearrangement-invariant spaces (up to equivalent norms).*

(i) *Suppose that $p_1 < \infty$. Then*

$$V^m L^{p_1,q_1;\alpha_1}(\mathbb{R}^n, \gamma_{n,\beta}) \hookrightarrow\hookrightarrow L^{p_2,q_2;\alpha_2}(\mathbb{R}^n, \gamma_{n,\beta}) \tag{4.39}$$

holds if and only if $p_1 > p_2$, or $p_1 = p_2$ and one of the following conditions is satisfied:

$$\begin{aligned} &q_1 \le q_2, \quad \alpha_1 + \frac{m(\beta-1)}{\beta} > \alpha_2; \\ &q_2 < q_1, \quad \alpha_1 + \frac{1}{q_1} + \frac{m(\beta-1)}{\beta} > \alpha_2 + \frac{1}{q_2}. \end{aligned}$$

(ii) *Suppose that $p_1 = \infty$. Then (4.39) holds if and only if $p_2 < \infty$, or*

$$p_2 = \infty, \quad \alpha_1 + \frac{1}{q_1} - \frac{m}{\beta} > \alpha_2 + \frac{1}{q_2}.$$

We shall first prove Theorem 4.13, the proof of Theorem 4.12 being an easy consequence of this result.

Proof of Theorem 4.13. Due to Theorem 4.10, condition (4.39) is equivalent to

$$(L^{p_1,q_1;\alpha_1})^G_{m,\beta}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2;\alpha_2}(0,1). \tag{4.40}$$

Furthermore, it follows from [3, Theorem 7.8] that

$$(L^{p_1,q_1;\alpha_1})^G_{m,\beta}(0,1) = \begin{cases} L^{p_1,q_1;\alpha_1+\frac{m(\beta-1)}{\beta}}(0,1) & \text{if } p_1 < \infty; \\ L^{\infty,q_1;\alpha_1-\frac{m}{\beta}}(0,1) & \text{if } p_1 = \infty. \end{cases}$$

By applying Proposition 4.2 we get the result. □

Proof of Theorem 4.12. Suppose that $p \in [1,\infty)$ and $q \in [1,\infty]$. Using Theorem 4.13 with $p_1 = q_1 = p$, $p_2 = q_2 = q$ and $\alpha_1 = \alpha_2 = 0$ we get that

$$V^m L^p(\mathbb{R}^n, \gamma_{n,\beta}) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^n, \gamma_{n,\beta}) \tag{4.41}$$

is satisfied if and only if $p > q$, or $p = q$ and $m(\beta - 1)/\beta > 0$. The last inequality is true provided that $\beta > 1$, so, in this case $L^p(\mathbb{R}^n, \gamma_{n,\beta})$ is indeed the smallest Lebesgue space into which $V^m L^p(\mathbb{R}^n, \gamma_{n,\beta})$ is compactly embedded. On the other hand, if $\beta = 1$ then (4.41) is fulfilled if and only if $p > q$.

Finally, suppose that $p = \infty$. Then, due to Theorem 4.13 once again, (4.41) is satisfied if and only if $q < \infty$, or $q = \infty$ and $-m/\beta > 0$. However, the last condition is never fulfilled, so (4.41) holds with $p = \infty$ if and only if $q < \infty$. This completes the proof. □

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