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DOCTORAL THESIS

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DESCRIPTIVE AND TOPOLOGICAL ASPECTS IN BANACH SPACE THEORY

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TITLE:
Descriptive and topological aspects in Banach space theory

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ABSTRACT:
The thesis consists of three papers of the author. In the first paper, it is shown that the sets of Fréchet subdifferentiability of Lipschitz functions on a Banach space X are Borel if and only if X is reflexive. This answers a question of L. Zajíček. In the second paper, a problem of G. Debs, G. Godefroy and J. Saint Raymond is solved. On every separable non-reflexive Banach space, equivalent strictly convex norms with the set of norm-attaining functionals of arbitrarily high Borel class are constructed. In the last paper, binormality, a separation property of the norm and weak topologies of a Banach space, is studied. A result of P. Holický is generalized. It is shown that every Banach space which belongs to a $\mathcal{P}$-class is binormal. It is also shown that the asplundness of a Banach space is equivalent to a related separation property of its dual space.

KEYWORDS:
Banach space, non-reflexive Banach space, Borel set, binormality, projectional resolution of identity

NÁZEV PRÁCE:
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ABSTRAKT:
Práce je složena ze tří autorových článků. V prvním článku je ukázáno, že množiny fréchetovské subdiferencovatelnosti lipschitzovských funkcí na Banachově prostoru X jsou borelovské právě tehdy, když X je reflexivní. Tím je zodpovězena otázka L. Zajíčka. V druhém článku je vyřešen problém, který položili G. Debs, G. Godefroy a J. Saint Raymond. Na každém separabilním nereflexivním Banachově prostoru jsou zkonstruovány ekvivalentní striktně konvexní normy s množinami normy nabývajících funkcionálů libovolně vysoké borelovské třídy. V posledním článku je studována binormalita, jistá oddělovací vlastnost normové a slabé topologie na Banachově prostoru. Je zobecněn výsledek P. Holického. Je ukázáno, že každý Banachův prostor patřící do nějaké $\mathcal{P}$-třídy je binormální. Je rovněž ukázáno, že asplundovost Banachova prostoru je ekvivalentní příbuzné oddělovací vlastnosti jeho duálního prostoru.

KLÍČOVÁ SLOVA:
Banachův prostor, nereflexivní Banachův prostor, borelovská množina, binormalita, projektivní rozklad identity

CONTENTS

PREFACE

The thesis consists of three author's papers.

- *Reflexivity and sets of Fréchet subdifferentiability*, Proc. Am. Math. Soc. **136**, No. 12 (2008), 4467–4473.
- *Structure of the set of norm-attaining functionals on strictly convex spaces*, accepted in Can. Math. Bull.
- *On binormality in non-separable Banach spaces*, J. Math. Anal. Appl. **371**, No. 2 (2010), 425–435.

The first two papers deal with descriptive set theory and its application in Banach space theory, while the third one deals with a relation of topologies on a Banach space.

We are interested in the descriptive complexity of sets. In the first paper, we study complexity of sets of (Fréchet) subdifferentiability. L. Zajíček [19] proved that the set $S(f)$ of subdifferentiability of a continuous function f on a Banach space X is a Suslin set. He posed the question if $S(f)$ is necessarily a Borel set. P. Holický and M. Laczkovich answered the question positively for X reflexive (p. 3, Theorem 1.1.2). Nevertheless, the answer in the general case is negative. In our paper, we construct a Lipschitz function with non-Borel set of subdifferentiability on every non-reflexive Banach space (p. 3, Theorem 1.1.3). Let us note that, in this moment, the question of possible complexity of sets of subdifferentiability of continuous functions is solved for every Banach space except the spaces of dimension 2.

Non-reflexivity plays a key role also in the second paper. Let X be a separable non-reflexive Banach space. It is not difficult to show that, if its norm $\|\cdot\|$ is strictly convex, then the set of norm-attaining functionals $\mathrm{NA}(\|\cdot\|)$ is Borel [12]. G. Debs, G. Godefroy and J. Saint Raymond proved that some better convexity assumptions provide sharper conclusions [1]. For example, if the dual norm $\|\cdot\|^*$ is Gâteaux differentiable, then $\mathrm{NA}(\|\cdot\|)$ is $F_{\sigma\delta}$. They asked whether only the assumption that $\|\cdot\|$ is strictly convex is sufficient for $\mathrm{NA}(\|\cdot\|)$ to belong to a fixed Borel class. We answer this question negatively (p. 11, Theorem 2.1.1).

Another object of our interest is binormality in Banach spaces. Let σ and τ be two topologies on a set X. We say that X is binormal with respect to σ and τ if, for every disjoint σ-closed A and τ-closed B, there are disjoint σ-open D and τ-open C with $A \subset C$ and $B \subset D$. We say that a Banach space X is binormal if X is binormal with respect to its norm and weak topologies.

It was shown by P. Holický [8] that every separable Banach space is binormal and that the space ℓ^∞ is not binormal. It was an open problem if there are some non-separable binormal spaces. In the third paper, we actually prove that there are many of them (p. 21, Theorem 3.1.1). Our method of proving that a Banach space is binormal is to decompose it into "smaller" ones through so-called projectional resolution of identity (defined on p. 29). The notion of a projectional resolution of identity is an important tool in the theory of non-separable Banach spaces, and our paper is an evidence for it. We are able to show that every Banach space which belongs to a $\mathcal{P}$-class is binormal (a $\mathcal{P}$-class is defined on p. 31). This provides numerous examples of binormal spaces (weakly compactly generated spaces, Plichko spaces, duals to Asplund spaces, $C([0,\mu])$ for an ordinal μ).

It is possible to study also the binormality of the norm and weak star topologies (which we call w^*-binormality). It was observed by O. Kalenda that it can be proved by an analogical decomposition method that the dual space of a weakly countably determined Asplund space is w^*-binormal (p. 36, Remark 3.6.5). We prove that a Banach space is necessarily Asplund if its dual is w^*-binormal but the converse does not hold. In fact, asplundness of a space is equivalent to a weaker form of w^*-binormality of its dual space (this weaker form is like w^*-binormality with the only difference that the norm-closed set A is assumed to be norm-separable, p. 22, Theorem 3.1.2).

We conclude with a characterization of scattered compact spaces (p. 37, Theorem 3.6.8). We do not know whether this characterization can be proved directly without using the methods presented in this work (namely, p. 35, Lemma 3.6.2 and Theorem 3.6.3).

I would like here to express my thanks to all the people who accompanied me throughout my mathematical education and researches. I am grateful to my supervisor Professor Petr Holický for numerous discussions on the problems, helpful suggestions, useful remarks on preliminary versions of my papers and also for abiding interest in my work.

1

REFLEXIVITY AND SETS OF FRÉCHET SUBDIFFERENTIABILITY

1.1 INTRODUCTION AND MAIN RESULT

Let X be a real normed linear space and f be a real function on X. Let $x \in X$. We say that $u \in X^*$ is a *Fréchet subgradient of f at x* if

$$\liminf_{y \to x} \frac{f(y) - f(x) - u(y-x)}{\|y - x\|} \geq 0.$$

The set of all Fréchet subgradients of f at x is called the *Fréchet subdifferential of f at x* and denoted by $\partial f(x)$. The set of all points $x \in X$ at which $\partial f(x) \neq \emptyset$ is called the *set of Fréchet subdifferentiability* and denoted by $S(f)$.

Further on, we omit "Fréchet" in the above notions and we suppose that all normed linear spaces are real.

At first, we recall some known results about the sets of subdifferentiability.

THEOREM 1.1.1. ([19, Section 4]) *Let f be a lower semicontinuous function on a normed linear space X. Then $S(f)$ is a Suslin set.*

We recall the definition of a Suslin set in Section 2.

L. Zajíček posed in [19, Section 4] the question whether $S(f)$ must be Borel for every lower semicontinuous function. We show in Theorem 1.1.3 below that the answer to Zajíček's question is negative in non-reflexive spaces. The situation in the reflexive case was clarified by an unpublished remark of P. Holický and M. Laczkovich. A proof of their result will be given at the end of this section.

THEOREM 1.1.2. (Holický, Laczkovich) *Let f be a lower semicontinuous function on a normed linear space X with a reflexive completion. Then $S(f)$ is an $F_{\sigma\delta\sigma}$ set.*

We note that there is a continuous function f on $\mathbb{R}^3$ such that $S(f)$ is not $G_{\delta\sigma\delta}$ (see [15]). We formulate the main result now. Its proof will be given in Section 1.2.

THEOREM 1.1.3. *Let X be a normed linear space with a non-reflexive completion. Then there is a Lipschitz function f on X such that $S(f)$ is not Borel.*

Remark 1.1.4. Theorem 1.1.1 can be generalized. M. Šmídek has proved that Theorem 1.1.1 holds for Borel functions (see [18]). It follows from his method and Theorem 1.1.2 that $S(f)$ is Borel if f is a Borel function on a space with a reflexive completion.

Proof of Theorem 1.1.2. By [19, Lemma 4], the set

$$A^K_{n_1,\dots,n_k} = \bigcup_{\|u\|\le K} \bigcap_{i=1}^{k} \Big\{ x \in X : \|y-x\| < \frac{1}{n_i} \Rightarrow (y) - f(x) \ge u(y-x) - \frac{1}{i}\|y-x\| \Big\}$$

is closed for $K, k, n_1, \dots, n_k \in \mathbb{N}$. It is enough to verify that

$$S(f) = \bigcup_{K=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{(n_1,\dots,n_k)\in\mathbb{N}^k} A^K_{n_1,\dots,n_k}.$$

Let $x \in S(f)$. There exists $u \in \partial f(x)$. For some $K \in \mathbb{N}$, $K \ge \|u\|$. By the definition of the subgradient, for every $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that $\|y-x\| < \frac{1}{n_i} \Rightarrow f(y) - f(x) \ge u(y-x) - \frac{1}{i}\|y-x\|$. Now, $x \in A^K_{n_1,\dots,n_k}$ for every $k \in \mathbb{N}$, which gives the inclusion "$\subset$". To prove the other inclusion, suppose that $K \in \mathbb{N}$ and $x \in \bigcap_{k=1}^{\infty} \bigcup_{(n_1,\dots,n_k)\in\mathbb{N}^k} A^K_{n_1,\dots,n_k}$. For every $k \in \mathbb{N}$, there exist $n_k \in \mathbb{N}$ and $u \in X^*, \|u\| \le K$, such that $\|y-x\| < \frac{1}{n_k} \Rightarrow f(y) - f(x) \ge u(y-x) - \frac{1}{k}\|y-x\|$. Consequently, for every $k \in \mathbb{N}$, the set

$$C_k = \Big\{ u \in X^* : \|u\| \le K, \liminf_{y\to x} \frac{f(y) - f(x) - u(y-x)}{\|y-x\|} \ge -\frac{1}{k} \Big\}$$

is non-empty. One can easily check that these sets are closed and convex. So they are w-closed, too. They are bounded at the same time. Since X^* is reflexive, $\{C_k\}_{k\in\mathbb{N}}$ is a decreasing system of non-empty w^*-compact sets. So its intersection is non-empty. The easy observation that $\bigcap_{k=1}^{\infty} C_k \subset \partial f(x)$ completes the proof. □

1.2 FUNCTIONS WITH NON-BOREL SETS OF FRÉCHET SUBDIFFERENTIABILITY

Let us recall some definitions and notation. By $\mathbb{N}^{<\omega}$ we will denote the set of all finite sequences of natural numbers, i.e., $\mathbb{N}^{<\omega} = \{\emptyset\} \cup \bigcup_{l=1}^{\infty} \mathbb{N}^l$. The closed unit ball of a Banach space X will be denoted by B_X. We use "co" for the convex hull, "$\overline{\text{co}}$" for its closure and "$\overline{\text{sp}}$" for the closed linear span. Given normed linear spaces X, Y, we define $X \oplus_\infty Y$ as the sum of X and Y with the norm $\|(x,y)\| = \max\{\|x\|, \|y\|\}, x \in X, y \in Y$. By c-Lipschitz we mean Lipschitz with constant c.

Let X be a metric space. We say that $M \subset X$ is *Suslin* if

$$M = \bigcup_{(n_1,n_2,\dots)\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} A_{n_1,\dots,n_k}, \tag{1.1}$$

where $A_{n_1,\dots,n_k}$, $(n_1,\dots,n_k) \in \mathbb{N}^{<\omega}$, are closed in X. Equivalently, we may consider $A_{n_1,\dots,n_k}$, $(n_1,\dots,n_k) \in \mathbb{N}^{<\omega}$, to be open (we have $\bigcap_{k=1}^{\infty} A_{n_1,\dots,n_k} = \bigcap_{k=1}^{\infty}\{x \in X : \mathrm{dist}(x, \bigcap_{i=1}^{k} A_{n_1,\dots,n_i}) < 1/k\}$). Let P be a countably infinite set. We note that $\{0,1\}^P$ can be identified with the set of subsets of P by $\nu \in \{0,1\}^P \mapsto \{p \in P : \nu(p) = 1\}$. We consider the subspace Tr of $\{0,1\}^{\mathbb{N}^{<\omega}}$ consisting of the *trees*, i.e., such subsets of $\mathbb{N}^{<\omega}$, which contain $\emptyset, (n_1), (n_1,n_2), \dots, (n_1,\dots,n_k)$ with every element $(n_1,\dots,n_k)$. We say that $T \in \mathrm{Tr}$ is *ill-founded* ($T \in \mathrm{IF}$), if there exists an infinite sequence of natural numbers $n_1, n_2, \dots$ such that $(n_1,\dots,n_k) \in T$ for every $k \in \mathbb{N}$. In the opposite case, we say that T is *well-founded* ($T \in \mathrm{WF}$).

The following lemma is an easy consequence of [9, Theorem 1].

Lemma 1.2.1. *If a Banach space X is not reflexive, then there exist $x_1, x_2, \dots$ in B_X and a bounded sequence $u_1, u_2, \dots$ in X^* such that, for every $k, j \in \mathbb{N}$,*

$$u_k(x_j) \ \geq\ 1 \quad \text{if } k \leq j, \qquad u_k(x_j) \ =\ 0 \quad \text{if } k > j.$$

Proposition 1.2.2. *Let X be a non-reflexive Banach space. Then there is a mapping $\theta : \mathbb{N}^{<\omega} \to B_X$ such that*

(i) *if $T \in \mathrm{IF}$, then there are distinct $\eta_1, \eta_2, \dots \in T$ such that the sequence $\theta(\eta_1), \theta(\eta_2), \dots$ is convergent, and so $\bigcap_{U\subset T, |U|<\infty} \overline{\mathrm{co}}(\theta(T \setminus U)) \neq \emptyset$,*

(ii) *if $T \in \mathrm{WF}$, then $\bigcap_{U\subset T, |U|<\infty} \overline{\mathrm{co}}(\theta(T \setminus U)) = \emptyset$.*

Proof. Let $x_1, x_2, \dots, u_1, u_2, \dots$ be as in Lemma 1.2.1. We define

$$\theta(n_1,\dots,n_k) \ =\ \sum_{i=1}^{k} 2^{-i} x_{n_i}$$

for $(n_1,\dots,n_k) \in \mathbb{N}^{<\omega}$. To prove (i), it is sufficient to realize that, for $n_1, n_2, \dots \in \mathbb{N}$, the sequence $\theta(\emptyset)$, $\theta(n_1)$, $\theta(n_1,n_2), \dots$ converges to $\sum_{i=1}^{\infty} 2^{-i} x_{n_i}$.

Assume that (ii) does not hold. Let $T \in \mathrm{WF}$ and let a be an element of $\bigcap_{U\subset T, |U|<\infty} \overline{\mathrm{co}}(\theta(T \setminus U))$. The sequence $u_1, u_2, \dots$ is bounded, so it is easy to check that $\{x \in X : \lim_{k\to\infty} u_k(x) = 0\}$ is closed. We have

$$a \in \overline{\mathrm{co}}\Big(\theta\big(\mathbb{N}^{<\omega}\big)\Big) \subset \overline{\mathrm{sp}}\{x_1, x_2, \dots\} \subset \Big\{x \in X : \lim_{k\to\infty} u_k(x) = 0\Big\},$$

and so $\lim_{k\to\infty} u_k(a) = 0$. We choose natural numbers $N_1, N_2, \dots$ such that

$$\sum_{i=1}^{l} 2^i u_{N_i}(a) \ <\ 1 \quad \text{for } l \in \mathbb{N}$$

(for example choose N_i such that $u_{N_i}(a) \leq 2^{-2i}$). The set

$$R = \Big\{(n_1, \dots, n_k) \in T \,:\, 1 \leq i \leq k \Rightarrow n_i \leq N_i\Big\}$$

(where k denotes the length of $(n_1, \dots, n_k)$) is finite by König's lemma (cf., [14, Exercise 4.12]). Thus there exists $l \in \mathbb{N}$ such that l is greater than the length of any element of R. We are going to prove the following implication:

$$(n_1, \dots, n_k) \in T \setminus R \quad \Rightarrow \quad \sum_{i=1}^{l} 2^i u_{N_i}\big(\theta(n_1, \dots, n_k)\big) \geq 1.$$

Let $(n_1, \dots, n_k) \in T \setminus R$. Let us realize that $n_j > N_j$ for some $j \leq \min\{k, l\}$. It is clear in the case that $k \leq l$. If $k > l$ and $n_j \leq N_j$ for every $j \leq l$, then the sequence $(n_1, \dots, n_l)$ of the first l members of $(n_1, \dots, n_k)$ would be an element of R, but its length would be l at the same time, which is impossible. We have:

$$\begin{aligned} \sum_{i=1}^{l} 2^i u_{N_i}\big(\theta(n_1, ..., n_k)\big) &\geq 2^j u_{N_j}\big(\theta(n_1, ..., n_k)\big) \\ &= \sum_{i=1}^{k} 2^{j-i} u_{N_j}(x_{n_i}) \geq u_{N_j}(x_{n_j}) \geq 1, \end{aligned}$$

and the implication holds. Now, as R is finite,

$$\begin{aligned} a &\in \bigcap_{U \subset T, |U| < \infty} \overline{\mathrm{co}}\Big(\theta\big(T \setminus U\big)\Big) \subset \overline{\mathrm{co}}\Big(\theta\big(T \setminus R\big)\Big) \\ &\subset \Big\{x \in X \,:\, \sum_{i=1}^{l} 2^i u_{N_i}(x) \geq 1\Big\}, \end{aligned}$$

which is a contradiction with the choice of $N_1, N_2, \dots$. □

LEMMA 1.2.3. *Let Y be an infinite-dimensional normed linear space and $(u_\gamma)_{\gamma \in \Gamma}$ be a system of elements of Y^*. Let $(\delta_\gamma)_{\gamma \in \Gamma}$ be a system of elements of $(0, \infty]$ such that $\{\gamma \in \Gamma : \delta_\gamma > \delta\}$ is finite for every $\delta > 0$ and $(\varepsilon_\gamma)_{\gamma \in \Gamma}$ be a system of positive numbers. If*

$$g(y) = \max\Big\{u_\gamma(y) - \varepsilon_\gamma \|y\| : \gamma \in \Gamma, \|y\| < \delta_\gamma\Big\}$$

$$\text{for } y \in Y,\ \|y\| < \max_{\gamma \in \Gamma} \delta_\gamma,$$

then

$$\partial g(0) \subset \bigcap_{U \subset \Gamma, |U| < \infty} \overline{\mathrm{co}}\{u_\gamma : \gamma \in \Gamma \setminus U\}.$$

In fact, if $\{\gamma \in \Gamma : \varepsilon_\gamma > \varepsilon\}, \varepsilon > 0$, are also finite, then the equality holds. We do not use the inclusion "$\supset$", but we prove an analogy of it elsewhere.

Proof. Suppose that $u \in Y^* \setminus \bigcap_{U \subset \Gamma, |U| < \infty} \overline{\text{co}}\{u_\gamma : \gamma \in \Gamma \setminus U\}$. We have to prove that $u \notin \partial g(0)$. For some finite $U \subset \Gamma$, u is not in $\overline{\text{co}}\{u_\gamma : \gamma \in \Gamma \setminus U\}$. By the Hahn-Banach theorem, there exist $F_0 \in Y^{**}$ and $\alpha > 0$ such that $F_0(u_\gamma - u) \geq \alpha$ for every $\gamma \in \Gamma \setminus U$. We can choose $\beta \in (0, 1/\|F_0\|]$ such that $-\beta F_0(u_\gamma - u) < \frac{1}{2}\varepsilon_\gamma$ for every $\gamma \in U$. We define $F = -\beta F_0, \varepsilon = \min\{\alpha\beta\} \cup \{\frac{1}{2}\varepsilon_\gamma : \gamma \in U\}$. We have $\|F\| \leq 1$ because $\|F\| = \beta\|F_0\| \leq (1/\|F_0\|)\|F_0\| = 1$. Let us verify that

$$F(u_\gamma - u) \;<\; \varepsilon_\gamma - \varepsilon, \quad \gamma \in \Gamma.$$

If $\gamma \in U$, then $F(u_\gamma - u) = -\beta F_0(u_\gamma - u) < \frac{1}{2}\varepsilon_\gamma = \varepsilon_\gamma - \frac{1}{2}\varepsilon_\gamma \leq \varepsilon_\gamma - \varepsilon$. If $\gamma \in \Gamma \setminus U$, then $F(u_\gamma - u) = -\beta F_0(u_\gamma - u) \leq -\beta\alpha \leq -\varepsilon < \varepsilon_\gamma - \varepsilon$.

Now, let $\delta > 0$ be given. Since $\{G \in Y^{**} : \delta < \delta_\gamma \Rightarrow G(u_\gamma - u) < \varepsilon_\gamma - \varepsilon\}$ is a neighbourhood of F in the w^*-topology on Y^{**}, by Goldstine's lemma, there exists $y_0 \in Y, \|y_0\| \leq 1$, such that $\delta < \delta_\gamma$ implies that $(u_\gamma - u)(y_0) < \varepsilon_\gamma - \varepsilon$. Since Y is infinite-dimensional, there exists $z \in Y, z \neq 0$, such that $\delta < \delta_\gamma$ implies that $(u_\gamma - u)(z) = 0$. For an appropriate $\lambda \in \mathbb{R}$, we have $\|y\| = \delta$, where $y = \delta y_0 + \lambda z$. Let $\gamma \in \Gamma$ be such that $\|y\| < \delta_\gamma$. It means that $\delta < \delta_\gamma$. We have $(u_\gamma - u)(y) = (u_\gamma - u)(\delta y_0 + \lambda z) < \delta(\varepsilon_\gamma - \varepsilon) = \|y\|(\varepsilon_\gamma - \varepsilon)$. Thus, $\|y\| < \delta_\gamma$ implies that $\frac{1}{\|y\|}(u_\gamma(y) - \varepsilon_\gamma\|y\| - u(y)) < -\varepsilon$. In other words, $\frac{1}{\|y\|}(g(y) - u(y)) < -\varepsilon$.

For arbitrary $\delta > 0$, we have found $y \in Y, \|y\| \leq \delta$, such that $\frac{1}{\|y\|}(g(y) - u(y)) < -\varepsilon$. So u is not a subgradient of g at 0, and the proof is finished. □

THEOREM 1.2.4. *Let X, Y be normed linear spaces such that the completion of Y is not reflexive. If $M \subset X$ is a Suslin set, then there exists a Lipschitz function f on $X \oplus_\infty Y$ such that, for every $a \in X$, $(a, 0) \in S(f)$ if and only if $a \in M$.*

Proof. Let $A_{n_1,\dots,n_k}, (n_1, \dots, n_k) \in \mathbb{N}^{<\omega}$, be a system of open subsets of X satisfying (1.1). We may suppose that, for $(n_1, \dots, n_k) \in \mathbb{N}^{<\omega}$ and $n_{k+1} \in \mathbb{N}$, $A_{n_1,\dots,n_k,n_{k+1}} \subset A_{n_1,\dots,n_k}$, i.e., that

$$T_a \;=\; \{\eta \in \mathbb{N}^{<\omega} \;:\; a \in A_\eta\}$$

is a tree for every $a \in X$ (we can take $\bigcap_{i=1}^{k} A_{n_1,\dots,n_i}$ instead of $A_{n_1,\dots,n_k}$). We observe that $T_a \in \mathbb{IF}$ if and only if $a \in M$.

Now, we are going to use the non-reflexivity of Y^*. Let $\theta : \mathbb{N}^{<\omega} \to B_{Y^*}$ be as in Proposition 1.2.2. It follows from (i), (ii) and from the observation that

$$a \in M \quad \Leftrightarrow \quad \bigcap_{U \subset T_a, |U| < \infty} \overline{\text{co}}\big(\theta(T_a \setminus U)\big) \;\neq\; \emptyset$$

for every $a \in X$. We choose two systems $(\delta_\eta)_{\eta \in \mathbb{N}^{<\omega}}$ and $(\varepsilon_\eta)_{\eta \in \mathbb{N}^{<\omega}}$ of elements of $(0,1)$ such that $\{\eta \in \mathbb{N}^{<\omega} : \delta_\eta > c\}$, $\{\eta \in \mathbb{N}^{<\omega} : \varepsilon_\eta > c\}$ are finite for every $c > 0$. For every $\eta \in \mathbb{N}^{<\omega}$, we define

$$\begin{aligned} D_\eta &= \big\{(x,y) \in X \times Y : x \in A_\eta, \\ &\qquad\qquad \|y\| < \operatorname{dist}(x, X \setminus A_\eta) \text{ and } \|y\| < \delta_\eta/2\big\}, \\ E_\eta &= \big\{(x,y) \in X \times Y : x \notin A_\eta \text{ or } \|y\| \geq \delta_\eta\big\}, \\ f_\eta(x,y) &= \begin{cases} \theta(\eta)(y) - \varepsilon_\eta\|y\| & (x,y) \in D_\eta \\ -2\|y\| & (x,y) \in E_\eta. \end{cases} \end{aligned}$$

We are going to prove that f_η is 6-Lipschitz on $D_\eta \cup E_\eta$. Obviously, f_η is 6-Lipschitz (in fact, 2-Lipschitz) on D_η and on E_η. Let $(x_1, y_1) \in D_\eta$ and $(x_2, y_2) \in E_\eta$. Since $|f_\eta(x_1,y_1) - f_\eta(x_2,y_2)| = |\theta(\eta)(y_1) - \varepsilon_\eta\|y_1\| + 2\|y_2\|| \leq 2\|y_1\| + 2\|y_2\|$, it remains to verify that $2\|y_1\| + 2\|y_2\| \leq 6\|(x_1,y_1) - (x_2,y_2)\|$. If $x_2 \in A_\eta$, then $\|y_2\| \geq \delta_\eta$, and thus $2\|y_1\| + 2\|y_2\| \leq -6\|y_1\| + 4\delta_\eta + 6\|y_2\| - 4\delta_\eta \leq 6\|(x_1,y_1) - (x_2,y_2)\|$. If $x_2 \notin A_\eta$, then $\|y_1\| < \operatorname{dist}(x_1, X \setminus A_\eta) \leq \|x_1 - x_2\|$, and thus $2\|y_1\| + 2\|y_2\| \leq 4\|x_1 - x_2\| + 2\|y_2\| - 2\|y_1\| \leq 6\|(x_1,y_1) - (x_2,y_2)\|$.

We recall that the supremum of a non-empty system of c-Lipschitz functions is a c-Lipschitz function unless it is identically equal to $+\infty$.

Now, f_η can be extended from $D_\eta \cup E_\eta$ to $X \times Y$ to be 6-Lipschitz and to satisfy

$$f_\eta(x,y) \leq \theta(\eta)(y) - \varepsilon_\eta\|y\|, \quad (x,y) \in X \times Y$$

(a 6-Lipschitz extension of f_η exists by the McShane-Whitney extension theorem ([17]), then we can take the minimum of this extension and the function $(x,y) \mapsto \theta(\eta)(y) - \varepsilon_\eta\|y\|$). We put

$$f = \sup\{f_\eta : \eta \in \mathbb{N}^{<\omega}\}.$$

Obviously, f is 6-Lipschitz. It remains to prove that, for every $a \in X$,

$$\bigcap_{U \subset T_a, |U| < \infty} \overline{\mathrm{co}}\big(\theta(T_a \setminus U)\big) \neq \emptyset \quad \Leftrightarrow \quad (a, 0) \in S(f).$$

Let us prove the implication "$\Leftarrow$". Assume that $a \in X$ and that $\bigcap_{U \subset T_a, |U| < \infty} \overline{\mathrm{co}}(\theta(T_a \setminus U)) = \emptyset$. We consider the function g on Y defined by

$$g(y) = \max\Big\{\theta(\eta)(y) - \varepsilon_\eta\|y\| : \eta \in T_a, \|y\| < \delta_\eta\Big\} \cup \Big\{-2\|y\|\Big\}.$$

By Lemma 1.2.3 (applied on $\Gamma = T_a \cup \{1\}, \delta_1 = \infty, \varepsilon_1 = 2, u_1 = 0, u_\eta = \theta(\eta)$ for $\eta \in T_a$), $\partial g(0) \subset \bigcap_{U \subset T_a, |U| < \infty} \overline{\mathrm{co}}(\theta(T_a \setminus U))$. So $\partial g(0) = \emptyset$. Let us verify that $f_\eta(a, \cdot) \leq g$ for every $\eta \in \mathbb{N}^{<\omega}$, and thus $f(a,\cdot) \le g$. If $\eta \notin T_a$, i.e. $a \notin A_\eta$, then $f_\eta(a,\cdot) = -2\|\cdot\| \le g$. If $\eta \in T_a$ and $\|y\| \ge \delta_\eta$, then $(a,y) \in E_\eta$, and thus $f_\eta(a,y) = -2\|y\| \le g(y)$ again. If $\eta \in T_a$ and $\|y\| < \delta_\eta$, then $f_\eta(a,y) \le \theta(\eta)(y) - \varepsilon_\eta\|y\| \le g(y)$. Now, the inequality $f(a,\cdot) \le g$ is verified. Since $f(a,0) = g(0) = 0$, we get $\partial(f(a,\cdot))(0) \subset \partial g(0) = \emptyset$. Hence, $\partial f(a,0) = \emptyset$, which proves the implication.

Let us prove the other implication. Assume that $a \in X$ and that $u \in \bigcap_{U \subset T_a, |U| < \infty} \overline{\mathrm{co}}(\theta(T_a \setminus U))$. Let $\varepsilon > 0$. Since $u \in \overline{\mathrm{co}}\{\theta(\eta) : \eta \in T_a, \varepsilon_\eta \le \varepsilon/2\}$, there is a finite subset V of T_a such that $\varepsilon_\eta \le \varepsilon/2$ for $\eta \in V$ and $\|u - v\| \le \varepsilon/2$ for some $v \in \mathrm{co}(\theta(V))$. We have $f(x,y) \ge f_\eta(x,y) = \theta(\eta)(y) - \varepsilon_\eta\|y\| \ge \theta(\eta)(y) - (\varepsilon/2)\|y\|$ for $\eta \in V$ and $(x,y) \in D_\eta$. So $f(x,y) \ge v(y) - (\varepsilon/2)\|y\|$ for $(x,y) \in \bigcap_{\eta \in V} D_\eta$. As $V \subset T_a$, we have $(a,0) \in \bigcap_{\eta \in V} D_\eta$. Consequently, $f(x,y) \ge u(y) - \varepsilon\|y\|$ on some neighbourhood of $(a,0)$ (D_η are open because A_η are open). Since $\varepsilon > 0$ was arbitrary, $(x,y) \mapsto u(y)$ is a subgradient of f at $(a,0)$, and the implication "$\Rightarrow$" is proved. □

Proof of Theorem 1.1.3. Let the completion of a normed linear space X is not reflexive. Then X is isomorphic to $\mathbb{R} \oplus_\infty Y$, where Y is a subspace of X of codimension 1. The completion of Y is not reflexive, too. A well-known fact says that there is $M \subset \mathbb{R}$, which is Suslin, but not Borel. By Theorem 1.2.4, there is a Lipschitz function f on $\mathbb{R} \oplus_\infty Y$ such that, for every $a \in \mathbb{R}$, $(a,0) \in S(f)$ if and only if $a \in M$. Since M is not Borel, $S(f)$ is not Borel, too. □

1.3 A BY-PRODUCT

As a consequence of Proposition 1.2.2, the non-Borelness of some natural sets of sequences in a non-reflexive space can be shown.

LEMMA 1.3.1. *Let X be a non-reflexive Banach space. Then there is a continuous mapping $\Theta : \mathrm{Tr} \to (B_X)^{\mathbb{N}}$ such that*

(i*) *if $T \in \mathrm{IF}$, then $\Theta(T)$ has a convergent subsequence,*

(ii*) *if $T \in \mathrm{WF}$, then $\bigcap_{n=1}^\infty \overline{\mathrm{co}}\{x_k : k \ge n\} = \emptyset$.*

Proof. Firstly, let T' be a fixed infinite well-founded tree. The mapping $T \mapsto T \cup T'$ is continuous and the image of each ill-founded (well-founded) tree is an infinite ill-founded (well-founded) tree. Secondly, let $\mathbb{N}^{<\omega}$ be ordered to a sequence. The mapping $f : \{T \in \mathrm{Tr} : |T| = \infty\} \to (\mathbb{N}^{<\omega})^{\mathbb{N}}$ induced by the restriction of this ordering to each infinite tree is continuous and the image of an infinite tree is a sequence of its elements. Let θ be as in Proposition 1.2.2. We define

$$\Theta(T) = \Big(\theta\big(f(T \cup T')(n)\big)\Big)_{n \in \mathbb{N}}, \quad T \in \mathrm{Tr}.$$

Now, Θ is continuous and the conditions (i*), (ii*) follow from (i), (ii). □

Proposition 1.3.2. *Let X be a non-reflexive Banach space. Then the following sets are not Borel in $(B_X)^{\mathbb{N}}$:*

$$\begin{aligned} A &= \{(x_1, x_2, \dots) : x_1, x_2, \dots \text{ has a convergent subsequence}\}, \\ B &= \{(x_1, x_2, \dots) : x_1, x_2, \dots \text{ has a } w\text{-convergent subsequence}\}, \\ C &= \{(x_1, x_2, \dots) : x_1, x_2, \dots \text{ has a } w\text{-cluster point}\}, \\ D &= \{(x_1, x_2, \dots) : \textstyle\bigcap_{n=1}^{\infty} \overline{\mathrm{co}}\{x_k : k \geq n\} \neq \emptyset\}. \end{aligned}$$

Proof. Taking Θ as in Lemma 1.3.1, we have $\Theta(\mathrm{IF}) \subset A \subset B \subset C \subset D \subset (B_X)^{\mathbb{N}} \setminus \Theta(\mathrm{WF})$. Thus, $\mathrm{IF} = \Theta^{-1}(A) = \Theta^{-1}(B) = \Theta^{-1}(C) = \Theta^{-1}(D)$, and the well-known fact that IF is not Borel in Tr (see, e.g., [14]) completes the proof. □

2 STRUCTURE OF THE SET OF NORM-ATTAINING FUNCTIONALS ON STRICTLY CONVEX SPACES

2.1 INTRODUCTION AND MAIN RESULT

R. Kaufman proved in [12] that every non-reflexive Banach space admits an equivalent norm such that the set of norm-attaining functionals is not Borel. He also observed that the set of norm-attaining functionals is Borel in the case that the space is separable and strictly convex. G. Debs, G. Godefroy and J. Saint Raymond asked in [1] whether there exist strictly convex norms with the set of norm-attaining functionals of arbitrarily high Borel class. We answer this question affirmatively in Theorem 2.1.1.

Let $(X, \|\cdot\|)$ be a real normed linear space. We denote by B_X and by S_X the closed unit ball and the unit sphere of X and we recall that the set of norm-attaining functionals with respect to the norm $\|\cdot\|$ is

$$\mathrm{NA}(\|\cdot\|) = \{f \in X^* : \exists x \in B_X \, (f(x) = \|f\|)\}.$$

The main result follows. Its proof is given at the end of the chapter.

THEOREM 2.1.1. *Let X be a separable non-reflexive Banach space and $\alpha < \omega_1$. Then there exists an equivalent strictly convex norm $|||\cdot|||$ on X such that $\mathrm{NA}(|||\cdot|||)$ is not of the additive Borel class α.*

Of course, it is not essential whether we consider additive or multiplicative class.

2.2 KAUFMAN'S METHOD

One of the ingredients of our construction of the new unit ball is the following result of R. Kaufman. By the Baire space we mean the countable topological product $\mathbb{N}^{\mathbb{N}}$ of natural numbers endowed with the discrete topology.

PROPOSITION 2.2.1 ([12, 13]). *Let Y be a closed linear subspace of a Banach space X. If Y is not reflexive, then there exists a continuous mapping $\psi : \mathbb{N}^{\mathbb{N}} \to B_Y$ such that*

(i) *if $(\lambda_m)_{m\in\mathbb{N}}$ is a sequence of probability measures on $\mathbb{N}^{\mathbb{N}}$ such that the integrals $\int_{\mathbb{N}^{\mathbb{N}}} \psi d\lambda_m$, $m \in \mathbb{N}$, belong to a compact subset of Y, then the sequence $(\lambda_m)_{m\in\mathbb{N}}$ is uniformly tight, i.e., for every $\varepsilon > 0$, there is a compact set $K \subset \mathbb{N}^{\mathbb{N}}$ such that $\lambda_m(K) > 1 - \varepsilon$ for all m,*

(ii) *if $F \subset \mathbb{N}^{\mathbb{N}}$ is closed, $\varrho : F \to X$ is a continuous mapping with $\varrho(F)$ relatively compact and θ denotes $\psi|_F + \varrho$, then, for every $x \in \overline{\text{co}}\,\theta(F)$, there is a probability measure λ_x on F such that*

$$x = \int_F \theta d\lambda_x.$$

In fact, (ii) is a consequence of (i). Since the mappings are continuous and $\mathbb{N}^{\mathbb{N}}$ is separable, it is not essential whether the integrals are understood in the Pettis or in the Bochner sense. We do not distinguish the Baire space and the Polish space of all infinite sets of natural numbers (denoted by J in [12] and by Σ in [13]) because they are homeomorphic (the topology on the space of all infinite sets of natural numbers is induced by the topology on $2^{\mathbb{N}}$).

The proof of the following proposition is given in the form of a series of claims. There are some connections between it and the main result from [13] (more details are discussed in Remark 2.2.7).

By an analytic set we mean a continuous image of a Polish space F (i.e., separable completely metrizable topological space). By [14, Theorem 7.9], we can consider F to be a closed subset of $\mathbb{N}^{\mathbb{N}}$.

Proposition 2.2.2. *Let X be a non-reflexive Banach space and $\varphi, \phi \in X^*$ be linearly independent. Let $M \subset [0, \pi/2]$ be analytic and dense in $[0, \pi/2]$. Then there is an absolutely convex closed bounded set $R \subset X$ such that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi$ has the supremum 1 on R, and it is attained if and only if $t \in M$.*

Since M is analytic, there are a closed subset F of $\mathbb{N}^{\mathbb{N}}$ and a continuous mapping $p : F \to [0, \pi/2]$ such that $p(F) = M$.

Notation 2.2.3. We denote

$$Y = \operatorname{Ker}\varphi \cap \operatorname{Ker}\phi.$$

The space X can be viewed as

$$X = Y \oplus \mathbb{R}^2,$$

where

$$\varphi(0;1,0) = 1, \quad \varphi(0;0,1) = 0,$$
$$\phi(0;1,0) = 0, \quad \phi(0;0,1) = 1$$

(for $y \in Y, r, s \in \mathbb{R}$, we use $(y;r,s)$ instead of $(y,(r,s))$). We put

$$u_t = (\cos t)\varphi + (\sin t)\phi \quad \text{for } t \in [0, 2\pi).$$

Since X is not reflexive, Y is not reflexive, too. Let $\psi : \mathbb{N}^{\mathbb{N}} \to B_Y$ be as in Proposition 2.2.1. We define

$$\theta(\eta) = \big(\psi(\eta); \cos p(\eta), \sin p(\eta)\big) \quad \text{for } \eta \in F,$$

$$P = \theta(F), \quad R = \overline{\text{co}}\,\big(P \cup (-P)\big).$$

Further on, we consider the Euclidean norm on $\mathbb{R}^n (n = 2, 3)$ and we denote it by $|\cdot|$.

CLAIM 2.2.4. *Let R' be such that $P \subset R' \subset Y \times B_{\mathbb{R}^2}$. If $t \in [0, \pi/2]$, then u_t has the supremum 1 on R', and it is attained if $t \in M$.*

Proof. For $x = (y; r\cos\alpha, r\sin\alpha) \in Y \times B_{\mathbb{R}^2}$, we have $u_t(x) = r(\cos\alpha\cos t + \sin\alpha\sin t) = r\cos(\alpha - t) \le 1$. Since $R' \subset Y \times B_{\mathbb{R}^2}$, the inequality $\sup u_t(R') \le 1$ holds. On the other hand, for $\eta \in F$, $\theta(\eta) \in P \subset R'$ and $u_t(\theta(\eta)) = u_t(\psi(\eta); \cos p(\eta), \sin p(\eta)) = \cos p(\eta)\cos t + \sin p(\eta)\sin t = \cos(p(\eta) - t)$. The opposite inequality $\sup u_t(R') \ge 1$ follows from the fact that $M = p(F)$ is dense in $[0, \pi/2]$.

Now, let $t \in M = p(F)$. For $\eta \in p^{-1}(t)$, we have $\theta(\eta) \in P \subset R'$ and $u_t(\theta(\eta)) = u_t(\psi(\eta); \cos p(\eta), \sin p(\eta)) = \cos^2 t + \sin^2 t = 1 = \sup u_t(R')$. □

CLAIM 2.2.5. *Let $t \in [0, 2\pi)$.*

(a) *If $x \in \overline{\text{co}}\, P$ satisfies $u_t(x) \ge 1$, then $x \in \overline{\text{co}}\,\theta(p^{-1}(t))$.*

(b) *If $t \notin M$, then $u_t(x) < 1$ for every $x \in \overline{\text{co}}\, P$.*

Proof. (a) Clearly, the image of the mapping $\varrho : \eta \in F \mapsto (0; \cos p(\eta), \sin p(\eta))$ is relatively compact. By the choice of ψ and P, there is a probability measure λ_x on F such that $x = \int_F \theta d\lambda_x$. We obtain $1 \le u_t(x) = \int_F u_t(\theta(\eta)) d\lambda_x = \int_F (\cos p(\eta)\cos t + \sin p(\eta)\sin t) d\lambda_x = \int_F \cos(p(\eta) - t) d\lambda_x$, and thus $\lambda_x(\{\eta \in F : \cos(p(\eta) - t) = 1\}) = 1$. Since $p(\eta) - t \in (-2\pi, \pi/2]$ for $\eta \in F$, $\cos(p(\eta) - t) = 1$ is the same as $p(\eta) = t$, i.e., $\eta \in p^{-1}(t)$. We get $x = \int_F \theta d\lambda_x = \int_{p^{-1}(t)} \theta d\lambda_x \in \overline{\text{co}}\,\theta(p^{-1}(t))$.

(b) If $t \notin M = p(F)$, then $\overline{\text{co}}\,\theta(p^{-1}(t))$ is empty. Considering (a), we see that $u_t(x) < 1$ for every $x \in \overline{\text{co}}\, P$. □

CLAIM 2.2.6. (a) $R \cap (Y \times S_{\mathbb{R}^2}) = (\overline{\text{co}}\, P \cup (-\overline{\text{co}}\, P)) \cap (Y \times S_{\mathbb{R}^2})$.

(b) *If $t \in [0, \pi/2] \setminus M$, then $u_t(x) < 1$ for every $x \in R$.*

Proof. For $t \in [0, \pi)$, we prove the implication

$$x \in R \;\&\; u_t(x) \ge 1 \quad \Rightarrow \quad x \in \overline{\text{co}}\, P. \tag{2.1}$$

Let $t \in [0, \pi), x \in R$ and $u_t(x) \ge 1$. We set $m = \min\{0, \cos t\} > -1$ and $M = \sup_{z \in \text{co}\, P} \|z\| < \infty$. Let $\varepsilon > 0$ be arbitrary. There are $a, b \in \text{co}\, P$ and $\lambda \in [0, 1]$ such that $\|x - (1 - \lambda)a - \lambda(-b)\| < \varepsilon$. For $\eta \in F$, we have $u_t(\theta(\eta)) = u_t(\psi(\eta); \cos p(\eta), \sin p(\eta)) = \cos p(\eta)\cos t + \sin p(\eta)\sin t = \cos(p(\eta) - t)$, and therefore $m \le u_t(\theta(\eta)) \le 1$ because $p(\eta) - t \in [-t, \pi/2]$. It follows that $m \le u_t(a) \le 1$ and $m \le u_t(b) \le 1$. We compute $1 \le u_t(x) < u_t((1 - \lambda)a + \lambda(-b)) + \|u_t\|\varepsilon \le (1 - \lambda) - \lambda m + \|u_t\|\varepsilon$. So $\lambda < \|u_t\|\varepsilon/(1 + m)$ and $\text{dist}(x, \text{co}\, P) \le \|x - a\| < \varepsilon + \|a - (1 - \lambda)a - \lambda(-b)\| <$

$(1 + 2\|u_t\|M/(1 + m))\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain $x \in \overline{\text{co}}\, P$, and (2.1) is proved.

(a) It is enough to prove the inclusion $R \cap (Y \times S_{\mathbb{R}^2}) \subset \overline{\text{co}}\, P \cup (-\overline{\text{co}}\, P)$. Let $x \in R \cap (Y \times S_{\mathbb{R}^2})$. For some $y \in Y$ and $t \in [0, 2\pi)$, we have $x = (y; \cos t, \sin t)$. We have $u_t(x) = \cos^2 t + \sin^2 t = 1$. If $t \in [0, \pi)$, then (2.1) says that $x \in \overline{\text{co}}\, P$. If $t \in [\pi, 2\pi)$, then (2.1) says that $x \in -\overline{\text{co}}\, P$ because $u_{t-\pi}(-x) = -u_t(-x) = u_t(x) = 1$.

(b) Let $t \in [0, \pi/2] \setminus M$ and $x \in R$ be such that $u_t(x) \geq 1$. Then (2.1) says that $x \in \overline{\text{co}}\, P$, which is impossible due to Claim 2.2.5(b). □

Now, Proposition 2.2.2 follows from Claims 2.2.4 and 2.2.6(b).

Remark 2.2.7. (a) If $\varepsilon > 0$ is small enough, then $\overline{\text{co}}\,(R \cup \varepsilon B_X)$ has the same property as R. Taking $|||\cdot|||$ as the norm which has $\overline{\text{co}}\,(R \cup \varepsilon B_X)$ for its unit ball, we get a norm such that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi \in \mathrm{NA}(|||\cdot|||)$ if and only if $t \in M$. Considering $M \subset [0, \pi/2]$ to be dense, analytic and non-Borel, we obtain the result from [12].

(b) Proposition 2.2.2 (and also Proposition 2.3.1 below) can be generalized as follows. It holds: *Let $(X, \|\cdot\|)$ be a non-reflexive Banach space and $\varphi_1, \varphi_2, \dots, \varphi_n \in X^*$ be linearly independent. Let $M \subset \mathrm{co}\{\varphi_1, \dots, \varphi_n\}$ be analytic. Then there is an equivalent norm $|||\cdot|||$ on X such that, for every $f \in \mathrm{co}\{\varphi_1, \dots, \varphi_n\}$, $f \in \mathrm{NA}(|||\cdot|||)$ if and only if $f \in M$.* Assuming that M is dense in $\mathrm{co}\{\varphi_1, \dots, \varphi_n\}$, we can prove this in a similar way as Proposition 2.2.2. In the general case, we realize that $M \cup (\mathrm{co}\{\varphi_1, \dots, \varphi_n, \varphi_{n+1}\} \setminus \mathrm{co}\{\varphi_1, \dots, \varphi_n\})$ is dense in $\mathrm{co}\{\varphi_1, \dots, \varphi_n, \varphi_{n+1}\}$, where $\varphi_{n+1} \in X^*$ is chosen so that $\varphi_1, \dots, \varphi_n, \varphi_{n+1}$ are linearly independent.

(c) In [1], the authors also ask whether every separable non-reflexive Banach space with separable dual admits a Fréchet smooth norm such that the set of norm-attaining functionals is not Borel. This question is answered affirmatively in [13]. There is a simple way how to give the positive answer with use of Proposition 2.2.2. We can proceed as follows. Let X be a separable non-reflexive Banach space with separable dual. We choose $M \subset [0, \pi/2]$ to be analytic, non-Borel and dense in $[0, \pi/2]$ and $\varphi, \phi \in X^*$ to be linearly independent. As M is not Borel, it is enough to find an equivalent Fréchet smooth norm $|||\cdot|||$ on X such that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi \in \mathrm{NA}(|||\cdot|||)$ if and only if $t \in M$.

By [2, Theorem II.2.6], there is an equivalent norm $\|\cdot\|$ on X such that the dual norm $\|\cdot\|$ is l.u.r. on X^*. Also, there is an equivalent norm $\|\cdot\|'$ on X such that the dual norm $\|\cdot\|'$ is l.u.r. on X^*, too, and, for every $t \in [0, \pi/2]$, $(x_n)_{n \in \mathbb{N}}$ is convergent in X whenever $\|x_n\|' \leq 1$ for $n \in \mathbb{N}$ and $((\cos t)\varphi + (\sin t)\phi)(x_n) \to \|(\cos t)\varphi + (\sin t)\phi\|'$. Indeed, this can be shown for the norm

$\|(y;r,s)\|' = |(\|y\|,r,s)|, (y;r,s) \in Y \times \mathbb{R}^2$, where Y is as in Notation 2.2.3.

Let R be as in Proposition 2.2.2. We define $|||\cdot|||$ to satisfy

$$B_{(X,|||\cdot|||)} = \overline{B_{(X,\|\cdot\|')} + R}.$$

For $u \in X^*$, we have $|||u||| = \|u\|' + \sup_{x\in R} u(x)$. From here, it can be shown that $|||\cdot|||$ is l.u.r. on X^*. Consequently, $|||\cdot|||$ is Fréchet smooth ([2, Proposition II.1.5]). It is straightforward to check that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi \in \mathrm{NA}(|||\cdot|||)$ if and only if $t \in M$. So the norm $|||\cdot|||$ works.

(d) In fact, this method is a simple analogy of the method from [13]. Our method allows us to choose which analytic subset of an arc will be the intersection of this arc with the set of norm-attaining functionals. In [13], these functionals are chosen from a considerably greater set. It is proved: *If X is a separable non-reflexive Banach space with separable dual, then there is a set $H \subset X^*$, homeomorphic to the Hilbert cube $[-1,1]^{\mathbb{N}}$, such that, for every analytic subset M of H, there is an equivalent Fréchet smooth norm $|||\cdot|||$ on X such that $H \cap \mathrm{NA}(|||\cdot|||) = M$.* In this case, to find the norm corresponding to our norm $\|\cdot\|'$ (mentioned in (c)) is much more complicated. One of the reasons is that the analogy of our space Y above has infinite codimension, and thus it does not have to be complemented.

2.3 THE ROTUNDING TECHNIQUE

PROPOSITION 2.3.1. *Let $(X, \|\cdot\|)$ be a strictly convex non-reflexive Banach space and $\varphi, \phi \in X^*$ be linearly independent. Let $M \subset [0, \pi/2]$ be Borel and dense in $[0, \pi/2]$. Then there is an equivalent strictly convex norm $|||\cdot|||$ on X such that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi \in \mathrm{NA}(|||\cdot|||)$ if and only if $t \in M$.*

The proof of the proposition is also given in the form of a series of claims.

Since M is Borel, there are a closed subset F of $\mathbb{N}^{\mathbb{N}}$ and a one-to-one continuous mapping $p : F \to [0, \pi/2]$ such that $p(F) = M$ ([14, Theorem 13.7]). We define $Y, u_t, \psi, \theta, P, R$ as in Notation 2.2.3. Clearly, Claims 2.2.4 – 2.2.6 hold. The condition that p is a one-to-one mapping makes the situation more concrete and allows us to improve some of them.

CLAIM 2.3.2. $(\overline{\mathrm{co}}\, P) \cap (Y \times S_{\mathbb{R}^2}) = P$.

Proof. It is enough to prove $(\overline{\mathrm{co}}\, P) \cap (Y \times S_{\mathbb{R}^2}) \subset P$ because the other inclusion is obvious. Let $x \in (\overline{\mathrm{co}}\, P) \cap (Y \times S_{\mathbb{R}^2})$. There are $y \in Y$ and $t \in [0, 2\pi)$ such that $x = (y; \cos t, \sin t)$. We have $u_t(x) = \cos^2 t + \sin^2 t = 1$. By Claim 2.2.5(a), $x \in \overline{\mathrm{co}}\, \theta(p^{-1}(t))$.

Let η be the only element of $p^{-1}(t)$. We obtain $x \in \overline{\text{co}}\,\theta(p^{-1}(t)) = \overline{\text{co}}\,\{\theta(\eta)\} = \{\theta(\eta)\} \subset P$. □

Claim 2.3.3. $R \cap (Y \times S_{\mathbb{R}^2}) = P \cup (-P)$.

Proof. It follows immediately from Claims 2.3.2 and 2.2.6(a). □

In the proof of the following claim, we need a continuous function $f : [0,2] \times [0,1] \to [0,1]$ with properties

(a) $f(x,y) \leq 1 - y$ for $(x,y) \in [0,2] \times [0,1]$,

(b) $f(\lambda a + (1-\lambda)b) > \lambda f(a) + (1-\lambda) f(b)$ for $a, b \in [0,2] \times [0,1), a \neq b, \lambda \in (0,1)$,

(c) $f(x_1, y) > f(x_2, y)$ when $x_1 < x_2$ and $y < 1$, $f(x, y_1) > f(x, y_2)$ when $y_1 < y_2$.

An explicit example of such a function is

$$f(x,y) = 1 - y - (1-y)^2 \left[\frac{1}{6} + \frac{1}{6-x}\right].$$

It is easy to check that the partial derivatives of f are negative on $[0,2] \times [0,1)$ and that

$$\frac{\partial^2 f}{\partial (r,s)^2}(x,y) = -\frac{2}{6-x}\left[s - \frac{1-y}{6-x} r\right]^2 - \frac{1}{3}s^2,$$

which is negative on $[0,2] \times [0,1)$ (by $\frac{\partial^2 f}{\partial (r,s)^2}(x,y)$ we mean the second derivative of f at (x,y) in the direction (r,s)).

Claim 2.3.4. *There is a continuous function $\rho : 2B_Y \times B_{\mathbb{R}^2} \to [0,1]$ with properties*

(a) $\rho(y; r, s) \leq 1 - |(r,s)|$ *for* $(y; r, s) \in 2B_Y \times B_{\mathbb{R}^2}$,

(b) $\rho(\lambda a + (1-\lambda) b) > \lambda \rho(a) + (1-\lambda)\rho(b)$ *for* $a, b \in 2B_Y \times (B_{\mathbb{R}^2} \setminus S_{\mathbb{R}^2}), a \neq b, \lambda \in (0,1)$,

(c) $\rho(x) = \rho(-x)$ *for* $x \in 2B_Y \times B_{\mathbb{R}^2}$.

Proof. We put

$$\rho(y; r, s) = f\big(\|y\|, |(r,s)|\big), \quad (y; r, s) \in 2B_Y \times B_{\mathbb{R}^2}.$$

Properties (a), (c) are obvious, let us check (b). Assume that $(y_1, z_1), (y_2, z_2) \in 2B_Y \times B_{\mathbb{R}^2}, (y_1, z_1) \neq (y_2, z_2), |z_1| < 1, |z_2| < 1, \lambda \in (0,1)$. We need to check the inequality

$$\begin{aligned} &f\big(\|\lambda y_1 + (1-\lambda) y_2\|, |\lambda z_1 + (1-\lambda) z_2|\big) \\ &\qquad > \lambda f\big(\|y_1\|, |z_1|\big) + (1-\lambda) f\big(\|y_2\|, |z_2|\big). \end{aligned}$$

If $\|y_1\| \neq \|y_2\|$ or $|z_1| \neq |z_2|$, then we have $f(\|\lambda y_1 + (1-\lambda) y_2\|, |\lambda z_1 + (1-\lambda) z_2|) \geq f(\lambda \|y_1\| + (1-\lambda)\|y_2\|, \lambda |z_1| + (1-\lambda)|z_2|) > \lambda f(\|y_1\|, |z_1|) + (1-\lambda) f(\|y_2\|, |z_2|)$ by the properties of the function f. If $\|y_1\| = \|y_2\|$ and $|z_1| = |z_2|$, then, by the strict convexity of $\|\cdot\|, |\cdot|$ and by $(y_1, z_1) \neq (y_2, z_2)$, we have $\|\lambda y_1 +$

$(1-\lambda)y_2\| < \lambda\|y_1\| + (1-\lambda)\|y_2\|$ or $|\lambda z_1 + (1-\lambda)z_2| < \lambda|z_1| + (1-\lambda)|z_2|$, and thus $f(\|\lambda y_1 + (1-\lambda)y_2\|, |\lambda z_1 + (1-\lambda)z_2|) > f(\lambda\|y_1\| + (1-\lambda)\|y_2\|, \lambda|z_1| + (1-\lambda)|z_2|) = \lambda f(\|y_1\|, |z_1|) + (1-\lambda)f(\|y_2\|, |z_2|)$. □

Let us take the function ρ from Claim 2.3.4. We denote

$$\|(y,z)\|_\infty = \max\{\|y\|, |z|\} \quad \text{for } (y,z) \in Y \oplus \mathbb{R}^2,$$

$$B(x,r) = \left\{(y,z) \in Y \oplus \mathbb{R}^2 : \|x - (y,z)\|_\infty \le r\right\}$$
$$\text{for } x \in Y \oplus \mathbb{R}^2,\ r \ge 0.$$

We choose a sequence of positive numbers $(\varepsilon_i)_{i\in\mathbb{N}}$ such that

$$\sum_{i=1}^\infty \varepsilon_i \le 1, \quad \prod_{i=1}^\infty (1-\varepsilon_i) > 0, \quad \lim_{n\to\infty} \frac{1}{\varepsilon_n} \sum_{i=n}^\infty \varepsilon_i = 1,$$

and define

$$R_0 = R,$$
$$R_n = \bigcup_{x\in R_{n-1}} B(x, \varepsilon_n\rho(x)), \quad n \in \mathbb{N},$$
$$R_\infty = \overline{\bigcup_{n=0}^\infty R_n}.$$

It is easy to verify by the induction that $R_n \subset (1 + \sum_{i=1}^n \varepsilon_i)B_Y \times B_{\mathbb{R}^2}$, and thus $R_n, n \in \mathbb{N}$, are well-defined. Besides this, the sets $R_n, n \in \mathbb{N}$, are absolutely convex.

Further on, by dist we mean the distance with respect to $\|\cdot\|_\infty$.

Claim 2.3.5. $R_\infty \cap (Y \times S_{\mathbb{R}^2}) = P \cup (-P)$.

Proof. Using Claim 2.3.3, we have $P \cup (-P) = R \cap (Y \times S_{\mathbb{R}^2}) \subset R_\infty \cap (Y \times S_{\mathbb{R}^2})$. It is enough to show that if $(y,z) \in Y \times S_{\mathbb{R}^2}$ and $(y,z) \notin R$, then $(y,z) \notin R_\infty$.

Let $(y,z) \in (Y \times S_{\mathbb{R}^2}) \setminus R$. We denote

$$d = \operatorname{dist}((y,z), R) > 0.$$

Let $n \in \mathbb{N}$. Given $x = (y',z') \in R_{n-1}$ and $(y'',z'') \in B(x, \varepsilon_n\rho(x))$, we have $\|(y'',z'') - (y,z)\|_\infty \ge \|x - (y,z)\|_\infty - \|x - (y'',z'')\|_\infty \ge \|x-(y,z)\|_\infty - \varepsilon_n\rho(x) \ge \|x - (y,z)\|_\infty - \varepsilon_n(1 - |z'|) = \|x - (y,z)\|_\infty - \varepsilon_n(|z| - |z'|) \ge \|x - (y,z)\|_\infty(1-\varepsilon_n)$. It means that $\operatorname{dist}((y,z), B(x, \varepsilon_n\rho(x))) \ge (1-\varepsilon_n)\|x - (y,z)\|_\infty$ for every $x \in R_{n-1}$. Consequently, $\operatorname{dist}((y,z), R_n) \ge (1-\varepsilon_n)\operatorname{dist}((y,z), R_{n-1})$ from the definition of R_n. By an easy induction argument,

$$\operatorname{dist}((y,z), R_n) \ge d\prod_{i=1}^n (1-\varepsilon_i), \quad n = 0, 1, \dots,$$

$$\operatorname{dist}((y,z), R_\infty) \ge d\prod_{i=1}^\infty (1-\varepsilon_i).$$

So $(y,z) \notin R_\infty$ by the choice of the sequence $(\varepsilon_i)_{i\in\mathbb{N}}$. □

CLAIM 2.3.6. *If a, b are two distinct points of R_∞, then $\lambda a + (1-\lambda)b$ is an element of the interior of R_∞ for every $\lambda \in (0,1)$.*

Proof. Given such a, b, λ, we denote $x = \lambda a + (1-\lambda)b$. Let us realize that $x \notin Y \times S_{\mathbb{R}^2}$. Assume that $x \in Y \times S_{\mathbb{R}^2}$. Since $a, b \in R_\infty \subset Y \times B_{\mathbb{R}^2}$, there is $z \in S_{\mathbb{R}^2}$ such that $a, b \in Y \times \{z\}$. By Claim 2.3.5, we have $a, b \in P \cup (-P)$. By the definition of P and by the fact that p is a one-to-one mapping, the set $(P \cup (-P)) \cap (Y \times \{z\})$ has at most one element. Thus $a = b$, which is a contradiction.

So $x \in Y \times (B_{\mathbb{R}^2} \setminus S_{\mathbb{R}^2})$. We may suppose that $a, b \in Y \times (B_{\mathbb{R}^2} \setminus S_{\mathbb{R}^2})$, too (we may take $(1/2)(a+x), (1/2)(b+x)$ instead of a, b). We have

$$\rho(x) = \rho(\lambda a + (1-\lambda)b) > \lambda\rho(a) + (1-\lambda)\rho(b).$$

We choose $r' > r > \rho(a)$ and $s' > s > \rho(b)$ such that

$$\rho(x) > \lambda r' + (1-\lambda)s'.$$

Since ρ is continuous, we can choose $u > 0$ and $v > 0$ such that $\rho \leq r$ on $B(a,u)$ and $\rho \leq s$ on $B(b,v)$. Let us prove that, for $n \in \mathbb{N}$,

$$\operatorname{dist}(a, R_n) \geq \min\left\{u - \varepsilon_n, \operatorname{dist}(a, R_{n-1}) - r\varepsilon_n\right\}.$$

If $y \in R_{n-1} \setminus B(a,u)$ and $z \in B(y, \varepsilon_n\rho(y))$, then $\|a - z\|_\infty \geq \|a - y\|_\infty - \|y - z\|_\infty \geq u - \varepsilon_n\rho(y) \geq u - \varepsilon_n$. If $y \in R_{n-1} \cap B(a,u)$ and $z \in B(y, \varepsilon_n\rho(y))$, then $\|a - z\|_\infty \geq \|a - y\|_\infty - \|y - z\|_\infty \geq \operatorname{dist}(a, R_{n-1}) - \varepsilon_n\rho(y) \geq \operatorname{dist}(a, R_{n-1}) - r\varepsilon_n$.

Now, since $\operatorname{dist}(a, R_n) \to 0$ and $u - \varepsilon_n \to u > 0$, there is n_0 such that $\operatorname{dist}(a, R_n) \geq \operatorname{dist}(a, R_{n-1}) - r\varepsilon_n$ for every $n \geq n_0$. For $n \geq n_0$, we have

$$\begin{aligned}
\operatorname{dist}(a, R_n) &\leq \operatorname{dist}(a, R_{n+1}) + r\varepsilon_{n+1} \\
&\leq \operatorname{dist}(a, R_{n+2}) + r\varepsilon_{n+1} + r\varepsilon_{n+2} \\
&\leq \cdots \leq r\sum_{i=n+1}^{\infty} \varepsilon_i.
\end{aligned}$$

By the same way, we find m_0 such that $\operatorname{dist}(b, R_n) \leq s\sum_{i=n+1}^{\infty} \varepsilon_i$ for $n \geq m_0$. We put $N = \max\{n_0, m_0\}$ and, for every $n \geq N$, we choose $a_n, b_n \in R_n$ such that $\|a - a_n\|_\infty \leq r'\sum_{i=n+1}^{\infty} \varepsilon_i$ and $\|b - b_n\|_\infty \leq s'\sum_{i=n+1}^{\infty} \varepsilon_i$. For $n \geq N$, we put $x_n = \lambda a_n + (1-\lambda)b_n$. Since ρ is continuous, we have $\rho(x_n) \to \rho(x)$. Since

$$\frac{\lambda r' + (1-\lambda)s'}{\rho(x_n)} \frac{1}{\varepsilon_{n+1}} \sum_{i=n+1}^{\infty} \varepsilon_i \to \frac{\lambda r' + (1-\lambda)s'}{\rho(x)} < 1,$$

we can choose $n \geq N$ such that $(\lambda r' + (1-\lambda)s') \sum_{i=n+1}^{\infty} \varepsilon_i < \rho(x_n)\varepsilon_{n+1}$. We have

$$\begin{aligned} \|x - x_n\|_\infty &\leq \lambda\|a - a_n\|_\infty + (1-\lambda)\|b - b_n\|_\infty \\ &\leq (\lambda r' + (1-\lambda)s') \sum_{i=n+1}^{\infty} \varepsilon_i \\ &< \rho(x_n)\varepsilon_{n+1}. \end{aligned}$$

So x is an element of the interior of $B(x_n, \varepsilon_{n+1}\rho(x_n))$, which is a subset of R_{n+1}. □

CLAIM 2.3.7. *If $t \in [0, \pi/2]$, then u_t attains its supremum on R_∞ if and only if $t \in M$.*

Proof. Considering Claim 2.2.4, it remains to prove that $u_t(x) < 1$ for every $x \in R_\infty$ in the case that $t \notin M$. Suppose that $t \notin M$, $x = (y; r\cos\alpha, r\sin\alpha) \in R_\infty$ and $u_t(x) = 1$. We have $1 = u_t(x) = r(\cos\alpha\cos t + \sin\alpha\sin t) = r\cos(\alpha - t)$, which is possible only if $r = 1$ and $\alpha = t$, i.e. $x \in Y \times \{(\cos t, \sin t)\}$. By Claim 2.3.5, $x \in P \cup (-P) \subset R$. By Claim 2.2.6(b), $u_t(x) < 1$, which is a contradiction. □

Now, we define $|||\cdot|||$ as the norm with the unit ball R_∞. Proposition 2.3.1 follows from Claims 2.3.6 and 2.3.7.

Proof of Theorem 2.1.1. Choose $\varphi, \phi \in X^*$ to be linearly independent. We take $M \subset [0, \pi/2]$, dense in $[0, \pi/2]$, which is Borel, but not of the additive Borel class α ([14, Theorem 22.4]). It is known that there is an equivalent strictly convex norm $\|\cdot\|$ on X ([2, Theorem II.2.6]). By Proposition 2.3.1, there is a strictly convex norm $|||\cdot|||$ on X such that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi \in \mathrm{NA}(|||\cdot|||)$ if and only if $t \in M$. Since M is not of the additive Borel class α, $\mathrm{NA}(|||\cdot|||)$ is not of the additive Borel class α, too ($t \in [0, \pi/2] \mapsto (\cos t)\varphi + (\sin t)\phi$ is a continuous mapping). □

3

ON BINORMALITY IN NON-SEPARABLE BANACH SPACES

3.1 Introduction and main results

Let σ and τ be two topologies on a set X. We say that (X, σ, τ) is *binormal* if, for every disjoint σ-closed $A \subset X$ and τ-closed $B \subset X$, there are disjoint σ-open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$. We say that a Banach space X is binormal if X is binormal with respect to its norm and weak topologies.

It is possible to meet the notion of binormality of (X, σ, τ) in the real analysis where it is more likely called Lusin-Menchoff property of τ in the case that the "second topology" τ is finer than σ. For example, it is known that both the density topology and the fine topology have the Lusin-Menchoff property with respect to the Euclidean topology (see, e.g., [16]). The situation in Banach spaces is somewhat opposite to that of real analysis because the finer topology is the metrizable one.

The question whether the weak topology has the corresponding "Lusin-Menchoff property" with respect to the norm topology was posed by L. Zajíček. This question was studied later by P. Holický who proved in [8] that every separable Banach space is binormal and that the space ℓ^∞ is not binormal. But it was not possible to decide what was the answer for many other non-separable Banach spaces, e.g. for non-separable Hilbert spaces.

In this work, we show that many non-separable Banach spaces are binormal. We prove the following result (see Theorem 3.5.2 and Theorem 3.4.2).

Theorem 3.1.1. *Every Plichko space is binormal. Every dual to an Asplund space is binormal. Generally, any Banach space which belongs to a $\mathcal{P}$-class is binormal.*

We give the necessary definitions below. Note that the class of Plichko spaces is quite wide and it contains all reflexive spaces or, more generally, all weakly compactly generated spaces. On the other hand, we show that there is a Banach space which admits a LUR norm but it is not binormal (Example 3.5.3).

Some results in this work are formulated for a general locally convex topology instead of the weak topology. If X is a Banach space and τ is a locally convex topology which is weaker than

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the norm topology, we say that X is *τ-binormal* if X is binormal with respect to its norm topology and τ. We prove characterizations of τ-binormality by another separation property and by an in-between condition (Proposition 3.2.6).

We are interested in the case of the w^*-topology. We prove the following theorem (which is covered by Theorem 3.6.3). Note that the separability of the set A cannot be dropped (Example 3.6.6).

THEOREM 3.1.2. *A Banach space E is Asplund if and only if, for every disjoint separable and closed $A \subset E^*$ and w^*-closed $B \subset E^*$, there are disjoint open $D \subset E^*$ and w^*-open $C \subset E^*$ with $A \subset C$ and $B \subset D$.*

Furthermore, our methods lead to the characterization of scattered compact spaces by a separation property (Theorem 3.6.8).

3.2 A CHARACTERIZATION OF BINORMALITY

We start with a well-known variant of the Urysohn lemma. The lemma follows from [16, Theorem 3.11] in the case that the topologies are comparable (which will be our case) but it holds in the general situation as well (see [16, exercise 3.B.5(e)]).

LEMMA 3.2.1. *Let (X, σ, τ) be binormal. If σ-closed $A \subset X$ and τ-closed $B \subset X$ are disjoint, then there is a lower σ-semicontinuous and upper τ-semicontinuous function h on X such that*

$$0 \leq h \leq 1, \quad h = 0 \text{ on } A, \quad h = 1 \text{ on } B.$$

We now prove an abstract version of our characterization.

LEMMA 3.2.2. *Let Y be a set with two topologies σ_Y and τ_Y with τ_Y weaker than σ_Y. Let*

$$X = Y \times \mathbb{R}$$

and let the products of σ_Y and τ_Y with the standard topology on $\mathbb{R}$ be denoted by σ and τ.

If the condition

$$(*) \qquad \forall U \in \tau\ \exists \{U_n\}_{n \in \mathbb{N}}, U_n \in \tau\ :\ U = \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} \overline{U_n}^{\sigma}$$

is satisfied, then the following assertions are equivalent:

(i) *(X, σ, τ) is binormal.*

(iia) *If $F_1 \supset F_2 \supset \dots$ are σ_Y-closed subsets of Y with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \dots$, τ_Y-open subsets of Y, such that $F_n \subset G_n$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma_Y} = \emptyset$.*

(iib) *If $F_1 \supset F_2 \supset \dots$ are σ-closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \dots$, τ-open subsets of X, such that $F_n \subset G_n$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma} = \emptyset$.*

(iii) *If $f : X \to (0, \infty)$ is lower σ-semicontinuous, then there exists $g : X \to (0, \infty)$, lower σ-semicontinuous and upper τ-semicontinuous, such that $g < f$.*

Remark 3.2.3. Binormality of (Y, σ_Y, τ_Y) is not sufficient for binormality of (X, σ, τ). If we take $Y = [0, 1]$, σ_Y the discrete topology on Y and τ_Y the standard topology, then (Y, σ_Y, τ_Y) is clearly binormal. Let us show that it does not satisfy (iia). Take pairwise distinct numbers $a_1, a_2, \dots \in [0, 1]$ which form a countable dense subset of $[0, 1]$ and put

$$F_n = \{a_n, a_{n+1}, \dots\}, \quad n \in \mathbb{N}.$$

Note that F_n is dense in $[0, 1]$ for every $n \in \mathbb{N}$. We have $\bigcap_{n=1}^{\infty} F_n = \emptyset$ but the Baire theorem guarantees that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$ whenever $G_1, G_2, \dots \subset [0, 1]$ are open sets with $F_n \subset G_n, n \in \mathbb{N}$.

We will use this simple idea in a general situation later (proof of Lemma 3.6.2).

Before proving the lemma, we prove

CLAIM 3.2.4 (cf. proof of [8, Theorem 1]). *Let σ and τ be two topologies on a set X and let the condition $(*)$ from Lemma 3.2.2 be satisfied. Let $A \subset X$ be σ-closed and $B \subset X$ be τ-closed. If there are σ-open $D_n \subset X, n \in \mathbb{N}$, such that $B \subset \bigcup_{n=1}^{\infty} D_n$ and $\overline{D_n}^{\tau} \cap A = \emptyset$ for all $n \in \mathbb{N}$, then there are disjoint σ-open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$.*

Proof. By $(*)$, there are τ-open sets $C_m \subset X, m \in \mathbb{N}$, such that $X \setminus B = \bigcup_{m=1}^{\infty} C_m$ and $\overline{C_m}^{\sigma} \cap B = \emptyset$ for all $m \in \mathbb{N}$. In particular, $A \subset \bigcup_{m=1}^{\infty} C_m$. Define

$$D = \bigcup_{n=1}^{\infty} \Big(D_n \setminus \bigcup_{m=1}^{n} \overline{C_m}^{\sigma} \Big),$$

$$C = \bigcup_{m=1}^{\infty} \Big(C_m \setminus \bigcup_{n=1}^{m} \overline{D_n}^{\tau} \Big).$$

It can be easily checked that C is τ-open, D is σ-open, $A \subset C, B \subset D$ and $C \cap D = \emptyset$. □

Proof of Lemma 3.2.2. (i) $\Rightarrow$ (iia) Put

$$A = \bigcup_{n=1}^{\infty} F_n \times [1/n, \infty), \quad B = Y \times \{0\}. \tag{3.1}$$

Clearly, A is σ-closed, B is τ-closed and $A \cap B = \emptyset$. By the assumption, there are disjoint σ-open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$. We have $A \cap \overline{D}^{\tau} \subset A \setminus C = \emptyset$. We define H_n as the set of points $y \in Y$ such that there is a σ_Y-open neighbourhood $U \ni y$ with $U \times [0, 1/n] \subset D$. Let G_n be defined as $Y \setminus \overline{H_n}^{\tau_Y}$. We have $\bigcup_{n=1}^\infty H_n = Y$, and so $\bigcap_{n=1}^\infty \overline{G_n}^{\sigma_Y} \subset \bigcap_{n=1}^\infty \overline{Y \setminus H_n}^{\sigma_Y} = \bigcap_{n=1}^\infty (Y \setminus H_n) = \emptyset$. Clearly, $G_1 \supset G_2 \supset \dots$. For $n \in \mathbb{N}$, we have $\overline{H_n}^{\tau_Y} \times [0,1/n] \subset \overline{D}^\tau \subset X \setminus A$, and so $F_n \times \{1/n\} = A \cap (Y \times \{1/n\}) \subset (Y \times \{1/n\}) \setminus (\overline{H_n}^{\tau_Y} \times [0,1/n]) = G_n \times \{1/n\}$.

(iia) $\Rightarrow$ (iib) For $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, we define

$$F_n^i = \{y \in Y : (y,r) \in F_n \text{ for some } r \in [i-1/2, i+1/2]\}. \tag{3.2}$$

Due to the compactness of $[i-1/2, i+1/2]$, the sets F_n^i are σ_Y-closed and $\bigcap_{n=1}^\infty F_n^i = \emptyset$ for all $i \in \mathbb{Z}$. By the assumption, there are, for all $i \in \mathbb{Z}$, τ_Y-open $G_1^i \supset G_2^i \supset \dots$ such that $F_n^i \subset G_n^i$ and $\bigcap_{n=1}^\infty \overline{G_n^i}^{\sigma_Y} = \emptyset$. Then the choice

$$G_n = \bigcup_{i \in \mathbb{Z}} \Big(G_n^i \times (i-1, i+1) \Big), \quad n \in \mathbb{N},$$

works. (We have $F_n \subset \bigcup_{i \in \mathbb{Z}} F_n^i \times [i-1/2, i+1/2] \subset G_n$ for $n \in \mathbb{N}$. Suppose that $(y,r) \in \bigcap_{n=1}^\infty \overline{G_n}^\sigma$. Put $U = Y \times (r-1, r+1)$. We have $U \cap (G_n^i \times (i-1, i+1)) = \emptyset$ whenever $|i-r| \geq 2$. There is $n \in \mathbb{N}$ such that $y \notin \overline{G_n^i}^{\sigma_Y}$ for all i with $|i-r| < 2$. If we take $V = (Y \setminus \bigcup_{|i-r|<2} \overline{G_n^i}^{\sigma_Y}) \times \mathbb{R}$, then $U \cap V$ is a σ-open neighbourhood of (y,r) which does not intersect G_n. This contradicts $(y,r) \in \overline{G_n}^\sigma$.)

(iib) $\Rightarrow$ (i) Let σ-closed $A \subset X$ and τ-closed $B \subset X$ satisfy $A \cap B = \emptyset$. We need to find disjoint σ-open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$. By $(*)$, there are τ-open sets $H_n \subset X, n \in \mathbb{N}$, such that $X \setminus B = \bigcup_{n=1}^\infty H_n$ and $\overline{H_n}^\sigma \cap B = \emptyset$ for all $n \in \mathbb{N}$. We may assume that $H_1 \subset H_2 \subset \dots$. The sets H_n are σ-open in particular. We put

$$F_n = A \setminus H_n \tag{3.3}$$

for $n \in \mathbb{N}$. The sets $F_n, n \in \mathbb{N}$, are σ-closed, $F_1 \supset F_2 \supset \dots$ and $\bigcap_{n=1}^\infty F_n = A \setminus \bigcup_{n=1}^\infty H_n = A \setminus (X \setminus B) = \emptyset$. By the assumption, there are τ-open $G_1 \supset G_2 \supset \dots$ such that $F_n \subset G_n, n \in \mathbb{N}$, and $\bigcap_{n=1}^\infty \overline{G_n}^\sigma = \emptyset$. For $n \in \mathbb{N}$, we put

$$C_n = G_n \cup H_n, \quad D_n = X \setminus \overline{C_n}^\sigma.$$

We obtain $A = F_n \cup (A \cap H_n) \subset G_n \cup (A \cap H_n) \subset C_n$, and so $\overline{D_n}^\tau \cap A \subset (X \setminus C_n) \cap C_n = \emptyset$, for $n \in \mathbb{N}$. Considering Claim 3.2.4, it remains to prove that $B \subset \bigcup_{n=1}^\infty D_n$. For $n \in \mathbb{N}$, we have

$$B \setminus D_n = B \cap \overline{C_n}^\sigma = (B \cap \overline{G_n}^\sigma) \cup (B \cap \overline{H_n}^\sigma) = B \cap \overline{G_n}^\sigma,$$

and so $B \setminus \bigcup_{n=1}^\infty D_n = \bigcap_{n=1}^\infty (B \setminus D_n) = \bigcap_{n=1}^\infty (B \cap \overline{G_n}^\sigma) = \emptyset$.

(iib) $\Rightarrow$ (iii) We have already proved (iib) $\Rightarrow$ (i). Therefore, assuming (iib), we can assume (i) as well.

We put $F_n = \{x \in X : f(x) \le 1/n\}$. By (iib), we take τ-open $G_1 \supset G_2 \supset \dots$ such that $F_n \subset G_n$ and $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma} = \emptyset$. By (i) and Lemma 3.2.1, there is, for every $n \in \mathbb{N}$, lower σ-semicontinuous and upper τ-semicontinuous function $g_n : X \to [0,1]$ such that $g_n = 0$ on F_n and $g_n = 1$ on $X \setminus G_n$. We have $g_n/n < f$ on X. Putting

$$g = \sum_{n=1}^{\infty} \frac{g_n}{2^n n},$$

we have $0 < g < f$ on X.

(iii) $\Rightarrow$ (iib) We may assume $F_1 = X$. We define $f(x) = 1/n$ for every $x \in F_n \setminus F_{n+1}$ (this defines a lower σ-semicontinuous function on whole space X). By (iii), there exists $g : X \to (0, \infty)$, lower σ-semicontinuous and upper τ-semicontinuous, such that $g < f$. For $n \in \mathbb{N}$, we take τ-open $G_n = \{x \in X : g(x) < 1/n\}$. We have $F_n = \{x \in X : f(x) \le 1/n\} \subset \{x \in X : g(x) < 1/n\} = G_n$. At the same time, $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma} \subset \bigcap_{n=1}^{\infty} \{x \in X : g(x) \le 1/n\} = \{x \in X : g(x) \le 0\} = \emptyset$. □

By an inspection of the proof of Lemma 3.2.2, we get the following modification.

LEMMA 3.2.5. *Let $Y, \sigma_Y, \tau_Y, X, \sigma, \tau$ be as in Lemma 3.2.2 and let $(*)$ be satisfied. Moreover, let σ be metrizable. Then the following assertions are equivalent:*

(i) *For every disjoint σ-separable and σ-closed $A \subset X$ and τ-closed $B \subset X$, there are disjoint σ-open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$.*

(iia) *If $F_1 \supset F_2 \supset \dots$ are σ_Y-separable and σ_Y-closed subsets of Y with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \dots$, τ_Y-open subsets of Y, such that $F_n \subset G_n, n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma_Y} = \emptyset$.*

(iib) *If $F_1 \supset F_2 \supset \dots$ are σ-separable and σ-closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \dots$, τ-open subsets of X, such that $F_n \subset G_n, n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma} = \emptyset$.*

Proof. The lemma can be proved in the same way as Lemma 3.2.2. The following should be mentioned.

- In the proof of (i) $\Rightarrow$ (iia), we realize that the set A defined by (3.1) is σ-separable because $F_1, F_2, \dots$ are assumed to be σ_Y-separable.
- In the proof of (iia) $\Rightarrow$ (iib), we realize that the sets F_n^i defined by (3.2) are σ_Y-separable because $F_1, F_2, \dots$ are assumed to be σ-separable (we use the metrizability of σ).

- In the proof of (iib) $\Rightarrow$ (i), we realize that the sets F_n defined by (3.3) are σ-separable because A is assumed to be σ-separable (we use the metrizability of σ again).

$\square$

The desired characterization and its variant follow.

PROPOSITION 3.2.6. *Let X be a Banach space and τ be a Hausdorff locally convex topology on X, weaker than the norm topology. Then the following assertions are equivalent:*

(i) *X is τ-binormal.*

(ii) *If $F_1 \supset F_2 \supset \dots$ are closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \dots$, τ-open subsets of X, such that $F_n \subset G_n, n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n} = \emptyset$.*

(iii) *If $f : X \to (0, \infty)$ is lower semicontinuous, then there exists $g : X \to (0, \infty)$, continuous and upper τ-semicontinuous, such that $g < f$.*

Proof. We may suppose that $X \neq \{0\}$. Then, by the Hahn-Banach theorem, there is a τ-continuous linear functional $f \neq 0$ on X. We define Y as the kernel of f, σ as the norm topology of X, σ_Y as the norm topology of Y and τ_Y as the restriction of τ on Y. We want to show that we are in the situation of Lemma 3.2.2. Fix an $x_0 \in X$ with $f(x_0) = 1$. We will identify a couple $(y, r) \in Y \times \mathbb{R}$ with the point $y + rx_0 \in X$ (so $x \in X$ will be identified with $(x - f(x)x_0, f(x)) \in Y \times \mathbb{R}$). It is easy to check that the mapping $(y, r) \in Y \times \mathbb{R} \mapsto y + rx_0$ is $(\tau_Y \times |\cdot|)$-τ-continuous and $(\sigma_Y \times |\cdot|)$-$\sigma$-continuous and that the mapping $x \in X \mapsto (x - f(x)x_0, f(x))$ is τ-$(\tau_Y \times |\cdot|)$-continuous and σ-$(\sigma_Y \times |\cdot|)$-continuous. So the products of σ_Y and τ_Y with the standard topology on $\mathbb{R}$ are σ and τ indeed.

It remains to show that $(*)$ is satisfied. Let $U \subset X$ be τ-open. We prove first that every $x \in U$ has a τ-open neighbourhood V such that $\operatorname{dist}(V, X \setminus U) > 0$. There are τ-continuous seminorms $p_1, p_2, \dots, p_n$ and $\varepsilon > 0$ such that $y \in U$ whenever $p_i(y - x) < \varepsilon$ for all $i \in \{1, 2, \dots, n\}$. The seminorms are continuous in particular, so we can take $C > 0$ such that $p_i(z) \leq C\|z\|$ for all $z \in X$ and $i \in \{1, 2, \dots, n\}$. We define τ-open

$$V = \{y \in X : p_i(y - x) < \varepsilon/2 \text{ for } i = 1, 2, \dots, n\}.$$

We are going to show that $\operatorname{dist}(V, X \setminus U) \geq \varepsilon/(2C)$. Let $a \in V$ and $b \in X \setminus U$. By the choice of $p_1, p_2, \dots, p_n$ and ε, there is $i \in \{1, 2, \dots, n\}$ such that $p_i(b - x) \geq \varepsilon$. We are computing $\|b - a\| \geq (1/C)p_i(b - a) \geq (1/C)(p_i(b - x) - p_i(a - x)) > (1/C)(\varepsilon - \varepsilon/2) = \varepsilon/(2C)$. So $\operatorname{dist}(V, X \setminus U) \geq \varepsilon/(2C)$.

Now, we define U_n as the set of all $x \in U$ for which there is a τ-open neighbourhood $V \ni x$ such that $\operatorname{dist}(V, X \setminus U) \geq 1/n$.

This is clearly a τ-open set. We know that every $x \in U$ belongs to U_n for a sufficiently large n. At the same time, $\overline{U_n} \subset U$ since $\operatorname{dist}(U_n, X \setminus U) \geq 1/n$. This completes the verification of $(*)$. □

Proposition 3.2.7. *Let X be a Banach space and τ be a Hausdorff locally convex topology on X, weaker than the norm topology. Then the following assertions are equivalent:*

(i) *For every disjoint separable and closed $A \subset X$ and τ-closed $B \subset X$, there are disjoint open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$.*

(ii) *If $F_1 \supset F_2 \supset \dots$ are separable and closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \dots$, τ-open subsets of X, such that $F_n \subset G_n, n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n} = \emptyset$.*

Proof. This has the same proof as Proposition 3.2.6 with the only difference that we use Lemma 3.2.5 instead of Lemma 3.2.2. □

3.3 a stronger property

We are going to introduce a property which is stronger than binormality. The notion of strong binormality plays a key role for us because our only method how to prove that a space is binormal is to prove that it is strongly binormal. Although we proved a characterization of binormality in the previous section, we still do not know too much about binormality itself. For example, we do not know whether $X \times Y$ is necessarily binormal when X and Y are binormal. However, there is no such a problem with strong binormality (Proposition 3.4.1).

Let X be a Banach space and τ be a locally convex topology on X, weaker than the norm topology. We say that X is *strongly τ-binormal* if there exists a system of τ-open neighbourhoods $U_x^n \ni x, x \in X, n \in \mathbb{N}$, such that

$$\bigcap_{n=1}^{\infty} \left(U_{x_n}^n + \varepsilon_n B_X \right) \neq \emptyset \quad \Longrightarrow \quad \{x_n : n \in \mathbb{N}\} \text{ is rel. compact}$$

whenever $\varepsilon_n \searrow 0$. We say that a Banach space X is strongly binormal if it is strongly w-binormal (where w denotes the weak topology of X).

We prove three easy lemmata about strong binormality.

Lemma 3.3.1. *If X is strongly τ-binormal, then it is τ-binormal.*

We do not know anything about the converse implication. The problem of the existence of a binormal space which is not strongly binormal does not seem to be easy.

Proof. We will use Proposition 3.2.6. Let $F_1 \supset F_2 \supset \dots$ be closed in X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$. We need to find τ-open $G_n \supset F_n$

with $\bigcap_{n=1}^{\infty} \overline{G_n} = \emptyset$ (the inclusions $G_1 \supset G_2 \supset \dots$ can be arranged by taking $\bigcap_{m \le n} G_m$ instead of G_n). Let $U_x^n \ni x, x \in X, n \in \mathbb{N}$, be a system witnessing the strong τ-binormality of X. Put

$$G_n = \bigcup_{x \in F_n} U_x^n, \quad n \in \mathbb{N}.$$

If now $a \in \bigcap_{n=1}^{\infty} \overline{G_n}$, then we find $a_n \in G_n$ with $\|a - a_n\| \le 1/n$ for every $n \in \mathbb{N}$. For some $x_n \in F_n$, we have $a_n \in U_{x_n}^n$. By the triangle inequality,

$$a \in \bigcap_{n=1}^{\infty} \Big(U_{x_n}^n + (1/n) B_X \Big).$$

It follows that $\{x_n : n \in \mathbb{N}\}$ is relatively compact. So we have a convergent subsequence $x_{n(k)}$. Its limit is an element of $\bigcap_{n=1}^{\infty} F_n$, which is a contradiction. □

Lemma 3.3.2. *Assume that there exist a dense subset Z of X and a system of τ-open neighbourhoods $U_z^n \ni z, z \in Z, n \in \mathbb{N}$, such that, for any sequence $z_n, n \in \mathbb{N}$, in Z,*

$$\bigcap_{n=1}^{\infty} \Big(U_{z_n}^n + \varepsilon_n B_X \Big) \neq \emptyset \quad \Longrightarrow \quad \{z_n : n \in \mathbb{N}\} \text{ is rel. compact}$$

whenever $\varepsilon_n \searrow 0$. Then X is strongly τ-binormal.

In other words, in the definition of strong τ-binormality, it is possible to require the neighbourhoods U_x^n for the elements of a dense set only.

Proof. Let $x \in X$ and $n \in \mathbb{N}$. There is some $z(x,n) \in Z$ for which $\|x - z(x,n)\| \le 1/n$. Put

$$V_x^n = U_{z(x,n)}^n + (1/n) B_X.$$

This is a τ-open neighbourhood of x. Now, suppose that $\varepsilon_n \searrow 0$ and that $a \in X$ and a sequence $x_n \in X, n \in \mathbb{N}$, satisfy

$$a \in \bigcap_{n=1}^{\infty} \Big(V_{x_n}^n + \varepsilon_n B_X \Big).$$

We obtain

$$a \in \bigcap_{n=1}^{\infty} \Big(U_{z(x_n,n)}^n + (\varepsilon_n + 1/n) B_X \Big).$$

By the property of the system $U_z^n, z \in Z, n \in \mathbb{N}$, the set $\{z(x_n, n) : n \in \mathbb{N}\}$ is relatively compact. Since $\|x_n - z(x_n,n)\| \le 1/n$, the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact, too. □

Lemma 3.3.3. *If X is separable and B_X is τ-closed, then X is strongly τ-binormal.*

Proof. Let $B_1, B_2, \dots$ be closed balls such that their interiors form a basis of the norm topology. Put

$$U_x^n = X \setminus \bigcup_{m \le n,\, x \notin B_m} B_m, \quad x \in X, n \in \mathbb{N}.$$

These sets are τ-open, as $B_1, B_2, \dots$ are τ-closed. Assume

$$a \in \bigcap_{n=1}^{\infty} \Big(U_{x_n}^n + \varepsilon_n B_X \Big).$$

We have to show that $\{x_n : n \in \mathbb{N}\}$ is relatively compact. We show that even $x_n \to a$. Let $m \in \mathbb{N}$ be such that a lies in the interior of B_m. Then there is n_0 such that $x_n \in B_m$ for $n \ge n_0$. Indeed, take n_0 with $n_0 \ge m$ and $\varepsilon_{n_0} < \operatorname{dist}(a, X \setminus B_m)$. Let $n \ge n_0$. There is $b \in U_{x_n}^n$ such that $\|b - a\| \le \varepsilon_n$. Since $\|b - a\| \le \varepsilon_n \le \varepsilon_{n_0} < \operatorname{dist}(a, X \setminus B_m)$, we have $b \in B_m$. Also, $x_n \in B_m$ (in the other case, $b \in U_{x_n}^n \subset X \setminus B_m$ because $n \ge n_0 \ge m$). So the choice of U_x^n works. □

3.4 binormality via decomposition

Let X be a non-separable Banach space, and let μ be the first ordinal with cardinality $\operatorname{dens}(X)$. We call a transfinite collection $\{P_\alpha\}_{\omega \le \alpha \le \mu}$ of projections in X a *projectional resolution of identity (PRI)* if

- $\|P_\alpha\| \le 1$ for $\alpha \in [\omega, \mu]$,
- $\operatorname{dens}(P_\alpha X) \le \operatorname{card}(\alpha)$ for $\alpha \in [\omega, \mu]$,
- $P_\alpha P_\beta = P_\beta P_\alpha = P_{\min\{\alpha,\beta\}}$ for $\alpha, \beta \in [\omega, \mu]$,
- $P_\omega = 0$ and P_μ is the identity on X,
- $\alpha \mapsto P_\alpha x$ is continuous on $[\omega, \mu]$ for every $x \in X$.

If the first condition is weakened to $\sup\{\|P_\alpha\| : \omega \le \alpha \le \mu\} < \infty$, we obtain the notion of a *bounded projectional resolution of identity*.

Our main tool for proving that a non-separable Banach space is binormal follows.

Proposition 3.4.1. *Let X be a Banach space and let $\{P_\alpha\}_{\omega \le \alpha \le \mu}$ be a bounded PRI in X. If $(P_{\alpha+1} - P_\alpha)X$ is strongly binormal for every $\alpha \in [\omega, \mu)$, then X is strongly binormal.*

Proof. We will denote

$$X_\alpha = (P_{\alpha+1} - P_\alpha)X, \quad \alpha \in [\omega, \mu),$$

$$Z = \bigoplus_{\omega \le \alpha < \mu} X_\alpha,$$

$$x(\alpha) = (P_{\alpha+1} - P_\alpha)x, \quad x \in X, \alpha \in [\omega, \mu),$$

where the direct sum $\oplus$ is meant in the algebraic sense (so Z is the linear span of $\bigcup_{\omega \le \alpha < \mu} X_\alpha$). We take some $M > 0$ such that $\|P_\alpha\| \le M$ for any $\alpha \in [\omega, \mu]$. By the assumption, there is, for every $\alpha \in [\omega, \mu)$, a system of weak neighbourhoods $U^n_{x,\alpha} \ni x, x \in X_\alpha, n \in \mathbb{N}$, in X_α, such that

$$\bigcap_{n=1}^{\infty} \left(U^n_{x_n,\alpha} + \varepsilon_n B_{X_\alpha} \right) \neq \emptyset \quad \Longrightarrow \quad \{x_n : n \in \mathbb{N}\} \text{ is rel. compact}$$

whenever $\varepsilon_n \searrow 0$.

Since Z is dense in X, considering Lemma 3.3.2, it is enough to find appropriate neighbourhoods on Z. Put

$$\begin{aligned} U^n_x = & \bigcap_{\alpha \in S(x)} (P_{\alpha+1} - P_\alpha)^{-1}(U^n_{x(\alpha),\alpha}) \\ & \cap \bigcap_{\gamma \le \beta;\, \beta,\gamma \in S(x)} (P_{\beta+1} - P_\gamma)^{-1}(X \setminus (\|(P_{\beta+1} - P_\gamma)x\|/2)B_X) \\ \text{for} \quad & x = \sum_{\alpha \in S(x)} x(\alpha) \in Z, \quad n \in \mathbb{N}, \end{aligned}$$

where $S(x) = \{\alpha : x(\alpha) \neq 0\}$ is finite.

Let us prove that the choice works. Let $\varepsilon_n \searrow 0$, let $x_n, n \in \mathbb{N}$, be a sequence in Z and let $a \in X$ satisfy

$$a \in \bigcap_{n=1}^{\infty} \left(U^n_{x_n} + \varepsilon_n B_X \right).$$

To show that $\{x_n : n \in \mathbb{N}\}$ is relatively compact, we prove by induction on $\lambda \in [\omega, \mu]$ that $\{P_\lambda x_n : n \in \mathbb{N}\}$ is relatively compact. This is clear for $\lambda = \omega$ because then $P_\lambda x_n = 0$ for $n \in \mathbb{N}$.

Let $\lambda = \alpha + 1$ for some $\alpha \in [\omega, \mu)$ and let $\{P_\alpha x_n : n \in \mathbb{N}\}$ be relatively compact. We have to show that $\{P_\lambda x_n : n \in \mathbb{N}\}$ is relatively compact. It is sufficient to show that $\{x_n(\alpha) : n \in \mathbb{N}\}$ is relatively compact because $P_\lambda x_n = P_\alpha x_n + x_n(\alpha)$ for $n \in \mathbb{N}$. Let us verify that, for every $n \in \mathbb{N}$,

$$x_n(\alpha) \neq 0 \quad \Rightarrow \quad a(\alpha) \in \left(U^n_{x_n(\alpha),\alpha} + (2M\varepsilon_n) B_{X_\alpha} \right).$$

Assume $x_n(\alpha) \neq 0$, i.e., $\alpha \in S(x_n)$. Choose $b \in U^n_{x_n}$ satisfying $\|b - a\| \le \varepsilon_n$. We have $b \in (P_{\alpha+1} - P_\alpha)^{-1}(U^n_{x_n(\alpha),\alpha})$, and so $b(\alpha) \in U^n_{x_n(\alpha),\alpha}$. Since $\|b(\alpha) - a(\alpha)\| = \|(P_{\alpha+1} - P_\alpha)(b - a)\| \le 2M\|b - a\| \le 2M\varepsilon_n$, we get $a(\alpha) \in U^n_{x_n(\alpha),\alpha} + (2M\varepsilon_n)B_{X_\alpha}$, and the verification is completed. Now, for $n \in \mathbb{N}$, we put

$$y_n = \begin{cases} x_n(\alpha), & x_n(\alpha) \neq 0, \\ a(\alpha), & x_n(\alpha) = 0. \end{cases}$$

We obtain

$$a(\alpha) \in \bigcap_{n=1}^{\infty} \left(U^n_{y_n,\alpha} + (2M\varepsilon_n) B_{X_\alpha} \right).$$

Therefore, $\{y_n : n \in \mathbb{N}\}$ is relatively compact. As $\{x_n(\alpha) : n \in \mathbb{N}\} \subset \{0\} \cup \{y_n : n \in \mathbb{N}\}$, the set $\{x_n(\alpha) : n \in \mathbb{N}\}$ is relatively compact, too. The inductive step $\alpha \to \alpha + 1$ is finished.

Let $\lambda \in (\omega, \mu]$ be a limit ordinal number and let $\{P_\alpha x_n : n \in \mathbb{N}\}$ be relatively compact for every $\alpha \in [\omega, \lambda)$. We have to show that $\{P_\lambda x_n : n \in \mathbb{N}\}$ is relatively compact. It is sufficient, given an $\varepsilon > 0$, to find n_0 and a sequence x'_n such that $\|P_\lambda x_n - x'_n\| < \varepsilon$ for $n \geq n_0$ and $\{x'_n : n \in \mathbb{N}\}$ is relatively compact. We show that the choice $x'_n = P_\alpha x_n, n \in \mathbb{N}$, for an $\alpha < \lambda$ so that

$$\|P_\lambda a - P_\beta a\| < \varepsilon/8, \quad \alpha \leq \beta \leq \lambda,$$

works. Fix such an α. We know that $\{P_\alpha x_n : n \in \mathbb{N}\}$ is relatively compact. It remains to find n_0 such that $\|P_\lambda x_n - P_\alpha x_n\| < \varepsilon$ for $n \geq n_0$. We choose n_0 so that $\varepsilon_{n_0} \leq \varepsilon/(8M)$. Let $n \geq n_0$ be given. If $S(x_n) \subset [\omega, \alpha) \cup [\lambda, \mu]$, then $P_\alpha x_n = P_\lambda x_n$, and so $\|P_\lambda x_n - P_\alpha x_n\| = 0 < \varepsilon$. Assume that $S(x_n) \cap [\alpha, \lambda) \neq \emptyset$ and denote by β and by γ the greatest and the least element of $S(x_n) \cap [\alpha, \lambda)$. We have

$$\begin{aligned} P_\lambda x_n - P_\alpha x_n &= \sum_{\nu \in S(x_n),\, \alpha \leq \nu < \lambda} x_n(\nu) \\ &= \sum_{\nu \in S(x_n),\, \gamma \leq \nu < \beta+1} x_n(\nu) = P_{\beta+1} x_n - P_\gamma x_n. \end{aligned}$$

Since $a \in U^n_{x_n} + \varepsilon_n B_X$, we can choose $b \in U^n_{x_n}$ satisfying $\|b - a\| \leq \varepsilon_n$. We have $b \in (P_{\beta+1} - P_\gamma)^{-1}(X \setminus (\|(P_{\beta+1} - P_\gamma)x_n\|/2)B_X)$, i.e., $\|(P_{\beta+1} - P_\gamma)b\| > \|(P_{\beta+1} - P_\gamma)x_n\|/2$. We obtain

$$\begin{aligned} \|P_\lambda x_n - P_\alpha x_n\| &= \|(P_{\beta+1} - P_\gamma)x_n\| \\ &< 2\|(P_{\beta+1} - P_\gamma)b\| \\ &\leq 2\|(P_{\beta+1} - P_\gamma)a\| + 4M\varepsilon_n \\ &\leq 2\|P_\lambda a - P_{\beta+1} a\| + 2\|P_\lambda a - P_\gamma a\| + 4M\varepsilon_n \\ &< 4(\varepsilon/8) + 4M\varepsilon_{n_0} \\ &\leq \varepsilon. \end{aligned}$$

The inductive step for a limit ordinal λ is finished. □

We say that a class $\mathcal{C}$ of Banach spaces is a *$\mathcal{P}$-class* if, for every non-separable $X \in \mathcal{C}$, there exists a PRI $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$ such that $(P_{\alpha+1} - P_\alpha)X \in \mathcal{C}$ for every $\alpha < \mu$, where μ is the first ordinal with cardinality dens(X).

There are several classes which are known to be $\mathcal{P}$-classes (see, e.g., [7]).

THEOREM 3.4.2. *Let $\mathcal{C}$ be a $\mathcal{P}$-class. Then every space in $\mathcal{C}$ is strongly binormal. In particular, every space in $\mathcal{C}$ is binormal.*

Proof. We prove by induction on the density of X that every $X \in \mathcal{C}$ is strongly binormal. If $\text{dens}(X) \leq \aleph_0$, then X is separable, and thus strongly binormal by Lemma 3.3.3. Let $X \in \mathcal{C}$ satisfy $\text{dens}(X) > \aleph_0$ and let every $Y \in \mathcal{C}$ with $\text{dens}(Y) < \text{dens}(X)$ be strongly binormal. Let μ be the first ordinal with cardinality $\text{dens}(X)$. There is a PRI $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$ such that $(P_{\alpha+1} - P_\alpha)X \in \mathcal{C}$ for every $\alpha < \mu$. The block $(P_{\alpha+1} - P_\alpha)X$ is strongly binormal for every $\alpha \in [\omega, \mu)$ because $\text{dens}((P_{\alpha+1} - P_\alpha)X) \leq \text{card}(\alpha) < \text{dens}(X)$. Now, X is strongly binormal by Proposition 3.4.1.

The second part of the statement follows from Lemma 3.3.1. □

3.5 EXAMPLES

EXAMPLE 3.5.1. *The space $C([0, \mu])$ is binormal for every ordinal μ.*

This can be proved directly from Proposition 3.4.1. We may assume that μ is an initial ordinal and that $\mu \geq \omega_1$ (recall that every separable Banach space is strongly binormal by Lemma 3.3.3). To define a suitable PRI, we take $P_\omega = 0$ and, for $\alpha \in (\omega, \mu]$, the projection

$$P_\alpha f(\nu) = \begin{cases} f(\nu), & 0 \leq \nu < \alpha, \\ f(\alpha), & \alpha \leq \nu \leq \mu \end{cases}$$

(then every block $(P_{\alpha+1} - P_\alpha)C([0, \mu])$ is strongly binormal — for $\alpha > \omega$, it is one-dimensional, for $\alpha = \omega$, it is isometric to $C([0, \omega + 1])$).

THEOREM 3.5.2. *Every Plichko space is binormal. Every dual to an Asplund space is binormal.*

For the definition of a *Plichko space*, see, e.g., [11]. For the definition of an *Asplund space*, see below.

Proof. We use Theorem 3.4.2. The class of 1-Plichko spaces is a $\mathcal{P}$-class by [11, Theorem 4.14]. Note that every Plichko space can be renormed to be 1-Plichko ([11, Theorem 4.16]). The class of duals to Asplund spaces is a $\mathcal{P}$-class by [3]. □

We say that a norm $\|\cdot\|$ is *locally uniformly rotund (LUR)* if $x_n \to x$ whenever $\|x_n\| \to \|x\|$ and $\|x + x_n\| \to 2\|x\|$. One may expect that every Banach space with a LUR norm is binormal because the norm and weak topologies coincide on the unit sphere. We are going to disprove this conjecture.

EXAMPLE 3.5.3. *There is a locally compact space T such that the function space $C_0(T)$ is Asplund and admits a LUR norm but it is not binormal.*

The presented example is the set

$$T = \Big(\bigcup_{n=1}^{\infty} \mathbb{N}^n \Big) \cup \mathbb{N}^{\mathbb{N}}$$

endowed with the coarsest topology in which $\{s \in T : s \subset t\}$ is clopen for every $t \in T$ (we write $s \subset t$ if s is an initial segment of t).

In fact, our space T is a tree. Function spaces on trees were widely studied in the article [6]. The fact that T is a tree is sufficient for $C_0(T)$ to be Asplund. By [6, Theorem 4.1], $C_0(T)$ has a LUR norm.

We denote by $\chi_{(0,t]}$ the characteristic function of the set $\{s \in T : s \subset t\}$. To show that $C_0(T)$ is not binormal, we put

$$F_n = \{\chi_{(0,t]} : n \leq \text{length}(t) < \infty\}, \quad n \in \mathbb{N}.$$

The sets F_n are closed because the functions $\chi_{(0,t]}$ form a discrete set. It is clear that $F_1 \supset F_2 \supset \dots$ and that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Considering Proposition 3.2.6, it is sufficient to prove the following claim. Note that the weak and the pointwise topologies coincide on the unit ball of $C_0(T)$ (this can be easily proved from [4, Theorem 12.28] which implies that the linear span of the Dirac measures is dense in the dual of $C_0(T)$).

CLAIM 3.5.4. *If $G_n \subset C_0(T), n \in \mathbb{N}$, are open sets in the pointwise topology such that $F_n \subset G_n, n \in \mathbb{N}$, then $B_{C_0(T)} \cap \bigcap_{n=1}^{\infty} G_n \neq \emptyset$.*

Proof. We construct a sequence $s_1, s_2, \dots$ of natural numbers such that

$$(s_1, s_2, \dots, s_{n+1}) \subset t \quad \Rightarrow \quad \chi_{(0,t]} \in G_n$$

for $n \in \mathbb{N}$. Choose $s_1 \in \mathbb{N}$ arbitrarily. Assume that $s_1, s_2, \dots, s_n$ are constructed. We have $\chi_{(0,(s_1,s_2,\dots,s_n)]} \in F_n \subset G_n$. There are finite $R \subset T$ and $\varepsilon > 0$ such that

$$\forall r \in R : |f(r) - \chi_{(0,(s_1,s_2,\dots,s_n)]}(r)| < \varepsilon \quad \Rightarrow \quad f \in G_n.$$

It is sufficient to choose s_{n+1} such that $(s_1, s_2, \dots, s_{n+1}) \not\subset r$ for any $r \in R$. Indeed, if $(s_1, s_2, \dots, s_{n+1}) \subset t$, then $\chi_{(0,t]}(r) \neq \chi_{(0,(s_1,s_2,\dots,s_n)]}(r)$ is possible only for r with $(s_1, s_2, \dots, s_{n+1}) \subset r$, and thus $\chi_{(0,t]}(r) = \chi_{(0,(s_1,s_2,\dots,s_n)]}(r)$ for every $r \in R$. Hence $\chi_{(0,t]} \in G_n$.

So the construction is done. Now, the function $\chi_{(0,s]}$, where $s = (s_1, s_2, \dots)$, belongs to G_n for every $n \in \mathbb{N}$. This proves the claim. ☐

3.6 ASPLUND SPACES AND w^*-BINORMALITY

A Banach space E is said to be an *Asplund space* provided every continuous convex function defined on a non-empty open convex subset D of E is Fréchet differentiable at each point of some dense G_δ subset of D.

A topological space (X, τ) is said to be *fragmented by a metric* ϱ if, for every $\varepsilon > 0$ and every non-empty $Y \subset X$, there is a non-empty relatively τ-open subset of Y of ϱ-diameter less than ε.

Further, a topological space (X, τ) is said to be *scattered* if every non-empty subset $Y \subset X$ has an isolated point in Y. In other words, (X, τ) is scattered if and only if it is fragmented by the discrete metric.

A metric ϱ on a topological space (X, τ) is said to be *lower τ-semicontinuous* if the set $\{(x, y) \in X \times X : \varrho(x, y) \leq r\}$ is closed in $(X, \tau) \times (X, \tau)$ for each $r \geq 0$.

We start with a separable reduction for non-fragmentability. The result may be known but we were not able to find a reference for it.

PROPOSITION 3.6.1. *Let (X, τ) be a compact Hausdorff space and ϱ be a lower τ-semicontinuous metric on X. If (X, τ) is not fragmented by ϱ, then there are an $\varepsilon > 0$ and a countable set $Y \subset X$ such that*

(1) *$\varrho(x_1, x_2) \geq \varepsilon$ whenever $x_1, x_2 \in Y$ and $x_1 \neq x_2$,*

(2) *$Y \cap U$ is infinite whenever $U \subset X$ is τ-open and $Y \cap U$ is non-empty.*

Proof. (cf. proof of [10, Lemma 4.4]) By the implication (d) $\Rightarrow$ (c) of [10, Theorem 4.1], there are an $\varepsilon > 0$, a τ-compact set $H \subset X$ and a continuous surjective mapping $p : (H, \tau) \to \{0, 1\}^{\mathbb{N}}$ with the inverse images of distinct points of $\{0, 1\}^{\mathbb{N}}$ separated by ϱ-distance at least ε.

By the Zorn lemma, we can take some minimal (in the sense of the inclusion) τ-compact set $K \subset H$ with $p(K) = \{0, 1\}^{\mathbb{N}}$. Let Σ be a countable dense subset of $\{0, 1\}^{\mathbb{N}}$. For every $\sigma \in \Sigma$, we choose some $x(\sigma) \in K \cap p^{-1}(\sigma)$. Let us verify that the choice

$$Y = \{x(\sigma) : \sigma \in \Sigma\}$$

works. The property (1) is an immediate consequence of the properties of p. Let us verify the property (2). Take a τ-open $U \subset X$ with $Y \cap U$ non-empty. From the minimality of K, we have $p(K \setminus U) \subsetneq \{0, 1\}^{\mathbb{N}}$. There are infinitely many pairwise distinct points $\sigma_1, \sigma_2, \dots \in \Sigma$ which are elements of the open set $\{0, 1\}^{\mathbb{N}} \setminus p(K \setminus U)$. Now, the points $x(\sigma_1), x(\sigma_2), \dots$ are pairwise distinct and they are elements of U. □

LEMMA 3.6.2. *Let (X,τ) be a compact Hausdorff space and ϱ be a lower τ-semicontinuous metric on X. If (X,τ) is not fragmented by ϱ, then there are $F_1 \supset F_2 \supset \dots$, ϱ-separable and ϱ-closed subsets of X with $\bigcap_{n=1}^\infty F_n = \emptyset$, such that $\bigcap_{n=1}^\infty G_n \neq \emptyset$ whenever $G_1, G_2, \dots$ are τ-open subsets of X with $F_n \subset G_n, n \in \mathbb{N}$.*

Proof. Let ε and Y be as in Proposition 3.6.1. Denote by $y_1, y_2, \dots$ the elements of Y (in such a way that every element of Y occurs exactly one time in the sequence $y_1, y_2, \dots$). We claim that the choice

$$F_n = \{y_n, y_{n+1}, \dots\}, \quad n \in \mathbb{N},$$

works. The sets F_n are ϱ-closed due to the property (1) and they are ϱ-separable because they are countable. Clearly, $\bigcap_{n=1}^\infty F_n = \emptyset$. Moreover,

$$Y \subset \overline{F_n}^\tau, \quad n \in \mathbb{N}.$$

Indeed, the set $Y \setminus \overline{F_n}^\tau$, being a subset of $\{y_1, y_2, \dots, y_{n-1}\}$, is finite, and so it is empty by the property (2).

Now, let $G_1, G_2, \dots$ be τ-open subsets of X with $F_n \subset G_n, n \in \mathbb{N}$. The sets $F_n, n \in \mathbb{N}$, are dense in $(\overline{Y}^\tau, \tau)$, so the sets $G_n \cap \overline{Y}^\tau, n \in \mathbb{N}$, are dense as well. Using the Baire theorem, we obtain $\bigcap_{n=1}^\infty G_n \cap \overline{Y}^\tau \neq \emptyset$. This proves the lemma. □

There is a connection between asplundness and w^*-binormality. We are ready to prove it now.

THEOREM 3.6.3. *For a Banach space E, the following assertions are equivalent:*

(i) *For every disjoint separable and closed $A \subset E^*$ and w^*-closed $B \subset E^*$, there are disjoint open $D \subset E^*$ and w^*-open $C \subset E^*$ with $A \subset C$ and $B \subset D$.*

(ii) *If $F_1 \supset F_2 \supset \dots$ are separable and closed subsets of E^* with $\bigcap_{n=1}^\infty F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \dots$, w^*-open subsets of E^*, such that $F_n \subset G_n, n \in \mathbb{N}$, and $\bigcap_{n=1}^\infty \overline{G_n} = \emptyset$.*

(iii) *E is an Asplund space.*

Proof. (i) $\Leftrightarrow$ (ii) This follows from Proposition 3.2.7.

(ii) $\Rightarrow$ (iii) Assume that E is not Asplund. Hence (B_{E^*}, w^*) is not fragmented by the norm ([2, Theorem I.5.2]). By Lemma 3.6.2, there are $F_1 \supset F_2 \supset \dots$, separable and closed subsets of B_{E^*} with $\bigcap_{n=1}^\infty F_n = \emptyset$, such that $\bigcap_{n=1}^\infty G_n \neq \emptyset$ whenever $G_1, G_2, \dots$ are relatively w^*-open subsets of B_{E^*} with $F_n \subset G_n, n \in \mathbb{N}$. This clearly disproves (ii).

(iii) $\Rightarrow$ (ii) There is a separable closed linear subspace M of E such that

$$\|f - g\| = \sup\Big\{|(f-g)(x)| : x \in M, \|x\| \leq 1\Big\}, \quad f, g \in F_1.$$

Indeed, we can take $M = \overline{\text{span}}\{x(f,g,k) : f,g \in P, k \in \mathbb{N}\}$ where P is a countable dense subset of F_1 and $x(f,g,k) \in B_E$ is chosen so that $|(f-g)(x(f,g,k))| > \|f-g\| - 1/k$. Denote by r the restriction map $r : E^* \to M^*, r(f) = f|_M$. By the choice of M, we have

$$\|f-g\| = \|r(f) - r(g)\|, \quad f,g \in F_1.$$

It follows that $r(F_1), r(F_2), \dots$ are closed in M^* and $\bigcap_{n=1}^{\infty} r(F_n) = \emptyset$. As E is Asplund, M^* is separable by [2, Theorem I.5.7]. So M^* is w^*-binormal (Lemma 3.3.3 and Lemma 3.3.1). There are $G_1' \supset G_2' \supset \dots$, w^*-open subsets of M^*, such that $r(F_n) \subset G_n', n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n'} = \emptyset$ (Proposition 3.2.6). Now, the choice

$$G_n = r^{-1}(G_n'), \quad n \in \mathbb{N},$$

works, as $\bigcap_{n=1}^{\infty} \overline{r^{-1}(G_n')} \subset \bigcap_{n=1}^{\infty} r^{-1}(\overline{G_n'}) = r^{-1}(\bigcap_{n=1}^{\infty} \overline{G_n'}) = \emptyset$. □

COROLLARY 3.6.4. *If the dual E^* of a Banach space E is w^*-binormal, then E is Asplund.*

Proof. The condition (i) in Theorem 3.6.3 is evidently weaker than w^*-binormality of E^*. □

One may ask whether the converse implication holds. Before proving that the answer is negative, we mention a positive result suggested by O. Kalenda.

Remark 3.6.5. It can be shown that E^* is w^*-binormal whenever E is an Asplund and weakly countably determined Banach space. To prove this, we can use the same method by which we proved Theorem 3.4.2 with the difference that we use the fact that the class of the duals to Asplund WCD spaces forms a $\mathcal{P}$-class with the special property that the projections are continuous with respect to the w^*-topology ([2, Theorem VI.4.3]).

EXAMPLE 3.6.6. *The space $C([0,\omega_1])$ is an Asplund space but its dual is not w^*-binormal.*

The space $C([0,\omega_1])$ is Asplund because $[0,\omega_1]$ is scattered ([4, Theorem 12.29]). To see that $C([0,\omega_1])^*$ is not w^*-binormal, it is sufficient to prove the following lemma. Indeed, the sets $F_1, F_2, \dots$ from the lemma form a counterexample to (ii) in Proposition 3.2.6 if we identify every point of $[0,\omega_1]$ with the appropriate Dirac measure (note that $[0,\omega_1]$ embeds topologically to $(C([0,\omega_1])^*, w^*)$ by this identification).

LEMMA 3.6.7. *There are $F_1 \supset F_2 \supset \dots$, subsets of $[0,\omega_1]$ satisfying $\bigcap_{n=1}^{\infty} F_n = \emptyset$, such that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$ whenever $G_1, G_2, \dots$ are open subsets of $[0,\omega_1]$ with $F_n \subset G_n, n \in \mathbb{N}$.*

Proof. Let us recall a definition first. We say that a set $S \subset [0, \omega_1)$ is *stationary* if $S \cap A \neq \emptyset$ for any $A \subset [0, \omega_1)$, unbounded and closed in $[0, \omega_1)$.

By the Fodor theorem [5], there are pairwise disjoint stationary sets $S_1, S_2, \dots \subset [0, \omega_1)$. We define

$$F_n = \bigcup_{i=n}^{\infty} S_i, \quad n \in \mathbb{N}.$$

Suppose that $G_n, n \in \mathbb{N}$, are open sets in $[0, \omega_1]$ for which $F_n \subset G_n, n \in \mathbb{N}$. We show that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$. Assume the opposite, i.e. that $\bigcap_{n=1}^{\infty} G_n = \emptyset$. If we denote $A_n = [0, \omega_1) \setminus G_n$, then we obtain $\bigcup_{n=1}^{\infty} A_n = [0, \omega_1)$. We have that A_n is closed and unbounded for some $n \in \mathbb{N}$. As S_n is stationary, we have $\emptyset \neq S_n \cap A_n \subset F_n \cap A_n \subset G_n \cap A_n = \emptyset$, which is a contradiction. □

THEOREM 3.6.8. *For a compact Hausdorff space X, the following assertions are equivalent:*

(i) *If $F_1 \supset F_2 \supset \dots$ are countable subsets of X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \dots$, open subsets of X, such that $F_n \subset G_n, n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} G_n = \emptyset$.*

(ii) *X is scattered.*

Proof. (i) ⇒ (ii) Assume that X is not scattered. It means that X is not fragmented by the discrete metric. Now, Lemma 3.6.2 disproves (i).

(ii) ⇒ (i) Assume that X is scattered. It means that $C(X)$ is an Asplund space ([4, Theorem 12.29]). If we identify every point of X with the appropriate Dirac measure, (i) follows straightforwardly from Theorem 3.6.3 (note that X embeds topologically to $(C(X)^*, w^*)$ by this identification). □

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