Charles University in Prague

Faculty of Social Sciences
Institute of Economic Studies

RIGOROSUS THESIS

Debt Contracts and Stochastic Default Barrier

Author: **Martin Dózsa**

Supervisor: **Prof. Ing. Karel Janda M.A., Dr., Ph. D.**

Academic Year: **2010/2011**

Declaration of Authorship

The author hereby declares that he compiled this thesis independently, using only the listed resources and literature.

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Prague, February 15, 2011

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Acknowledgments

This thesis could not have been written without Karel Janda who not only served as my supervisor but also helped in finding the proper topic, and guided me throughout the work. I am also heartily thankful to Jakub Seidler for his comments and advices, and to Karel Sladký for his help with the mathematical part of the model.

Abstract

This thesis focuses on the theory of asset pricing models and their usage in the design of credit contracts. We describe the evolution of structural models starting from the basic Mertonian framework through the introduction of a default barrier, and ending with stochastic interest rate environment. Further, with the use of game theory analysis, the parameters of an optimal capital structure and safety covenants are examined. To the author's best knowledge, the first EBIT-based structural model is built up that considers stochastic default barrier. This set-up is able to catch the different optimal capital structures in various business cycle periods, as well as bankruptcy decisions dependent on the state of the economy. The effects of an exogenous change in the risk-free interest rate on the asset value, probability of default, and optimal debt ratio are also explained.

JEL Classification	C73, G12, G32, G33
Keywords	credit contracts, stochastic default barrier, asset pricing, EBIT-based models, structural models

Author's e-mail	`martin@dozsa.cz`
Supervisor's e-mail	`Karel-Janda@seznam.cz`

Abstrakt

Tato práce se zabývá teoretickými modely pro oceňování finančních aktiv a jejich použitím při návrhu optimálních úvěrových smluv mezi dlužníky a věřiteli. V první části je popsán vývoj strukturálních modelů od základního Mertonova modelu, který byl dále rozšířen o defaultní bariéru a zasazen do prostředí se stochastickou úrokovou mírou. Práce dále pomocí teorie her hledá parametry optimálního zadlužení vzhledem k existenci dluhových kovenantů. Hlavní přidanou hodnotou práce je navržení modelu se stochatickou bariérou defaultu, který využívá EBIT jako stavovou proměnnou a který tak lépe zohledňuje aktuální makroekonomickou situaci při hledání optimální půjčky a rozhodování o možném úpadku. Práce také podrobně diskutuje následky exogenních změn úrokové míry na hodnotu aktiv, pravděpodobnost úpadku a optimální míru zadlužení.

Klasifikace JEL	C73, G12, G32, G33
Klíčová slova	dluhové kontrakty, stochastická bariéra úpadku, oceňování cenných papírů, modely založené na EBITu, strukturální modely

E-mail autora	`martin@dozsa.cz`
E-mail vedoucího práce	`Karel-Janda@seznam.cz`

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Acronyms

APR Absolute Priority Rule

DB Default Barrier

EBIT Earnings Before Interest and Taxes

FPT First Passage Time

FV Face Value of debt

GBM Geometric Brownian Motion

LGD Loss Given Default

M-M Modigliani-Miller

PD Probability of Default

PDE Partial Differential Equation

RR Recovery Rate

SDE Stochastic Differential Equation

TS Tax Shield

Chapter 1

Introduction

The present Rigorosus thesis is based on the author's Diploma thesis defended at the Institute of Economic Studies, Faculty of Social Sciences, Charles University in Prague in 2010. During the winter semester of the academic year of 2010/2011 an article has been written using the results of the original Diploma thesis. Being presented both within the university and on the 6th Biennial Conference 2010 of the Czech Economic Society, there was enough scope for the article to absorb the colleague's advices. An extra care has been taken on the remarks of the Diploma thesis opponent, especially with respect to the structure and grammar. The mentioned article is attached to the end of this Rigorosus thesis.

In the past decades financial markets rapidly gain on complexity due to an increased demand for risk diversification and hedging. A number of sophisticated instruments was developed that capture various aspects of price movements, correlations of assets, macroeconomical developments, and other changes that might affect the future income generated by the considered securities. The pricing of these securities was not sufficiently accurate using the traditional asset pricing models. In the search for new methods two different approaches appeared. One stream of literature (called the reduced-form approach) focused on finding a purely mathematical way of asset pricing, without the effort of finding any economical intuition behind the models. In contrast, the other group of academics studied the firm and its evolution. These, so-called structural models have an intuitive connection to the underlying economics, and therefore they can be helpful in understanding the reasons of price movements.

This work fits in the category of structural approaches. First it gives a brief overview to the development of these models, and proposes their extension to a stochastic interest rate environment. Second, it uses these models to examine the effects of parameter settings in debt contracts, and therefore gives a guidance for the design of an optimal credit contract that maximizes firm value. With the introduction of a stochastic interest rate environment, it is possible to consider the implications of the business cycle period on the optimal debt ratio, and—using stochastic default barrier—on the bankruptcy decision as well. Game theory is also invoked, therefore agency costs arising from asymmetric information are predicted and minimized with the help of safety covenants and properly chosen parameters.

The thesis is structured as follows: Chapter 2 reviews the literature of structural models. Starting with Merton's (1974) path-breaking article, it discusses the basis of this framework. Section 2.2 begins with the contribution of Black & Cox (1976), who modelled early bankruptcy by defining a default barrier, and continues with subsequent works of Leland (1994), Longstaff & Schwartz (1995), and Briys & de Varenne (1997). The mathematical properties of these models are described in the last two sections of the chapter. Chapter 3 focuses on the types and design of credit contracts, describing the reasons for debt financing and the wide range of available debt securities that might be used. Section 3.5 uses game theory analysis to describe how a credit contract should be specified to maximize the overall firm value, and to mitigate the agency costs arising from asymmetric information. Chapter 4 presents—to the author's best knowledge—the first EBIT-based model with stochastic interest rate and default barrier. The main advantages of this set-up are a self-consistent description of the cash flows to different claim holders, and the projection of the current macro-situation to the distribution of the firm's future earnings, and therefore to the value of the different assets (i.e. equity and debt). Chapter 5 concludes the findings and suggests areas where research should continue.

Chapter 2

Asset Pricing Models

The purpose of this chapter is to review asset pricing models developed during the last four decades. An important factor that favoured the development of these models is the availability of a sophisticated mathematical tool called stochastic calculus. It allows continuous time modelling and together with the idea of risk-neutral measure it is able to provide closed form solutions for pricing risky assets.

We live in a risk-averse world, where the price of an asset is also dependent on its riskiness (that is, on the volatility of its future returns), since investors price assets below their expected payoff if they bare some risk. However, the idea of a risk-neutral probability measure deals with this issue: it is possible to adjust the probabilities of future states for risk in a way that assets are priced at their expected values.[1] To derive this risk-neutral probability measure we need the assumption that market prices include all available information, since known fair prices are needed in order to create a measure that produces expected values equal to these fair prices. Furthermore this risk-neutral probability measure is unique if markets are complete.

Models that require the assumption that market prices incorporate all available information are called market information based models. They can be further divided to structural and reduced-form models.

Models representing the first category are based on Merton's (1974) framework that employs the option pricing theory presented by Black & Scholes

[1] The probability measure that reflects the true probabilities is called the physical measure.

(1973). In Merton's work a company defaults at the maturity of its debt if the value of its assets is below the sum of its liabilities. Default prior maturity is not possible. The subsequent models relaxed this assumption as well as others taken by Merton. The common attribute of these models is that they concentrate on the structural characteristics of a company, including asset volatility and financial leverage.

By contrast, reduced-form (aka hazard rate) models ignore structural characteristics, and treat bankruptcy as a possible exogenous event that is described as the first jump time of a point process, without trying to explain the reason of default. This approach was first proposed by Jarrow & Turnbull (1995) and later extended in several works, for example Jarrow *et al.* (1997), Madan & Unal (1998) or Duffie & Singleton (1999).

As our model demonstrated in Chapter 4 fits in the category of structural models, we will focus on the description of this approach in the following sections. Since the understanding of the original Merton's framework is crucial for following its extensions, we will start with its description.

2.1 Merton's Structural Model

In his pathbreaking pater, Merton (1974) paralleled the value of equity in a leveraged firm to a European call option on the firm's assets and used the option pricing theory developed by Black & Scholes (1973) to value it. A corresponding debt is a zero-coupon bond with finite maturity with a promised terminal payoff B. This rather simplified description has many unrealistic restrictions, however, because of its simplicity and new perspective Merton built the basics of the framework used in structural models.

A large and growing body of literature has relaxed one or more assumptions posed by Merton. Some of the most important extensions are: more complex capital structure and safety covenants (Black & Cox 1976), interest paying debt (Geske 1977), Bankruptcy costs and tax benefits (Leland 1994), short and long term debt types (Vasicek 1984), or stochastic interest rate (Longstaff & Schwartz 1995; Hull & White 1995; Briys & de Varenne 1997; Collin-Dufresne & Goldstein 2001).

The original framework's assumptions, mainly coming from the Black & Scholes (1973) option pricing theory are:[2]

(A.1) there are no transactions costs, taxes, or problems with indivisibilities of assets.

(A.2) there are a sufficient number of investors with comparable wealth levels so that each investor believes that he can buy and sell as much of an asset as he wants at the market price.

(A.3) there exists an exchange market for borrowing and lending at the same rate of interest.

(A.4) short-sales of all assets, with full use of the proceeds, is allowed.

(A.5) trading in assets takes place continuously in time.

(A.6) the Modigliani-Miller theorem that the value of the firm is invariant to its capital structure obtains.

(A.7) the Term-Structure is "flat" and known with certainty. I.e., the price of a riskless discount bond which promises a payment of one dollar at time τ in the future is $P(\tau) = e^{-r\tau}$ where r is the (instantaneous) riskless rate of interest, the same for all time.

(A.8) The dynamics for the value of the firm, V, through time can be described by a diffusion-type stochastic process with Stochastic Differential Equation (SDE)[3]

$$dV = (\mu V - C)dt + \sigma V dW \tag{2.1}$$

where μ is the instantaneous expected rate of return on the firm per unit time, C is the total dollar payout by the firm per unit time to either its shareholders or liability-holders (e.g., dividends or interest payments) if positive, and it is the net dollars received by the firm from new financing if negative; σ^2 is the instantaneous variance of the return on the firm per unit time; dW is a standard Gauss-Wiener process.

[2] The assumptions are written exactly in a way as Merton wrote them, except for the symbols used

[3] This process is called Geometric Brownian Motion. Its basic properties are described in section 2.3.

Suppose a security with market value, Y dependent on the value of a firm. More specifically, its price can be written as a function of the firm value V, and time t: $Y = F(V,t)$. The dynamics of this security can be formally written using a SDE as

$$dY = [\mu_Y Y - C_Y]dt + \sigma_Y Y dW_Y, \tag{2.2}$$

where μ_Y, C_Y, σ_y and W_Y and defined similarly as in (2.1). Using the stochastic equivalent of chain-rule, the so-called Itō's Lemma we also have:

$$\begin{aligned} dY &= F_V dV + \frac{1}{2}F_{VV}(dV)^2 + F_t \\ &= \left[\frac{1}{2}\sigma^2 V^2 F_{VV} + (\mu V - C)F_V + F_t\right] dt + \sigma V F_V dW, \end{aligned} \tag{2.3}$$

where subscripts denote partial derivatives, and the second equation comes from (2.1). Comparing terms in (2.2) and (2.3) we have

$$\mu_Y Y \equiv \frac{1}{2}\sigma^2 V^2 F_{VV} + (\mu V - C)F_V + F_t + C_Y \tag{2.4a}$$

$$\sigma_Y Y \equiv \sigma V \tag{2.4b}$$

$$dW_Y \equiv dW \tag{2.4c}$$

The last equation indicates that Y_t and V_t are perfectly correlated, as they are driven by the same stochastic parameter. This implies the existence of such linear combination of these securities that the resulting payoff is non-stochastic. Using this fact Merton constructed a portfolio of three securities V, Y and riskless debt in a way that the initial investment was zero.[4] He showed that for any security Y whose value can be written as a function of the firm value and time has to satisfy the following equation:

$$0 = \frac{1}{2}\sigma^2 V^2 F_{VV} + (\mu V - C)F_V - rF + F_t + C_Y \tag{2.5}$$

As we can see, F depends on the value of the firm, time, interest rate, the volatility of the firm's value, the payout policy of the firm and the payout policy to the holders of Y. It does not depend on the expected rate of return neither the risk preference of the investors. This is the result where the idea of risk-neutral valuation comes from. Also it should be noted, that the only

[4] For the details about the construction of this portfolio, and for the complete derivation of equation (2.5) see Merton (1974) pp. 451–452.

thing that distinguishes one security from the other (debt vs. equity) is a pair of boundary conditions.

For pricing a simple corporate bond Merton took four further assumptions:

(A.9) The corporation has two classes of claims, a single homogeneous class of debt and the residual claim, called equity.

(A.10) The firm commits to pay \$B to the bondholders at date T

(A.11) If the payment is not met at T, the bondholders immediately take over the company, and so the shareholders receive nothing.

(A.12) The firm cannot issue any new claims that are not junior to the original one nor can pay dividends or do share repurchase before T.

As it can be seen this set-up ensures no default prior to maturity. Using equation (2.5) for the value of the debt, D, setting $C = C_Y = 0$ in line with the assumptions and defining $\tau = T - t$, so thus $D_t = -D_\tau$ we can write

$$0 = \frac{1}{2}\sigma^2 V^2 D_{VV} + rVD_V - rD - D_\tau \tag{2.6}$$

Denoting the value of equity as E and using (2.1), we have $V = D(V,\tau) + E(V,\tau)$. As E and D are non-negative, we know:

$$D(0,\tau) = E(0,\tau) = 0$$

and also $D(V,\tau) \leq V$, that is for $V > 0$ we have the other boundary condition

$$D(V,\tau)/V \leq 1$$

As the payment is made exactly when $V(T) > B$, the initial condition for the debt at $\tau = 0$ is

$$D(V,0) = \min[V,B]$$

The function $D(V,\tau)$ can be found using (2.6) and the above boundary conditions using standard methods as separation of variables. However, as Merton noticed, the problem can be transformed to another, already solved. For the value of equity holds $E(V,\tau) = V - D(V,\tau)$, so the solution for equity is given by (2.5):

$$0 = \frac{1}{2}\sigma^2 V^2 E_{VV} + rVE_V - rE - E_\tau \tag{2.7}$$

with a corresponding initial condition

$$E(V, 0) = \max[0, V - B]$$

and the boundary conditions $E(0, \tau) = 0$ and $E(V, \tau)/V \leq 1$. This is identical to the equations for an European call option on a non-dividend-paying stock in the Black-Scholes option pricing model. The firm value corresponds to the stock price, the equity to the option value and B to the exercise price.

Therefore the equity price is

$$E(V, \tau) = V\Phi(d_1) - Be^{-r\tau}\Phi(d_2), \tag{2.8}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(V/B) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\ d_2 &= d_1 - \sigma\sqrt{\tau} \end{aligned}$$

and $\Phi(\cdot)$ is the cumulative standard normal distribution.

As $D = V - E$, the debt value can be expressed as

$$D(V, \tau) = V\Phi(-d_1) + Be^{-r\tau}\Phi(d_2) \tag{2.9}$$

with same d_1 and d_2 as in (2.8).

2.2 First Passage Time Approach

The original Merton (1974) model described in the previous section uses several assumptions that limit its practical implementability. One of the most unrealistic restriction is the impossibility of default before maturity. To solve this problem Black & Cox (1976) came with a set-up where default occurs if the firm value touches a threshold level. This level is called the Default Barrier (DB), and generally can be constant (Leland 1994; Longstaff & Schwartz 1995), deterministic (Black & Cox 1976; Leland & Toft 1996) or stochastic (Briys & de Varenne 1997; Collin-Dufresne & Goldstein 2001) function of time. Models with a DB not only explain early default, but are also able to produce a large variety of Recovery Rates (RRs) and therefore reflect more precisely factors as bond covenants, bankruptcy costs or taxes.

The name of the First Passage Time (FPT) models corresponds to the method how the default is described mathematically: since the evolution of the firm value is represented using a Geometric Brownian Motion (GBM), it is possible to transform the probability distribution of the default to the FPT of a Wiener process.[5] These models can be also divided to two groups in dependence on the determination of the DB: it can be set exogenously (Black & Cox 1976; Longstaff & Schwartz 1995), or be an endogenous result of an optimization process (Leland 1994; Ziegler 2004). This section gives a brief overview these FPT models in order to describe the background of our model presented in Chapter 4. The notation used throughout the section follows the one introduced in the description of Merton's model, unless it is explicitly defined otherwise.

2.2.1 Black and Cox Model

Black & Cox (1976) extended the original Merton (1974) framework to include several features of debt contracts, namely safety covenants, subordinated bonds, and restriction on asset sales. Since this chapter discusses basic asset pricing methods, only the introduction of a DB will be described.[6]

The evolution of the firm value is the the same as in Merton (1974), except a restriction that the continuous dividend payment received by the stockholders is a constant fraction of the firm value. Therefore equation (2.1) takes the form

$$dV = V(\mu - c)dt + \sigma V dW \tag{2.10}$$

with a constant $c = C/V$ representing the payout ratio received by the equity holders. Again, the short-term interest rate is assumed to be constant, and so the interest-rate risk is disregarded. The original case described in Black & Cox (1976) also assumes zero bankruptcy costs.

A safety covenant, that provides a right for the bondholders to force bankruptcy if the firm is performing poorly, is introduced. This poor performance

[5] A derivation that uses such transformation is described in section A.2

[6] For pricing of more complex capital structures and the issue of contractural design see the original work of Black & Cox (1976).

is signalled by the fall of the firm value under a time-dependent default barrier defined as $\bar{v}(t) = Ke^{-\gamma(T-t)}, t \in [0, T)$ for some constants $K > 0$ and γ. The creditors take over the firm as soon as the firm value hits this barrier. Consequently default could be triggered in two ways: prior to maturity (by reaching the threshold level) or at maturity, if the firm value was above the DB but is below the debt principal at T. To simplify the notation let us set the default barrier as one function:

$$v_t = \begin{cases} \bar{v}(t) & \text{for } t < T, \\ B & \text{for } t = T. \end{cases}$$

The default time τ is

$$\tau = \inf \{t \in [0, T] : V_t < v_t\} .$$

We also assume the following:

$$V_0 > \bar{v}(0)$$

$$Ke^{-\gamma(T-t)} \leq Be^{-r(T-t)}, \quad \forall t \in [0, T]$$

i.e. the firm is not in default initially and the default barrier (and hence the payment to the bondholder) is never higher than the present value of the principal amount. This holds also for $t = T$, therefore $K \leq L$.

Zero-Coupon Bond

In Merton's model the debt pricing function solved equation (2.6). The analogous Partial Differential Equation (PDE) for zero-coupon debt value with default barrier is

$$0 = \frac{1}{2}\sigma^2 V^2 D_{VV} + (r - c)V D_V - rD + D_t \tag{2.11}$$

with the boundary condition

$$D(Ke^{-\gamma(T-t)}, t) = Ke^{-\gamma(T-t)}$$

and terminal condition

$$D(V, T) = \min(V, B).$$

Equation (2.11) can be solved using the classical methods used for PDEs or with a probabilistic approach.[7]

Note, that similarly as the equity value in Merton (1974) corresponds to a call option, it corresponds to a down-and-out barrier option here. Using the in-out parity (i.e. the plain vanilla option price equals to the sum of down-and-out and down-and-in barrier options price, all having the same strike price, underlining asset, maturity and the last two having the same barrier as well), the equity has a lower value by the price of a down-and-in barrier option in the presence of a DB. As there are no bankruptcy cots, this value is transferred to the bondholder.

Perpetual Coupon Bond

A perpetual coupon bond has infinite maturity and continuous coupon payment at a constant rate c_D.[8] The net cost of the coupon is financed by issuing additional equity. Its price $D_{c_D}(t)$ equals

$$D_{c_D}(t) = \lim_{T\to\infty} E\left(\int_t^T c_D e^{-r(s-t)} \mathbf{1}_{\{s<\bar{\tau}\}} ds\right) + \lim_{T\to\infty} E\left(K e^{\gamma(\bar{\tau}-T)} e^{-r(\bar{\tau}-t)} \mathbf{1}_{\{t<\bar{\tau}<T\}}\right)$$

under risk-neutral probability measure with $\mathbf{1}$ used as a symbol for indicator function. Since the coupon payments are constant it is straightforward to define the default barrier constant as well, i.e. set $\gamma = 0$. With the assumption that dividends paid to equity holders are zero (that is $c = 0$) D_{c_D} can be written as[9]

$$D_{c_D} = \frac{c_D}{r}\left(1 - \left(\frac{\bar{v}}{V_t}\right)^{\alpha}\right) + \bar{v}\left(\frac{\bar{v}}{V_t}\right)^{\alpha}, \tag{2.12}$$

with $\alpha = 2r/\sigma^2$.

2.2.2 Leland's model

Leland (1994) extended the perpetual coupon bond model described above by incorporating bankruptcy costs and tax benefits. Now V is a variable for the "asset value" of the firm; the total firm value is V less the expected costs

[7] The solution of (2.11) can be found in Black & Cox (1976) p. 356

[8] Here we use the subscript D in order to distinguish this pay-out from c, which was the payout ratio to equity holders.

[9] For the mathematical derivation see Bielecki & Rutkowski (2002) p. 81 and the preceding calculations.

of bankruptcy plus the value of the tax shield. V follows the same diffusion process as in (2.10) with no dividend payments ($c = 0$):

$$dV = V\mu dt + \sigma V dW,$$

hence V is not affected by the financial structure of the firm, thus the difference between coupon payments and tax benefits is financed by equity dilution.

When bankruptcy occurs at level $V_t = V_B$ a fraction $0 \leq \omega \leq 1$ is lost as costs due to bankruptcy, and the debt holders receive the remaining $(1-\omega)V_B$ leaving the equity holders with nothing. The value of the bond can be written as

$$D_{c_D}(V_t) = \frac{c_D}{r}\left(1 - \left(\frac{\bar{v}}{V_t}\right)^{\alpha}\right) + (1-\omega)\bar{v}\left(\frac{\bar{v}}{V_t}\right)^{\alpha}. \tag{2.13}$$

Note that with $\omega = 0$ this is identical to (2.12). If we denote $p_B = (\bar{v}/V_t)^{\alpha}$ (2.13) becomes

$$D_{c_D}(V_t) = \frac{c_D}{r}(1 - p_B) + (1-\omega)\bar{v}p_B.$$

p_B represents the value of a contingent claim that pays \$1 when bankruptcy occurs, $\omega\bar{v}p_B$ is the present value of expected bankruptcy costs, and $c_D/r\,(1-p_B)$ is the present value of expected coupon payments. Consequently the value of the tax benefits is equal to:

$$TS = T_c\frac{c_D}{r}(1-p_B),$$

where T_c is the corporate tax rate.

The total value of the firm, denoted by $G(V_t)$ is therefore equal to

$$G(V_t) = V_t - \omega \cdot \bar{v} \cdot p_B + T_c\frac{c_D}{r}(1-p_B).$$

Since the total value of the firm is equal to the sum of its equity and debt value, the shareholders' claim can be found as

$$\begin{aligned} E(V_t) &= G(V_t) - D_{c_D}(V_t) \\ E(V_t) &= V_t - (1-T_c)\frac{c_D}{r}(1-p_B) - \bar{v} \cdot p_B. \end{aligned}$$

Intuitively the value of equity is equal to the value of firm's assets less the present value of expected coupon payments reduced by tax and the contingent claim on $\bar{v}$. Note that the value of equity is not dependent on the bankruptcy costs, and so that is paid in full by the bondholders.

2.2.3 Models with Stochastic Interest Rates

One of the shortcomings of the Black & Cox (1976) model is the assumption of constant and know risk-free interest rate. This restriction is relaxed in models with stochastic interest rates. Because our work[10] assumes stochastic interest rate as well, we will make a review of the relevant literature at this point.

Longstaff and Schwartz

Longstaff & Schwartz (1995) price corporate bonds reflecting both interest rate risk and credit risk using risk-neutral probability measure for both stochastic processes. The evolution of the short-term interest rate is inherited from the Vasicek (1977) model:

$$dr_t = (a - br_t)dt + \sigma_r d\tilde{W}_t,$$

and the firm-s value is driven by the

$$dV_t = V_t(r_t dt + \sigma_V dW_t^*)$$

SDE. As we can see the constant drift from the Leland (1994) model is replaced by the stochastically evolving short-term interest. Furthermore, following Longstaff & Schwartz we have the following properties:

- Browninan motions $\tilde{W}$ and W^* are correlated with the instantaneous correlation $\rho_{V,r}$.
- DB is represented as a constant threshold level $\bar{v}$.
- Recovery Rate (RR) is independent on the default time, proportional to the face value of the bond and paid out at maturity.
- $\bar{v} \geq B$, hence the debt is repaid in full if default does not occur prior maturity.[11]

[10] See Chapter 4

[11] In fact this inequality is not explicitly wrote down by Longstaff & Schwartz, however it is implicitly assumed.

- The firm has one or more debt classes with different recovery rates $(1-\omega_i)$, where ω_i is the writedown rate for the ith class. The seniority of the claims is already reflected in the writedown rates, and therefore does not play essential role.[12] It is natural to suppose the following relationship: $\bar{v} = \sum_{i=1}^{k}(1-\omega_i)B_i$ with B_i $(\sum_{i=1}^{k} B_i = B)$ representing the total face value of debt from the ith class.

It we define τ, the time of default in the traditional way, that is

$$\tau - \inf\{t \in [0,T] : V_t < \bar{v}\},$$

than the bond's payoff at T can be written as

$$D_i(V_T, T) = B(1 - \omega_i \mathbf{1}_{\{\tau \leq T\}}).$$

For finding an analytical solution of the bond value at time $t < T$ with given V_t there are basically two ways: by solving the fundamental PDE with the corresponding boundary and terminal conditions, or alternatively, by probabilistic approach. A closed-form solution however, according to the best knowledge of the author, has not yet been produced using any of them. For this reason—even if some quasi-explicit results can be obtained analytically—numerical computations are required in order to obtain the results of the model. Such computations were made by the authors as well as others (Lehrbass 1997; Collin-Dufresne & Goldstein 2001). A shortcoming of this model is, that it produces credit spreads close to zero for low debt maturities.

Briys and de Varenne

Briys & de Varenne (1997) submitted a model that addressed some restrictive features and assumptions of the then available literature. For example, the previously analysed Longstaff & Schwartz (1995) model cannot work with a default barrier that would be lower than the present value of the debt principal. Their work also assumes stochastic default barrier, as it is derived from the instantaneous short-term interest.

[12] Note that this set-up can easily catch Absolute Priority Rule (APR) violations.

The short-term rate dynamics follows the so-called generalized Vasicek model, which is a mean-reverting stochastic function:

$$dr_t = a(t)(b(t) - r_t)dt + \sigma(t)d\tilde{W}_t,$$

where $a, b, \sigma : [0, T] \to \mathbb{R}$ are known, deterministic functions. Consequently the price of a default-free zero-coupon bond, P follows the dynamics

$$dP(t, T) = P(t, T)(r_t dt + b(t, T)d\tilde{W}_t)$$

for some deterministic $b(\cdot, T) : [0, T] \to \mathbb{R}$. The firm value V is assumed to follow the process

$$\frac{dV_t}{V_t} = r_t dt + \sigma_V(\rho d\tilde{W}_t + \sqrt{1-\rho^2}d\hat{W}_t),$$

with constant $\sigma_V > 0$, and mutually independent Brownian motions $\tilde{W}$ and $\hat{W}$. The local correlation coefficient between the risk-free rate and firm value is $\rho = \rho_{V,r}$. If we denote $W^* = \rho d\tilde{W}_t + \sqrt{1-\rho^2}d\hat{W}_t$, it is visible that the firm value process is defined in the same fashion as in Leland (1994).

The DB is defined as the price of a default-free bond with the same maturity and some face value $K \in (0; B]$ not greater than the defaultable bond principal:

$$v_t = \begin{cases} K \cdot P(t, T) & \text{for } t < T, \\ B & \text{for } t = T. \end{cases}$$

The default time is, as usually,

$$\tau = \inf\{t \in [0, T] : V_t < v_t\}.$$

The payoff at default is dependent on τ: for $\tau < T$ the bondholders receive a $(1 - \omega_2)$ part of the remaining assets, whereas for $\tau = T$ this payoff ratio may be different, and is represented as $(1 - \omega_1)$. The remaining ω_1 respectively ω_1 part is lost as bankruptcy cost and/or paid out to equity holders (APR). The bond's final cash flow at T is therefore

$$D(V_t, T) = (1 - \omega_2)B\mathbf{1}_{\{\tau<T\}} + (1 - \omega_1)V_T\mathbf{1}_{\{\tau=T\}} + B\mathbf{1}_{\{\tau>T\}}$$

If the bond price volatility function $b(t,T)$ is known, than the price of a defaultable corporate bond can be derived as a closed-from solution:

$$D(t,T) = P(t,T) \cdot [B - D_1 + D_2 - \omega_2 R_2 - \omega_1 R_1], \tag{2.14}$$

where $F_t = V_t / P(t,T)$

$$\begin{aligned} D_1 &= B\Phi(d_1) - F_t\Phi(d_2), \\ D_2 &= K\Phi(d_5) - (F_t L/K)\Phi(d_6), \\ R_2 &= F_t\Phi(d_4) + K\Phi(d_3), \\ R_1 &= F_t\big(\Phi(d_2) - \Phi(d_4)\big) + K\big(\Phi(d_5) - \Phi(d_3)\big), \end{aligned}$$

with

$$\begin{aligned} d_1 &= \frac{\ln(B/F_t) + \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)} = d_2 + \sigma(t,T), \\ d_3 &= \frac{\ln(K/F_t) + \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)} = d_4 + \sigma(t,T), \\ d_5 &= \frac{\ln(K^2/(F_t B)) + \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)} = d_6 + \sigma(t,T), \end{aligned}$$

and

$$\sigma^2(t,T) = \int_t^T \left((\rho\sigma_V - b(u,T))^2 + (1-\rho^2)\sigma_V^2\right) du.$$

Let us analyse (2.14) here: $B - D_1$ corresponds to the Mertonian valuation (i.e. risk-free bond less put-to-default option), D_2 is associated with the value brought to the debt holders by the possibility of early default triggered by safety covenant. The last two terms, $\omega_2 R_2$ and $\omega_1 R_1$, are both positive,[13] and represent the costs of early default and default at maturity respectively. It is therefore clear that the bond's price is decreasing in ω_1 and ω_2.

2.3 Geometric Brownian Motion

As the reader have probably noticed, the Geometric Brownian Motion (GBM) is often employed in modelling financial assets and is included in all of the models described in this text. Consequently it is crucial to understand what GBM is, and it is beneficial to be familiar with its basic characteristics.

[13] See Bielecki & Rutkowski (2002) pp. 105–106

Geometric Brownian Motion is defined as

$$S_t = S_0 \cdot \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_T\right\}$$

and corresponds to a SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$, where S_0 is an arbitrary initial value, μ is called the (percentage) drift , σ is the (percentage) volatility and W_t is a Standard Wiener Process. It has the following properties:

$$E(S_t) = S_0 \cdot e^{\mu t}$$

$$Var(S_t) = S_0^2 \cdot e^{2\mu t}(e^{\sigma^2 t} - 1)$$

As the Wiener process is symmetric with respect to the origin, we have

$$\forall p : P\left(S_t \geq S_0 \cdot \exp\left\{(\mu - \frac{\sigma^2}{2})t\right\} \cdot p\right) = P\left(S_t \leq S_0 \cdot \exp\left\{(\mu - \frac{\sigma^2}{2})t\right\} \cdot \frac{1}{p}\right)$$

and thus

$$P\left(S_t \geq S_0 \cdot \exp\left\{(\mu - \frac{\sigma^2}{2})t\right\}\right) = 0.5$$

that is, keeping the expected value constant and rising the percentage volatility σ the median is decreasing. If the earnings are divided in a way that one party (the borrower) earns on extremely high states and the extreme lows are suffered by the other one (the lender), as it is the case of credit contracts, it is clear that the expected earnings can be shifted if the volatility is changed. Actually, for one of the parties (the borrower) it is desirable to set a higher volatility even if the expected value decreases.

2.4 Alternatives to Geometric Brownian Motion

In the previous section we have seen the properties of the GBM. However, understanding the mathematical models is only a part of our way towards understanding how financial markets work. It is similarly important to realize the limitations of these models, and to recognize how their predictions differ from the asset prices observed on the markets.

A broadly employed method of testing and calibrating theoretical schemes is to use their computations backwards, that is to estimate the basic parameters of the model with the help of the observed market data. Applying this idea to the Black & Scholes (1973) model, it is possible to estimate the asset volatility: the resulting figure is called "implied volatility". It was shown (see, for example Dumas *et al.* 1998), that options with different strike prices and expirations have different Black-Scholes implied volatilities—this is the so called "volatility smile"—on the same asset. To explain this phenomenon, new, more sophisticated stochastic processes were introduced, that can be used instead of the relatively simple GBM.

2.4.1 Stochastic volatility

An easily observable weak point of the GBM modelling is the assumption of constant volatility: following the evolution of the asset prices on market, it can be noticed that there are time intervals with relatively small deviations from the trend line, and on the contrary there are periods with large movements (market crashes are a good example). Figure 2.1 shows the frequency distribution of SPX daily log returns from 1/1/1990 to 31/12/1999 compared with the Normal distribution. A high central peak and fat tails—that can be recognized in the comparison with the normal distribution—are characteristic for mixtures of distributions with different variations. These observations motivate to use models with stochastic volatility.

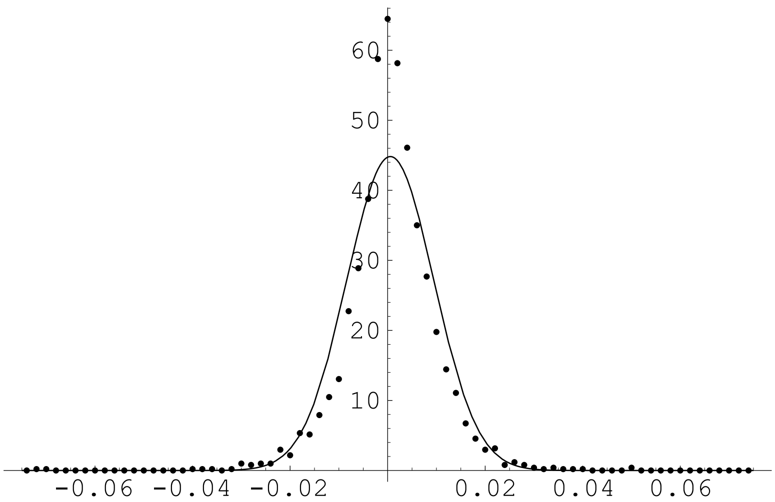

Figure 2.1: Frequency distribution of SPX daily log returns compared with the normal distribution
Source: Gatheral (2002)

Following Wilmott (1998)[14], we suppose that the stock price S and it's variance are described by the following SDEs:

$$dS(t) = \mu(t)S(t)dt + \sqrt{v(t)}S(t)dZ_1 \tag{2.15}$$

$$dv(t) = \alpha(S, v, t)dt + \eta\beta(S, v, t)\sqrt{v(t)}dZ_2 \tag{2.16}$$

with

$$\langle dZ_1\, dZ_2 \rangle = \rho dt$$

where $\mu(t)$ is the instantaneous drift of stock price, η is the volatility of volatility and ρ is the correlation coefficient between the Wiener processes Z_1 and Z_2 representing the stochastic factor of stock price returns and changes in $v(t)$.

2.4.2 Local Volatility

Since it is usually impossible to find closed-form solution for the stochastic volatility models and the numerical computation is complex, simpler models of option pricing were searched for. Breeden & Litzenberger (1978) derived a risk-neutral probability distribution function form market prices of European options. Later Dupire (1994) and Derman & Kani (1994) found a unique diffusion process consistent with these distributions using a state-dependent diffusion coefficient $\sigma_L(S, t)$ called local volatility function.

Even if the local volatility does not describe how the real volatilities actually evolve, it is a a good proxy representing some kind of average. It is rather a simplification that allows to price exotic options consistently with the observed prices of vanilla options.

2.4.3 Heston Model

A special case of stochastic volatility described in equations (2.15) and (2.16) is the Heston model (Heston 1993). It defines functions α and β as $\alpha(S, v(t), t) = -\lambda(v(t) - \overline{v})$ and $\beta(S, v, t) = 1$:

$$dS(t) = \mu(t)S(t)dt + \sqrt{v(t)}S(t)dZ_1$$

$$dv(t) = -\lambda(v(t) - \overline{v})dt + \eta\sqrt{v(t)}dZ_2$$

[14] Wilmott (1998), chap. 23, pp. 299–304

with

$$\langle dZ_1 \, dZ_2 \rangle = \rho dt$$

where ρ is the rate of reversion of $v(t)$ to its long-term mean $\overline{v}$.

Heston presented a "closed-form solution for options on assets with stochastic volatility".[15] It is able to incorporate the first four moments of the spot return compared the first two moments incorporated in the Black-Scholes model. In the case of at-the-money options the two models produce identical option prices. As the options are usually traded near-the-money, the Black-Scholes model is able to get empirical support. However, in the case of far-from-the-money options the two models predict significantly different prices. As bond (and debt) valuation corresponds to far-from-the-money option pricing, this is an important conclusion for the topic of this work.

[15] Heston (1993) p. 339

Chapter 3

Credit Contracts

This chapter explains the types of corporate financing, the reasons for issuing debt, and gives an insight to the design of credit contracts. The most basic issue of this design is the maximization of firm value and the prevention of unexpected losses in the contracting parties' claims. The answer to this problem is given using the tools described in the previous chapter, where we briefly introduced theoretical works that help us in pricing the two basic types of claims on the firm's assets: debt and equity.

The Modigliani-Miller theorem is the basic cornerstone of the corporate financing theory, therefore the following text starts with its explanation. Second, a wide range of debt types is listed and characterized in short, to demonstrate the available ways of financing. We continue with the explanation of the possible firm states that are given by the financial condition of the company, and with the description of the bankruptcy procedure. This knowledge will be employed in the closing section, where the game theory aspects of credit contracting are taken into account.

3.1 Capital Structure

The capital structure of a firm refers to the proportion of securities that ensure the needed funds for financing the firm's projects. These securities have two basic types: a riskier asset called equity and a relatively safe one, the debt. Equity has two further sub-groups (preferred and common), debt has many flavours, and furthermore there exists a group called "hybrid securities" including, for example convertible bonds. In this work we will concentrate on the two basic types only, however the model presented in Chapter 4 can be easily extended to more complex capital structures as well.

The value of the firm is therefore the sum of the market value of its debts and its equity: $V = D + E$. Proposition I of the Modigliani-Miller (M-M) theorem (Modigliani & Miller 1958) says that the market value of the firm is not dependent on its capital structure, if the following assumptions hold:

- There are no taxes
- The market is efficient (and consequently the bankruptcy costs are zero)
- Absence of asymmetric information

Therefore under these assumptions capital structure does not matter. On the contrary, when capital structure matters, at least one of the M-M assumptions is violated. Consequently the M-M assumptions can guide us in finding the determinants of an optimal capital structure.

The M-M theorem can be extended to an environment with taxes, where interest payments are a tax deductible item. The amount saved on taxes due to leverage is called the Tax Shield (TS) and can be expressed as $TS = T_C \cdot D$, where T_C is the corporate tax rate and D is the value of a perpetual debt. The tax shield is therefore increasing in the debt/equity ratio.

It was showed[1] that the second assumption is violated as well: financial distress and bankruptcy have direct and indirect costs, such as loss of costumers, suppliers, and employees due to uncertain future, need of immediate sale of assets at lower prices, expenses on experts, and so on. As higher leverage means higher interest payments and thus higher probability of not meeting them and falling into financial distress, the expected distress costs are increasing in with higher leverage. The effect on the overall firm value is therefore the opposite as for the tax shield.

Asymmetric information—i.e. the violation of the third assumption—implies agency costs, when the conflict of interest between different groups of stakeholders cause suboptimal investment decisions.[2] The typical examples of agency

[1] See Opler & Titman (1994), or Bris *et al.* (2006)

[2] More on this see, for example Ang *et al.* (2000)

costs are over-investment, under-investment, and cashing-out problem, all of them gaining in significance in states of (or close to) financial distress. The negative effects of agency costs are increasing in leverage, and therefore shifting the optimal indebtedness to lower values.

3.2 Classification of Corporate Debt

A company can choose several ways of debt financing, according to its size other corporation-specific needs. Corporate debt can be classified[3] by several attributes:

by ownership

public debt traded on open market

private debt, usually a loan provided by a bank or a group of banks

by security

unsecured debt

secured debt - specific assets are pledged as collateral

by seniority

senior debt with higher priority of claims if bankruptcy occurs

junior debt with lower priority

subordinated debt issued with lower priority than the outstanding debt at the time of issue

by residency of bondholders

domestic bond, with local issuer, traded in a local market, denominated in local currency

foreign bond, with foreign issuer, traded in a local market, denominated in local currency

Eurobond, denominated in foreign currency and not under national regulation

by rating

investment-grade bonds, graded by rating agencies (Standard & Poor's AAA-BBB) as bearing low risk of default

speculative bonds with high risk of default (graded by Standard & Poor's as BB-D)

[3] The structure of this section is based on Dedek (2009)

Whereas the first four classifications are determined in the moment of debt issue, the last one reflects the current situation, and therefore gives an estimate for the current value of a specific bond.

3.3 States of a Firm

A company with financial obligations (e.g. interest payments and payables) can be in several states, depending on its financial condition:

Solvent

The firm is able to meet its obligations on time.

Financial distress

The firm has difficulties to meet its obligations on time. In this state usually costs of distress arise. The situation can be handled through selling assets, equity dilution, merger, reducing capital spending, renegotiation of debt, or filing for bankruptcy.

Insolvency

The debtor is unable to meet its payment obligations.

Default

The debtor has not met its payment obligations.

Bankruptcy

Bankruptcy is legally declared inability of payment to creditors, usually initialized by the debtor itself. In the United States it is regulated by the Bankruptcy Code[4], where two chapters are available for corporate entities: protection under Chapter 11 or Chapter 7. Both types invoke automatic stay that provides a period of time in witch any type of debt collection is suspended.

Chapter 11 begins with filling a petition either by the debtor or by creditors, if they meet certain requirements. The debtor in possession receives the rights and powers of a Chapter 11 trustee. The trustee has the right, with the court's approval, employ professional persons to assist during the bankruptcy procedure as well as acquire new loans with higher priority than the outstanding debt. Some

[4] Source: the official web pages of United States Bankruptcy Courts

contracts (such as contracts with trade unions or leases) can be rejected if it is financially favourable to the entity.

After filling for Chapter 11 bankruptcy a plan of reorganization needs to be accepted by the creditors. During the first 120 days the debtor has exclusive right to file a plan of reorganization, after this period any of the creditors in interest has the right to file a plan. If the plan is confirmed by vote of creditors, it becomes binding and so determines the treatment of debts. A liquidating plan is also possible, which allows the debtor to liquidate the business with higher return as it would be possible under Chapter 7 bankruptcy.

Chapter 7 provides for liquidation. A trustee is appointed that administers the liquidation of the debtor's assets. After Chapter 7 liquidation the company ceases to exists and therefore stops to operate. In the case of larger entities complete divisions can be sold and so the recovered value might be higher.

3.4 Absolute Priority Rule

Absolute Priority Rule (APR) is a concept that describes how the assets should be divided among stakeholders after the event of bankruptcy. The basic order of the APR is, that a junior creditor receives some fraction of the remaining assets only in the case when senior creditors are paid in full. Similarly, equity holders receive nothing, unless all the creditors (both secured and unsecured) get the whole amount of their claim. Furthermore, when a class of stakeholders have the same seniority, they all receive the same ratio of their principal.

A considerable amount of literature[5] has been published on the violations of the APR: while under Chapter 7 liquidation absolute priority is generally enforced, in the case of Chapter 11 reorganizations[6] violation of APR is rather a rule than an exception. The reason is, that equity holders have the power to enforce APR deviation during workout negotiations due to the structure of Chapter 11 rules. The management can put the firm in Chapter 11 at a moment when it is in the best interest of equity holders. As there is an automatic stay

[5] See, for example, Meckling (1977), Miller (1977), Warner (1977), and Jackson (1986)

[6] See Franks & Torous (1989) and Weiss (1990)

on payouts to claimants under Chapter 11, a renegotiation could enhance the situation of both equity and debt holders. In addition, the reorganization plan needs to be accepted by the shareholders as well, and therefore they can prolong the bargaining process, and therefore increase the costs of default. This is clearly not in the interest of the senior claimants, and so they rather distribute some value to equity holders and avoid long negotiations.

A large amount of empirical research have been done in the past two decades about the consequences of these absolute priority violations, and the result showed that APR deviations are beneficial ex ante. They decrease the severity of over-investment in assets requiring managers' special skills and under-investment in firm-specific human capital (Bebchuk & Picker 1993), might improve the timing of bankruptcy (Povel 1999), hold back excessive risk taking (Gertner & Scharfstein 1991) and help to resolve under-investment problem (White 1989). On the other hand, negative effects of absolute priority violation arise through the problem of moral hazard with respect to investment decisions (Bebchuk 2002).

3.5 Game Theory Analysis of Credit Contracts

As a typical company of our interest has complex capital structure with many parties of interest, it is reasonable to examine the problem of financing from the perspective of Game Theory. This section is therefore dedicated to this topic, and is particularly based on the work of Ziegler (2004).

The method combines game theory and option pricing, so the maximized value of an option (note the parallel of options and credit contracts) can be calculated. The essence of the method is a three-step procedure:

1. The game between players is defined. The game tree is constructed.
2. The uncertain payoffs are valued using option pricing theory, where the parameters are the player's possible actions.
3. The game is solved using backward induction or subgame perfection.

The strengths of such a method are: taking into account the time value of money and the market price of risk, and separating the valuation problem from the analysis of strategic interaction.

3.5.1 Credit and Collateral

In financial contracting two forms of moral hazard occur: risk-shifting in the situation of hidden action, and observability problem in the situation of hidden information. In the following text these two basic problems are analysed, whereas more complicated issues will be addressed in the upcoming parts of the section.

The Risk-Shifting Problem

The origin of the risk-shifting problem is the borrower's incentive to influence the risk of the project, as he could increase his expected payoff on the expense of the lender. If he is able to change the risk of the project without the creditor's notice, we are talking about hidden action. The lender usually anticipates such behaviour, and requires higher interest rate that leads to adverse selection (see Stiglitz & Weiss 1981). An alternative solution is to closely monitor the activities of the borrower, however this increases the costs of lending and therefore the interest rate. The best option would be a contract designed in a way that the borrower has no incentive for risk-shifting without the need of monitoring.

Ziegler (2004) examined the situation when the borrower is able to set the riskiness of the project after the debt contract have been signed and the final payoff is observable to both parties with no cost. As it turned out, there exists an infinite number of contracts that preclude risk-shifting, however only contracts with proportional payout are renegotiation-proof (i.e. a situation, when a renegotiation is desirable for both the creditor and the debtor cannot occur). Renegotiation usually involves costs, and therefore both parties will have an incentive to agree on a contract that is not changed over its whole life. This means, that in the case of hidden action, only all-equity financing avoids risk-shifting.

The Observability Problem

When the terminal value of the investment is not observable by both parties, a problem arises how the final transfer should be determined. In fact, it can be expected in many situations, that the borrower will have more accurate information about the terminal value, and therefore he can report distorted figures to minimize his payout to the lender.

According to Townsend's (1979) costly state verification model—where the lender and the borrower agree in advance on situations when the verification should be taken—the optimal contract has the following properties (pure strategies allowed only):

- If verification does not take place, the payment to the lender is equal to some constant amount D.
- Verification should be taken when the terminal value is below some predefined threshold.

This contract is similar to a debt contract with fixed payment D and verification as a parallel to declaration of bankruptcy. Thus the observability problem can be addressed with constant promised payment in no-bankruptcy states. As risk-shifting can be solved only by proportional payment, there is no contract that could avoid both problems simultaneously.

Collateral

Collateral is an asset, that can be—according to the credit contract—seized in the event of default to limit the lender's losses. A considerable amount of literature has been published on the role of collateral in providing motivation for the borrower to avoid default. For instance, in Barro (1976), the loan repayment decision is dependent entirely on the relative values of the collateral and the amount of outstanding debt, default occurring if the value of the collateral at maturity is below the amount due. An inverse relationship between agency costs and the amount of collateral available to borrowers has been shown by Bernanke & Gertler (1989).

Chan & Kanatas (1985) mentioned two types of collateral: it is an existing asset (for example the financed project) or it is an additional asset, normally not available to the lender. Ziegler's model examines the effects of the latter, and concludes that risk-shifting problem disappears only when the loan is fully collateralized, resulting riskless loan. However, collateral protects the lender in two ways: grants higher recovery after bankruptcy and reduces the borrowers incentives to risk-shifting behaviour.

3.5.2 Endogenous Bankruptcy and Capital Structure

In the previous section the credit was a finite maturity contract with a single payment to the lender at maturity. Although such approach is good to understand project financing, it is less useful to model corporate financing. In reality firms keep operating by issuing new debt to finance their new projects, or to repay the maturing debt and therefore keep the ongoing projects alive. Bankruptcy happens, when the entity is unable to meet its contractual payments. In fact equity holders can decide at any point in time whether they want the firm to make the agreed payments or default and trigger bankruptcy. Thereby bankruptcy is an endogenous decision made by equity holders, even if it might be initiated in principle by the creditor.

Ziegler (2004) analyses endogenous bankruptcy building on the base of Leland's (1994) infinite horizon model with the introduction of several modifications. First, interest on the loan is divided to two distinct types, a continuous effective payment and an increase in the face value of the loan. This division allows to investigate the role of these two components in finding market equilibrium. Second, endogenous bankruptcy is discussed as a principal-agent problem and the agency costs of the equity holder's socially suboptimal behaviour are quantified. Third, the effect of loan covenants and information asymmetry are considered. Fourth, the properties of optimal capital structure are studied, and finally, an incentive contract is developed that could influence equity holder's bankruptcy choice.

The Model

A lender and a borrower signs the following contract: at initial time the lender transfers a loan of F_0,[7] and in exchange the borrower pays instantaneous interest of $\phi D(t)dt$, where $D(t) = D_0 e^{\kappa t}$ is the face value of the debt at time t and ϕ is the instantaneous interest rate to be effectively paid on the perpetual debt. Asset sales are prohibited, therefore net cash outflows on interest payments are financed by equity dilution. As κ is the rate of increase in the face value of debt (and therefore the rate of increase in interest payments as well), it is assumed, that $\kappa < r$, where r is the risk-free interest rate.[8] Sinking fund corresponds to the setting $\kappa < 0$.

[7] F_0 denotes the fair value of the loan at time 0, as it will be described in more details later.

[8] Otherwise the present value of the interest payments would converge to infinity.

If (and only if) the debtor defaults on his interest payments, the firm is liquidated with costs proportional to the asset value. The creditor therefore receives $(1 - \omega)S_B$ in the event of default, where ω is the proportion lost due to liquidation and S_B is asset value at the time of bankruptcy.

The game has the following structure:

1. The amount of debt, D_0, and interest rates κ and ϕ are determined, the contract is signed. In exchange for its promised obligations the firm receives the fair value of the loan, F_0.

2. The firm makes its investment decision with the associated risk, represented by the volatility rate, σ. In the financing of additional (later) projects under-investment problem might occur.

3. Equity holders choose their default strategy S_B. In the event of bankruptcy ωS_B is lost, $(1 - \omega)S_B$ is received by debt holders, and nothing remains to the equity holders.

The management is assumed to fully represent the equity holder's interest, hence there is no conflict of interest between these two parties. Ziegler (2004) assumes the asset value, S to follow the usual geometric Brownian motion, and estimates the firm, equity and debt value using the standard framework based on Merton.

In line with the principle of backward induction, the last stage of the game is examined at first. In this step the equity holders choose optimal asset level S_B for triggering bankruptcy. This level can be found using first-order condition, and is equal to

$$S_B = \frac{(1 - \theta)\phi D(t)}{r - \kappa + \sigma^2/2},$$

where θ is the corporate tax rate.

As it can be noted, this optimal level is linear in $\phi D(t)$, and is independent on current asset value S. Furthermore, higher asset risk (σ) implies lower optimal bankruptcy boundary.

The Principal-Agent Problem and Agency Costs

The principal-agent problem stems from the fact that the debtor (agent) adopts a different bankruptcy barrier than it would be optimal from the creditor's (principal's) view.[9] The creditor would choose a default boundary either to zero (to make his claim riskless) or as high as possible (to receive the firm's assets when they have a high value). The socially optimal bankruptcy strategy turns out to be the one with the lowest possible level of bankruptcy triggering, i.e. $S_B = 0$. This comes from the positive cost of bankruptcy for any asset value higher than zero.

In order to construct an incentive contract that would lead to socially optimal bankruptcy the effectively paid interest on debt, ϕ has to be zero, since for any other value the equity holders would trigger bankruptcy at a positive asset level. However, setting $\phi = 0$ means that the claim is worthless, as no interest is paid out. In other words, because of the borrower's limited liability, socially optimal default level can not be reached.

Armed with the above results the agency costs arising from endogenous bankruptcy can be expressed. The agency cost represents the expected deadweigth loss caused by the expected costs of bankruptcy. Intuitively, these costs are in direct relationship with the probability of bankruptcy (increasing in S_B and $\phi D(t)$), and with the proportional loss due to liquidation, ω.

The Investment Decision - Under-investment and Risk-shifting

Once we have investigated the equity holder's optimal bankruptcy decision S_B, we should examine their investment choices. Two main issues are studied in the following paragraphs: under-investment and risk-shifting. Myers (1977) highlighted that firms may abandon profitable projects in the existence of debt by refusing recapitalization of the firm. The reason of doing so is, that although equity holders would bear the full costs of the project, debt holders also benefit from this investment as the debt becomes less risky.

Ziegler (2004) analyses the under-investment problem with a model that represents new investment as a scale up of the existing operations by some

[9] The optimal default levels from the debtor's and the creditor's points of view are derived in Ziegler (2004), pp. 48–49.

factor $w > 0$. The investment requires therefore additional wS of funding and increases the value of the firm's assets to $(1 + w)S$. Since additional (equity funded) investment reduces expected bankruptcy costs and increases tax shield,[10] it always increases the overall firm value.

The model's calculated change in the value of the equity shows, that it is always lower than the costs of the investment, and therefore the overall return to equity holders is negative. Hence under-investment always arises. This problem can be addressed by renegotiation of the debt (reduction of D, ϕ, or κ) in order to ensure positive expected return on investment for the equity holders, or alternatively by sharing the costs of the new investment.

So far in the model of endogenous bankruptcy constant and know asset risk σ was considered, however in some cases this assumption might not hold. The question is, whether the agent has an incentive to increase the asset risk if the principal can not observe (and therefore control) his action. To answer this, Ziegler (2004) examined the partial derivative of the equity value with respect to σ^2. The result shows, that a leveraged firm has always incentives to increase asset risk. This has an implication for the optimal behaviour of the lender: he should focus on monitoring asset risk instead of asset value, as the risk is the relevant variable for the borrowers' bankruptcy decision.

Agency costs of risk-shifting can be expressed as a difference between the firm value at the social optimum less the firm value with the possibility of risk-shifting. Since firm value decreases with bankruptcy costs, it can be maximized by setting these costs to zero by approaching σ to zero. Agency cost is therefore equal to

$$C = \lim_{\sigma \to 0} W(S) - \lim_{\sigma \to \infty} W(S),$$

where, again firm value is W. As Ziegler showed, the difference in the above limits is

$$C = \frac{\theta \phi D(t)}{r - \kappa},$$

i.e. to the value of the (safe) tax shields.

[10] Note that early bankruptcy means no tax deductibility in the future, and therefore it decreases the current value of the tax shield.

Effects of Loan Covenants

It was shown in the previous sections, that under certain conditions, a "plain vanilla" debt contract[11] might imply deadweigth loss that moves the resulting firm value below its socially optimal level. To mitigate these losses, loan covenants might be introduced. A loan covenant is a condition agreed at debt issue that has to be fulfilled by the debtor. Covenants can take many forms, regulating operating activity, asset sale, cash payout and others. Here, so-called safety covenants are analysed which give the bondholder the right to force bankruptcy if certain conditions are met. More specifically, suppose a covenant that forces the firm into bankruptcy, if its asset value falls below some specified level $\overline{S_B}$. Reaching this level means transfer the ownership of the assets to the lender. As it turned out, the risk-shifting incentive depends on the level of this barrier: for low levels risk-shifting incentive is still present, however for higher values the situation changes and the debtor will have an incentive to decrease the risk of the investment. The breakpoint is naturally higher than S_B, the endogenous bankruptcy barrier set by the equity holders only.[12] Concluding the effects of such loan covenant, we should remark that they protect the lenders in two ways:

First, they reduce losses of the creditors by setting the default barrier higher, and

Second, they mitigate or even eliminate equity holder's risk-shifting incentives.

Hence, setting a safety covenant with an agreed level has similar effects as using collateral.

3.5.3 The Financing Decision

Using the results derived, we can investigate the way a firm should be financed. We will analyse—under endogenous bankruptcy—the optimal capital structure of a firm, and the effects of the way how the interest is divided between the interest effectively paid and growth rate in the face value of debt.

[11] Here "plain vanilla" refers to the absence of additional clauses defining loan covenants.

[12] For the mathematical derivation of this statement see Ziegler (2004), pp. 58–59.

Optimal Capital Structure

Assume that the asset risk is known to the lender and risk-shifting is not possible, or alternatively, it is possible only within certain bounds. In the latter case the lender would anticipate the borrower's risk-shifting behaviour, and therefore he will use the maximal available volatility value in his loan pricing calculations, $\bar{\sigma}$. We assume that the face value of the loan cannot be changed after the initial agreement, and that the borrower takes the offered interest rates κ and ϕ as given when selecting the initial face value of debt, D_0.

The financing decision is made with respect to the equity holders' effort to maximize the value of their holdings after the initial investment, I. Ziegler's calculations show, that there exists an interior maximum of the net equity value (that is the difference between the value of equity after the debt is taken and the equity holders' initial investment) in terms of optimal capital structure. As the rate of effective interest payments, ϕ rises—and consequently so does the cost of the debt service—the optimal face value of debt decreases. Similarly a higher growth rate in the face value of debt, κ, means lower optimal face value of debt. It also turns out, that changes in ϕ are perfectly offset by the endogenously chosen face value of the debt, and so the continuously paid coupon remains the same. Thus ϕ affects the nominal leverage (D_0/S_0), however it does not affect the leverage in market terms (F_0/S_0).

Interest Payments vs. Increase in the Face Value of Debt

A natural question is, how the debt service should be divided between the interest payments ϕ, and the growth rate of face value of the debt κ. As the optimal leverage in market terms is not affected by ϕ, the borrower is indifferent to the interest rate effectively paid. On contrary, the rate κ does affect the optimal capital structure and the net equity value: with increasing κ the optimal leverage ratio and the net equity value decreases. Consequently equity holders prefer to pay higher effective interest instead of higher growth in the face value of debt.

Expected Life of Companies

As the optimal capital structure and the conditions of the loan are given, it is possible to express the mean time of default. Using the analysis of Ingersoll (1987), we know that the mean time of passing the origin for a standard geometric Brownian motion $dx = \mu dt + \sigma dW_t$ with initial value x_0 is given by

$$\bar{\tau} = \frac{x_0}{\mu}$$

With the help of this formula—after some computations—the mean time of default under endogenous bankruptcy can be revealed[13]. This value turns out to be independent on the parameter ϕ, in line with the finding that the borrower offsets the changes in the effective payout rate by changing the face value of debt. Again, the important parameter is κ, that influences mean time to bankruptcy.

An Incentive Contract

It is worth to consider whether the lender can set the contract parameters ϕ and κ in a way that influences the borrower's bankruptcy strategy S_B. As it is in the lender's interest to have a higher default barrier, we will examine the possibilities of an incentive contract that induces the borrower to declare bankruptcy at a higher asset value. Early bankruptcy is interesting for the borrower for several reasons. First, the lender might be himself an agent and so he might have restrictions on the maximum he can take. Second, early liquidation may increase beliefs about the lender's solvency and therefore avoid some problems such as bank runs. Third, it enables the lender to save on monitoring costs as he can use early information provided by default on interest payments.

At first, the effective interest rate's influence on bankruptcy level is considered. As the optimal instantaneous coupon payment $\phi D(t)$ is independent on ϕ, in has no effect on time to bankruptcy either. However, it has influence on the nominal losses, and therefore it is possible to set ϕ to a hight level, and therefore imply low face value of debt. More specifically, zero nominal losses in the event of bankruptcy can be reached by setting ϕ in a way that $S_B = D(t)/(1-\alpha)$. Of course such a contract protects only in nominal terms, and has no effect on the losses in market value terms, as well as on the amount initially received by the borrower.

[13] See Ziegler (2004), pp. 67–68

Unlike the effective interest rate, the rate of growth in the face value of debt, κ does influence the borrower's optimal bankruptcy strategy. As a lower κ means faster debt repayment (through higher face value or equivalently higher ϕ), the resulting optimal bankruptcy triggering level is higher. It is important to note, that the rate of growth in debt affects the evolution of the default triggering level as well. Consequently, as time passes, this barrier will be lower in absolute terms as it would be with a higher κ. However, in relative terms $D(t)$ and S_B growths at a same rate, therefore this should be of no concern to the lender.

Chapter 4

The Model's Framework

Chapter 3 gave an insight to the design of credit contracts, and showed the usability of game theory in pricing of corporate assets and predictions of rational actions taken by the parties concerned. Here, we extend the available literature of asset pricing models introduced in Chapter 2, and build up a framework with stochastic interest rate. This framework than serves as a valuation method for a similar game theory analysis as was introduced in Section 3.5. The starting-point of this work is the Goldstein *et al.* (2001) EBIT-based model, that will be extended by the relaxation of the constant (or deterministic) interest rate requirement.

Sections 4.1–4.5 define the model and take the necessary assumptions. Sections 4.6–4.8 explain the basic implications of this model and compare this results with the available literature. Finally sections 4.9 and 4.10 demonstrate the contributions of a stochastic interest-rate environment, showing the added value of our construction.

4.1 Assumptions

First of all we take the following assumptions:

(i) The management fully represents the equity holders' interest.

(ii) The APR is never violated.

(iii) Asset sales are prohibited, interest payments are financed by earnings and equity dilution.

(iv) When the earnings are above the paid interest, the difference is paid out as dividend.

(v) Paid interest is a tax deductible item, however no tax carry-back or carry-forward exists.

(vi) There is a sufficiently large number of investors, and only a limited amount of projects.

Assumptions (iii), (iv), and (v) imply the unimportance of the historical cash flow in the asset pricing. The current values of the two memoryless processes—the risk-free interest rate and the EBIT—are the only two stochastic variables that affect the debt, equity and firm value. Assumption (vi) has the consequence that the provided loan is always fairly priced, since the financial institutions perfectly compete with each other. Next to these initial assumptions we will use further suppositions in the subsequent sections, particularly during the description of the stochastic evolution of the variables: the risk-free interest follows an Ornstein-Uhlenbeck process, the Earnings Before Interest and Taxes (EBIT) is supposed to follow a GBM, and so on.

4.2 Risk-free Interest Rate

Most of the models assume constant risk-free interest rate in order to simplify the calculation. However, in reality this interest rate does change in time, reflecting the situation of the overall economy. Modelling the interest rate stochastically allows us to include the possibility of a macro-level change and catch the correlation between the overall market and the modelled asset. Using this correlation the model could be extended to a risk averse measure, where higher return is expected just for the market risk—the one that can not be diversified (in line with Modern portfolio theory, see Markowitz 1952).

The risk-free interest rate $r(t)$ follows an Ornstein-Uhlenbeck process suggested by Vasicek (1977), and used by, for example in the Longstaff & Schwartz (1995) approach:

$$dr = \alpha(\gamma - r)dt + \sigma_r dW_t \tag{4.1}$$

where $\alpha > 0$ indicates the force pulling the interest rate back to its long-term mean γ at speed $\alpha(\gamma - r)$ per unit of time. The stochastic element is a standard Wiener process W_t times the volatility σ_r.

The expected value and variance at time s given $r(t)$ are

$$E_t[r(s)] = \gamma + (r(t) - \gamma)e^{-\alpha(s-t)}, \quad t \leq s \tag{4.2}$$
$$Var_t[r(s)] = \frac{\sigma_r^2}{2\alpha}(1 - e^{-2\alpha(s-t)}), \quad t \leq s$$

respectively. The distribution of $r(s)$ given $r(t)$, $t \leq s$ can be written as

$$r(s) = r(t)e^{-\alpha(s-t)} + \gamma(1 - e^{-\alpha(s-t)}) + \frac{\sigma_r}{\sqrt{2\alpha}} W_t(e^{2\alpha(s-t)} - 1)e^{-\alpha(s-t)}$$

Having the assumption of risk-neutral measure (i.e. the yield to maturity is not dependent on the maturity date and thus there is no risk premium), the value of \$1 received at time $s \geq t$ has the value of

$$P(t,s) = E_t\left[\exp\left\{-\int_t^s r(\tau)d\tau\right\}\right] \tag{4.3}$$

received at t. Vasicek (1977) gave a closed-form solution for the above expression:

$$P(t,s,r(t)) =$$
$$\exp\left[\frac{1}{\alpha}(1 - e^{-\alpha(s-t)})(R(\infty) - r) - (s-t)R(\infty) - \frac{\sigma_r^2}{4\alpha^3}(1 - e^{-\alpha(s-t)})^2\right],$$

where

$$R(\infty) = \gamma + \frac{\sigma_r^2}{2\alpha^2}.$$

Unfortunately we cannot use this solution, as the earnings are correlated with the interest rate[1], and therefore we can not simply discount by the expected value.

Figure 4.1 shows a possible evolution of the risk-free interest rate with different initial values. This evolution was simulated using our base values $\gamma = 0.03$, $\alpha = 0.25$, and $\sigma_r = 0.005$. As it can be noted, the effect of the initial value disappears in 10 to 15 years.

[1] See equation (4.4)

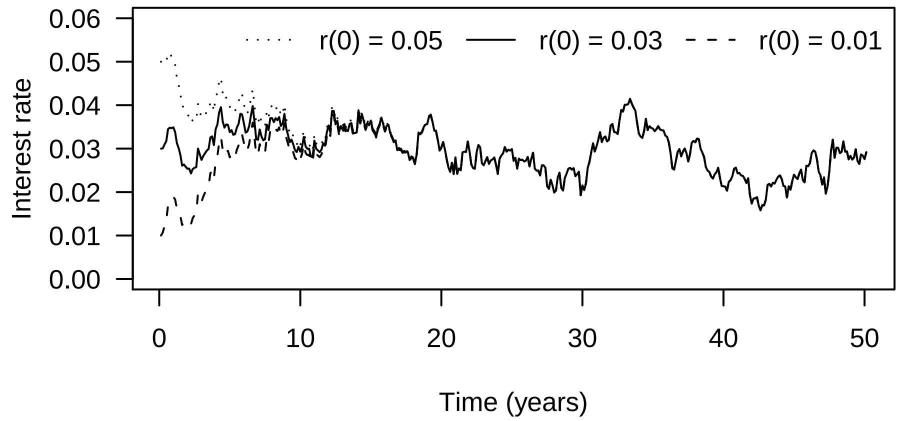

Figure 4.1: Interest rate evolution with different initial values

4.3 Earnings Before Interest and Taxes

Traditional models—building on the basis of Merton's (1974) framework, including those introduced in Chapter 2—take unlevered equity as primitive variable with log-normal dynamics. However, for some models it seems to be more straightforward to use earnings instead of unlevered equity. Mella-Barral & Perraudin (1997) considers a firm that produces output and sells it on the market, where the price of the sold product follows a geometric Brownian motion. Mello & Parsons (1992) use a similar framework with a mining company and stochastic commodity price movements. Graham (2000) models EBIT flow as a pseudo-random walk with drift, Goldstein *et al.* (2001) and Broadie *et al.* (2007) use geometric Brownian motion for the evolution of EBIT.

To see the advantages of such approach, we should review some of the main shortcomings of the traditional framework. **First**, unlevered equity ceases to exist as a traded asset when debt is issued. This problem is one of the motivating factors behind the frameworks of Kane *et al.* (1984; 1985) and Fischer *et al.* (1989). **Second**, they treat tax payments in a different fashion as they deal with cash flows to debt and equity holders. In fact, they count tax benefit as capital inflow instead of using it for reduction of outflows. This implicitly assumes that it is always possible to deduce fully the interest costs from the tax payments, however, this is not the case when the cost of debt service is higher than the current EBIT. Leland (1994) deals with this issue introducing an asset level under which there is zero deductibility. This is basically a hybrid approach that converts firm value to current EBIT, however it ignores partial deductibility. Another problem with the tax benefit approach is, that it implies higher firm value through higher tax shield as the tax rate increases. This is not only contra-intuitive, but it has been also found to be invalid by Lang & Shackelford (2000), who investigated the stock price movements during a decision process of change in capital gains tax. **Third,** as Goldstein *et al.* (2001) noted, these models may significantly overestimate the risk-neutral drift, consequently underestimate the probability of bankruptcy and so the optimal leverage ratio.

Our model assumes an EBIT evolution with log-normal dynamics, and therefore is able to address the mentioned issues. It abolishes the problem of unobservable and multiple (unlevered and levered) equity values, in treats the different claims (coupon payments, dividends and tax) in a self-consistent fashion and it is more flexible in implementing different set-ups, such as more sophisticated capital structures.

As mentioned, the evolution of the firm's instantaneous EBIT, δ_t is modeled using geometric Brownian motion with risk-neutral measure $\mathbb{Q}$, similarly as Broadie *et al.* (2007):

$$\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dX_t(\mathbb{Q}), \tag{4.4}$$

where

$$X_t = \rho W_t + \sqrt{(1-\rho^2)} Z_t.$$

W_t is the same process as in (4.1), Z_t is a standard Wiener process and ρ is the correlation coefficient between the risk-free interest rate and EBIT.

If the δ_t is known at $t = 0$, the differential equation (4.4) has the solution

$$\delta_t = \delta_0 \cdot \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma X_t \right\} \tag{4.5}$$

Assuming no taxes and zero leverage, the value of the firm is the sum of discounted earnings. Using the notation V_t^0 for unlevered equity value at time t,

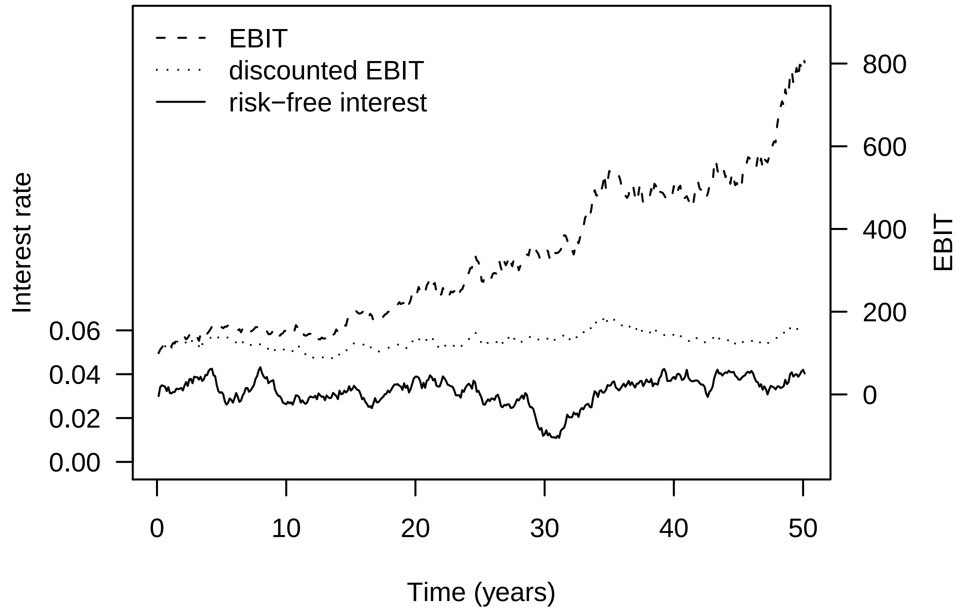

Figure 4.2: An example firm with high growth: the evolution of EBIT, discounted EBIT, and risk-free interest rate

we have

$$V_t^0 = \int_t^\infty \delta_t \cdot \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right)(s-t) + \sigma X_s - \int_t^s r(\tau)d\tau \right\} ds,$$

in line with (4.3).

Figure 4.2 plots an example of a high-growth firm; Figure 4.3, in contrast, is an example of a poor performance.[2] In the low performance firm, for instance, it is visible how between the 5th and 10th years the EBIT and the risk-free interest rate move together. It can be also noted, that the discounted EBIT is rather stable even if the firm performs well. In an average case—as it could be expected—it is decreasing.

[2] In order to produce telling plots, we have modified the basic parameters: $\mu = 0.02$, $\sigma = 0.1$, and $\rho = 0.4$.

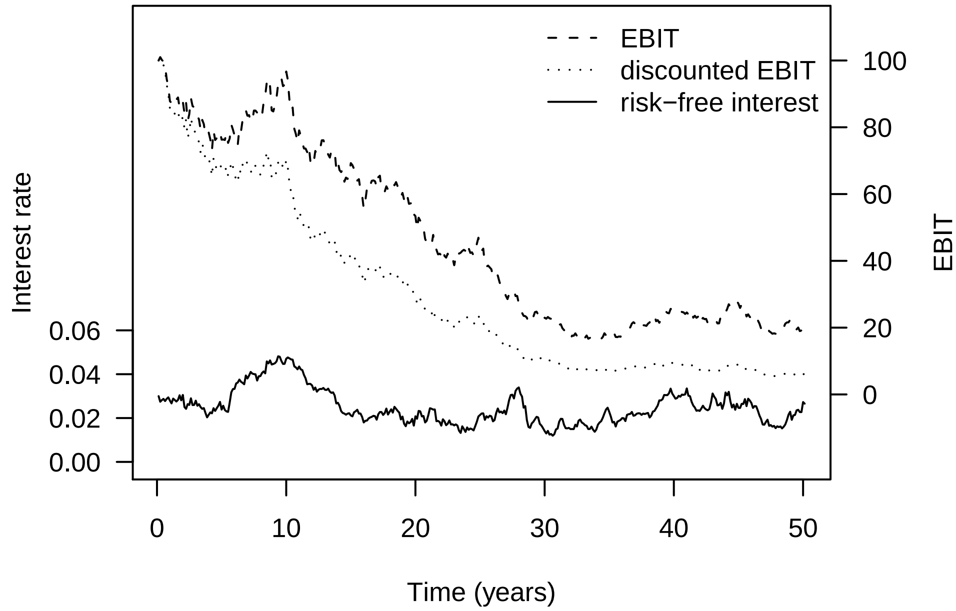

Figure 4.3: An example firm with poor growth: the evolution of EBIT, discounted EBIT, and risk-free interest rate

4.4 Debt

The debt issuance and repayment is similar as in Ziegler's (2004) model with endogenous bankruptcy, although several modifications are implemented. Most importantly, as the risk-free interest rate is considered to be stochastic, the interest payments are stochastic as well. Second, Ziegler considered a debt service divided between effective interest payments and growth in Face Value of debt (FV). As he proved that changes in effective interest rate are compensated by changes in face value of debt, its scalability will be left out from our model.

The debt is therefore set up it the following way:

1. The rate of growth in face value of debt, κ is chosen
2. The borrower (i.e. the firm) chooses the initial face value of debt, FV_0
3. The lender calculates the fair value of this debt, given the face value and κ, and provides a transfer to the borrower equal to this fair value.

Table 4.1: Notation

Symbol	Explanation	Base value
	Interest rate	
$r(t)$	Risk-free interest rate	$r(0) = \gamma$
γ	Long-term mean of risk-free interest rate	3%
α	Speed of expected risk-free interest rate convergence to γ	0.25
σ_r	The volatility of risk-free interest rate	0.5%
$P(t,s)$	The price of a \$1 face value riskless zero-coupon bond at time t, maturing at time s	
	Firm	
δ_t	EBIT	$\delta_0 = 100$
μ	Drift of EBIT under $\mathbb{Q}$	0.01
σ	Volatility of EBIT	20%
ρ	Correlation coefficient between $r(t)$ and δ_t	0.2
V^0	Firm value with no leverage and the assumption of zero taxes	
T_C	Corporate tax rate	35%
	Debt	
FV_t	Face value of debt	
κ	Growth rate of the face value of debt FV_t	1%
$D(\delta_t)$	Debt value	
c_t	Coupon rate, equals to $FV_t \cdot r(t)$	
	Default	
DB_t	Default Barrier	
τ	Time of default	
RR	Recovery rate defined as a multiple of yearly EBIT	$10\times$

After receiving the funds, the borrower starts to serve the interest payments. The FV at any point in time is given as:

$$FV_t = FV_0 \cdot e^{\kappa t}$$

The interest is continuously paid out at a rate $c_t = FV_t \cdot r(t)$ (coupon rate) with infinite horizon. We assume $\kappa < \gamma$, similarly as Ziegler, otherwise the discounted FV, and consequently the interest payments would growth to infinity.

The economic intuition behind this model is a floating coupon perpetual bond issue, where this corporate bond is (usually) sold below par. In order to catch constructions as a sinking fund, or alternatively a growth in debt principal, the parameter κ is introduced as well.

4.5 Default

The event of default corresponds to the situation, when the firm does not meet its obligation on interest payments. We assume, similarly as Ziegler, that creditors take over the firm immediately after the default. Such default is associated with losses (due to liquidation, reorganization or other costs associated with the takeover). Absolute priority rule is enforced, therefore after bankruptcy equity holders receive nothing, whereas creditors have to pay the (either Chapter 7 or 11) bankruptcy costs.

As the state variable is the instantaneous EBIT, it is convenient to define the recovery value as a multiple of the EBIT at the moment of default. Since a firm effectively becomes unlevered after bankruptcy (as its debt holders become the new equity holders), and we calculate the unlevered value during the iterations, this multiplier can be easily transformed to Loss Given Default (LGD)—a ratio that expresses the asset value lost due to bankruptcy.

4.5.1 Default Barrier

It is sensible to define the Default Barrier (DB) on the state (primitive) variable, since all the other values can be written as a function of this state variable. As we have an EBIT based model, DB will be defined on earnings. When the primitive variable is firm (or unlevered equity) value, DB is usually a function of the face value of debt, optionally with some other parameters involved as well (see Ziegler's 2004 Endogenous Bankruptcy model, or Briys & de Varenne 1997 for stochastic interest rate environment). A straightforward modification for our model is to make the DB dependent on the instantaneous coupon rate.

Such modification would imply a lower barrier in recession (low risk-free rate), and thus work counter-cyclically. There are several facts that support this design: in recession the number of bankruptcies increases (see, for example Altman *et al.* 2005), thus banks experience losses in connection with other loans and might prefer immediate payments instead of triggering bankruptcy that yields uncertain income later. Furthermore as Altman *et al.* (2005) also showed, the recovery rate is significantly lower in recession. Exactly the opposite holds for economic boom and high interest rates, therefore higher default barrier is reasonable.

For the above mentioned reasons (even if in our model the recovery rate is assumed to be constant and therefore independent from the risk-free interest rate) we decided to search optimal default barrier level as a linear function of the actual coupon rate c_t. To justify that this decision is consistent with our model, we have run simulations with a default barrier that is dependent only on the actual face value of debt, and therefore is not influenced by the interest rate. For the results of these simulations, see Section 4.10, where this deterministic default barrier is compared with the otherwise used stochastic barrier.

4.5.2 The Bankruptcy Decision

The decision of bankruptcy; i.e., the determination of the default barrier can be made in several ways. The concrete realization is dependent on the transparency of the firm, on the credit contract, and possibly on other factors.

When the state variable is not publicly observable, the firm's management (who represents the equity holders interests, as we assume no conflict of interest between these two parties) is the one who makes the decision whether to default on interest payments—and therefore trigger bankruptcy—or keep the equity holder's option on firm's assets alive. Note, that if the EBIT is not sufficiently large to cover the interest payments, they need to be financed through equity dilution (as asset sales are prohibited) in order to avoid bankruptcy. This is modelled as negative dividend, since it effectively lowers equity holders' payoff by diluting their claim.

On the contrary, when the state variable is observable, bankruptcy decision can be declared in the credit contract, and therefore support more favourable debt financing. This is in fact a safety covenant for the creditors, that ensures them the right to force bankruptcy if the firm performs poorly. This poor performance is indicated by crossing the DB in our case. The last, rather theoretical option is to set up a socially optimal default barrier, one that maximizes the aggregate payoff of all involved parties.

4.6 Method and Calculations

Since the model is so complex, that it is hardly possible to find closed form solutions to determine the values of the claims, the probability of default and other properties, we decided to use numerical calculations in order to uncover the model's sensitivity on its parameters. The core of the Monte-Carlo simulations is the following: after the parameters are set (see Table 4.1 for their base values) a large number[3] of iterations is run. Every iteration calculates a randomly[4] chosen EBIT trajectory and a correlated interest rate evolution. Following the realizations, the discounted sum of cash and asset flows is calculated for both debt and equity holders. In order to observe the payoffs' sensitivity on the DB and FV, several combinations of these parameters are examined in each iteration. Consequently every iteration produces matrices, where every matrix contains the result of one output parameter:[5] different rows correspond to different debt face values while different columns correspond to different default barrier levels. The generated matrices are then averaged and so the expected values are obtained. These results are then used as payoff valuation for game trees analysed in Section 4.7. A sample result matrix for equity values can be found in Table 4.2, for the complete output see Section B.1. An illustrative pseudo-code is presented in Section B.3.

Since our model has infinite time horizon, that cannot be calculated with the numerical approach, we had to approximate the results using finite number of years considered. We decided to encounter 150 years in our calculations, as the earnings in these first 150 years represent approximately 99% of the firm value.[6]

Because simulating 150 years would require time-consuming computations, we divided this time period into two parts: while the first 50 years are computed

[3] The number of iterations is set in a way to produce stable results. It is typically between 5,000 and 120,000, depending mainly on σ, the variance of the EBIT process.

[4] The probability distributions that drive the simulated random values are described in equations (4.1) and (4.4).

[5] These output parameters are: Debt payoff, Equity payoff, Total payoff, Debt ratio, and Default time (zero indicates no default).

[6] The discount of 150 years with constant 3% continuously compounded interest rate is $1/\exp(0.03 \cdot 150) \approx 0.011$. This is a rough estimate only, as the EBIT is expected to grow, and on the other hand default in the first 150 years is possible. Considering the calculated default rate, that is above 30% in the first 50 years even for firms with low leverage, the time horizon of 150 years is sufficiently high.

using high-precision simulations[7], the last 100 years are calculated using lower precision and then added to the first 50. Such division is faster for a given number of iterations, and produces results with smaller deviation, consequently a lower amount of iterations is sufficient.

4.6.1 The Effects of Debt Face Value

The Face Value of debt (FV) is the most basic parameter of a corporate loan: it is the figure that appears on the firm's balance sheet and in other reports and statistics. It is also the exclusive right of the borrower to specify the loan's FV directly or through the amount of borrowed funds. The main questions addressed in the following lines are, whether it pays off to issue debt at all, whether there is a maximal firm value an if so, what level of FV corresponds to this maximum, and how this optimal value is dependent on the DB.[8]

Since the obtained matrices contain a large amount of figures—and therefore it is hard to follow the key numbers—we use line charts to produce lucid output. Figure 4.4 illustrates the dependence of debt, equity and firm values on credit contracts with different face values.

As it is visible, when the leverage is low, firm value can be enhanced if a debt with higher face value is issued. The reason behind this observation is the increasing tax shield, in conformance with the theory known from corporate finance. However, after some point the rising bankruptcy costs offset and later exceed the growth rate of tax savings. Consequently there is an optimal face value of debt that maximizes the overall firm value. With a low DB[9] equal to 0.3, for example the firm value can reach 35 times the yearly EBIT if a debt is issued with face value between 20 and 30 yearly earnings. This means an optimal debt ratio of circa $60-80\%$. As the DB rises, this optimal ratio declines due to higher Probability of Default (PD): with $DB = 0.7$ the maximal firm

[7] Here precision refers to the sampling frequency of the generated Wiener processes. "High-precision" calculations are sampled every trading day (i.e. 250 times a year), "lower precision" calculations are sampled once per ten days (i.e. 25 times a year). The two methods produce similar results with small differences in the produced output.

[8] At this point we do not concentrate on the problem how the DB is chosen; that issue will be covered in Section 4.7.

[9] Recall that a default barrier of 0.3 means triggering default when the instantaneous earnings are at 30 percent of the coupon rate.

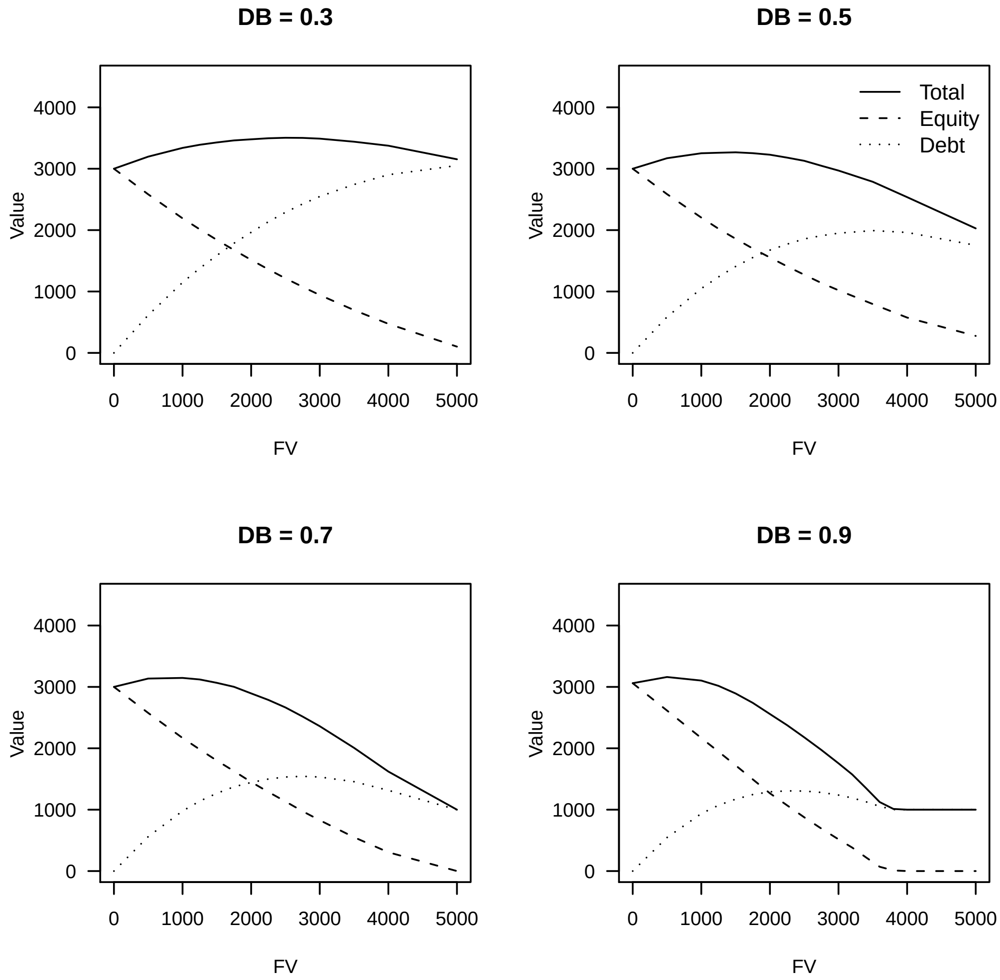

Figure 4.4: Debt, equity and total value with different face values of debt

value declines below 3200 (i.e. 32 times the yearly EBIT) with debt ratio of 30% only. The effects of changes in the DB are described in details in Section 4.6.2.

From the point of the debt value, there are two FV levels that might be interesting. The first, rather symbolic one is at which the bond value is equal to the par value. This equality is at approx. at 2000 for $DB = 0.3$, at 1500 for $DB = 0.4$ and at 1000 for $DB = 0.6$. Lower debt values are priced above par and vice versa. The second, and more important level of FV is where the debt value reaches its maximum: this is the highest possible amount of money that could be reached with debt financing only. Consequently this is the maximum reasonable FV of the debt contract, as higher values would increase the interest payments and the PD, but it would cut back the amount of money received.

The third examined output parameter is the equity value, which is a strictly monotonically decreasing function of the FV. This might be misleading, since the equity holders do not necessarily book loss with increase in debt: they are compensated with capital inflow from the creditors. To illustrate this, assume a simple example: an unlevered firm with value of $1000 issues debt in volume $400 and the obtained funds are paid out as dividends. The levered firm has the same assets, however, due to a tax shield it has a higher value, say $1100 with $400 debt and $700 equity value. Even if the nominal equity value has declined, the equity holders' payoff is

$$\$400 \text{ (dividends)} + \$700 \text{ (new value of equity)} = \$1100.$$

As the above example illustrates, it is in the equity holders' interest to sign a credit contract that maximizes the overall firm value. Intuitively, as the debt is fairly priced, the only party who could gain on debt issue is the equity holder.[10]

4.6.2 The Effects of Default Barrier Level

Next, we should explore how the output variables react on different levels of default barriers. To do so, we have plotted our basic calculation,[11] where no extreme values distort the picture. Figure 4.5 shows how the level of default barrier affects the equity, debt and overall firm value.

The overall firm value has the most unequivocal trend: it is declining as the barrier rises: the FV affects only the slope, not the tendency. Intuitively, setting the DB lower implies drop in the number of bankruptcies, later occurrence of the expected bankruptcy, and shrink of the LGD in absolute terms. Recall that the expected costs of bankruptcy equal to the product of these three factors: PD, LGD and the discount.

The value of debt is rising with lower DB level. Again, this is intuitive, since default occurs later, therefore more money flows to creditors through equity dilution. If we examine the curves of the debt value on Figure 4.5, a convergence in this value can be observed, as the DB rises. Because the initial EBIT is set to 100 and the base value of the RR multiple is 10, the debt value

[10] This holds only at the moment when the contract is signed. Later on both the debt and equity holders profit from an increase in the firm value.

[11] That is the one with parameters set to their base levels.

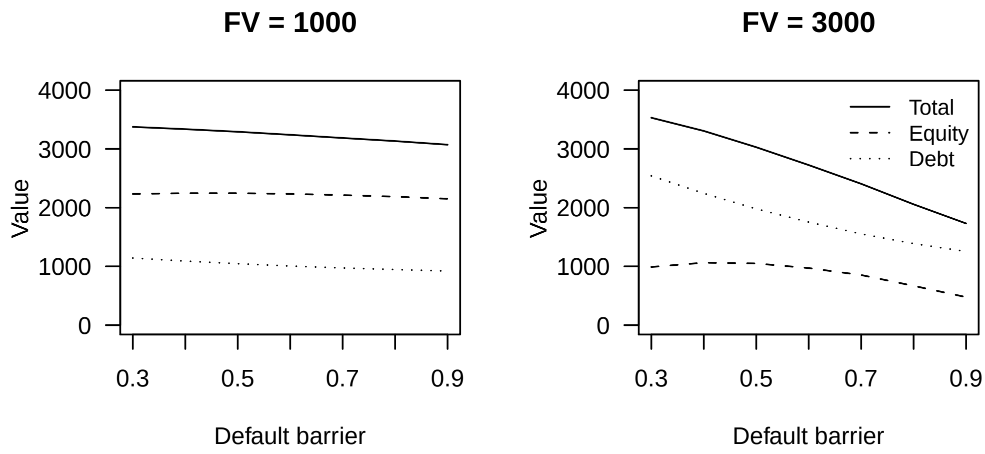

Figure 4.5: Debt, equity and total value dependence on the DB with FV 1000 and 2750

needs to be 1000 for sufficiently high DB that triggers default immediately. Consequently this needs to be the level where debt value converges to. In fact the same holds for a given DB and sufficiently high FV, as it can be seen on Figure 4.4.

The third curve—the one that demonstrates the equity value sensitivity on shifts in the DB—is somewhat different: it has a "quadratic" shape with a maximum around 0.5. This means that, from the equity holders' point of view, there exists an optimal non-zero default decision. This result is highly important for our game theory analysis in Section 4.7, where we examine the rational behaviour of the involved parties. This conclusion, as well as the results related to the firm and debt values, is in line with Ziegler's (2004) findings derived using closed form calculations in constant interest rate environment.

4.7 Agency Costs

4.7.1 Observable Actions

With observable actions, the creditor is able to control the parameters that affect the probability distribution of the EBIT flow, most importantly σ, which is determined by the riskiness of the firm's projects. This situation significantly simplifies the arrangement of the credit contract, since the lender does not need to study the set of possible actions that might be done by the debtor. In other words, the probability distribution of the payoffs is given, and therefore risk-shifting is not possible.[12]

Observable State Variable

The simplest situation is, when the firm is completely transparent, and therefore the creditor can observe the management's actions and also the state of the firm. In this case a debt contract can be signed with such covenants that enforce both an agreed volatility and defines a default barrier at which bankruptcy will be triggered.

In this case such a combination of debt face value and default barrier will be chosen that maximizes firm value. (In our basic calculation with results printed in Table 4.3, this corresponds to the setting $DB = 0.3$[13], $FV = 3000$, with total firm value of circa 35.5 yearly EBITs.) There is however, one natural limitation: logically, both the resulting equity and debt value need to be positive. This leads to a highly leveraged firm (to maximize the value of tax shield), and to low default barrier (to minimize the bankruptcy costs). Note, that it might be not always possible to specify an arbitrarily low DB: when the EBIT decreases so drastically, that the equity becomes worthless, it is not possible to finance the interest payments trough equity dilution. In a stock company the shareholders cannot be forced to transfer additional funds to the distressed firm. In contrast, when the considered firm is owned by a parent company, the interest payments can be guaranteed by the mother.

Not Observable State Variable

Similarly as in the previous case, actions are observable, and therefore risk shifting is not possible. However, as the state variable is not followed by the creditor, a bankruptcy barrier as safety covenant can not be included in the credit contract, because it would be impossible to enforce it. Consequently the debtor will choose the default barrier in a way that maximizes its equity

[12] More about risk shifting in the next section, where—in contrast with the present situation—it is possible.

[13] We did not calculate cases with even lower barrier. These would have produced higher total values, however it is hard to imagine that the firm would be kept alive with extremely low earnings. Furthermore there are usually some fixed assets owned by the company (immovable property, etc.) that cannot lose their values completely. Consequently the RR might be higher for firms with extremely low EBIT flow. Since we assume constant RR, we decided to leave out these extreme cases.

holders' value under the given circumstances. This decision is the bottom level of the game tree, and therefore it determines the expected payoffs under certain credit contract parameters. Table 4.2 shows an equity value matrix for several debt face values calculated using the base parameter setting.[14] As it can seen, the equity holders will choose to default on interest payments when the EBIT will be between 40 and 50% of the coupon rate (bold values in Table 4.2).

Table 4.2: Equity values - Basic parameters

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	3037	3037	3037	3037	3037	3037	3037
500	2618	**2623**	**2623**	2619	2614	2604	2593
1000	2233	**2246**	**2246**	2234	2213	2187	2151
1500	1881	**1904**	**1903**	1881	1848	1792	1727
2000	1558	**1595**	**1595**	1562	1504	1404	1281
2500	1264	**1319**	**1316**	1257	1162	1037	865
3000	990	**1063**	**1050**	971	853	669	477
3500	742	**837**	**813**	725	563	354	121
4000	510	**623**	**610**	483	304	73	0

Default barrier on the X-axis and debt face value on the Y-axis

As the lender anticipates the borrower's behaviour in the bankruptcy triggering decision, he prices the loan according to this action. We have discussed in Section 4.6.1, that the equity holders want to maximize the overall firm value, and so they will choose FV that implies this highest possible value. auto4.3 gives the valuation of this step in the game: the creditor offers loans priced according to the equity holders's default decision, therefore the equity holders' can choose total firm value only within the column specified by the planned (by shareholders) respectively assumed (by bondholders) DB. In this case the optimal face value of debt is 2000 for $DB = 0.4$ and 1500 for $DB = 0.5$. The corresponding firm values are 3400 and 3300 respectively.[15] The resulting total value, equal to 33–34 yearly EBITs is significantly higher than the unlevered

[14] See Table 4.1

[15] All these values are rounded: as we want to illustrate the decision process, the accurate numbers are not important. In real the DB is one number (between the mentioned 0.4 and 0.5) not an interval, and the FV that corresponds to the maximal firm value given this DB is determined unambiguously as well.

value with 30 EBITs only. On the other hand, the maximally possible 3550 is not reached due to agency costs caused by asymmetric information.

Table 4.3: Total firm values - Basic parameters

	0.3	**0.4**	**0.5**	0.6	0.7	0.8	0.9
0	3037	3037	3037	3037	3037	3037	3037
500	3234	3222	3207	3191	3175	3156	3137
1000	3375	3336	3292	3240	3186	3133	3072
1500	3471	3393	**3306**	3211	3118	3010	2897
2000	3525	**3404**	3270	3122	2966	2778	2582
2500	3543	3375	3173	2952	2714	2463	2179
3000	3531	3306	3029	2725	2406	2057	1731
3500	3499	3203	2839	2454	2049	1632	1203
4000	3429	3045	2613	2123	1650	1147	1000

Default barrier on the X-axis and debt face value on the Y-axis

Paradoxically, the equity holders' ex post effort to increase the value of their claim decreases the total firm value (and so their total payoff) ex ante. This problem can be solved if they manage to ensure the lender, that they will default on their payments when the EBIT truly crosses the DB. Such contract requires monitoring with some associated costs, however if these costs are below the agency costs then monitoring should be introduced.

4.7.2 Hidden Actions

When the management's actions are not observable, the debtor is able to modify the parameters driving the EBIT flow, and so to change the expected payoffs of the involved parties. More specifically, he is able to shift the risk to the creditor, and consequently to enhance the value of his claim on the creditor's costs. Such behaviour is called risk-shifting or, in a wider sense, moral hazard.

To demonstrate this problem, recall section 2.1, where we described how Merton (1974) proved that the value of equity in a leveraged firm can be expressed as European call option, and (using put-call parity) the value of debt is equal to a riskless bond with appropriate parameters less the value of a European put option. When the volatility of the asset's value rises, both options become more valuable, and therefore the equity value rises while the debt value declines. This model is valid only when there is no default prior debt maturity (and other assumptions made by Merton hold), however it illustrates the principle of risk-shifting.

To find out whether risk-shifting appears in our model, and if so, what are its consequences, we have run simulations[16] with several different EBIT volatility parameters. With higher σ values we observed the following (see Figure 4.6):

Equity value was rising, with steeper slopes for lower DB settings. In consequence the equity holders try to increase the EBIT volatility as much as they can, however they have a lower incentive to do so when the DB is higher. This means that if there are some additional costs of higher volatility paid by the equity holders[17], than they will not set the volatility to such high levels as they would so with lower DB.

Debt value was declining, however this decline was moderate for high DB settings. There are two reasons that support lower losses in debt value: First, and most importantly, default occurs at higher firm value, and therefore the firm has higher residual value after the bankruptcy that is transferred to the creditor. Second, default occurs earlier, therefore the asset value received has a smaller discount.

Probability of default rose.

Total firm value was decreasing due to increased PD.

Default barrier chosen by the equity holders was decreasing: their option on the firm's assets become more valuable with the increased volatility.

All of these observations are in line with the conclusions of Ziegler (2004), who based his analysis on game theory and gave closed-form results for his model with constant risk-free interest rate. Next we examine how the observability of the instantaneous EBIT affects the credit contract's design and the behaviour of the involved parties.

[16] For some of the results, see Subsection B.2.2.

[17] This could be lower expected EBIT growth, or some risk of being exposed, for example.

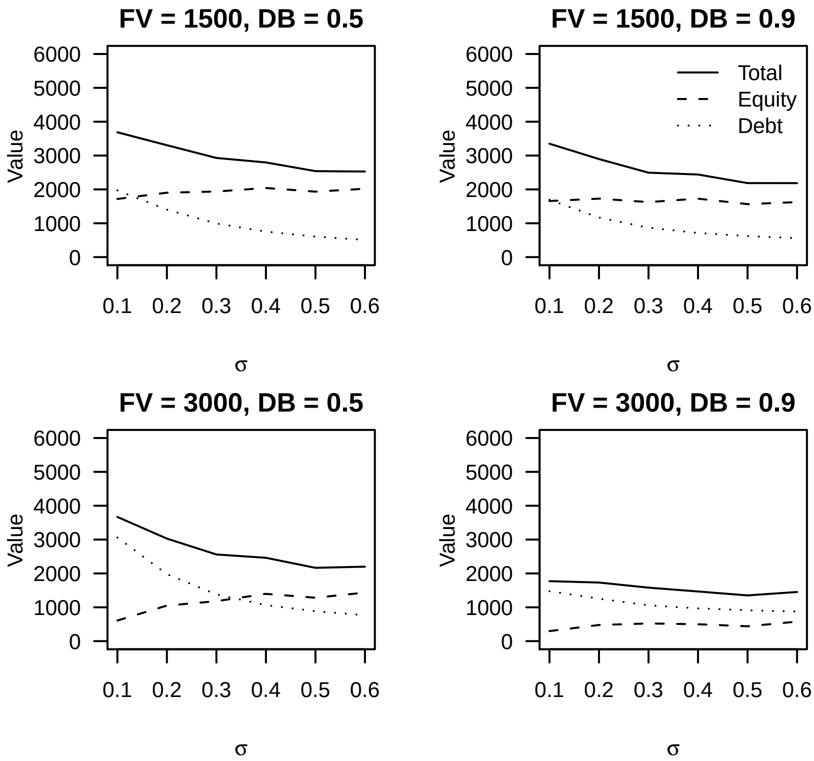

Figure 4.6: Firm value dependence on σ

Observable State Variable

If the state variable is observable, it is feasible to mitigate the equity holders' risk-shifting incentive by setting a sufficiently high DB as a safety covenant. For a better understanding of the mechanism of this safety covenant we extend the Mertonian parallel of the equity value and a European call option. After the introduction of an exogenous default barrier the European call option is replaced by a down-and-out call barrier option.

Such an option has a similar price as a plain vanilla option if the DB is far below the spot price, and the volatility is not extremely high. However, as the spot price approaches the barrier, the option values begin to significantly differ. Figure 4.7 shows[18] the prices of down-and-out barrier and plain vanilla call options as a function of the volatility, assuming a strike price 1000, barrier 900, constant risk-free interest 3% and time to maturity 1 year. As we can see, the equity holders' incentive to increase the volatility is mitigated when the firm value approaches the DB.

[18] Source: author's calculations using Financial Derivatives Toolbox

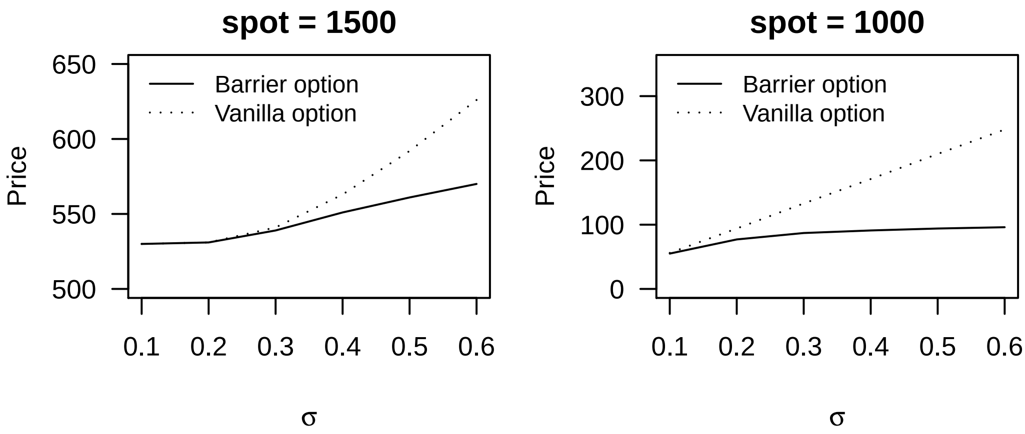

Figure 4.7: Barrier option price dependency on volatility, barrier 90% of strike

Our model shows a similar behaviour: when the DB is high (80–90% of the coupon rate), the equity value is not increasing significantly with higher volatility. A high DB can be used therefore as a safety covenant in order to avoid risk-shifting. This implies a loan with low FV (about 5 yearly EBITs in our basic setting; recall Figure 4.4), and consequently results a total firm value of only circa 3150 (31.5 yearly EBITs). Comparing this number with the theoretical maximum of a fully transparent firm (3550), the losses caused by risk-shifting are equal to the firm's four yearly earnings. Similarly as in the case of not observable state variable, it might pay off to introduce monitoring on the management's actions, and therefore to avoid risk-shifting.

Not Observable State Variable

If the state variable is not observable, equity holders will increase the EBIT volatility and default on interest payments later. Since the creditor anticipates such behaviour, he prices the loan with respect to higher expected volatility. Consequently the resulting firm value (as it is depicted in Figure 4.6) is lower than the value of the unlevered firm. The shareholders' ex-post behaviour therefore disables debt financing, and hence making the possible tax benefits unavailable.

4.8 Sensitivity on Parameters

In the following section we will investigate the reactions of the model to changes in different parameters. This is important for several reasons:

First, it helps to understand the model and its implications more properly.

Second, effects of possible or expected changes in macro environment can be predicted.

Third, it facilitates to ascertain the equity holders' incentives to change these parameters if it rises their expected payoff. Therefore effective safety covenants can be introduced to avoid risk-shifting and incentive contracts can be developed that mitigate motives to such behaviour.

4.8.1 Growth Rate in Face Value of Debt

The growth rate in face value of debt is denoted as the κ parameter, and next to the face value is one of the two exogenously set parameters that determine the interest payments and therefore present their value. To see the influence of κ on the rate of growth in coupon payments recall the formula that determines them:

$$c(t) = FV_t \cdot r(t) = FV_0 \cdot e^{\kappa t} \cdot r(t)$$

The only stochastic variable in this equation is $r(t)$, therefore using (4.2) we can express the expected value of $c(t)$ as

$$E[c(t)] = FV_0 \cdot (\gamma + (r(0) - \gamma)e^{-\alpha t}) \cdot e^{\kappa t}$$

and so

$$\lim_{t \to \infty} E[c(t)] = FV_0 \cdot \gamma \cdot e^{\kappa t}$$

The growth rate in expected values of interest payments thus converges to κ as time passes, however in the early years it is dependent on the initial risk-free interest rate level: with rate below the long term average (γ) the growth in expected interest payments is higher and vice versa. A precise calculation can be found in the appendix, Section A.1.

The value of κ also influences the probability of default: it occurs when

$$\delta_t \leq \overline{DB} \cdot \overline{FV_0} \cdot e^{\kappa t} \cdot r(t)$$

holds for the first time (as both sides are continuous, equality can be used as well). Assuming constant $r(t) = r$, the probability of default is[19]:

[19] For the derivation of this result see A.2

$$P\left\{\tau_b < \infty\right\} = \begin{cases} 1 & \text{if } \mu - \frac{\sigma^2}{2} \le \kappa \\ \exp\left[-2\hat{\mu}\hat{b}\right] & \text{if } \mu - \frac{\sigma^2}{2} > \kappa \end{cases} \tag{4.6}$$

where

$$\hat{b} = -\frac{\ln\left(\overline{DB} \cdot \overline{FV_0} \cdot r/\delta_0\right)}{\sigma}$$

$$\hat{\mu} = \frac{\left(\mu - \frac{\sigma^2}{2} - \kappa\right)}{\sigma}$$

and τ_b is the time of default.

As equation (4.6) shows, default is sure if $\mu - \frac{\sigma^2}{2} \le \kappa$ and is otherwise increasing in the DB, initial face value of debt, volatility and growth rate in face value of debt, κ.

On the other hand it seems to be reasonable to keep κ above some level: as the EBIT and the total firm value are supposed to growth at a rate μ, the leverage ratio is expected to decline for $\kappa < \mu$. Since the equity value, debt value and total firm value are homogeneous function of degree one with respect to the instantaneous EBIT, the optimal proportion of EBIT to debt face value is constant. The question is, what is the breakpoint of κ at which the gain from smaller expected distance to optimal leverage in the future is offset by increased probability of bankruptcy.

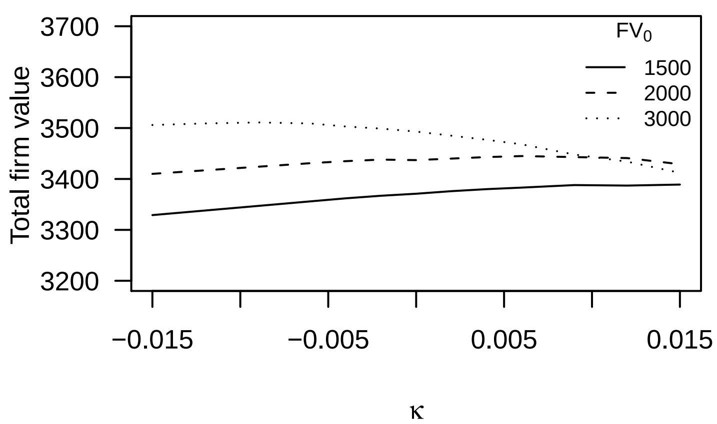

Figure 4.8: Firm value dependence on κ, $DB = 0.3$

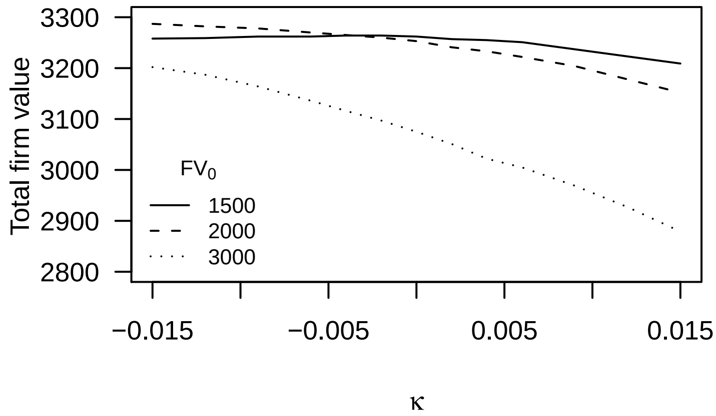

Figure 4.9: Firm value dependence on κ, $DB = 0.5$

We have run simulations in order to see the effect of changes in the κ parameter in our model. The results are consistent with the theoretical calculations and our expectations. Figures 4.8 and 4.9 illustrate the evolution of the firm value as κ changes for two different DBs and three different initial FVs. As it noticeable, the κ maximizing total firm value is lower for higher DB and higher initial FV. This means that when the other parameters are increasing the probability of default it is optimal to offset this by lower κ value. Consequently, for companies with high probability of default (due to high leverage, default barrier, volatility or any other factors) it pays off to establish a sinking fund.

4.8.2 Tax Rate

Debt financing in our model exists only because of the presence of a positive corporate tax:[20] interest payments are not taxed, and therefore debt financing creates a tax shield that increases the value of the firm. Consequently a higher tax rate implies an incentive for higher debt issue in order to reduce tax payments. The other natural effect of a higher tax is decrease in overall firm value. Even if the increase in tax payments is compensated by lower leverage (or by changing other parameters), the maximum firm value is lower in situations with higher corporate tax.[21]

[20] However, there might be other reasons for preferring debt financing to equity issue. An important example is the situation when the current owner wants to keep his full control over the company, however he has not enough funds to finance the ongoing or new projects.

[21] This statement can be easily proved: consider a firm with optimized parameters in a country with some given tax rate. If the tax rate suddenly decreases, the firm becomes more valuable due to the reduced tax burden. Later, when the firm optimizes its parameters for the new tax environment, the firm value will not be lower than without optimization.

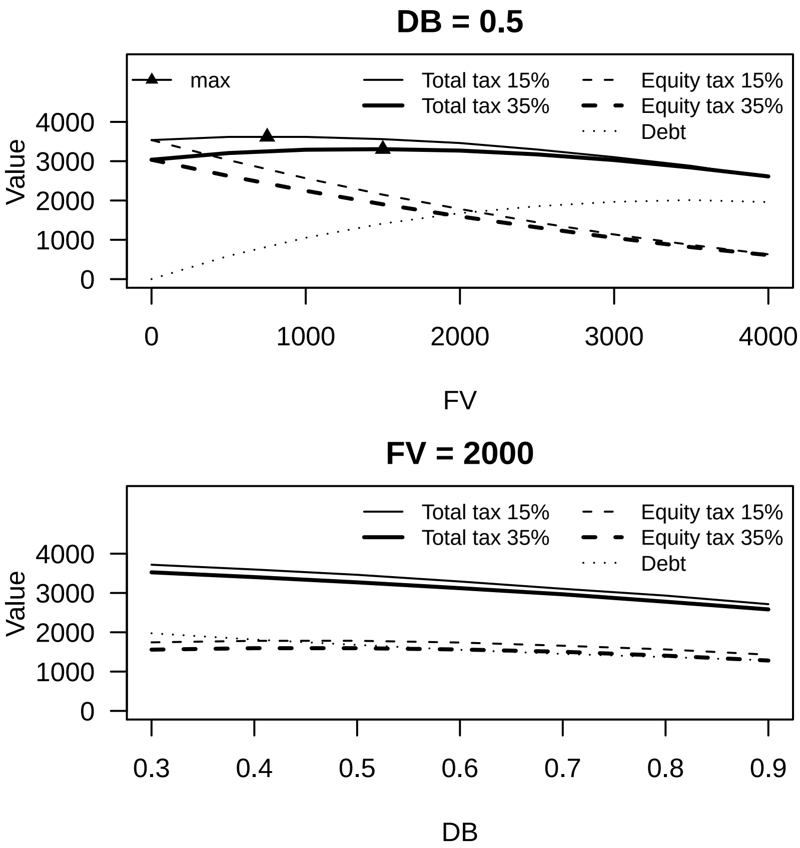

Figure 4.10: Firm value dependence on tax rate

Figure 4.10 depicts the sensitivity of firm, debt and equity values on the rate of corporate tax. The top chart suggests, that the results are in line with the theoretical expectations. The FV maximizing the firm value is increasing in the tax rate, whilst the maximal firm value decreases with higher tax. The bottom chart shows how the values of the involved parties react on a change in the corporate tax for different DBs. As it was demonstrated in Section 4.7, the peak in the equity value curve determines the bankruptcy level chosen by the debtor. This DB level is the same for both tax rates, and therefore the bankruptcy decision is not affected by the tax burden. This is an important reason, since the probability of default is not affected by a change in corporate tax rate, and therefore leaves the debt value unaffected. According to this, the debt is priced independently on the tax burden, as the debt value is the same for given FV and DB. This is the reason why there is just one curve for debt.

According to this two-step procedure the maximal possible value of a firm with lower tax rate needs to be higher.

4.8.3 Recovery Rate

In traditional models[22] the Recovery Rate (RR) represents the fraction of firm value that remains to the owners after the the costs of a bankruptcy are booked. In our case, however, it is more convenient to define RR as a multiple of the yearly EBIT at the moment of default in order to simplify the calculations. This can be done without the loss of generality, as our RR can be easily transformed to the classical one: the asset value is equal to the value of an unlevered firm, what is calculated in our simulations as an EBIT multiple. For example, our basic set-up has unlevered equity value 30 yearly EBITs (see Table 4.2), therefore $RR = 5$, $RR = 10$, and $RR = 20$ corresponds to "classical" recovery rate of 17%, 33%, and 66% respectively.

Since we assume no APR violations, the bankruptcy costs are born solely by the debt holders, similarly as in Leland's (1994) model. Consequently the value of the share holders' claim is independent on the RR whereas the debt and so the total value are increasing in RR.

Since equity holder's bankruptcy decision is not affected by the RR (as the value of their claim is independent on RR), we will not deal with changing DB for different RRs. On the contrary, the firm value is dependent of the RR, and so FV that maximizes its might be sensitive as well. To find out the optimal debt ratio dependence on the RR we have simulated firms with three different (5,10, and 20) recovery rates, all other variables leaving unchanged. The obtained values are plotted on Figure 4.11, with maximal firm values visualised. According to the calculated results, the optimal debt ratio declines as the RR decreases. This is an intuitive outcome: the expected costs of default are decreasing in the RR, and therefore (leaving the tax rate constant) the optimal FV shifts to higher levels.

4.9 Initial Interest Rate Level

An important advantage of the introduced mean-reverting interest rate environment is, that it can deal with a risk-free interest rate that is not on its long-term average (γ). In such case the interest rate is expected to return to γ, however, this takes some (random) time. In models with constant interest

[22] For example those presented in Chapter 2.

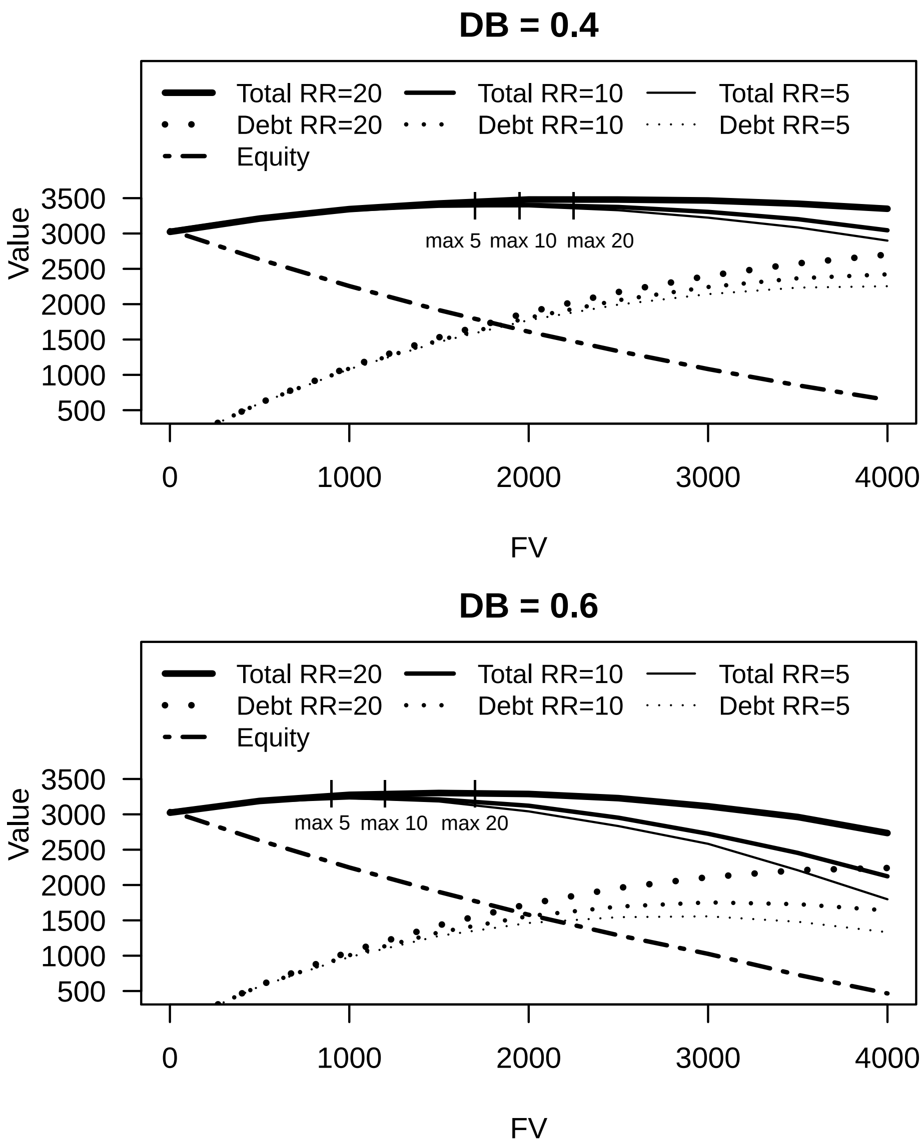

Figure 4.11: Firm value dependence on Recovery rate, $DB = 0.4, 0.6$

rate it is not possible to cover this situation. With a stochastic interest rate model though, it is just a question of different initial value $r(0)$ in the SDE (4.1). Furthermore, the effects of exogenous changes in this initial level can be examined. These exogenous changes in the risk-free interest rate correspond to the decisions of the central bank, and therefore we are able to predict the effects of the monetary policy on microeconomical level.

To see the effects of changes in the initial interest rate, we have run calculations with $r(0) = 1\%$, $r(0) = 3\%$, and $r(0) = 5\%$. Figure 4.12 demonstrates the obtained results for two different FVs. The tick lines show the total firm

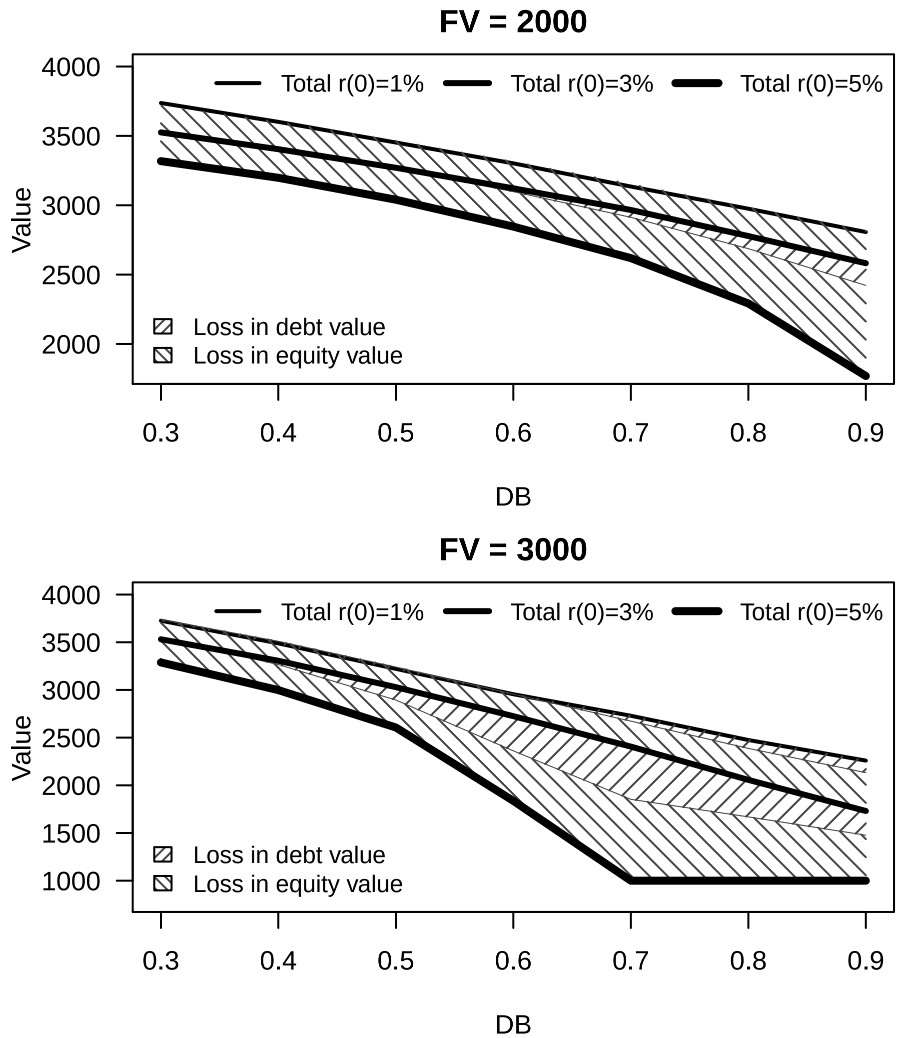

Figure 4.12: Firm value dependence on initial interest rate

value dependence on the DB for three different initial interest rate levels. The gap between these lines represent the loss—ceteris paribus—when the interest rate suddenly increases to the next examined level. This drop in firm value is caused by two factors: higher discount for all future earnings and increased PD due to higher interest payments.[23] The mentioned gap is a sum of declines in equity and debt value, and therefore we can divide this area to distinguish the losses of the two involved parties.[24]

[23] Higher interest payments imply higher DB in absolute terms. The DB of the x axis on Figure 4.12 is a ratio of the instantaneous interest payments.

[24] For the calculated debt and equity values, see Subsection B.2.1.

For a better understanding of the forces driving these changes, we have plotted the PD in the first 50 years[25] and the mean times of defaults happened before year 50 ($E[\,\tau \mid \tau < 50\,]$, where τ is the time of default, as usually). As it can be seen on Figure 4.13, both the PD and the expected time of default seem to be insensitive to changes in the initial interest rate, when the FV and the DB are low.[26] On contrary, when the default probability is high due to other parameters, both PD and $E[\tau]$ become sensitive to initial interest rate movements.

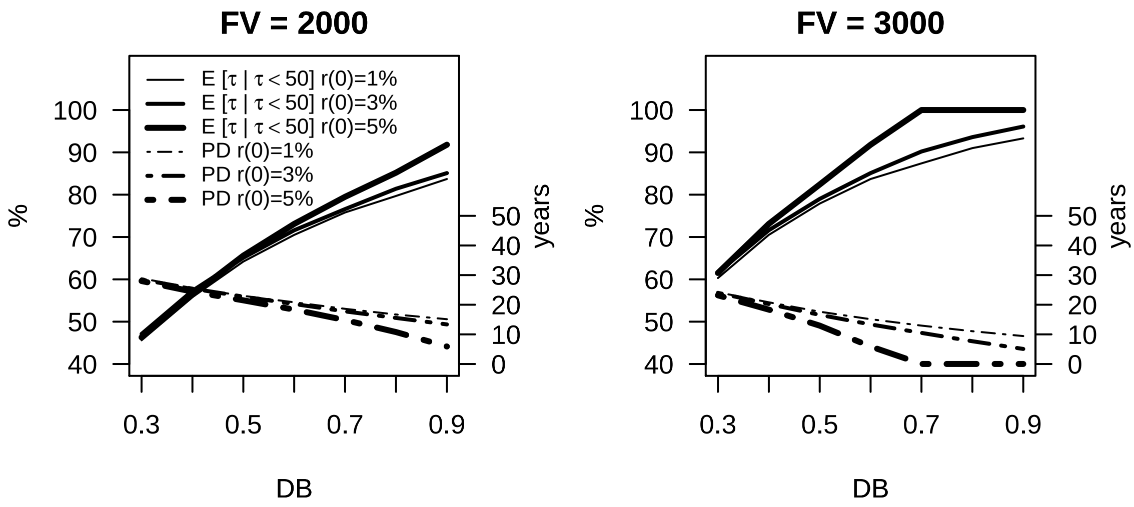

Figure 4.13: PD and default time dependence on initial interest rate

A larger fraction of the firm losses is booked by the equity holders (recall Figure 4.12). Their claim is depreciated by the factors that affect the firm value (i.e. higher discount of future income and increased PD), and also by one additional: higher interest paid out to debt holders.

We can see that the debt value is insensitive to changes in initial interest rate, when the probability of early default is close to zero due to low FV and DB. Our conclusion is, that increased coupon payments perfectly offset higher discount on future cash flows.[27] Consequently the only factor that decreases

[25] It is mentioned in Section 4.6 where this 50 comes from. Also we saw on Figure 4.1, that the effect of different initial interest rate disappears in circa 10 to 15 years, therefore it is sufficient to deal with defaults in the first 50 years only.

[26] A more precise description would be, that the difference of these values is below the level of significance.

[27] For $\kappa = 0$ this is intuitive: the defaultable corporate bond can be represented as a risk-free bond with the same parameters minus the expected losses caused by default. Since the

the bond's value is the increased default probability and its earlier expected occurrence.

Note, that this section explains how the central bank's interventions work. In economical downturn the monetary policy can support the companies by targeting a lower short-term rate. This increases the value of both traded and non-traded assets, reduces the number of defaults, and supports debt financing through the decrease of interest paid on the outstanding principal. The latter is favoured by two factors: the risk-free interest is low, and the risk-premium drops due to lower PD. On the contrary, an overheated can be cooled down with higher risk-free interest.

4.10 Comparison of Stochastic and Deterministic Default Barrier

Stochastic risk-free interest rate and DB are the two features of our model that distinguish it from other EBIT-based works (Goldstein *et al.* 2001; Broadie *et al.* 2007). The contribution of a stochastic interest rate is intuitive: a constant or deterministic risk-free rate is hardly acceptable. Its usefulness was presented also in Section 4.9, where our model have easily dealt with different initial interest rate levels and it was able to predict the implications of such macro-level shocks. The benefits of a stochastic DB were however not proved. In the description of the DB for our model (see Section 4.5.1) we mentioned why banks might prefer a DB that is dependent on the interest rate. We saw however, that it is not the bank who sets the default triggering level: it is the debtor or it is specified in the debt contract, that is designed by both parties.

In order to examine whether it is correct to base our model on stochastic DB we simulated two firms with identical parameters[28] but different DB settings: one stochastic, driven by the instantaneous risk-free interest rate, and one deterministic DB, dependent only on FV_t.

price of a riskless bond that pays continuous interest is always equal to its face value, it is not dependent on the current interest rate.

[28] These parameters were the same as in the basic setting, with the exception of lower recovery rate (5 yearly EBITs), and higher correlation between the EBIT and interest rate processes ($\rho = 0.5$). These modifications were made in order to make the results more sensible on the selection of the DB. Furthermore the number of iterations was doubled to increase the significance of small deviations between the two settings.

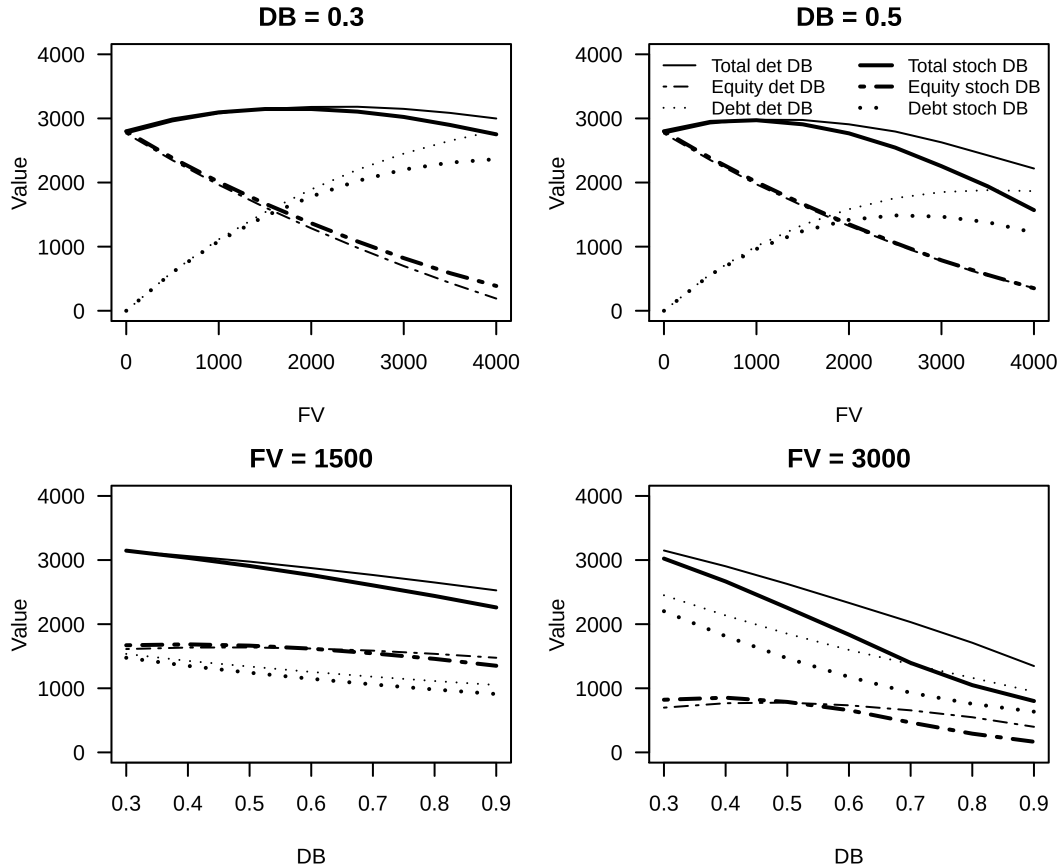

Figure 4.14: Stochastic vs. Deterministic DB

The default triggering levels were therefore set to $FV_t \cdot r(t) \cdot DB$ in the stochastic case and to $FV_t \cdot \gamma \cdot DB$ in the deterministic case, where $DB > 0$ is the same variable in both cases. Figure 4.14 visualizes the comparison of results obtained by stochastic and deterministic DB setting. For the first sight it is apparent that the total firm value is higher when the DB is defined as a deterministic function.

To understand the reason of this better performance, let us take an example macro-level shock. Assume a firm with our basic parameters, that is near the defined DB with the parameter $DB = 0.3$: set the EBIT to $\delta_t = 100$, the Face Value of debt to $FV_t = 10,000$ and the risk-free interest rate to $r(t) = 3\%$. The default triggering level is in both cases $10,000 \cdot 0.03 \cdot 0.3 = 90$, as $r(t) = \gamma = 0.03$. Now take a jump in the W Wiener process, so that $dW_t = 1$ for a very short time frame, that is dt is close to zero. Using (4.1) and (4.4) we can calculate the changes in the earnings and the interest payments:

$$dr = \alpha(\gamma - r)dt + \sigma_r dW_t = 0 + 0.005 \cdot 1 = 0.005$$

and

$$d\delta_t = \delta_t(\mu dt + \sigma dX_t) = 100 \cdot (0 + 0.2 \cdot dX_t),$$

as $dt \approx 0$. Using $X_t = \rho W_t + \sqrt{(1-\rho^2)}Z_t$ we can write

$$d\delta_t = 100 \cdot 0.2 \cdot (\rho dW_t + \sqrt{(1-\rho^2)}dZ_t).$$

Knowing that $E[dZ_t] = 0$, we obtain the expected change in the EBIT process:

$$E[d\delta_t] = 100 \cdot 0.2 \cdot 0.2 \cdot 1 = 4$$

If this shock was positive, the new interest rate is 3.5%, and therefore the stochastic default triggering level increases to $10,000 \cdot 0.035 \cdot 0.3 = 105$. The EBIT grows to 104,[29] as we calculated, and consequently default occurs with stochastic DB, whereas it does not occur with deterministic DB that remains at level 90, independently on the interest rate.

A macro-shock with the same magnitude, but opposite direction produces $\delta_t = 96$ and stochastic default barrier level of 75, using similar calculations as above. Therefore there is no default neither with stochastic nor with deterministic DB.

A deterministic DB therefore softens the default triggering bound, and hence increases the firm value. The problem is however, that when the primitive variable is not observable[30], default is triggered by the equity holders in a way to maximize the value of their claim. Recall Figure 4.14: a stochastic DB bears higher equity value for barrier ratios below 0.5. Since the equity-maximizing DB is below 0.5 (as we have seen in Sections 4.6.2 and 4.7), the equity holders will prefer triggering default according to a stochastic barrier. In fact this is a logical conclusion: the situation of the overall economy, as well as the size of the interest payments is taken into account.

[29] It is expected to grow to 104. However, as the considered time interval approaches to zero, the grow will converge to 4 independently on the realization of Z_t.

[30] As it was discussed in Section 4.7, observable primitive variable implies low default triggering level. Consequently there is insignificant difference in the values produced by the two DB types.

Chapter 5

Conclusion

Our work extends the available literature of asset pricing by an Earnings Before Interest and Taxes (EBIT) based model with stochastic interest rate. This framework is able to price equity and debt in a way consistent with the cash flow of the firm, and therefore to address some defects of the current frameworks. It solves the "delicate" issue of Leland (1994), that the unlevered firm value might not be a traded asset, and deals with the problem of partial tax deductibility. The stochastic interest rate assumption contributes the possibility of analysing the effects of changes in the central bank's monetary policy, and it is able to answer the question how the macroeconomical situation affects the optimal capital structure. The default is triggered using a stochastic interest barrier, that is shown to be more accurate then its deterministic equivalent.

We also analyse the design of credit contracts, focusing on the finding of firm-value maximizing parameters and safety covenants. With the help of the game theory apparatus actions taken by the involved parties can be predicted. Using this scheme the agency costs arising due to asymmetric information are computed, and methods are suggested for the minimization of these losses.

Since we use numerical calculations, the model can be easily extended and modified in many aspects. A natural candidate is a more complex capital structure, with several debt classes, contracts with finite horizon and absolute priority violations. Also, following Broadie *et al.* (2007) it would be fruitful to examine a two-barrier model, where reorganization and liquidation are distinguished.

A weak point in our design is the assumption that the EBIT process is driven by a GBM, and therefore it cannot handle negative earnings. It might be argued that employing arithmetic Brownian motion would be a better choice for this reason, however it should be noted that our model has an infinite time horizon. As the prices of the commodities grow exponentially, it is hard to accept a linear model for the EBIT evolution. Finding better alternatives for the EBIT process will be the subject of further research. A promising idea is to model the earnings as a difference of two correlated GBMs (representing revenues and expenses): it has a clear economic intuition, it is able to produce negative values, has an exponential expected evolution, and works with observable figures.

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Appendix A

Calculations

A.1 Growth Rate of Interest Payments

We know, that

$$E[c(t)] = FV_0 \cdot (\gamma + (r(0) - \gamma)e^{-\alpha t}) \ \cdot \ e^{\kappa t},$$

therefore the growth rate can be calculated as

$$\frac{\frac{\partial E[c(t)]}{\partial t}}{E[c(t)]} = \frac{FV_0 \cdot (\gamma\kappa + (r(0) - \gamma)(\kappa - \alpha)e^{-\alpha t}) \ \cdot \ e^{\kappa t}}{FV_0 \cdot (\gamma + (r(0) - \gamma)e^{-\alpha t}) \ \cdot \ e^{\kappa t}} =$$

$$\frac{\gamma\kappa + (r(0) - \gamma)(\kappa - \alpha)e^{-\alpha t}}{\gamma + (r(0) - \gamma)e^{-\alpha t}} = \kappa + \frac{-\alpha(r(0) - \gamma) \cdot e^{-\alpha t}}{\gamma + (r(0) - \gamma) \cdot e^{-\alpha t}}.$$

A.2 Probability of Default as a Function of κ

Recall the definition of processes involved:

Risk-free interest rate:

$$dr = \alpha(\gamma - r)dt + \sigma_r dW_t$$

$$r(t) = r(0)e^{-\alpha t} + \gamma(1 - e^{-\alpha t}) + \frac{\sigma_r}{\sqrt{2\alpha}} W_t(e^{2\alpha t} - 1)e^{-\alpha t}$$

EBIT:

$$\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dX_t$$

$$X_t = \rho W_t + \sqrt{(1 - \rho^2)} Z_t$$

$$\delta_t = \delta_0 \cdot \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma X_t\right\}$$

Default occurs at:

$$\delta_t = \overline{DB} \cdot \overline{FV_0} \cdot e^{\kappa t} \cdot r(t)$$

$$\delta_0 \cdot \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma X_t\right\} = \overline{DB} \cdot \overline{FV_0} \cdot e^{\kappa t} \cdot r(t)$$

$$\exp\left\{\left(\mu - \frac{\sigma^2}{2} - \kappa\right)t + \sigma X_t\right\} = \frac{\overline{DB} \cdot \overline{FV_0} \cdot r(t)}{\delta_0}$$

$$\left(\mu - \frac{\sigma^2}{2} - \kappa\right)t + \sigma X_t = \ln\left(\frac{\overline{DB} \cdot \overline{FV_0} \cdot r(t)}{\delta_0}\right)$$

$$X_t = \frac{\ln\left(\overline{DB} \cdot \overline{FV_0} \cdot r(t)/\delta_0\right)}{\sigma} - \frac{\left(\mu - \frac{\sigma^2}{2} - \kappa\right)}{\sigma} \cdot t$$

Note that the first term is negative as the initial EBIT is supposed to be higher than the initial level of the DB (otherwise default would occur immediately). Now assume a constant $r(t) = r$, and see the probability that this equality will hold within a finite time horizon. As X_t is a standard Wiener process, and so has a symmetric probability density function with respect to the origin, we can multiply the right side without changing the calculated probability. Therefore the first constant term will be positive.

Denote

$$\hat{b} = -\frac{\ln\left(\overline{DB} \cdot \overline{FV_0} \cdot r/\delta_0\right)}{\sigma}$$

$$\hat{\mu} = \frac{\left(\mu - \frac{\sigma^2}{2} - \kappa\right)}{\sigma}$$

and the first time of reaching the barrier as $\tau_b = \inf\left\{t : X_t = \hat{b} + \hat{\mu}t\right\}$. We want to calculate the probability

$$P\left\{\tau_b < \infty\right\}.$$

This is a simple boundary crossing problem, and has the following solution:

$$P\left\{\tau_b < \infty\right\} = \begin{cases} 1 & \text{if } \hat{\mu} \leq 0 \text{, i.e.; } \mu - \frac{\sigma^2}{2} \leq \kappa \\ \exp\left[-2\hat{\mu}\hat{b}\right] & \text{if } \hat{\mu} > 0 \text{, i.e.; } \mu - \frac{\sigma^2}{2} > \kappa \end{cases}$$

Appendix B

Simulations

The simulations were run in the GNU R software environment on several computers with Gentoo Linux operating system. The number of iterations was set in a way to produce stable (and therefore significant) results, and it was typically 5,000. In simulations with higher asset volatility (i.e. $\sigma > 0.2$) the number of necessary iterations was higher: for $\sigma = 0.6$ we iterated 120,000 times.

Section B.1 contains the results produced by the basic setting. In the subsequent section some alternative settings are presented, that produced important or interesting output. For further results see the enclosed media,[1] where both the R source code and its output are provided.

The tables present the referred values for different FV and DB settings. The rows represent the different FVs (see the first column), whereas the columns correspond to different DB ratios (see the first row). For more information about the calculations, see Section 4.6, and Section B.3 for a pseudo-code illustrating the calculation.

B.1 Basic Setting

Parameters:

$\rho = 0.2$, $\sigma = 0.2$, $\mu = 0.01$, $\kappa = 0.01$, $RR = 10$, $T_C = 35\%$,

$\alpha = 0.25$, $\gamma = 0.03$, $\sigma_r = 0.005$, $r(0) = \gamma$

For equity and total values, see Table 4.2 and Table 4.3.

[1] Or contact the author via e-mail on address martin@dozsa.cz.

Table B.1: Debt value

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0
500	616	599	584	572	561	552	544
1000	1142	1090	1046	1006	973	946	921
1500	1590	1489	1403	1330	1270	1218	1170
2000	1967	1809	1675	1560	1462	1374	1301
2500	2279	2056	1857	1695	1552	1426	1314
3000	2541	2243	1979	1754	1553	1388	1254
3500	2757	2366	2026	1729	1486	1278	1082
4000	2919	2422	2003	1640	1346	1074	1000

Table B.2: Debt ratio (Debt value/Total value)

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00
500	0.19	0.19	0.18	0.18	0.18	0.17	0.17
1000	0.34	0.33	0.32	0.31	0.31	0.30	0.30
1500	0.46	0.44	0.42	0.41	0.41	0.40	0.40
2000	0.56	0.53	0.51	0.50	0.49	0.49	0.50
2500	0.64	0.61	0.59	0.57	0.57	0.58	0.60
3000	0.72	0.68	0.65	0.64	0.65	0.67	0.72
3500	0.79	0.74	0.71	0.70	0.73	0.78	0.90
4000	0.85	0.80	0.77	0.77	0.82	0.94	1.00

Table B.3: Percentage defaulted

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
500	11.0	16.6	21.2	25.6	29.9	33.8	37.4
1000	25.6	33.8	40.8	47.1	52.7	57.3	61.6
1500	37.4	47.1	55.0	61.6	66.8	71.6	75.5
2000	47.1	57.3	65.1	71.6	76.6	81.4	85.1
2500	55.0	65.1	73.1	79.0	84.1	88.5	92.2
3000	61.6	71.6	79.0	85.1	90.2	93.6	96.1
3500	66.8	76.6	84.1	90.2	94.0	96.8	99.2
4000	71.6	81.4	88.5	93.6	96.8	99.4	100.0

Table B.4: Average time of default during the first 50 years

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0							
500	37.9	36.7	35.2	34.2	33.2	32.2	31.1
1000	34.2	32.2	30.1	28.5	26.9	25.4	24.0
1500	31.1	28.5	26.1	24.0	22.1	20.2	18.3
2000	28.5	25.4	22.7	20.2	17.8	15.6	13.4
2500	26.1	22.7	19.5	16.6	13.9	11.5	9.1
3000	24.0	20.2	16.6	13.4	10.5	7.7	5.1
3500	22.1	17.8	13.9	10.5	7.2	4.2	1.4
4000	20.2	15.6	11.5	7.7	4.2	1.0	0.0

B.2 Modified Parameters

These calculations generally differ from the basic setting in one parameter only. For the sake of simplicity, we will note explicitly only this one different parameter.

B.2.1 Different Initial Interest Rate

Initial level 1% $r(0) = 0.01$

Table B.5: Debt value

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0
500	625	608	593	579	567	557	548
1000	1162	1104	1059	1019	983	951	924
1500	1604	1506	1411	1329	1262	1210	1160
2000	1986	1810	1663	1548	1445	1365	1293
2500	2287	2038	1847	1681	1548	1447	1366
3000	2530	2221	1960	1753	1609	1476	1380
3500	2728	2339	2028	1805	1609	1479	1374
4000	2894	2420	2062	1792	1600	1451	1334

Table B.6: Equity value

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	3249	3249	3249	3249	3249	3249	3249
500	2824	2829	2829	2825	2817	2807	2794
1000	2430	2442	2442	2432	2409	2379	2342
1500	2074	2099	2098	2075	2035	1980	1905
2000	1751	1791	1789	1754	1690	1610	1514
2500	1460	1517	1513	1461	1381	1283	1181
3000	1196	1267	1261	1202	1120	999	879
3500	958	1043	1039	982	871	771	665
4000	736	846	847	778	688	589	487

Initial level 5% $r(0) = 0.05$

Table B.7: Debt value

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0
500	608	592	580	568	559	550	543
1000	1135	1089	1047	1012	981	952	928
1500	1587	1493	1409	1335	1264	1206	1151
2000	1971	1813	1669	1536	1409	1283	1141
2500	2295	2054	1827	1598	1329	1017	1000
3000	2554	2203	1844	1390	1000	1000	1000
3500	2739	2241	1627	1000	1000	1000	1000
4000	2864	2132	1049	1000	1000	1000	1000

Table B.8: Equity value

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	2820	2820	2820	2820	2820	2820	2820
500	2408	2412	2412	2408	2404	2395	2385
1000	2025	2036	2036	2027	2010	1980	1935
1500	1674	1698	1698	1666	1615	1536	1435
2000	1347	1385	1371	1310	1210	1009	628
2500	1034	1079	1057	954	657	53	0
3000	734	795	762	447	-0	0	0
3500	470	559	425	-0	0	0	0
4000	231	341	28	0	0	0	0

B.2.2 High Asset Volatility

40% volatility $\sigma = 0.4$

Table B.9: Results

(a) Debt value

	0.5	0.9
0	0	0
1500	755	713
3000	1065	968

(b) Equity value

	0.5	0.9
0	2901	2901
1500	2042	1727
3000	1397	499

(c) Total value

	0.5	0.9
0	2901	2901
1500	2797	2440
3000	2462	1467

(d) Debt ratio

	0.5	0.9
0	0.00	0.00
1500	0.27	0.29
3000	0.43	0.66

(e) % defaulted

	0.5	0.9
0	0.0	0.0
1500	92.6	96.5
3000	97.1	99.3

(f) Avg. def. time

	0.5	0.9
0		
1500	14.3	9.2
3000	8.2	2.4

60% volatility $\sigma = 0.6$

Table B.10: Results

(a) Debt value

	0.5	0.9
0	0	0
1500	508	560
3000	765	876

(b) Equity value

	0.5	0.9
0	2682	2682
1500	2020	1625
3000	1434	576

(c) Total value

	0.5	0.9
0	2682	2682
1500	2528	2185
3000	2199	1452

(d) Debt ratio

	0.5	0.9
0	0.00	0.00
1500	0.21	0.26
3000	0.36	0.61

(e) % defaulted

	0.5	0.9
0	0.0	0.0
1500	99.1	99.6
3000	99.6	99.9

(f) Avg. def. time

	0.5	0.9
0		
1500	8.1	5.1
3000	4.5	1.4

B.3 Illustrative Code

For a better understanding, how the simulations are calculated, we present here a simple pseudo-code.[2]

```
repeat the iterations
  generate a random EBIT and interest rate evolution
  follow the evolutions in time starting at t=0
    set the possible values of DB to 0.3, 0.4 .. 0.9
    set the possible values of FV to 0, 500, .. 5000
    for all possible combinations of DB and FV do
       if EBIT > DB then
          if EBIT > interest payment
             pay out the coupon to debt holders,
             pay out the rest of the EBIT as dividends
          else
             pay out the coupon to debt holders,
             compensate cash deficit by equity dilution
          end if
       else
          trigger default,
          transfer the remaining assets (EBIT*RR)
                                      to the debt holder
       end if
    end of for all possible...
  end of follow
end of repeat
average the results of all iterations
print out this average
```

[2] A pseudo-code is a compact description of a programming algorithm. It is easier for understanding, however it cannot be run by the computer.