**Charles University in Prague**

Faculty of Mathematics and Physics

Department of Algebra

# Algebraic methods in multivalued logics (Orthocomplemented lattices with a symmetric difference)

PhD thesis

Author: RNDr. Milan Matoušek

Supervisor: Prof. RNDr. Pavel Pták, DrSc.

Specialization: M1 - Algebra, number theory and mathematical logic

**Název:** Algebraické metody ve vícehodnotových logikách (Ortokomplementární svazy se symetrickou diferencí)

**Autor:** RNDr. Milan Matoušek

**Katedra:** Katedra algebry
Matematicko-fyzikální fakulta
Univerzita Karlova v Praze

**Školitel:** Prof. RNDr. Pavel Pták, DrSc.

**E-mail autora:** matmilan@email.cz

**E-mail školitele:** ptak@math.feld.cvut.cz

**Abstrakt:** Pedkldan disertan prce studuje binrn operaci, kter byla zskna abstrakc vlastnost mnoinov symetrick diference. Pidnm tto operace k teorii ortokomplementrnch svaz byla zskna nov tda algeber, takzvan ortokomplementrn diferenn svazy (znaen ODL). Nejdve je ukzno, e tda ODL obsahuje Booleovy algebry a je obsaena v td ortomodulrnch svaz. Dle byly studovny rozlin algebraick vlastnosti tdy ODL (identity platn v nkterch podtdch tdy ODL, zvltnosti volnch ODL, atd.) a byla nalezena charakterizace mnoinov-reprezentovatelnch ODL. V dalm textu je poloena pirozen otzka, kdy lze dan ortomodulrn svaz vnoit nebo rozit do ODLu. Je ukzna metoda konstrukce specifickch typ ODL, kter prohlubuje chpn vnitnch vlastnost tchto algeber (nap. monost jejich zskn ze systmu podalgeber Booleovy algebry). Dle je ukzna ponkud pekvapiv souvislost mezi tdou ODL a $Z_2$-hodnotovmi mrami. V zvru prce je uinno zobecnn - msto svaz se uvauj ortokomplementrn diferenn posety. V tto sti jsou formulovny a rozebrny nkter otzky souvisejc s nesvazovmi kvantovmi logikami.

**Klíčová slova:** Booleova algebra, symetrick diference, ortokomplementrn diferenn svaz, ortomodulrn svaz, mnoinov reprezentovatelnost, Greechieho diagram

**Title:** Algebraic methods in multivalued logics (Orthocomplemented lattices with a symmetric difference)

**Author:** RNDr. Milan Matoušek

**Department:** Katedra algebry
Matematicko-fyzikální fakulta
Univerzita Karlova v Praze

**Supervisor:** Prof. RNDr. Pavel Pták, DrSc.

**Author's e-mail address:** matmilan@email.cz

**Supervisor's e-mail address:** ptak@math.feld.cvut.cz

**Abstract:** In the thesis we deal with a binary operation that acts as abstract "symmetric difference". We endow orthocomplemented lattices with this operation and obtain a new class of algebras. We call these algebras orthocomplemented difference lattices (ODLs). We first see that the ODLs form a class that contains Boolean algebras and is contained in orthomodular lattices (OMLs). In the subsequent analysis we study algebraic properties of ODLs (identities valid in classes of ODLs, peculiarities connected with free ODLs, etc.) and find a characterization of set-representable ODLs. We then ask a natural question of which OML can be made (resp. can be enlarged to) an ODL. We exhibit several constructions - quite involved in places - that deepen the understanding of intrinsic properties of ODLs. As a rather surprising result in this line we find a connection with $Z_2$-valued measures. In the end we relax the lattice condition imposed on ODLs. We obtain orthocomplemented difference posets. We then formulate and clarify several questions related to non-lattice "quantum logics".

**Key words:** Boolean algebra, symmetric difference, orthocomplemented difference lattice, orthomodular lattice, set-representability, Greechie diagram

# Contents

# The summary of the PhD thesis

Important questions related to the study of multivalued logics are problems that come into existence in connection with the "logic" of quantum mechanical experiments. In 1936, G. Birkhoff and J. von Neumann postulated that a suitable semantics for such a logic are modular lattices. Later on K. Husimi, 1937, suggested that a more appropriate semantics is the class of orthomodular lattices. On these lattices one can contemplate an analogy of classical logical connectives. For instance, J. Abbott, 1967, studied a type of implication connective. An important connective, with a broader mathematical bearing, is the "either..or" connective (also known as "alternative" or "exclusive or"). The treatment of this connective could be done either by considering an appropriate term in orthomodular lattices or by adding a new operation symbol as well as new axioms to the theory of orthomodular lattices. The latter approach together with initially obtained results is the contents of this thesis.

The connective "either..or" considered on Boolean algebras is expressed by means of the symmetric difference. Thus, extracting properties of the Boolean symmetric difference, we obtain a class of orthocomplemented lattices endowed with a "symmetric difference". This class has, as we believe, rather interesting properties in its own right. Besides, this class seems to be worth investigating in view of its relation to orthomodular lattices and Boolean algebras. Considered purely algebraicly, a variety of algebras has therefore been introduced that enriches the realm of orthomodular structures.

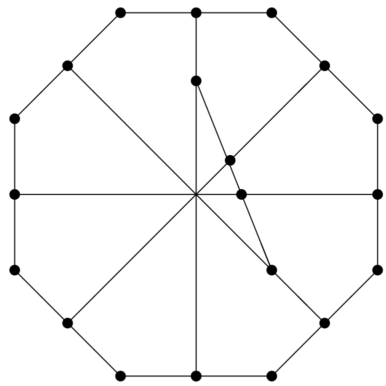
Let us come to the review of technical results. The thesis is based on the papers [1] - [5] enclosed to which we shall refer in the sequel (they are also listed at the end of this summary).

Let $L$ be a 7-tuple, $L = (X, \wedge, \vee, ^{\perp}, 0, 1, \triangle)$, where $(X, \wedge, \vee, ^{\perp}, 0, 1)$ is an orthocomplemented lattice and $\triangle : X^2 \to X$ is a binary operation. Then $L$ is said to be an *orthocomplemented difference lattice* (an ODL, [1], Def. 3.1), if $L$ is subject to the following requirements:
(ODL$_1$) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$,
(ODL$_2$) $x \triangle 1 = x^{\perp}$, $1 \triangle x = x^{\perp}$,
(ODL$_3$) $x \triangle y \leq x \vee y$.

Obviously, the notion of ODL generalizes the notion of Boolen algebra, when $\triangle$ generalizes the notion of symmetric difference. The initial questions we first have to ask (and answer) are the characterization of Boolean algebras among ODLs and the description of the compatibility. This is done in [1], Prop. 3.6. Then we prove an important result that an ODL is always orthomodular, i.e. if $L$ is an ODL and if we forget the operation $\triangle$, we obtain an orthomodular lattice (an OML). It is worth noting already now that not all OMLs are induced by ODLs in this way - a question that deserves attention and will be discussed later in the thesis.

Analysing further intrinsic properties of ODLs, a natural question occurs of whether (when) an ODL allows for a set representation. Main results of [1] provide the answer: Not all ODLs are set-representable. In fact, it can be proved that every Boolean algebra can be embedded in an ODL that is not set-representable ([1] Thm. 1.18). In the study of the set-representable ODLs, however, it is proved that they constitute a rather huge class - they form a variety ([1], Thm. 6.12). A characterization of this vatiety in terms of two-valued ODL-states is provided ([1], Thm. 6.7). To obtain this characterization, a thorough analysis of ODL-ideals is presented ([1], Thm. 6.10).

As indicated above, the other papers further pursue the relationship of OMLs and ODLs. One asks the following question: Given an OML, can it always be made an ODL ? Though sometimes it is the case (e.g. for $L(R^2)$ or for $\mathrm{MO}_k$ with $k = 2^{n-1}$, [1], Thm. 8.7), it is not so in general (e.g. for $\mathrm{MO}_k$ with $k \neq 2^{n-1}$, [1], Thm. 8.7). A more combinatorially involved question then reads: Which OML is ODL-embeddable ? It would be instrumental for the study of OMLs if such was every OML. However, it is not the case - in the paper [2] a construction is exhibited which produces OMLs that are not ODL-embeddable. When analysed more in depth, a rather surprising necessary condition for the embedding to exist can be derived: If an OML is ODL-embeddable then it has to have an abundance of $Z_2$-valued measures ([2], Thm. 3.2). This result allows us to see that some OMLs familiar within the theory of orthomodular lattices cannot be ODL-embeddable. A finite example of this kind is exhibited bellow (a prerequisity here is the Greechie paste job; out of the other important examples of non ODL-embeddable OMLs, such are $L(R^n)$ for $n \geq 4$ as the result of [2], Thm. 3.8 establishes).



The paper [3] and [4] iniciate the investigation of ODLs along the universal algebraic line. The first result of [3] characterizes the free ODL with 2 generators - it is the (set-representable) ODL $\mathrm{MO}_3 \times 2^4$ ([3], Thm. 2.3). The characterization of the free ODL with 3 generators remains open - so far it can only be proved that this ODL is not set-representable. In fact, a rather involved construction is presented in ([3], Thm. 3.11) that manufactures a non set-representable ODL with 3 generators. Here we may deal with a formidable problem as the comparison with OMLs suggests. As regards the note [4], it concerns identities in ODLs. It is shown that relatively simple identities distinguish Boolean algebras among ODLs ([4], Thm. 2.2). In addition, an identity in ODLs is found that is valid only in the set-representable ODLs ([4], Thm. 2.6). This identity could be potentially utilized in constructing ODLs with preassigned properties (e.g. there is a non set-representable ODL that induces a set-representable non-modular OML ([4], Obs. 2.9).

In the last paper included, [5], the condition ($\mathrm{ODL}_3$) of $L$ being a lattice is relaxed - instead we require

($\mathrm{ODP}_3$) $x \leq z \ \& \ y \leq z \Rightarrow x \bigtriangleup y \leq z$,

the other two axioms we leave unchanged. We obtain orthocomplemented posets with a symmetric difference (ODPs). Out of the results obtained, let us mention only a few. The class of set-representable ODPs forms a quasivariety ([5], Thm. 5.4). It should be observed that this quasivariety is rather big and contains several (non-lattice) examples familiar within the quantum logic theory - notably it contains the standard examples built up on the ODPs of all even-cardinality subsets of an even-cardinality set. Furthermore, different construction techniques of ODPs are obtained ([5], Prop. 4.13). Finally, a modification of Frink ideals is then adopted that allows one to show that a pseudocomplemented ODP must be set-representable ([5], Thm. 6.8).

The results of the thesis appeared very recently (see the references bellow). They therefore present a topical line in lattice theory and constitute a research area lying between the theory of orthomodular lattices and Boolean algebras. The solutions of open questions related to this research area (see again [1] - [5] for some of these open questions) are supposed to indicate further prospects and applications of the affort iniciated in the thesis.

# References


[1] Matoušek M., *Orthocomplemented lattices with a symmetric difference*, Algebra Universalis **60** (2009), 185-215.

[2] Matoušek M., Pták P., *Symmetric difference on orthomodular lattices and $Z_2$-valued states*, Comment. Math. Univ. Carolin. **50, 4** (2009), 535-547.

[3] Matoušek M., Pták P., *Orthocomplemented difference lattices with few generators*, submitted for publication.

[4] Matoušek M., Pták P., *On identities in orthocomplemented difference lattices*, Mathematica Slovaca, to appear.

[5] Matoušek M., Pták P., *Orthocomplemented posets with a symmetric difference*, Order **26** (2009), 1-21.


Prague 2010

*Milan Matoušek*

Algebra univers. **60** (2009) 185–215
0002-5240/09/020185 – 31, *published online* February 27, 2009
DOI 10.1007/s00012-009-2105-5

**Algebra Universalis**

# Orthocomplemented lattices with a symmetric difference

Milan Matoušek

Abstract. Modelling an abstract version of the set-theoretic operation of symmetric difference, we first introduce the class of orthocomplemented difference lattices ($\mathcal{ODL}$). We then exhibit examples of ODLs and investigate their basic properties finding, for instance, that any ODL induces an orthomodular lattice (OML) but not all OMLs can be converted to ODLs. We then analyse an appropriate version of ideals and valuations in ODLs and show that the set-representable ODLs form a variety. We finally investigate the question of constructing ODLs from Boolean algebras and obtain, as a by-product, examples of ODLs that are not set-representable but that "live" on set-representable OMLs.

## 1. Introduction and basic notions

The class of orthocomplemented lattices (OCLs) was intensely studied in recent years. The investigation was often motivated by the theory of quantum logic (see, e.g., [8, 15, 22, 24, 25]). The algebraic theory of OCLs was carried out in several papers and monographs (see, e.g., [1, 2, 3, 4, 5, 16]). In this paper we shall study orthocomplemented lattices endowed with an abstract symmetric difference, obtaining thus a natural class of algebras that properly contains the class of Boolean algebras. It should be noted that certain attempts to model the symmetric difference have been made (see [10, 9, 23]). However, in contrast to the previously used attitude based on term operations within OMLs, our approach presents an algebraic abstraction of the symmetric difference of sets and gives rise to a completely new variety of algebras. This variety (denoted by $\mathcal{ODL}$) has rather interesting properties in its own right as well as in relation to the formerly studied variety of OMLs.

In the present paper we first study algebraic properties of ODLs. Then we focus on the question of when an ODL is set-representable. We find that such are many ODLs—a large subvariety of $\mathcal{ODL}$—but not all. In fact, an ODL may induce a set-representable OML without being itself set-representable. Related questions are also investigated.

Presented by S. Pulmannova.
Received April 10, 2007; accepted in final form February 12, 2008.
2000 *Mathematics Subject Classification*: 06C15, 03G12.
*Key words and phrases*: Boolean algebra, orthomodular lattice, symmetric difference, variety.
This work is a part of the research plan MSM 0021620839 that is financed by the Ministry of Education of the Czech Republic.

Let us introduce some notation and notions as we shall use them in the sequel. The logical connectives will be denoted by standard symbols: & (conjunction), $\vee$ (disjunction), $\Rightarrow$ (implication) and $\Leftrightarrow$ (equivalence). Identities will be denoted by $s \approx t$, where $s, t$ are terms. If $X, Y$ are classes, the expression $X \subset Y$ will mean $X \subseteq Y$ & $X \neq Y$. If $f\colon X \to Y$ is a mapping, $S \subseteq X$, then we write $f[S] = \{f(x);\ x \in S\}$. The cardinality of a set $X$ will be denoted by $|X|$.

Let $A$ be a structure of the type $\mathcal{L}$ (see [6, p. 192]). Then its underlying set will be denoted by $\dot{A}$. If there is no danger of confusion, we simply write $x \in A$, resp. $X \subseteq A$ instead of the more correct $x \in \dot{A}$, resp. $X \subseteq \dot{A}$. Analogously, we shall write $|A|$ instead of $|\dot{A}|$. A structure $A$ will be called *trivial* provided $|A| = 1$, and $A$ will be called *nontrivial* if $|A| \geq 2$. If $F \in \mathcal{L}$ is an operation symbol, we shall denote the corresponding $n$-ary operation by $F_A$ if there is a need for specification. By $\mathrm{Sub}(A)$ we shall denote the set of all substructures of $A$. If $A$, $B$ are two isomorphic structures of the same type, we shall write $A \cong B$. The notation $f\colon A \cong B$ will mean that $f$ is an isomorphism of the structure $A$ onto $B$.

Let us recall some standard notions of the theory of orthocomplemented lattices. The reader may consult the monographs [1] and [16] for more details.

**Definition 1.1.** Let $L = (X, \wedge, \vee, {}^{\perp}, \mathbf{0}, \mathbf{1})$ be an algebra of the type $(2, 2, 1, 0, 0)$. Then $L$ is said to be an *orthocomplemented lattice* (abbr., an OCL) if $(X, \wedge, \vee)$ is a lattice and if the following formulas hold in $L$:

$$x \wedge x^{\perp} \approx \mathbf{0}, \quad x \vee x^{\perp} \approx \mathbf{1}, \quad (x^{\perp})^{\perp} \approx x, \quad x \leq y \Rightarrow y^{\perp} \leq x^{\perp}.$$

If, moreover, $L$ satisfies the orthomodular law,

$$x \leq y \Rightarrow y \approx x \vee (y \wedge x^{\perp}),$$

then $L$ is called an *orthomodular lattice* (abbr., an OML).

Let us denote by $\mathcal{OCL}$, resp. $\mathcal{OML}$, resp. $\mathcal{BA}$, the variety of orthocomplemented lattices, resp. orthomodular lattices, resp. Boolean algebras. Of course, $\mathcal{BA} \subset \mathcal{OML} \subset \mathcal{OCL}$.

It should be noted that instead of "orthocomplemented lattice" one often uses the term *ortholattice*.

**Proposition 1.2.** *Let $L$ be an OCL. Then $L$ is an OML if and only if the following formula holds in $L$: $x \leq y$ & $y \wedge x^{\perp} \approx \mathbf{0} \Rightarrow x \approx y$.*

*Proof.* It reduces to a routine verification (see, e.g., [16, p. 22]). □

**Definition 1.3.** Let $L$ be an OML. For $x, y \in L$, let $\mathrm{com}(x, y)$ denote the *commutator* of $x, y$, i.e., let us write $\mathrm{com}(x, y) = (x \vee y) \wedge (x \vee y^{\perp}) \wedge (x^{\perp} \vee y) \wedge (x^{\perp} \vee y^{\perp})$. The elements $x, y$ of $L$ are called *commutative* (abbr., $x\ C\ y$, or more specifically

$x\ C_L\ y$) if $\mathrm{com}(x,y) = \mathbf{0}$. If $x\ C\ y$ is false, we say that the elements $x, y$ are not commutative and we shall write $x \mid y$ (or, more specifically, $x \mid_L y$).

For any $x \in L$, let us write $C(x) = \{a \in L;\ x\ C\ a\}$.

**Proposition 1.4.** *Let $L$ be an OML and let $x, y \in L$. Then the following conditions are equivalent:*

(a) $x\ C\ y$,
(b) $x = (x \wedge y) \vee (x \wedge y^{\perp})$,
(c) $x = (x \vee y) \wedge (x \vee y^{\perp})$,
(d) *$x, y$ are contained in a Boolean subalgebra of $L$.*

*Proof.* See, e.g., [16, p. 26]. □

**Proposition 1.5.** *Let $L$ be an OML and let $x \in L$. Then the set $C(x)$ is a subalgebra in $L$. Moreover, if $y \in L$ and the elements $x, y$ are comparable (i.e., $x \leq y$ or $y \leq x$), then $y \in C(x)$.*

*Proof.* See, e.g., [16, p. 24]. □

**Definition 1.6.** Let $L$ be an OML. A maximal Boolean subalgebra of $L$ is called a *block* of $L$. The collection of all blocks of $L$ will be denoted by $\mathrm{Bl}(L)$. The set $C(L) = \bigcap_{B \in \mathrm{Bl}(L)} B$ is called the *centre* of $L$ and the elements of $C(L)$ are called *central.*

**Proposition 1.7.** *Let $L$ be an OML. Let $M$ be such a subset of $L$ that $x\ C\ y$ for any $x, y \in M$. Then there exists a block, $B$, in $\mathrm{Bl}(L)$ such that $M \subseteq B$.*

*Proof.* See, e.g., [16, p. 38]. □

**Convention 1.8.** Let $\kappa \geq 0$ be a cardinal. Let us denote by $\mathrm{B}_{(\kappa)}$ the Boolean algebra of all finite and co-finite subsets of $\kappa$. Thus, if $\kappa$ is finite, then $|\mathrm{B}_{(\kappa)}| = 2^{\kappa}$, and if $\kappa$ is infinite, then $|\mathrm{B}_{(\kappa)}| = \kappa$.

## 2. Difference algebras

In this section we introduce auxiliary algebras and list their basic properties. They will be used later.

**Definition 2.1.** Let $D = (X, \diamond, \mathbf{0}, \mathbf{1})$ be an algebra of the type $(2, 0, 0)$. Then $D$ is said to be a *difference algebra* (abbr., a DA) if the following formulas hold in $D$:

($\mathrm{d}_1$) $x \diamond (y \diamond z) \approx (x \diamond y) \diamond z$,
($\mathrm{d}_2$) $x \diamond \mathbf{0} \approx x$,
($\mathrm{d}_3$) $x \diamond x \approx \mathbf{0}$,
($\mathrm{d}_4$) $(\exists x : x \not\approx \mathbf{0}) \Rightarrow \mathbf{1} \not\approx \mathbf{0}$.

Let us observe that the difference algebra $D = (X, \diamond, \mathbf{0}, \mathbf{1})$ is nontrivial if and only if $\mathbf{1} \neq \mathbf{0}$. This follows immediately from ($d_4$).

**Proposition 2.2.** *Let $D = (X, \diamond, \mathbf{0}, \mathbf{1})$ be a DA. Then the operation $\diamond$ is commutative.*

*Proof.* Let us first prove that $\mathbf{0} \diamond x = x$ for any $x \in X$ . We see that

$$\mathbf{0} \diamond x = (x \diamond x) \diamond x = x \diamond (x \diamond x) = x \diamond \mathbf{0} = x.$$

Now, we have

$$\begin{aligned} x \diamond y &= (x \diamond y) \diamond \mathbf{0} = (x \diamond y) \diamond [(y \diamond x) \diamond (y \diamond x)] = x \diamond y \diamond y \diamond x \diamond (y \diamond x) \\ &= x \diamond \mathbf{0} \diamond x \diamond (y \diamond x) = x \diamond x \diamond (y \diamond x) = \mathbf{0} \diamond (y \diamond x) = y \diamond x. \end{aligned}$$ □

**Proposition 2.3.** *Let $D$ be a finite DA. Then $|D| = 2^n$, where $n \geq 0$ is a natural number.*

*Proof.* Let us introduce the operation $-\colon X \to X$ by putting $-x = x$. Then we infer that the algebra $G = (X, \diamond, -, \mathbf{0})$ is a group such that each element of $G$ has order 2. Thus $G$ is a 2-group and the number of elements of $G$ must be a natural power of 2 (see, e.g., [17]). □

It should be noted that the notion of a difference algebra is synonymous with the notion of a 2-group, together with a distinguished element 1 (not the identity, in the nontrivial case). If DAs are viewed as 2-groups, many of the proofs in this section can be viewed as exercises in groups. If $D = (X, \diamond, \mathbf{0}, \mathbf{1})$ is a DA, then, for each $x \in D$ with $x \neq \mathbf{0}, \mathbf{1}$, the algebra $D_x = (X, \diamond, \mathbf{0}, x)$ is also a DA. Moreover, it is isomorphic to $D$ (Proposition 7.7).

**Example 2.4.** Let $B$ be a Boolean algebra and let $x, y \in B$. Let $x \,\Delta_B\, y$ be the standard symmetric difference of $x$ and $y$. Thus, $x \,\Delta_B\, y = (x \wedge y^\perp) \vee (y \wedge x^\perp) = (x \vee y) \wedge (x \wedge y)^\perp$. Then $D_B = (\dot{B}, \Delta_B, \mathbf{0}_B, \mathbf{1}_B)$ is a DA.

As we shall see later, all difference algebras can be obtained in this way (see Proposition 7.8).

Prior to the next proposition, let us make the following convention: If $D$ is a difference algebra, $x \in D$, then write $x^\perp = x \diamond \mathbf{1}$.

**Proposition 2.5.** *Let $D$ be a DA. Then for any $x, y, z \in D$ the following statements hold true:*

(a) $\mathbf{0}^\perp = \mathbf{1}$, $\mathbf{1}^\perp = \mathbf{0}$,
(b) $x^{\perp\perp} = x$,
(c) $x^\perp = y^\perp \Rightarrow x = y$,
(d) $x \diamond y^\perp = x^\perp \diamond y = (x \diamond y)^\perp$,

(e) $x^{\perp} \diamond y^{\perp} = x \diamond y$,
(f) $x \diamond x^{\perp} = \mathbf{1}$,
(g) $x \diamond y = z \Leftrightarrow x \diamond z = y \Leftrightarrow x \diamond y^{\perp} = z^{\perp}$,
(h) $x \diamond y = x \diamond z \Rightarrow y = z$,
(i) $x \diamond y = \mathbf{0} \Leftrightarrow x = y$,
(j) $x \diamond y = \mathbf{1} \Leftrightarrow x = y^{\perp}$.

*Proof.* (a): $\mathbf{0}^{\perp} = \mathbf{0} \diamond \mathbf{1} = \mathbf{1}$, $\mathbf{1}^{\perp} = \mathbf{1} \diamond \mathbf{1} = \mathbf{0}$.

(b): $x^{\perp\perp} = (x \diamond \mathbf{1}) \diamond \mathbf{1} = x \diamond (\mathbf{1} \diamond \mathbf{1}) = x \diamond \mathbf{0} = x$.

(c): It follows from (b).

(d): $x \diamond y^{\perp} = x \diamond (y \diamond \mathbf{1}) = (x \diamond y) \diamond \mathbf{1} = (x \diamond y)^{\perp}$.

(e): Using (d) we obtain $x^{\perp} \diamond y^{\perp} = (x^{\perp} \diamond y)^{\perp} = (x \diamond y)^{\perp\perp} = x \diamond y$.

(f): $x \diamond x^{\perp} = (x \diamond x)^{\perp} = \mathbf{0}^{\perp} = \mathbf{1}$.

(g): Let $x \diamond y = z$. Then $x \diamond z = x \diamond (x \diamond y) = (x \diamond x) \diamond y = \mathbf{0} \diamond y = y$. The reverse implication can be proved analogously. The second equivalence follows from (d) and (b).

(h): Suppose $x \diamond y = x \diamond z$. Then $x \diamond (x \diamond y) = x \diamond (x \diamond z)$, and hence $(x \diamond x) \diamond y = (x \diamond x) \diamond z$. It follows that $\mathbf{0} \diamond y = \mathbf{0} \diamond z$ and therefore $y = z$.

(i): If $x = y$, then $x \diamond y = \mathbf{0}$ by condition (d$_3$). Conversely, suppose $x \diamond y = \mathbf{0}$. Then $(x \diamond y) \diamond y = \mathbf{0} \diamond y$. This means that $x \diamond (y \diamond y) = y$ and therefore $x = y$.

(j): $x \diamond y = \mathbf{1} \Leftrightarrow x \diamond y^{\perp} = \mathbf{0} \Leftrightarrow x = y^{\perp}$. □

**Proposition 2.6.** *Let $D$ be a nontrivial DA and let $x \in D$. Then $x \neq x^{\perp}$.*

*Proof.* Let $x = x^{\perp}$. Then $x \diamond x = x \diamond x^{\perp}$ and therefore $\mathbf{0} = \mathbf{1}$, which we excluded. □

## 3. Orthocomplemented difference lattices

In this section we define the basic notion of this article—the notion of an orthocomplemented difference lattice (ODL). We investigate basic properties of ODLs and find their relationships to other orthocomplemented structures.

**Definition 3.1.** Let $L = (X, \wedge, \vee, {}^{\perp}, \mathbf{0}, \mathbf{1}, \triangle)$, where $(X, \wedge, \vee, {}^{\perp}, \mathbf{0}, \mathbf{1})$ is an OCL and $\triangle\colon X^2 \to X$ is a binary operation. Then $L$ is said to be an *orthocomplemented difference lattice* (abbr., an ODL) if the following formulas hold in $L$:

(D$_1$) $x \triangle (y \triangle z) \approx (x \triangle y) \triangle z$,
(D$_2$) $x \triangle \mathbf{1} \approx x^{\perp}$, $\mathbf{1} \triangle x \approx x^{\perp}$,
(D$_3$) $x \triangle y \leq x \vee y$.

Obviously, the class of all ODLs forms a variety. We will denote it by $\mathcal{ODL}$.

Let $L = (X, \wedge, \vee, {}^{\perp}, \mathbf{0}, \mathbf{1}, \triangle)$ be an ODL. Then the OCL $(X, \wedge, \vee, {}^{\perp}, \mathbf{0}, \mathbf{1})$ will be denoted by $L_{\text{supp}}$ and called the *support* of $L$. Occasionally, the ODL $L$ will be identified with the pair $(L_{\text{supp}}, \triangle)$.

Let us adopt the convention that in writing a formula with $\triangle$, $\wedge$, $\vee$ we will give preference to the operation $\triangle$ over the operations $\wedge$ and $\vee$. Thus, for instance, $x \wedge y \triangle z$ means $x \wedge (y \triangle z)$.

**Proposition 3.2.** *Let $L = (X, \wedge, \vee, {}^{\perp}, \mathbf{0}, \mathbf{1}, \triangle)$ be an ODL. Then the algebra $D_L = (X, \triangle, \mathbf{0}, \mathbf{1})$ is a difference algebra. Moreover, for any $x \in L$ we have $x^{\perp_{D_L}} = x^{\perp_L}$.*

*Proof.* Let us first observe that the property ($D_2$) yields $\mathbf{1} \triangle \mathbf{1} = \mathbf{1}^{\perp} = \mathbf{0}$. Let us verify the properties ($d_2$), ($d_3$) and ($d_4$) of Definition 2.1. Suppose that $x \in L$.

($d_2$): $x \triangle \mathbf{0} = x \triangle (\mathbf{1} \triangle \mathbf{1}) = (x \triangle \mathbf{1}) \triangle \mathbf{1} = x^{\perp} \triangle \mathbf{1} = (x^{\perp})^{\perp} = x$.

($d_3$): Let us first show that $x^{\perp} \triangle x^{\perp} = x \triangle x$. We consecutively obtain $x^{\perp} \triangle x^{\perp} = (x \triangle \mathbf{1}) \triangle (\mathbf{1} \triangle x) = (x \triangle (\mathbf{1} \triangle \mathbf{1})) \triangle x = (x \triangle \mathbf{0}) \triangle x = x \triangle x$. Moreover, we have $x \triangle x \le x \vee x = x$ as well as $x \triangle x = x^{\perp} \triangle x^{\perp} \le x^{\perp} \vee x^{\perp} = x^{\perp}$. This implies that $x \triangle x \le x \wedge x^{\perp} = \mathbf{0}$.

($d_4$): If there exists an element $x \in L$ such that $x \ne \mathbf{0}$, then $L$ is a nontrivial OCL and therefore $\mathbf{1} \ne \mathbf{0}$.

As for the rest of the proof of Proposition 3.2, we have $x^{\perp_{D_L}} = x \triangle_{D_L} \mathbf{1} = x \triangle_L \mathbf{1} = x^{\perp_L}$. □

**Corollary 3.3.** *Let $L$ be a finite ODL. Then $|L| = 2^n$, where $n$ is a natural number.*

*Proof.* It follows from Propositions 3.2 and 2.3. □

**Proposition 3.4.** *Let $L$ be an ODL and let $x, y \in L$. Then*

$$(x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) \le x \triangle y \le (x \vee y) \wedge (x \wedge y)^{\perp}.$$

*Proof.* The property ($D_3$) together with the properties (d), (e) of Proposition 2.5 imply that $x \triangle y \le x \vee y$, $x \triangle y \le x^{\perp} \vee y^{\perp} = (x \wedge y)^{\perp}$, $x \wedge y^{\perp} \le x \triangle y$ and $x^{\perp} \wedge y \le x \triangle y$. □

The following result shows that we do not leave the realm of orthomodular lattices when we deal with ODLs.

**Theorem 3.5.** *Let $L$ be an ODL. Then its support $L_{\mathrm{supp}}$ is an OML.*

*Proof.* Suppose that $x, y \in L$, $x \le y$, $y \wedge x^{\perp} = \mathbf{0}$. Let us prove that $x = y$ (see Proposition 1.2). Since $x \le y$, we conclude that $(x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = y \wedge x^{\perp} = \mathbf{0}$ and $(x \vee y) \wedge (x \wedge y)^{\perp} = y \wedge x^{\perp} = \mathbf{0}$. By Proposition 3.4, we see that $x \triangle y = \mathbf{0}$. By Proposition 2.5(i), we infer that $x = y$. □

Following our convention, all notions defined for OMLs or difference algebras can be transferred to any ODL $L$ by considering the support $L_{\mathrm{supp}}$ or the difference algebra $D_L$. Thus, for instance, if $L$ is an ODL and $x, y \in L$ then we say that the elements $x, y$ *commute* (in $L$) if they commute in the OML $L_{\mathrm{supp}}$.

**Proposition 3.6.** *Let $L$ be an ODL and let $x, y \in L$. Then the following conditions are equivalent:*

(a) $x\, C\, y$,
(b) $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp})$,
(c) $x \triangle y = (x \vee y) \wedge (x \wedge y)^{\perp}$,
(d) $x\, C\, (x \triangle y)$.

*Proof.* Let us first prove that (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c). Suppose therefore that $x\, C\, y$. Then $(x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = (x \vee y) \wedge (x \wedge y)^{\perp}$, and the equalities $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = (x \vee y) \wedge (x \wedge y)^{\perp}$ follow from Proposition 3.4.

Further, let us prove that (b) $\Rightarrow$ (d) and (c) $\Rightarrow$ (d). For example, let $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp})$. As $x\, C\, x \wedge y^{\perp}$, $x\, C\, y \wedge x^{\perp}$, we have $x\, C\, ((x \wedge y^{\perp}) \vee (y \wedge x^{\perp}))$ and therefore $x\, C\, (x \triangle y)$.

Finally, let us prove that (d) $\Rightarrow$ (a). Let $x\, C\, (x \triangle y)$. From the implication (a) $\Rightarrow$ (d) we have $x\, C\, (x \triangle (x \triangle y))$ but $x \triangle (x \triangle y) = y$. $\square$

Let us exhibit some simple examples of ODLs. Firstly, each Boolean algebra can be made into an ODL as the following proposition shows.

**Proposition 3.7.** *Let $B$ be a BA. There is exactly one mapping $\triangle \colon \dot{B} \times \dot{B} \to \dot{B}$ which fulfils all the conditions* ($D_1$), ($D_2$) *and* ($D_3$) *of Definition* 3.1.

*Proof.* To prove the existence, take for $\triangle$ the standard symmetric difference in Boolean algebras. In other words, let us set $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp})$. The properties ($D_1$), ($D_2$) and ($D_3$) of Definition 3.1 are then obviously fulfilled.

Let us prove the uniqueness of $\triangle$. Let $\triangle_1 \colon \dot{B} \times \dot{B} \to \dot{B}$ be a mapping that fulfils conditions ($D_1$), ($D_2$) and ($D_3$). Thus, the pair $(B, \triangle_1)$ is an ODL. If $x, y \in B$, then $x\, C\, y$, and so $x \triangle_1 y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = x \triangle y$ (Proposition 3.6). $\square$

In view of Proposition 3.7 we can (and shall) view any Boolean algebra as an ODL with uniquely defined operation $\triangle$.

**Example 3.8.** Let $L$ be the OML $MO_3$ (see, e.g., [16]) with the elements $\{0, 1, x, x^{\perp}, y, y^{\perp}, z, z^{\perp}\}$. Then one can easily show that there is exactly one mapping $\triangle \colon \dot{L} \times \dot{L} \to \dot{L}$ such that $x \triangle y = z$ and $(L, \triangle)$ is an ODL. The ODL obtained in this way will again be denoted by $MO_3$.

There are much more involved examples of ODLs than those depicted in Proposition 3.7 and Example 3.8, of course. We will meet them later. For the reader's intuition, Construction 8.5 dealt with at the end of the paper—a kind of horizontal sum—provides a large class of ODLs and gives some more idea about their intrinsic properties (see also Proposition 8.7). The construction is independent of the other text and can be read now, but Section 8 also brings certain applications.

Let us shortly consider the independence of the conditions required for the operation $\triangle$ to define an ODL (Definition 3.1). First, let us recall some notions from model theory. If $\varphi$ is a (first-order) formula, let us denote by $\mathrm{Lng}(\varphi)$ (*language* of $\varphi$) the set of all relational and operational symbols which occur in the formula $\varphi$. If $S$ is a set of (first-order) formulas, let us put $\mathrm{Lng}(S) = \bigcup_{\varphi \in S} \mathrm{Lng}(\varphi)$.

Let $T$ be a (first-order) theory and let $S$ be a set of formulas. Let us write $\mathcal{L} = \mathrm{Lng}(T \cup S)$. Then the set $S$ is called *independent relative to the theory $T$* (abbr., *$T$-independent*) if, for any formula $\varphi \in S$, there is a structure $A$ of $\mathcal{L}$ such that $A \models T \cup (S \setminus \{\varphi\})$ but $A \not\models \varphi$. Obviously, if $T_1 \subseteq T_2$ and $S$ is $T_2$-independent, then $S$ is $T_1$-independent as well.

Let $\mathsf{BA}$ be the theory of Boolean algebras ($\mathsf{BA}$ is a theory of the language $\{\wedge, \vee, {}^{\perp}, \mathbf{0}, \mathbf{1}\}$, thus the symbol $\triangle$ does not occur in the language of $\mathsf{BA}$). As usual, the formula $s \le t$ is an abbreviation of the formula $s \wedge t \approx s$.

**Proposition 3.9.** *Suppose that*

$$S = \{x \triangle (y \triangle z) \approx (x \triangle y) \triangle z,\ x \triangle \mathbf{1} \approx x^{\perp},\ \mathbf{1} \triangle x \approx x^{\perp},\ x \triangle y \le x \vee y\}.$$

*Then $S$ is $\mathsf{BA}$-independent.*

*Proof.* Let us first deal with the formula $\mathbf{1} \triangle x \approx x^{\perp}$. Let $B = \{\mathbf{0}, \mathbf{1}, a, a^{\perp}\}$ be a four-element Boolean algebra. Let $\triangle\colon B^2 \to B$ be the mapping defined as follows:

| | $\mathbf{0}$ | $a$ | $a^{\perp}$ | $\mathbf{1}$ |
|---|---|---|---|---|
| $\mathbf{0}$ | $\mathbf{0}$ | $a$ | $a^{\perp}$ | $\mathbf{1}$ |
| $a$ | $a$ | $a$ | $a^{\perp}$ | $a^{\perp}$ |
| $a^{\perp}$ | $a^{\perp}$ | $a$ | $a^{\perp}$ | $a$ |
| $\mathbf{1}$ | $\mathbf{1}$ | $a$ | $a^{\perp}$ | $\mathbf{0}$ |

In particular, $a \triangle \mathbf{1} = a^{\perp}$ and $\mathbf{1} \triangle a = a$. Let us denote by $A$ the algebra $(B, \triangle)$. Then $A$ is of type $\{\wedge, \vee, {}^{\perp}, \mathbf{0}, \mathbf{1}, \triangle\}$ and, moreover, $A \models \mathsf{BA}$. Further, because $\mathbf{1} \triangle a = a \ne a^{\perp}$, we see that $A \not\models \mathbf{1} \triangle x \approx x^{\perp}$. It remains to prove that all the formulas $x \triangle (y \triangle z) \approx (x \triangle y) \triangle z$, $x \triangle \mathbf{1} \approx x^{\perp}$, $x \triangle y \le x \vee y$ hold in $A$. Let $x, y, z \in A$ be arbitrary elements. The validity of the conditions $x \triangle \mathbf{1} = x^{\perp}$, $x \triangle y \le x \vee y$ is immediately seen from the table that defines the operation $\triangle$. We have to check the associativity of $\triangle$. Let us distinguish three possibilities.

- If $z = \mathbf{0}$, then the equality follows from the neutrality of $\mathbf{0}$ with respect to $\triangle$.

- If $z = a$ or $z = a^{\perp}$, then the equality follows from the "right aggressiveness" of the elements $a, a^{\perp}$.
- Finally, let $z = \mathbf{1}$. Then $x \triangle (y \triangle z) = x \triangle (y \triangle \mathbf{1}) = x \triangle y^{\perp}$ and $(x \triangle y) \triangle z = (x \triangle y) \triangle \mathbf{1} = (x \triangle y)^{\perp}$. It remains to be checked that $x \triangle y^{\perp} = (x \triangle y)^{\perp}$, but it can be easily seen from the table.

As a result, the formula $\mathbf{1} \triangle x \approx x^{\perp}$ is "independent" of the other formulas.

The "independence" of the formula $x \triangle \mathbf{1} \approx x^{\perp}$ can be derived by considering the dually defined operation $\triangle$. Finally, the "independence" of formulas $x \triangle (y \triangle z) \approx (x \triangle y) \triangle z$ and $x \triangle y \leq x \vee y$ is obvious. □

## 4. Symmetric difference and OML-term operations

Let us show that, for an ODL $L$ and for $a, b \in L$, the element $a \triangle b$ is generally not expressible by means of elements $a, b$ and the operations $\wedge, \vee, {}^{\perp}, \mathbf{0}, \mathbf{1}$. This implies that the present approach essentially differs from the attempts made in [10, 9, 23]. Recall that the term $t$ is called an *OML-term* if $t$ contains only operation symbols from the set $\{\wedge, \vee, {}^{\perp}, \mathbf{0}, \mathbf{1}\}$.

**Proposition 4.1.** *Let $L$ be an ODL and let $a, b \in L$. Then there is an OML-term $t(x, y)$ such that $a \triangle b = t_L(a, b)$ if and only if $a \, C_L \, b$.*

*Proof.* If $a \, C_L \, b$, then by Proposition 3.6 we can take $t(x, y) = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp})$. Conversely, assume that $a \triangle b = t_L(a, b)$ for some OML-term $t(x, y)$. Let $\mathrm{F}_2$ be the free OML over the set $\{x, y\}$. We can assume that $t(x, y) \in \mathrm{F}_2$. Let $f \colon \mathrm{F}_2 \to L_{\mathrm{supp}}$ be the uniquely determined homomorphism such that $f(x) = a$, $f(y) = b$. Then $f(t(x, y)) = t_{L_{\mathrm{supp}}}(f(x), f(y)) = t_L(a, b) = a \triangle b$. Since $\mathrm{F}_2 \cong B_{(4)} \times \mathrm{MO}_2$ (see [1, p. 80]), $\mathrm{F}_2$ possesses exactly two blocks, say $B_1$ and $B_2$. Since $x \not\!C_{\mathrm{F}_2} \, y$, we can assume that $x \in B_1, y \in B_2$. Now, if, e.g., $t(x, y) \in B_1$, then $x \, C_{\mathrm{F}_2} \, t(x, y)$. But then $f(x) \, C_L \, f(t(x, y))$, i.e., $a \, C_L \, (a \triangle b)$. By Proposition 3.6 we see that $a \, C_L \, b$. □

Making use of Proposition 4.1 we are in a position to prove the following characterization of Boolean algebras in the classes $\mathcal{ODL}$ and $\mathcal{OML}$.

**Proposition 4.2.** *Let $L$ be an ODL. Then there is an OML-term $t$ such that $a \triangle b = t_L(a, b)$ for any $a, b \in L$ exactly when $L$ is a Boolean algebra.*

*Proof.* If $L$ is a Boolean algebra, then $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp})$. Conversely, suppose that there is an OML-term $t$ such that $a \triangle b = t_L(a, b)$ for any $a, b \in L$. Then, according to Proposition 4.1, $a \, C_L \, b$ holds for any $a, b \in L$. Hence $L$ is a Boolean algebra. □

**Proposition 4.3.** *Let $L$ be an OML. For $x, y \in L$, set $x \,\Delta_1\, y = (x \wedge y^\perp) \vee (y \wedge x^\perp)$ and $x \,\Delta_2\, y = (x \vee y) \wedge (x \wedge y)^\perp$. Then, for each $i \in \{1, 2\}$, the operation $\Delta_i$ is associative exactly when $L$ is a Boolean algebra.*

*Proof.* If $L$ is a Boolean algebra, then both $\Delta_1$, $\Delta_2$ are obviously associative. Conversely, suppose that $\Delta_1$ is associative. This means that $\Delta_1$ fulfils the condition ($\mathrm{D}_1$). The conditions ($\mathrm{D}_2$) and ($\mathrm{D}_3$) are fulfilled automatically. Hence the pair $(L, \Delta_1)$ is an ODL. As a consequence, $x \; C \; y$ for any $x, y \in L$ (Proposition 4.1). This shows that $L$ is a BA. The associativity of $\Delta_2$ is argued similarly. □

It should be noted that Proposition 4.3 can be proved directly without using the notion of ODL, see, e.g., [1, p. 272], [9] or [23].

## 5. Intrinsic properties of ODLs

Let us go on with the analysis of the algebraic properties of the operation $\triangle$. We shall be mainly interested in the commutativity relation in ODLs.

**Proposition 5.1.** *Let $L$ be an ODL and let $x, y \in L$. Then the following two statements hold true:*

$$x \le y \Leftrightarrow x \triangle y = y \wedge x^\perp, \qquad x \perp y \Leftrightarrow x \triangle y = x \vee y$$

*(where $x \perp y$ stands for $x \le y^\perp$).*

*Proof.* Suppose first that $x \le y$. Then $x \; C \; y$ and $x \wedge y^\perp = \mathbf{0}$. As a consequence (Proposition 3.6), $x \triangle y = (x \wedge y^\perp) \vee (y \wedge x^\perp) = y \wedge x^\perp$. Conversely, suppose that $x \triangle y = y \wedge x^\perp$. Then

$$x = x \triangle \mathbf{0} = x \triangle (y \triangle y) = (x \triangle y) \triangle y \le (x \triangle y) \vee y = (y \wedge x^\perp) \vee y = y.$$

Let us take up the second equivalence. Using the equalities $(x \triangle y^\perp)^\perp = x \triangle y^{\perp\perp} = x \triangle y$, we have

$$\begin{aligned} x \perp y &\Leftrightarrow x \le y^\perp \Leftrightarrow x \triangle y^\perp = y^\perp \wedge x^\perp \Leftrightarrow (x \triangle y^\perp)^\perp = (y^\perp \wedge x^\perp)^\perp \\ &\Leftrightarrow x \triangle y = y \vee x. \end{aligned}$$ □

**Lemma 5.2.** *Let $L$ be an OML. Let $x, y, x_1, x_2 \in L$ and let $y = x_1 \vee x_2$, $x_1 \le x$, $x_2 \le x^\perp$. Then $x \; C \; y$ and $x_1 = y \wedge x$, $x_2 = y \wedge x^\perp$.*

*Proof.* Since $x_1 \le x$ and $x_2 \le x^\perp$, we see that $x_1 \le x \le x_2^\perp$. Thus, the elements $x_1, x_2, x$ are mutually commutative. By Proposition 1.7 we infer that $x \; C \; (x_1 \vee x_2)$ and therefore $x \; C \; y$. Moreover,

$$\begin{aligned} y \wedge x &= (x_1 \vee x_2) \wedge x = (x_1 \wedge x) \vee (x_2 \wedge x) = x_1 \vee \mathbf{0} = x_1, \quad \text{and} \\ y \wedge x^\perp &= (x_1 \vee x_2) \wedge x^\perp = (x_1 \wedge x^\perp) \vee (x_2 \wedge x^\perp) = \mathbf{0} \vee x_2 = x_2. \end{aligned}$$ □

**Proposition 5.3.** *Let $L$ be an ODL. Let $x, y, z \in L$ with $x \, C \, y$ and $x \, C \, z$. Then $x \, C \, (y \triangle z)$ and $x \wedge (y \triangle z) = (x \wedge y) \triangle (x \wedge z)$.*

*Proof.* The commutativity of the pair $x \, C \, y$ and $x \, C \, z$ yields the equations

$$y = (y \wedge x) \vee (y \wedge x^{\perp}) \quad \text{and} \quad z = (z \wedge x) \vee (z \wedge x^{\perp}).$$

Since $(y \wedge x) \perp (y \wedge x^{\perp})$ and $(z \wedge x) \perp (z \wedge x^{\perp})$, we see by Proposition 5.1 that $y = (y \wedge x) \triangle (y \wedge x^{\perp})$ and $z = (z \wedge x) \triangle (z \wedge x^{\perp})$. But we also have

$$\begin{aligned} y \triangle z &= [(y \wedge x) \triangle (y \wedge x^{\perp})] \triangle [(z \wedge x) \triangle (z \wedge x^{\perp})] \\ &= [(y \wedge x) \triangle (z \wedge x)] \triangle [(y \wedge x^{\perp}) \triangle (z \wedge x^{\perp})]. \end{aligned}$$

Let us write $x_1 = (y \wedge x) \triangle (z \wedge x)$ and $x_2 = (y \wedge x^{\perp}) \triangle (z \wedge x^{\perp})$. Then we have $x_1 \leq (y \wedge x) \vee (z \wedge x) \leq x$. Analogously, $x_2 \leq x^{\perp}$. This implies that $x_1 \perp x_2$. By Proposition 5.1, $y \triangle z = x_1 \vee x_2$. Finally, the proof is completed by using Lemma 5.2. □

**Proposition 5.4.** *Let $L$ be an ODL and let $x \in L$. Then the set $C(x)$ (see Definition* 1.3*) is a subalgebra of $L$.*

*Proof.* According to Proposition 1.5, the set $C(x)$ is a subalgebra of $L_{\text{supp}}$. The closedness of the operation $\triangle$ on $C(x)$ follows from Proposition 5.3. □

**Proposition 5.5.** *Let $L$ be an ODL and let $x, y \in L$. Then*

(a) $x \vee (x \triangle y) = x \vee y$,
(b) $x \wedge (x \triangle y) = x \wedge y^{\perp}$.

*Proof.* Recall the convention of the preference of $\triangle$ over the operations $\wedge$ and $\vee$ (thus, for instance, $x \vee y \triangle z$ means $x \vee (y \triangle z)$).

(a): The inequality $x \vee x \triangle y \leq x \vee y$ is obvious. We have to show that $x \vee y \leq x \vee x \triangle y$. But $x \leq x \vee x \triangle y$ and therefore we need to show $y \leq x \vee x \triangle y$. According to ($D_3$), we have $x \vee x \triangle y \geq x \triangle (x \triangle y) = y$.

(b): The equality follows from (a) via the following calculation:

$$x \wedge x \triangle y = (x \wedge x \triangle y)^{\perp\perp} = (x^{\perp} \vee x^{\perp} \triangle y)^{\perp} = (x^{\perp} \vee y)^{\perp} = x \wedge y^{\perp}. \qquad \square$$

It is worthwhile observing that the equality (a) of Proposition 5.5 can be viewed as a strengthening of the condition ($D_3$) from the definition of ODLs.

**Proposition 5.6.** *Let $L$ be an ODL and let $x \in L$. Then either $x$ lies in exactly one block or $x$ lies in at least three blocks.*

*Proof.* Seeking a contradiction, let $x$ lie in exactly two blocks $B_1$ and $B_2$. Then there exist elements $y \in B_1$, $z \in B_2$ such that $y, z$ do not commute. Since $x \, C \, y$, $x \, C \, z$, we see that $x \, C \, (y \triangle z)$ (Proposition 5.3). As a consequence, either $y \triangle z \in B_1$ or $y \triangle z \in B_2$. In view of the symmetry in the role of $y, z$, let us assume that

$y \triangle z \in B_1$. Since $y \in B_1$, we infer that $y \; C \; (y \triangle z)$. By Proposition 3.6, we see that $y \; C \; z$, which is absurd. □

Let us take up the intervals in ODLs. Consider first the situation in OMLs. Let $K$ be an OML and let $a \in K$. Let us write $[\mathbf{0}, a]_K = \{x \in K;\ x \leq a\}$. As known, the interval $[\mathbf{0}, a]$ constitutes an OML. We will denote it by $K^a$. Let us shortly recall the construction of $K^a$ (see, for example, [16, p. 20]): If $x, y \in [\mathbf{0}, a]$, then $x \wedge y \in [\mathbf{0}, a]$ and $x \vee y \in [\mathbf{0}, a]$. The element $\mathbf{0}$, resp. $a$, is the least, resp. the greatest, element of $K^a$. The orthocomplement of $x$ in $K^a$, $x^{\perp_a}$, is defined by setting $x^{\perp_a} = x^{\perp_K} \wedge a$. It can be easily seen that $K^a = ([\mathbf{0}, a], \wedge, \vee, {}^{\perp_a}, \mathbf{0}, a)$ is an OML.

Now let $L$ be an ODL and let $a \in L$. If $x, y \in [\mathbf{0}, a]$, then $x \triangle y \in [\mathbf{0}, a]$. Let us consider the algebra $L^a = ([\mathbf{0}, a], \wedge, \vee, {}^{\perp_a}, \mathbf{0}, a, \triangle) = ((L_{\mathrm{supp}})^a, \triangle)$.

**Proposition 5.7.** *Let $L$ be an ODL and let $a \in L$. Then the algebra $L^a$ is again an ODL. Moreover, $(L^a)_{\mathrm{supp}} = (L_{\mathrm{supp}})^a$.*

*Proof.* It is sufficient to show that the conditions ($\mathrm{D}_1$), ($\mathrm{D}_2$) and ($\mathrm{D}_3$) of Definition 3.1 hold in $L^a$. The conditions ($\mathrm{D}_1$) and ($\mathrm{D}_3$) can be verified easily. It remains to check the condition ($\mathrm{D}_2$). For that, suppose $x \in [\mathbf{0}, a]$. Then $x \triangle_{L^a} \mathbf{1}_{L^a} = x \triangle a$. From Proposition 5.1 we obtain $x \triangle a = a \wedge x^{\perp} = x^{\perp_a}$. The equality $\mathbf{1}_{L^a} \triangle_{L^a} x = x^{\perp_a}$ follows from the commutativity of $\triangle$. The equality $(L^a)_{\mathrm{supp}} = (L_{\mathrm{supp}})^a$ is then obvious. □

In the following proposition we show that an ODL can be decomposed with the help of a central element by the way analogous to the situation known in OMLs.

**Proposition 5.8.** *Suppose that $L$ is an ODL and $a \in C(L)$. Then $L \cong L^a \times L^{a^{\perp}}$. More explicitly, the mapping $h \colon L \to L^a \times L^{a^{\perp}}$ defined by $h(x) = (x \wedge a,\, x \wedge a^{\perp})$ is an isomorphism of $L$ onto $L^a \times L^{a^{\perp}}$.*

*Proof.* The mapping $h$ is an isomorphism between the OMLs $L_{\mathrm{supp}}$ and $(L^a)_{\mathrm{supp}} \times (L^{a^{\perp}})_{\mathrm{supp}}$, see [16, p. 20]. It remains to show that the mapping $h$ preserves the operation $\triangle$. Suppose that $x, y \in L$. Then by Proposition 5.3 we consecutively obtain

$$\begin{aligned} h(x \triangle_L y) &= ((x \triangle_L y) \wedge a, (x \triangle_L y) \wedge a^{\perp}) \\ &= ((x \wedge a) \triangle_L (y \wedge a), (x \wedge a^{\perp}) \triangle_L (y \wedge a^{\perp})) \\ &= ((x \wedge a) \triangle_{L^a} (y \wedge a), (x \wedge a^{\perp}) \triangle_{L^{a^{\perp}}} (y \wedge a^{\perp})) \\ &= (x \wedge a, x \wedge a^{\perp}) \triangle_{L^a \times L^{a^{\perp}}} (y \wedge a, y \wedge a^{\perp}) = h(x) \triangle_{L^a \times L^{a^{\perp}}} h(y). \end{aligned}$$ □

## 6. Set-representable ODLs

A natural question occurs if (when) an ODL allows for a set representation. We will see later that there are some ODLs which are not set-representable and we will present a method how to construct them (Section 7). In this section, however, we concentrate on those ODLs which are set-representable. We will show that these ODLs form a variety and we will characterize this variety. Prior to that, let us introduce a few notions of the theory of orthomodular lattices. As usual, if $A, B$ are sets, we write $A \setminus B = \{x \in A;\ x \notin B\}$ and $A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$.

**Definition 6.1.** Let $X$ be a set and let $\Omega \subseteq \mathcal{P}(X)$, where $\mathcal{P}(X)$ stands for the set of all subsets of $X$. Then the pair $(X, \Omega)$ is said to be a *D-ring* if

(1) $X \in \Omega$,
(2) for any $A, B \in \Omega$ we have $A \,\Delta\, B \in \Omega$.

**Proposition 6.2.** *Let $(X, \Omega)$ be a D-ring. Then*

(a) $\emptyset \in \Omega$,
(b) *for any $A \in \Omega$ we have $A^c \in \Omega$ (where $A^c$ is the complement of $A$ in $X$, $A^c = X \setminus A$),*
(c) *for any $A, B \in \Omega$ the following implication holds: if $A \cap B = \emptyset$, then $A \cup B \in \Omega$.*

*Proof.* A routine verification. □

**Definition 6.3.** Let $L$ be an ODL. Let us say that $L$ is a *set-representable ODL* (abbr., *SRODL*) if there is a $D$-ring $(X, \Omega)$ such that $(\dot{L}, \leq, \mathbf{0}, \mathbf{1}, \triangle)$ is isomorphic to $(\Omega, \subseteq, \emptyset, X, \Delta)$.

Let us denote by $\mathcal{SRODL}$ the class of all set-representable ODLs.

**Proposition 6.4.** *Let $L$ be an ODL. Then $L$ is an SRODL if and only if there is a set $M$ and a mapping $f\colon \dot{L} \to \mathcal{P}(M)$ such that the following two conditions hold true for any $x, y \in L$:*

$$x \leq y \Leftrightarrow f(x) \subseteq f(y), \qquad f(x \triangle_L y) = f(x) \,\Delta\, f(y).$$

*Proof.* Let us prove that the conditions are sufficient; the rest is obvious. Set $X = f(\mathbf{1}) \subseteq M$, $\Omega = f[\dot{L}] = \{f(x);\ x \in L\}$. Then the second condition implies that the pair $(X, \Omega)$ is a $D$-ring. Further, the first condition implies that $f$ is an isomorphism of the poset $(L, \leq)$ onto the poset $(\Omega, \subseteq)$. Finally, $f(\mathbf{0}) = f(\mathbf{0} \triangle_L \mathbf{0}) = f(\mathbf{0}) \,\Delta\, f(\mathbf{0}) = \emptyset$. □

The class of SRODLs is considerably large and contains some rather complex as well as some rather simple ODLs. The simplest ones are presented in the following example.

**Example 6.5.** (a) Every Boolean algebra is an SRODL.
(b) The ODL $\mathrm{MO}_3$ (see Example 3.8) is an SRODL.

*Proof.* (a) This follows from the Stone representation of Boolean algebras.

(b) Let $S = \{\mathbf{0}, \mathbf{1}, x, x^\perp, y, y^\perp, z, z^\perp\}$ be the underlying set of $\mathrm{MO}_3$. Set $M = \{1,2,3,4\}$. Then the mapping $f\colon S \to \mathcal{P}(M)$ defined by putting $f(\mathbf{0}) = \emptyset$, $f(\mathbf{1}) = M$, $f(x) = \{1,2\}$, $f(x^\perp) = \{3,4\}$, $f(y) = \{1,3\}$, $f(y^\perp) = \{2,4\}$, $f(z) = \{2,3\}$, $f(z^\perp) = \{1,4\}$ has both properties of Proposition 6.4. (Let us note that $f[S] = \{A \subseteq M;\ |A| \text{ is an even number}\}$.) $\square$

In the rest of this section we shall be proving that the class $\mathcal{SRODL}$ is a variety. It should be noted that the central strategic line of the investigation of set-representable OMLs as used in [11, 19, 20, 21] was instrumental in places. However, the presence of the operation $\triangle$ required to invent some new techniques. These techniques—in particular those concerning the ODL evaluations—seem to be of interest in their own right.

Let $\oplus$ stand for addition modulo 2 on the set $\{0,1\}$ (i.e., $0 \oplus 0 = 1 \oplus 1 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1$). The following notion is crucial in characterizing SRODLs:

**Definition 6.6.** Let $L$ be an ODL and let $e\colon L \to \{0,1\}$. Then $e$ is said to be an *ODL-evaluation* (abbr., *evaluation*) on $L$ if the following properties are fulfilled for any $x, y \in L$:

($\mathrm{E}_1$) $e(\mathbf{1}_L) = 1$,
($\mathrm{E}_2$) $x \le y \Rightarrow e(x) \le e(y)$,
($\mathrm{E}_3$) $e(x \triangle y) = e(x) \oplus e(y)$.

Let $\mathcal{E}(L)$ be the set of all evaluations on $L$.

The following result provides a characterization of SRODLs in terms of $\mathcal{E}(L)$.

**Theorem 6.7.** *Let $L$ be an ODL. Then $L$ is an SRODL if and only if*

$$\forall a, b \in L, a \not\le b\ \exists e \in \mathcal{E}(L) : e(a) = 1,\ e(b) = 0.$$

*Proof.* ($\Rightarrow$): Suppose that $(\dot{L}, \le, \mathbf{0}, \mathbf{1}, \triangle) = (\Omega, \subseteq, \emptyset, X, \Delta)$, where $(X, \Omega)$ is a $D$-ring.

Choosing an $x \in X$, define a mapping $e_x \colon \Omega \to \{0,1\}$ by

$$e_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Let us check that the mapping $e_x$ is an evaluation on $L$. Obviously, $e_x(\mathbf{1}_L) = e_x(X) = 1$. If $A, B \in \Omega$ and $A \leq_L B$, then $A \subseteq B$, and therefore $e_x(A) \leq e_x(B)$. That $e_x(A \triangle B) = e_x(A) \oplus e_x(B)$ follows from a straightforward case analysis.

Finally, if $A, B \in \Omega$ and $A \not\subseteq B$, then there exists an $x \in A$ such that $x \notin B$. Then $e_x(A) = 1$ and $e_x(B) = 0$.

($\Leftarrow$): Let us assume that the condition on evaluations is fulfilled. For $x \in L$, let us write $f(x) = \{e \in \mathcal{E}(L);\ e(x) = 1\}$. We will check that this mapping $f \colon L \to \mathcal{P}(\mathcal{E}(L))$ fulfills both conditions of Proposition 6.4. First suppose that $x \leq y$. Let $e \in f(x)$. Thus, $e(x) = 1$. According to ($\mathrm{E}_2$) we have $e(x) \leq e(y)$. As a result, $e(y) = 1$ and hence $e \in f(y)$. We have shown that $f(x) \subseteq f(y)$. Conversely, suppose that $x \not\leq y$. Then there is $e \in \mathcal{E}(L)$ such that $e(x) = 1$, $e(y) = 0$. We see that $f(x) \not\subseteq f(y)$. To complete the proof, we use the equalities

$$\begin{aligned} f(x \triangle_L y) &= \{e \in \mathcal{E}(L);\ e(x \triangle_L y) = 1\} = \{e \in \mathcal{E}(L);\ e(x) \oplus e(y) = 1\} \\ &= \big\{e \in \mathcal{E}(L);\ (e(x) = 1 \ \&\ e(y) = 0) \vee (e(x) = 0 \ \&\ e(y) = 1)\big\} \\ &= f(x) \,\Delta\, f(y). \qquad \square \end{aligned}$$

Let $L$ be an ODL. Consider the discrete topology on the set $\{0,1\}$ and form the topological product $\{0,1\}^L$. Then (Tichonov's Theorem) $\{0,1\}^L$ is a compact topological space.

**Lemma 6.8.** *$\mathcal{E}(L)$ is a closed subset in $\{0,1\}^L$.*

*Proof.* For any $x \in L$, let us denote by $\pi_x$ the $x$-th projection of $\{0,1\}^L$ onto $\{0,1\}$, i.e., for $e \in \{0,1\}^L$ let us have $\pi_x(e) = e(x)$. Then for $x \in L$ both the sets $\pi_x^{-1}(0)$ and $\pi_x^{-1}(1)$ are closed subsets in $\{0,1\}^L$.

Let us write $L^2_{\leq} = \{(x,y) \in L^2;\ x \leq y\}$. For $(x,y) \in L^2_{\leq}$, let us further write $\mathcal{R}_{(x,y)} = \{e \in \{0,1\}^L;\ e(x) \leq e(y)\}$ and, for $(x,y) \in L^2$, let us write $\mathcal{S}_{(x,y)} = \{e \in \{0,1\}^L;\ e(x \triangle y) = e(x) \oplus e(y)\}$. It is easy to see that

$$\mathcal{E}(L) = \pi_{\mathbf{1}_L}^{-1}(1) \cap \Big( \bigcap_{(x,y) \in L^2_{\leq}} \mathcal{R}_{(x,y)} \Big) \cap \Big( \bigcap_{(x,y) \in L^2} \mathcal{S}_{(x,y)} \Big).$$

It remains to show that the sets $\mathcal{R}_{(x,y)}$ and $\mathcal{S}_{(x,y)}$ are closed subsets in $\{0,1\}^L$. For $(x,y) \in L^2_{\leq}$ we have

$$\begin{aligned} \mathcal{R}_{(x,y)} &= \{e \in \{0,1\}^L;\ e(x) \leq e(y)\} \\ &= \big\{e \in \{0,1\}^L;\ e(x) = 0 \vee e(y) = 1\big\} = \pi_x^{-1}(0) \cup \pi_y^{-1}(1). \end{aligned}$$

For $(x, y) \in L^2$ we have

$$\begin{aligned}\mathcal{S}_{(x,y)} &= \{e \in \{0,1\}^L;\ e(x \bigtriangleup y) = e(x) \oplus e(y)\}\\ &= \big\{e \in \{0,1\}^L;\ (e(x) = 0\ \&\ e(y) = 0\ \&\ e(x \bigtriangleup y) = 0)\\ &\qquad \vee\ (e(x) = 0\ \&\ e(y) = 1\ \&\ e(x \bigtriangleup y) = 1)\\ &\qquad \vee\ (e(x) = 1\ \&\ e(y) = 0\ \&\ e(x \bigtriangleup y) = 1)\\ &\qquad \vee\ (e(x) = 1\ \&\ e(y) = 1\ \&\ e(x \bigtriangleup y) = 0)\big\}\\ &= \big[\pi_x^{-1}(0) \cap \pi_y^{-1}(0) \cap \pi_{x \triangle y}^{-1}(0)\big] \cup \big[\pi_x^{-1}(0) \cap \pi_y^{-1}(1) \cap \pi_{x \triangle y}^{-1}(1)\big]\\ &\qquad \cup \big[\pi_x^{-1}(1) \cap \pi_y^{-1}(0) \cap \pi_{x \triangle y}^{-1}(1)\big] \cup \big[\pi_x^{-1}(1) \cap \pi_y^{-1}(1) \cap \pi_{x \triangle y}^{-1}(0)\big]. \qquad \square\end{aligned}$$

Let us recall the definition of an ideal (resp. filter) in an OML [1, 16]:

**Definition 6.9.** Let $L$ be an OML and let $I \subseteq L$, $F \subseteq L$.

(A) Let us call $I$ an *ideal* in $L$ if the following are satisfied for any $x, y \in L$:
   (a) $\mathbf{0}_L \in I$,
   (b) $x \in I,\ y \leq x \Rightarrow y \in I$,
   (c) $x, y \in I \Rightarrow x \vee y \in I$.
   The ideal $I$ is called *proper* if $\mathbf{1}_L \notin I$ (or, equivalently, $I \neq \dot{L}$).

(B) Let us call $F$ a *filter* in $L$ if the following are satisfied for any $x, y \in L$:
   (a) $\mathbf{1}_L \in F$,
   (b) $x \in F,\ x \leq y \Rightarrow y \in F$,
   (c) $x, y \in F \Rightarrow x \wedge y \in F$.
   The filter $F$ is called *proper* if $\mathbf{0}_L \notin F$ (or, equivalently, $F \neq \dot{L}$).

In accord with our convention, if $L$ is an ODL and $I$, resp. $F$, is an ideal, resp. filter, in $L_{\text{supp}}$ we shall refer to it as an ideal, resp. filter, in the ODL $L$.

**Proposition 6.10.** *Let $L$ be an SRODL. Let $I$ be an ideal in $L$ and let $F$ be a filter in $L$ with $I \cap F = \emptyset$. Then there is $e \in \mathcal{E}(L)$ such that $e[I] = \{0\}$ and $e[F] = \{1\}$.*

*Proof.* Let $(x, y) \in I \times F$. Set $\mathcal{E}_{(x,y)} = \{e \in \mathcal{E}(L);\ e(x) = 0,\ e(y) = 1\}$. Since $\mathcal{E}_{(x,y)} = \mathcal{E}(L) \cap \pi_x^{-1}(0) \cap \pi_y^{-1}(1)$, $\mathcal{E}_{(x,y)}$ is a closed subset in $\{0,1\}^L$.

Let us now go over all choices of $(x, y) \in I \times F$. We claim that $\{\mathcal{E}_{(x,y)}\}_{(x,y) \in I \times F}$ is a centered system of sets, i.e., each finite subsystem has a non-empty intersection. Indeed, let $(x_1, y_1), \ldots, (x_n, y_n) \in I \times F$. Put $x = x_1 \vee \cdots \vee x_n \in I$ and $y = y_1 \wedge \cdots \wedge y_n \in F$. Since $y \not\leq x$, so by Theorem 6.7 we see that $\mathcal{E}_{(x,y)} \neq \emptyset$. Take some $e \in \mathcal{E}_{(x,y)}$. Then the property ($\mathrm{E}_2$) implies that $e(x_1) = \cdots = e(x_n) = 0$ and $e(y_1) = \cdots = e(y_n) = 1$. It follows that $e \in \mathcal{E}_{(x_1,y_1)} \cap \cdots \cap \mathcal{E}_{(x_n,y_n)}$.

We have verified that $\{\mathcal{E}_{(x,y)}\}_{(x,y) \in I \times F}$ is a centered system of closed subsets in the compact topological space $\{0,1\}^L$. As a result, $\bigcap_{(x,y) \in I \times F} \mathcal{E}_{(x,y)} \neq \emptyset$. Take an arbitrary $e \in \bigcap_{(x,y) \in I \times F} \mathcal{E}_{(x,y)}$. Then $e$ is the evaluation we looked for. $\square$

**Proposition 6.11.** *Let $K, L$ be ODLs. Let $f\colon K \to L$ be a surjective homomorphism. Let $e \in \mathcal{E}(K)$ be such that $e(x) = 0$ for any $x \in f^{-1}(\mathbf{0}_L)$. Then there exists $\widetilde{e} \in \mathcal{E}(L)$ such that $e = f \circ \widetilde{e}$.*

*Proof.* If $y \in L$ with $y = f(x)$, then we set $\widetilde{e}(y) = e(x)$. Let us show that the definition of $\widetilde{e}$ is correct. To this end, suppose that $y = f(x_1) = f(x_2)$. Then $f(x_1) \bigtriangleup_L f(x_2) = \mathbf{0}_L$ and therefore $f(x_1 \bigtriangleup_K x_2) = \mathbf{0}_L$. Hence $e(x_1 \bigtriangleup_K x_2) = 0$. This implies that $e(x_1) \oplus e(x_2) = 0$ and therefore $e(x_1) = e(x_2)$. Let us prove in the next step that $\widetilde{e} \in \mathcal{E}(L)$. We have to verify the conditions of Definition 6.6.

($E_1$) $\widetilde{e}(\mathbf{1}_L) = \widetilde{e}(f(\mathbf{1}_K)) = e(\mathbf{1}_K) = 1$.

($E_2$) Suppose that $y_1, y_2 \in L$, $y_1 \leq y_2$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2) = y_1 \wedge y_2 = y_1$. Moreover, $x_1 \wedge x_2 \leq x_2$. This yields that $e(x_1 \wedge x_2) \leq e(x_2)$. As a consequence, $\widetilde{e}(y_1) = e(x_1 \wedge x_2) \leq e(x_2) = \widetilde{e}(y_2)$.

($E_3$) Suppose that $y_1, y_2 \in L$ and $y_1 = f(x_1)$, $y_2 = f(x_2)$. Then we see that

$$\begin{aligned}\widetilde{e}(y_1 \bigtriangleup_L y_2) &= \widetilde{e}(f(x_1) \bigtriangleup_L f(x_2)) = \widetilde{e}(f(x_1 \bigtriangleup_K x_2)) = e(x_1 \bigtriangleup_K x_2) \\ &= e(x_1) \oplus e(x_2) = \widetilde{e}(y_1) \oplus \widetilde{e}(y_2).\end{aligned}$$ □

**Theorem 6.12.** *The class $\mathcal{SRODL}$ of all set-representable ODLs forms a variety.*

*Proof.* We shall show that the class $\mathcal{SRODL}$ is closed under the formation of subalgebras, products and homomorphic images.

(a) Closedness under subalgebras:
Suppose that $L \in \mathcal{SRODL}$ and $K$ is a subalgebra of $L$. Suppose $a, b \in K$, $a \not\leq b$. Then there exists $e \in \mathcal{E}(L)$ such that $e(a) = 1$ and $e(b) = 0$. It suffices to observe that the restriction of $e$ to $K$ is an evaluation on $K$.

(b) Closedness under products:
Suppose $L_i \in \mathcal{SRODL}$, $i \in I$. For any $j \in I$, let us denote by $\pi_j$ the $j$-th projection $\prod_{i\in I} L_i \to L_j$. Suppose that $\mathbf{a}, \mathbf{b} \in \prod_{i\in I} L_i$ and $\mathbf{a} \not\leq \mathbf{b}$. Then there exists an index $i_0 \in I$ such that $a_{i_0} \not\leq b_{i_0}$, where $a_{i_0} = \pi_{i_0}(\mathbf{a})$, $b_{i_0} = \pi_{i_0}(\mathbf{b})$. Take an evaluation $e \in \mathcal{E}(L_{i_0})$ such that $e(a_{i_0}) = 1$ and $e(b_{i_0}) = 0$. Consider the evaluation $\pi_{i_0} \circ e$ on $\prod_{i\in I} L_i$. Then $(\pi_{i_0} \circ e)(\mathbf{a}) = e(\pi_{i_0}(\mathbf{a})) = e(a_{i_0}) = 1$ and $(\pi_{i_0} \circ e)(\mathbf{b}) = e(\pi_{i_0}(\mathbf{b})) = e(b_{i_0}) = 0$.

(c) Closedness under homomorphic images:
Suppose that $f\colon K \to L$ is a surjective homomorphism and $K \in \mathcal{SRODL}$. Suppose that $x, y \in L$ and $x \not\leq y$. Then there are $x_1, y_1 \in K$ such that $x = f(x_1)$, $y = f(y_1)$. Write $I = \{a \in K;\ a \leq b \vee y_1 \text{ for some } b \in f^{-1}(\mathbf{0}_L)\}$ and write $F = [x_1, \mathbf{1}_L] = \{a \in K;\ x_1 \leq a\}$. Then $I$ is an ideal in $K$ and $F$ is a filter in $K$. Let us prove that $I \cap F = \emptyset$. Looking for a contradiction, let $a \in I \cap F$. Then $a \leq b \vee y_1$ for some $b \in f^{-1}(\mathbf{0}_L)$ and $x_1 \leq a$. This implies that $x_1 \leq b \vee y_1$ and therefore $f(x_1) \leq f(b \vee y_1) = f(b) \vee f(y_1) = \mathbf{0}_L \vee f(y_1) = f(y_1)$. This is a contradiction.

By Proposition 6.10 we can find an $e \in \mathcal{E}(K)$ such that $e[I] = \{0\}$ and $e[F] = \{1\}$. Since $f^{-1}(\mathbf{0}_L) \subseteq I$, we infer by Proposition 6.11 that there is an $\widetilde{e} \in \mathcal{E}(L)$ such that $e = f \circ \widetilde{e}$. In particular, we obtain $\widetilde{e}(x) = \widetilde{e}(f(x_1)) = e(x_1) = 1$ and $\widetilde{e}(y) = e(y_1) = 0$. This concludes the proof. $\square$

## 7. Quasiideals and d-ideals

We have shown in the previous section that the possibility to find a set representation of an ODL is closely related to the existence of evaluations with certain properties. We have also seen that, if $e$ is an evaluation on an ODL $L$, then the set $e^{-1}(0)$ shares certain properties with the prime ideals in Boolean algebras. It is these properties which serve as another characterization of set representation (see Theorem 7.17). This characterization will be then applied in the constructions of Section 8.

Through this section, let $D$ denote a difference algebra (see Definition 2.1).

**Definition 7.1.** Let $I \subseteq D$. Let us say that $I$ is a *quasiideal* in $D$ provided the following two conditions are satisfied:

(a) $\mathbf{0} \in I$,
(b) $\forall x, y \in I : x \diamond y \in I$.

If, moreover, $\mathbf{1} \notin I$, let us say that $I$ is a *proper quasiideal.*

**Proposition 7.2.** *Let $I \subseteq D$ be a quasiideal. Then $I$ is proper exactly when, for any $x \in D$, at most one of the elements $x$, $x^\perp$ belongs to $I$.*

*Proof.* Let $I$ be proper. Suppose that both $x$ and $x^\perp$ belong to $I$. Then $x \diamond x^\perp$ belongs to $I$ and therefore $\mathbf{1} \in I$. This is a contradiction.

Conversely, suppose that at most one of the elements $x$, $x^\perp$ belongs to $I$. Since $\mathbf{0} \in I$, then $\mathbf{0}^\perp \notin I$, and therefore $\mathbf{1} \notin I$. $\square$

**Proposition 7.3.** *Let $I \subseteq D$ be a proper quasiideal. Let $x \in D$ be such an element that $x^\perp \notin I$. Set $J = I \cup \{i \diamond x;\ i \in I\}$. Then $J$ is also a proper quasiideal in $D$ and, moreover, $x \in J$ and $x^\perp \notin J$.*

*Proof.* Obviously, $J$ is a quasiideal. Let us show that $\mathbf{1} \notin J$. Suppose that $\mathbf{1} \in J$. Then $\mathbf{1} = i \diamond x$ for some $i \in I$. Since the equality $\mathbf{1} = i \diamond x$ implies $x^\perp = i$, we see that $x^\perp \in I$, which is a contradiction. Further, $x = \mathbf{0} \diamond x \in J$. If $x^\perp \in J$, then $\mathbf{1} = x \diamond x^\perp \in J$, which is also a contradiction. $\square$

**Proposition 7.4.** *Let $I \subseteq D$ be a quasiideal. Then $I$ is a maximal proper quasiideal in $D$ exactly when, for any $x \in D$, either $x \in I$ or $x^\perp \in I$.*

*Proof.* ($\Rightarrow$): Suppose $I$ is a maximal proper quasiideal. Let $x \in D$. Seeking a contradiction, suppose that $x \notin I$, $x^\perp \notin I$. According to Proposition 7.3, there is a proper quasiideal, $J$, $I \subset J$, which contradicts the maximality of $I$. It follows that at least one of the elements $x$, $x^\perp$ belongs to $I$. Since $I$ is proper, both of $x$, $x^\perp$ cannot belong to $I$.

($\Leftarrow$): Suppose that the right-hand side condition is fulfilled. Since $\mathbf{0} \in I$, we see that $\mathbf{0}^\perp \notin I$, and therefore $\mathbf{1} \notin I$. As a result, $I$ is a proper quasiideal. Suppose further that $I \subset J$ and $J$ is a quasiideal in $D$. Let us show that $\mathbf{1} \in J$. Since $I \subset J$, there is an $x$ such that $x \in J$, $x \notin I$. Since $x \notin I$, we have $x^\perp \in I$ in view of our condition. Summarizing, we see that $x \in J$ and $x^\perp \in J$, and therefore $\mathbf{1} = x \diamond x^\perp \in J$. □

The property "for any $x \in D$, either $x \in I$ or $x^\perp \in I$" shall be referred to as "*the selectivity property*".

**Proposition 7.5.** *Let $I_0 \subseteq D$ be a proper quasiideal, $b \in D$ and $b^\perp \notin I_0$. Then there exists a maximal proper quasiideal $J$ in $D$ such that $I_0 \subseteq J$ and $b \in J$.*

*Proof.* Let $J_0 = I_0 \cup \{i \diamond b;\ i \in I_0\}$. By Proposition 7.3, $J_0$ is a proper quasiideal in $D$ and $b \in J_0$. Write $\mathcal{I} = \{I \subseteq \dot{D};\ I \text{ is a proper quasiideal in } D, J_0 \subseteq I\}$. Since $J_0 \in \mathcal{I}$, we see that $\mathcal{I} \neq \emptyset$. Further, the system $\mathcal{I}$ is closed under the formation of the union of its chains. By Zorn's Lemma the set $\mathcal{I}$ contains a maximal element, say some quasiideal $J$. Since $b \in J_0$ and $J_0 \subseteq J$, we have $b \in J$. Thus, $J$ is the maximal proper quasiideal we looked for. □

**Proposition 7.6.** *Let $I$ be a maximal proper quasiideal in $D$. Then $|D| = 2 \cdot |I|$.*

*Proof.* It is easily seen (Propositions 2.6 and 7.4) that $^\perp$ is a bijection between $I$ and $\dot{D} \setminus I$. □

**Proposition 7.7.** *Let $D_1$ and $D_2$ be DAs with $|D_1| = |D_2|$. Then $D_1 \cong D_2$ (i.e., the algebras $D_1, D_2$ are isomorphic).*

*Proof.* Choose two maximal proper quasiideals $I$, resp. $J$ in $D_1$, resp. $D_2$. Then $|I| = |J|$ (Proposition 7.6). Further, $(I, \diamond_{D_1}, \mathbf{0}_{D_1})$ and $(J, \diamond_{D_2}, \mathbf{0}_{D_2})$ are commutative groups the elements of which have order 2. But the theory of commutative groups in which each element has the order of a given prime number is categorical in each cardinality (see, for instance, [7, p. 40]). Hence there exists a (group) isomorphism $f$ of $(I, \diamond_{D_1}, \mathbf{0}_{D_1})$ onto $(J, \diamond_{D_2}, \mathbf{0}_{D_2})$. Let us define $g\colon \dot{D}_1 \to \dot{D}_2$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in I, \\ (f(x^{\perp_{D_1}}))^{\perp_{D_2}} & \text{if } x \in \dot{D}_1 \setminus I. \end{cases}$$

Then one can easily check that $g$ is an isomorphism of the algebras $D_1$, $D_2$. □

**Proposition 7.8.** *Let $D$ be a DA. Then there is a Boolean algebra, $B$, such that $D \cong D_B$.*

*Proof.* By Proposition 7.7 it is sufficient to show that there is a Boolean algebra $B$ such that $|D| = |B|$. If $D$ is finite, then $|D| = 2^n$ (Proposition 2.3). In this case one takes the Boolean algebra $B_{(n)}$ for $B$. If $D$ is infinite and $|D| = \kappa$, one takes $B_{(\kappa)}$ for $B$ (compare with Convention 1.8). □

The following notion seems quite useful in the study of DAs and ODLs.

**Definition 7.9.** Let $X \subseteq D$. We shall say that $X$ is an *independent set* in $D$ if $x_1 \diamond \cdots \diamond x_n \neq \mathbf{1}$ for any choice $x_1, \ldots, x_n \in X$.

**Proposition 7.10.** *Let $X \subseteq D$. Write*

$$I_{(X)} = \{\mathbf{0}\} \cup \{x_1 \diamond \cdots \diamond x_n;\ n \geq 1,\ x_1, \ldots, x_n \in X\}.$$

*Then $I_{(X)}$ is the smallest quasiideal in $D$ which includes $X$. Moreover, if $D$ is nontrivial then the following result holds true:*

*$I_{(X)}$ is a proper quasiideal in $D$ if and only if $X$ is an independent set in $D$.*

*Proof.* One can easily show that $I_{(X)}$ is the smallest quasiideal in $D$ including $X$. Suppose that $D$ is nontrivial. If $I_{(X)}$ is a proper quasiideal in $D$, then $\mathbf{1} \notin I_{(X)}$, and therefore $\mathbf{1} \neq x_1 \diamond \cdots \diamond x_n$ for any $x_1, \ldots, x_n \in X$. This implies that $X$ is independent. Conversely, suppose that $X$ is independent. Then $\mathbf{1} \notin \{x_1 \diamond \cdots \diamond x_n;\, n \geq 1,\, x_1, \ldots, x_n \in X\}$. Since $D$ is nontrivial, we have $\mathbf{1} \neq \mathbf{0}$. This implies that $\mathbf{1} \notin I_{(X)}$. □

**Remark 7.11.** Let $X \subseteq D$. If $X = \emptyset$, then $I_{(X)} = \{\mathbf{0}\}$. If $X \neq \emptyset$, then $I_{(X)} = \{x_1 \diamond \cdots \diamond x_n;\ n \geq 1,\ x_1, \ldots, x_n \in X\}$. Moreover, if $X$ is infinite, then $|I_{(X)}| = |X|$.

**Corollary 7.12.** *Let $D$ be nontrivial and let $X \subseteq D$. Then $X$ is an independent set in $D$ if and only if there exists a proper quasiideal $I$ in $D$ such that $X \subseteq I$.*

*Proof.* It follows from Proposition 7.10. □

**Definition 7.13.** Let $L$ be an ODL and let $I \subseteq L$. Let us call $I$ a *difference-ideal* (abbr., a *d-ideal*) in $L$ if the following conditions are satisfied for any $x, y \in L$:

(a) $\mathbf{0} \in I$,
(b) $x \in I,\ y \leq x \Rightarrow y \in I$,
(c) $x, y \in I \Rightarrow x \triangle y \in I$.

If, moreover, $I$ has the selectivity property (i.e., for any $x \in L$ either $x \in I$ or $x^{\perp} \in I$), then we call $I$ a *prime d-ideal* in $L$.

In the following propositions we find an explicit relation between ideals and d-ideals.

**Proposition 7.14.** *Let $L$ be an ODL. Then every ideal in $L$ is also a d-ideal in $L$.*

*Proof.* Let $I$ be an ideal in $L$. Let us check condition (c) of Definition 7.13. Let $x, y \in I$. Then $x \vee y \in I$. Since $x \triangle y \leq x \vee y$, we also have $x \triangle y \in I$. $\square$

**Proposition 7.15.** *Let $B$ be a Boolean algebra and let $I \subseteq B$. Then $I$ is a d-ideal in $B$ if and only if $I$ is an ideal in $B$.*

*Proof.* If $I$ is an ideal in $B$, then $I$ is a d-ideal as the previous proposition states. Conversely, suppose that $I$ is a d-ideal in $B$. Let $x, y \in B$. Since $B$ is Boolean, we have $x \vee y = x \triangle (y \wedge x^{\perp})$. As $y \wedge x^{\perp} \leq y$ and $y \in I$, we see that $y \wedge x^{\perp} \in I$. Condition (c) of Definition 7.13 then implies that $x \triangle (y \wedge x^{\perp}) \in I$. It means that $x \vee y \in I$. $\square$

Let us note that in general the d-ideals do not have to be ideals. For instance, take the ODL $\mathrm{MO}_3$ of Example 3.8. In this ODL the set $I = \{\mathbf{0}, x, y, z\}$ constitutes a d-ideal (indeed, it constitutes a prime d-ideal) but $I$ is not an ideal.

**Proposition 7.16.** *Let $L$ be an ODL.*

(1) *If $e$ is an evaluation on $L$, then the set $I_e = \{x \in L;\ e(x) = 0\}$ is a prime d-ideal in $L$.*
(2) *Conversely, if $I$ is a prime d-ideal in $L$, then the mapping $e_I$ defined by the requirement*

$$e_I(x) = 0 \text{ for } x \in I, \quad e_I(x) = 1 \text{ for } x \in L \setminus I$$

*is an evaluation on $L$.*

*Proof.* The only fact that needs to be checked is that the set $I_e$ has the selectivity property, the rest is a simple direct verification. Let $x \in L$ with $x \notin I_e$. Then $e(x) = 1$. Further, $e(x^{\perp}) = e(x \triangle \mathbf{1}) = e(x) \oplus e(\mathbf{1}) = 1 \oplus 1 = 0$. Thus $x^{\perp} \in I_e$. $\square$

In conclusion of this section, let us formulate a result that characterizes the set-representable ODLs in terms of prime d-ideals. The proof of this theorem follows immediately from the previous proposition and from Theorem 6.7.

**Theorem 7.17.** *Let $L$ be an ODL. Then $L$ is an SRODL (i.e., $L$ is set-representable) if and only if the following condition holds true:*

*Whenever $x, y \in L$ and $x^{\perp} \not\leq y$, then there exists a prime d-ideal $I$ in $L$ such that $x, y \in I$.*

## 8. Horizontal sums of Boolean algebras as ODLs

In this section we will exhibit a construction of ODLs with the help of which we can produce examples of ODLs with specific or rather peculiar properties. In particular, we exhibit a construction of ODLs which are not set-representable. The construction is based on the horizontal sum of Boolean algebras borrowed from the theory of OMLs.

**Definition 8.1.** Let $L$ be a nontrivial OML. We say that $L$ is the *horizontal sum* of its blocks if $B_1 \cap B_2 = \{\mathbf{0}_L, \mathbf{1}_L\}$ for all blocks $B_1, B_2 \in \mathrm{Bl}(L)$ with $B_1 \neq B_2$.

Let us denote by $\mathcal{HOR}$ the class of ODLs $L$ such that $L_{\mathrm{supp}}$ is the horizontal sum of its blocks.

**Lemma 8.2.** *Let $L$ be the horizontal sum of its blocks.*

(a) *Suppose that $x, y \in L$, $x \neq \mathbf{1}$, $y \leq x$ and $x \in B \in \mathrm{Bl}(L)$. Then $y \in B$.*
(b) *If $x \in L$, $x \notin \{\mathbf{0}, \mathbf{1}\}$, then the element $x$ lies in exactly one block of $L$.*
(c) *Let $x, y \in L$ and let $x \mid_L y$. Then there exists a unique block $B_1 \in \mathrm{Bl}(L)$ and a unique block $B_2 \in \mathrm{Bl}(L)$ such that $x \in B_1$ and $y \in B_2$.*

*Proof.* The statements (a), (b) are trivial. To show the statement (c), observe that the assumption $x \mid_L y$ implies that neither of the elements $x, y$ can be equal to $\mathbf{0}$ or $\mathbf{1}$. The rest follows from the statement (b). □

Let $\mathcal{B}$ be a nonempty set of Boolean algebras such that $B_1 \cap B_2 = \{\mathbf{0}, \mathbf{1}\}$ for all $B_1, B_2 \in \mathcal{B}$ with $B_1 \neq B_2$. Then $L = \bigcup \mathcal{B}$ carries in a natural way the structure of an orthomodular lattice which is the horizontal sum of its blocks (see [16, p. 59]). Let us call the OML $L$ the *horizontal sum* of the system $\mathcal{B}$.

Let $\kappa \geq 1$ be a cardinal number. Recall that $\mathrm{MO}_\kappa$ denotes the horizontal sum of $\kappa$ copies of the four-element Boolean algebra $B_{(2)}$. Obviously, $\mathrm{MO}_\kappa$ is an OML and $|\mathrm{MO}_\kappa| = 2\kappa + 2$.

**Definition 8.3.** Let $B$ be a nontrivial Boolean algebra and let $\mathcal{B} \subseteq \mathrm{Sub}(B)$. Let us say that $\mathcal{B}$ is a *disjoint system of subalgebras* of $B$ if for all $B_1, B_2 \in \mathcal{B}$ with $B_1 \neq B_2$ we have $B_1 \cap B_2 = \{\mathbf{0}, \mathbf{1}\}$ and the algebras $B_1$, $B_2$ are not comparable (i.e., $B_1 \nsubseteq B_2$ and $B_2 \nsubseteq B_1$). Moreover, if $\bigcup \mathcal{B} = B$, then the system $\mathcal{B}$ is said to be a *partition* of the algebra $B$. Let us denote by $\mathrm{Part}(B)$ the set of all partitions of $B$.

**Proposition 8.4.** *Let $\sqsubseteq$ be the following relation on $\mathrm{Part}(B)$: $\mathcal{B}_1 \sqsubseteq \mathcal{B}_2$ if for any $B_1 \in \mathcal{B}_1$ there exists $B_2 \in \mathcal{B}_2$ such that $B_1 \subseteq B_2$. Then the relation $\sqsubseteq$ is a partial ordering on $\mathrm{Part}(B)$.*

*Proof.* Reflexivity and transitivity of $\sqsubseteq$ is obvious. Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathrm{Part}(B)$ with $\mathcal{B}_1 \sqsubseteq \mathcal{B}_2$ and $\mathcal{B}_2 \sqsubseteq \mathcal{B}_1$. We are going to show that $\mathcal{B}_1 = \mathcal{B}_2$. Let $B_1 \in \mathcal{B}_1$. Since $\mathcal{B}_1 \sqsubseteq \mathcal{B}_2$, there is $B_2 \in \mathcal{B}_2$ such that $B_1 \subseteq B_2$. Since $\mathcal{B}_2 \sqsubseteq \mathcal{B}_1$, there is $B_3 \in \mathcal{B}_1$ such that $B_2 \subseteq B_3$. From the inclusions $B_1 \subseteq B_2$ and $B_2 \subseteq B_3$ we infer that $B_1 \subseteq B_3$. Hence the algebras $B_1$ and $B_3$ are comparable. According to the definition of a disjoint system of subalgebras, $B_1 = B_3$ holds. Since $B_1 \subseteq B_2 \subseteq B_3$ and $B_1 = B_3$, we have $B_1 = B_2 \in \mathcal{B}_2$. It gives us that $\mathcal{B}_1 \subseteq \mathcal{B}_2$. The opposite inclusion $\mathcal{B}_2 \subseteq \mathcal{B}_1$ follows analogously. □

**Construction 8.5.** Let $B$ be a Boolean algebra and let $\mathcal{B}$ be a disjoint system of subalgebras of $B$. Let us construct an OML, $K$, and the mapping $\triangle_K \colon K^2 \to K$ as follows:

In the first step we construct a system $\mathcal{B}' \subseteq \mathrm{Sub}(B)$ determined by the following requirement: If $\bigcup \mathcal{B} = B$ (i.e., if $\mathcal{B}$ is a partition of $B$), then we set $\mathcal{B}' = \mathcal{B}$. If $\bigcup \mathcal{B} \subset B$, then we add to $\mathcal{B}$ all necessary four-element subalgebras of B such that the resulting system $\mathcal{B}'$ is a partition of $B$. In the second step we take for $K$ the horizontal sum of the system $\mathcal{B}'$. And finally, if $x, y \in \dot{K}$ $(= \dot{B})$, let us set $x \triangle_K y = x \triangle_B y$. The pair $(K, \triangle_K)$ so obtained will be denoted by $L^{B,\mathcal{B}}$ (abbr., $L^{\mathcal{B}}$).

**Proposition 8.6.** *The algebra $L^{\mathcal{B}}$ is an ODL. Thus, $L^{\mathcal{B}} \in \mathcal{HOR}$.*

*Proof.* Conditions $(\mathrm{D}_1)$ and $(\mathrm{D}_2)$ are obvious. Let us verify condition $(\mathrm{D}_3)$. Let $x, y \in \dot{B}$. If there is $B_1 \in \mathcal{B}$ such that $x, y \in B_1$, then $x \vee_K y = x \vee_B y$. As a result, $x \triangle y = x \triangle_B y \le x \vee_B y = x \vee_K y$. If there is no block $B_1$ such that $x, y \in B_1$, then $x \vee_K y = \mathbf{1}$. The inequality $x \triangle y \le x \vee_K y$ is then valid automatically and the proof is complete. □

Let $B$ be a Boolean algebra, $|B| \ge 4$. Let us take the least element of $\mathrm{Part}(B)$ in the ordering $\sqsubseteq$. Obviously, this element consists of all four-element subalgebras of $B$. Let us consider the algebra $L^{\mathcal{B}}$. Obviously, the OML $L^{\mathcal{B}}_{\mathrm{supp}}$ is isomorphic to $\mathrm{MO}_\kappa$. We want to take up the question of the existence and uniqueness of an ODL $L$ with $L_{\mathrm{supp}} \cong \mathrm{MO}_\kappa$.

**Proposition 8.7.** *Let $\kappa \ge 1$ be a cardinal number. Then the necessary and sufficient condition for the existence of an ODL $L$ such that $L_{\mathrm{supp}} \cong \mathrm{MO}_\kappa$ is that $\kappa$ is either finite with $\kappa = 2^n - 1$ for some natural number $n \ge 1$ or $\kappa$ is infinite.*

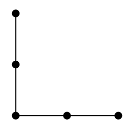
*Proof.* Let there be an ODL $L$ such that $L_{\mathrm{supp}} \cong \mathrm{MO}_\kappa$ and let $\kappa$ be finite. Then $|L| = |\mathrm{MO}_\kappa| = 2\kappa + 2$. On the contrary, the result of Corollary 3.3 implies that $|L| = 2^m$ $(m \ge 2)$. This gives us that $\kappa = 2^{m-1} - 1$.

Conversely, let $\kappa$ be finite and $\kappa = 2^n - 1$ for some natural $n \ge 1$ or let $\kappa$ be infinite. Choose a Boolean algebra, $B$, such that $|B| = 2\kappa + 2$. In other words, let us choose $B$ with $|B| = 2^{n+1}$ if $\kappa = 2^n - 1$, and let us choose $B$ with $|B| = \kappa$ if $\kappa$ is infinite. Further, take a prime ideal, $I$, in $B$. Consider the partition $\mathcal{B} = \big\{\{\mathbf{0}_B, \mathbf{1}_B, x, x^\perp\};\ x \in I \setminus \{\mathbf{0}_B\}\big\}$ of $B$. Then $|\mathcal{B}| = |I \setminus \{\mathbf{0}_B\}| = \kappa$ and therefore $L^{\mathcal{B}}_{\mathrm{supp}} \cong \mathrm{MO}_\kappa$. □

**Proposition 8.8.** *Let $K, L$ be ODLs with $K_{\mathrm{supp}} \cong L_{\mathrm{supp}} \cong \mathrm{MO}_\kappa$. Then $K \cong L$.*

*Proof.* By Proposition 7.7 there is an isomorphism, $f$, of the difference algebra $D_K$ onto the difference algebra $D_L$. Let us show that $f$ is also an isomorphism of $K$ onto $L$. Observe first that $f(\mathbf{0}_K) = \mathbf{0}_L$, $f(\mathbf{1}_K) = \mathbf{1}_L$. Further, for any $x \in K$ we have $f(x^{\perp_K}) = f(x \bigtriangleup_K \mathbf{1}_K) = f(x) \bigtriangleup_L f(\mathbf{1}_K) = f(x) \bigtriangleup_L \mathbf{1}_L = (f(x))^{\perp_L}$. Let us finally show that $f(x \wedge_K y) = f(x) \wedge_L f(y)$ for any $x, y \in K$. Suppose that $x, y \in K$. If $x = y$ or if at least one of the elements $x, y$ belongs to $\{\mathbf{0}_K, \mathbf{1}_K\}$, then the equality we are proving is trivial. Suppose therefore that $x \neq y$, $\mathbf{0}_K < x < \mathbf{1}_K$ and $\mathbf{0}_K < y < \mathbf{1}_K$. Then $x \wedge_K y = \mathbf{0}_K$ and therefore $f(x \wedge_K y) = f(\mathbf{0}_K) = \mathbf{0}_L$. On the other hand, since $f$ is injective we have $f(x) \neq f(y)$, $\mathbf{0}_L < f(x) < \mathbf{1}_L$ and $\mathbf{0}_L < f(y) < \mathbf{1}_L$. Hence $f(x) \wedge_L f(y) = \mathbf{0}_L$, which we wanted to show. $\square$

In the rest of this section let us demonstrate some applications of the construction of horizontal sums. The first application concerns a potential converse to Corollary 3.3: Is every OML of $2^n$ elements a support for some ODL? The answer is no. Indeed, let us take for $L$ the horizontal sum of two copies of four-element Boolean algebras with the OML given by the following Greechie diagram (see [14, 16]):




Then the element $a$ lies in exactly two blocks of $L$. By Proposition 5.6, $L$ cannot be the support of any ODL.

In the second application we will see that the class $\mathcal{HOR}$ contains a proper class of ODLs which are set-representable and a proper class of ODLs which are not. Before formulating the main results, we need to derive some more properties of d-ideals in the ODLs that belong to $\mathcal{HOR}$.

**Proposition 8.9.** *Let $L \in \mathcal{HOR}$ and let $I \subseteq L$. Then the following statements hold true:*

(A) *$I$ is a d-ideal in $L$ if and only if*
  (1) *$I$ is a quasiideal in the difference algebra $D_L$, and*
  (2) $\forall B \in \mathrm{Bl}(L)\ \forall x, y \in B : (x \in I \ \&\ y \leq x) \Rightarrow y \in I$.

(B) *$I$ is prime d-ideal in $L$ if and only if*
  (1) *$I$ is a maximal proper quasiideal in $D_L$, and*
  (2) $\forall B \in \mathrm{Bl}(L)\ \forall x, y \in B : (x \in I \ \&\ y \leq x) \Rightarrow y \in I$.

*Proof.* (A): If $I$ is a d-ideal in $L$, then conditions (1) and (2) are obviously satisfied. Conversely, let conditions (1) and (2) be satisfied. Let $x, y \in L$, $x \in I$ and $y \leq x$. Let $B \in \mathrm{Bl}(L)$ be such a block that $x, y \in B$. From condition (2) we have $y \in I$.

(B): If $I$ is a prime d-ideal in $L$, then conditions (1) and (2) are again satisfied. Conversely, if (1) and (2) are satisfied, then it follows from part (A) that $I$ is a d-ideal in $L$. According to Proposition 7.4, $I$ has the selectivity property and therefore $I$ is a prime d-ideal in $L$. $\square$

**Theorem 8.10.** *Let $L \in \mathcal{HOR}$. Then $L$ is an SRODL if and only if the following condition holds true:*

$$\forall x, y \in L\text{: if } x \mid_L y\text{, then there exists a prime d-ideal } I \text{ in } L \text{ such that } x, y \in I.$$

*Proof.* ($\Rightarrow$): Let $L$ be an SRODL of $\mathcal{HOR}$ and let $x, y \in L$ be such elements that $x \mid_L y$. Then $x^\perp \leq y$ cannot hold (if $x^\perp \leq y$, then $y\ C\ x$, which is not the case). According to Theorem 7.17, there is a prime d-ideal $I$ in $L$ such that $x, y \in I$.

($\Leftarrow$): If $L$ is Boolean, then $L$ is obviously an SRODL. Suppose that $L$ is not Boolean. Let $x, y \in L$ be such elements that $x^\perp \not\leq y$. According to Theorem 7.17 we want to show that there exists a prime d-ideal $I$ in $L$ such that $x, y \in I$. If $x \mid_L y$, then the existence of $I$ follows directly from our assumption. Let us assume that $x\ C\ y$. Let $B$ be a block in $L$ such that $x, y \in B$. Since $x^\perp \not\leq y$, we have $x \vee y < \mathbf{1}$. Let us write $a = x \vee y \in B$. We will show that there exists a prime d-ideal $I$ in $L$ such that $a \in I$. The case $a = \mathbf{0}$ is trivial. Otherwise let $b \in L$ such that $a$ and $b$ do not lie in the same block (such an element $b \in L$ exists because $L$ is not Boolean). Then $a \mid_L b$ and, according to our assumption, there exists a prime d-ideal $I$ in $L$ such that $a, b \in I$. $\square$

**Theorem 8.11.** *Let $L \in \mathcal{HOR}$. Assume there exists a block $B_0 \in \mathrm{Bl}(L)$ such that $|B| = 4$ for any block $B \in \mathrm{Bl}(L)$, $B \neq B_0$. Then $L$ is an SRODL.*

Before giving the proof of Theorem 8.11, let us prove the following lemma:

**Lemma 8.12.** *Let $L$ be an ODL that satisfies all the assumptions of Theorem 8.11. Let $I_0$ be a prime ideal in the Boolean algebra $B_0$ and let $J$ be an independent set in $L$ such that $I_0 \subseteq J$. Then there exists a prime d-ideal $I$ in $L$ such that $J \subseteq I$.*

*Proof.* Let us choose such a maximal proper quasiideal $I$ in the difference algebra $D_L$ that $J \subseteq I$. (Such a quasiideal exists as can be seen from Corollary 7.12 by applying Zorn's Lemma.) We will show that $I$ is a prime d-ideal we are looking for. It is sufficient to prove that $I$ fulfils condition (B2) of Proposition 8.9. Consider a block $B \in \mathrm{Bl}(L)$. We will prove that the set $B \cap I$ is an ideal in the Boolean algebra $B$. There are two cases to be argued.

(1): $B = B_0$. In this case we will show that $B \cap I = I_0$. The inclusion $I_0 \subseteq B_0 \cap I$ is obvious. Let us assume that $x \in B_0 \cap I$ but $x \notin I_0$. Since $I_0$ is a prime ideal in $B_0$ and $x \in B_0$, $x \notin I_0$, it must hold $x^\perp \in I_0$. Now, $x \in I$, $x^\perp \in I_0 \subseteq I$, and therefore both of the elements $x, x^\perp$ belong to $I$. This is a contradiction because $I$ is a proper quasiideal.

(2): $B \neq B_0$. Then $|B| = 4$. Let $B = \{\mathbf{0}, a, a^{\perp}, \mathbf{1}\}$. Since $I$ has the selectivity property, we have either $I \cap B = \{\mathbf{0}, a\}$ or $I \cap B = \{\mathbf{0}, a^{\perp}\}$. In both cases the set $B \cap I$ is an ideal in $B$. □

*Proof of Theorem* 8.11. Let $x, y \in L$ with $x \mid_L y$. Then $x \notin \{\mathbf{0}, \mathbf{1}\}$ and $y \notin \{\mathbf{0}, \mathbf{1}\}$. We will prove that there is a prime d-ideal $I$ in $L$ such that $x, y \in I$. To this end, let us distinguish the following two cases.

(I): $\{x, y\} \cap B_0 \neq \emptyset$. We may assume that $x \in B_0$ and $y \notin B_0$.
Let $I_0$ be a prime ideal in the algebra $B_0$ such that $x \in I_0$ (such a prime ideal exists because $x \neq \mathbf{1}$). Then the set $J = I_0 \cup \{y\}$ is an independent set in $D_L$ (otherwise we would have $y \in B_0$). The existence of $I$ follows directly from the previous lemma.

(II): $\{x, y\} \cap B_0 = \emptyset$, i.e., $x \notin B_0, y \notin B_0$.
Let us distinguish another two cases.

(II a): Suppose $x \triangle y \in B_0$. Let us write $z = x \triangle y$. From case (I) specified to the elements $z$ and $x$ it follows that there exists a prime d-ideal $I$ in $L$ such that $x, z \in I$. Then $y = x \triangle z \in I$.

(II b): Suppose $x \triangle y \notin B_0$. Let us choose a prime ideal $I_0$ in the Boolean algebra $B_0$. The set $J = I_0 \cup \{x, y\}$ is now an independent set in $D_L$. According to Lemma 8.12, there exists a prime d-ideal $I$ in $L$ such that $J \subseteq I$. As a consequence, $x, y \in I$. □

**Corollary 8.13.** *Let $L$ be an ODL such that $L_{\mathrm{supp}} = \mathrm{MO}_{\kappa}$. Then $L$ is a modular SRODL.*

*Proof.* According to Theorem 8.11, $L$ is an SRODL. Moreover, $L$ is modular since $L$ does not contain the pentagon as a sublattice (see [16, p. 16]). □

**Corollary 8.14.** *Let $L \in \mathcal{HOR}$ such that $L$ is not a Boolean algebra. Let us suppose that there exists a block $B_0 \in \mathrm{Bl}(L)$ such that $|B_0| \geq 8$ and $|B| = 4$ for any block $B \in \mathrm{Bl}(L)$, $B \neq B_0$. Then $L$ is a non-modular SRODL.*

*Proof.* According to Theorem 8.11, $L$ is an SRODL. It remains to prove that $L$ is not modular. Let us choose the elements $x, y$ in the algebra $B_0$ such that $\mathbf{0} < x < y < \mathbf{1}$ (this choice is possible because $|B_0| \geq 8$). Since $L$ is not Boolean, there is an element $z \in L$ such that $z \notin B_0$. But the set $\{\mathbf{0}, x, y, z, \mathbf{1}\}$ is a pentagon in $L$. □

**Example 8.15.** We will construct an ODL $L$, $L \in \mathcal{HOR}$, such that $L$ is not an SRODL. Let $B$ be the Boolean algebra of all subsets of the set $\{1, \ldots, 5\}$. Then

$\mathbf{0}_B = \emptyset$ and $\mathbf{1}_B = \{1, \dots, 5\}$. Let us consider the following elements of $B$:

$$\begin{array}{llll} x = \{1,2,4,5\}, & a_1 = \{1,2\}, & a_2 = \{4,5\}, & a_3 = \{3\}, \\ y = \{2,3,4\}, & b_1 = \{3,4\}, & b_2 = \{2\}, & b_3 = \{1,5\}, \\ c_1 = \{1,3\}, & c_2 = \{2,4\}, & c_3 = \{5\}. & \end{array}$$

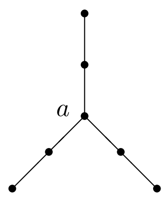
Let us consider the following subalgebras of $B$:

$$B_1 = \{\mathbf{0}_B, \mathbf{1}_B, a_1, a_2, a_3, a_1^\perp, a_2^\perp, a_3^\perp\}, \quad B_2 = \{\mathbf{0}_B, \mathbf{1}_B, b_1, b_2, b_3, b_1^\perp, b_2^\perp, b_3^\perp\},$$
$$B_3 = \{\mathbf{0}_B, \mathbf{1}_B, c_1, c_2, c_3, c_1^\perp, c_2^\perp, c_3^\perp\}.$$

It is easy to show that $B_i \cap B_j = \{\mathbf{0}_B, \mathbf{1}_B\}$ for any $i \neq j$. Moreover, $x = a_3^\perp \in B_1$, $y = b_3^\perp \in B_2$. If we write $\mathcal{B} = \{B_1, B_2, B_3\}$, then $\mathcal{B}$ is a disjoint system of subalgebras of the algebra $B$. Let us set $L = L^{\mathcal{B}}$ (see Construction 8.5). Then $x \mid_L y$. In order to show that $L$ is not an SRODL, it is sufficient to show that there is no prime d-ideal $I$ in $L$ such that $x, y \in I$. Indeed, let $I$ be a d-ideal in $L$ such that $x, y \in I$. Since $a_1 \leq x, a_2 \leq x$ and $b_1 \leq y, b_2 \leq y$, we have $a_1, a_2, b_1, b_2 \in I$. Since the set $I$ is closed with respect to the operation $\triangle$, we also have $a_1 \triangle b_1, a_2 \triangle b_2 \in I$. But $a_1 \triangle b_1 = \{1,2,3,4\} = c_3^\perp$ and $a_2 \triangle b_2 = \{2,4,5\} = c_1^\perp$. Because $c_1 \leq c_3^\perp \in I$, we have $c_1 \in I$. Now, $c_1, c_1^\perp \in I$ and therefore $I$ is not a prime d-ideal.

By examining the foregoing example we see that the possibility to have $L_{\mathrm{supp}}$ set-representable (as an OML) does not imply that $L$ is set-representable as an ODL. Indeed, for the ODL $L$ of Example 8.15, $L_{\mathrm{supp}}$ is obviously a set-representable OML since it is a horizontal sum of Boolean algebras (see [24]). As a matter of fact, Example 8.15 is the smallest ODL for this circumstance to occur as Proposition 8.17 shows.

**Lemma 8.16.** *Let $L$ be an OML with $|L| = 16$. Suppose that there exist an atom $a \in L$ and mutually distinct blocks $B_1, B_2, B_3 \in \mathrm{Bl}(L)$ such that $a \in B_1 \cap B_2 \cap B_3$, $|B_1| = |B_2| = |B_3| = 8$. Then $L$ is the OML with the following Greechie diagram:*



*Proof.* Since $B_1, B_2, B_3$ are 8-element blocks, they can have at most one atom in common. It means that $B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3 = \{\mathbf{0}, \mathbf{1}, a, a^\perp\}$. The set $\dot{B}_1 \cup \dot{B}_2 \cup \dot{B}_3$ contains exactly 16 elements, i.e., $\dot{L} = \dot{B}_1 \cup \dot{B}_2 \cup \dot{B}_3$. Let us prove that $B_1, B_2, B_3$ are precisely all blocks in $L$. Let $B \in \mathrm{Bl}(L)$. We will prove that $B = B_k$ for some $k \in \{1, 2, 3\}$. First, let us exclude the case $|B| = 4$. Let us suppose that $B = \{\mathbf{0}, \mathbf{1}, b, b^\perp\}$. Then there is an $i \in \{1, 2, 3\}$ such that $b \in B_i$. This means $B \subset B_i$, which is a contradiction with the definition of a block. Therefore $|B| = 8$. Let $b_1, b_2, b_3$ be all atoms in $B$. Since $\{b_1, b_2, b_3\}$ is a three-element set, at least two of its elements are distinct from $a$. Let, e.g., $b_1 \neq a$, $b_2 \neq a$. Let us suppose that $b_1 \in B_k$, $b_2 \in B_l$, $k, l \in \{1, 2, 3\}$. Then $b_1, a$ are distinct atoms in the algebra $B_k$ and therefore $b_1 \leq a^\perp$. Analogously, $b_2 \leq a^\perp$. It follows that $b_1 \vee b_2 \leq a^\perp$. Let us suppose that $b_1 \vee b_2 < a^\perp$. Then the chain $\mathbf{0} < b_1 < b_1 \vee b_2 < a^\perp < \mathbf{1}$ is contained in some block, $B'$, such that the cardinality of $B'$ is at least 16. But this is a contradiction. Hence, $b_1 \vee b_2 = a^\perp$. Because $b_1, b_2 \in B$, we see that $a^\perp \in B$. Thus $a \in B$. The blocks $B_k, B$ have the atoms $b_1, a$ in common, and because both the blocks are eight-element, we have $B = B_k$. Thus, $B \in \{B_1, B_2, B_3\}$. Hence, $\mathrm{Bl}(L) = \{B_1, B_2, B_3\}$. □

**Proposition 8.17.** *Let $L$ be an ODL such that $|L| \leq 16$. Then $L$ is an SRODL.*

*Proof.* We may suppose that $L$ is not Boolean. Then either $|L| = 8$ or $|L| = 16$. If $|L| = 8$, then $L = \mathrm{MO}_3$ and $L$ is an SRODL. Suppose that $|L| = 16$. Let us write $\mathrm{Bl}_8(L) = \{B \in \mathrm{Bl}(L);\ |B| = 8\}$. Then there are only three cases possible.

(1): $\mathrm{Bl}_8(L) = \emptyset$. Then $L = \mathrm{MO}_7$ and $L$ is an SRODL.

(2): $|\mathrm{Bl}_8(L)| = 1$. Let $\mathrm{Bl}_8(L) = \{B\}$. Because the other blocks of $L$ are four-element blocks, $L$ is a horizontal sum of its blocks. According to Theorem 8.11, $L$ is an SRODL.

(3): $|\mathrm{Bl}_8(L)| \geq 2$. Let us choose $B_1, B_2 \in \mathrm{Bl}_8(L)$ with $B_1 \neq B_2$. We will prove that $B_1 \cap B_2 \neq \{\mathbf{0}, \mathbf{1}\}$. Seeking a contradiction, let us assume that $B_1 \cap B_2 = \{\mathbf{0}, \mathbf{1}\}$. Then $|B_1 \cup B_2| = 14$. Let $\{a_1, a_2, a_3\}$ be the set of all atoms in the algebra $B_1$. Let us choose an atom $b \in B_2$. Then the elements $a_1 \triangle b$, $a_2 \triangle b$, $a_3 \triangle b$ are pairwise distinct because if, for instance, $a_1 \triangle b = a_2 \triangle b$ then $(a_1 \triangle b) \triangle b = (a_2 \triangle b) \triangle b$ and therefore $a_1 = a_2$ which is a contradiction. Moreover, the elements $a_1 \triangle b$, $a_2 \triangle b$, $a_3 \triangle b$ belong to the set $L \setminus (B_1 \cup B_2)$. Indeed, let e.g. $a_1 \triangle b \in B_1$. Since $a_1 \in B_1$, we see that $a_1 \triangle (a_1 \triangle b) \in B_1$. But $a_1 \triangle (a_1 \triangle b) = b$, and therefore $b \in B_1$ which we excluded. Obviously, $|L \setminus (B_1 \cup B_2)| = 16 - 14 = 2$. We have obtained a contradiction. Therefore $B_1 \cap B_2 = \{\mathbf{0}, \mathbf{1}, a, a^\perp\}$, where $a$ is an atom. According to Proposition 5.6, the element $a$ lies in at least three blocks. Therefore there exists another block $B_3 \in \mathrm{Bl}_8(L)$ such that $a \in B_3$. Then Lemma 8.16 implies that $\mathrm{Bl}(L) = \{B_1, B_2, B_3\}$. This means that $a \in C(L)$. According to Proposition 5.8, $L \cong L^a \times L^{a^\perp}$. Moreover, $|L^a| = |[\mathbf{0}, a]_L| = 2$ and $|L^{a^\perp}| = |[\mathbf{0}, a^\perp]_L| = 8$. Therefore both ODLs $L^a$ and $L^{a^\perp}$ are SRODLs. Because the class $\mathcal{SRODL}$ is closed under the formation of products, we infer that $L$ is an SRODL. □

In the final result, let us revisit Example 8.15 to demonstrate that there are non-set-representable ODLs that contain preassigned Boolean algebras. As a consequence, there are "as many" non-set-representable ODLs as Boolean algebras.

**Theorem 8.18.** *Let $B$ be a nontrivial BA. Then there is an ODL, $M$, such that $B$ is a subalgebra of $M$ and $M$ is not set-representable.*

*Proof.* Let $L$ be the ODL of Example 8.15. Let $M = B \times L$ and let us denote by $\pi_L$ the projection of $M$ onto $L$. Because $L$ is not set-representable and $L$ is a homomorphic image of $M$ ($L = \pi_L(M)$), $M$ cannot be set-representable. It remains to show that there is an embedding of $B$ into $M$. For this purpose, let us fix some prime ideal, $I$, in $B$. (The existence of $I$ follows from the nontriviality of $B$.) Let us set $F = \dot{B} \setminus I$. Obviously, $F$ is an ultrafilter in $B$. Now, we can define a mapping $f\colon B \to M$ as follows:

$$f(x) = \begin{cases} (x, \mathbf{0}_L) & \text{if } x \in I, \\ (x, \mathbf{1}_L) & \text{if } x \in F. \end{cases}$$

We are going to show that $f$ is the embedding we are looking for. Obviously, $f$ is an injective mapping. Further, $f(\mathbf{0}_B) = (\mathbf{0}_B, \mathbf{0}_L) = \mathbf{0}_M$, $f(\mathbf{1}_B) = (\mathbf{1}_B, \mathbf{1}_L) = \mathbf{1}_M$. Suppose $x, y \in B$. We will prove that $f(x \vee_B y) = f(x) \vee_M f(y)$ and $f(x \bigtriangleup_B y) = f(x) \bigtriangleup_M f(y)$.

We have to distinguish three possibilities.

- Firstly, $x, y \in I$. Then $x \vee_B y \in I$, $x \bigtriangleup_B y \in I$ and therefore

$$f(x \vee_B y) = (x \vee_B y, \mathbf{0}_L) = (x, \mathbf{0}_L) \vee_M (y, \mathbf{0}_L) = f(x) \vee_M f(y),$$
$$f(x \bigtriangleup_B y) = (x \bigtriangleup_B y, \mathbf{0}_L) = (x, \mathbf{0}_L) \bigtriangleup_M (y, \mathbf{0}_L) = f(x) \bigtriangleup_M f(y).$$

- Secondly, exactly one of the elements $x, y$ lies in $I$. Let, e.g., $x \in I$, $y \in F$. In this case $x \vee_B y \in F$, $x \bigtriangleup_B y \in F$ and therefore

$$f(x \vee_B y) = (x \vee_B y, \mathbf{1}_L) = (x, \mathbf{0}_L) \vee_M (y, \mathbf{1}_L) = f(x) \vee_M f(y),$$
$$f(x \bigtriangleup_B y) = (x \bigtriangleup_B y, \mathbf{1}_L) = (x, \mathbf{0}_L) \bigtriangleup_M (y, \mathbf{1}_L) = f(x) \bigtriangleup_M f(y).$$

- Thirdly, $x, y \in F$. Then $x \vee_B y \in F$, $x \bigtriangleup_B y \in I$ and therefore

$$f(x \vee_B y) = (x \vee_B y, \mathbf{1}_L) = (x, \mathbf{1}_L) \vee_M (y, \mathbf{1}_L) = f(x) \vee_M f(y),$$
$$f(x \bigtriangleup_B y) = (x \bigtriangleup_B y, \mathbf{0}_L) = (x, \mathbf{1}_L) \bigtriangleup_M (y, \mathbf{1}_L) = f(x) \bigtriangleup_M f(y).$$

Finally, because the equalities $x^\perp = x \bigtriangleup \mathbf{1}$ and $x \wedge y = (x^\perp \vee y^\perp)^\perp$ hold in any ODL, the mapping $f$ preserves both operations $^\perp$ and $\wedge$. Hence, $f$ is an injective homomorphism of the algebra $B$ into $M$. (Let us note that $M_{\text{supp}} = B_{\text{supp}} \times L_{\text{supp}}$ and therefore $M_{\text{supp}}$ is a set-representable OML.) □

Concluding the article, let us finally formulate some open questions related to the investigation presented.

(1) Which OMLs are embeddable into $L_{\text{supp}}$ for some ODL $L$? This interesting question also deserves attention because of a potential application of ODLs within quantum theories. In particular, it seems desirable to clarify the question for set-representable OMLs and for OMLs $\mathcal{L}(H)$ of projections in a Hilbert space $H$. It should be noted in connection with the latter class that for $\mathcal{L}(\mathbf{R}^1)$ and $\mathcal{L}(\mathbf{R}^2)$ the answer is obviously yes—these OMLs can be even converted to ODLs (Proposition 8.7). Further, each $\mathcal{L}(\mathbf{R}^n)$, $3 \leq n < \infty$, cannot be converted to an ODL as one finds out easily upon testing the axioms of ODLs, but it seems conceivable that $\mathcal{L}(\mathbf{R}^n)$ allows for an ODL extension.

(2) Is the variety $\mathcal{SRODL}$ finitely based? We conjecture it is not.

**Acknowledgements.** The author thanks Professor P. Pták for introducing him to the problem and for many valuable discussions. Thanks also go to anonymous referees for valuable suggestions which helped in improving the paper.

## References

[1] L. Beran, *Orthomodular Lattices, Algebraic Approach*, D. Reidel, Dordrecht, 1985.
[2] G. Bruns and R. Greechie, *Blocks and commutators in orthomodular lattices*, Algebra Universalis **27** (1990), 1–9.
[3] G. Bruns and J. Harding, *Algebraic aspects of orthomodular lattices*, in B. Coecke, D. Moore and A. Wilce, eds., Current Research in Operational Quantum Logic, Kluwer Academic Publishers, Dordrecht, 2000, pp. 37–65.
[4] G. Bruns and M. Roddy, *Projective orthomodular lattices*, Canad. Math. Bull. **37** (1994), 145–153.
[5] G. Bruns and M. Roddy, *Projective orthomodular lattices II*, Algebra Universalis **37** (1997), 143–153.
[6] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer, New York, 1981.
[7] C. C. Chang and H. J. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973.
[8] M. L. Dalla Chiara, R. Giuntini and R. Greechie, *Reasoning in Quantum Theory: Sharp and Unsharp Quantum Logics*, Kluwer Academic Publishers, Dordrecht, 2004.
[9] G. Dorfer, *Non-commutative symmetric differences in orthomodular lattices*, Internat. J. Theoret. Phys. **44** (2005), 885–896.
[10] G. Dorfer, A. Dvurečenskij and H. M. Länger, *Symmetric difference in orthomodular lattices*, Math. Slovaca **46** (1996), 435–444.
[11] R. Godowski, *Varieties of orthomodular lattices with a strongly full set of states*, Demonstratio Math. **14** (1981), 725–732.
[12] R. Godowski, *States on orthomodular lattices*, Demonstratio Math. **15** (1982), 817–822.
[13] R. Godowski and R. Greechie, *On some equations related to states on orthomodular lattices*, Demonstratio Math. **17** (1984), 41–250.
[14] R. Greechie, *Orthomodular lattices admitting no states*, J. Combinat. Theory **10** (1971), 119–132.
[15] S. P. Gudder, *Stochastic Methods in Quantum Mechanics*, North-Holland, New York, 1979.
[16] G. Kalmbach, *Orthomodular Lattices*, Academic Press, London, 1983.

[17] S. Mac Lane and G. Birkhoff, *Algebra*, Macmillan, New York, 1968.

[18] M. Matoušek, *Boolean carried homomorphisms in orthomodular lattices*, Demonstratio Math. **37** (2004), 255–265.

[19] R. Mayet, *Equational bases for some varieties of orthomodular lattices related to states*, Algebra Universalis **23** (1986), 167–195.

[20] R. Mayet and P. Pták, *Quasivarieties of OML determined by conditions on states*, Algebra Universalis **42** (1999), 155–164.

[21] R. Mayet and P. Pták, *Orthomodular lattices with state-separated noncompatible pairs*, Czechoslovak Math. J. **50** (2000), 359–366.

[22] M. Navara and J. Tkadlec, *Automorphisms of concrete logics*, Comment. Math. Univ. Carolinae **32** (1991), 15–25.

[23] E. Park, M. M. Kim and J. Y. Chung, *A note on symmetric differences of orthomodular lattices*, Commun. Korean Math. Soc. **18** (2003), 207–214.

[24] P. Pták P and S. Pulmannová, *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publishers, Dordrecht, 1991.

[25] V. S. Varadarajan, *Geometry of Quantum Theory I, II*, Van Nostrand, Princeton, 1968, 1970.

Milan Matoušek

Department of Logic, Philosophical Faculty of the Charles University, Celetná 20, 116 42 Prague 1, Czech Republic
*e-mail*: matmilan@email.cz

# Symmetric difference on orthomodular lattices and $Z_2$-valued states

Milan Matoušek, Pavel Pták

*Abstract.* The investigation of orthocomplemented lattices with a symmetric difference initiated the following question: Which orthomodular lattice can be embedded in an orthomodular lattice that allows for a symmetric difference? In this paper we present a necessary condition for such an embedding to exist. The condition is expressed in terms of $Z_2$-valued states and enables one, as a consequence, to clarify the situation in the important case of the lattice of projections in a Hilbert space.

*Keywords:* orthomodular lattice, quantum logic, symmetric difference, Boolean algebra, group-valued state

*Classification:* 06A15, 03G12, 28E99, 81P10

## 1. Introduction and preliminaries

In the paper [11] the author introduces algebras that can be viewed as "orthomodular lattices with a symmetric difference". Their definition is as follows (the standard definition of an orthocomplemented lattice can be found in [9], [10], [16], etc.).

**Definition 1.1.** Let $L = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$, where $(X, \wedge, \vee, ^\perp, 0, 1)$ is an orthocomplemented lattice and $\triangle : X^2 \to X$ is a binary operation. Then $L$ is said to be an *orthocomplemented difference lattice* (abbr., an ODL) if the following formulas hold in $L$:

(D$_1$) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$,
(D$_2$) $x \triangle 1 = x^\perp$, $1 \triangle x = x^\perp$,
(D$_3$) $x \triangle y \leq x \vee y$.

Let us first formulate basic properties of ODLs as we shall use them in the sequel (see also [11]). We shall adopt the convention that in writing a formula with $\triangle$ and $^\perp$, we give the preference to the operation $^\perp$ over the operation $\triangle$. Thus, for instance, $x \triangle y^\perp$ means $x \triangle (y^\perp)$, etc.

**Proposition 1.2.** *Let $L = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$ be an ODL. Then the following statements hold true:*

The authors acknowledge the support of the research plans MSM 0021620839 and MSM 6840770038 that are financed by the Ministry of Education of the Czech Republic.

(1) $x \triangle 0 = x$, $0 \triangle x = x$,
(2) $x \triangle x = 0$,
(3) $x \triangle y = y \triangle x$,
(4) $x \triangle y^\perp = x^\perp \triangle y = (x \triangle y)^\perp$,
(5) $x^\perp \triangle y^\perp = x \triangle y$,
(6) $x \triangle y = 0 \Leftrightarrow x = y$,
(7) $(x \wedge y^\perp) \vee (y \wedge x^\perp) \leq x \triangle y \leq (x \vee y) \wedge (x \wedge y)^\perp$.

Proof: Suppose that $x, y \in L$ and verify the properties (1)–(7).

(1) Let us first see that the property ($D_2$) yields $1 \triangle 1 = 1^\perp = 0$. Using this, we have $x \triangle 0 = x \triangle (1 \triangle 1) = (x \triangle 1) \triangle 1 = x^\perp \triangle 1 = (x^\perp)^\perp = x$. Analogously, $0 \triangle x = (1 \triangle 1) \triangle x = 1 \triangle (1 \triangle x) = 1 \triangle x^\perp = (x^\perp)^\perp = x$.

(2) Let us first show that $x^\perp \triangle x^\perp = x \triangle x$. We consecutively obtain $x^\perp \triangle x^\perp = (x \triangle 1) \triangle (1 \triangle x) = (x \triangle (1 \triangle 1)) \triangle x = (x \triangle 0) \triangle x = x \triangle x$. Moreover, we have $x \triangle x \leq x$ as well as $x \triangle x = x^\perp \triangle x^\perp \leq x^\perp$. This implies that $x \triangle x \leq x \wedge x^\perp = 0$.

(3) $x \triangle y = (x \triangle y) \triangle 0 = (x \triangle y) \triangle [(y \triangle x) \triangle (y \triangle x)] = x \triangle (y \triangle y) \triangle x \triangle (y \triangle x) = x \triangle 0 \triangle x \triangle (y \triangle x) = x \triangle x \triangle (y \triangle x) = 0 \triangle (y \triangle x) = y \triangle x$.

(4) $x \triangle y^\perp = x \triangle (y \triangle 1) = (x \triangle y) \triangle 1 = (x \triangle y)^\perp$. The equality $x^\perp \triangle y = (x \triangle y)^\perp$ follows from $x \triangle y^\perp = (x \triangle y)^\perp$ by using the equality (3).

(5) Applying (4), we obtain $x^\perp \triangle y^\perp = (x^\perp \triangle y)^\perp = (x \triangle y)^{\perp\perp} = x \triangle y$.

(6) If $x = y$, then $x \triangle y = 0$ by the condition (2). Conversely, suppose that $x \triangle y = 0$. Then $x = x \triangle 0 = x \triangle (y \triangle y) = (x \triangle y) \triangle y = 0 \triangle y = y$.

(7) The property ($D_3$) together with the properties (4), (5) imply that $x \triangle y \leq x \vee y$, $x \triangle y \leq x^\perp \vee y^\perp = (x \wedge y)^\perp$, $x \wedge y^\perp \leq x \triangle y$, $x^\perp \wedge y \leq x \triangle y$. □

Our interest in this paper is the relationship of ODLs to orthomodular lattices (OMLs). Let us recall the definition of OML (the acquaintance with basic facts about OMLs will be helpful in the sequel — see [1], [9], [10], etc.).

**Definition 1.3.** Let $L$ be an orthocomplemented lattice. If $L$ satisfies the orthomodular law,

$$x \leq y \Rightarrow y = x \vee (y \wedge x^\perp),$$

then $L$ is said to be an *orthomodular lattice* (abbr., an OML).

Though the orthomodular law is not explicitly stated among the axioms of ODL, it can be easily shown ([11]) that an ODL is automatically orthomodular. More precisely, if $K$ is an ODL and $K_{\mathrm{supp}}$ is the orthocomplemented lattice obtained from $K$ by forgetting $\triangle$, then $K_{\mathrm{supp}}$ is an OML. A question arises: Given an OML, $L$, can $L$ be made an ODL? Or, in case the above question answers in the negative too often, can $L$ be at least enlarged to an ODL? If $L$ allows for such an enlargement, the algebraic "calculus" of $L$ would be enriched and these ODL-enlargeable OMLs might find an application in quantum logic theory, or elsewhere (see [3], [6], [18], etc.).

Let us comment on "the state of art" in this line of problems and agree on some terminology. In [11] the author shows that several OMLs are *ODL-convertible*, i.e. they are such OMLs that can be endowed with $\triangle$ to become ODLs. Such are, for instance, the lattices $MO_\kappa$ for $\kappa = 2^n - 1$, the lattice $MO_\kappa$ for any infinite cardinal $\kappa$, certain pastings of Boolean algebras (this will also be commented on later), several "non-concrete" OMLs, etc. On the other hand, there are OMLs that are far from being ODL-convertible (such as, for instance, each finite OML the cardinality of which differs from $2^n$). In fact, there are even OMLs that are not ODL-embeddable (an OML, $L$, is said to be *ODL-embeddable* if there is an ODL, $K$, such that $L$ is a sub-OML of $K_{\mathrm{supp}}$) — a rather elaborate construction presented in [12] provides such an example. In considering the ODL-embeddable OMLs a rather interesting connection came into existence. It turned out that if $L$ is ODL-embeddable then it has to possess an abundance of $Z_2$-states. This allows us to show, in an interplay with [15], that if $n \geq 4$ then the projection lattice $L(R^n)$ is not ODL-embeddable. The same question about $L(R^3)$ remains open (see also [8], [15]). However, a purely ODL consideration (Theorem 3.10) clarifies the ODL-convertibility of $L(R^3)$: The lattice $L(R^3)$ is not ODL-convertible (Theorem 3.11). The lattice $L(R^2)$ is ODL-convertible and, of course, so is $L(R^1)$.

Let $L$ be an OML. Let us recall that two elements $a, b \in L$ are called *compatible* in $L$ ($a\ C\ b$) if they lie in a Boolean subalgebra of $L$ (see [1] and [9] for the properties of compatible pairs). If $a, b \in L$ are not compatible, we write $a \neg\ C\ b$. Further, let us recall that by *a block* in $L$ we mean a maximal Boolean subalgebra of $L$. Finally, let us call the set $C(L) = \{c \in L;\ c\ C\ a \text{ for any } a \in L\}$ the *centre* of $L$ (i.e., $C(L)$ is the set of all "absolutely compatible" elements of $L$). Obviously, $C(L)$ is the intersection of all blocks of $L$.

It is convenient to adopt the following convention.

**Convention 1.4.** Let $L$ be an ODL. Then any OML notion can be referred to $L$ as well by applying this notion to the corresponding OML $L_{\mathrm{supp}}$.

**Proposition 1.5.** *Let $L$ be an ODL and let $a, b \in L$ with $a\ C\ b$. Then $a \triangle b = (a \wedge b^\perp) \vee (b \wedge a^\perp) = (a \vee b) \wedge (a \wedge b)^\perp$. A corollary: If $a\ C\ b$, then $a\ C\ a \triangle b$.*

PROOF: It follows from Proposition 1.2(7). □

In concluding this paragraph let us observe the following consequence of Proposition 1.5: For each block $B$ of $L$, the operation $\triangle$ on $L$ acts on $B$ as the standard symmetric difference.

## 2. OMLs with 8-element blocks

In this section we shall be interested in some intrinsic properties of the OMLs whose blocks are of cardinality 8 and whose pairs of atoms, $a$ and $b$, satisfy the inequality $a \vee b < 1$. We will then apply the results obtained in the constructions enabling us to prove our main result formulated in Theorem 3.10. (It should be noted that the class of OMLs considered in this section contains, as an important example, the lattice $L(R^3)$ of projections of $R^3$. The paper [17] studies, with the motivation coming from theoretical physics, the existence of sub-orthoposets of $L(R^3)$. Incidentally, our result of Theorem 2.5 adds to Proposition 6.5 of [17].)

**Proposition 2.1.** *Let $L$ be an OML such that the cardinality of each block of $L$ is 8. Then*

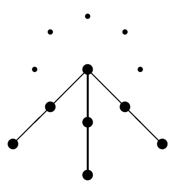
(i) *for any pair $a$ and $b$ of atoms in $L$, the following statement holds true: $a \vee b < 1$ if and only if there is an atom $c$ such that $a\; C\; c$ and $b\; C\; c$;*

(ii) *for any pair of distinct atoms $a$ and $b$ in $L$ there is at most one atom $c$ such that $a\; C\; c$ and $b\; C\; c$.*

Proof: The statement (i) is trivial. For the statement (ii) suppose that $a, b$ are atoms and $a \neq b$. Suppose that $c, d$ are such atoms that $c\; C\; a$, $c\; C\; b$, $d\; C\; a$ and $d\; C\; b$. Then we have $0 < a < a \vee b \leq c^{\perp} \wedge d^{\perp} \leq c^{\perp} < 1$. Since each block of $L$ has 8 elements, we infer that $c^{\perp} \wedge d^{\perp} = c^{\perp}$. Thus, $c^{\perp} \leq d^{\perp}$ and therefore $d \leq c$. As $c, d$ are atoms, it follows that $c = d$. □

**Definition 2.2.** An OML $L$ is said to be a 3-*star* if $L$ is isomorphic with the product $\{0,1\} \times MO_{\kappa}$ for $\kappa \geq 1$.

The figure below indicates the Greechie diagram of the 3-star $\{0,1\} \times MO_{\kappa}$. Note that the number of blocks of this 3-star is $\kappa$.



**Proposition 2.3.** *Let $L$ be an OML. Then $L$ is a 3-star if and only if the cardinality of each block of $L$ is 8 and $C(L) \neq \{0,1\}$.*

Proof: The proof is evident. □

Prior to the main result of this section, let us recall some notions of orthomodular combinatorics (see also [4] and [16]).

**Definition 2.4.** Let $L$ be an OML such that the cardinality of each block of $L$ is 8. For three mutually distinct and compatible atoms $a_1, a_2, a_3$ of $L$, let us denote by $[a_1, a_2, a_3]_L$ the block of $L$ generated by these atoms.

An *n-path* in $L$ ($n \geq 1$) is a sequence $B_1, \ldots, B_n$ of blocks of $L$ such that there are pairwise distinct atoms $b_1, a_1, b_2, \ldots, a_n, b_{n+1} \in L$ with $B_i = [b_i, a_i, b_{i+1}]_L$, $i = 1, \ldots, n$.

An *n-loop* in $L$ ($n \geq 3$) is a sequence $B_1, \ldots, B_n$ of blocks of $L$ such that there are pairwise distinct atoms $b_1, a_1, b_2, \ldots, a_n \in L$ with $B_i = [b_i, a_i, b_{i+1}]_L$, $i = 1, \ldots, n-1$, $B_n = [b_n, a_n, b_1]_L$.

We shall also need the following corollary of Greechie's lemma ([4]): An OML satisfying the assumptions of Def. 2.4 cannot contain any $n$-loop for $n \leq 4$.

**Theorem 2.5.** *Let $L$ be an OML. Let the cardinality of each block of $L$ be 8 and let $C(L) = \{0, 1\}$. Let for any pair $a, b$ of atoms in $L$ the inequality $a \vee b < 1$ hold true. Then any block of $L$ is contained in a 5-loop.*

Proof: We shall need three lemmas (the OML $L$ dealt with in the lemmas satisfies the assumptions of Theorem 2.5).

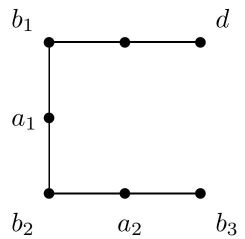
**Lemma 1.** *Each block in $L$ is contained in a 2-path.*

Proof: Consider a block $B = [a_1, a_2, a_3]_L$. Since $L$ is not a Boolean algebra, we see that $L \neq B$. Hence there is an atom $b \in L$ with $b \notin B$. The assumptions required for $L$ obviously guarantee the existence of an atom $c \in L$ such that $a_1\ C\ c$ and $b\ C\ c$. Let us complete the lemma arguing by cases. If $c \in \{a_1, a_2, a_3\}$, then the couple $[a_1, a_2, a_3]_L, [c, b, c^\perp \wedge b^\perp]_L$ is a 2-path. If $c \notin \{a_1, a_2, a_3\}$, then the couple $[a_1, a_2, a_3]_L, [c, a_1, c^\perp \wedge a_1^\perp]_L$ is a 2-path. The proof is done. □

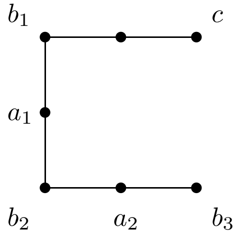
**Lemma 2.** *Each 2-path in $L$ is contained in a 3-path.*

Proof: Consider a 2-path, some $B_1 = [b_1, a_1, b_2]_L, B_2 = [b_2, a_2, b_3]_L$. Since $b_2 \notin C(L)$, there is an atom $d \in L$ such that $b_2 \neg\ C\ d$. It follows that $d \notin \{b_1, a_1, a_2, b_3\}$. We have two possibilities to argue.

(I) First, $d$ is compatible with some of the atoms $b_1, a_1, a_2, b_3$. Without any loss of generality, suppose that $d\ C\ b_1$. Then $a_1 \neg\ C\ d$, $a_2 \neg\ C\ d$ and $b_3 \neg\ C\ d$. Indeed, if $a_1\ C\ d$ then $d = b$. If $a_2\ C\ d$ or $b_3\ C\ d$ then $L$ contains a 4-loop which is excluded by the Greechie lemma. Thus, we obtain the following Greechie diagram:



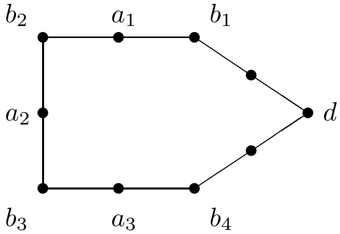

(II) Second, $d$ is not compatible with any of the elements $b_1, a_1, a_2, b_3$. By our assumption, there is an atom $c \in L$ such that $b_1\ C\ c$ and $d\ C\ c$. Since $d$ is not compatible with any of the elements $b_1, a_1, b_2, a_2, b_3$ and since $d\ C\ c$, we see that $c \notin \{b_1, a_1, b_2, a_2, b_3\}$. Mimicking the reasoning of the part (I) we obtain a 3-path portrayed below:




This completes the proof of Lemma 2. □

**Lemma 3.** *Each 3-path in $L$ is contained in a 5-loop.*

Proof: Consider a 3-path, some $B_1 = [b_1, a_1, b_2]_L$, $B_2 = [b_2, a_2, b_3]_L$, $B_3 = [b_3, a_3, b_4]_L$. By our assumption on $L$, there is an atom $d \in L$ such that $d \ C \ b_1$ and $d \ C \ b_4$. Obviously, $d \notin \{a_1, b_2, a_2, b_3, a_3\}$. In other words, we have completed the proof of Lemma 3 by constructing a 5-loop in $L$ with the following Greechie diagram:




□

Let us return to the proof of Theorem 2.5. Let us choose a block $B$ of $L$. Then a consecutive application of Lemma 1, Lemma 2 and Lemma 3 allows us to obtain the desired 5-loop. □

## 3. Results

Let $Z_2$ stand for the group $\{0, 1\}$ understood with the modulo 2 addition $\oplus$ (thus, $1 \oplus 1 = 0 \oplus 0 = 0$, $1 \oplus 0 = 0 \oplus 1 = 1$). Let $L$ be an OML and let $s : L \to Z_2$ be a mapping. Then $s$ is said to be a *$Z_2$-valued state* (abbr., a *$Z_2$-state*) provided $s(1) = 1$ and $s(x \vee y) = s(x) \oplus s(y)$ whenever $x, y \in L$, $x \leq y^\perp$. The following definition is a variant of "fullness" dealt with in the quantum logic theory ([7]) and it is crucial in our consideration.

**Definition 3.1.** Let $L$ be an OML. Then $L$ is called *$Z_2$-full* if for any $x, y \in L$, $x \neq y$, $x \neq 0$, $y \neq 1$ there exists a $Z_2$-state, $s$, on $L$ such that $s(x) = 1$ and $s(y) = 0$.

Our first result reads as follows.

**Theorem 3.2.** *Let $L$ be an OML. If $L$ is ODL-embeddable then $L$ is $Z_2$-full.*

The proof of Theorem 3.2 will be obtained in a series of propositions. Let us first examine a certain type of ideals in ODLs. They will correspond to $Z_2$-states.

**Definition 3.3.** Let $K$ be an ODL and let $I$ be a subset of $K$. Then $I$ is said to be a $\triangle$-*ideal* if $0 \in I$ and whenever $a, b \in I$, then $a \triangle b \in I$. Further, if $1 \notin I$, then $I$ is called a *proper* $\triangle$-*ideal*. Finally, $I$ is called *maximal* if $I$ is proper and for any proper $\triangle$-ideal $J$ with $I \subseteq J$ we have $I = J$.

**Proposition 3.4.** *Suppose that $K$ is an ODL and $I$ is a proper $\triangle$-ideal in $K$. Suppose that $x \in K$ and neither $x$ nor $x^\perp$ belongs to $I$. Let us write $J = I \cup \{a \triangle x;\ a \in I\}$. Then $J$ is also a proper $\triangle$-ideal in $K$ and, moreover, $x \in J$ and $x^\perp \notin J$.*

Proof: The set $J$ is obviously a $\triangle$-ideal. Let us see that $1 \notin J$. Suppose on the contrary that $1 \in J$. Then $1 = a \triangle x$ for some element $a \in I$. The equality $1 = a \triangle x$ implies that $a = x^\perp$ (indeed, by Proposition 1.2 we have $0 = (a \triangle x)^\perp = a \triangle x^\perp$ and therefore $a = x^\perp$). But $x^\perp$ does not belong to $I$ which is a contradiction. Thus, $1 \notin J$. Further $x = 0 \triangle x \in J$. If $x^\perp \in J$, then $1 = x \triangle x^\perp \in J$ — a contradiction again. □

**Proposition 3.5.** *Let $K$ be an ODL and let $I$ be a maximal $\triangle$-ideal in $K$. Then* $\mathrm{card}(\{x, x^\perp\} \cap I) = 1$ *for any $x \in K$.*

Proof: Suppose that $I$ is maximal and $x \in K$. Suppose further that $x \notin I$ and, also $x^\perp \notin I$. Then (Proposition 3.4) there is a $\triangle$-ideal, $J$, such that $I \subseteq J$ and $I \neq J$. As a result, at least one element of the set $\{x, x^\perp\}$ belongs to $I$. Looking for a contradiction, suppose that $\{x, x^\perp\} \subseteq I$. Then $x \triangle x^\perp = 1$ which means that $1 \in I$ — a contradiction ($I$ is supposed to be proper). □

**Proposition 3.6.** *Let $K$ be an ODL and let $a, b \in K$, $a \neq b$, $a < 1$ and $0 < b$. Then there is a maximal $\triangle$-ideal, $J$, such that $a \in J$ and $b \notin J$.*

Proof: Write $\mathcal{I} = \{I \subseteq K;\ I$ is a proper $\triangle$-ideal, $a \in I$ and $b \notin I\}$. Then $\{0, a\} \in \mathcal{I}$ and therefore $\mathcal{I} \neq \emptyset$. By a standard application of Zorn's lemma, the set $\mathcal{I}$ ordered by inclusion contains a maximal element, $J$. Of course, $J$ is a proper $\triangle$-ideal. Moreover, $b^\perp \in J$ (otherwise the $\triangle$-ideal $J' = J \cup \{c \triangle b^\perp;\ c \in J\}$ extends $J$, Proposition 3.4, and $J'$ belongs to the system $\mathcal{I}$). Let us show that $J$ is maximal. Suppose therefore that $J \subseteq I$ for a proper $\triangle$-ideal $I$, $J \neq I$. Thus, $I$ is strictly larger than $J$ and therefore $I \notin \mathcal{I}$. Therefore $b \in I$ and since $b^\perp \in J \subseteq I$, we see that $1 = b \triangle b^\perp \in I$. This means that $I$ is not proper and the proof is complete. □

**Proposition 3.7.** *Let $K$ be an ODL and $I$ be a maximal $\triangle$-ideal in $K$. Let us define a mapping $s : K \to Z_2$ as follows: $s(a) = 0$ (resp., $s(a) = 1$) if $a \in I$ (resp., $a \notin I$). Then $s(x \triangle y) = s(x) \oplus s(y)$ for any $x, y \in L$. A consequence: The mapping $s$ is a $Z_2$-state on $K_{\mathrm{supp}}$.*

Proof: Let us consider two elements $x, y \in K$. We are to prove the equality $s(x \triangle y) = s(x) \oplus s(y)$. We will argue by cases. If both $x$ and $y$ belong to $I$, then $x \triangle y \in I$ and therefore $s(x \triangle y) = 0 = 0 \oplus 0 = s(x) \oplus s(y)$. If $x \in I$ and $y \notin I$, then $x \triangle y \notin I$ (indeed, should $x \triangle y$ be an element of $I$, then $y = x \triangle (x \triangle y) \in I$ which is a contradiction). Hence $s(x \triangle y) = 1 = 0 \oplus 1 = s(x) \oplus s(y)$. The case of $x \notin I$ and $y \in I$ argues analogously. Let us suppose that $x \notin I$ and $y \notin I$. Since $I$ is a maximal $\triangle$-ideal, we infer that $x^\perp \in I$ and $y^\perp \in I$. Then $x^\perp \triangle y^\perp \in I$. But $x^\perp \triangle y^\perp = x \triangle y$ (Proposition 1.2(5)) and therefore $x \triangle y \in I$. Hence $s(x \triangle y) = 0 = 1 \oplus 1 = s(x) \oplus s(y)$.

It remains to show that the mapping $s$ defined above is a $Z_2$-state on $K_{\text{supp}}$. Of course, $s(1) = 1$. Let us take $x, y \in K$ with $x \le y^\perp$. Then $x\, C\, y$ and therefore (Proposition 1.5) we see that $x \triangle y = (x \vee y) \wedge (x \wedge y)^\perp = (x \vee y) \wedge 0^\perp = x \vee y$. Then $s(x \vee y) = s(x \triangle y) = s(x) \oplus s(y)$ by the analysis above. The proof of Proposition 3.7 is complete. □

Proof of Theorem 3.2: Let $L$ be an ODL-embeddable OML. Then there is an ODL, $K$, such that $L$ is a sub-OML of $K_{\text{supp}}$. Let $x, y$ be elements of $L$ with $x \ne y$, $x \ne 0$ and $y \ne 1$. According to Proposition 3.6 there is a maximal $\triangle$-ideal $J$ in $K$ such that $y \in J$ and $x \notin J$. Let us set $s(a) = 0$ for $a \in J$ and $s(a) = 1$ for $a \in K$, $a \notin J$. Then, according to Proposition 3.7, the mapping $s$ is a $Z_2$-state on $K_{\text{supp}}$. If we denote by $s_1$ the restriction of $s$ to the OML $L$, then $s_1$ is a $Z_2$-state on $L$. Moreover, $s_1(x) = s(x) = 1$ and $s_1(y) = s(y) = 0$. □

The link of ODL-embeddable OMLs with $Z_2$-states revealed in Theorem 3.2 allows us to shed light on the ODL embeddability of the lattice $L(H)$ of projections in a (real) Hilbert space $H$.

**Theorem 3.8.** *Let $H$ be a Hilbert space. If $\dim H \ge 4$, then $L(H)$ is not ODL-embeddable.*

Proof: In [15] it is shown that for $\dim H \ge 4$ the OML $L(H)$ does not allow for any $Z_2$-state. The rest follows from Theorem 3.2. □

The case of $L(R^3)$ remains open — it seems still open whether or not $L(R^3)$ possesses a $Z_2$-state (see [8] and [15]). However, it is not difficult to show that $L(R^3)$ cannot be made an ODL (i.e., it can be proved that $L(R^3)$ is not ODL-convertible). In fact, even relatively mild lattice-theoretic conditions shared by $L(R^3)$ prevent us from introducing $\triangle$ on $L(R^3)$. We are going to prove this by deriving a characterization of 3-stars — a result which may be of separate interest in the theory of ODLs.

Recall first a result already referred to in the introduction (for a detailed proof, see [11]; let us provide a sketch for the convenience of the reader).

**Proposition 3.9.** *Let $\kappa$ be a cardinal number. Let $\kappa = 2^n - 1$ for a natural number $n \in \mathbb{N}$ or let $\kappa$ be infinite. Then the horizontal sum $MO_\kappa$ is, up to an ODL-isomorphism, uniquely ODL-convertible.*

Proof: Let $\kappa = 2^n - 1$ (resp. $\kappa$ be infinite). Then there is a Boolean algebra, $B$, with $\mathrm{card}(B) = 2^{n+1}$ (resp. $\mathrm{card}(B) = \kappa$). Take a prime-ideal on $B$, some $I$ and set, for any $a \in I \setminus \{0\}$, $B_a = \{0, a, a^\perp, 1\}$. Since $\mathrm{card}(I \setminus \{0\}) = \kappa$, we see that $MO_\kappa$ is OML-isomorphic with the horizontal sum of $B_a$, $a \in I \setminus \{0\}$. Moreover, $MO_\kappa$ and $B$ have the same underlying set. Thus, elements $c, d \in MO_\kappa$ can be viewed as elements of $B$ and hence we can define $c \triangle d$ as the corresponding symmetric difference in $B$ (understood in $MO_\kappa$ this time). It can be shown that $MO_\kappa$ endowed with this symmetric difference is an ODL and that $\triangle$ is (up to an ODL-isomorphism) the only one which converts $MO_\kappa$ to an ODL. □

Before we formulate the main result of this section let us again make use of Convention 1.4 allowing ourselves to call an ODL $K$ a 3-star provided so is $K_{\mathrm{supp}}$.

**Theorem 3.10.** *Let $K$ be an ODL. Then the following two statements are equivalent:*

(i) *$K$ is a 3-star,*
(ii) *the cardinality of each maximal Boolean subalgebra of $K$ is 8, and for any pair $a, b \in K$ of atoms in $K$ the inequality $a \vee b < 1$ holds true.*

Proof: The implication (i)⇒(ii) is obvious. Let us launch on (ii)⇒(i). Let us first formulate and prove a few auxiliary propositions.

**Lemma 1.** *Suppose that $K$ is as in Theorem 3.10(ii). Let $a, b$ be atoms of $K$. Then*

(i) *$a \triangle b$ is a co-atom of $K$ if and only if $a \neq b$ and $a\ C\ b$,*
(ii) *if $a$ is not compatible with $b$, then $a \triangle b$ is an atom of $K$.*

Proof: (i) If $a \neq b$ and $a\ C\ b$, then $a \leq b^\perp$ and therefore $a \triangle b = a \vee b$. Since both $a, b$ belong to an 8-element Boolean subalgebra of $K$, the element $a \triangle b$ must be a co-atom.

Suppose for the reverse implication that $a \triangle b = d^\perp$ for an atom $d \in K$. Choose an atom, $c$, such that $a\ C\ c$ and $b\ C\ c$. Then $a \leq c^\perp$ and $b \leq c^\perp$. It follows that $a \triangle b \leq a \vee b \leq c^\perp$. Thus, $d^\perp \leq c^\perp$ and therefore $c \leq d$. Since $c, d$ are atoms, we see that $c = d$. The equality $a \triangle b = c^\perp$ gives us $a \triangle a \triangle b = a \triangle c^\perp$. According to Proposition 1.2 we have $b = a \triangle c^\perp$. Since $a\ C\ c^\perp$, we see in view of Proposition 1.5 that $a\ C\ a \triangle c^\perp$. Hence $a\ C\ b$.

(ii) Suppose that $a \neg\ C\ b$. As known ([1] and [9]), $a\ C\ b$ precisely when $a\ C\ b^\perp$. It follows that $a \neq b^\perp$ and $a \neq b$. Then $a \triangle b \neq 1$ and $a \triangle b \neq 0$. If $a \triangle b$ were a co-atom, the part (i) gives us $a\ C\ b$. This implies that $a \triangle b$ is an atom in $K$. □

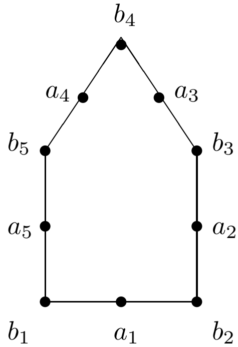
**Lemma 2.** *Suppose that $K$ is as in Theorem 3.10(ii). Let $a, b, c$ be atoms in $K$. Then $a \triangle b \triangle c = 1$ if and only if the atoms $a, b, c$ are pairwise distinct and pairwise compatible.*

Proof: If $a, b, c$ are pairwise distinct and pairwise compatible, they must be the atoms of a block of $K$. In this case $a \triangle b \triangle c = 1$.

Suppose that $a \triangle b \triangle c = 1$. Then $a, b, c$ are pairwise distinct. Indeed, if e.g. $a = b$, then $a \triangle b \triangle c = a \triangle a \triangle c = 0 \triangle c = c \neq 1$. Further, $a \triangle b = c^{\perp}$ and therefore $a \triangle b$ is a co-atom. It follows that $a \, C \, b$ (Lemma 1). Analogously, $a \, C \, c$ and $b \, C \, c$ and this completes the proof. □

**Lemma 3.** *Suppose that $K$ is as in Theorem 3.10(ii). Then $K$ does not contain a 5-loop.*

Proof: Suppose that it is not the case. Then there must be a configuration of blocks indicated by the following figure.




We see that we obtain the following collection of identities:
$b_1 \triangle a_1 \triangle b_2 = 1$, $b_2 \triangle a_2 \triangle b_3 = 1$, $b_3 \triangle a_3 \triangle b_4 = 1$, $b_4 \triangle a_4 \triangle b_5 = 1$, and $b_5 \triangle a_5 \triangle b_1 = 1$.
As a result, we have the equality
$(b_1 \triangle a_1 \triangle b_2) \triangle (b_2 \triangle a_2 \triangle b_3) \triangle (b_3 \triangle a_3 \triangle b_4) \triangle (b_4 \triangle a_4 \triangle b_5) \triangle (b_5 \triangle a_5 \triangle b_1) = 1 \triangle 1 \triangle 1 \triangle 1 \triangle 1$. Since $x \triangle x = 0$ for any $x$ in $K$, the right-hand side of the equality above equals to 1 and the left-hand side equals to $a_1 \triangle a_2 \triangle a_3 \triangle a_4 \triangle a_5$. Thus, $a_1 \triangle a_2 \triangle a_3 \triangle a_4 \triangle a_5 = 1$. Let us rewrite the last equality as follows: $(a_1 \triangle a_2) \triangle (a_3 \triangle a_4) \triangle a_5 = 1$. Lemma 1 gives us that $a_1 \triangle a_2$ as well as $a_3 \triangle a_4$ are atoms in $K$. Further, Lemma 2 implies that $a_1 \triangle a_2$ and $a_5$ are compatible atoms. Moreover, $a_1 \leq b_2^{\perp}$ and $a_2 \leq b_2^{\perp}$. This means that $a_1 \triangle a_2 \leq a_1 \vee a_2 \leq b_2^{\perp}$. We therefore see that $b_2 \, C \, (a_1 \triangle a_2)$. But then $b_1$ and $a_1 \triangle a_2$ are distinct atoms that are compatible with $a_5$ and $b_2$. This contradicts Proposition 2.1(ii). The proof of Lemma 3 is complete. □

Proof of Theorem 3.10: It is easily seen that the proof of Theorem 3.10 can be obtained as an interplay of the Lemma 3 and Theorem 2.5. Indeed, suppose $K$ satisfies the conditions of Theorem 3.10(ii). Then as $K$ does not contain a 5-loop, to avoid a contradiction with Theorem 2.5 we must have $C(K) \neq \{0, 1\}$. But this means that $K$ is a 3-star (Proposition 2.3). □

**Theorem 3.11.** *The OML $L(R^3)$ is not ODL-convertible.*

PROOF: Suppose that $L(R^3)$ is ODL-convertible. Then $L(R^3)$ must be a 3-star (Theorem 3.10). But $C(L(R^3)) = \{0,1\}$ and we have reached a contradiction. The proof is complete. □

**Theorem 3.12.** *The OMLs $L(R^2)$ and $L(R^1)$ are ODL-convertible.*

PROOF: Of course, $L(R^1) = \{0,1\}$ and there is nothing to prove. Let us consider $L(R^2)$. Obviously, $L(R^2)$ is nothing but $MO_\kappa$, where $\kappa = 2^{\omega_0}$ (= the cardinality of continuum). This OML is ODL-convertible (Proposition 3.9). □

We have seen that a lack of $Z_2$-states on $L$ prevents $L$ from being ODL-embeddable (and, in turn, from being ODL-convertible). It should be noted that in [14] and [19] the authors construct finite OMLs without any group-valued state at all. Their technique therefore provides another type of OMLs that are not ODL-embeddable. However, the technique is very involved and even computer-proved in places. A relatively simple OML without any $Z_2$-states can be constructed on the ground of the following proposition. This proposition allows us to extend the class of non-embeddable OMLs, and it also slightly adds to the area of orthomodular peculiarities (see [4], [13], etc.). It should be noted that the result generalizes Proposition 7.2 of the paper [12].

**Proposition 3.13.** *Suppose that $L$ is an OML. Suppose that there are blocks $B_1, B_2, \ldots, B_n$ of $L$ such that the following two conditions are satisfied:*

(1) *each $B_i$, $1 \le i \le n$ is finite and $n$ is an odd number,*
(2) *if $a \in L$ is an atom in $L$, then $a$ lies in an even number of blocks $B_1, B_2, \ldots, B_n$ (i.e. the cardinality of the set $\{i;\ a \in B_i\}$ is even).*

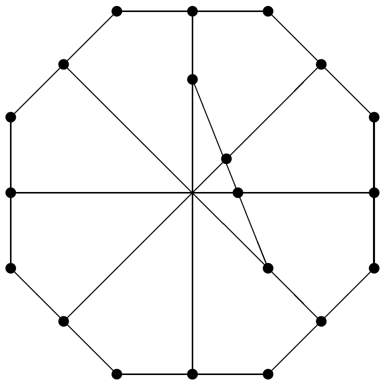
*Then there is no $Z_2$-state on $L$.*

PROOF: Seeking a contradiction, let $s : L \to Z_2$ be a $Z_2$-state. Let $\{a_{i,1}, \ldots, a_{i,k_i}\}$ be the set of all atoms of the algebra $B_i$, $i = 1, \ldots, n$. Then the elements $a_{i,1}, \ldots, a_{i,k_i}$ are mutually orthogonal and, moreover, $a_{i,1} \vee \ldots \vee a_{i,k_i} = 1_L$. Since $s$ is a $Z_2$-state, we have $s(a_{i,1} \vee \ldots \vee a_{i,k_i}) = s(a_{i,1}) \oplus \ldots \oplus s(a_{i,k_i})$. Since $a_{i,1} \vee \ldots \vee a_{i,k_i} = 1_L$, we obtain $s(a_{i,1} \vee \ldots \vee a_{i,k_i}) = s(1_L) = 1$. Summarizing, $s(a_{i,1}) \oplus \ldots \oplus s(a_{i,k_i}) = 1$ for any $i \in \{1, \ldots, n\}$. As a consequence,

$$(s(a_{1,1}) \oplus \ldots \oplus s(a_{1,k_1})) \oplus \ldots \oplus (s(a_{n,1}) \oplus \ldots \oplus s(a_{n,k_n})) = 1 \oplus \ldots \oplus 1.$$

The right-hand side of the latter identity contains the element 1 exactly $n$-many times. Since $n$ is odd, the right-hand side equals to 1. Moreover, if $a$ is an arbitrary atom of $L$, then the assumption of Proposition 3.13 gives us that the left-hand side of the identity contains the expression $s(a)$ an even number of times. By the property of the operation $\oplus$, the left-hand side must be equal to 0. We have derived a contradiction and the proof is complete. □

This result enables us to construct OMLs that do not possess a $Z_2$-state (and, as a consequence, the OMLs that are not ODL-embeddable). Let us conclude our paper by exhibiting a simple example of an OML in this class (the OML portrayed below by its Greechie diagram obviously satisfies the assumptions of Proposition 3.13; a proper class of such OMLs can be constructed in an analogous manner).



## References


[1] Beran L., *Orthomodular Lattices, Algebraic Approach*, D. Reidel, Dordrecht, 1985.

[2] Bruns G., Harding J., *Algebraic aspects of orthomodular lattices*, in Coecke B., Moore D. and Wilce A., Eds., Current Research in Operational Quantum Logic, 2000, pp. 37–65.

[3] Dvurečenskij A., Pulmannová S., *New Trends in Quantum Structures*, Kluwer Academic Publishers, Dordrecht, and Ister Science, Bratislava, 2000.

[4] Greechie R.J., *Orthomodular lattices admitting no states*, J. Combinatorial Theory **10** (1971), 119–132.

[5] Hamhalter J., *Quantum Measure Theory*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.

[6] *Handbook of Quantum Logic and Quantum Structures*, ed. by K. Engesser, D.M. Gabbay and D. Lehmann, Elsevier, 2007.

[7] Gudder S.P., *Stochastic Methods in Quantum Mechanics*, North-Holland, New York-Oxford, 1979.

[8] Harding J., Jager E., Smith D., *Group-valued measures on the lattice of closed subspaces of a Hilbert space*, Internat. J. Theoret. Phys. **44** (2005), 539–548.

[9] Kalmbach G., *Orthomodular Lattices*, Academic Press, London, 1983.

[10] Maeda F., Maeda S., *Theory of Symmetric Lattices*, Springer, Berlin-Heidelberg-New York, 1970.

[11] Matoušek M., *Orthocomplemented lattices with a symmetric difference*, Algebra Universalis **60** (2009), 185–215.

[12] Matoušek M., Pták P., *Orthocomplemented posets with a symmetric difference*, Order **26** (2009), 1–21.

[13] Navara M., Pták P., Rogalewicz V., *Enlargements of quantum logics*, Pacific J. Math. **135** (1988), 361–369.

[14] Navara M., *An orthomodular lattice admitting no group-valued measure*, Proc. Amer. Math. Soc. **122** (1994), 7–12.

[15] Navara M., Pták P., *For $n \geq 5$ there is no nontrivial $Z_2$-measure on $L(R^n)$*, Internat. J. Theoret. Phys. **43** (2004), 1595–1598.

[16] Pták P., Pulmannová S., *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1991.
[17] Svozil K., Tkadlec J., *Greechie diagrams, noexistence of measures in quantum logics, and Kochen–Specker-type constructions*, J. Math. Phys. **37** (1996), 5380–5401.
[18] Varadarajan V.S., *Geometry of Quantum Theory I, II*, Van Nostrand, Princeton, 1968, 1970.
[19] Weber H., *There are orthomodular lattices without non-trivial group-valued states: A computer-based construction*, J. Math. Anal. Appl. **183** (1994), 89–93.

Ruská 22, 101 00 Prague 10, Czech Republic
*Email:* matmilan@email.cz

Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University, 166 27 Prague 6 Czech Republic
*Email:* ptak@math.feld.cvut.cz

(*Received* May 25, 2009, *revised* November 16, 2009)

# Orthocomplemented difference lattices with few generators

**Milan Matoušek** and **Pavel Pták** [1]

Abstract. The algebraic theory of quantum logics overlaps in places with certain areas of cybernetics, notably with the field of artificial intelligence (see, e.g., [19, 20]). Recently an affort has been exercised to advance with logics that possess a symmetric difference ([13, 14]) - with so called orthocomplemented difference lattices (ODLs). This paper further contributes to this affort. In [13] the author constructs an ODL that is not set-representable. This example is quite elaborate. A main result of this paper somewhat economizes on this construction: There is an ODL with 3 generators that is not set-representable (and so the free ODL with 3 generators cannot be set-representable). The result is based on a specific technique of embedding orthomodular lattices into ODLs. The ODLs with 2 generators are always set-representable as we show by characterizing the free ODL with 2 generators - this ODL is $\mathrm{MO}_3 \times 2^4$.

*AMS Class:* 06C15, 03G12, 81B10.
*Key words:* orthomodular poset, quantum logic, symmetric difference, Gödel's coding, Boolean algebra, free algebra.

## 1 Introduction. Basic notions.

The notion of ODL has been introduced in [13] and further studied in [14] and [15]. The axiomatic setup of ODLs came into existence by taking an abstract form of set theoretic *symmetric difference* as a primitive operation (see Def. 1.1). As it turns out, an ODL is automatically orthomodular and therefore it forms an orthomodular lattice (an OML). This situates the variety of ODLs between OMLs and Boolean algebras. In a potential application, the ODLs add to the instances considered previously as quantum logics (see [4, 7, 10, 19] etc.). In this paper we find a minimal number of generators of an ODL that is not set-representable. This number is 3. We shall make use of the Greechie's paste job for OMLs together with certain techniques of embeddings of OMLs into ODLs. An acquitance with the theory of OMLs is assumed in places (see, e.g., [1, 12, 19] for basics on OMLs). For some specific properties of ODLs, let us refer the reader to [13].

Let us first recall the definition of an ODL.

**Definition 1.1** *Let* $L = (X, \wedge, \vee, {}^{\perp}, 0, 1, \triangle)$*, where* $(X, \wedge, \vee, {}^{\perp}, 0, 1)$ *is an OCL and* $\triangle : X^2 \to X$ *is a binary operation. Then* $L$ *is said to be an* orthocomplemented difference lattice *(abbr., an ODL) if the following identities hold in* $L$*:*
$(\mathrm{D}_1)$ $x \triangle (y \triangle z) = (x \triangle y) \triangle z$,
$(\mathrm{D}_2)$ $x \triangle 1 = x^{\perp}$, $1 \triangle x = x^{\perp}$,
$(\mathrm{D}_3)$ $x \triangle y \leq x \vee y$.

[1]The authors acknowledge the support of the research plans MSM 0021620839 and MSM 6840770038 that are financed by the Ministry of Education of the Czech Republic.

Obviously, the class of all ODLs forms a variety. We will denote it by $\mathcal{ODL}$.

Let $L = (X, \wedge, \vee, ^{\perp}, 0, 1, \triangle)$ be an ODL. Then the OCL $(X, \wedge, \vee, ^{\perp}, 0, 1)$ will be denoted by $L_{\mathrm{supp}}$ and called the *support* of $L$. Occasionally, the ODL $L$ will be identified with the couple $(L_{\mathrm{supp}}, \triangle)$.

Let us list basic properties of ODLs as we shall use them in the sequel.

**Proposition 1.2** *Let $L$ be an ODL. Then the following statements hold true:*
(1) $x \triangle 0 = x$, $0 \triangle x = x$,
(2) $x \triangle x = 0$,
(3) $x \triangle y = y \triangle x$,
(4) $x \triangle y^{\perp} = x^{\perp} \triangle y = (x \triangle y)^{\perp}$ ,
(5) $x^{\perp} \triangle y^{\perp} = x \triangle y$ ,
(6) $x \triangle y = 0 \Leftrightarrow x = y$,
(7) $(x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) \leq x \triangle y \leq (x \vee y) \wedge (x \wedge y)^{\perp}$.

**Proof**. Let us first observe that the property (D$_2$) yields $1 \triangle 1 = 1^{\perp} = 0$. Let us verify the properties (1)-(7). Suppose that $x, y \in L$.
(1) $x \triangle 0 = x \triangle (1 \triangle 1) = (x \triangle 1) \triangle 1 = x^{\perp} \triangle 1 = (x^{\perp})^{\perp} = x$. Further, $0 \triangle x = (1 \triangle 1) \triangle x = 1 \triangle (1 \triangle x) = 1 \triangle x^{\perp} = (x^{\perp})^{\perp} = x$ .
(2) Let us first show that $x^{\perp} \triangle x^{\perp} = x \triangle x$. We consecutively obtain $x^{\perp} \triangle x^{\perp} = (x \triangle 1) \triangle (1 \triangle x) = (x \triangle (1 \triangle 1)) \triangle x = (x \triangle 0) \triangle x = x \triangle x$. Moreover, we have $x \triangle x \leq x$ as well as $x \triangle x = x^{\perp} \triangle x^{\perp} \leq x^{\perp}$. This implies that $x \triangle x \leq x \wedge x^{\perp} = 0$.
(3) $x \triangle y = (x \triangle y) \triangle 0 = (x \triangle y) \triangle [(y \triangle x) \triangle (y \triangle x)] = x \triangle (y \triangle y) \triangle x \triangle (y \triangle x) = x \triangle 0 \triangle x \triangle (y \triangle x) = x \triangle x \triangle (y \triangle x) = 0 \triangle (y \triangle x) = y \triangle x$.
(4) $x \triangle y^{\perp} = x \triangle (y \triangle 1) = (x \triangle y) \triangle 1 = (x \triangle y)^{\perp}$. The equality $x^{\perp} \triangle y = (x \triangle y)^{\perp}$ follows from $x \triangle y^{\perp} = (x \triangle y)^{\perp}$ by applying the equality (3).
(5) Using (4) we obtain $x^{\perp} \triangle y^{\perp} = (x^{\perp} \triangle y)^{\perp} = (x \triangle y)^{\perp\perp} = x \triangle y$.
(6) If $x = y$, then $x \triangle y = 0$ by the condition (2). Conversely, suppose that $x \triangle y = 0$. Then $x = x \triangle 0 = x \triangle (y \triangle y) = (x \triangle y) \triangle y = 0 \triangle y = y$.
(7) The property (D$_3$) together with the properties (4), (5) imply that $x \triangle y \leq x \vee y$, $x \triangle y \leq x^{\perp} \vee y^{\perp} = (x \wedge y)^{\perp}$, $x \wedge y^{\perp} \leq x \triangle y$, $x^{\perp} \wedge y \leq x \triangle y$.

**Theorem 1.3** *Let $L$ be an ODL. Then its support $L_{\mathrm{supp}}$ is an OML.*

**Proof**. Suppose that $x, y \in L$, $x \leq y$, $y \wedge x^{\perp} = 0$. Let us prove that $x = y$. Since $x \leq y$, we conclude that $(x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = y \wedge x^{\perp} = 0$ and $(x \vee y) \wedge (x \wedge y)^{\perp} = y \wedge x^{\perp} = 0$. By Prop. 1.2, (6), (7) we see that $x \triangle y = 0$ and therefore $x = y$.

In view of the above proposition, all notions of OMLs can be referred to in ODLs, too. In particular, we shall say that two elements $x, y$ in an ODL $L$ *commute* (in symbols, $x\ C\ y$) if they commute in $L_{\mathrm{supp}}$. Similarly, we shall denote by $C(L)$ the set of all elements of $L$ that commute with all elements of $L$. Let us call $C(L)$ the *centre* of $L$. It can be easily shown that $C(L)$ is a subalgebra of $L$ ([13]).

Let us suppose that $B$ is a Boolean algebra. Let us denote by $\Delta_B$ the standard symmetric diffrence on $B$. Thus, if $x, y \in B$ then $x\ \Delta_B\ y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = (x \vee y) \wedge (x \wedge y)^{\perp}$.

**Proposition 1.4** *Let $L$ be an ODL. Let $x, y \in L$ with $x\ C\ y$. Then $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = (x \vee y) \wedge (x \wedge y)^{\perp}$.*

*A consequence: If $B$ is a Boolean sub-algebra of $L$ and $x_1, \dots, x_n \in B$, then $x_1 \triangle \dots \triangle x_n = x_1 \Delta_B \dots \Delta_B x_n$.*

**Proof**. According to Prop. 1.2, (7), we have the inequalities $(x \wedge y^\perp) \vee (y \wedge x^\perp) \leq x \triangle y \leq (x \vee y) \wedge (x \wedge y)^\perp$. Since the elements $x, y$ commute, the left-hand side of the previous inequality coincides with the right-hand side and therefore $x \triangle y = (x \wedge y^\perp) \vee (y \wedge x^\perp) = (x \vee y) \wedge (x \wedge y)^\perp$.

Let us exhibit some simple examples of ODLs. Firstly, each Boolean algebra can be understood as an ODL which the following proposition shows.

**Proposition 1.5** *Let $B$ be a BA. Then there exists exactly one mapping $\triangle : B \times B \to B$ which fulfils the conditions* $(\mathrm{D}_1)$, $(\mathrm{D}_2)$ *and* $(\mathrm{D}_3)$ *of Def. 1.1.*

**Proof**. To prove the existence, take for the operation $\triangle$ the standard symmetric difference $\Delta_B$ in $B$. The properties $(\mathrm{D}_1)$, $(\mathrm{D}_2)$ and $(\mathrm{D}_3)$ of Def. 1.1 are then obviously fulfilled.

Let us prove the uniqueness of $\triangle$. Let $\triangle_1 : B \times B \to B$ be a mapping that fulfils the conditions $(\mathrm{D}_1)$, $(\mathrm{D}_2)$ and $(\mathrm{D}_3)$. So the couple $(B, \triangle_1)$ is an ODL. If $x, y \in B$, then $x \; C \; y$, and therefore $x \triangle_1 y = x \; \Delta_B \; y = x \triangle y$ (Prop. 1.4).

**Example 1.6** Let $\mathrm{MO}_3$ be the OML obtained as the horizontal sum of three 4-element BA's (see, e.g., [12]). Write $\mathrm{MO}_3 = \{0, 1, x, x^\perp, y, y^\perp, z, z^\perp\}$. Then one can easily show that there is exactly one mapping $\triangle : \mathrm{MO}_3 \times \mathrm{MO}_3 \to \mathrm{MO}_3$ such that $x \triangle y = z$ and $(\mathrm{MO}_3, \triangle)$ is an ODL. The ODL obtained in this way will again be denoted by $\mathrm{MO}_3$. Obviously, the ODL $\mathrm{MO}_3$ is generated by the elements $x, y$. (It might be noted that $\mathrm{MO}_k$ can be viewed as an ODL exactly when $k = 2^n + 1$, [13]. We shall only use $\mathrm{MO}_3$ in this paper.)

**Proposition 1.7** *Let $L$ be an ODL and let $x, y \in L$. Then*
(a) $x \vee (x \triangle y) = x \vee y$ ,
(b) $x \wedge (x \triangle y) = x \wedge y^\perp$.

**Proof**. Before verifying the equalities, recall the convention of the preference of $\triangle$ over the operations $\wedge$ and $\vee$ (thus, for instance, $x \vee y \triangle z$ means $x \vee (y \triangle z)$ etc.).
(a) The inequality $x \vee x \triangle y \leq x \vee y$ is obvious. We have to show that $x \vee y \leq x \vee x \triangle y$. But $x \leq x \vee x \triangle y$ and therefore we need to check $y \leq x \vee x \triangle y$. According to $(\mathrm{D}_3)$, we have $x \vee x \triangle y \geq x \triangle (x \triangle y) = y$. (It is worthwhile observing that this equality can be viewed as a strenghtening of the condition $(\mathrm{D}_3)$ from the definition of ODL's.)
(b) The equality follows from (a) via the following calculation: $x \wedge x \triangle y = (x \wedge x \triangle y)^{\perp\perp} = (x^\perp \vee x^\perp \triangle y)^\perp = (x^\perp \vee y)^\perp = x \wedge y^\perp$.

**Proposition 1.8** *Let $L$ be an ODL and let $x, y \in L$. Then $x \leq y \Leftrightarrow x \triangle y \leq y$.*

**Proof**. Let us suppose that $x \leq y$. As $y \leq y$, the condition $(\mathrm{D}_3)$ implies that $x \triangle y \leq y$. Conversely, suppose $x \triangle y \leq y$. Making again use of $y \leq y$, the condition $(\mathrm{D}_3)$ implies that $(x \triangle y) \triangle y \leq y$. But $(x \triangle y) \triangle y = x \triangle (y \triangle y) = x \triangle 0 = x$.

We shall need the following simple fact on OMLs.

**Lemma 1.9** *Let $L$ be an OML. Let $x, y, x_1, x_2 \in L$ and let $y = x_1 \vee x_2$, $x_1 \leq x$, $x_2 \leq x^\perp$. Then $x \; C \; y$ and $x_1 = y \wedge x$, $x_2 = y \wedge x^\perp$.*

**Proof**. Since $x_1 \leq x$ and $x_2 \leq x^\perp$, we see that $x_1 \leq x \leq x_2^\perp$. Thus, the elements $x_1, x_2, x$ are mutually commutative. As known, $x\ C\ (x_1 \vee x_2)$ and therefore $x\ C\ y$. Moreover,
$y \wedge x = (x_1 \vee x_2) \wedge x = (x_1 \wedge x) \vee (x_2 \wedge x) = x_1 \vee 0 = x_1$, and
$y \wedge x^\perp = (x_1 \vee x_2) \wedge x^\perp = (x_1 \wedge x^\perp) \vee (x_2 \wedge x^\perp) = 0 \vee x_2 = x_2$.

**Proposition 1.10** *Let $L$ be an ODL. Let $x, y, z \in L$ with $x\ C\ y$ and $x\ C\ z$. Then $x\ C\ (y \triangle z)$ and $x \wedge (y \triangle z) = (x \wedge y) \triangle (x \wedge z)$.*

**Proof**. The commutativity of the pair $x\ C\ y$ and $x\ C\ z$ yields the equations $y = (y \wedge x) \vee (y \wedge x^\perp)$, $z = (z \wedge x) \vee (z \wedge x^\perp)$. Since $(y \wedge x) \perp (y \wedge x^\perp)$ and $(z \wedge x) \perp (z \wedge x^\perp)$, we see by Prop. 1.8 that $y = (y \wedge x) \triangle (y \wedge x^\perp)$ and $z = (z \wedge x) \triangle (z \wedge x^\perp)$. But we also have $y \triangle z = [(y \wedge x) \triangle (y \wedge x^\perp)] \triangle [(z \wedge x) \triangle (z \wedge x^\perp)] = [(y \wedge x) \triangle (z \wedge x)] \triangle [(y \wedge x^\perp) \triangle (z \wedge x^\perp)]$. Let us write $x_1 = (y \wedge x) \triangle (z \wedge x)$, $x_2 = (y \wedge x^\perp) \triangle (z \wedge x^\perp)$. Then $x_1 \leq (y \wedge x) \vee (z \wedge x) \leq x$. Analogously, $x_2 \leq x^\perp$. This implies that $x_1 \perp x_2$. By Prop. 1.8, $y \triangle z = x_1 \vee x_2$. The proof is completed by using Lemma 1.9.

Let us take up the intervals in ODLs. We will need them for the decomposition property with respect to a central element. Consider first the situation in OMLs. Let $K$ be an OML and let $a \in K$. Let us write $[0, a]_K = \{x \in K;\ x \leq a\}$. As known, the interval $[0, a]$ constitutes an OML. We will denote it by $K^a$. Let us shortly recall the construction of $K^a$ (see, for example, [12], p. 20): If $x, y \in [0, a]$, then $x \wedge y \in [0, a]$ and $x \vee y \in [0, a]$. The element 0, resp. $a$, is a least, resp. a greatest, element of $K^a$. The orthocomplement of $x$ in $K^a$, $x^{\perp_a}$, is defined by setting $x^{\perp_a} = x^{\perp_K} \wedge a$. It can be easily seen that $K^a = ([0, a], \wedge, \vee, ^{\perp_a}, 0, a)$ is an OML.

Let $L$ be an ODL and let $a \in L$. If $x, y \in [0, a]$ then $x \triangle y \in [0, a]$. Let us consider the algebra $L^a = ([0, a], \wedge, \vee, ^{\perp_a}, 0, a, \triangle) = ((L_{\text{supp}})^a, \triangle)$.

**Proposition 1.11** *Let $L$ be an ODL and let $a \in L$. Then the algebra $L^a$ is again an ODL. Moreover, if $a \in C(L)$, then the mapping $\pi_a : L \to [0, a]$ defined by putting $\pi_a(x) = x \wedge a$ is a surjective homomorphism of $L$ onto $L^a$.*

**Proof**. In order for $L^a$ to be an ODL, it is sufficient to check that the conditions ($D_1$), ($D_2$) and ($D_3$) of Def. 1.1 hold in $L^a$. The conditions ($D_1$) and ($D_3$) can be easily verified . It remains to check the condition ($D_2$). For that, suppose $x \in [0, a]$. Then $x \triangle_{L^a} 1_{L^a} = x \triangle a$. From Prop. 1.8 we obtain the equalities $x \triangle a = a \wedge x^\perp = x^{\perp_a}$. The equality $1_{L^a} \triangle_{L^a} x = x^{\perp_a}$ follows from the commutativity of $\triangle$.

Suppose further that $a \in C(L)$. Then the mapping $\pi_a$ is an OML-homomorphism $L_{\text{supp}} \to (L^a)_{\text{supp}}$ (see [12], p. 20). It remains to show that the mapping $\pi_a$ preserves the operation $\triangle$. Suppose that $x, y \in L$. Then by Prop. 1.10 we consecutively obtain
$\pi_a(x \triangle_L y) = (x \triangle_L y) \wedge a = (x \wedge a) \triangle_L (y \wedge a) = \pi_a(x) \triangle_{L^a} \pi_a(y)$. This completes the proof.

In the final auxiliary result, let us show that an ODL can be decomposed with the help of a central element in the way analogous to the situation known in OMLs.

**Proposition 1.12** *Suppose that $L$ is an ODL and $a \in C(L)$. Then the mapping $i : L \to [0, a] \times [0, a^\perp]$ defined by putting $i(x) = (\pi_a(x), \pi_{a^\perp}(x))$ is an isomorphism of $L$ onto $L^a \times L^{a^\perp}$.*

**Proof**. The mapping $i$ is an isomorphism between the OMLs $L_{\mathrm{supp}}$ and $(L^a)_{\mathrm{supp}} \times (L^{a^\perp})_{\mathrm{supp}}$ (see again [12], p. 20). Since both the mappings $\pi_a, \pi_{a^\perp}$ preserve the operation $\triangle$, so does the mapping $i$ and the proof is done.

In the conclusion of preliminaries, let us recall an important class of ODLs - the ODLs that are *set-representable*. They form a variety ([13]) and represent some 'nearly Boolean' ODLs. Though the name itself suggests their definition, let us recall it in more formal terms. Let $X$ be a set and let $\mathcal{D}$ a family of subsets of $X$ such that
(1) $X \in \mathcal{D}$,
(2) the family $\mathcal{D}$ forms a lattice with respect to the inclusion relation, and
(3) $\mathcal{D}$ is closed under the formation of the set symmetric difference.
Obviously, $\mathcal{D}$ constitutes an ODL. Let us call it *concrete*. If $L$ is an ODL that is isomorphic with a concrete one, then $L$ is said to be set-representable.

## 2 Each ODL with two generators is set-representable (a characterization of the free ODL with two generators).

Let us show in this section that the free ODL on 2 generators coincides with $2^4 \times \mathrm{MO}_3$ (where, as usual, $2^4$ stands for the Boolean algebra with 4 atoms). Since the ODL $2^4 \times \mathrm{MO}_3$ is set-representable, and since a homomorphic image of a set-representable ODL is again set-representable ([13]), we see that any ODL with two generators is set-representable.

In order to characterize the free ODL with 2 generators, we shall need two auxiliary results. For the sake of a transparent formulation of these results, let us assume that the generators of the Boolean algebra $2^4$ are elements $x_1, y_1$ and the generators of the ODL $\mathrm{MO}_3$ are elements $x_2, y_2$ (compare with Example 1.6 - we have renamed $x, y$ of Example 1.6 with $x_2, y_2$).

**Proposition 2.1** *Let $L$ be an ODL and let $a, b \in L$. Let us suppose that $a \wedge b = a \wedge b^\perp = a^\perp \wedge b = a^\perp \wedge b^\perp = 0$. Then there exists a homomorphism $h : \mathrm{MO}_3 \to L$ with $h(x_2) = a$, $h(y_2) = b$.*

**Proof**. Let us denote $z_2 = x_2 \triangle y_2$. Let us set $h(0_{\mathrm{MO}_3}) = 0_L$, $h(1_{\mathrm{MO}_3}) = 1_L$, $h(x_2) = a$, $h(x_2^\perp) = a^\perp$, $h(y_2) = b$, $h(y_2^\perp) = b^\perp$, $h(z_2) = a \triangle b$, $h(z_2^\perp) = a \triangle b^\perp$, where $z_2 = x_2 \triangle y_2$ in $\mathrm{MO}_3$.
The definition of $h$ implies that $h$ preserves the least and greatest element. Also, the operations $^\perp$ and $\triangle$ are obviously preserved. Let us check that $h$ preserves the operation $\wedge$, too. Suppose therefore that $x, y \in \mathrm{MO}_3$ and let us ask whether or not we have $h(x \wedge y) = h(x) \wedge h(y)$. If $x, y$ commute in $\mathrm{MO}_3$, this equality is obvious. Suppose that $x, y$ do not commute. Without any loss of generality, it is sufficient to consider the images of the elements $x_2 \wedge y_2$ and $x_2 \wedge z_2$. We firstly see that $h(x_2 \wedge y_2) = h(x_2 \wedge z_2) = h(0_{\mathrm{MO}_3}) = 0_L$, and further we have $h(x_2) \wedge h(y_2) = a \wedge b = 0_L$ as well as, by Prop. 1.7, $h(x_2) \wedge h(z_2) = a \wedge (a \triangle b) = a \wedge b^\perp = 0_L$. The preservation of the operation $\vee$ is a simple consequence of de Morgan's law. The proof is complete.

**Proposition 2.2** *Let $L$ be an ODL with two generators $s, t$. Let us set $a = (s \wedge t) \vee (s \wedge t^{\perp}) \vee (s^{\perp} \wedge t) \vee (s^{\perp} \wedge t^{\perp})$. Then $a \in C(L)$ and there exist homomorphisms $g : 2^4 \to L^a$, $h : \mathrm{MO}_3 \to L^{a^{\perp}}$ such that*

$g(x_1) = \pi_a(s), g(y_1) = \pi_a(t),$

$h(x_2) = \pi_{a^{\perp}}(s), h(y_2) = \pi_{a^{\perp}}(t).$

**Proof**. It is obvious that the element $s \wedge t$ commutes with both $s$ and $t$. Since $s, t$ generate the ODL $L$, we see that $s \wedge t \in C(L)$. Analogously, all the elements $s \wedge t^{\perp}$, $s^{\perp} \wedge t$ and $s^{\perp} \wedge t^{\perp}$ belong to $C(L)$. As a consequence, $(s \wedge t) \vee (s \wedge t^{\perp}) \vee (s^{\perp} \wedge t) \vee (s^{\perp} \wedge t^{\perp}) = a \in C(L)$.

Let us go on with the proof. Since the elements $s, t$ generate $L$ and since $\pi_a$ is a surjective homomorphism onto $L^a$, it follows that the elements $\pi_a(s)$, $\pi_a(t)$ generate the ODL $L^a$. Making use of the Foulis-Holland theorem ([12]) we infer that

$\pi_a(s) = s \wedge a = (s \wedge t) \vee (s \wedge t^{\perp}),$

$\pi_a(t) = t \wedge a = (s \wedge t) \vee (s^{\perp} \wedge t).$

As a consequence of the above identities we see that the elements $\pi_a(s), \pi_a(t)$ commute and therefore $L^a$ is a Boolean algebra. Since $2^4$ is a free Boolean algebra on the set $\{x_1, y_1\}$, the existence of the homorphism $g$ is evident.

Let us take up the construction of the morphism $h$. It is sufficient to check (Prop. 2.1) that

$\pi_{a^{\perp}}(s) \wedge \pi_{a^{\perp}}(t) = \pi_{a^{\perp}}(s) \wedge (\pi_{a^{\perp}}(t))^{\perp} = (\pi_{a^{\perp}}(s))^{\perp} \wedge \pi_{a^{\perp}}(t) = (\pi_{a^{\perp}}(s))^{\perp} \wedge (\pi_{a^{\perp}}(t))^{\perp} = 0.$

Let us prove that $\pi_{a^{\perp}}(s) \wedge (\pi_{a^{\perp}}(t))^{\perp} = 0$, the other equalities can be derived analogously. Since $\pi_{a^{\perp}} : L \to L^{a^{\perp}}$ preserves the operation ${}^{\perp}$, we see that $(\pi_{a^{\perp}}(t))^{\perp} = (\pi_{a^{\perp}}(t))^{\perp_{L^{a^{\perp}}}} = \pi_{a^{\perp}}(t^{\perp_L})$. As a consequence we obtain

$\pi_{a^{\perp}}(s) \wedge (\pi_{a^{\perp}}(t))^{\perp} = \pi_{a^{\perp}}(s) \wedge \pi_{a^{\perp}}(t^{\perp_L}) = \pi_{a^{\perp}}(s \wedge t^{\perp_L}) = (s \wedge t^{\perp}) \wedge a^{\perp} = (s \wedge t^{\perp}) \wedge (s^{\perp} \vee t^{\perp}) \wedge (s^{\perp} \vee t) \wedge (s \vee t^{\perp}) \wedge (s \vee t) = (s \wedge t^{\perp}) \wedge (s^{\perp} \vee t) = (s \wedge t^{\perp}) \wedge (s \wedge t^{\perp})^{\perp} = 0.$

**Theorem 2.3** *Suppose that the elements $x_1, y_1$ are generators of the free Boolean algebra $2^4$ and suppose that the elements $x_2, y_2$ are generators of the ODL $\mathrm{MO}_3$. Then the product $2^4 \times \mathrm{MO}_3$ is a free ODL on the set $\{x, y\}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.*

**Proof**. Write $F = 2^4 \times \mathrm{MO}_3$. Let us first show that the set $\{x, y\}$ generates $F$. Let us denote by $S$ the subalgebra of $F$ generated by $\{x, y\}$. Suppose that $a \in F$. Then $a = (a_1, a_2) = (a_1, 0) \vee (0, a_2)$, where $a_1 \in 2^4$ and $a_2 \in \mathrm{MO}_3$. We therefore have to show that all elements of the form $(a_1, 0), (0, a_2)$ lie in $S$. Since $x, x^{\perp}, y, y^{\perp}$ are elements of $S$, so are the elements $x \wedge y$, $x \wedge y^{\perp}$, $x^{\perp} \wedge y$, $x^{\perp} \wedge y^{\perp}$. Taking into account that $x_2 \wedge y_2 = x_2 \wedge y_2^{\perp} = x_2^{\perp} \wedge y_2 = x_2^{\perp} \wedge y_2^{\perp} = 0$, we infer that all elements $(x_1 \wedge y_1, 0), (x_1 \wedge y_1^{\perp}, 0), (x_1^{\perp} \wedge y_1, 0), (x_1^{\perp} \wedge y_1^{\perp}, 0)$ belong to $S$. But $x_1 \wedge y_1$, $x_1 \wedge y_1^{\perp}$, $x_1^{\perp} \wedge y_1$, $x_1^{\perp} \wedge y_1^{\perp}$ are precisely all atoms of the Boolean algebra $2^4$. This implies that $(a_1, 0) \in S$. As a consequence, $(a_1, 1) \in S$. Further, observing $(0, x_2) = (x_1^{\perp}, 1) \wedge (x_1, x_2)$, we see that $(0, x_2) \in S$. Analogously, $(0, y_2) \in S$ and, also, $(0, z_2) = (0, x_2) \triangle (0, y_2) \in S$. We have shown that $S = F$.

In order to show that $F$ is free, let $K$ be an ODL and let $f_0 : \{x, y\} \to K$ be a mapping. We have to show that $f_0$ can be extended as a homomorphism $f : F \to K$. Write $s = f_0(x)$, $t = f_0(y)$ and suppose that $L$ is the subalgebra of $K$ generated by the set $\{s, t\}$. Set $a = (s \wedge t) \vee (s \wedge t^{\perp}) \vee (s^{\perp} \wedge t) \vee (s^{\perp} \wedge t^{\perp})$. By Prop. 2.2 we have that $a \in C(L)$ and, moreover, there exist homomorphisms $g : 2^4 \to L^a$, $h : \mathrm{MO}_3 \to L^{a^{\perp}}$ such that

$g(x_1) = \pi_a(s), g(y_1) = \pi_a(t),$

$h(x_2) = \pi_{a^\perp}(s), h(y_2) = \pi_{a^\perp}(t)$.
Let $i : L \to L^a \times L^{a^\perp}$ be the isomorphism of Prop. 1.12. Let us consider the mapping $g \times h$ defined by setting $(g \times h)(p, q) = (g(p), h(q))$, where $(p, q) \in 2^4 \times \mathrm{MO}_3$. Obviously, $g \times h : 2^4 \times \mathrm{MO}_3 \to L^a \times L^{a^\perp}$ is a homomorphism. Let us set $f = (g \times h) \circ i^{-1}$, i.e. for any $(p, q) \in 2^4 \times \mathrm{MO}_3$ let us set $f(p, q) = i^{-1}(g(p), h(q))$. Then $f : 2^4 \times \mathrm{MO}_3 \to L$ is a homomorphism and since $L$ is a subalgebra of $K$, we see that $f : 2^4 \times \mathrm{MO}_3 \to K$ is a homomorphism, too. Moreover, $f(x) = i^{-1}(g(x_1), h(x_2)) = i^{-1}(\pi_a(s), \pi_{a^\perp}(s)) = i^{-1}(i(s)) = s = f_0(x)$. Analogously, $f(y) = f_0(y)$. We have verified that $f$ extends $f_0$ and the proof is complete.

It should be noted in the conclusion of this paragraph that the result of Thm. 2.3 has already been obtained in [11] (a student thesis under the supervision of the authors of this paper). However, the methods used here differ considerably from those of [11] and allow us to prove the result in a simpler way.

# 3 There is an ODL with three generators that is not set-representable (so the free ODL with three generators is not set-representable).

In this section we develop an embedding technique of OMLs into ODLs. This will allow us to prove the assertion stated in the heading of this paragraph. Let us start with a few conventions.
Let $N$ stand for the set of all natural numbers, $N = \{0, 1, 2, \ldots\}$. Let $\mathcal{B}$ be the Boolean algebra of all finite and cofinite subsets of $N$. Let us denote by $\Delta$ the standard set-theoretic difference on $\mathcal{B}$. In considering countable ODLs we can visualise, with the help of $\mathcal{B}$, the operation $\triangle$ set-theoretically . The following proposition formalizes it.

**Proposition 3.1** *If $L = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$ is at most countable ODL, then the algebra $(X, \triangle, 0, 1)$ can be embedded into the algebra $(\mathcal{B}, \Delta, 0_\mathcal{B}, 1_\mathcal{B})$, where $0_\mathcal{B} = \emptyset$, $1_\mathcal{B} = N$.*

**Proof**. Let us choose a Boolean algebra $B'$ such that $B'$ is a sub-algebra of $\mathcal{B}$ and $\mathrm{card}(B') = \mathrm{card}(X)$. Obviously, the algebra $(B', \Delta, 0_\mathcal{B}, 1_\mathcal{B})$ is a sub-algebra of the algebra $(\mathcal{B}, \Delta, 0_\mathcal{B}, 1_\mathcal{B})$. Moreover, the study of ultrafilters in $(X, \triangle, 0, 1)$ made in [13], Prop. 7.7 implies that the algebras $(B', \Delta, 0_\mathcal{B}, 1_\mathcal{B})$ and $(X, \triangle, 0, 1)$ are isomorphic.

The above result will be frequently used in the sequel. Before, let us agree on a convention. In order to avoid rather inconvenient referring to finite and cofinite subsets of $N$, let us make use of the standard coding of finite subsets of $N$ by natural numbers. If $A$ is finite, $A \subset N$, let us assign to $A$ the number $k(A)$ as follows: $k(\emptyset) = 0$, $k(\{a_1, \ldots, a_n\}) = 2^{a_1} + \ldots + 2^{a_n}$. Thus, so defined $k$ is nothing but the famous *Gödel's coding.* As known, the assignment $k$ is injective. Take copies of $0, 1, 2, \ldots$, some $\bar{0}, \bar{1}, \bar{2}, \ldots$. For any cofinite $B$, $B = N \setminus A$ with $A$ finite, let us set $k(B) = \overline{k(A)}$. Denoting by $D_\infty$ the set $\{0, \bar{0}, 1, \bar{1}, 2, \bar{2}, \ldots\}$, we see that $k$ is a bijection of $\mathcal{B}$ onto $D_\infty$. Further, let us introduce an operation, $\oplus$, on the set $D_\infty$ by setting $x \oplus y = k(k^{-1}(x) \ \Delta \ k^{-1}(y))$. The following two assertions bring the properties of the operation $\oplus$. The proofs are not difficult and we omit them.

**Lemma 3.2** *The mapping $k$ is an isomorphism of the algebra $(\mathcal{B}, \Delta, 0_{\mathcal{B}}, 1_{\mathcal{B}})$ onto the algebra $(D_\infty, \oplus, 0, \bar{0})$.*

**Lemma 3.3** *If $n \in N$, then the set $\{0, \bar{0}, 1, \bar{1}, \ldots, 2^n - 1, \overline{2^n - 1}\}$ is a subalgebra of the algebra $(D_\infty, \oplus, 0, \bar{0})$.*

We shall utilize the following mapping $^\star : D_\infty \to D_\infty$. If $n \in N$, we set $n^\star = \bar{n}$ and $(\bar{n})^\star = n$.

**Lemma 3.4** *If $x \in D_\infty$, then $x^\star = x \oplus \bar{0} = \bar{0} \oplus x$.*

**Proof**. We have $x \oplus \bar{0} = k(k^{-1}(x) \ \Delta \ k^{-1}(\bar{0})) = k(k^{-1}(x) \ \Delta \ N) = k(N \setminus k^{-1}(x))$. If $x = n$, then $k(N \setminus k^{-1}(x)) = \overline{k(k^{-1}(x))} = \bar{x} = x^\star$. Alternatively, suppose that $x = \bar{n}$. Take a set $A$ such that $k(A) = n$. We then see that $x = \bar{n} = \overline{k(A)} = k(N \setminus A)$, i.e. $k^{-1}(x) = N \setminus A$. Consequently, $k(N \setminus k^{-1}(x)) = k(N \setminus (N \setminus A)) = k(A) = n = x^\star$.

Since $\oplus$ is commutative, we infer that $\bar{0} \oplus x = x^\star$ and this completes the proof.

The objective of the following consideration is to show that mappings into $D_\infty$ allow us to embed certain OMLs into ODLs. Let us first introduce a few new notions.

**Definition 3.5** *Let $K$ be an OML and let $L$ be an ODL. Let us agree to write $K \ll L$ if $K$ is a sub-OML of $L_{\mathrm{supp}}$ and $a \triangle b \in K$ for any $a, b \in K$ with $a \vee b < 1$.*

**Lemma 3.6** (1) *Let $K$ be an OML and let $L$ be an ODL. Let us suppose that there is an OML $M$ such that $L_{\mathrm{supp}}$ is a horizontal sum of OMLs $K$ and $M$. Then $K \ll L$.*
(2) *Let $L$ be an ODL and let $K$ be a sub-ODL of $L$. Then $K_{\mathrm{supp}} \ll L$.*

**Proof**. It is routinne and we will omit it.

Let $K$ be an OML and let $\mathrm{Bl}(K)$ be the set of all blocks (= the set of all maximal Boolean subalgebras) of $K$. Let $\mathrm{At}(K)$ stand for the set of all atoms of $K$. Let us denote by $\mathcal{OML}_8$ the class of all OMLs $K$ such that $\mathrm{card}(B) = 8$ for any $B \in \mathrm{Bl}(K)$. So, for instance, each horizontal sum of 8-element Boolean algebras belongs to $\mathcal{OML}_8$ and so does the projection lattice $L(R^3)$. More involved examples will be encountered in the sequel.

Let $K \in \mathcal{OML}_8$ and let $p, q \in \mathrm{At}(K)$. Let us write $p \approx q$ provided $p \neq q$ and $p \ C \ q$. Further, let us write $p \sim q$ if there exists an $r \in \mathrm{At}(K)$ such that $p \ C \ r$ and $r \ C \ q$ and, moreover, $p$ does not commute with $q$.

**Lemma 3.7** *Let $K \in \mathcal{OML}_8$ and let $p, q \in \mathrm{At}(K)$ with $p \sim q$. Let $r \in \mathrm{At}(K)$ such that $r \ C \ p$ and $r \ C \ q$. Then $r = p^\perp \wedge q^\perp$.*

**Proof**. Since $p$ does not commute with $q$, we have $p \neq q$. Further, the element $r \in \mathrm{At}(K)$ with $r \ C \ p$ and $r \ C \ q$ must be different from both $p$ and $q$. Since $p \ C \ r$ and $p \neq r$ we see that $r \leq p^\perp$. For an analogous reason, $r \leq q^\perp$. As a result, $r \leq p^\perp \wedge q^\perp < p^\perp$. We infer that $r = p^\perp \wedge q^\perp$ and this completes the proof.

**Definition 3.8** *Let $K \in \mathcal{OML}_8$ be finite and let $l : \mathrm{At}(K) \to D_\infty$ be a mapping. We say that $l$ is a* labelling *of the atoms in $K$ if*
(1) *for any pair $a, b \in \mathrm{At}(K)$ with $a \neq b$ we have $l(a) \neq l(b)$, $l(a) \neq l(b)^\star$,*
(2) *for any $B \in \mathrm{Bl}(K)$ such that $\mathrm{At}(B) = \{a, b, c\}$ we have $l(a) \oplus l(b) \oplus l(c) = \bar{0}$,*
(3) *for any pair $a, b \in \mathrm{At}(K)$ with $a \sim b$ there is an $s$, $s \in \mathrm{At}(K)$ such that $s \approx a^\perp \wedge b^\perp$ and $l(s) = l(a) \oplus l(b)$.*

Before justifying this definition in the next theorem, let us explicitely formulate the following simple fact.

**Lemma 3.9** *Suppose that $l$ is a labelling of the atoms of $K$ and suppose that $a \in \mathrm{At}(K)$. Then $l(a) \notin \{0, \bar{0}\}$.*

**Proof**. Let $B$ be a block in $K$ with $a \in B$. Let $\mathrm{At}(B) = \{a, b, c\}$. Then $l(a) \oplus l(b) \oplus l(c) = \bar{0}$. If $l(a) = 0$, then $l(b) \oplus l(c) = \bar{0}$. This means that $l(b) = l(c)^\star$ which is absurd. If $l(a) = \bar{0}$, then $l(b) \oplus l(c) = 0$. This means that $l(b) = l(c)$ which is again absurd.

**Theorem 3.10** *Let $K \in \mathcal{OML}_8$ be finite. Then the following two statements are equivalent:*
*(1) There is a finite ODL, $L$, such that $K \ll L$,*
*(2) there is a labelling of the atoms of $K$.*

**Proof**. Suppose first that there is a finite ODL, $L$, such that $K \ll L$. Then there is an embedding, $f$, of the algebra $(L, \triangle, 0, 1)$ into the algebra $(D_\infty, \oplus, 0, \bar{0})$ (see Prop. 3.1 and Lemma 3.2). Let $l$ be the restriction of $f$ to the set $\mathrm{At}(K)$. In order to show that $l$ is a labelling, we are to verify three conditions.
(1) Suppose that $a, b \in \mathrm{At}(K)$ with $a \neq b$. Then $a \neq b^\perp$ and the rest follows from the injectivity of $f$.
(2) Let $B \in \mathrm{Bl}(K)$. Write $\mathrm{At}(B) = \{a, b, c\}$. Then $a\ \Delta_B\ b\ \Delta_B\ c = 1$. Since $B$ is a Boolean sub-algebra of $L$ we have $a \triangle b \triangle c = a\ \Delta_B\ b\ \Delta_B\ c$ (see Prop. 1.4). It means that $a \triangle b \triangle c = 1$, and therefore $f(a) \oplus f(b) \oplus f(c) = \bar{0}$.
(3) Suppose that $a, b \in \mathrm{At}(K)$ with $a \sim b$. Obviously, $a \vee b < 1$. Set $s = a \triangle b \in L$. Then $s \in K$ (compare the Def. 3.5). If $s = a \vee b$, then $a \perp b$ in view of $a \vee b = a \triangle b$. This is a contradiction. If $s = 0$, then $a = b$ - a contradiction again. Summarizing the previous considerations, we conclude that $0 < s < a \vee b$. And this implies that $s \in \mathrm{At}(K)$. Since $s \leq a \vee b$, we have $s\ C\ (a \vee b)$ and therefore $s\ C\ (a^\perp \wedge b^\perp)$. If $s = a^\perp \wedge b^\perp$, then $(a \vee b)^\perp \leq a \vee b$ which cannot be the case since this would imply $a \vee b = 1$. We conclude that $s \approx a^\perp \wedge b^\perp$ and therefore $l(s) = f(s) = f(a \triangle b) = f(a) \oplus f(b) = l(a) \oplus l(b)$. So the implication (1) $\Rightarrow$ (2) has been verified.

Conversaly, assume that there is a labelling $l : \mathrm{At}(K) \to D_\infty$. We can suppose that $K \cap D_\infty = \emptyset$. Choose an $n$, $n \in N$, such that $l[\mathrm{At}(K)] \subseteq \{1, \bar{1}, 2, \bar{2}, \ldots, 2^n - 1, \overline{2^n - 1}\}$. Rewrite the set $\{1, \bar{1}, 2, \bar{2}, \ldots, 2^n - 1, \overline{2^n - 1}\} \setminus \{l(a), l(a)^\star;\ a \in \mathrm{At}(K)\}$ as $\{i_1, \overline{i_1}, i_2, \overline{i_2}, \ldots, i_m, \overline{i_m}\}$. Let $M$ be the copy of $\mathrm{MO}_m$, where $M = \{0, 1, i_1, \overline{i_1}, i_2, \overline{i_2}, \ldots, i_m, \overline{i_m}\}$ under the understanding of $\overline{i_k} = i_k^\perp$, $1 \leq k \leq m$. Consider the horizontal sum $K$ with $M$ and denote it by $L'$. It remains to show that there is an ODL $L$ such that $L_{\mathrm{supp}} = L'$.

Let $e : L' \to \{0, \bar{0}, 1, \bar{1}, \ldots, 2^n - 1, \overline{2^n - 1}\}$ be the mapping that is defined as follows:
$e(0_{L'}) = 0$, $e(1_{L'}) = \bar{0}$,
$e(a) = l(a)$, $e(a^\perp) = l(a)^\star$ for $a \in \mathrm{At}(K)$,
and $e$ acts as identity on $\{i_1, \overline{i_1}, i_2, \overline{i_2}, \ldots, i_m, \overline{i_m}\}$.
Obviously, $e$ is a bijection of $L'$ onto $\{0, \bar{0}, 1, \bar{1}, \ldots, 2^n - 1, \overline{2^n - 1}\}$. Let us set, for $x, y \in L'$, $x \triangle y = e^{-1}(e(x) \oplus e(y))$ and verify that $L = (L', \triangle)$ is an ODL.

The associativity as well as commutativity follows immediately from the algebra isomorphism of $(L', \triangle)$ on $(\{0, \bar{0}, 1, \bar{1}, \ldots, 2^n - 1, \overline{2^n - 1}\}, \oplus)$. Further, $x \triangle 1_{L'} = e^{-1}(e(x) \oplus e(1_{L'})) = e^{-1}(e(x) \oplus \bar{0}) = e^{-1}(e(x)^\star)$. Consider now the possibilities for $x$ in order. If $x \in K$, then $e^{-1}(e(x)^\star) = e^{-1}(l(x)^\star) = e^{-1}(l(x^\perp)) = e^{-1}(e(x^\perp)) = x^\perp$. If $x = i_k$, then $x \triangle 1_{L'} = e^{-1}(e(i_k)^\star) = e^{-1}(i_k^\star) = e^{-1}(\overline{i_k})) = \overline{i_k} = i_k^\perp$. Finally, if $x = \overline{i_k}$, then $x \triangle 1_{L'} = e^{-1}(e(\overline{i_k})^\star) = e^{-1}(\overline{i_k}^\star) = e^{-1}(i_k) = i_k = \overline{i_k}^\perp$. Thus, $x \triangle 1_{L'} = x^\perp$ for any $x \in L'$.

It remains to check the last axiom of ODL's, $x \triangle y \leq x \vee y$. Let $x, y \in L'$. If $x \vee y = 1$, there is nothing to check. If $x = y$, then $x \triangle y = 0_{L'}$ and the inequality in question is clear. Let us finally suppose that $x \vee y < 1$ with $x \neq y$. Then $x, y \in K$. Let us discuss the possibilities for $x, y \in K$ which may occur.

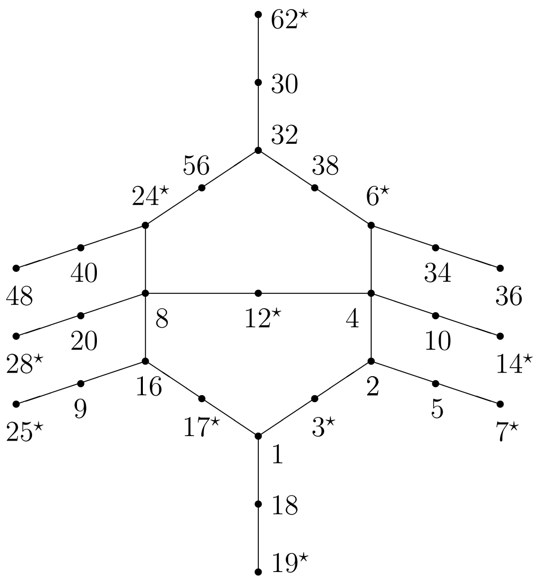
First, suppose that $x \ C \ y$. Let us choose a block $B$, $B \in \mathrm{Bl}(K)$ such that $x, y \in B$. Then either both $x, y$ are atoms or not. In the former case, when $\mathrm{At}(B) = \{x, y, z\}$, we have $l(x) \oplus l(y) \oplus l(z) = \bar{0}$. It means that $x \triangle y = \varphi^{-1}(\varphi(x) \oplus \varphi(y)) = \varphi^{-1}(l(x) \oplus l(y)) = \varphi^{-1}(l(z)^\star) = \varphi^{-1}(\varphi(z^\perp) = z^\perp = x \vee y$. In the latter case, when at least one of $x$ and $y$ is not an atom, we have $x \neq y$ and $x \vee y < 1$. Thus, exactly one of $x$ and $y$ is a coatom. Suppose, for instance, that $x$ is an atom and $y$ a coatom. Then $x \leq y$. Suppose that $\mathrm{At}(B) = \{x, y^\perp, z\}$. Then $l(x) \oplus l(y^\perp) \oplus l(z) = \bar{0}$ and we obtain $x \triangle y = e^{-1}(e(x) \oplus e(y)) = e^{-1}(e(x) \oplus e(y^{\perp\perp})) = e^{-1}(l(x) \oplus l(y^\perp)^\star)) = e^{-1}(l(z)) = e^{-1}(e(z)) = z \leq y = x \vee y$. Again, $x \triangle y \leq x \vee y$.

Secondly, $x \neg \ C \ y$. Then neither of $x$ and $y$ coincides with 0 or 1. We are going to show that both $x$ and $y$ are atoms. Looking for a contradiction, suppose that $x$ is a coatom. Then $x \leq x \vee y < 1$ and therefore $x = x \vee y$ and this means that $y \leq x$ - a contradiction with $x \neg \ C \ y$. We see that both $x$ and $y$ are atoms. So $x < x \vee y < 1$ and therefore $x \vee y$ is a coatom. If we set $z = (x \vee y)^\perp$, we obtain that $x \sim y$. According to the condition (3) in the definition of labelling, an element $s \in \mathrm{At}(K)$ is guarranteed such that $s \approx z$ and $l(s) = l(x) \oplus l(y)$. Consequently, one derives the equalities $x \triangle y = e^{-1}(e(x) \oplus e(y)) = e^{-1}(l(x) \oplus l(y)) = e^{-1}(l(s)) = e^{-1}(e(s)) = s \leq z^\perp = x \vee y$. This completes the proof.

The previous result will be applied in our final construction to provide a proof of a main result of this paper.

**Theorem 3.11** *There is an ODL $L$ with 3 generators that is not set-representable. A consequence: The free ODL on 3 generators is not set-representable.*

**Proof**. Consider the ODL $K$ portrayed by the following figure. Let us make use in the figure the conventions of the Greechie paste job ([9, 12]) and the labelling notation agreed on in Thm. 3.10.



As shown in [8], in each set-representable OML (and, in turn, in each set-representable ODL) the following inequality holds true: $x \wedge (y \vee z) \leq \varphi_x(y) \vee \varphi_{y^\perp}(z)$, where $\varphi_a(b) = (b \vee a^\perp) \wedge a$ is the well-known Sasaki projection ([1, 12]).

Let us see that this inequality fails in the OML $K$ depicted by the figure. Indeed, let us take $x = 16^\star$, $y = 4$ and $z = 6^\star$. Then $x \wedge (y \vee z) = 16^\star \wedge (4 \vee 6^\star) = 16^\star \wedge 2^\star = 1$ whereas $\varphi_x(y) = (4 \vee 16) \wedge 16^\star = 8^\star \wedge 16^\star = 24^\star$ and $\varphi_{y^\perp}(z) = (6^\star \vee 4) \wedge 4^\star = 2^\star \wedge 4^\star = 6^\star$ which gives us $\varphi_x(y) \vee \varphi_{y^\perp}(z) = 24^\star \vee 6^\star = 32^\star < 1$. By Thm. 3.10, there is an ODL, $L$, such that $K \ll L$. Let $L_1$ be the sub-ODL of $L$ generated by $x, y$ and $z$. Then the inequality $x \wedge (y \vee z) \leq \varphi_x(y) \vee \varphi_{y^\perp}(z)$ does not hold true in $L_1$ and therefore $L_1$ is not set-representable. Obviously, $L_1$ has 3 generators and we have completed the proof.

In the series of papers [13] - [16] together with this note we have iniciated a systematic study of axiomatic symmetric difference. The algebras which came into existence, the ODLs, lie between orthomodular lattices and Boolean algebras and might therefore find application in quantum logic theory or elsewhere in algebra. In the former area of application it would be desirable to investigate 'states' on ODLs. In the latter area, a natural step in the effort to understand the intristic structure of ODLs is the investigation of free objects in the variety $\mathcal{ODL}$ (the complexity of this problem indicates the analogous study in OMLs, see [2]). A problem linked with the last question is whether this variety is locally finite. Though we conjecture it is not, the problem is still open to us.

# References


[1] Beran L., *Orthomodular Lattices, Algebraic Approach*, D. Reidel, Dordrecht, 1985.

[2] Bruns G., Harding J., *Algebraic aspects of orthomodular lattices*, in Coecke B., Moore D. and Wilce A. eds. Current Research in Operational Quantum Logic, 2000, 37-65.

[3] Burris S., Sankappanavar H.P., *A Course in Universal Algebra*, Springer-Verlag, New York Inc., 1981.

[4] Dalla Chiara, M.L., Giuntini, R., Greechie, R., *Reasoning in Quantum Theory: Sharp and Unsharp Quantum Logics*, Kluwer Academic Publishers, Dordrecht, Boston, London, (2004).

[5] Dorfer G., Dvurečenskij A., Länger H.M., *Symmetric difference in orthomodular lattices*, Math. Slovaca **46** (1996), 435-444.

[6] Dorfer G., *Non-commutative symmetric differences in orthomodular lattices*, International Journal of Theoretical Physics **44** (2005), 885-896.

[7] Dvurečenskij A., Pulmannová S., *New Trends in Quantum Structures*, Kluwer Acad. Publ., Dordrecht, and Ister Science, Bratislava, 2000.

[8] Godowski R., Greechie R.J., *Some equations related to states on orthomodular lattices*, Demonstratio Mathematica, Vol. XVII (1984), No 1, 241-250.

[9] Greechie R.J., *Orthomodular lattices admitting no states*, J. Combinat. Theory, **10** (1971), 119-132.

[10] *Handbook of Quantum Logic and Quantum Structures*, ed. by K.Engesser, D.M.Gabbay and D.Lehmann, Elsevier 2007.

[11] Havlík F., *Ortokomplementární diferenční svazy* (Czech, Mgr Thesis, Department of Logic, Faculty of Arts, Charles University in Prague, 2007).

[12] Kalmbach G., *Orthomodular Lattices*, Academic Press, London, 1983.

[13] Matoušek M., *Orthocomplemented lattices with a symmetric difference*, Algebra Universalis **60** (2009), 185-215.

[14] Matoušek M., Pták P., *Orthocomplemented posets with a symmetric difference*, Order **26** (2009), 1-21.

[15] Matoušek M., Pták P., *On identities in orthocomplemented difference lattices*, Mathematica Slovaca, to appear.

[16] Matoušek M., Pták P., *Symmetric difference on orthomodular lattices and $Z_2$-valued states*, Comment. Math. Univ. Carolin. **50, 4** (2009), 535-547.

[17] Navara M., Pták P., *Almost Boolean orthomodular posets*, J. Pure Applied Algebra **60** (1989), 105-111

[18] Park E., Kim M.M., Chung J.Y., *A note on symmetric differences of orthomodular lattices*, Commun. Korean Math. Soc. **18** (2003), No 2, 207-214.

[19] Pták P., Pulmannová S., *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1991.

[20] Watanabe S., *Modified concepts of logic, probability and integration based on generalized continuous characteristic function*, Information and Control 2 (1969), 121.

Milan Matoušek
Ruská 22
101 00 Prague 10
Czech Republic
*e-mail*: matmilan@email.cz

Pavel Pták
Department of Mathematics
Czech Technical University, Faculty of Electrical Eng.
166 27 Prague 6
Czech Republic
*e-mail*: ptak@math.feld.cvut.cz

To Sylvia Pulmannová with compliments and admiration

# On identities in orthocomplemented difference lattices

**Milan Matoušek** and **Pavel Pták** [1]

Abstract. In this note we continue the investigation of algebraic properties of orthocomplemented (symmetric) difference lattices (ODLs) as initiated in [10] and further studied in [11, 12]. We take up a few identities that we came across in the previous considerations. We first see that some of them characterize, in a somewhat non-trivial manner, the ODLs that are Boolean. In the second part we select an identity peculiar for set-representable ODLs. This identity allows us to present another construction of an ODL that is not set-representable. We then give the construction a more general form and consider algebraic properties of the 'orthomodular support'.

*AMS Class:* 06C15, 03G12, 81B10.
*Key words:* orthomodular poset, quantum logic, symmetric difference, Boolean algebra.

## 1 Basic notions and preliminaries

Let us first recall the definition of ODL.

**Definition 1.1** *Let $L = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$, where $(X, \wedge, \vee, ^\perp, 0, 1)$ is an orthocomplemented lattice and $\triangle : X^2 \to X$ is a binary operation. Then $L$ is said to be an* orthocomplemented difference lattice *(abbr., an ODL) if the following identities hold in $L$:*
$(\mathrm{D}_1)$ $x \triangle (y \triangle z) = (x \triangle y) \triangle z$,
$(\mathrm{D}_2)$ $x \triangle 1 = x^\perp$, $1 \triangle x = x^\perp$,
$(\mathrm{D}_3)$ $x \triangle y \le x \vee y$.

Obviously, the class of all ODLs forms a variety. We will denote it by $\mathcal{ODL}$. (It should be noted that a certain version of symmetric difference has been dealt with in the area of orthomodular lattices - see [4, 5, 13]. Our approach essentially differs from the above quoted papers since we take the operation $\triangle$ as primitive.)

Let $L = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$ be an ODL. Then the orthocomplemented lattice $(X, \wedge, \vee, ^\perp, 0, 1)$ will be denoted by $L_{\mathrm{supp}}$ and called the *support* of $L$. Occasionally, we allow ourselves to harmlessly abuse the notation by identifying an ODL $L$ with the couple $(L_{\mathrm{supp}}, \triangle)$.

[1]The authors acknowledge the support of the research plan MSM 0021620839 that is financed by the Ministry of Education of the Czech Republic and the grant GAČR 201/07/1051 of the Czech Grant Agency.

Let us list basic properties of ODLs as we shall use them in the sequel. Let us note that in this list (and in other results of preliminary nature like Thm. 1.3 and Thm. 2.5) this paper overlaps with [10]. Main novelties lie in Thm. 2.2 and in the proof technique of Thm. 2.8.

**Proposition 1.2** *Let $L$ be an ODL. Then the following statements hold true ($x, y \in L$):*
(1) $x \triangle 0 = x$, $0 \triangle x = x$,
(2) $x \triangle x = 0$,
(3) $x \triangle y = y \triangle x$,
(4) $x \triangle y^\perp = x^\perp \triangle y = (x \triangle y)^\perp$ ,
(5) $x^\perp \triangle y^\perp = x \triangle y$ ,
(6) $x \triangle y = 0 \Leftrightarrow x = y$,
(7) $(x \wedge y^\perp) \vee (y \wedge x^\perp) \leq x \triangle y \leq (x \vee y) \wedge (x \wedge y)^\perp$.

**Proof**. Let us first observe that the property ($D_2$) yields $1 \triangle 1 = 1^\perp = 0$. Let us verify the properties (1)-(7). Suppose that $x \in L$.
(1) $x \triangle 0 = x \triangle (1 \triangle 1) = (x \triangle 1) \triangle 1 = x^\perp \triangle 1 = (x^\perp)^\perp = x$. Further, $0 \triangle x = (1 \triangle 1) \triangle x = 1 \triangle (1 \triangle x) = 1 \triangle x^\perp = (x^\perp)^\perp = x$ .
(2) Let us first show that $x^\perp \triangle x^\perp = x \triangle x$. We consecutively obtain $x^\perp \triangle x^\perp = (x \triangle 1) \triangle (1 \triangle x) = (x \triangle (1 \triangle 1)) \triangle x = (x \triangle 0) \triangle x = x \triangle x$. Moreover, we have $x \triangle x \leq x$ as well as $x \triangle x = x^\perp \triangle x^\perp \leq x^\perp$. This implies that $x \triangle x \leq x \wedge x^\perp = 0$.
(3) If $y \in L$, then $x \triangle y = (x \triangle y) \triangle 0 = (x \triangle y) \triangle [(y \triangle x) \triangle (y \triangle x)] = x \triangle (y \triangle y) \triangle x \triangle (y \triangle x) = x \triangle 0 \triangle x \triangle (y \triangle x) = x \triangle x \triangle (y \triangle x) = 0 \triangle (y \triangle x) = y \triangle x$.
(4) $x \triangle y^\perp = x \triangle (y \triangle 1) = (x \triangle y) \triangle 1 = (x \triangle y)^\perp$. The equality $x^\perp \triangle y = (x \triangle y)^\perp$ follows from $x \triangle y^\perp = (x \triangle y)^\perp$ by applying the equality (3).
(5) Using (4) we obtain $x^\perp \triangle y^\perp = (x^\perp \triangle y)^\perp = (x \triangle y)^{\perp\perp} = x \triangle y$.
(6) If $x = y$, then $x \triangle y = 0$ by the condition (2). Conversely, suppose that $x \triangle y = 0$. Then $x = x \triangle 0 = x \triangle (y \triangle y) = (x \triangle y) \triangle y = 0 \triangle y = 0$.
(7) The property ($D_3$) together with the properties (4), (5) imply that $x \triangle y \leq x \vee y$, $x \triangle y \leq x^\perp \vee y^\perp = (x \wedge y)^\perp$, $x \wedge y^\perp \leq x \triangle y$, $x^\perp \wedge y \leq x \triangle y$.

The following observation links ODLs with orthomodular lattices (OMLs) and, in turn, with quantum logics (for a link of quantum logics with theoretical physics, see [3, 6, 8]).

**Theorem 1.3** *Let $L$ be an ODL. Then its support $L_{\text{supp}}$ is an OML.*

**Proof**. Assume that $x, y \in L$ and $x \leq y$, $y \wedge x^\perp = 0$. Let us prove that $x = y$. Since $x \leq y$, we conclude that $(x \wedge y^\perp) \vee (y \wedge x^\perp) = y \wedge x^\perp = 0$ and $(x \vee y) \wedge (x \wedge y)^\perp = y \wedge x^\perp = 0$. By Prop. 1.2, (6), (7), we infer that $x \triangle y = 0$ and therefore $x = y$.

In view of the above proposition, all notions of OMLs can be referred to in ODLs, too. In particular, we may say that two elements $x, y$ in an ODL commute (in symbols, $x\ C\ y$) if they commute in $L_{\text{supp}}$ (for the notion of commutativity in OMLs, see [1, 9, 14]).

The following proposition shows that for the commutative pairs the operation $\triangle$ in $L$ can be recovered from $L_{\text{supp}}$.

**Proposition 1.4** *Let $L$ be an ODL. Let $x, y \in L$ with $x\ C\ y$. Then $x \triangle y = (x \wedge y^\perp) \vee (y \wedge x^\perp) = (x \vee y) \wedge (x \wedge y)^\perp$.*

**Proof**. According to Prop. 1.2, (7), we have the inequalities $(x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) \leq x \triangle y \leq (x \vee y) \wedge (x \wedge y)^{\perp}$. Since the elements $x, y$ commute, the left-hand side of the previous inequality coincides with the right-hand side and therefore $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = (x \vee y) \wedge (x \wedge y)^{\perp}$.

Let us note that each Boolean algebra can be viewed as an ODL (more general ODLs will be met later, see also [10, 12]).

**Proposition 1.5** *Let $B$ be a BA. Then there exists exactly one mapping $\triangle : \dot{B} \times \dot{B} \to \dot{B}$ which fulfils all the conditions* $(\mathrm{D}_1)$, $(\mathrm{D}_2)$ *and* $(\mathrm{D}_3)$ *of Def. 1.1.*

**Proof**. To prove the existence, take for $\triangle$ the standard symmetric difference in Boolean algebras. In other words, let us set $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp})$. The properties $(\mathrm{D}_1)$, $(\mathrm{D}_2)$ and $(\mathrm{D}_3)$ of Def. 1.1 are then obviously fulfilled.

Let us prove the uniqueness of $\triangle$. Let $\triangle_1 : \dot{B} \times \dot{B} \to \dot{B}$ be a mapping that fulfils conditions $(\mathrm{D}_1)$, $(\mathrm{D}_2)$ and $(\mathrm{D}_3)$. Thus, the couple $(B, \triangle_1)$ is an ODL. If $x, y \in B$, then $x\ C\ y$, and therefore $x \triangle_1 y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = x \triangle y$ (Prop. 1.4).

# 2 Results

In view of Prop. 1.5 we can (and shall) understand any Boolean algebra as an ODL with the uniquely defined operation $\triangle$. A natural question arises how to characterize Boolean algebras (= Boolean ODLs) among ODLs in terms of the operation $\triangle$. The departure point is the following result (observe that what we claim is that a strenghtening of the condition $(\mathrm{D}_3)$ makes the ODL in question Boolean).

**Proposition 2.1** *Let $L$ be an ODL. Then $L$ is a Boolean algebra exactly when the formula $x \triangle y \leq x \vee (y \wedge x^{\perp})$ is valid in $L$.*

**Proof**. If $L$ is a Boolean algebra, then for any pair of elements $x, y \in L$ we have $x \vee (y \wedge x^{\perp}) = x \vee y \geq x \triangle y$. Conversely, let $L$ fulfil the above formula. In order to prove that $L$ is Boolean, let us use [9], p. 31. Consider elements $x, y \in L$ with $x \wedge y = 0$. According to our assumption, $x^{\perp} \triangle y \leq x^{\perp} \vee (y \wedge (x^{\perp})^{\perp}) = x^{\perp} \vee (y \wedge x) = x^{\perp}$. Since $x^{\perp} \leq x^{\perp}$, we see in view of the condition $(\mathrm{D}_3)$ that we have $x^{\perp} \triangle (x^{\perp} \triangle y) \leq x^{\perp}$. But $x^{\perp} \triangle (x^{\perp} \triangle y) = y$. Therefore $y \leq x^{\perp}$ and we find that $L$ is Boolean.

The identity of Prop. 2.1 inspires one to consider other natural identities with the potential to be "Boolean". The following result summarizes this effort. In a certain sense it provides a definition of Boolean algebra in terms of 'abstract symmetric difference'.

**Theorem 2.2** *Let $L$ be an ODL. Then $L$ is a Boolean algebra exactly when $L$ fulfils any of the following four identities:*
(a) $(x \vee z) \triangle (y \vee z) \leq x \triangle y$,
(b) $x \triangle (x \vee y) \leq x \triangle y$,
(c) $x \vee y = x \triangle y \triangle (x \wedge y)$,
(d) $x \triangle y \triangle (x \vee y) \leq x \triangle y \triangle (x \wedge y)$,

**Proof**. Evidently, if $L$ is Boolean, then all identities (a)-(d) hold true. In proving the vice versa part, we first prove that (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ $L$ is Boolean. Let us suppose the condition (a). By setting $z = x$, we obtain $x \triangle (y \vee x) \leq x \triangle y$, which is the condition (b). Assuming the condition (b) and taking into account $x \, C \, x \vee y$, we see that $x \triangle y \geq x \triangle (x \vee y) = (x \vee y) \wedge x^\perp$. Then $(x \triangle y)^\perp \leq ((x \vee y) \wedge x^\perp)^\perp$. It follows that $(x \triangle y)^\perp = x \triangle y^\perp \leq x \vee (y^\perp \wedge x^\perp)$. Writing $y$ instead of $y^\perp$, we obtain $x \triangle y \leq x \vee (y \wedge x^\perp)$ which is the identity of Prop. 2.1.

To complete the the proof, let us verify (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ $L$ is Boolean. Let us suppose the condition (c). Then $x \triangle y \triangle (x \vee y) = x \triangle y \triangle (x \triangle y \triangle (x \wedge y))$ and therefore $x \triangle y \triangle (x \vee y) = x \wedge y$. It follows that $x \triangle y \triangle (x \vee y) = x \wedge y \leq x \vee y = x \triangle y \triangle (x \wedge y)$ and so we have derived the condition (d). Finally, having the condition (d), it is sufficient to show that any pair $x, y \in L$ with $x \wedge y = 0$ satisfies $x \leq y^\perp$. But then $x \triangle y \triangle (x \vee y) \leq x \triangle y$. Since $x \triangle y \leq x \triangle y$, we utilize ($D_3$) to obtain $x \triangle y \triangle (x \triangle y \triangle (x \vee y)) \leq x \triangle y$. As $x \triangle y \triangle (x \triangle y \triangle (x \vee y)) = x \vee y$, we infer that $x \vee y \leq x \triangle y$. But $x \triangle y \leq x \vee y$ and therefore $x \triangle y = x \vee y$. This implies that $y \leq x \triangle y$ and therefore $x \triangle y^\perp \leq y^\perp$. Since $y^\perp \leq y^\perp$, we utilize ($D_3$) to obtain $(x \triangle y^\perp) \triangle y^\perp \leq y^\perp$. But $(x \triangle y^\perp) \triangle y^\perp = x \triangle (y^\perp \triangle y^\perp) = x$ and the proof is complete.

In the next considerations we take up 'nearly Boolean ODLs' - the ODLs that are set-representable. We will find out that there is a formula which allows us to see that not all ODLs are nearly Boolean. With the help of Boolean algebras we will first introduce a class of ODLs. We will utilize it in the crucial example of the next section. Prior to that, let us fix some notation. Let $B$ be a non-trivial Boolean algebra and let $\mathcal{B}$ be a system of subalgebras of $B$. Let us say that $\mathcal{B}$ is a *disjoint system of subalgebras* of $B$ if for all $B_1, B_2 \in \mathcal{B}$ with $B_1 \neq B_2$ we have $B_1 \cap B_2 = \{0, 1\}$, and neither of the inclusions $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$ is valid. Moreover, if $\bigcup \mathcal{B} = B$, then the system $\mathcal{B}$ is said to be a *partition* of the algebra $B$.

Let $B$ be a Boolean algebra and let $\mathcal{B}$ be a disjoint system of subalgebras of $B$. Let us construct an OML, $K$, and the mapping $\triangle_K : K^2 \to K$ as follows:

In the first step we construct a partition $\mathcal{B}'$ of $B$ determined by the following requirement: If $\mathcal{B}$ is a partition of $B$, then we set $\mathcal{B}' = \mathcal{B}$. Otherwise, we add to $\mathcal{B}$ all necessary four-element subalgebras of B such that the resulting system $\mathcal{B}'$ is a partition of $B$. In the second step we take for $K$ the horizontal sum of the system $\mathcal{B}'$ (the horizontal sum alias the $\{0, 1\}$-pasting is a standard construction in OMLs, see [9, 14]). And finally, if $x, y \in K$, let us set $x \triangle_K y = x \triangle_B y$ (note that $K$ and $B$ live on the same set).

The couple $(K, \triangle_K)$ so obtained will be denoted by $L^{\mathcal{B}}$.

**Proposition 2.3** *The algebra $L^{\mathcal{B}}$ is an ODL.*

**Proof**. Conditions ($D_1$) and ($D_2$) are obvious. Let us verify condition ($D_3$). Let $x, y \in B$. If there is $B_1 \in \mathcal{B}$ such that $x, y \in B_1$, then $x \vee_K y = x \vee_B y$. As a result, $x \triangle y = x \triangle_B y \leq x \vee_B y = x \vee_K y$. If there is no $B_1$ such that $x, y \in B_1$, then $x \vee_K y = 1$. The inequality $x \triangle y \leq x \vee_K y$ is obvious and the proof is done.

Let $B$ be a Boolean algebra, $|B| \geq 4$. Let us take the finest partition of $B$, $\mathcal{B}$. Thus, the elements of $\mathcal{B}$ consist of all four-element subalgebras of $B$. Let us consider the algebra $L^{\mathcal{B}}$. Obviously, the OML $L^{\mathcal{B}}_{\text{supp}}$ coincides with the familiar $\mathrm{MO}_\kappa$ for an aproppriate cardinal number $\kappa$ (in fact, if $B$ is finite, it is easily seen that $\kappa = 2^n - 1$ for some $n \in \mathbf{N}$). We will allow ourselves to denote the ODL $L^{\mathcal{B}}$ by $\mathrm{MO}_\kappa$, too.

Let us return to the ODLs that are set-representable. They form a variety ([10]) and in view of the Stone set representation for Boolean algebras they could be seen as nearly Boolean. Though the name itself suggests their definition, let us recall it in more formal terms.

Let $X$ be a set and let $\mathcal{D}$ a family of subsets of $X$ such that
(1) $X \in \mathcal{D}$,
(2) the family $\mathcal{D}$ forms a lattice with respect to the inclusion relation, and
(3) $\mathcal{D}$ is closed under the formation of the set symmetric difference.
Obviously, $\mathcal{D}$ constitutes an ODL. Let us call it concrete. If $L$ is an ODL that is isomorphic with a concrete one, then $L$ is said to be *set-representable* (abbr., a SRODL). Let us denote by $\mathcal{SRODL}$ the class of all such ODLs.

The set-representable ODLs can be characterized in terms of certain evaluations. Let $\oplus$ stand for the addition modulo 2 on the set $\{0, 1\}$ (i.e., $0 \oplus 0 = 1 \oplus 1 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1$).

**Definition 2.4** *Let $L$ be an ODL and let $e : L \to \{0, 1\}$. Then $e$ is said to be an* ODL-evaluation *(abbr.,* evaluation*) on $L$ if the following properties are fulfilled for any $x, y \in L$:*
*($\mathrm{E}_1$) $e(1_L) = 1$,*
*($\mathrm{E}_2$) $x \leq y \Rightarrow e(x) \leq e(y)$,*
*($\mathrm{E}_3$) $e(x \bigtriangleup y) = e(x) \oplus e(y)$.*

Let $\mathcal{E}(L)$ be the set of all ODL-evaluations on $L$. The following result provides a characterization of $\mathcal{SRODL}$ in terms of $\mathcal{E}(L)$. The proof is straightforward ([10]) and we will omit it.

**Theorem 2.5** *Let $L$ be an ODL. Then $L$ is a SRODL if and only if*
$$\forall a, b \in L, a \not\leq b\ \exists e \in \mathcal{E}(L) : e(a) = 1, e(b) = 0.$$

The variety of SRODLs is rather large. For instance, the ODLs $\mathrm{MO}_\kappa$ are SRODLs. We will see that in general $L^{\mathcal{B}}$ does not have to be a SRODL (though $L^{\mathcal{B}}_{\mathrm{supp}}$ is always a set-representable OML !). It is the objective of this section to show this - it will be established as a consequence of a certain identity valid in SRODLs.

Let us start off with the following result that concerns the intrinsic property of SRODLs. It could be viewed, in a sense, as a contribution to a general research plan indicated in [7].

**Theorem 2.6** *Every SRODL $L$ satisfies the following formula ($x, y, z_1, z_2 \in L$):*

$$x \perp y \;\Rightarrow\; (x \bigtriangleup z_1) \wedge (y \bigtriangleup z_2) \leq z_1 \vee z_2 \ .$$

**Proof**. Let us suppose that there are elements $x, y, z_1, z_2 \in L$ with $x \perp y$ but $(x \bigtriangleup z_1) \wedge (y \bigtriangleup z_2) \not\leq z_1 \vee z_2$. As $L$ is set-representable, there is an ODL-evaluation $e$ such that $e((x \bigtriangleup z_1) \wedge (y \bigtriangleup z_2)) = 1$, $e(z_1 \vee z_2) = 0$. Since $z_1, z_2 \leq z_1 \vee z_2$ it has to be $e(z_1) = e(z_2) = 0$. By the same reasoning, $e(x \bigtriangleup z_1) = e(y \bigtriangleup z_2) = 1$. Because $1 = e(x \bigtriangleup z_1) = e(x) \oplus e(z_1)$ and $e(z_1) = 0$, we have $e(x) = 1$. Analogously, $e(y) = 1$. But this is absurd in view of the orthogonality of elements $x$ and $y$.

Let us note that the previous result allows us to formulate the following identity valid in $\mathcal{SRODL}$.

**Proposition 2.7** *Let $L$ be an ODL. Then the formula of Thm. 2.6 holds in $L$ exactly when the following identity holds in $L$:*

$$(x \triangle z_1) \wedge ((x^\perp \wedge y) \triangle z_2) \leq z_1 \vee z_2 \ .$$

**Proof**. It is sufficient to take into account that $x \perp y$ is equivalent with $y = x^\perp \wedge y$.

The identity of Thm. 2.6 allows us to prove the following result.

**Theorem 2.8** *There is a Boolean algebra $B$ and a disjoint system $\mathcal{B}$ of subalgebras of $B$ such that $L^\mathcal{B}$ is not set-representable ODL.*

**Proof**. Take $B = \exp\{1, 2, 3, 4, 5\}$. Let us make use of the following notation. Set $0_B = \emptyset$ and $1_B = \{1, 2, 3, 4, 5\}$. Let us denote by $\underline{n_1 \dots n_k}$, where $n_1 < \dots < n_k$ and $k \leq 5$, the element $a \subseteq \{1, 2, 3, 4, 5\}$ such that $a = \{n_1, \dots, n_k\}$. For any $a \subseteq \{1, 2, 3, 4, 5\}$, let us write $a^\perp = \{1, 2, 3, 4, 5\} \setminus a$. Thus, for instance, $\underline{12}^\perp = \{3, 4, 5\}$.

Let us go on with the construction. Consider the following subalgebras of $B$:

$B_1 = \{0_B, \underline{12}, \underline{3}, \underline{45}, \underline{12}^\perp, \underline{3}^\perp, \underline{45}^\perp, 1_B\}$,

$B_2 = \{0_B, \underline{15}, \underline{2}, \underline{34}, \underline{15}^\perp, \underline{2}^\perp, \underline{34}^\perp, 1_B\}$,

$B_3 = \{0_B, \underline{13}, \underline{24}, \underline{5}, \underline{13}^\perp, \underline{24}^\perp, \underline{5}^\perp, 1_B\}$.

Let us set $\mathcal{B} = \{B_1, B_2, B_3\}$. It is easily seen that $\mathcal{B}$ is a disjoint system of subalgebras of $B$. Consider $L^\mathcal{B}$ and test this ODL for the formula of Thm. 2.6. Set $x = \underline{12}, y = \underline{3}, z_1 = \underline{34}, z_2 = \underline{234}$ $(= \underline{15}^\perp)$. Then $x \perp y$, $x \triangle z_1 = \underline{1234}$ $(= \underline{5}^\perp)$ and $y \triangle z_2 = \underline{24}$. We see that both elements $x \triangle z_1$ and $y \triangle z_2$ lie in $B_3$, and $(x \triangle z_1) \wedge (y \triangle z_2) = \underline{24}$. But $z_1 \vee z_2 = \underline{34} \not\geq \underline{24}$. It follows that $L^\mathcal{B}$ does not satisfy the formula of Thm. 2.6 and therefore $L^\mathcal{B}$ is not a set-representable ODL.

The following fact given by the previous construction (one takes the ODL $L^\mathcal{B}$ exhibited above) could be of a mild separate interest.

**Observation 2.9** *There is a non set-representable ODL, $L$, such that $L_{\mathrm{supp}}$ is a non-modular set-representable OML.*

The above construction of $L^\mathcal{B}$ allows one not only to find an ODL with rather surprising properties but also to show that a certain class of OMLs (the horizontal sums of Boolean algebras) are embeddable into ODLs. The following proposition clarifies this situation in general.

**Proposition 2.10** *Let $L$ be an OML obtained as a horizontal sum of Boolen algebras. Then $L$ is OML-embeddable into an ODL.*

**Proof**. Let $L$ be a horizontal sum of Boolean algebras $B_\alpha$, $\alpha \in I$. As known (see e.g. [15]), there exists a Boolean algebra, $B$, such that each $B_\alpha$ $(\alpha \in I)$ is a subalgebra of $B$ and, moreover, if $\alpha_1 \neq \alpha_2$ then $B_{\alpha_1} \cap B_{\alpha_2} = \{0, 1\}$. As a result, the system $B_\alpha$, $\alpha \in I$ constitutes a disjoint system of subalgebras of $B$. It is clear that $L$ is embeddable into $L^\mathcal{B}$ and this completes the proof.

The horizontal sums of Boolean algebras constitute an important class of OMLs, [2]. It would be therefore desirable, in connection with the interplay between OMLs and ODLs, to answer the following questions. We will formulate them in the conclusion of this paper.

1. Could any horizontal sum of Boolean algebras be OML-embedded in a set-representable ODL ?

2. If $L_{\mathrm{supp}}$ is a set-representable and modular OML, does the ODL $L$ have to be set-representable ?

# References


[1] Beran L., *Orthomodular Lattices, Algebraic Approach*, D. Reidel, Dordrecht, 1985.

[2] Bruns G., Harding J., *Algebraic aspects of orthomodular lattices*, in Coecke B., Moore D. and Wilce A. eds. Current Research in Operational Quantum Logic, 2000, 37-65.

[3] Dalla Chiara M.L., Giuntini R., Greechie R., *Reasoning in Quantum Theory: Sharp and Unsharp Quantum Logics*, Kluwer Academic Publishers, Dordrecht, Boston, London, (2004).

[4] Dorfer G., Dvurečenskij A., Länger H.M., *Symmetric difference in orthomodular lattices*, Math. Slovaca **46** (1996), 435-444.

[5] Dorfer G., *Non-commutative symmetric differences in orthomodular lattices*, International Journal of Theoretical Physics **44** (2005), 885-896.

[6] Dvurečenskij A., Pulmannová S., *New Trends in Quantum Structures*, Kluwer Acad. Publ., Dordrecht, and Ister Science, Bratislava, 2000.

[7] Godowski R., Greechie R., *Some equations related to states on orthomodular lattices*, Demonstratio Mathematica **17** (1984), 241-250.

[8] *Handbook of Quantum Logic and Quantum Structures*, ed. by K.Engesser, D.M.Gabbay and D.Lehmann, Elsevier 2007.

[9] Kalmbach G., *Orthomodular Lattices*, Academic Press, London, 1983.

[10] Matoušek M., *Orthocomplemented lattices with a symmetric difference*, Algebra Universalis **60** (2009), 185-215.

[11] Matoušek M., Pták P., *Orthocomplemented posets with a symmetric difference*, Order **26** (2009), 1-21.

[12] Matoušek M., Pták P., *Symmetric difference on orthomodular lattices and $Z_2$-valued states*, to appear.

[13] Park E., Kim M.M., Chung J.Y., *A note on symmetric differences of orthomodular lattices*, Commun. Korean Math. Soc. **18** (2003), No 2, 207-214.

[14] Pták P., Pulmannová S., *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1991.

[15] Sikorski, R., *Boolean algebras*, Berlin, Heidelberg, New York, Springer 1964.

Milan Matoušek
Ruská 22
101 00 Prague 10
Czech Republic
*e-mail*: matmilan@email.cz

Pavel Pták
Department of Mathematics
Czech Technical University, Faculty of Electrical Eng.
166 27 Prague 6
Czech Republic
*e-mail*: ptak@math.feld.cvut.cz

Order (2009) 26:1–21
DOI 10.1007/s11083-008-9102-8

# Orthocomplemented Posets with a Symmetric Difference

**Milan Matoušek · Pavel Pták**

Received: 7 April 2008 / Accepted: 17 November 2008 / Published online: 13 February 2009


**Abstract** We endow orthocomplemented posets with a binary operation–an abstract symmetric difference of sets–and we study algebraic properties of this class, $\mathcal{ODP}$. Denoting its elements by ODP, we first investigate on the features related to compatibility in ODPs. We find, among others, that any ODP is orthomodular. This explicitly links $\mathcal{ODP}$ with the theory of quantum logics. By analogy with Boolean algebras, we then ask if (when) an ODP is set representable. Though we find that general ODPs do not have to be set representable, many natural ODPs are shown to be. We characterize the set-representable ODPs in terms of two valued morphisms and prove that they form a quasivariety. This quasivariety contains the class of pseudocomplemented ODPs as we show afterwards. At the end we ask whether any orthomodular poset can be converted or, more generally, enlarged to an ODP. By countre-examples we answer these questions to the negative.

**Keywords** Orthomodular poset · Quantum logic · Symmetric difference · Boolean algebra · Quasivariety · Frink ideal

**Mathematics Subject Classifications (2000)** 06C15 · 03G12 · 81B10

The authors acknowledge the support of the research plan MSM 0021620839 that is financed by the Ministry of Education of the Czech Republic and the grant GAČR 201/07/1051 of the Czech Grant Agency.

M. Matoušek (✉)
Department of Logic, Philosophical Faculty, Charles University,
Celetná 20, 116 42 Prague 1, Czech Republic
e-mail: matmilan@email.cz

P. Pták
Department of Mathematics, Faculty of Electrical Eng.,
Czech Technical University,
166 27 Prague 6, Czech Republic
e-mail: ptak@math.feld.cvut.cz

## 1 Introduction

In this paper we introduce certain "quantum logics with a symmetric difference". So we enrich the area of quantum logics (see e.g. [1, 9, 13, 14, 22, 24], etc.) with some "enriched" quantum logics (called ODPs). We analyze intrinsic properties of ODPs, proving results highlighted in the abstract.

Two remarks are in place. First, there have been attempts to model the operation of symmetric difference within orthomodular structures (see [7, 8] and [21]). Our approach differs essentially from the previous ones–we start with the abstract symmetric difference as an extra operation and obtain orthomodularity as a consequence. Second, this paper complements the paper [17] in a sense–the paper [17] studies lattice ODPs in the universal algebra vein whereas this paper mostly pursues natural (non-lattice) questions of quantum logics. The present paper and [17] overlap a little, typically when one generalizes results obtained in the lattice ODPs to general ODPs (Sections 2 and 3). The Sections 4–7 are essentially non-lattice though an application of the method used in Proposition 7.2 allows one to give a partial answer to an open question on lattice ODPs formulated in [17].

## 2 Orthocomplemented Difference Posets

In this section we shall define the notion of orthocomplemented difference poset (ODP). We shall collect basic properties of ODPs and find their relationships to other orthocomplemented structures. Before formulating the basic definition, let us recall that by an orthocomplemented poset (OCP, [22]) we mean a 5-tuple $(X, \leq, ^{\perp}, 0, 1)$ such that $\leq$ is a partial ordering on the set $X$ with a smallest element, 0, and a greatest element, 1, and $^{\perp}$ is a unary operation on $X$ with $x \wedge x^{\perp} = 0$, $x \vee x^{\perp} = 1$, $(x^{\perp})^{\perp} = x$ and $x \leq y \Rightarrow y^{\perp} \leq x^{\perp}$ $(x, y \in X)$.

**Definition 2.1** Let $P = (X, \leq, ^{\perp}, 0, 1, \triangle)$, where $(X, \leq, ^{\perp}, 0, 1)$ is an OCP and $\triangle : X^2 \to X$ is a binary operation. Then $P$ is said to be an *orthocomplemented difference poset* (abbr., an ODP) if $P$ is subject to the following axioms:

($D_1$) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$,
($D_2$) $x \triangle 1 = x^{\perp}$, $1 \triangle x = x^{\perp}$,
($D_3$) $x \leq z \;\&\; y \leq z \Rightarrow x \triangle y \leq z$.

Obviously, the class of all ODPs is defined by a set of quasiidentities and therefore it forms a quasivariety ([4]). Let us denote this quasivariety by $\mathcal{ODP}$.

Let $P = (X, \leq, ^{\perp}, 0, 1, \triangle)$ be an ODP. Then the orthocomplemented poset $(X, \leq, ^{\perp}, 0, 1)$ will be denoted by $P_{\mathrm{supp}}$ and called the *support* of $P$. Occasionally, the ODP $P$ will be identified with the couple $(P_{\mathrm{supp}}, \triangle)$ when a misunderstanding cannot occur.

*Example 2.2* Let $\Omega = \{1, 2, \ldots, 2k-1, 2k\}$ be a set, $k \in \mathbf{N}$. Let $\Omega_{\mathrm{even}}$ be the collection of all subsets of $\Omega$ consisting of an even number of elements. Then $\Omega_{\mathrm{even}}$ endowed with the inclusion ordering and with the standard symmetric difference is an ODP. (Note that $\Omega_{\mathrm{even}}$ is a lattice exactly when $k \leq 2$.)

*Example 2.3* Let $P$ be the well known "chinese lantern" lattice $MO_3$ (see, e.g., [14]), $P = \{0, 1, x, x^\perp, y, y^\perp, z, z^\perp\}$. Then there is exactly one mapping $\triangle : P \times P \to P$ such that $x \triangle y = z$ and $(P, \triangle)$ is an ODP. The ODP obtained in this way will be again denoted by $MO_3$. (More generally, it can be proved with a slightly more involved combinatorial argument that any $MO_{2^n-1}$ allows for converting into an ODP, see [17] for more on this type of lattice ODPs.)

Before we formulate our first result, let us adopt the convention that in writing the formula with $\triangle, {}^\perp$ we will give the preference to the operation $\perp$ over the operation $\triangle$. Thus, for instance, $x \triangle y^\perp$ means $x \triangle (y^\perp)$, etc.

**Proposition 2.4** *Let $P = (X, \leq, ^\perp, 0, 1, \triangle)$ be an ODP. Then the following statements hold true:*

(1) $x \triangle 0 = x$,
(2) $x \triangle x = 0$,
(3) $0 \triangle x = x$,
(4) $x \triangle y = y \triangle x$,
(5) $x \triangle y^\perp = x^\perp \triangle y = (x \triangle y)^\perp$,
(6) $x^\perp \triangle y^\perp = x \triangle y$,
(7) $x \triangle y = 0 \Leftrightarrow x = y$.

*Proof* Let us first observe that the property $(D_2)$ yields $1 \triangle 1 = 1^\perp = 0$. Let us verify the properties (1)–(7). Suppose that $x, y \in L$.

(1) $x \triangle 0 = x \triangle (1 \triangle 1) = (x \triangle 1) \triangle 1 = x^\perp \triangle 1 = (x^\perp)^\perp = x$.
(2) Let us first show that $x^\perp \triangle x^\perp = x \triangle x$. We consecutively obtain $x^\perp \triangle x^\perp = (x \triangle 1) \triangle (1 \triangle x) = (x \triangle (1 \triangle 1)) \triangle x = (x \triangle 0) \triangle x = x \triangle x$. Moreover, we have $x \triangle x \leq x$ as well as $x \triangle x = x^\perp \triangle x^\perp \leq x^\perp$. This implies that $x \triangle x \leq \inf_P\{x, x^\perp\} = 0$.
(3) $0 \triangle x = (x \triangle x) \triangle x = x \triangle (x \triangle x) = x \triangle 0 = x$ according to (2).
(4) $x \triangle y = (x \triangle y) \triangle 0 = (x \triangle y) \triangle [(y \triangle x) \triangle (y \triangle x)] = x \triangle (y \triangle y) \triangle x \triangle (y \triangle x) = x \triangle 0 \triangle x \triangle (y \triangle x) = x \triangle x \triangle (y \triangle x) = 0 \triangle (y \triangle x) = y \triangle x$.
(5) $x \triangle y^\perp = x \triangle (y \triangle 1) = (x \triangle y) \triangle 1 = (x \triangle y)^\perp$. The equality $x^\perp \triangle y = (x \triangle y)^\perp$ follows from $x \triangle y^\perp = (x \triangle y)^\perp$ by applying the equality (4).
(6) Using (5) we obtain $x^\perp \triangle y^\perp = (x^\perp \triangle y)^\perp = (x \triangle y)^{\perp\perp} = x \triangle y$.
(7) If $x = y$, then $x \triangle y = 0$ by the condition (2). Conversely, suppose that $x \triangle y = 0$. Then $x = x \triangle 0 = x \triangle (y \triangle y) = (x \triangle y) \triangle y = 0 \triangle y = y$. □

**Corollary 2.5** *Let $P$ be a finite ODP. Then* $\mathrm{card}(P) = 2^n$*, where $n$ is a natural number.*

*Proof* Let us introduce the operation $- : P \to P$ by putting $-x = x$. Then we see that the algebra $G = (P, \triangle, -, 0)$ is a group such that each element of $G$ has the order 2. Thus $G$ is a 2-group and the number of elements of $G$ must be a natural power of 2 (see [16]). □

**Proposition 2.6** *Let $P$ be an ODP and let $x, y \in P$. Then $x \leq y \Leftrightarrow x \triangle y \leq y$.*

*Proof* Let us suppose that $x \leq y$. As $y \leq y$, the condition ($D_3$) implies that $x \triangle y \leq y$. Conversely, suppose $x \triangle y \leq y$. Making again use of $y \leq y$, the condition ($D_3$) implies that $(x \triangle y) \triangle y \leq y$. But $(x \triangle y) \triangle y = x \triangle (y \triangle y) = x \triangle 0 = x$. □

**Proposition 2.7** *Let $P$ be an ODP and let $x, y, z \in P$. Then $z$ is an upper bound of the set $\{x, y\}$ if and only if $z$ is an upper bound of the set $\{x, x \triangle y\}$. A consequence: The supremum $x \vee y$ exists in $P$ if and only if the supremum $x \vee (x \triangle y)$ exists in $P$ and if either of the suprema exist, we have the equality $x \vee (x \triangle y) = x \vee y$.*

*Proof* Suppose first that $x \leq z$ and $y \leq z$. According to ($D_3$), we see that $x \triangle y \leq z$. Conversely, let $x \leq z$ and $x \triangle y \leq z$. According to ($D_3$) again, we obtain $x \triangle (x \triangle y) \leq z$. But $x \triangle (x \triangle y) = (x \triangle x) \triangle y = y$. □

In what follows, instead of writing $x \leq y^\perp$ we shall often use an equivalent expression $x \perp y$ (and say that $x$ is *orthogonal* to $y$) as customary in quantum logic theories.

**Proposition 2.8** *Let $P$ be an ODP and let $x, y \in P$. Then*

(a) *$x \perp y$ if and only if the supremum $x \vee y$ exists in $P$ and $x \vee y = x \triangle y$,*
(b) *$x \leq y$ if and only if the infimum $y \wedge x^\perp$ exists in $P$ and $y \wedge x^\perp = x \triangle y$.*

*Proof*

(a) Let us suppose that $x \perp y$. We shall show that $x \triangle y$ is the least upper bound of $\{x, y\}$.

  (i) First, let us show that $x \triangle y$ is an upper bound of $\{x, y\}$. Because of the symmetric role of the elements $x$ and $y$, it is sufficient to verify the inequality $x \leq x \triangle y$. Since $x \perp y$, we have $y \leq x^\perp$. The condition ($D_3$) then implies that $y \triangle x^\perp \leq x^\perp$. This means that $(x^\perp)^\perp \leq (y \triangle x^\perp)^\perp$. Moreover, $(x^\perp)^\perp = x$, $(y \triangle x^\perp)^\perp = y \triangle (x^\perp)^\perp = y \triangle x = x \triangle y$.
  (ii) Second, the element $x \triangle y$ is the least upper bound of $\{x, y\}$. Suppose that some $z \in P$ is an upper bound of $\{x, y\}$. Thus, $x \leq z$ and $y \leq z$. But then the condition ($D_3$) gives us $x \triangle y \leq z$ which we were to show.

  Let us verify the reverse implication. Let us suppose that $x \triangle y = x \vee y$. Then $y \leq x \triangle y$. Hence, $x \triangle y^\perp \leq y^\perp$. By Proposition 2.6 this yields that $x \leq y^\perp$ and therefore $x \perp y$.

(b) Suppose that $x \leq y$. Then $x \leq (y^\perp)^\perp$ which means that $x \perp y^\perp$. By the previous statement (a) we see that $x \triangle y^\perp = x \vee y^\perp$. This implies that $(x \triangle y^\perp)^\perp = (x \vee y^\perp)^\perp$. As a result, $(x \triangle y^\perp)^\perp = x \triangle y$ and $(x \vee y^\perp)^\perp = x^\perp \wedge y$.

Conversely, let $x \triangle y = y \wedge x^\perp$. Then $x \triangle y \leq y$. By Proposition 2.6 we obtain that $x \leq y$. □

We are going to show (Theorem 2.10) that the supports of ODPs are *orthomodular* posets (hence the abbreviation ODP could be read "orthomodular difference poset"). Recall that an OCP is said to be orthomodular ([22]) if the following implication holds true:

If $x \leq y$, then $y = x \vee (y \wedge x^\perp)$.

In proving Theorem 2.10, we will find it convenient to use the following characterization of orthomodularity.

**Proposition 2.9** *Let $P$ be an OCP such that the supremum $x \vee y$ exists in $P$ whenever $x \perp y$. Then $P$ is orthomodular if and only if the following implication holds true: If $x \leq y$ and $y \wedge x^{\perp} = 0$, then $x = y$.*

*Proof* If $P$ is orthomodular then the condition obviously holds true. Conversely, suppose $x \leq y$. Then $x \perp y^{\perp}$ and therefore the element $x \vee y^{\perp}$ exists in $P$. Moreover, $x \leq x \vee y^{\perp}$. This implies that $x \perp (x \vee y^{\perp})^{\perp}$. Therefore the element $x \vee (x \vee y^{\perp})^{\perp}$ exists in $P$. But $(x \vee y^{\perp})^{\perp} = y \wedge x^{\perp}$. It remains to show that $y = x \vee (y \wedge x^{\perp})$. Obviously, $x \vee (y \wedge x^{\perp}) \leq y$. Further, $y \wedge (x \vee (y \wedge x^{\perp}))^{\perp} = y \wedge x^{\perp} \wedge (y \wedge x^{\perp})^{\perp} = 0$. This means that $y = x \vee (y \wedge x^{\perp})$ and therefore $P$ is orthomodular. □

**Theorem 2.10** *Let $P$ be an ODP. Then its support $P_{\text{supp}}$ is an orthomodular poset.*

*Proof* Since $P$ is an ODP, its support $P_{\text{supp}}$ is an OCP. Moreover, the supremum $x \vee y$ exists in $P$ whenever $x \perp y$ (Proposition 2.8). So the characterization of orthomodularity in Proposition 2.9 can be used. Suppose that $x, y \in P$ with $x \leq y$ and $y \wedge x^{\perp} = 0$. Let us show that $x = y$. According to Proposition 2.8, we see that $x \triangle y = y \wedge x^{\perp} = 0$. By the condition (7) of Proposition 2.4 we infer that $x = y$. □

Obviously, all notions defined for OMPs can be transferred to ODPs by considering the supports. Thus, for instance, if $P$ is an ODP and $x, y \in P$ then we say that the elements $x, y$ are *compatible* (in $P$) if they are compatible in the OMP $P_{\text{supp}}$. Recall that two elements $x, y$ of an OMP $P$ are said to be compatible, in symbols $x\ C\ y$, if they lie in a Boolean subalgebra of $P$. As known ([22]), $B$ is a Boolean subalgebra of an OMP $P$ if and only if (1) $0 \in B$, (2) if $x \in B$, then $x^{\perp} \in B$, (3) if $x, y \in B$ and $x \perp y$, then $x \vee y \in B$ and (4) $B$ is a distributive lattice.

In the rest of this paragraph we will situate Boolean algebras within ODPs. Let us first formulate an auxiliary proposition.

**Proposition 2.11** *Let $P$ be an OMP. Let $x, y \in P$ with $x\ C\ y$. Let $B$ be a Boolean subalgebra of $P$ such that $x, y \in B$. Then both $x \vee y$ and $x \wedge y$ exist in $P$ and, moreover, $x \vee y = x \vee_B y$ and $x \wedge y = x \wedge_B y$.*

*Proof* Let us only argue the case of $x \vee y$, the case of $x \wedge y$ follows then by de Morgan's laws. Since $B$ is a Boolean algebra, we have $x \vee_B y = x \vee_B (y \wedge_B x^{\perp})$. The elements $x$ and $y \wedge_B x^{\perp}$ are orthogonal and they lie in $B$. But then the element $x \vee (y \wedge_B x^{\perp})$ exists in $P$ and since $B$ is a Boolean subalgebra of $P$, we have $x \vee (y \wedge_B x^{\perp}) \in B$. Then one easily sees that $x \vee_B (y \wedge_B x^{\perp}) = x \vee (y \wedge_B x^{\perp})$. It remains to show that the element $x \vee_B y$ is the supremum of $x, y$ in $P$. To this aim, let us consider an upper bound $z \in P$ of $x$ and $y$. Then $x \leq z$, $y \wedge_B x^{\perp} \leq z$ and hence $x \vee (y \wedge_B x^{\perp}) \leq z$. But $x \vee (y \wedge_B x^{\perp}) = x \vee_B (y \wedge_B x^{\perp}) = x \vee_B y$. □

The following proposition recalls the well-known characterization of compatibility in OMPs (see e.g. [22] for a detailed proof) which we shall frequently use. Leaving aside the routine completion of the argument, let us only indicate the basic line of the reasoning.

**Proposition 2.12** *Let $P$ be an OMP and let $x, y \in P$. Then the following conditions are equivalent:*

(1) $x\, C\, y$,
(2) *there exists an element $c \in P$ such that $c \le x$, $c \le y$ and $x \wedge c^\perp \le y^\perp$,*
(3) *there exist pairwise orthogonal elements $a, b, c \in P$ such that $x = a \vee c$ and $y = b \vee c$.*

*Moreover, in cases* (2) *and* (3)*, the element $c$ is determined uniquely and $c = x \wedge y$.*

*Proof* (1) $\Rightarrow$ (2) One takes $c = x \wedge y$. (2) $\Rightarrow$ (3) One takes $a = x \wedge c^\perp$, $b = y \wedge c^\perp$. (3) $\Rightarrow$ (1) It is easy to see that the elements $a, b, c$ generate in $P$ a Boolean subalgebra in view of the orthogonality of $a, b, c$. □

**Proposition 2.13** *Let $P$ be an ODP. Let $x, y \in P$ with $x\, C\, y$. Then $x \triangle y = (x \wedge y^\perp) \vee (y \wedge x^\perp) = (x \vee y) \wedge (x \wedge y)^\perp$.*

*Proof* Let $B$ be a Boolean subalgebra in $P$ such that $x, y \in B$. The property ($D_3$) together with the properties (5), (6) of Proposition 2.4 imply that $x \triangle y \le x \vee_B y$, $x \triangle y \le x^\perp \vee_B y^\perp = (x \wedge_B y)^\perp$, $x \wedge_B y^\perp \le x \triangle y$, $x^\perp \wedge_B y \le x \triangle y$. Then $(x \wedge_B y^\perp) \vee_B (y \wedge_B x^\perp) \le x \triangle y \le (x \vee_B y) \wedge_B (x \wedge_B y)^\perp$. Since $B$ is a Boolean algebra, the left-hand side of the previous inequality coincides with the right-hand side and therefore $x \triangle y = (x \wedge_B y^\perp) \vee_B (y \wedge_B x^\perp) = (x \vee_B y) \wedge_B (x \wedge_B y)^\perp$. To complete the proof, it is sufficient to observe that the infimum and the supremum of two elements in $B$ equals to the infimum, resp. supremum in the entire $P$ (Proposition 2.11). □

**Proposition 2.14** *Let $B$ be a Boolean algebra. Then there exists exactly one mapping $\triangle : B \times B \to B$ which fulfils all the conditions* ($D_1$), ($D_2$) *and* ($D_3$) *of Definition* 2.1.

*Proof* To prove the existence, take for $\triangle$ the standard symmetric difference in Boolean algebras. In other words, set $x \triangle y = (x \wedge y^\perp) \vee (y \wedge x^\perp)$. The properties ($D_1$), ($D_2$) and ($D_3$) of Definition 2.1 are then obviously fulfilled. □

Let us prove the uniqueness of $\triangle$. Let $\triangle_1 : B \times B \to B$ be a mapping that fulfils the conditions ($D_1$), ($D_2$) a ($D_3$). Then the couple $(B, \triangle_1)$ is an ODP. If $x, y \in B$, then $x\, C\, y$ and therefore $x \triangle_1 y = (x \wedge y^\perp) \vee (y \wedge x^\perp) = x \triangle y$ (Proposition 2.13).

In view of Proposition 2.14 we can (and shall) understand any Boolean algebra as an ODP with a uniquely defined operation $\triangle$. This ODP is a lattice, of course. The following result strengthens Proposition 2.13 and allows us to characterize Boolean algebras among ODPs.

**Proposition 2.15** *Let $P$ be an ODP and let $x, y \in P$. Then the following conditions are equivalent:*

(a) $x\, C\, y$,
(b) *the elements $x \wedge y^\perp$ and $y \wedge x^\perp$ exist in $P$ and $x \triangle y = (x \wedge y^\perp) \vee (y \wedge x^\perp)$,*
(c) *the elements $x \vee y$ and $x \wedge y$ exist in $P$ and $x \triangle y = (x \vee y) \wedge (x \wedge y)^\perp$,*
(d) $x\, C\, (x \triangle y)$.

*Proof* The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) have been verified in Proposition 2.13. Let us show (b) $\Rightarrow$ (d). Since $x \wedge y^{\perp} \leq x$, we obtain $x = (x \wedge y^{\perp}) \vee [x \wedge (x \wedge y^{\perp})^{\perp}]$. We have assumed $x \mathbin{\triangle} y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp})$. It suffices to show that the elements $x \wedge (x \wedge y^{\perp})^{\perp}$ and $y \wedge x^{\perp}$ are orthogonal, i.e., $x \wedge (x \wedge y^{\perp})^{\perp} \leq (y \wedge x^{\perp})^{\perp}$. But $x \wedge (x \wedge y^{\perp})^{\perp} \leq x \leq y^{\perp} \vee x = (y \wedge x^{\perp})^{\perp}$.

Further, let us prove that (c) $\Rightarrow$ (d). Assuming (c), the infima $x^{\perp} \wedge y^{\perp}$ and $x \wedge y$ exist and, moreover, $(x \mathbin{\triangle} y)^{\perp} = (x^{\perp} \wedge y^{\perp}) \vee (x \wedge y)$. This implies that $x \mathbin{\triangle} y^{\perp} = (x \wedge (y^{\perp})^{\perp}) \vee (y^{\perp} \wedge x^{\perp})$. Since (b) $\Rightarrow$ (d), we see that $x \, C \, (x \mathbin{\triangle} y^{\perp})$, and therefore $x \, C \, (x \mathbin{\triangle} y)$.

Finally, let us show that (d) $\Rightarrow$ (a). Let $x \, C \, (x \mathbin{\triangle} y)$. From the implication (a) $\Rightarrow$ (d) we have $x \, C \, (x \mathbin{\triangle} (x \mathbin{\triangle} y))$ but $x \mathbin{\triangle} (x \mathbin{\triangle} y) = y$. □

**Proposition 2.16** *Let $P$ be an ODP. Then $P$ is Boolean algebra if and only if $P$ is a lattice and any of the following three condition is satisfied:*

(1) $x \, C \, y$ *for any* $x, y \in P$,
(2) $x \mathbin{\triangle} y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp})$ *for any* $x, y \in P$,
(3) $x \mathbin{\triangle} y = (x \vee y) \wedge (x \wedge y)^{\perp}$ *for any* $x, y \in P$.

*Proof* It follows from Proposition 2.15. □

## 3 Further Algebraic Properties of ODPs

Let us go on with the analysis of properties of the operation $\triangle$. Let us first recall a result in OMPs.

**Proposition 3.1** *Let $P$ be an OMP. Let $x, x_1, x_2 \in P$ with $x \, C \, x_1$, $x \, C \, x_2$, $x_1 \perp x_2$. Then $x \, C \, (x_1 \vee x_2)$ and $x \wedge (x_1 \vee x_2) = (x \wedge x_1) \vee (x \wedge x_2)$.*

*Proof* Since the elements $x_1, x_2$ are orthogonal, so are the elements $x \wedge x_1$, $x \wedge x_2$. As a consequence, the element $c = (x \wedge x_1) \vee (x \wedge x_2)$ exists in $P$. Obviously, $c \leq x$ and $c \leq x_1 \vee x_2$. It is easy to see that $x \wedge c^{\perp} = x \wedge (x \wedge x_1)^{\perp} \wedge (x \wedge x_2)^{\perp} = x \wedge (x^{\perp} \vee x_1^{\perp}) \wedge (x^{\perp} \vee x_2^{\perp})$. Since $x \, C \, x_1$, we have $x \wedge (x^{\perp} \vee x_1^{\perp}) = x \wedge x_1^{\perp}$. Analogously, $x \wedge (x^{\perp} \vee x_2^{\perp}) = x \wedge x_2^{\perp}$. The previous two equalities imply that $x \wedge c^{\perp} = x \wedge x_1^{\perp} \wedge x_2^{\perp} \leq x_1^{\perp} \wedge x_2^{\perp} = (x_1 \vee x_2)^{\perp}$. By Proposition 2.12, (2), where one sets $y = x_1 \vee x_2$, we see that $x \, C \, (x_1 \vee x_2)$ and $c = x \wedge (x_1 \vee x_2)$. □

The previous result allows us to derive a certain form of distributivity in ODPs.

**Theorem 3.2** *Let $P$ be an ODP. Let $x, y, z \in P$ and $x \, C \, y$, $x \, C \, z$. Then $x \, C \, (y \mathbin{\triangle} z)$ and $x \wedge (y \mathbin{\triangle} z) = (x \wedge y) \mathbin{\triangle} (x \wedge z)$.*

*Proof* The compatibility of the pairs $x \, C \, y$ and $x \, C \, z$ gives us the equations $y = (y \wedge x) \vee (y \wedge x^{\perp})$, $z = (z \wedge x) \vee (z \wedge x^{\perp})$. Since $(y \wedge x) \perp (y \wedge x^{\perp})$ and $(z \wedge x) \perp (z \wedge x^{\perp})$, we see by Proposition 2.8 that $y = (y \wedge x) \mathbin{\triangle} (y \wedge x^{\perp})$ and $z = (z \wedge x) \mathbin{\triangle} (z \wedge x^{\perp})$.

But we also have

$$y \triangle z = \left[(y \wedge x) \triangle (y \wedge x^{\perp})\right] \triangle \left[(z \wedge x) \triangle (z \wedge x^{\perp})\right]$$
$$= \left[(y \wedge x) \triangle (z \wedge x)\right] \triangle \left[(y \wedge x^{\perp}) \triangle (z \wedge x^{\perp})\right].$$

Moreover, $(y \wedge x) \triangle (z \wedge x) \leq x$. Analogously, $(y \wedge x^{\perp}) \triangle (z \wedge x^{\perp}) \leq x^{\perp}$. This implies that $[(y \wedge x) \triangle (z \wedge x)] \perp [(y \wedge x^{\perp}) \triangle (z \wedge x^{\perp})]$. By Proposition 2.8,

$$y \triangle z = [(y \wedge x) \triangle (z \wedge x)] \vee \left[\left(y \wedge x^{\perp}\right) \triangle \left(z \wedge x^{\perp}\right)\right].$$

Finally, since we have $(y \wedge x) \triangle (z \wedge x) \leq x$ and $(y \wedge x^{\perp}) \triangle (z \wedge x^{\perp}) \leq x^{\perp}$, the proof can be completed by using Proposition 3.1 (one sets $x_1 = (y \wedge x) \triangle (z \wedge x)$, $x_2 = (y \wedge x^{\perp}) \triangle (z \wedge x^{\perp})$). □

There are a few consequences of Theorem 3.2. The first consequence asserts that those orthomodular posets which came into existence as supports of ODPs have a rather curious block property (in accord with the terminology of OMPs, let us call a maximal Boolean subalgebra of an ODP *a block*).

**Proposition 3.3** *Let $P$ be an ODP and let $x \in P$. Then either $x$ lies in exactly one block or $x$ lies in at least three blocks.*

*Proof* Looking for a contradiction, let $x$ lie in exactly two blocks $B_1$ and $B_2$. Then there exist elements $y \in B_1$, $z \in B_2$ such that $y, z$ are not compatible. Since $x\ C\ y$, $x\ C\ z$, we have $x\ C\ (y \triangle z)$ (Theorem 3.2). As a consequence, either $y \triangle z \in B_1$ or $y \triangle z \in B_2$. In view of the symmetry in the role of $y, z$, let us assume that $y \triangle z \in B_1$. Since $y \in B_1$, we infer that $y\ C\ (y \triangle z)$. By Proposition 2.15, we see that $y\ C\ z$, which is a contradiction. □

In the sequel we shall need to specify the subsets of ODPs which are ODPs in their own right.

**Definition 3.4** Let $P$ be a ODP and let $Q \subseteq P$. Then $Q$ is said to be a *sub-ODP of $P$* if

(1) $1 \in Q$,
(2) for any $x, y \in Q$ we have $x \triangle y \in Q$.

**Proposition 3.5** *Let $Q$ be a sub-ODP of $P$. Then*

(a) $0 \in Q$,
(b) *if $x \in Q$, then $x^{\perp} \in Q$,*
(c) *if $x, y \in Q$, $x \perp y$, then $x \vee y \in Q$.*

*A consequence: A sub-ODP of $P$ is an ODP when endowed with the ordering and operations inherited from $P$.*

*Proof* The property (a) is trivial. The property (b) follows from the equality $x^{\perp} = x \triangle 1$. According to Proposition 2.8 and the property (c), we easily see that if $x \perp y$, then $x \vee y = x \triangle y$. □

For the last consequence, let us introduce two notions related to compatibility in OMPs. If $x \in P$, write $C(x) = \{y \in P;\ x\, C\, y\}$. Further, write $C(P) = \bigcap_{x \in P} C(x)$ and call $C(P)$ the *centre* of $P$. Obviously, $C(P)$ is a Boolean subalgebra of $P$.

**Proposition 3.6** *Let $P$ be an ODP and let $x \in P$. Then the set $C(x)$ is a sub-ODP of $P$.*

*Proof* Obviously, $1 \in C(x)$. The rest of the proof follows from Theorem 3.2. □

Let us take up the intervals in ODPs. Consider first the situation in OMPs. Let $P$ be an OMP and let $a \in P$. Let us write $[0, a] = \{x \in P;\ x \leq a\}$. As known, the interval $[0, a]$ can be viewed as an OMP. We will denote it by $P^a$. Let us shortly recall the construction of $P^a$ (see, for example, [22]): If $x, y \in [0, a]$, then we put $x \leq_a y$ exactly when $x \leq y$. The element 0, resp. $a$, is the least, resp. the greatest, element of $P^a$. The complement of $x$ in $P^a$, $x^{\perp_a}$, is defined by setting $x^{\perp_a} = x^{\perp} \wedge a$ (since $x \leq a$, the elements $x, a$ are compatible in $P$ and therefore $x^{\perp} \wedge a$ exists). It can be easily seen that $P^a = ([0, a], \leq_a, {}^{\perp_a}, 0, a)$ is an OMP.

Let $P$ be an ODP and let $a \in P$. If $x, y \in [0, a]$ then $x \triangle y \in [0, a]$. Let us consider the structure $P^a = ([0, a], \leq_a, {}^{\perp_a}, 0, a, \triangle) = \left(\left(P_{\mathrm{supp}}\right)^a, \triangle\right)$.

**Proposition 3.7** *Let $P$ be an ODP and let $a \in P$. Then the structure $P^a$ is again an ODP. Moreover, $(P^a)_{\mathrm{supp}} = \left(P_{\mathrm{supp}}\right)^a$.*

*Proof* It is sufficient to show that the conditions $(D_1)$, $(D_2)$ and $(D_3)$ of Definition 2.1 hold in $P^a$. The conditions $(D_1)$ and $(D_3)$ can be verified easily. It remains to check the condition $(D_2)$. For that, suppose $x \in [0, a]$. Then $x \triangle 1_{P^a} = x \triangle a = a \wedge x^{\perp} = x^{\perp_a}$. The equality $1_{P^a} \triangle x = x^{\perp_a}$ follows from the commutativity of $\triangle$. The equality $(P^a)_{\mathrm{supp}} = (P_{\mathrm{supp}})^a$ is then obvious. □

In the following proposition we show that an ODP can be decomposed with the help of the central element by the way analogous to the situation in OMPs.

**Proposition 3.8** *Suppose that $P$ is an ODP and $a \in C(P)$. Then $P \cong P^a \times P^{a^{\perp}}$. More explicitly, the mapping $h : P \to P^a \times P^{a^{\perp}}$ defined by putting $h(x) = (x \wedge a, x \wedge a^{\perp})$ is an isomorphism of $P$ onto $P^a \times P^{a^{\perp}}$.*

*Proof* The mapping $h$ is an isomorphism between the OMPs $P_{\mathrm{supp}}$ and $(P^a)_{\mathrm{supp}} \times (P^{a^{\perp}})_{\mathrm{supp}}$, see [22]. It remains to show that the mapping $h$ preserves the operation $\triangle$. Suppose that $x, y \in P$. Then by Theorem 3.2 we consecutively obtain

$$\begin{aligned}
h(x \triangle_P y) &= \left((x \triangle_P y) \wedge a, (x \triangle_P y) \wedge a^{\perp}\right) \\
&= \left((x \wedge a) \triangle_P (y \wedge a), \left(x \wedge a^{\perp}\right) \triangle_P \left(y \wedge a^{\perp}\right)\right) \\
&= \left((x \wedge a) \triangle_{P^a} (y \wedge a), \left(x \wedge a^{\perp}\right) \triangle_{P^{a^{\perp}}} \left(y \wedge a^{\perp}\right)\right) \\
&= \left(x \wedge a, x \wedge a^{\perp}\right) \triangle_{P^a \times P^{a^{\perp}}} \left(y \wedge a, y \wedge a^{\perp}\right) \\
&= h(x) \triangle_{P^a \times P^{a^{\perp}}} h(y).
\end{aligned}$$

□

## 4 Set-representable ODPs

In this section we characterize those ODPs that are set-representable. We formulate this characterization in terms of two-valued morphisms and use the characterization in constructing the ODPs that are *not* set-representable.

**Definition 4.1** Let $B$ be a Boolean algebra and $\Omega$ be a subset of $B$. Then the set $\Omega$ is said to be a *d-subring of* $B$ if $\Omega$ is a sub-ODP of $B$, i.e.

(1) $1 \in \Omega$,
(2) for any $x, y \in \Omega$ we have $x \triangle y \in \Omega$.

Let us denote by $\mathsf{Subring}(B)$ the collection of all d-subrings of $B$.

**Proposition 4.2** *Let $B$ be a Boolean algebra and let $\Omega \subseteq B$ be a d-subring. Then* $0 \in \Omega$ *and* $\Omega_B = (\Omega, \leq, 0, 1, ^{\perp}, \triangle)$ *is an ODP.*

*Proof* Since for all orthogonal elements $a, b \in \Omega$ we have $a \vee b = a \triangle b$, we see that $\Omega_B$ is obviously an ODP. □

Let us recall two standard set-theoretic operations. If $A$, $B$ are sets, let us write $A \setminus B$ for the set $\{x \in A;\, x \notin B\}$ and $A \Delta B$ for the set $(A \setminus B) \cup (B \setminus A)$.

**Definition 4.3** Let $P$ be an ODP. Let us say that $P$ is a *set-representable ODP* (abbr. *SRODP*) if there is a set $X$ and a d-subring $\Omega$ of the power Boolean algebra $\exp(X)$ such that $P \cong \Omega_{\exp(X)}$, i.e. the structures $P$ and $\Omega_{\exp(X)}$ are isomorphic.
Let us denote by $\mathcal{SRODP}$ the class of all set-representable ODPs.

**Proposition 4.4** *Let $P$ be an ODP. Then $P$ is a SRODP if and only if there is a set $M$ and a mapping $f$ of $P$ to* $\exp(M)$ *such that the following two conditions hold for any* $x, y \in P$*:*

$$x \leq y \Leftrightarrow f(x) \subseteq f(y),$$

$$f(x \triangle_P y) = f(x) \Delta f(y).$$

*Proof* Let us prove that the conditions are sufficient, the rest is obvious. Set $X = f(1) \subseteq M$, $\Omega = f[P] = \{f(x);\ x \in P\}$. Then the second condition implies that $\Omega$ is a d-subring of the Boolean algebra $\exp(X)$. Further, the first condition implies that $f$ is an isomorphism of the poset $(P, \leq)$ onto the poset $(\Omega, \subseteq)$. Finally, $f(0) = f(0 \triangle 0) = f(0) \Delta f(0) = \emptyset$. □

**Proposition 4.5** *The SRODPs are exactly the d-subrings of Boolean algebras. More precisely: Let $B$ be a Boolean algebra and let $\Omega$ be a d-subring of $B$. Then $\Omega_B$ is a SRODP. Vice versa, let $P$ be a SRODP. Then there exists a Boolean algebra, $B$, and a d-subring $\Omega$ of $B$ such that $P$ is isomorphic with $\Omega_B$.*

*Proof* It follows from the Stone representation theorem for Boolean algebras. □

Let $\oplus$ stand for the addition modulo 2 on the set $\{0, 1\}$ (i.e., $0 \oplus 0 = 1 \oplus 1 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1$).

**Definition 4.6** Let $P$ be an ODP and let $e : P \to \{0, 1\}$ be a mapping. Then $e$ is said to be an *ODP-evaluation* (abbr. *evaluation*) on $P$ if the following properties are fulfilled ($x, y \in P$):

($E_1$) $e(1) = 1$,
($E_2$) $x \leq y \Rightarrow e(x) \leq e(y)$,
($E_3$) $e(x \triangle y) = e(x) \oplus e(y)$.

Let $\mathcal{E}(P)$ be the set of all evaluations on $P$. The following result characterizes SRODPs in terms of $\mathcal{E}(P)$.

**Theorem 4.7** *Let $P$ be an ODP. Then $P$ is a SRODP if and only if for any $a, b \in P$ with $a \not\leq b$ there exists an $e \in \mathcal{E}(P)$ such that $e(a) = 1$ and $e(b) = 0$.*

*Proof*

($\Rightarrow$) Write $P = \Omega_{\exp(X)}$, where $\Omega$ is a d-subring of the Boolean algebra $\exp(X)$. Choosing an $x \in X$, let us define a mapping $e_x : \Omega \to \{0, 1\}$ as follows:
If $A \in \Omega$, then $e_x(A) = 1$ if $x \in A$, otherwise $e_x(A) = 0$.
Let us check that the mapping $e_x$ is an evaluation on $P$. Obviously, $e_x(1_P) = e_x(X) = 1$. If $A, B \in \Omega$ and $A \leq_P B$, then $A \subseteq B$, and therefore $e_x(A) \leq e_x(B)$. In order to show that $e_x(A \triangle B) = e_x(A) \oplus e_x(B)$, we have four cases to argue: $x \in A$, $x \notin A$ and $x \in B$, $x \notin B$. But in any of these cases the latter equality obviously holds.
Finally, if $A, B \in \Omega$ and $A \not\subseteq B$, then there exists an $x \in A$ such that $x \notin B$. Then $e_x(A) = 1$ and $e_x(B) = 0$.

($\Leftarrow$) Let us assume that the condition on evaluations is fulfilled. Choosing $x \in P$, let us write $f(x) = \{e \in \mathcal{E}(P); e(x) = 1\}$. We will check that this mapping $f : P \to \mathcal{P}(\mathcal{E}(P))$ fulfils both conditions of Proposition 4.4. First, suppose that $x \leq y$. Let $e \in f(x)$. This means that $e(x) = 1$. According to ($E_2$) we have $e(x) \leq e(y)$. As a result, $e(y) = 1$ and hence $e \in f(y)$. We have shown that $f(x) \subseteq f(y)$. Conversely, suppose that $x \not\leq y$. Then there is $e \in \mathcal{E}(P)$ such that $e(a) = 1$, $e(b) = 0$. We see that $f(x) \not\subseteq f(y)$. To complete the proof, we use the equalities $f(x \triangle_P y) = \{e \in \mathcal{E}(L); e(x \triangle_P y) = 1\} = \{e \in \mathcal{E}(P); e(x) \oplus e(y) = 1\} =$
$= \{e \in \mathcal{E}(P); (e(x) = 1 \,\&\, e(y) = 0) \bigvee (e(x) = 0 \,\&\, e(y) = 1)\} = f(x) \triangle f(y)$.
□

The following simple consequence of the previous theorem will be repeatedly used in the sequel.

**Proposition 4.8** *The class $\mathcal{SRODP}$ is closed under the formation of substructures.*

*Proof* Suppose that $P \in \mathcal{SRODP}$ and $Q$ is a sub-ODP of $P$. Suppose $a, b \in Q$, $a \not\leq b$. Then there exists $e \in \mathcal{E}(P)$ such that $e(a) = 1$ and $e(b) = 0$. It suffices to observe that the restriction of $e$ on $Q$ is an evaluation on $Q$. □

The SRODPs will be revisited in the next section in order to study their structure properties. It will be seen that the SRODPs form quite a large class (they are closed under the formation of products, etc.). However, not all ODPs are set-representable. To show that, let us shortly examine a construction with ODPs analogous to the horizontal sum of orthomodular posets. Let $\mathcal{B}$ be a nonempty set of OMPs such that $P_1 \cap P_2 = \{0, 1\}$ for all $P_1, P_2 \in \mathcal{B}$ with $P_1 \neq P_2$. Then $P = \bigcup \mathcal{B}$ carries in a natural way the structure of an orthomodular poset (see [22]). Let us call this OMP $P$ the *horizontal sum* of the system $\mathcal{B}$.

**Definition 4.9** Let $B$ be a nontrivial Boolean algebra and let $\mathcal{B} \subseteq \mathsf{Subring}(B)$. Let us say that $\mathcal{B}$ is a *disjoint system of d-subrings* of $B$ if for all $\Omega_1, \Omega_2 \in \mathcal{B}$ with $\Omega_1 \neq \Omega_2$ we have $\Omega_1 \cap \Omega_2 = \{0, 1\}$ and, moreover, if $\mathrm{card}(B) \geq 4$ then $\mathrm{card}(\Omega) \geq 4$ for any $\Omega \in \mathcal{B}$. In addition, if $\bigcup \mathcal{B} = B$, then the system $\mathcal{B}$ is said to be a *partition* of the algebra $B$.

**Construction 4.10** Let $B$ a Boolean algebra and let $\mathcal{B}$ be a disjoint system of d-subrings of $B$. Let us construct an OMP, $P$, and the mapping $\triangle_P : P^2 \to P$ as follows:

In the first step, let us construct a system $\mathcal{B}' \subseteq \mathsf{Subring}(B)$ determined by the following requirement: If $\bigcup \mathcal{B} = B$ (i.e., if $\mathcal{B}$ is a partition of $B$), then we set $\mathcal{B}' = \mathcal{B}$. If it is not the case, meaning that $\bigcup \mathcal{B}$ is a proper subset of $B$, we add to $\mathcal{B}$ all necessary four-element subalgebras of B such that the resulting system $\mathcal{B}'$, $\mathcal{B}' \supseteq \mathcal{B}$, be a partition of $B$. In the second step, let us take for $P$ the horizontal sum of the system $\mathcal{B}'$. And finally, if $x, y \in P$, let us set $x \triangle_P y = x \triangle_B y$. The couple $(P, \triangle_P)$ so obtained will be denoted by $P^{B,\mathcal{B}}$ or simply by $P^{\mathcal{B}}$ if there is no need to refer to $B$.

**Proposition 4.11** *The structure $P^{\mathcal{B}}$ is an ODP.*

*Proof* The conditions ($D_1$) and ($D_2$) are obvious. Let us verify the condition ($D_3$). Let $x, y \in B$. If there is $B_1 \in \mathcal{B}$ such that $x, y \in B_1$, then $x \vee_P y = x \vee_B y$. As a result, $x \triangle y = x \triangle_B y \leq x \vee_B y = x \vee_P y$. If there is no block $B_1$ such that $x, y \in B_1$, then $x \vee_P y = 1$. The inequality $x \triangle y \leq x \vee_P y$ is then valid automatically and the proof is complete. □

**Proposition 4.12** *Let $B_1$ be a subalgebra of $B_2$. Let $\mathcal{B}$ be a disjoint system of d-subrings of $B_1$. Then $P^{B_1,\mathcal{B}}$ is a sub-ODP of $P^{B_2,\mathcal{B}}$.*

*Proof* The proof is straightforward. □

In the rest of this section we will apply the above construction to obtain ODPs that are not set-representable. In fact, we will show that *any* ODP is contained in a non-set-representable ODP. To this aim, we need to prove a few propositions.

**Proposition 4.13** *Let $P$ and $Q$ be ODPs and let $P$ possess an evaluation. Then $P$ can be embedded into $P \times Q$.*

*Proof* Let $e : P \to \{0, 1\}$ be an evaluation. Then $e$ could be viewed as a homomorphism of $P$ into $Q$ (we only identify 0 and 1 with $0_Q$ and $1_Q$). Then the mapping $f : P \to P \times Q$ such that $f(p) = (p, e(p))$ will do the job. □

**Proposition 4.14** *Let $B$ be a Boolean algebra and $\mathcal{B}$ be a disjoint system of d-subrings of $B$. Let $P = P^{\mathcal{B}}$ and $a \in P$ with $a >_P 0_P$. Then there is an evaluation $e$ on $P$ such that $e(a) = 1$.*

*Proof* Let us choose an ultrafilter, $F$, on $B$ such that $a \in F$. Then we set $e(x) = 1$ provided $x \in F$ and $e(x) = 0$ otherwise. □

The following proposition shows that in general we cannot hope for having "distinguishable" set of evaluations on $P^{\mathcal{B}}$. This will show that $P^{\mathcal{B}}$ are generally not set-representable (Proposition 4.7).

**Proposition 4.15** *Let $B$ be a Boolean algebra with* $\mathrm{card}(B) \geq 32$*. Then there is such a disjoint system $\mathcal{B}$ of d-subrings of $B$ that $P^{\mathcal{B}}$ is not set-representable.*

*Proof* Let $B_{32}$ be the 32-element Boolean algebra. Since $\mathrm{card}(B) \geq 32$, the algebra $B_{32}$ can be viewed as a subalgebra of $B$. By Proposition 4.8 and Proposition 4.12, it is sufficient to find a disjoint system of d-subrings on the algebra $B_{32}$ such that $P^{B_{32},\mathcal{B}}$ is not set-representable.

Let us identify the Boolean algebra $B_{32}$ with the algebra of all subsets of the set $\{1, \ldots, 5\}$. Then $0_B = \emptyset$ and $1_B = \{1, \ldots, 5\}$. Let us consider the following elements of $B$: $a_1 = \{1, 2\}$, $a_2 = \{4, 5\}$, $a_3 = \{3\}$, $b_1 = \{3, 4\}$, $b_2 = \{2\}$, $b_3 = \{1, 5\}$, $c_1 = \{1, 3\}$, $c_2 = \{2, 4\}$, $c_3 = \{5\}$. Further, take the following d-subrings $\Omega_1$, $\Omega_2$, $\Omega_3$ of $B$:

$$\Omega_1 = \left\{0_B, 1_B, a_1, a_2, a_3, a_1^{\perp}, a_2^{\perp}, a_3^{\perp}\right\}, \Omega_2 = \left\{0_B, 1_B, b_1, b_2, b_3, b_1^{\perp}, b_2^{\perp}, b_3^{\perp}\right\},$$
$$\Omega_3 = \left\{0_B, 1_B, c_1, c_2, c_3, c_1^{\perp}, c_2^{\perp}, c_3^{\perp}\right\}.$$

It is easy to show that $\Omega_i \cap \Omega_j = \{0_B, 1_B\}$ for any $i \neq j$. Writing $\mathcal{B} = \{\Omega_1, \Omega_2, \Omega_3\}$, we see that $\mathcal{B}$ is a disjoint system. Let us set $P = P^{\mathcal{B}}$. Further put $a = \{3\} = a_3$ and $b = \{2, 3, 4\} = b_3^{\perp}$. Then $a \not\leq b$ in the poset $P$. In order to show that $P$ is not set-representable, it is sufficient to prove that there is no evaluation $e$ on $P$ such that $e(a) = 1$ and $e(b) = 0$. Seeking a contradiction, let $e$ be an evaluation on $P$ with $e(a) = 1$ and $e(b) = 0$. Then $e(a^{\perp}) = 0$. Observing that $a_1 \leq a^{\perp}, a_2 \leq a^{\perp}, b_1 \leq b, b_2 \leq b$ in $P$, we infer that $e(a_1) = e(a_2) = e(b_1) = e(b_2) = 0$. Since $c_3^{\perp} = \{1, 2, 3, 4\} = a_1 \triangle b_1$ and $c_1^{\perp} = \{2, 4, 5\} = a_2 \triangle b_2$, we see that $e(c_3^{\perp}) = e(a_1 \triangle b_1) = e(a_1) \oplus e(b_1) = 0$. Analogously, $e(c_1^{\perp}) = 0$. As $c_1 \leq c_3^{\perp}$ , we obtain that $e(c_1) = 0$. It means that $e(c_1) = e(c_1^{\perp}) = 0$, which is absurd. □

**Theorem 4.16** *Let $P$ be an ODP. Then there is an ODP, $Q$, such that $Q$ is not set-representable and $P$ is embeddable into $Q$.*

*Proof* If $P$ is not set-representable then there is nothing to prove. Let $P$ be set-representable. By Proposition 4.14 and 4.15 there is a non-set-representable ODP $R$ such that there exists an evaluation $e$ on $R$. Set $Q = P \times R$. Since $R$ is embeddable into $Q$ (Proposition 4.13), $Q$ is not set-representable (Proposition 4.8). Since there is an evaluation on $P$, we see that $P$ is embeddable into $Q$ (Proposition 4.13). This completes the proof. □

## 5 Set-representable ODPs form a Quasivariety

In this section we shall show that the class $\mathcal{SRODP}$ is a quasivariety (i.e. the class $\mathcal{SRODP}$ is defined by a set of quasiidentities). Hence the class $\mathcal{SRODP}$ is rather large and algebraically "stylish". It should be noted that in showing that $\mathcal{SRODP}$ is a quasivariety the investigation of the set-representation of orthomodular posets was instrumental (see [18] and [19]). However, the presence of the extra operation $\triangle$ required here a somewhat different reasoning in places.

We shall deal with the ultraproduct of ODPs. Let us first recall the notions and results we shall need.

**Proposition 5.1** *Let $I$ be a non-empty set and let $\mathcal{F}$ be an ultrafilter on $I$. Let $f : I \to \{0, 1\}$ be a mapping. Then there exists exactly one value $v \in \{0, 1\}$ such that $f^{-1}(v) \in \mathcal{F}$ (where $f^{-1}(v) = \{i \in I;\ \ f(i) = v\}$).*

*Proof* Obviously, $f^{-1}(1) = I \setminus f^{-1}(0)$. Since $\mathcal{F}$ is an ultrafilter on $I$, it is clear that exactly one of the sets $f^{-1}(0)$, $f^{-1}(1)$ belongs to $\mathcal{F}$. □

The value $v$ uniquely determined by the previous proposition will be denoted by $v(\mathcal{F}, f)$. If $f_1, f_2 : I \to \{0, 1\}$, let us agree to write $f_1 \sim_{\mathcal{F}} f_2$, provided $\mathrm{Eq}(f_1, f_2) \in \mathcal{F}$, where $\mathrm{Eq}(f_1, f_2) = \{i \in I;\ \ f_1(i) = f_2(i)\}$.

**Proposition 5.2** *Let $I$ be a nonempty set and let $\mathcal{F}$ be an ultrafilter on $I$. Suppose that $f_1, f_2 : I \to \{0, 1\}$ are mappings with $f_1 \sim_{\mathcal{F}} f_2$. Then $v(\mathcal{F}, f_1) = v(\mathcal{F}, f_2)$.*

*Proof* Suppose that $a = v(\mathcal{F}, f_1)$. Then $f_1^{-1}(a) \cap \mathrm{Eq}(f_1, f_2) \subseteq f_2^{-1}(a)$. Since $f_1^{-1}(a) \in \mathcal{F}$ and $\mathrm{Eq}(f_1, f_2) \in \mathcal{F}$, we see that $f_2^{-1}(a) \in \mathcal{F}$. It follows that $a = v(\mathcal{F}, f_2)$. □

Let $X_i$ be nonempty sets, where $i \in I$ for a nonempty set $I$. Let $X = \prod_{i \in I} X_i$ be the Cartesian product of the sets $X_i$, $i \in I$. Let $\mathcal{F}$ be an ultrafilter on $I$ and let $Y = \prod_{i \in I}^{\mathcal{F}} X_i$ be the corresponding ultraproduct (i.e., let $Y$ be the set of all classes of the equivalence $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \{i \in I;\ \ \mathbf{x}(i) = \mathbf{y}(i)\} \in \mathcal{F}$). Let us suppose that we are given mappings $e_i : X_i \to \{0, 1\}$, $i \in I$. Then we can construct a mapping $e : Y \to \{0, 1\}$ as follows:

If $\mathbf{a} \in X$, let us denote by $g_{\mathbf{a}} : I \to \{0, 1\}$ the mapping defined by setting $g_{\mathbf{a}}(i) = e_i(\mathbf{a}(i))$. Put $h(\mathbf{a}) = v(\mathcal{F}, g_{\mathbf{a}})$. We have therefore obtained a mapping $h : X \to \{0, 1\}$. Let $\alpha \in Y$ and $\alpha = [\mathbf{a}]_{\mathcal{F}}$. It means that $\mathbf{a}$ is such an element of $X$ that $\mathbf{a} \in \alpha$. Let us set $e(\alpha) = h(\mathbf{a})$. We have to verify the correctness of this definition. Suppose that $\alpha = [\mathbf{a}]_{\mathcal{F}} = [\mathbf{b}]_{\mathcal{F}}$. Then $\{i \in I;\ \ \mathbf{a}(i) = \mathbf{b}(i)\} \in \mathcal{F}$. It follows that $\{i \in I;\ \ g_{\mathbf{a}}(i) = g_{\mathbf{b}}(i)\} \in \mathcal{F}$. This implies that $g_{\mathbf{a}} \sim_{\mathcal{F}} g_{\mathbf{b}}$. By Proposition 5.2, $v(\mathcal{F}, g_{\mathbf{a}}) = v(\mathcal{F}, g_{\mathbf{b}})$. This means that $h(\mathbf{a}) = h(\mathbf{b})$ and we have found that the definition is correct.

Let us denote by $\prod_{i \in I}^{\mathcal{F}} e_i$ the mapping $e : Y \to \{0, 1\}$ constructed above.

**Proposition 5.3** *Let $P_i$ be ODPs, $i \in I$, $I \neq \emptyset$. Let $\mathcal{F}$ be an ultrafilter on $I$. Let $P = \prod_{i \in I}^{\mathcal{F}} P_i$ be the ultraproduct of $P_i$, $i \in I$. For any $i \in I$, let $e_i : P_i \to \{0, 1\}$ be such* *mappings that* $\{i \in I;\ e_i$ *is an evaluation on* $P_i\} \in \mathcal{F}$. *Then the mapping* $e = \prod_{i \in I}^{\mathcal{F}} e_i$ *is an evaluation on* $P$.

*Proof* The proof reduces to a straightforward verification. □

**Theorem 5.4** *The class* $\mathcal{SRODP}$ *of all set-representable ODPs forms a quasivariety.*

*Proof* We shall show that the class $\mathcal{SRODL}$ is closed under the formation of substructures, products and ultraproducts.

(a) The closedness under the formation of substructures has been proved in Proposition 4.8.

(b) The closedness under the formation of products: Suppose $P_i \in \mathcal{SRODP}$, $i \in I$. Write $P = \prod_{i \in I} P_i$. For any $j \in I$, let us denote by $\pi_j$ the $j$-th projection $P \to P_j$. Suppose that $\mathbf{a}, \mathbf{b} \in P$ and $\mathbf{a} \not\leq \mathbf{b}$. Then there exists an index $i_0 \in I$ such that $a_{i_0} \not\leq b_{i_0}$, where $a_{i_0} = \pi_{i_0}(\mathbf{a})$, $b_{i_0} = \pi_{i_0}(\mathbf{b})$. Take an evaluation $e \in \mathcal{E}(P_{i_0})$ such that $e(a_{i_0}) = 1$ and $e(b_{i_0}) = 0$. Consider the evaluation $\pi_{i_0} \circ e$ on $P$. Then $(\pi_{i_0} \circ e)(\mathbf{a}) = e(\pi_{i_0}(\mathbf{a})) = e(a_{i_0}) = 1$ and $(\pi_{i_0} \circ e)(\mathbf{b}) = e(\pi_{i_0}(\mathbf{b})) = e(b_{i_0}) = 0$.

(c) The closedness under the formation of ultraproducts: Suppose $P_i \in \mathcal{SRODP}$, $i \in I$ and suppose that $\mathcal{F}$ is an ultrafilter on $I$. Write $Q = \prod_{i \in I}^{\mathcal{F}} P_i$. Suppose that $\alpha, \beta \in Q$, $\alpha \not\leq_Q \beta$, and, moreover, suppose that $\alpha = [\mathbf{a}]_{\mathcal{F}}$, $\beta = [\mathbf{b}]_{\mathcal{F}}$, $\mathbf{a}, \mathbf{b} \in \prod_{i \in I} P_i$. Then $\{i \in I;\ \mathbf{a}(i) \leq \mathbf{b}(i)\} \notin \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, we infer that $\{i \in I;\ \mathbf{a}(i) \not\leq \mathbf{b}(i)\} \in \mathcal{F}$. Write $J = \{i \in I;\ \mathbf{a}(i) \not\leq \mathbf{b}(i)\}$. Suppose that, for any $i \in J$, $e_i : P_i \to \{0, 1\}$ be such evaluations that $e_i(\mathbf{a}(i)) = 1$ and $e_i(\mathbf{b}(i)) = 0$. If $i \in I \setminus J$ let $e_i : P_i \to \{0, 1\}$ be arbitrary mappings. Let us set $e = \prod_{i \in I}^{\mathcal{F}} e_i$. Then $e$ is an evaluation on $P$. Furthermore, $e(\alpha) = h(\mathbf{a}) = v(\mathcal{F}, g_{\mathbf{a}})$. If $i \in J$, then $g_{\mathbf{a}}(i) = e_i(\mathbf{a}(i)) = 1$. It follows that $J \subseteq \{i \in I;\ g_{\mathbf{a}}(i) = 1\}$ and therefore $\{i \in I;\ g_{\mathbf{a}}(i) = 1\} \in \mathcal{F}$. Then $v(\mathcal{F}, g_{\mathbf{a}}) = 1$. The equalities $e(\beta) = h(\mathbf{b}) = v(\mathcal{F}, g_{\mathbf{b}}) = 0$ can be shown analogously. This concludes the proof of Theorem 5.4. □

In view of the last result we see that the class $\mathcal{SRODP}$ is defined by a set of quasiidentities. It would be desirable to know if one can do here with a finite number of these quasiidentities but this seems to be a rather difficult problem.

## 6 Pseudocomplemented ODPs

The main result we want to prove in this section says that the pseudocomplemented ODPs are set-representable, generalizing thus the situation in OMPs (see [15], [20] and [23]). Let us start with the following standard definition of the theory of complemented structures.

**Definition 6.1** Let $P$ be an OMP. Let us say that $P$ is *pseudocomplemented* if for any $x, y \in P$ the following implication is true: If $0 = x \wedge y$, then $x \leq y^{\perp}$.

It should be remarked that the pseudocomplemented OMPs play a noteworthy role in the study of compatibility – they are exactly those "quantum logics" in which the compatibility relation allows for the following lattice characterization. Let us present a simple proof of this fact. It does not seem to be contained in monographs though it is related to the question of whether the compatibility can be described lattice-theoretically – a question relevant to quantum axiomatics.

**Proposition 6.2** *Let $P$ be an OMP. Then the following conditions are equivalent:*

(1) *$P$ is pseudocomplemented,*
(2) *for any $a, b \in P$, the infimum $a \wedge b$ exists in $P$ exactly when $a\, C\, b$.*

*Proof* Suppose first that $P$ is pseudocomplemented. Let us consider two elements $a, b \in P$ such that $a \wedge b$ exists in $P$. Write $d = a \wedge b$. Then $a = d \vee (a \wedge d^{\perp})$, $b = d \vee (b \wedge d^{\perp})$. It is sufficient to show that $a \wedge d^{\perp}$ and $b \wedge d^{\perp}$ are orthogonal. Since $P$ is pseudocomplemented, it is enough to show that $(a \wedge d^{\perp}) \wedge (b \wedge d^{\perp}) = 0$. Suppose that $c$ is such an element that $c \leq a \wedge d^{\perp}$ and $c \leq b \wedge d^{\perp}$. Then $c \leq a$ and $c \leq b$. It follows that $c \leq d$. On the other hand, the inequality $c \leq a \wedge d^{\perp}$ implies $c \leq d^{\perp}$. Since $c \leq d$ and $c \leq d^{\perp}$, we see that $c = 0$ and the implication (1) $\Rightarrow$ (2) is proved.

Let us take up the reverse implication (2) $\Rightarrow$ (1). Suppose that $a \wedge b = 0$. By the assumption, $a\, C\, b$. Then $a = (a \wedge b) \vee (a \wedge b^{\perp})$. Since $a \wedge b = 0$, we have $a = a \wedge b^{\perp}$, and therefore $a \leq b^{\perp}$. □

Let $P$ be an ODP. In accord with our convention that all notions of $P_{\text{supp}}$ can be attributed to the original $P$, let us say that $P$ is *pseudocomplemented* if so is $P_{\text{supp}}$. We want to prove that the pseudocomplemented ODPs are set-representable. Before doing so, let us introduce a few notions of ordered sets.

Let $P$ be a poset with the least and the greatest element, 0 and 1. Let us agree to write, for $A \subseteq P$, $A^{\wedge} = \{x \in P;\ \forall y \in A:\ y \leq x\}$, $A^{\vee} = \{x \in P;\ \forall y \in A:\ x \leq y\}$.

**Definition 6.3** Suppose that $I \subseteq P$. We say that I is a *Frink ideal* in $P$ if for any finite subset $J \subseteq I$ we have $J^{\wedge\vee} \subseteq I$ (as usual, $J^{\wedge\vee}$ stands for $(J^{\wedge})^{\vee}$). We say that $I$ is proper if $I \neq P$ (equivalently, $1 \notin I$).

It should be noted that a Frink ideal, $I$, in $P$ has the standard properties of an ideal in a poset: $0 \in I$, and for any $x, y \in P$ such that $x \in I$, $y \leq x$ we have $y \in I$. Moreover, a Frink ideal in an ODP respects the operation $\triangle$ as follows.

**Proposition 6.4** *Let $P$ be an ODP and let $I$ be a Frink ideal in $P$. Then for any pair of elements $x, y \in I$ we have $x \triangle y \in I$.*

*Proof* By the condition ($D_3$) of the Definition 2.1, $x \triangle y \in \{x, y\}^{\wedge\vee} \subseteq I$, which proves the result. □

The following simple proposition gives us an information on how a set generates a Frink ideal.

**Proposition 6.5** *Let $P$ be an OMP. Let $A \subseteq P$. Then the smallest Frink ideal that contains $A$ is the following union $I_A$:*

$$I_A = \bigcup_{X \subseteq A,\ X finite} X^{\wedge\vee} .$$

*Further, let $I$ be a Frink ideal in $P$ and let $a \in P$. Then the smallest Frink ideal that contains $I$ and the element $a$ is the set $I[a]$, where*

$$I[a] = \bigcup_{X \subseteq I,\ X finite} (X \cup \{a\})^{\wedge\vee} .$$

**Proposition 6.6** *Let $P$ be a pseudocomplemented OMP, let $I$ be a Frink ideal in $P$ and let $a \in P$. Then the following statements are equivalent:*

(1) *there is a proper Frink ideal $J$ in $P$ such that $I \subseteq J$ and $a \in J$,*
(2) *$a^\perp \notin I$.*

*Proof*

(1) $\Rightarrow$ (2) Suppose that $I \cup \{a\} \subseteq J$. Further, suppose that $a^\perp \in I$. Then $a, a^\perp \in J$. By the definition of Frink ideal, we see that $\{a, a^\perp\}^{\wedge\vee} \subseteq J$. But $\{a, a^\perp\}^{\wedge\vee} = (\{a, a^\perp\}^\wedge)^\vee = \{1_P\}^\vee = P$. This means that $J = P$ and therefore $J$ is not proper.

(2) $\Rightarrow$ (1) Suppose that $a^\perp \notin I$. Let us set $J = I[a]$. We want to show that $1 \notin J$. By Proposition 6.5, it suffices to show that $1 \notin (X \cup \{a\})^{\wedge\vee}$ for any finite $X \subseteq I$. Take an arbitrary finite $X \subseteq I$. Since $X^{\wedge\vee} \subseteq I$, $a^\perp \notin I$, we infer that $a^\perp \notin X^{\wedge\vee}$. It follows that there is an element $b \in X^\wedge$ with $a^\perp \not\leq b$. The definition of pseudocomplementarity gives us that $0 \neq \inf_P\{a^\perp, b^\perp\}$. Hence, there is an element $c \in P$ such that $c \leq a^\perp$ and $c \leq b^\perp$ and, moreover, $c > 0$. This implies that $a \leq c^\perp$, $b \leq c^\perp$ and $c^\perp < 1$. We conclude that $c^\perp$ is an upper bound of $X \cup \{a\}$. But $c^\perp < 1$, which implies $1 \notin (X \cup \{a\})^{\wedge\vee}$ and the proof is complete. □

**Theorem 6.7** *Let $P$ be a pseudocomplemented OMP. Let $I \subseteq P$ be a maximal proper Frink ideal in $P$. Then $I$ has the selectivity property: For any $x \in P$ either $x \in I$ or $x^\perp \in I$.*

*Proof* Let $a \in P$. Suppose that $a^\perp \notin I$. We want to show that $a \in I$. By Proposition 6.6, the set $I[a]$ is a proper Frink ideal. Since $I \subseteq I[a]$ and since $I$ is a maximal, we obtain $I = I[a]$. Thus, $a \in I$, which was to show. □

**Theorem 6.8** *Each pseudocomplemented ODP is set-representable.*

*Proof* Let $P$ be a pseudocomplemented ODP. Let $a, b \in P$ with $a \not\leq b$. Since $a \not\leq b$, it follows that $a \neq 0$. Thus, $a^\perp \neq 1$. This implies that the set $I = \{x \in P;\ x \leq a^\perp\}$ is a proper Frink ideal in $P$. Moreover, $b^\perp \notin I$. By Proposition 6.6 and an obvious application of Zorn's lemma, we obtain that there is a maximal proper Frink ideal $J$ in $P$ such that $I \cup \{b\} \subseteq J$. Let us define a mapping $e : P \to \{0, 1\}$ as follows: If $x \in J$ then $e(x) = 0$, $e(x) = 1$ otherwise. Let us show that this mapping $e$ is an evaluation on $P$. The property ($E_1$) is implied by the fact that $1 \notin J$. Further, suppose that $x, y \in P$ with $x \leq y$. If $y \in J$, then $x \in J$, and therefore $e(x) = e(y) = 0$. If $y \notin J$, then $e(y) = 1$ and the inequality $e(x) \leq e(y)$ is automatically valid. Finally, suppose that $x, y \in P$. Let us try to show that $e(x \triangle y) = e(x) \oplus e(y)$. We have to distinguish three possibilities.

- Firstly, suppose that $x, y \in J$. Then $x \triangle y \in J$ (Proposition 6.4) and the inequality holds true.
- Secondly, suppose that exactly one of the elements $x, y$ lies in $J$. Without a loss of generality, let $x \in J$ and $y \notin J$. If $x \triangle y \in J$, then $x \triangle (x \triangle y) \in J$ and therefore $y \in J$. But this is absurd. It follows that $x \triangle y \notin J$. As a result, $e(x) = 0$, $e(y) = 1$ and $e(x \triangle y) = 1$.
- Thirdly, $x \notin J$ and $y \notin J$. By Theorem 6.7, we see that $x^{\perp} \in J$ and $y^{\perp} \in J$. This implies that $x^{\perp} \triangle y^{\perp} \in J$. But $x^{\perp} \triangle y^{\perp} = x \triangle y$. We therefore see that $e(x) = 1$, $e(y) = 1$ and $e(x \triangle y) = 0$.

This shows that $e$ is an evaluation. To complete the proof, we observe that $a^{\perp}$, $b \in J$, and therefore $e(a^{\perp}) = 0$, $e(b) = 0$. In other words, $e(a) = 1$ and the proof is done. □

In concluding this paragraph, let us see that there are non-Boolean pseudocomplemented ODPs and that there are set-representable ODPs which are not pseudocomplemented. Putting it in interplay with the results obtained above, let us formulate the result in the following manner.

**Proposition 6.9** *The following inclusions are proper: $\mathcal{BA} \subset$ pseudocomplemented ODPs $\subset \mathcal{SRODP}$.*

*Proof* The first inclusion is obvious, the second follows from Theorem 6.8. Obviously, $MO_3$ is a SRODP which is not pseudocomplemented. It remains to construct a pseudocomplemented ODP which is not Boolean. Let $N$ denote the set of all natural numbers. For $i \in \{0, 1, 2, 3, 4, 5\}$, put $N_i = \{n \in N;\ n = 6k + i$ for some natural $k\}$. Further, let us put $A_1 = N_0 \cup N_2 \cup N_4$ (i.e., $A_1$ is the set of all even numbers), $A_2 = N_0 \cup N_3$, $A_3 = A_1 \triangle A_2 = N_2 \cup N_3 \cup N_4$. For $m \in \{1, 2, 3\}$ put $B_m = N \setminus A_m$. Consider $\mathcal{A} = \{\emptyset, A_1, B_1, A_2, B_2, A_3, B_3, N\}$. Then the set $\mathcal{A}$ is a d-subring in $\exp(N)$. Going on with the construction put $\Omega = \{X \subseteq N;\ X \triangle D$ is finite for some $D \in \mathcal{A}\}$. Then $\Omega$ is again a d-subring. We will show that $P = (\Omega, \subseteq, \emptyset, N, {}^c, \triangle)$, where $X^c = N \setminus X$, is a pseudocomplemented ODP that is not Boolean.

Suppose that $X, Y \in \Omega$ such that $\inf\{X, Y\} = 0 = \emptyset$. Suppose that $X \cap Y \neq \emptyset$ and look for a contradiction. Pick an element $a \in X \cap Y$ and put $A = \{a\}$. Then $A \in \Omega$. Moreover, $A \subseteq X$ and $A \subseteq Y$. But this contradicts the condition that $\inf\{X, Y\} = \emptyset$. As a result, $X \cap Y = \emptyset$, and therefore $X \subseteq Y^c$.

In order to show that $P$ is not Boolean, it suffices to prove that the elements $A_1, A_2$ do not possess an infimum in $P$. Take a set $X \in \Omega$ such that $X \subseteq A_1$, $X \subseteq A_2$. If $X$ were infinite, then $X \triangle D$ is also infinite for any $D \in \mathcal{D}$. But this would imply that $X \notin \Omega$. We therefore see that $X$ is finite. Choose a finite set $Y$ such that $X \subset Y \subset N_0$. Then $Y \in \Omega$ and $Y \subseteq A_1$, $Y \subseteq A_2$. Since $Y$ is strictly larger then $X$, the set $X$ cannot be the infimum of $A_1, A_2$ in $P$ and this completes the proof. □

## 7 The OMPs not Embeddable into ODPs

In this paragraph we ask if (when) an OMP is induced by an ODP. The very first question one would pose is whether each OMP is a $P_{\text{supp}}$ for some ODP $P$. Obviously, this question answers to the negative if only because of Proposition 3.3. A more natural and more challenging question is whether each OMP is a *sub*-OMP of some $P_{\text{supp}}$ for an ODP $P$. This question answers to the negative, too, but with a slightly less trivial argument. This argument is contained in Proposition 7.1 that follows. To formulate it, we need a definition.

**Definition 7.1** Let $P_1$, $P_2$ be orthomodular posets and let $f : P_1 \to P_2$ be a mapping. Then $f$ is said to be an *orthomorphism* if

(1) $f(1_{P_1}) = 1_{P_2}$,
(2) $f(x^{\perp}) = (f(x))^{\perp}$ for any $x \in P_1$,
(3) $f(x \vee y) = f(x) \vee f(y)$ for any pair of orthogonal elements $x, y \in P_1$.

Note that if $f$ is an orthomorphism, then the equation in the above condition (3) can be generalized to any finite number of mutually orthogonal elements. Note also that (a) if $K$, $L$ are lattice OMPs and $f : K \to L$ is a lattice homomorphism then $f$ is an orthomorphism, and (b) if $P$, $Q$ are ODPs and $f : P \to Q$ is a homomorphism then $f$ is an orthomorphism as mapping between $P_{\text{supp}}$ and $Q_{\text{supp}}$.

**Proposition 7.2** *Let $P$ be a finite OMP. Let $P$ possess an odd number of blocks and let each atom of $P$ lie precisely in two blocks. If $Q$ is an ODP which consists of at least two elements, then there is no orthomorphism $P \to Q_{\text{supp}}$. A corollary: Any OMP with the assumptions of this proposition is not embeddable into an ODP.*

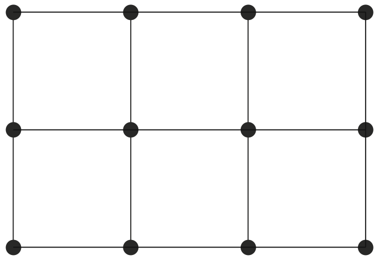
*Proof* Seeking a contradiction, let $f : P \to Q_{\text{supp}}$ be an orthomorphism. Suppose that $B_1, \ldots, B_n$ are all blocks of $P$. By our assumption the number $n$ is odd. Let $\{a_{i,1}, \ldots, a_{i,k_i}\}$ be the set of all atoms of the algebra $B_i$, $i = 1, \ldots, n$. Then the elements $a_{i,1}, \ldots, a_{i,k_i}$ are mutually orthogonal and, moreover, $a_{i,1} \vee \ldots \vee a_{i,k_i} = 1_P$. Since $f$ is an orthomorphism, we have $f(a_{i,1}) \vee \ldots \vee f(a_{i,k_i}) = 1_Q$. Again, the elements $f(a_{i,1}), \ldots, f(a_{i,k_i})$ are mutually orthogonal, and therefore $f(a_{i,1}) \vee \ldots \vee f(a_{i,k_i}) = f(a_{i,1}) \triangle \ldots \triangle f(a_{i,k_i})$ (see Proposition 2.8). This shows that $f(a_{i,1}) \triangle \ldots \triangle f(a_{i,k_i}) = 1_Q$ for any $i \in \{1, \ldots, n\}$. As a consequence,

$$(f(a_{1,1}) \triangle \ldots \triangle f(a_{1,k_1})) \triangle \ldots \triangle (f(a_{n,1}) \triangle \ldots \triangle f(a_{n,k_n})) = 1_Q \triangle \ldots \triangle 1_Q.$$

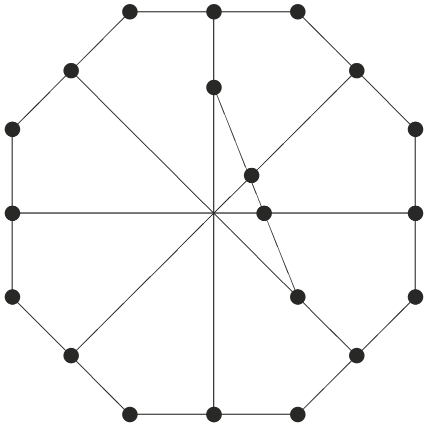
The right-hand side of the latter identity contains the element $1_Q$ exactly $n$-many times. Since $n$ is odd, the right-hand side equals to $1_Q$. Moreover, if $a$ is an arbitrary atom of $P$, then our assumption gives us that the left-hand side of the identity contains the expression $f(a)$ exactly two times. By the property of the operation $\triangle$, the left-hand side must be equal to $0_Q$. Since $Q$ is non-trivial, $0_Q$ is distinct from $1_Q$, and we have derived a contradiction. □

With the help of the Greechie paste job ([11, 22]) it is not difficult to construct OMPs with the assumptions of Proposition 7.2. Thus, in figure (a) below we present the simplest OMP non-embeddable in an ODP. In figure (b) we present a lattice

OMP non-embeddable in a ODP (this provides a partial answer to a problem formulated in [17]).



(a)



(b)

The characterisation of the OMPs embeddable in ODPs–a problem of some importance within quantum logic theory–seems open. A step forward in understanding this problem and some further link with quantum theories would be the answer to the question of whether or not the projection OMP $\mathcal{L}(R^3)$ is embeddable in an ODP. Even this remains open to us for the time being though we would rather conjecture that such an embedding of $\mathcal{L}(R^3)$ into an ODP exists.

## References


1. Beran, L.: Orthomodular Lattices, Algebraic Approach. D. Reidel, Dordrecht (1985)
2. Bruns, G., Greechie, R.: Blocks and commutators in othomodular lattices. Algebra Univers. **27**, 1–9 (1990)
3. Bruns, G., Harding, J.: Algebraic aspects of orthomodular lattices. In: Coecke, B., Moore, D., Wilce, A. (eds.) Current Research in Operational Quantum Logic, pp. 37–65. Springer, New York (2000)
4. Burris, S., Sankappanavar, H.P.: A Course in Universal Algebra. Springer, New York (1981)
5. Chang, C.C., Keisler, H.J.: Model Theory. North-Holland, Amsterdam (1973)
6. Dalla Chiara, M.L., Giuntini, R., Greechie, R.: Reasoning in Quantum Theory: Sharp and Unsharp Quantum Logics. Kluwer, Dordrecht (2004)
7. Dorfer, G., Dvurečenskij, A., Länger, H.M.: Symmetric difference in orthomodular lattices. Math. Slovaca **46**, 435–444 (1996)
8. Dorfer, G.: Non-commutative symmetric differences in orthomodular lattices. Int. J. Theor. Phys. **44**, 885–896 (2005)
9. Dvurečenskij, A., Pulmannová, S.: New Trends in Quantum Structures. Kluwer, Dordrecht, and Ister Science, Bratislava (2000)
10. Godowski, R.: States on orthomodular lattices. Demonstr. Math. **XV**(3), 817–822 (1982)
11. Greechie, R.J.: Orthomodular lattices admitting no states. J. Combinat. Theory **10**, 119–132 (1971)
12. Engesser, K., Gabbay, D.M., Lehmann, D. (eds.): Handbook of Quantum Logic and Quantum Structures. Elsevier, Amsterdam (2007)
13. Gudder, S.P.: Stochastic Methods in Quantum Mechanics. Dover, Amsterdam (1979)
14. Kalmbach, G.: Orthomodular Lattices. Academic Press, London (1983)

15. Klukowski, J.: On the representations of Boolean orthomodular partially ordered sets. Demonstr. Math. **8**, 405–423 (1975)
16. MacLane, S., Birkhoff, G.: Algebra. Macmillan, New York (1968)
17. Matoušek, M.: Orthocomplemented lattices with a symmetric difference. Algebra Univers. (2008, in press)
18. Mayet, R., Pták, P.: Quasivarieties of OML determined by conditions on states. Algebra Univers. **42**, 155–164 (1999)
19. Mayet, R., Pták, P.: Orthomodular lattices with state-separated noncompatible pairs. Czechoslov. Math. J. **50**(125), 359–366 (2000)
20. Navara, M., Pták, P.: Almost Boolean orthomodular posets. J. Pure Appl. Algebra **60**, 105–111 (1989)
21. Park, E., Kim, M.M., Chung, J.Y.: A note on symmetric differences of orthomodular lattices. Commun. Korean Math. Soc. **18**(2), 207–214 (2003)
22. Pták, P., Pulmannová, S.: Orthomodular Structures as Quantum Logics. Kluwer, Dordrecht (1991)
23. Tkadlec, J.: Partially additive measures and set representations of orthoposets. J. Pure Appl. Algebra **86**, 79–94 (1993)
24. Varadarajan, V.S.: Geometry of Quantum Theory I, II. Van Nostrand, Princeton (1970)