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DISERTAČNÍ PRÁCE

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Množinově-teoretické metody v teorii modulů

Katedra algebry

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Mé poděkování patří především školiteli prof. Janu Trlifajovi, bez jehož podpory a důvěry by tato práce nejspíš nikdy nevznikla. Velice děkuji rovněž spoluautorovi jednoho z článků, Honzovi Šťovíčkovi, za všechny ty hodiny strávené inspirativními diskusemi nad různými algebraickými problémy. Přátelům a rodině děkuji za to, že to se mnou doposud vydrželi bez vážné újmy na zdraví.

Prohlašuji, že jsem svou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 23. 7. 2010 Jan Šaroch

Contents

Introduction **4**

1 On the telescope conjecture for module categories **13**

2 The countable telescope conjecture for module categories **29**

3 Kaplansky classes, finite character, and $\aleph_1$–projectivity **55**

Název práce: Množinově-teoretické metody v teorii modulů
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Abstrakt: Disertační práce shrnuje mé dosavadní příspěvky k teorii kotorzních párů modulů, s užším zaměřením na aplikaci množinově-teoretických metod v této oblasti. Sestává z úvodu a tří článků se spoluautory. První dva — již publikované — se věnují tzv. teleskopické hypotéze pro kategorie modulů. Dokazujeme zde například, že dědičný kotorzní pár $(\mathcal{A}, \mathcal{B})$ s třídou $\mathcal{B}$ uzavřenou na direktní limity je již úplný a generovaný množinou spočetně prezentovaných modulů. Je-li navíc i třída $\mathcal{A}$ uzavřena na direktní limity, je $(\mathcal{A}, \mathcal{B})$ kogenerován množinou nerozložitelných čistě-injektivních modulů. Ve třetím článku se blíže zabýváme jednak kotorzními páry, jež poskytují netriviální příklady abstraktních elementárních tříd (v Shelahově smyslu), a dále zkoumáme třídu $\mathcal{D}$ všech $\aleph_1$-projektivních modulů, přicemž kupříkladu ukazujeme, že se — nezávisle na okruhu — jedná vždy o Kaplanského třídu.
Klíčová slova: kotorzní pár, teleskopická hypotéza, množinově-teoretické metody, abstraktní elementární třída, $\aleph_1$-projektivní modul

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Abstract: The thesis collects my actual contributions to the theory of cotorsion pairs, with closer attention paid to the application of set-theoretic methods in this area. It consists of an introduction and three papers with coauthors. The first two, already published, deal with the so-called Telescope Conjecture for Module Categories. We prove here, for instance, that a hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ with the class $\mathcal{B}$ closed under direct limits is generated by a set of countably presented modules. Moreover, if the class $\mathcal{A}$ is closed under direct limits too, then the pair $(\mathcal{A}, \mathcal{B})$ is cogenerated by a set of indecomposable pure-injective modules. In the third paper, we deal with the cotorsion pairs which provide us with non-trivial examples of abstract elementary classes (in the sense of Shelah). Then we study the class $\mathcal{D}$ of all $\aleph_1$-projective modules, proving e.g. that—regardless of the ring—it always forms a Kaplansky class.
Keywords: cotorsion pair, Telescope Conjecture, set-theoretic methods, abstract elementary class, $\aleph_1$-projective module

Introduction

The thesis consists of this Introduction and three papers I am a coauthor of. Two of them are already published, the last one was submitted to Forum Mathematicum in January 2010:

(1) L. Angeleri Hügel, J. Šaroch and J. Trlifaj, *On the telescope conjecture for module categories*, J. Pure Appl. Algebra **212** (2008), 297–310.

(2) J. Šaroch and J. Šťovíček, *The countable telescope conjecture for module categories*, Adv. Math. **219** (2008), 1002–1036.

(3) J. Šaroch and J. Trlifaj, *Kaplansky classes, finite character, and $\aleph_1$–projectivity*, preprint (2010).

In more detail, the contribution of myself in the papers above is as follows. In (1), the vast majority of material of the fourth section including the main results—Theorem 4.6 and Theorem 4.10. In (2), all sections except the fourth and the sixth. In (3), concerning the first section, slight generalization of Proposition 1.7 and (consequently) Theorem 1.8; in the second section, everything up until Theorem 2.9.

0.1 Set-theoretic methods

Until the beginning of seventies, the tools of set theory appearing in papers dealing with ring and module theory reduced more or less to using Zorn's lemma or one of its equivalents. Although the classical results obtained (e.g. the existence and uniqueness of the algebraic closure of a field; the existence of maximal ideals in commutative unital rings; the existence of a basis in a vector space) have been of essential importance in modern algebra, it had not convinced a majority of algebraic community to turn their attention more closely to other basic concepts of set theory (stationary sets, almost disjoint systems, properties of singular cardinals etc.).

However, when Saharon Shelah published ([22]) his solution to the Whitehead problem, i.e. a description of the class of all abelian groups A with $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0$ (these are called Whitehead groups), it became obvious that, in algebra, one would need to count much more with the set theory. Not only in the sense that its tools and methods proved themselves to be strong and fruitful, but even in the awkward sense that the answers to natural questions arising in, say, homological algebra might depend on the set theory one is working in. Recall that accepting the Axiom of Constructibility, every Whitehead group is free, however there are (relatively consistent) extensions of ZFC allowing non-free Whitehead groups.

It was in this work by Shelah, where, for the first time, the full-bodied singular compactness argument appeared (only particular cases of which had been known, due to Hill, see [18] and [19]). It has turned out to be a powerful tool for proving generalized freeness conditions for classes of modules. Vaguely said, if you are proving certain type of statement by transfinite induction on the cardinality or the number of generators of a module, you get the singular step for free if you did slightly harder on the regular rungs below. A simplified version of the Shelah's result together with a typical application is presented in (2). For a more detailed description of applications of the argument in module theory, see [6].

Apart from Shelah's singular compactness and some fine using of stationary sets and infinitary combinatorial principles in his Whitehead-related papers, another important set-theoretic argument appeared in algebra during 70s. It was in the work by Hunter ([20]), where he used for the first time completely non-algebraic, cardinal argument to prove that some *Ext* group vanishes. His technique was later successfully adopted to prove various other results involving vanishing of *Ext*, notably it played the crucial role in the proof that all n-cotilting modules are pure-injective ([2] and [26]). We will illustrate its strength at the end of this introduction.

In the 1980s, set-theoretic methods firmly settled in the theory of modules, most notably in works of Eklof, Mekler, Shelah and Göbel (e.g. [23], [8], [13] and the monograph [7]). The fundamental paper by Ziegler from 1984 ([31]) draw the attention to the possibility of studying modules over a fixed ring by means of the first-order logic. Thereafter, not only set-theoretic tools but also the ones from the model theory slowly emerged in the theory of modules. However, the model theory of modules with its first-order language is far from being able to answer all algebraically relevant questions. The point is that there are many interesting classes of modules which are worth studying but not first-order axiomatisable (in the language of modules over a ring), for example: the class of all injective modules over a non-noetherian ring, or the class of all pure-injective modules over a ring which is not pure-semisimple. In some cases, another Shelah's invention, from 1987 ([24]), called *abstract elementary classes* might be a useful generalization of the classical model theory, but even this logical concept does not have enough strength for algebraic purposes. The way it emerges in the approximation theory of modules (see [1] and (3)) is pretty interesting though, from both, logical and algebraic, points of view.

This short historical introduction is not meant at all to be exhaustive. It is just a draft which should justify the object of this thesis—to study set-theoretic methods and their applications to the theory of modules. To conclude, let us mention that these methods allowed to prove, at the break of the century, the existence of a flat cover of any module over any ring ([5]). This result has boosted the study of approximation properties of classes of modules by means of cotorsion pairs, a concept which had appeared a long time ago in [21]. The recent knowledge concerning the approximation theory of modules together with various results from another fruitful field of algebra for set-theoretic methods—realization theorems—is collected in the monograph [16].

It should be noted, that the relation between set theory, or mathematical logic in general, and algebra is not a one-way one. There are several examples of non-trivial algebraic tools being successfully used to solve problems in logic. Recently, for instance, deep results from algebraic number theory (class field theory) were applied by Poonen and Koenigsmann to prove that $\mathbb{Z}$ is definable by a universal formula (in the language of rings) in $\mathbb{Q}$, thus improving the classic result by Julia Robinson. However, it is not an object of this thesis to deal with this, reciprocal, application of algebra in logic.

0.2 Cotorsion pairs — preliminaries

Now, let us recall in more detail some basic definitions and properties concerning cotorsion pairs. They are almost omnipresent in the three papers presented in this thesis, and it is in the context of cotorsion pairs and their deconstruction where several applications of set-theoretic methods are shown.

Let R be a (unital) ring and denote by Mod-R the category of all (right R-)modules. Further by mod-R, we denote the class of all modules possessing a projective resolution consisting of finitely generated projective modules. Assuming R is right coherent, it is just the class of all finitely presented modules.

For a class $\mathcal{C}$ of R-modules, let $\mathcal{C}^\perp = \{M \in \text{Mod-}R \mid \text{Ext}^1_R(C, M) = 0 \text{ for all } C \in \mathcal{C}\}$. Similarly, ${}^\perp\mathcal{C} = \{M \in \text{Mod-}R \mid \text{Ext}^1_R(M, C) = 0 \text{ for all } C \in \mathcal{C}\}$. Recall that for $A, B \in$ Mod-R, $\text{Ext}^1_R(A, B) = 0$ iff any short exact sequence of the form $0 \to B \to C \to A \to 0$ splits. For example, if $R = \mathbb{Z}$, then ${}^\perp\{\mathbb{Z}\}$ is the class of all Whitehead groups, already mentioned in the historical overview.

We say, that a pair $(\mathcal{A}, \mathcal{B})$ of classes of modules is *a cotorsion pair* provided that $\mathcal{A} = {}^\perp\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$. Otherwise said, $(\mathcal{A}, \mathcal{B})$ is a $\subseteq$-maximal pairwise Ext^1-orthogonal pair of classes of modules.

Indeed, for any class $\mathcal{C}$ of modules, one has $\mathcal{C} \subseteq {}^\perp(\mathcal{C}^\perp)$ and $\mathcal{C} \subseteq ({}^\perp\mathcal{C})^\perp$. Moreover, every class of modules $\mathcal{C}$ determines two distinguished cotorsion pairs—the cotorsion pair *generated* by $\mathcal{C}$, that is $({}^\perp(\mathcal{C}^\perp), \mathcal{C}^\perp)$, and the cotorsion pair *cogenerated* by $\mathcal{C}$—the one equal to $({}^\perp\mathcal{C}, ({}^\perp\mathcal{C})^\perp)$. So any cotorsion pair is generated by its left-hand class and cogenerated by its right-hand class. In applications, one usually tries to find the minimal possible subclass (or even a subset) which (co)generates the respective cotorsion pair.

The close relation between cotorsion pairs and approximation theory of modules has been established by the result of Eklof and Trlifaj from [11] saying that any cotorsion pair $(\mathcal{A}, \mathcal{B})$ generated by a set is *complete* which means that $\mathcal{A}$ is *a special precovering class*, or equivalently $\mathcal{B}$ is *a special preenveloping class*, that is: for every $M \in$ Mod-R, there is a short exact sequence $0 \to B \to A \to M \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and a short exact sequence $0 \to M \to B' \to A' \to 0$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$, respectively. The map $A \to M$ is then called *a special $\mathcal{A}$-precover* of M, because any homomorphism from a module in $\mathcal{A}$ to M can be factorized through this map. Dually, the map $M \to B'$ is called *a special $\mathcal{B}$-preenvelope*. It could happen that there exist special precovers, called *covers*, which are minimal in the following sense: if $f : A \to M$ is a special $\mathcal{A}$-precover of M, then it is an $\mathcal{A}$-cover provided that any endomorphism g of A with $f = fg$ is actually an automorphism. Dually, $\mathcal{B}$-envelopes are defined.

Finally, we say that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *hereditary* if $\mathcal{A}$ is closed under taking kernels of epimorphisms, or equivalently $\mathcal{B}$ is closed under taking cokernels of monomorphisms.

Now, we sum up several properties of cotorsion pairs; from the basic ones to the more recent results.

Proposition 1. *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair of right R-modules. Then the following holds:*

(i) *$\mathcal{A}$ and $\mathcal{B}$ are closed under extensions.*

(ii) *$\mathcal{A}$ is closed under arbitrary direct sums, and $\mathcal{B}$ is closed under arbitrary products.*

(iii) *If $\mathfrak{C}$ is cogenerated by a class of pure-injective modules, then it is generated by a set ([11]).*

(iv) *If $\mathfrak{C}$ is generated by a set of modules, then $\mathfrak{C}$ is complete ([12]).*

(v) *If $\mathcal{A}$ is closed under direct limits and $\mathfrak{C}$ is complete, then every module has an $\mathcal{A}$-cover and a $\mathcal{B}$-envelope ([30]).*

(vi) *If $\mathcal{A}$ and $\mathcal{B}$ are both closed under direct limits and $\mathfrak{C}$ is complete, then $\mathfrak{C}$ is cogenerated by a set of indecomposable pure-injective modules (in (2)).*

Let $\mathfrak{C}$ be hereditary. Then we have also:

(vii) *If $\mathcal{B}$ is closed under unions of well-ordered chains, then $\mathfrak{C}$ is complete, generated by a set of countably presented modules, and $\mathcal{B}$ is first-order axiomatizable (in (2)).*

(viii) *If $\mathcal{A}$ consists of modules of finite projective dimension and $\mathcal{B}$ is closed under direct sums, then $\mathfrak{C}$ is generated by a set of modules from* mod-R *([28], [4]).*

(ix) *If $\mathcal{B}$ consists of modules of finite injective dimension and $\mathcal{A}$ is closed under direct products, then $\mathfrak{C}$ is cogenerated by a set of pure-injective modules ([3], [26], [25]).*

The reader could ask in what places the set-theoretic methods mentioned play their important role. For example, the proof of (ix) relies on Hunter's argument and a model-theoretic description of pure-injective modules. Stationary sets and Shelah's compactness are used in the proofs of (vii) and (viii). The proof of (vi) involves a tree construction on finite sequences of ordinal numbers, and also tools and results from the model theory of modules are heavily used. In most cases, set-theoretic methods help when one wants to show that some module belongs to $\mathcal{A}$, or even that it can be build up (i.e. filtered, see (1) or (2) for the definition) from sufficiently small modules belonging to $\mathcal{A}$—this process is called *the deconstruction of a cotorsion pair*.

We conclude this subsection with some basic examples of cotorsion pairs. We denote by $\mathcal{P}_0, \mathcal{I}_0$ and $\mathcal{F}_0$ the classes of all projective, injective and flat (i.e. direct limits of projective) modules, respectively. Further the class $\mathcal{F}_0^{\perp}$ of all Enochs cotorsion modules will be denoted by $\mathcal{EC}$.

Examples.

(i) The trivial cotorsion pairs $(\mathcal{P}_0, \text{Mod-}R)$ and $(\text{Mod-}R, \mathcal{I}_0)$.

The first of them is generated by $\{R\}$, and by classic results of Chase and Bass, we have that $\mathcal{P}_0$ is closed under arbitrary products iff R is a left coherent ring, and $\mathcal{P}_0$ is closed under direct limits iff $\mathcal{P}_0 = \mathcal{F}_0$ iff R is right perfect iff every module has a projective cover.

In the latter pair, we have that $\mathcal{I}_0$ is closed under direct limits iff $\mathcal{I}_0$ is first-order axiomatizable iff R is right noetherian (cf. [9]).

(ii) The cotorsion pair $(\mathcal{F}_0, \mathcal{EC})$.

It is cogenerated by the class of all pure-injective modules, so complete by (iii) and (iv) from Proposition 1. From (v), it follows the existence of flat covers and cotorsion envelopes.

(iii) Over a semisimple artinian ring R, there is, indeed, only one cotorsion pair—the trivial one. However, if R is a non-right perfect ring, then there is a proper class of cotorsion pairs over R. The consistence of this fact follows from [29] where it is shown that there is an extension of ZFC such that no module (over a non-right perfect ring) tests projectivity, i.e. for any N there exists a non-projective module M such that $\text{Ext}^1(M, N) = 0$. The proof in ZFC follows, for example, from the non-deconstructibility of the class of all $\aleph_1$-projective modules (see [17]) over non-right perfect rings. (For abelian groups, it followed already from [15].)

(iv) There is no known example of a cotorsion pair which would not be generated by a set, in ZFC. From the Shelah's papers on Whitehead groups, only consistence results follow: there is an extension of ZFC such that the cotorsion pair (over a ring $\mathbb{Z}$) cogenerated by $\{\mathbb{Z}\}$ is not even complete ([10], it is shown that $\mathbb{Q}$ has no Whitehead precover); and so it is not generated by a set, by (iv) from Proposition 1. It has to be noted however that the direct proof of the consistency of the latter conclusion is much more easier than showing that there may not be any Whitehead precover of $\mathbb{Q}$.

0.3 Some remarks to the papers

The three papers which form the main part of this thesis are presented with unchanged content, as they were published (the first two of them) or submitted for publishing (the case of (3)). Only the amount of pages may differ because of slightly different formatting and font type used in the journals and in this work.

As for the material contained, the papers (1) and (2) share the same subject—the Telescope Conjecture for Module Categories, (1) being sort of a prologue for (2). Also being the oldest one, a segment from (1) appeared already in the author's diploma thesis; and unlike in the two subsequent papers and this introduction, the meaning of the terms *generated* and *cogenerated* (talking about cotorsion pairs) is swapped there.

The Telescope Conjecture for Module Categories remains unsolved. The only recent progress in this field is a partial positive solution [27] for Artin algebras with zero transfinite radical due to Šťovíček.

The content of (3) differs from the first two papers. Abstract elementary classes and $\aleph_1$-projective modules are studied. It is a continuation of the work begun in [1] and [17].

In (3), arguably one of the first applications of the singular cardinal hypothesis (SCH) in algebra appears. However, it turned out shortly after submission of the paper that this additional set-theoretic assumption is actually redundant. This follows from the recent manuscript by Bazzoni and Šťovíček. We state the result and sketch the proof here. It will serve as the promised application of Hunter's argument.

In what follows, $\mathcal{D}$ denotes the class of all $\aleph_1$-projective modules, that is the modules M with the property that each countable subset of M is contained in a countably generated projective submodule of M which is pure in M. Whenever $\mathcal{P}_0 \neq \mathcal{F}_0$, i.e. R is not a right perfect ring, we have the strict inclusions $\mathcal{P}_0 \subsetneq \mathcal{D} \subsetneq \mathcal{F}_0$. It means that $\mathcal{D}$ is not closed under direct limits. Nevertheless Bazzoni and Šťovíček managed to prove, in ZFC, that countable direct limits of modules in $\mathcal{D}$ belong to ${}^\perp(\mathcal{D}^\perp)$; by the way, from this, it already follows that $\mathcal{D} \neq {}^\perp(\mathcal{D}^\perp)$, and so $\mathcal{D}$ does not form a left-hand class of a cotorsion pair (over a non-right perfect ring). To be able to profit from the Hunter's argument, they needed the following simple combinatorial fact from set theory.

Lemma 2. *For every cardinal μ, there exists an infinite cardinal $\lambda \geq \mu$ and $J \subseteq \lambda^\omega$ of cardinality 2^λ such that the values of any pair of distinct maps $f, g : \omega \to \lambda$ from J coincide on a finite segment of ω.*

The promised result with a sketch of the proof follows.

Theorem 3. *Let R be a ring and $\mathcal{D}$ be the class of all $\aleph_1$-projective right R-modules. Given any countable chain*

$$F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow \cdots$$

of homomorphisms such that $F_n \in \mathcal{D}$ for all $n < \omega$, we have $\varinjlim F_n \in {}^\perp(\mathcal{D}^\perp)$.

Sketch of the proof. Put $F = \varinjlim F_n$ and fix a module $C \in \mathcal{D}^\perp$. We must show that $\operatorname{Ext}^1_R(F, C) = 0$.

Lemma 2 provides us with an infinite λ such that $\lambda \geq |\operatorname{Hom}_R(F_n, C)|$, for each $n < \omega$, and with $J \subseteq \lambda^\omega$ with the appropriate properties. We denote, for each ordinal $\alpha < \lambda$, by $F_{n,\alpha}$ a copy of F_n, and by P the direct sum $\bigoplus_{n<\omega, f\in J} F_{n,f(n)}$.

Cleverly using the almost disjoint property of the set J of maps, and some pushout construction, Bazzoni and Šťovíček obtained a short exact sequence

$$0 \longrightarrow P \longrightarrow E \longrightarrow F^{(2^\lambda)} \longrightarrow 0 \qquad (\dagger)$$

such that $E \in \mathcal{D}$. We skip this rather technical step here.

At this point, Hunter's argument is applied. Using the functor $\operatorname{Hom}_R(-, C)$ on the sequence (†), we get the long exact sequence

$$\cdots \longrightarrow \operatorname{Hom}_R(P, C) \longrightarrow \operatorname{Ext}^1_R(F^{(2^\lambda)}, C) \longrightarrow \operatorname{Ext}^1_R(E, C) \longrightarrow \cdots,$$

where $\mathrm{Ext}^1_R(E,C)=0$. This means that the map $\mathrm{Hom}_R(P,C)\longrightarrow \mathrm{Ext}^1_R(F^{(2^\lambda)},C)$ is an epimorphism. Notice that P is a summand in $\bigoplus_{n<\omega} F_n^{(\lambda)}$, so we have

$$|\mathrm{Hom}_R(P,C)|\leq \big|\mathrm{Hom}_R(\bigoplus_{n<\omega} F_n^{(\lambda)},C)\big| \leq \prod_{n<\omega} |\mathrm{Hom}_R(F_n,C)|^\lambda \leq \big(\lambda^\lambda\big)^\omega = 2^\lambda.$$

Since $\mathrm{Ext}^1_R\big(F^{(2^\lambda)},C\big)\cong \big(\mathrm{Ext}^1_R(F,C)\big)^{2^\lambda}$, we obtain

$$\mathrm{Ext}^1_R(F,C)\neq 0 \implies \Big|\mathrm{Ext}^1_R\big(F^{(2^\lambda)},C\big)\Big| \geq 2^{2^\lambda},$$

which would contradict the existence of the epimorphism. So $\mathrm{Ext}^1_R(F,C)=0$ and our proof is complete. □

As illustrated above, the point of Hunter's argument is to prove that some Ext group is zero by showing that the opposite would lead to a contradiction on the level of cardinality; no algebraic structure of the group $\mathrm{Hom}_R(P,C)$ played role, it was sufficient to know that it is small enough.

When comparing this proof with the one used in (3) under the additional hypothesis of SCH, one can find certain similarities. For instance, in both cases, the argument is carried out on a singular (typically strong limit singular) cardinal. This should come as no surprise since, unlike at the beginning of the Twentieth Century, singular cardinals are recognized today as the important ones telling much more about the behaviour of the universe of all sets than the regular ones do. Going on with comparing the proofs, we can observe another typical phenomenon: while the construction of Bazzoni and Šťovíček is made-to-measure, the proof from (3) uses only selected properties of a construction from [17] designed originally for another purpose; the additional set-theoretic hypothesis (to be precise, the combinatorial power it implies) then stands in for the missing algebraic information on a fine structure of the module hereby constructed. Another example of this feature, additional set-theoretic tool standing in for a missing algebraic quality, is provided by Lemma 10.1.1 and Theorem 4.3.2 from [16].

We conclude our Introduction by stating several open problems as the possible directions for further research.

Open problems.

(i) Telescope Conjecture for Module Categories. In its full generality, or at least for the original setting of self-injective Artin algebras.

(ii) Does there exist a hereditary cotorsion pair $(\mathcal{A},\mathcal{B})$ with $\mathcal{A}$ closed under direct limits but not closed under taking pure-epimorphic images?

This is mentioned in Remark at the end of the fifth section of (2). The positive solution to this problem would have an interesting implications. If such a $(\mathcal{A},\mathcal{B})$ was not generated by a set, it would be the first example (in ZFC) of such a pair, even with the additional nice property of $\mathcal{A}$ being closed under direct limits. This might seem really unlike to happen but the other alternative is no less interesting. If $(\mathcal{A},\mathcal{B})$ was generated by a set, it would provide us with a negative solution to the following open problem.

(iii) Is any hereditary cotorsion pair $(\mathcal{A},\mathcal{B})$ which is generated by a set and such that $\mathcal{A}$ is closed under direct limits cogenerated by a class of pure-injective modules?

This is the setting when $\mathcal{A}$ forms an abstract elementary class in the sense explained in the first section of (3). The positive solution to this problem would immediately imply that all elementary classes emerging in this way have finite character (see (3) for the definition).

(iv) Does the class ${}^{\perp}(\mathcal{D}^{\perp})$ always coincide with $\mathcal{F}_0$?

We have a positive answer for all countable rings by the result proved above (and trivially, for all right perfect rings).

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ON THE TELESCOPE CONJECTURE FOR MODULE CATEGORIES

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Abstract. In [22], the Telescope Conjecture was formulated for the module category Mod R of an artin algebra R as follows: "If $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in Mod R with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits, then $\mathcal{A} = \varinjlim(\mathcal{A} \cap \text{mod} R)$". We extend this conjecture to arbitrary rings R, and show that it holds true if and only if the cotorsion pair $\mathfrak{C}$ is of finite type. Then we prove the conjecture in the case when R is right noetherian and $\mathcal{B}$ has bounded injective dimension (thus, in particular, when $\mathfrak{C}$ is any cotilting cotorsion pair). We also focus on the assumptions that $\mathcal{A}$ and $\mathcal{B}$ are closed under direct limits and on related closure properties, and detect several asymmetries in the properties of $\mathcal{A}$ and $\mathcal{B}$.

In the late 1970's, Bousfield and Ravenel formulated a telescope conjecture for the stable homotopy category. Later on, Neeman extended it to compactly generated triangulated categories $\mathcal{T}$. In this generality, the conjecture said that any smashing localizing subcategory $\mathcal{L}$ of $\mathcal{T}$ is of finite type, cf. [10], [22], [26]. Keller [19] gave an example disproving the conjecture in the case when $\mathcal{T}$ is the (unbounded) derived category of the module category over a particular (non-noetherian) commutative ring.

However, it appears open whether the conjecture holds true when $\mathcal{T}$ is the stable module category of a self-injective artin algebra R. In that case, the conjecture was shown to be equivalent to a certain property of cotorsion pairs of R-modules, cf. [22, §7]. This lead Krause and Solberg to the following version of the telescope conjecture for module categories of arbitrary artin algebras:

[22, 7.9] "Let R be an artin algebra, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a complete hereditary cotorsion pair in Mod R with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits. Then $\mathcal{A} = \varinjlim(\mathcal{A} \cap \text{mod} R)$."

The latter conjecture is known to hold when $\mathfrak{C}$ is a tilting cotorsion pair by [9] (see also [18, §5]), when $\mathfrak{C}$ is a 1-cotilting cotorsion pair by [11], and when $\mathcal{A} \cap \text{mod} R$ is a contravariantly finite subcategory of mod$R$ by [22].

In the present paper, we deal with the following general version of the Krause-Solberg conjecture, formulated for arbitrary rings:

0.1. Telescope Conjecture for Module Categories. "Let R be ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a complete hereditary cotorsion pair in Mod R with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits. Then $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$."

Here $\mathcal{A}^{<\omega} = \mathcal{A} \cap \text{mod} R$ where mod R denotes the class of all modules possessing a projective resolution consisting of finitely generated modules.

Recall that a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is of *finite type* provided there is a set $\mathcal{C} \subseteq \text{mod} R$ such that $\mathcal{B} = \text{Ker Ext}^1_R(\mathcal{C}, -)$. It is known that 0.1 holds for all cotorsion pairs $\mathfrak{C}$ of finite type (with $\mathcal{A}$ closed under direct limits). In Corollary 4.7, we prove the converse: any cotorsion pair $\mathfrak{C}$ for which the conclusion of 0.1 holds is necessarily of finite type.

Moreover, in Theorem 4.10, we prove that 0.1 holds for any right noetherian ring under the additional assumption that all modules in $\mathcal{B}$ have bounded injective dimension.

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This yields 0.1 in the particular case when $\mathfrak{C}$ is a cotilting cotorsion pair over a right noetherian ring.

The proofs of these theorems rely on a number of results on deconstruction and completeness of cotorsion pairs from [31], [8], [9] and [28] which were essential for the recent rapid progress in infinite dimensional tilting and cotilting theory. Unfortunatelly, most of these preprints are not published yet, so we supplement the original references below with quotations of the corresponding results in the recent monograph [18]. The latter was submitted for publication only in Spring 2006, but thanks to the rapid publication policy of Walter de Gruyter, it is paradoxically available in printed form much earlier then the papers submitted in 2005.

The assumptions made in 0.1 that $\mathcal{A}$ and $\mathcal{B}$ are closed under direct limits also lead us to an investigation of closure properties of cotorsion pairs, with special emphasis on tilting and cotilting cotorsion pairs (Sections 2 and 3). Finally, the last section is devoted to some asymmetries that can occur in the properties of $\mathcal{A}$ and $\mathcal{B}$. In 5.2(3) we exhibit an example of a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ of infinite type with $\mathcal{B}$ not being closed under coproducts and $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$. We also show that in general the validity of 0.1 does not imply $\mathcal{B} = \varinjlim \mathcal{B}^{<\omega}$, see 5.2(1) and Theorem 5.3.

1. Preliminaries

Notation. Let R be a ring. Denote by $\operatorname{Mod} R$ the category of all (right R-) modules, and by $\operatorname{mod} R$ the subcategory of all modules possessing a projective resolution consisting of finitely generated modules. (If R is right coherent then $\operatorname{mod} R$ is just the category of all finitely presented modules).

Given an infinite cardinal κ and a class of modules $\mathcal{A}$, the symbol $\mathcal{A}^{<\kappa}$ ($\mathcal{A}^{\leq\kappa}$) denotes the subclass of $\mathcal{A}$ consisting of all modules possessing a projective resolution consisting of $<\kappa$-generated ($\leq\kappa$-generated) modules. For example, $\operatorname{mod} R = (\operatorname{Mod} R)^{<\omega}$.

We denote by $\mathcal{P}$ and $\mathcal{I}$ the class of all modules of finite projective and injective dimension, respectively. For $n < \omega$, $\mathcal{P}_n$ ($\mathcal{I}_n$, $\mathcal{F}_n$) is the class of all modules of projective (injective, flat) dimension $\leq n$.

Let $\mathcal{M}$ be a subcategory of $\operatorname{Mod} R$. We always assume that $\mathcal{M}$ is *full and that it is closed under direct summands and isomorphic images.*

We denote by $\operatorname{Add}\mathcal{M}$ (respectively $\operatorname{add}\mathcal{M}$) the subcategory of all modules isomorphic to a direct summand of a (finite) direct sum of modules of $\mathcal{M}$, and by $\operatorname{Prod}\mathcal{M}$ the subcategory of all modules isomorphic to a direct summand of a product of modules of $\mathcal{M}$. If $\mathcal{M} = \{M\}$, we write $\operatorname{Add}M$, $\operatorname{add}M$, $\operatorname{Prod}M$.

Furthermore, $\varinjlim \mathcal{M}$ denotes the class of all modules D such that $D = \varinjlim_{i\in I} M_i$ where $\{M_i \mid i \in I\}$ is a direct system of modules from $\mathcal{M}$. We will use repeatedly the following characterization of $\varinjlim \mathcal{M}$ due to Lenzing.

Lemma 1.1. [25, 2.1] (see also [18, 1.2.9]) Assume that $\mathcal{M}$ is an additive subcategory of $\operatorname{mod} R$. Then the following statements are equivalent for a module A_R.
(1) $A \in \varinjlim \mathcal{M}$.
(2) There is a pure epimorphism $\coprod_{k\in K} X_k \to A$ for some modules X_k in $\mathcal{M}$.
(3) Every homomorphism $h : F \to A$ where F is finitely presented factors through a module in $\operatorname{add}\mathcal{M}$.

Resolving subcategories. A class $\mathcal{S} \subseteq \operatorname{Mod}R$ (or $\mathcal{S} \subseteq \operatorname{mod}R$) is said to be a *resolving* subcategory of $\operatorname{Mod}R$ (respectively, of $\operatorname{mod}R$) if it satisfies the following conditions:

(R1) $\mathcal{S}$ contains all (finitely generated) projective modules,
(R2) $\mathcal{S}$ is closed under extensions,
(R3) $\mathcal{S}$ is closed under kernels of epimorphisms.

Coresolving subcategories are defined by the dual conditions (CR1), (CR2), (CR3).

Orthogonal classes. For a class $\mathcal{C} \subseteq \mathrm{Mod}R$ and for $i > 0$, we define

$$\mathcal{C}^{\perp_i} = \mathrm{Ker}\ \mathrm{Ext}^i_R(\mathcal{C}, -) \qquad {}^{\perp_i}\mathcal{C} = \mathrm{Ker}\ \mathrm{Ext}^i_R(-, \mathcal{C})$$

$$\mathcal{C}^{\perp} = \bigcap_{i>0} \mathcal{C}^{\perp_i} \qquad {}^{\perp}\mathcal{C} = \bigcap_{i>0} {}^{\perp_i}\mathcal{C}$$

Similarly, we define the classes $\mathcal{C}^{\intercal_i}$, ${}^{\intercal_i}\mathcal{C}$, $\mathcal{C}^{\intercal}$, and ${}^{\intercal}\mathcal{C}$, replacing Ext by Tor.

We collect here some well-known facts often used in the sequel.

Remark 1.2. (1) If $\mathcal{S}$ is resolving, then $\mathcal{S}^{\perp_1} = \mathcal{S}^{\perp}$ and $\mathcal{S}^{\intercal_1} = \mathcal{S}^{\intercal}$. Coresolving classes have the dual properties.
(2) For any $\mathcal{M} \subseteq \mathrm{Mod}R$, the classes ${}^{\perp}\mathcal{M}$, $\mathcal{M}^{\intercal}$ are resolving, and $\mathcal{M}^{\perp}$ is coresolving.
(3) (cf. [17, 10.2.4, and 3.2.26]) If $\mathcal{C} \subseteq \mathrm{mod}\,R$ and $i > 0$, then $\mathcal{C}^{\perp_i}$ and $\mathcal{C}^{\intercal_i}$ are closed under direct products and direct limits.

Approximations. Let $\mathcal{M}$ be a subcategory of $\mathrm{Mod}R$, and let A be a right R-module. A morphism $f \in \mathrm{Hom}_R(A, X)$ with $X \in \mathcal{M}$ is an $\mathcal{M}$-*preenvelope* (or a left $\mathcal{M}$-approximation) of A provided that the abelian group homomorphism $\mathrm{Hom}_R(f, M)$: $\mathrm{Hom}_R(X, M) \longrightarrow \mathrm{Hom}_R(A, M)$ is surjective for each $M \in \mathcal{M}$.

An $\mathcal{M}$-preenvelope $f \in \mathrm{Hom}_R(A, X)$ of A is said to be *special* if f is a monomorphism and $\mathrm{Ext}^1_R(\mathrm{Coker}\, f, M) = 0$ for all $M \in \mathcal{M}$.

An $\mathcal{M}$-*envelope* of A is an $\mathcal{M}$-preenvelope $f \in \mathrm{Hom}_R(A, X)$ which is left minimal, that is, h is an automorphism of X whenever $h \in \mathrm{End}_R(X)$ satisfies $hf = f$. Note that $\mathcal{M}$-envelopes may not exist in general, but they are always unique up to isomorphism.

The notions of an $\mathcal{M}$-*cover* and a (*special*) $\mathcal{M}$-*precover* are defined dually.

A subcategory $\mathcal{S}$ of $\mathrm{mod}R$ is said to be *covariantly* (respectively, *contravariantly*) *finite* in $\mathrm{mod}R$ if every module in $\mathrm{mod}R$ has an $\mathcal{S}$-preenvelope (respectively, an $\mathcal{S}$-precover). A class of modules $\mathcal{M}$ is *definable* if it is closed under direct products, direct limits, and pure submodules. We will frequently use the following relationship between covariantly finite subcategories of $\mathrm{mod}R$ and definable classes.

Theorem 1.3. [13, 4.2], [20, 3.11] Let $\mathcal{S}$ be a full additive subcategory of $\mathrm{mod}R$. The following statements are equivalent.
(1) $\mathcal{S}$ is covariantly finite in $\mathrm{mod}\,R$.
(2) $\varinjlim \mathcal{S}$ is closed under products.
(3) $\varinjlim \mathcal{S}$ is definable.

Cotorsion pairs. Let $\mathcal{A}, \mathcal{B} \subseteq \mathrm{Mod}R$ (or $\mathcal{A}, \mathcal{B} \subseteq \mathrm{mod}R$) be classes of modules. Then $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a *cotorsion pair in* $\mathrm{Mod}R$ (respectively, a *cotorsion pair in* $\mathrm{mod}R$) provided $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$ (respectively, provided $\mathcal{A} = ({}^{\perp_1}\mathcal{B})^{<\omega}$ and $\mathcal{B} = (\mathcal{A}^{\perp_1})^{<\omega}$).

A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in $\mathrm{Mod}R$ (or in $\mathrm{mod}R$) is *complete* if every module (respectively, every module in $\mathrm{mod}R$) has a special $\mathcal{A}$-precover and a special $\mathcal{B}$-preenvelope. Moreover, $\mathfrak{C}$ is *perfect* if every module (respectively, every module in $\mathrm{mod}R$) has an $\mathcal{A}$-cover and a $\mathcal{B}$-envelope. Note that a complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathrm{Mod}R$ is perfect provided $\mathcal{A}$ is closed under direct limits [17, 7.2.6]. It is an open problem whether the converse holds true. For artin algebras, the following result was established by Auslander and Reiten.

Lemma 1.4. [6] Let Λ be an artin algebra, and let $\mathcal{A}, \mathcal{B}$ be subcategories of modΛ. The following statements are equivalent.
(1) $\mathcal{A}$ is a contravariantly finite subcategory of modΛ satisfying conditions (R1) and (R2), and $\mathcal{B} = (\mathcal{A}^{\perp_1})^{<\omega}$.
(2) $\mathcal{B}$ is a covariantly finite subcategory of modΛ satisfying conditions (CR1) and (CR2), and $\mathcal{A} = ({}^{\perp_1}\mathcal{B})^{<\omega}$.
(3) $(\mathcal{A}, \mathcal{B})$ is a perfect cotorsion pair in modΛ.

The existence of approximations cannot be omitted in the result above, as shown by the following example.

Example 1.5. Let Λ be an artin algebra such that the big (left) finitistic dimension of Λ equals $n > 1$, but its little (left) finitistic dimension is $< n$, see [32]. Consider the class $\mathcal{A}'$ of all n-th syzygies of cyclic right Λ-modules, and set $\mathcal{A} = ({}^{\perp_1}(\mathcal{A}'^{\perp_1}))^{<\omega}$. Then $\mathcal{A}$ is a resolving subcategory of modΛ, and by Baer's Lemma $\mathcal{A}^{\perp_1} = \mathcal{A}'^{\perp_1} = \mathcal{I}_n$. On the other hand, $\mathcal{B} = (\mathcal{A}^{\perp_1})^{<\omega} \subseteq \mathcal{I}_{n-1}$.
We deduce that $\mathcal{A}$ is properly contained in $({}^{\perp_1}\mathcal{B})^{<\omega}$. In fact, if we choose $M \in \Lambda$Mod with pdim$M = n$, and $N \in$ modΛ such that $\mathrm{Ext}^n_R(N, D(M)) \neq 0$, then it is easy to see that the module $X = \Omega^{n-1}(N)$ is contained in $({}^{\perp_1}\mathcal{B})^{<\omega}$. On the other hand, X is not contained in $\mathcal{A}$, because $D(M) \in \mathcal{I}_n = \mathcal{A}^{\perp_1}$, and $\mathrm{Ext}^1_R(X, D(M)) \neq 0$.
This shows that $(\mathcal{A}, \mathcal{B})$ is not a cotorsion pair in modΛ. □

We will need further terminology on cotorsion pairs.

Lemma 1.6. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in Mod$R$ (or in mod$R$). Then $\mathcal{A}$ is resolving if and only if $\mathcal{B}$ is coresolving, and this is further equivalent to $\mathrm{Ext}^i_R(A, B) = 0$ for all $A \in \mathcal{A}, B \in \mathcal{B}, i \geq 2$. In this case $(\mathcal{A}, \mathcal{B})$ is called *hereditary*.

Let $\mathcal{C}$ be a class of modules. A module M is called *$\mathcal{C}$-filtered* provided there exist an ordinal σ and an increasing chain, $(M_\alpha \mid \alpha < \sigma)$, consisting of submodules of M such that $M_0 = 0$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha < \sigma$, $M = \bigcup_{\alpha<\sigma} M_\alpha$, and $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for each $\alpha + 1 < \sigma$.

Theorem 1.7. [14] Let $\mathcal{C}$ be a class of modules and let $\mathcal{B} = \mathcal{C}^{\perp_1}$ and $\mathcal{A} = {}^{\perp_1}(\mathcal{C}^{\perp_1})$. Then $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair, called the *cotorsion pair cogenerated by* $\mathcal{C}$. If the isomorphism classes of $\mathcal{C}$ form a set, then $(\mathcal{A}, \mathcal{B})$ is complete. Moreover, in this case $\mathcal{A} = {}^{\perp_1}(\mathcal{C}^{\perp_1})$ consists of all direct summands of $\mathcal{C} \cup \{R\}$-filtered modules.

Theorem 1.8. [15] Let $\mathcal{C}$ be a class of modules and let $\mathcal{A} = {}^{\perp_1}\mathcal{C}$ and $\mathcal{B} = ({}^{\perp_1}\mathcal{C})^{\perp_1}$. Then $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair, called the *cotorsion pair generated by* $\mathcal{C}$. If $\mathcal{C}$ consists of pure injective modules, then $(\mathcal{A}, \mathcal{B})$ is perfect.

The theorems above together with Remark 1.2 apply to the following situations.

Cotorsion pairs of (co)finite type. A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in Mod$R$ is of *finite type* provided it is cogenerated by a class $\mathcal{S} \subseteq$ mod$R$. Then $\mathfrak{C}$ is complete and $\mathcal{B}$ is definable.
Dually, a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in RMod is of *cofinite type* provided there is a class $\mathcal{S} \subseteq$ mod$R$ such that $\mathcal{A} = \mathcal{S}^{\intercal_1}$. Then $\mathfrak{C}$ is perfect. Moreover, $\mathcal{A}$ is definable provided $\mathfrak{C}$ is hereditary. In fact, by the well-known Ext-Tor relation [17, 3.2.1], $(\mathcal{A}, \mathcal{B})$ is of cofinite type iff it is generated by the class $\mathcal{S}^* = \{S^* \mid S \in \mathcal{S}\}$ where S^* denotes the dual module, e. g. $S^* = \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})$, or over artin algebras $S^* = D(S)$ for the usual duality D.
If $\mathfrak{C}$ is hereditary, then in both cases, we can assume w.l.o.g. that $\mathcal{S}$ is resolving.

(Co)smashing cotorsion pairs. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair in Mod$R$. Observe that the class $\mathcal{A}$ is always closed under coproducts, and $\mathcal{B}$ is always closed under products. We will say that $\mathfrak{C}$ is *smashing* if $\mathcal{B}$ is also closed under coproducts, and *cosmashing* if $\mathcal{A}$ is also closed under products.

For further properties of the notions defined above we refer to [18] (note however, that the terminology in [18] occasionally differs from the one used here).

2. Closure under direct limits

We start by recalling a result from [4]:

Theorem 2.1. [4, 2.3 and 2.4] Let $\mathcal{S}$ be a subcategory of mod$R$ with properties (R1) and (R2), and let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by $\mathcal{S}$. Then the following hold true:
(1) $\mathcal{A} \subseteq \varinjlim \mathcal{S} = {}^{\intercal}(\mathcal{S}^{\intercal})$, and $\mathcal{S} = \mathcal{A}^{<\omega} = (\varinjlim \mathcal{S})^{<\omega}$.
(2) There is a perfect cotorsion pair $(\varinjlim \mathcal{S}, \mathcal{Y})$ which is generated by the class of all pure-injective modules from $\mathcal{B}$.

Theorem 2.1 has a number of consequences concerning Conjecture 0.1.

Corollary 2.2. (1) The (perfect) hereditary cotorsion pairs $(\mathcal{X}, \mathcal{Y})$ in Mod$R$ satisfying $\mathcal{X} = \varinjlim \mathcal{X}^{<\omega}$ correspond bijectively to the resolving subcategories of mod$R$.
The correspondence is given by the mutually inverse assignments

$$\begin{array}{rccc} \alpha: & (\mathcal{X}, \mathcal{Y}) & \mapsto & \mathcal{X}^{<\omega} \\ \beta: & \mathcal{S} & \mapsto & (\varinjlim \mathcal{S}, \mathcal{Y}) \end{array}$$

(2) The hereditary cotorsion pairs of finite type in Mod$R$ correspond bijectively to the resolving subcategories of mod$R$.
The correspondence is given by the mutually inverse assignments

$$\begin{array}{rccc} \alpha: & (\mathcal{A}, \mathcal{B}) & \mapsto & \mathcal{A}^{<\omega} \\ \beta': & \mathcal{S} & \mapsto & ({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp}) \end{array}$$

Proof: This follows by Lemma 1.6 and Theorem 2.1. □

Corollary 2.3. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair of finite type in Mod$R$. Let moreover $\mathcal{B}'$ be the class of all pure-injective modules from $\mathcal{B}$. Then the following statements are equivalent.
(1) $\mathcal{A}$ is closed under direct limits.
(2) $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$.
(3) The cotorsion pair $(\varinjlim \mathcal{A}^{<\omega}, \mathcal{Y})$ is of finite type.
(4) The class $\mathcal{Y} = ({}^{\perp_1}\mathcal{B}')^{\perp_1}$ is definable.
(5) Every pure embedding into a module $M \in \mathcal{A} \cap \mathcal{B}$ splits.

Proof: First, notice that any cotorsion pair of finite type is complete by Theorem 1.7.
The equivalence of (1), (2), (3) follows from Theorem 2.1. Of course, (3) implies (4).
(4) $\Rightarrow$ (3): Since $\mathcal{B}' \subseteq \mathcal{Y} \subseteq \mathcal{B}$, the two definable classes $\mathcal{B}$ and $\mathcal{Y}$ contain the same pure-injective modules, and so they coincide.
To prove the equivalence of (2) and (5), we generalize an argument from [4, 4.2]:
First, if $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$ then $\mathcal{A}$ is closed under pure-epimorphic images by Lemma 1.1. Since $\mathfrak{C}$ is of finite type, $\mathcal{B}$ is closed under pure submodules. So, if $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ is a pure-exact sequence with $M \in \mathcal{A} \cap \mathcal{B}$, then $P \in \mathcal{A}$ and $N \in \mathcal{B}$, and the sequence splits.

For the converse, note first that $\mathfrak{C}$ being of finite type implies $\mathcal{A} \subseteq \varinjlim \mathcal{A}^{<\omega}$ by Theorem 2.1. Next, we claim $\mathcal{A} \cap \mathcal{B} = \varinjlim \mathcal{A}^{<\omega} \cap \mathcal{B}$. Let $N \in \varinjlim \mathcal{A}^{<\omega} \cap \mathcal{B}$. Since $\mathfrak{C}$ is complete, there is a special $\mathcal{A}$-precover $\mathcal{E} : 0 \longrightarrow B \longrightarrow A \longrightarrow N \longrightarrow 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then $A \in \mathcal{A} \cap \mathcal{B}$. Moreover, since $N \in \varinjlim \mathcal{A}^{<\omega}$ is a pure-epimorphic image of an element of $\mathcal{A}$, $\mathcal{E}$ is pure-exact. So by (5), $\mathcal{E}$ splits, proving our claim.

Let us now take an arbitrary module $N \in \varinjlim \mathcal{A}^{<\omega}$ and a special $\mathcal{B}$-preenvelope $0 \longrightarrow N \longrightarrow B' \longrightarrow A' \longrightarrow 0$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$. Then A' and therefore also B' belong to $\varinjlim \mathcal{A}^{<\omega}$. So, by the claim above, $B' \in \mathcal{A} \cap \mathcal{B}$, which yields $N \in \mathcal{A}$ as $\mathcal{A}$ is resolving. This shows that $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$, so (2) holds. □

In particular, we infer that all cotorsion pairs of finite type with $\mathcal{A}$ closed under direct limits satisfy Conjecture 0.1. In Section 4, we will prove that also the converse is true in the sense that any cotorsion pair satisfying 0.1 is necessarily of finite type.

Corollary 2.4. Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair of finite type in Mod$R$. Then the following statements are equivalent.
(1) $\mathcal{A}$ is definable.
(2) $\mathcal{A}$ is closed under direct limits, and $\mathcal{A}^{<\omega}$ is covariantly finite in mod$R$.

Proof: By Theorem 1.3 and Corollary 2.3.□

3. Closure properties of tilting and cotilting cotorsion pairs

Before we continue our discussion of Conjecture 0.1, let us apply the considerations above to tilting theory, which will be a source of interesting examples in Section 5.

Let $n < \omega$. A module T is *n-tilting* provided
(T1) $T \in \mathcal{P}_n$,
(T2) $\mathrm{Ext}^i_R(T, T^{(I)}) = 0$ for each $i \geq 1$ and all sets I, and
(T3) there exist $r \geq 0$ and a long exact sequence $0 \longrightarrow R \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_r \longrightarrow 0$ such that $T_i \in \mathrm{Add}T$ for each $0 \leq i \leq r$.

Every n-tilting module T induces a complete hereditary smashing cotorsion pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{B} = T^{\perp}$ and $\mathcal{A} \subseteq \mathcal{P}_n$, see [1]. Such cotorsion pairs are called *n-tilting cotorsion pairs*. By [9] (see also [18, §5]), tilting cotorsion pairs are always of finite type.

Dually, a module C is *n-cotilting* provided that
(C1) $C \in \mathcal{I}_n$,
(C2) $\mathrm{Ext}^i_R(C^I, C) = 0$ for each $i \geq 1$ and all sets I, and
(C3) there exists $r \geq 0$ and an exact sequence $0 \longrightarrow C_r \longrightarrow \cdots \longrightarrow C_0 \longrightarrow W \longrightarrow 0$ where W is an injective cogenerator for Mod$R$ and $C_i \in \mathrm{Prod}C$ for each $0 \leq i \leq r$.

Every n-cotilting module C is pure-injective by [30], and so it induces a perfect hereditary cosmashing cotorsion pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A} = {}^{\perp}C$ and $\mathcal{B} \subseteq \mathcal{I}_n$, see [1]. Such cotorsion pairs are called *n-cotilting cotorsion pairs*. Cotilting cotorsion pairs are not always of cofinite type [7], however, the class $\mathcal{A}$ is always definable.

Finally, we recall from [21] that a module M with Add$M$ being closed under products is said to be *product-complete*. Note that M is product-complete iff Add$M$ = Prod$M$. Moreover, every product-complete module is Σ-pure-injective.

Proposition 3.1. Let T be a tilting module with corresponding tilting cotorsion pair $(\mathcal{A}, \mathcal{B})$. Then the following statements are equivalent.
(1) $\mathcal{A}$ is definable.
(2) $(\mathcal{A}, \mathcal{B})$ is cosmashing.
(3) T is product-complete.
If R has finite global dimension, then (1)–(3) are further equivalent to
(4) T is a cotilting module such that $\mathcal{A} = {}^{\perp}T$.

Proof: For the equivalence of (1)–(3), we generalize an argument from [4, 4.3], which we include for the reader's convenience. Clearly, (1) $\Rightarrow$ (2). Moreover, we know from [1, 2.4] that $\mathrm{Add}T = \mathcal{A} \cap \mathcal{B}$, and that $\mathcal{A}$ consists of the modules A having a long exact sequence $0 \to A \to T_0 \to \ldots \to T_n \to 0$ with $T_0, \ldots, T_n \in \mathrm{Add}T$. We then deduce that $\mathcal{A}$ is closed under direct products iff so is $\mathrm{Add}T$, which means that (2) and (3) are equivalent. Moreover, under the assumption (3), the module T is Σ-pure-injective. Then every pure submodule of a module $M \in \mathcal{A} \cap \mathcal{B}$ is a direct summand of M, and thus $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$ by Corollary 2.3. So (3) $\Rightarrow$ (1) holds by Theorem 1.3.
Assume now (4). Then we have from [1, 2.4] that $\mathrm{Add}T = \mathcal{A} \cap \mathcal{B} = \mathrm{Prod}T$, so (3) holds true.
Conversely, if (3) holds true and $\mathrm{gldim}R < \infty$, then T satisfies conditions (C1) and (C2) in the definition of a cotilting module. Moreover, if W is an injective cogenerator for $\mathrm{Mod}R$, then $W \in \mathcal{B}$, and since W has finite projective dimension, we deduce from [1, 2.4] that there is a long exact sequence $0 \to T_m \to \ldots \to T_0 \to W \to 0$ with $T_0, \ldots, T_m \in \mathrm{Add}T = \mathrm{Prod}T$. So, also (C3) is satisfied, and T is a cotilting module. Furthermore, $T \in \mathcal{B}$ implies that $\mathcal{A} \subseteq {}^{\perp}T$.
It remains to prove that every module $X \in {}^{\perp}T$ belongs to $\mathcal{A}$. To this end, we consider a special $\mathcal{B}$-preenvelope $0 \longrightarrow X \longrightarrow B \longrightarrow A \longrightarrow 0$. Then since $A \in \mathcal{A}$ belongs to ${}^{\perp}T$, we have $B \in \mathcal{B} \cap {}^{\perp}T$. As above, we consider a long exact sequence $0 \to T_m \stackrel{f}{\longrightarrow} T_{m-1} \to \ldots \to T_0 \to B \to 0$ with $T_0, \ldots, T_m \in \mathrm{Add}T = \mathrm{Prod}T$, and we choose it of minimal length m. Assume $m > 0$. Since $B, T_0, \ldots, T_m$ all belong to the resolving subcategory ${}^{\perp}T$, it follows that $\mathrm{Coker}f$ also belongs to ${}^{\perp}T$. But then $\mathrm{Ext}^1_R(\mathrm{Coker}f, T_m) = 0$, so $\mathrm{Coker}f$ even belongs to $\mathrm{Add}T$, contradicting the minimality of m. We conclude that $m = 0$, that is, that B belongs to $\mathrm{Add}T \subseteq \mathcal{A}$. Since $\mathcal{A}$ is resolving, this completes the proof. □

Dually, one obtains the following result for cotilting cotorsion pairs, see also [12, 3.4].

Proposition 3.2. Let C be a cotilting module with corresponding cotilting cotorsion pair $(\mathcal{A}, \mathcal{B})$. Then the following statements are equivalent.
(1) $(\mathcal{A}, \mathcal{B})$ is smashing.
(2) C is Σ-pure-injective.
(3) There is a product-complete cotilting module C' such that $\mathcal{A} = {}^{\perp}C'$.
If R has finite global dimension, then (1)–(3) are further equivalent to
(4) There is a tilting (and cotilting) module C' such that $\mathcal{B} = C'^{\perp}$.

Proof: With arguments dual to those used in 3.1, we see that $\mathcal{B}$ is closed under direct sums iff so is $\mathrm{Prod}C$. Since C is pure-injective, the latter implies that the module C is Σ-pure-injective. So, we have (1)$\Rightarrow$(2) and (3)$\Rightarrow$(1). Moreover, (4)$\Rightarrow$(1) because tilting cotorsion pairs are smashing, and if $\mathrm{gldim}R < \infty$, then the module C' in (3) satisfies (4) dually to Proposition 3.1.
It remains to prove (2)$\Rightarrow$(3): Assume that C is Σ-pure-injective. By [24, 8.1], there is a cardinal κ such that every product of copies of C is a direct sum of modules of cardinality at most κ. Of course, the isomorphism classes of all κ-generated modules lying in $\mathrm{Prod}\, C$ form a set $\mathcal{K}$. Let C' be the direct sum of all modules in $\mathcal{K}$, and P the direct product of all modules in $\mathcal{K}$. We then have $\mathrm{Prod}\, C \subseteq \mathrm{Add}\, C'$. Moreover, $P \in \mathrm{Prod}\, C$ is Σ-pure-injective. Hence the pure submodule C' of P is a direct summand of P. This proves $\mathrm{Prod}\, C' \subseteq \mathrm{Prod}\, C$, and further, by Σ-pure-injectivity, $\mathrm{Add}\, C' \subseteq \mathrm{Prod}\, C'$. We then conclude that $\mathrm{Add}\, C' = \mathrm{Prod}\, C' = \mathrm{Prod}\, C$, so C' is a product-complete cotilting module such that $\mathcal{A} = {}^{\perp}C'$.□

Corollary 3.3. Let R be right noetherian and hereditary, and let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. The following statements are equivalent.
(1) $\mathcal{A}$ and $\mathcal{B}$ are definable.
(2) $\mathcal{A}$ and $\mathcal{B}$ are closed under direct limits.
(3) $(\mathcal{A}, \mathcal{B})$ is smashing and cosmashing.
(4) There is a product-complete tilting module M such that $\mathcal{B} = \mathrm{Gen}M$.
(4') There is a product-complete cotilting module M such that $\mathcal{A} = \mathrm{Cogen}M$.
(5) There is a tilting and cotilting module M such that $\mathcal{A} = \mathrm{Cogen}M$ and $\mathcal{B} = \mathrm{Gen}M$.

Proof: Clearly, (1) implies (2). Moreover, by [1, 4.1 and 4.2] we know that $(\mathcal{A}, \mathcal{B})$ is (co)smashing iff it is a (co)tilting cotorsion pair. So, (2) implies that $(\mathcal{A}, \mathcal{B})$ is smashing, and therefore tilting, thus a hereditary cotorsion pair of finite type. Further, $\mathcal{A}^{<\omega}$ is a resolving subcategory of $\mathcal{P}_1^{<\omega}$. As R is right noetherian, it follows from [2, 2.5] that $\mathcal{A}^{<\omega}$ is covariantly finite in mod$R$. Since $\mathcal{A}$ closed under direct limits, we then conclude from 2.4 that $\mathcal{A}$ is definable. In particular, $(\mathcal{A}, \mathcal{B})$ is cosmashing, so we have shown (2)⇒(3). The implication (5)⇒(1) follows from the fact that (co)tilting classes are always definable. The remaining implications hold by Propositions 3.1 and 3.2.□

Example 3.4. Let Λ be a tame hereditary artin algebra (w.l.o.g. basic indecomposable). Reiten and Ringel have shown in [27] that there is a cotorsion pair $(\mathcal{C}, \mathcal{D})$ in ModΛ which is generated by the class $\mathbf{q}$ of all indecomposable preinjective modules and is cogenerated by the class $\mathbf{t}$ of all indecomposable regular modules. In other words, $(\mathcal{C}, \mathcal{D})$ is a hereditary cotorsion pair of finite and cofinite type. In particular, $\mathcal{C}$ and $\mathcal{D}$ are definable. Now let S_λ, $\lambda \in \mathbb{P}$, be a complete irredundant set of quasi-simple modules and let $S_\lambda[\infty]$, $\lambda \in \mathbb{P}$, be the corresponding Prüfer modules. Let further G be the generic module. Then $W = \bigoplus_{\lambda \in \mathbb{P}} S_\lambda[\infty] \oplus G$ is a tilting and cotilting module such that $\mathcal{C} = \mathrm{Cogen}W$ and $\mathcal{D} = \mathrm{Gen}W$; for details see [27]. □

Remark 3.5. The additional hypothesis in Propositions 3.1 and 3.2 is necessary. In fact, if there is an n-tilting-cotilting cotorsion pair $(\mathcal{A}, \mathcal{B})$, then every module M has a long exact sequence $0 \to A_m \xrightarrow{f} \ldots \to A_0 \to M \to 0$ where $A_0, \ldots, A_m \in \mathcal{A}$ and $m \leq n$, and moreover, $\mathcal{A} \subseteq \mathcal{P}_n$, see [1]. But then gldim$R \leq 2n$.

4. The telescope conjecture for module categories

In this section, we deal in detail with Conjecture 0.1. We have seen in Corollary 2.3 that 0.1 holds for any cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ of finite type such that $\mathcal{A}$ is closed under direct limits. Our first main result shows that the converse is also true, that is, the cotorsion pairs satisfying 0.1 must be of finite type. We start with some preliminary results.

Proposition 4.1. Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by a class $\mathcal{C}$ of countably presented modules. Assume that $B^{(\omega)} \in \mathcal{B}$ whenever $B \in \mathcal{B}$. Then
(1) $\mathfrak{C}$ is smashing, and $\mathcal{B}$ is closed under pure submodules.
(2) If $\mathfrak{C}$ is hereditary then $\mathcal{B}$ is definable.
(3) If $\mathcal{C} \subseteq \varinjlim \mathcal{A}^{<\omega}$ then $\mathfrak{C}$ is of finite type.

Proof: (1) First, our assumption on the class $\mathcal{B}$ implies that $\mathcal{B}$ is closed under pure submodules by [8, Theorem 2.5] (see also [18, 5.2.16]). Since $\mathcal{B}$ is closed under arbitrary direct products, and direct sums are pure submodules in direct products, we infer that $\mathfrak{C}$ is smashing.

(2) Since $\mathcal{B}$ is coresolving, (1) also implies that $\mathcal{B}$ is closed under pure-epimorphic images, thus in particular under direct limits. This shows that $\mathcal{B}$ is definable.
(3) It suffices to verify that $\mathcal{B} = (\mathcal{A}^{<\omega})^{\perp_1}$. Clearly $\mathcal{B} \subseteq (\mathcal{A}^{<\omega})^{\perp_1}$. For the reverse inclusion, we first show that the classes $\mathcal{B}$ and $(\mathcal{A}^{<\omega})^{\perp_1}$ contain the same pure-injective modules. Indeed, for any pure-injective module I, the functor $\mathrm{Ext}^1_R(-, I)$ takes direct limits into inverse limits by [5]. So, the assumption $\mathcal{C} \subseteq \varinjlim \mathcal{A}^{<\omega}$ implies that any pure-injective module $I \in (\mathcal{A}^{<\omega})^{\perp_1}$ belongs to $\mathcal{B}$. Now, let $M \in (\mathcal{A}^{<\omega})^{\perp_1}$, and let P be the pure-injective envelope of M. Since the class $(\mathcal{A}^{<\omega})^{\perp_1}$ is definable, $P \in (\mathcal{A}^{<\omega})^{\perp_1}$. But then $P \in \mathcal{B}$, and thus $M \in \mathcal{B}$ since M is a pure submodule of P. This proves that $\mathcal{B} = (\mathcal{A}^{<\omega})^{\perp_1}$. □

Remark 4.2. Let R be a right $\aleph_0$-noetherian ring. Then for each $n < \omega$, the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is cogenerated by a class of countably presented modules (see e.g. [17, §7.4]).

Denote by $(\mathcal{A}, \mathcal{B})$ the cotorsion pair (of finite type) cogenerated by $\mathcal{P}^{<\omega}$. Let $\mathrm{fdim}(R)$ and $\mathrm{Fdim}(R)$ denote the little and the big finitistic dimensions of R, respectively.

Clearly, $\mathfrak{C}_F = (\mathcal{P}, \mathcal{P}^{\perp})$ is a cotorsion pair iff $\mathrm{Fdim}(R) < \infty$. Like in [3], we infer from Proposition 4.1 and [9, Theorem 4.2] (see also [18, 5.2.20]) that $\mathcal{A} = \mathcal{P}$ iff $\mathfrak{C}_F$ is a tilting cotorsion pair iff $\mathfrak{C}_F$ is a cotorsion pair of finite type iff $\mathrm{Fdim}(R) < \infty$ and $B^{(\omega)} \in \mathcal{P}^{\perp}$ whenever $B \in \mathcal{P}^{\perp}$.

By Theorem 1.7, the condition $\mathcal{A} = \mathcal{P}$ is also equivalent to (i) $\mathrm{Fdim}(R) < \infty$ and (ii) each module of finite projective dimension is a direct summand in a $\mathcal{P}^{<\omega}$-filtered module, see [3, 3.2].

Of course, (ii) implies $\mathrm{fdim}(R) = \mathrm{Fdim}(R)$ (but the converse fails, even when $\mathrm{fdim}(R) = \mathrm{Fdim}(R) = 1$, for the IST-algebra R from [23], see [4]).

Note that this is the way the equality $\mathrm{fdim}(R) = \mathrm{Fdim}(R)$ was proved for artin algebras with $\mathcal{P}^{<\omega}$ contravariantly finite in [3], and for all Iwanaga-Gorenstein rings in [2].

In view of Proposition 4.1, our strategy will consists in proving that every cotorsion pair $(\mathcal{A}, \mathcal{B})$ satisfying 0.1 is cogenerated by the class of countably presented modules from $\mathcal{A}$. To this end, we need results which enable us to filter modules from $\mathcal{A}$ by "smaller" modules which still belong to $\mathcal{A}$. The following two lemmas are the first step in this direction.

Lemma 4.3. Let C be an injective cogenerator in $\mathrm{Mod}\, R$. Define $F(X) = C^{\mathrm{Hom}_R(X,C)}$ and $F(\varphi)(f) = f(- \circ \varphi)$, for all $X, Y \in \mathrm{Mod}\, R$, every $\varphi \in \mathrm{Hom}_R(X, Y)$ and $f \in F(X)$. Then F is an endofunctor of $\mathrm{Mod}\, R$ preserving monomorphisms. Moreover, the family $\iota = (\iota_X \mid X \in \mathrm{Mod}\, R)$ consisting of canonical embeddings $\iota_X : X \to F(X)$ is a natural transformation from the identity functor to F.

Proof: It is straightforward to check that F is a functor and ι is a natural transformation. ι_X is an embedding since C is a cogenerator, and the injectivity of C implies that F preserves monomorphisms. □

Lemma 4.4. Let R be an arbitrary ring, and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{B}$ is closed under direct limits. Let λ be a regular uncountable cardinal, $\kappa \geq \lambda$, $A \in \mathcal{A}$ a κ-presented module, and X a subset of A with $\mathrm{card}\, X < \lambda$. Then there exists a $< \lambda$-presented module $\bar{X}$ such that $X \subseteq \bar{X} \subseteq A$. Moreover, $\bar{X}$ can be taken of the form $\pi(R^{(I)})$ where $\pi : R^{(\kappa)} \to A$ is an epimorphism and I is a subset of κ of cardinality $< \lambda$.

Proof: By assumption, A has a presentation

$$0 \longrightarrow K \stackrel{\subseteq}{\longrightarrow} R^{(\kappa)} \stackrel{\pi}{\longrightarrow} A \longrightarrow 0$$

with $\mathrm{gen}(K) \leq \kappa$, and there is $I_0 \subseteq \kappa$ of cardinality $< \lambda$ such that $X \subseteq \pi(R^{(I_0)})$. Let $\mathcal{L}$ be the set consisting of all $< \lambda$-generated submodules of K. We claim that $K \cap R^{(I_0)} \subseteq L_0$ for some $L_0 \in \mathcal{L}$.

Let $\mathcal{D} = \{\langle L', L\rangle \in \mathcal{L} \times \mathcal{L} \mid L \nsubseteq L'\}$. Using the notation from Lemma 4.3, for each $\langle L', L\rangle \in \mathcal{D}$, we define $\tau_{\langle L',L\rangle} : L \to F((L+L')/L')$ as the composition of the canonical projection $L \to (L+L')/L'$ with the embedding $\iota_{(L+L')/L'}$. For $L \in \mathcal{L}$, put $\mathcal{L}_L = \{L' \in \mathcal{L} \mid \langle L', L\rangle \in \mathcal{D}\}$. Note that for every $L, \tilde{L} \in \mathcal{L}$, $L \subseteq \tilde{L}$ implies $\mathcal{L}_L \subseteq \mathcal{L}_{\tilde{L}}$.

Now for each $L \in \mathcal{L}$, we put

$$G(L) = \prod_{L' \in \mathcal{L}_L} F((L+L')/L'),$$

notice that $G(L) \in \mathcal{I}_0$, and for every $\varepsilon : L \subseteq \tilde{L}$ ($\in \mathcal{L}$), we define

$$G(\varepsilon) = \prod_{L' \in \mathcal{L}_L} F(\varepsilon_{L'})$$

where $\varepsilon_{L'}$ is the inclusion $(L+L')/L' \subseteq (\tilde{L}+L')/L'$. Then G is a functor from the small category $\mathcal{L}$, morphisms of which are just inclusions, to $\mathrm{Mod}\, R$. Moreover, G preserves monomorphisms (since F does), and there is the natural transformation $\tau = (\tau_L \mid L \in \mathcal{L})$ from the canonical emdedding $\mathcal{L} \hookrightarrow \mathrm{Mod}\, R$ to G where τ_L is a fibred product of $(\tau_{\langle L',L\rangle} \mid L' \in \mathcal{L}_L)$: it is routine to check that the square

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\tau_{\tilde{L}}} & G(\tilde{L}) \\ {\scriptstyle\varepsilon}\uparrow & & \uparrow{\scriptstyle G(\varepsilon)} \\ L & \xrightarrow{\tau_L} & G(L) \end{array}$$

commutes for each $L, \tilde{L} \in \mathcal{L}$ and $\varepsilon : L \subseteq \tilde{L}$ (one needs the fact that ι is a natural transformation).

Let E be a direct limit of the directed system $G(\mathcal{L})$. For every $L \in \mathcal{L}$, denote by ν_L the colimit injection $G(L) \hookrightarrow E$. Since K is a directed union of its $< \lambda$-generated submodules, it follows from the preceding paragraph that there exists the unique homomorphism $f : K \to E$ such that $f \restriction L = \nu_L \tau_L$ for all $L \in \mathcal{L}$. Note that $\mathcal{L}$ is λ-directed since λ is a regular cardinal, so $G(\mathcal{L})$ has the same property.

Using the assumption put on $\mathcal{B}$, we have $E \in \mathcal{B}$, which allows us to extend f to some $g : R^{(\kappa)} \to E$. Since $\mathrm{card}\, I_0 < \lambda$ and $G(\mathcal{L})$ is λ-directed, there exists $L_0 \in \mathcal{L}$ such that $g \restriction R^{(I_0)}$ factorizes through ν_{L_0}. We deduce then that $K \cap R^{(I_0)} \subseteq L_0$; if not, there exist $x \in K \cap R^{(I_0)}$ and $L \in \mathcal{L}$ such that $x \in L \setminus L_0$, whence $\tau_{\langle L_0, L+L_0\rangle}(x) \neq 0 \neq \tau_{L+L_0}(x)$ contradicting $f \restriction (K \cap R^{(I_0)})$ being factorized through ν_{L_0}. Our claim is proved.

Since L_0 is a $< \lambda$-generated module, $L_0 \subseteq R^{(I_1)}$ for some $I_0 \subseteq I_1 \subseteq \kappa$ with $\mathrm{card}\, I_1 < \lambda$. Iterating this construction, we obtain a set $I = \bigcup_{n<\omega} I_n$ such that $K \cap R^{(I)} = L$ for some $L \in \mathcal{L}$, and $\bar{X} = \pi(R^{(I)}) \cong R^{(I)}/L$ has the desired properties. □

Lemma 4.5. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$. Let λ be a regular uncountable cardinal, $\kappa \geq \lambda$, $A \in \mathcal{A}$ a κ-presented module, and X be a subset of A of cardinality $< \lambda$. Assume that either (i) R is a right $\aleph_0$-noetherian ring, or (ii) $\mathcal{B}$ is closed under direct limits. Then there is a $< \lambda$-presented module $A' \in \mathcal{A}$ such that $X \subseteq A' \subseteq A$.

Proof: *Step 1:* For any $<\lambda$-presented submodule B of A, we construct a $<\lambda$-generated submodule B' of A containing B with the property that any homomorphism of the form $D \xrightarrow{h} B \subseteq B'$ with D finitely presented factors through a module in $\mathcal{A}^{<\omega}$.

To this end, we fix a pure-exact sequence $0 \longrightarrow \operatorname{Ker} \pi \longrightarrow \bigoplus_{i \in I} D_i \xrightarrow{\pi} B \longrightarrow 0$ with D_i finitely presented for all $i \in I$. Since B is $<\lambda$-presented, we will w.l.o.g. assume that I has cardinality $<\lambda$. For F a non-empty finite subset of I, let $D_F = \bigoplus_{i \in F} D_i$, and $\pi_F = \pi \restriction D_F$. By induction on $\operatorname{card}(F)$, we define finitely generated modules $A_F \in \mathcal{A}^{<\omega}$ and $C_F \subseteq A$ such that there is a commutative diagram

$$\begin{array}{ccc} D_F & \xrightarrow{\pi_F} & B \\ {\scriptstyle f_F}\big\downarrow & & \big\downarrow{\scriptstyle \subseteq} \\ A_F & \xrightarrow{g_F} & A \end{array}$$

and $\pi(D_F) \subseteq C_F = \operatorname{Im} g_F$. Hereby we proceed as follows:

If $\operatorname{card}(F) = 1$, then the existence of A_F and C_F follows immediately from Lemma 1.1 since $A \in \varinjlim \mathcal{A}^{<\omega}$. If $\operatorname{card}(F) > 1$, we take $M = D_F \oplus \bigoplus_{\emptyset \neq G \subsetneq F} A_G$ and let $g = \pi_F \oplus \bigoplus_{\emptyset \neq G \subsetneq F} g_G$. By Lemma 1.1, there exist $A_F \in \mathcal{A}^{<\omega}$, $h_F : M \to A_F$ and $g_F : A_F \to A$ such that $g = g_F h_F$, and we put $C_F = \operatorname{Im}\, g_F$ and $f_F = h_F \restriction D_F$. Note that C_F contains C_G for each $\emptyset \neq G \subsetneq F$.

Now let B' be the union of all C_F where F runs through all non-empty finite subsets of I. This is a directed union of $<\lambda$-many finitely generated submodules of A, so B' is a $<\lambda$-generated submodule of A containing B. Moreover, if $h : D \to B$ is a homomorphism with D finitely presented, then there is a factorization f of h through the pure epimorphism π. But then $\operatorname{Im}\, f \subseteq D_F$ for a non-empty finite subset $F \subseteq I$, and $D \xrightarrow{h} B \subseteq B'$, which equals $g_F f_F f$, factors through $A_F \in \mathcal{A}^{<\omega}$ as required.

Step 2: Consider now the presentation of A from Lemma 4.4. We will define A' as the union of an increasing chain $(B_n \mid n < \omega)$ of $<\lambda$-presented submodules in A of the form $\pi(R^{(J_n)})$ for some J_n of cardinality $<\lambda$ (where $J_0 \subseteq J_1 \subseteq \dots$). The chain will be defined by induction on n:

Take $B_0 = \pi(R^{(J_0)})$ $<\lambda$-presented and such that $X \subseteq B_0$ (this is clearly possible in case (i), and it is possible by Lemma 4.4 in case (ii)). If B_n is defined, there is a $<\lambda$-generated submodule B'_n of A containing B_n constructed as in Step 1. Let $B_{n+1} = \pi(R^{(J_{n+1})})$ be a $<\lambda$-presented submodule of A containing B'_n (again, obtained using the $\aleph_0$-noetherian property of R in case (i), and Lemma 4.4 in case (ii)).

It remains to prove that $A' \in \mathcal{A}$. By Lemma 1.1, it suffices to show that every R-homomorphism $h : D \to A'$ with D finitely presented has a factorization through a module in $\mathcal{A}^{<\omega}$. However, $\operatorname{Im}\, h \subseteq B_n$ for some $n < \omega$, and the claim then follows by construction of B'_n in Step 1. □

Theorem 4.6. Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a smashing cotorsion pair such that $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$. Assume that either (i) R is a right $\aleph_0$-noetherian ring, or (ii) $\mathcal{B}$ is closed under direct limits. Then $\mathfrak{C}$ is of finite type.

Proof: We denote by $\mathcal{A}_0$ the class of all countably presented modules in $\mathcal{A}$. Let $A \in \mathcal{A}$, and let $\kappa \geq \aleph_0$ be such that A is a κ-presented module. By induction on κ, we will prove that A is $\mathcal{A}_0$-filtered. There is nothing to prove for $\kappa = \aleph_0$.

If κ is a regular uncountable cardinal then Lemma 4.5 yields a κ-filtration, $\mathcal{F} = (A_\alpha \mid \alpha < \kappa)$, of A such that $A_\alpha \in \mathcal{A}$ is $<\kappa$-presented for each $\alpha < \kappa$. By [31, Theorem 8] (see also [18, 4.3.2]), there is a subfiltration, $\mathcal{G}$, of $\mathcal{F}$ such that all successive factors in $\mathcal{G}$ are $<\kappa$-presented modules from $\mathcal{A}$, so they are $\mathcal{A}_0$-filtered by inductive premise. Hence A is $\mathcal{A}_0$-filtered.

If κ is singular, we use Shelah's Singular Compactness Theorem [16, IV.3.7] as follows: first, call a module M "free" if M is $\mathcal{A}_0$-filtered. For each regular uncountable cardinal $\lambda < \kappa$, we let S_λ denote the set of all $< \lambda$-presented submodules $A' \subseteq A$ with $A' \in \mathcal{A}$. Clearly, $0 \in S_\lambda$, and S_λ is closed under unions of well-ordered chains of length $< \lambda$ since $\mathcal{A}$ is closed under arbitrary direct limits. By Lemma 4.5, each subset of A of cardinality $< \lambda$ is contained in an element of S_λ. By inductive premise, S_λ consists of "free" modules for all regular $\omega < \lambda < \kappa$, so A is "free" by [16, IV.3.7]. This proves that each $A \in \mathcal{A}$ is $\mathcal{A}_0$-filtered.

So, we infer from the Eklof Lemma [16, XII.1.5] (see also [18, 3.1.2]) that $\mathcal{B} = (\mathcal{A}_0)^{\perp_1}$. Finally, Proposition 4.1 shows that $\mathfrak{C}$ is of finite type. □

Corollary 4.7. Let R be an arbitrary ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair such that $\mathcal{A}$ and $\mathcal{B}$ are closed under direct limits. Then $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$ if and only if $\mathfrak{C}$ is of finite type.

Now, we are going to prove a particular case of Conjecture 0.1 for arbitrary right noetherian rings. This will imply validity of 0.1 in the particular case $\mathfrak{C}$ is a cotilting cotorsion pair over a right noetherian ring.

By Theorem 4.6, the proof of Conjecture 0.1 amounts to showing that $\mathfrak{C}$ is of finite type. First, we need a lemma which is implicit already in [7], and the dual version of which appears in [28]:

Lemma 4.8. Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a smashing cotorsion pair cogenerated by a class $\mathcal{C}$ such that $\mathcal{C}^\perp$ contains all direct sums of injective modules. Then $\mathcal{C}^{\perp_n}$ is closed under arbitrary direct sums for each $n \geq 1$.

Proof: By induction on n. The case of $n = 1$ is clear since $\mathfrak{C}$ is smashing. Let $(M_\alpha \mid \alpha < \kappa)$ be a family of modules in $\mathcal{C}^{\perp_{n+1}}$. Consider short exact sequences

$$0 \longrightarrow M_\alpha \longrightarrow I_\alpha \longrightarrow C_\alpha \longrightarrow 0$$

with I_α injective for each $\alpha < \kappa$. Since $0 = \operatorname{Ext}_R^{n+1}(A, M_\alpha) \cong \operatorname{Ext}_R^n(A, C_\alpha)$ for all $A \in \mathcal{C}$, the inductive premise gives $\bigoplus_{\alpha<\kappa} C_\alpha \in \mathcal{C}^{\perp_n}$, so our assumption on $\mathcal{C}^\perp$ yields $\bigoplus_{\alpha<\kappa} M_\alpha \in \mathcal{C}^{\perp_{n+1}}$. □

Proposition 4.9. Let R be a right coherent ring. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a smashing hereditary cotorsion pair cogenerated by a class $\mathcal{C} \subseteq (\operatorname{Mod} R)^{\leq\omega}$ and such that $\mathcal{B} \subseteq \mathcal{I}_n$ for some $n \geq 0$. Then $\mathfrak{C}$ is of finite type.

Proof: We will construct cotorsion pairs $\mathfrak{C}_i = (\mathcal{A}_i, \mathcal{B}_i)$, $1 \leq i \leq n+1$, such that

$$\mathcal{B} = \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots \subseteq \mathcal{B}_n \subseteq \mathcal{B}_{n+1}$$

and by reverse induction on i, we will show that $\mathfrak{C}_i$ is of finite type for each $1 \leq i \leq n+1$.

Let us start with the cotorsion pair $\mathfrak{C}_{n+1} = (\mathcal{A}_{n+1}, \mathcal{B}_{n+1})$ cogenerated by the class $\mathcal{S}_{n+1}$ of all modules that are k-th syzygies of modules from $\operatorname{mod} R$ for some $k \geq n$. Then $\mathcal{B}_{n+1} = \bigcap_{k \geq n+1} (\operatorname{mod} R)^{\perp_k}$, and we claim that $\mathcal{B}_{n+1} \subseteq \bigcap_{k \geq n+1} \mathcal{A}^{\perp_k}$. In fact, $(\operatorname{mod} R)^{\perp_1}$ coincides with the class of all pure submodules of injective modules since R is right coherent. Moreover, $\operatorname{mod} R$ is resolving, so $(\operatorname{mod} R)^{\perp_1} = (\operatorname{mod} R)^\perp$ by Remark 1.2. Since $\mathcal{B}$ is definable by Proposition 4.1(2), we deduce that $(\operatorname{mod} R)^\perp \subseteq \mathcal{B}$, and our claim follows by dimension shifting.

We now set $\mathcal{B}_i = \mathcal{B}_{n+1} \cap \bigcap_{k \geq i} \mathcal{A}^{\perp_k}$ for $1 \leq i \leq n$. Then, as $\mathcal{B} \subseteq \mathcal{I}_n \subseteq \mathcal{B}_{n+1}$, we have $\mathcal{B} = \mathcal{B}_1$. Moreover, all $\mathcal{B}_i$ are obviously coresolving. Further, applying Lemma 4.8 to $\mathfrak{C}$ (which is possible because $\mathfrak{C}$ is cogenerated by $\mathcal{A}$ and $\mathcal{A}^\perp = \mathcal{B}$ contains all direct sums of injective modules), we infer that all $\mathcal{B}_i$ are closed under direct sums.

For each $1 \le i \le n$, we thus obtain a hereditary smashing cotorsion pair $\mathfrak{C}_i = (\mathcal{A}_i, \mathcal{B}_i)$ which is cogenerated by a class of countably presented modules, namely by $\mathcal{S}_i = \mathcal{S}_{n+1} \cup \mathcal{C}_i$, where $\mathcal{C}_i$ denotes the class of all modules that are k-th syzygies of modules from $\mathcal{C}$ for some $k \ge i - 1$.

Of course, $\mathfrak{C}_{n+1}$ is of finite type. Let $1 \le i \le n$, and let $M \in \mathcal{S}_i$. We have a short exact sequence

$$0 \longrightarrow K \longrightarrow R^{(\omega)} \longrightarrow M \longrightarrow 0.$$

We claim that $K \in \mathcal{A}_{i+1}$. Indeed, if $N \in \mathcal{B}_{i+1} = \mathcal{B}_{n+1} \cap \bigcap_{k \ge i+1} \mathcal{A}^{\perp_k}$ then its first cosyzygy C belongs to $\mathcal{B}_i$, so $\operatorname{Ext}^2_R(\mathcal{A}_i, N) = 0$, and in particular, $\operatorname{Ext}^2_R(M, N) = 0$, hence $\operatorname{Ext}^1_R(K, N) = 0$. This proves the claim.

By inductive premise, $\mathfrak{C}_{i+1}$ is of finite type, hence cogenerated by $\mathcal{A}_{i+1}^{<\omega}$. By Theorem 1.7, it follows that K is a direct summand in a $\mathcal{A}_{i+1}^{<\omega}$-filtered module. Using [9, Lemma 3.3] (see also [18, 5.2.20, p.215]), we obtain the exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow M \longrightarrow 0$$

with H and G countably generated $\mathcal{A}_{i+1}^{<\omega}$-filtered modules. W.l.o.g. we can assume that H is a submodule of G. As in the proof of [9, Lemma 3.6], we show that $M \in \varinjlim \mathcal{A}_i^{<\omega}$. We state here the argument for the reader's convenience.

By [9, Corollary 3.2] (see also [18, 4.2.6]), we can write $H = \bigcup_{k<\omega} H_k$ and $G = \bigcup_{k<\omega} G_k$ where, for every $k < \omega$, H_k and G_k are finitely presented $\mathcal{A}_{i+1}^{<\omega}$-filtered modules, and H/H_k, G/G_k are $\mathcal{A}_{i+1}^{<\omega}$-filtered. Given $k < \omega$, there is j_k such that $H_k \subseteq G_{j_k}$. Moreover, we can choose the sequence $(j_k \mid k < \omega)$ to be strictly increasing.

We claim that $G_{j_k}/H_k \in \mathcal{A}_i^{<\omega}$. Clearly, G_{j_k}/H_k finitely presented and R right coherent implies $G_{j_k}/H_k \in \operatorname{mod} R$, thus we have to show that $\operatorname{Ext}^1_R(G_{j_k}/H_k, B) = 0$ for each $B \in \mathcal{B}_i$. Since $G_{j_k} \in \mathcal{A}_{i+1} \subseteq \mathcal{A}_i$, we need only to check that every $f \in \operatorname{Hom}_R(H_k, B)$ can be extended to a homomorphism from G_{j_k} to B. We have $\operatorname{Ext}^1_R(H/H_k, B) = 0$ because $H/H_k \in \mathcal{A}_{i+1}$, thus we may extend f to a homomorphism f' from H to B, and then, since $G/H \cong M \in \mathcal{A}_i$, to a homomorphism g from G to B. The restriction of g to G_{j_k} obviously induces an extension of f to G_{j_k}. Our claim is proved.

Set $C_k = G_{j_k}/H_k$. Since $(j_k \mid k < \omega)$ is increasing and unbounded in ω, the inclusions $G_{j_k} \subseteq G_{j_{k+1}}$ induce maps $f_k : C_k \to C_{k+1}$, and M is a direct limit of the direct system $((C_k, f_k) \mid k < \omega)$.

But then, since $M \in \mathcal{S}_i$ was arbitrary, it follows that $\mathcal{S}_i \subseteq \varinjlim \mathcal{A}_i^{<\omega}$, and so $\mathfrak{C}_i$ is of finite type by Proposition 4.1(3). □

Theorem 4.10. Let R be a right noetherian ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary smashing cotorsion pair. If either

(i) $\mathcal{A}$ consists of modules of bounded projective dimension, or

(ii) $\mathcal{B}$ consists of modules of bounded injective dimension,

then $\mathfrak{C}$ is of finite type. [1]

Proof: By [31], (i) implies that $\mathfrak{C}$ is a tilting cotorsion pair, hence $\mathfrak{C}$ is of finite type by [9] (Indeed, this holds for an arbitrary ring R, cf. [18, 5.1.16 and 5.2.20].)

Assume (ii). Then it follows from [28, Corollary 1.10] that $\mathfrak{C}$ is cogenerated by a class of countably presented modules, so it is of finite type by Proposition 4.9. □

Corollary 4.11. Let R be a right noetherian ring, and $(\mathcal{A}, \mathcal{B})$ an n-cotilting cotorsion pair. Then the following statements are equivalent.

[1] *Added in proof.* The bounds on the homological dimension can be removed in the sense that if R is any ring and $\mathfrak{C}$ satisfies the assumptions of 0.1, then $\mathfrak{C}$ is of countable type, $\mathcal{A}$ is closed under pure-epimorphic images, and $\mathcal{B}$ is definable. This is proved in a recent manuscript by the second author and J. Šťovíček, entitled "The countable telescope conjecture for module categories."

(1) $(\mathcal{A},\mathcal{B})$ is of finite type.
(2) $\mathcal{B}$ is definable.
(3) There is a Σ-pure-injective cotilting module C such that $\mathcal{A} = {}^{\perp}C$.

Proof: By Proposition 3.2, condition (3) means that $(\mathcal{A},\mathcal{B})$ is smashing. So, we have (1)⇒(2)⇒(3). (3)⇒(1) is an immediate consequence of Theorem 4.10. □

5. Extensions of small cotorsion pairs.

We close the paper by pointing out some asymmetries that can occur in the behaviour of the classes involved in a cotorsion pair. Throughout this section Λ denotes an artin algebra.

Definition. [29] Let $(\mathcal{S},\mathcal{T})$ be a cotorsion pair in modΛ. A cotorsion pair $(\mathcal{X},\mathcal{Y})$ in ModΛ is said to be an *extension* of $(\mathcal{S},\mathcal{T})$ if $\mathcal{X}^{<\omega} = \mathcal{S}$ and $\mathcal{Y}^{<\omega} = \mathcal{T}$.

We have seen above three different ways of extending $(\mathcal{S},\mathcal{T})$.

Proposition 5.1. Let $(\mathcal{S},\mathcal{T})$ be a cotorsion pair in modΛ. The following cotorsion pairs are extensions of $(\mathcal{S},\mathcal{T})$:
(1) the complete cotorsion pair $(\mathcal{A},\mathcal{B})$ cogenerated by $\mathcal{S}$,
(2) the perfect cotorsion pair $(\varinjlim \mathcal{S},\mathcal{Y})$,
(3) the perfect cotorsion pair $(\mathcal{C},\mathcal{D})$ generated by $\mathcal{T}$.
They are related by the inclusions $\mathcal{A} \subseteq \varinjlim \mathcal{S} \subseteq \mathcal{C}$ and $\mathcal{D} \subseteq \mathcal{Y} \subseteq \mathcal{B}$.

Proof: We already know that the first cotorsion pair is complete, and the third is perfect since it is of cofinite type. Observe further that $\mathcal{S}$ has properties (R1) and (R2). By Theorem 2.1 we then have that the second cotorsion pair is perfect, generated by the pure-injective modules from $\mathcal{B}$, and moreover, $\mathcal{S} = \mathcal{A}^{<\omega} = (\varinjlim \mathcal{S})^{<\omega}$. Furthermore, $\mathcal{B}^{<\omega} = (\mathcal{S}^{\perp_1})^{<\omega} = \mathcal{T}$ since $(\mathcal{S},\mathcal{T})$ is a cotorsion pair in modΛ, and similarly $\mathcal{C}^{<\omega} = ({}^{\perp_1}\mathcal{T})^{<\omega} = \mathcal{S}$. In particular, $\mathcal{T}$ consists of pure-injective modules from $\mathcal{B}$. We then infer that $\mathcal{A} \subseteq \varinjlim \mathcal{S} \subseteq \mathcal{C}$, and thus $\mathcal{T} \subseteq \mathcal{D} \subseteq \mathcal{Y} \subseteq \mathcal{B}$. But this implies that $\mathcal{D}^{<\omega} = \mathcal{Y}^{<\omega} = \mathcal{B}^{<\omega} = \mathcal{T}$, and the proof is complete. □

Let us look at some examples.

Examples 5.2. (1) Let Λ be a tame hereditary artin algebra (w.l.o.g. basic indecomposable), and let the notation be as in Example 3.4. We set $\mathcal{S} = \mathrm{add}\,(\mathbf{p} \cup \mathbf{t})$ where $\mathbf{p}$ denotes the class of all indecomposable preprojective modules, and $\mathcal{T} = \mathrm{add}\,(\mathbf{q})$. Then $(\mathcal{S},\mathcal{T})$ is a cotorsion pair in modΛ, and the three extensions in Proposition 5.1 coincide. Note however that $(\mathcal{S},\mathcal{T})$ is not complete in modΛ, and that the generic module G belongs to $\mathcal{D} \setminus \varinjlim \mathcal{T}$, so $\mathcal{D} \neq \varinjlim \mathcal{D}^{<\omega}$. In particular, we see that the validity of Conjecture 0.1 for $(\mathcal{A},\mathcal{B})$ does not imply $\mathcal{B} = \varinjlim \mathcal{B}^{<\omega}$.

(2) If $(\mathcal{S},\mathcal{T})$ is a cotorsion pair in modΛ with $\mathcal{S} \subseteq (\mathcal{P}_1)^{<\omega}$, then the second and the third cotorsion pair in Proposition 5.1 coincide. In fact, in this case $\mathcal{B}$ is closed under epimorphic images. Since moreover every module over an artin algebra is a pure submodule of a direct product of its finitely generated factor modules, it follows that the modules from $\mathcal{B}$ are pure submodules of a product of modules from $\mathcal{T}$. Now remember that $(\varinjlim \mathcal{S},\mathcal{Y})$ is generated by the class $\mathcal{B}'$ of all pure-injective modules from $\mathcal{B}$. But $\mathcal{B}'$ consists of direct summands of products of modules from $\mathcal{T}$, and so $\varinjlim \mathcal{S} = \mathcal{C}$.

(3) The following example shows that the assumption "smashing" in Theorem 4.6 is essential. Let Λ be the algebra from [23]. We set $\mathcal{S} = (\mathcal{P}_1)^{<\omega}$, and $\mathcal{T} = (\mathcal{S}^{\perp_1})^{<\omega}$. As above we have ${}^{\perp_1}\mathcal{T} = \varinjlim \mathcal{S} = \mathcal{P}_1$, so $({}^{\perp_1}\mathcal{T})^{<\omega} = \mathcal{S}$, and $(\mathcal{S},\mathcal{T})$ is a cotorsion pair

in modΛ. Here again, $(\mathcal{S}, \mathcal{T})$ is not complete. Moreover, although $\mathcal{C} = \varinjlim \mathcal{C}^{<\omega}$, the cotorsion pair $(\mathcal{C}, \mathcal{D}) = (\varinjlim \mathcal{S}, \mathcal{Y})$ is not of finite type. This follows from 2.3, since we know from [4] that the first two cotorsion pairs in Proposition 5.1 do not coincide. In particular, $(\mathcal{C}, \mathcal{D})$ is not smashing (because it cannot be a tilting cotorsion pair, see [1] and [8], see also [18, §5.2]). However, it is of cofinite type, hence cosmashing. □

As a consequence of a result of Krause and Solberg in [22], we can now describe when a cotorsion pair has the shape $(\varinjlim \mathcal{S}, \varinjlim \mathcal{T})$ for some cotorsion pair $(\mathcal{S}, \mathcal{T})$ in modΛ.

Theorem 5.3. The following statements are equivalent for a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in ModΛ.
(1) $(\mathcal{X}, \mathcal{Y}) = (\varinjlim \mathcal{X}^{<\omega}, \varinjlim \mathcal{Y}^{<\omega})$.
(2) $(\mathcal{X}, \mathcal{Y})$ is the unique extension of some complete cotorsion pair $(\mathcal{S}, \mathcal{T})$ in modΛ.
(3) $(\mathcal{X}, \mathcal{Y})$ is of finite type and $\mathcal{X}^{<\omega}$ is contravariantly finite in modΛ.
If $(\mathcal{X}, \mathcal{Y})$ is hereditary and $\mathcal{X} \subseteq \mathcal{P}$, then (1)–(3) are further equivalent to
(4) $\mathcal{Y} = T^{\perp}$ for a tilting module $T \in$ modΛ.

Proof: (1)⇒(2): Set $\mathcal{T} = \mathcal{Y}^{<\omega}$. First of all, since $\mathcal{Y} = \varinjlim \mathcal{T}$ is closed under products, it follows from 1.3 and 1.4 that there is a complete cotorsion pair $(\mathcal{S}, \mathcal{T})$ in modΛ. We then know from [22, 2.4] that the three extensions of $(\mathcal{S}, \mathcal{T})$ in Proposition 5.1 coincide with $(\mathcal{X}, \mathcal{Y})$. Suppose now that $(\mathcal{E}, \mathcal{F})$ is a further extension of $(\mathcal{S}, \mathcal{T})$. Then $\mathcal{S} \subseteq \mathcal{E}$, thus $\mathcal{F} \subseteq \mathcal{Y} = \varinjlim \mathcal{T}$, hence $\varinjlim \mathcal{S} \subseteq \mathcal{E}$. Further $\mathcal{T} \subseteq \mathcal{F}$, thus $\mathcal{E} \subseteq \mathcal{X} = \varinjlim \mathcal{S}$. This shows $(\mathcal{E}, \mathcal{F}) = (\mathcal{X}, \mathcal{Y})$, so there is a unique extension.
(2)⇒(3): Consider the complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ cogenerated by $\mathcal{S}$. Then $\mathcal{B}^{<\omega} = (\mathcal{S}^{\perp_1})^{<\omega} = \mathcal{T}$, and by Theorem 2.1(1) we have $\mathcal{A}^{<\omega} = \mathcal{S}$. So $(\mathcal{A}, \mathcal{B})$ is an extension of $(\mathcal{S}, \mathcal{T})$ and therefore coincides with $(\mathcal{X}, \mathcal{Y})$.
(3)⇒(1): $(\mathcal{X}, \mathcal{Y})$ is cogenerated by $\mathcal{S} = \mathcal{X}^{<\omega}$, and there is a complete cotorsion pair $(\mathcal{S}, \mathcal{T})$ in modΛ. By [22, 2.4] it follows that $(\mathcal{X}, \mathcal{Y}) = (\varinjlim \mathcal{S}, \varinjlim \mathcal{T})$ and $\mathcal{T} = \mathcal{Y}^{<\omega}$.
The equivalence with (4) follows from a well-known result of Auslander and Reiten [6], see [3, 4.1]. □

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THE COUNTABLE TELESCOPE CONJECTURE FOR MODULE CATEGORIES

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Abstract. By the Telescope Conjecture for Module Categories, we mean the following claim: "Let R be any ring and $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in Mod-R with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits. Then $(\mathcal{A}, \mathcal{B})$ is of finite type."

We prove a modification of this conjecture with the word 'finite' replaced by 'countable'. We show that a hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ of modules over an arbitrary ring R is generated by a set of strongly countably presented modules provided that $\mathcal{B}$ is closed under unions of well-ordered chains. We also characterize the modules in $\mathcal{B}$ and the countably presented modules in $\mathcal{A}$ in terms of morphisms between finitely presented modules, and show that $(\mathcal{A}, \mathcal{B})$ is cogenerated by a single pure-injective module provided that $\mathcal{A}$ is closed under direct limits. Then we move our attention to strong analogies between cotorsion pairs in module categories and localizing pairs in compactly generated triangulated categories.

Motivated by the paper [30] of Krause and Solberg, the first author with Lidia Angeleri Hügel and Jan Trlifaj started in [4] an investigation of the *Telescope Conjecture for Module Categories* (TCMC) stated as follows (see Section 1 for unexplained terminology):

Telescope Conjecture for Module Categories. Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in Mod-R with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits. Then $\mathcal{A} = \varinjlim(\mathcal{A} \cap \text{mod-}R)$.

The term 'Telescope Conjecture' is used here because the particular case of TCMC when R is a self-injective artin algebra and $(\mathcal{A}, \mathcal{B})$ is a projective cotorsion pair was shown in [30] to be equivalent to the following telescope conjecture for compactly generated triangulated categories (in this case—for the stable module category over R) which originates in works of Bousfield [12] and Ravenel [38] and has been extensively studied by Krause in [29, 27]:

Telescope Conjecture for Triangulated Categories. Every smashing localizing subcategory of a compactly generated triangulated category is generated by compact objects.

Under some restrictions on homological dimensions of modules in the cotorsion pair $(\mathcal{A}, \mathcal{B})$, TCMC is known to hold. The first author and co-authors showed in [4] that the conclusion of TCMC amounts to saying that the given cotorsion pair is of finite type. If all modules in $\mathcal{A}$ have finite projective dimension, then the cotorsion pair is tilting [42], hence of finite type [9]. If R is a right noetherian ring and $\mathcal{B}$ consists of modules of finite injective dimension, then $(\mathcal{A}, \mathcal{B})$ is of finite type, too [4]. Therefore, TCMC holds true for example for any cotorsion pair over a ring with finite global dimension. Unfortunately, the interesting connection with triangulated categories introduced in [30] works for self-injective artin algebras, where the only cotorsion pairs satisfying the former conditions are the trivial ones.

The aim of this paper is twofold. First, we prove the Countable Telescope Conjecture in Theorem 3.5: any cotorsion pair satisfying the hypotheses of TCMC is of countable type—that is, the class $\mathcal{B}$ is the Ext^1-orthogonal class to the class of all (strongly) countably presented modules from $\mathcal{A}$. This is a weaker version of TCMC. We will also show that this result easily implies a more direct argument for a large part of the proof that all tilting classes are of finite type [7, 8, 42, 9].

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The second goal is to systematically analyze analogies between approximation theory for cotorsion pairs and results about localizations in compactly generated triangulated categories. Considerable efforts have been made on both sides. Cotorsion pairs were introduced by Salce in [40] where he noticed a homological connection between special preenvelopes and precovers—or left and right approximation in the terminology of [6]. In [16], Eklof and Trlifaj proved that any cotorsion pair generated by a set of modules provides for these approximations. This turns out to be quite a usual case and the related theory with many applications is explained in the recently issued monograph [19]. Localizations of triangulated categories have, on the other hand, motivation in algebraic topology. The telescope conjecture above was introduced by Bousfield [12, 3.4] and Ravenel [38, 1.33]. Compactly generated triangulated categories and their localizations were studied by Neeman [34, 35] and Krause [29, 27]. Even though the telescope conjecture is known to be false for general triangulated categories [26], it is still open for the important and topologically motivated stable homotopy category as well as for stable module categories over self-injective artin algebras.

Although it should not be completely unexpected that there are some analogies between the two settings, as the derived unbounded category is triangulated compactly generated and provides a suitable language for homological algebra, the extent to which the analogies work is rather surprising. Roughly speaking, it is sufficient to replace an Ext^1-group in a module category by a Hom-group in a triangulated category, and we obtain a valid result. However, there are also substantial differences here—for instance special precovers and preenvelopes provided by cotorsion pairs are, unlike adjoint functors coming from localizations, not functorial.

In Section 4, we prove in Theorem 4.9 that if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair meeting the assumptions of TCMC, then $\mathcal{B}$ is defined by finite data in the sense that it is the Ext^1-orthogonal class to a certain ideal of maps between finitely presented modules. Moreover, we characterize the countably generated modules in $\mathcal{A}$ as direct limits of systems of maps from this ideal (Theorem 4.8). In Section 5, we prove in Theorem 5.13 that $\mathcal{A} = \mathrm{Ker}\,\mathrm{Ext}^1(-, E)$ for a single pure-injective module E.

Finally, in Section 6, we give the triangulated category analogues of all of the main results for module categories. Some of them come from our analysis, while the others were originally proved by Krause in [29] and served as a source of inspiration for this paper.

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1. Preliminaries

Throughout this paper, R will always stand for an associative ring with unit, and all modules will be (unital) right R-modules. We call a module *strongly countably presented* if it has a projective resolution consisting of countably generated projective modules. *Strongly finitely presented* modules are defined in the same manner with the word 'countably' replaced by 'finitely'. We denote the class of all modules by Mod-R and the class of all strongly finitely presented modules by mod-R.

We note that the notation mod-R is often used in the literature for the class of *finitely presented modules*; that is, the modules M possessing a presentation $P_1 \to P_0 \to M \to 0$ where P_0 and P_1 are finitely generated and projective. We have digressed a little from this de-facto standard for the sake of keeping our notation simple, and we believe that this should not cause much confusion. We remind that if R is a right coherent ring, then the class of strongly finitely presented modules coincides with the class of finitely presented ones. Moreover, one typically restricts oneself to coherent rings in various applications.

1.1. **Continuous directed sets and associated filters.** Let $(I, \leq)$ be a partially ordered set and λ be an infinite regular cardinal. We say that I is λ-*complete* if every well-ordered ascending chain $(i_\alpha \mid \alpha < \tau)$ of elements from I of length $< \lambda$ has a supremum in I. If this is the case, we call a subset $J \subseteq I$ λ-*closed* if, whenever such a chain is contained in J, its supremum is in J as well. For instance for any set X, the power set $\mathfrak{P}(X)$ ordered by inclusion is λ-complete and the set $\mathfrak{P}^{<\lambda}(X)$ of all subsets of X of cardinality $< \lambda$ is λ-closed in $\mathfrak{P}(X)$.

Recall that a subset $J \subseteq I$ is called *cofinal* if for every $i \in I$ there is $j \in J$ such that $i \leq j$. Note that if I is a totally ordered set, then the cofinal subsets of I are precisely the unbounded ones.

From now on, we assume that $(I, \leq)$ is a directed set. If $(M_i, f_{ji} : M_i \to M_j \mid i, j \in I \;\&\; i \leq j)$ is a direct system of modules, we call it λ-*continuous* if the index set I is λ-complete and for each well-ordered ascending chain $(i_\alpha \mid \alpha < \tau)$ in I of length $< \lambda$ we have

$$M_{\sup i_\alpha} = \varinjlim_{\alpha<\tau} M_{i_\alpha}.$$

It is well-known that every module is the direct limit of a direct system of finitely presented modules. But if we want the direct system to be λ-continuous, we have to pass to $< \lambda$-presented modules in general. The following lemma is a slight modification of [24, Proposition 7.15].

Lemma 1.1. *Let M be any module and λ an infinite regular cardinal. Then M is the direct limit of a λ-continuous direct system of $< \lambda$-presented modules.*

Proof. Fix a free presentation

$$R^{(X)} \xrightarrow{f} R^{(Y)} \to M \to 0$$

of M and let I be the following set:

$$\big\{(X', Y') \in \mathfrak{P}(X) \times \mathfrak{P}(Y) \mid |X'| + |Y'| < \lambda \;\&\; f\big[R^{(X')}\big] \subseteq R^{(Y')}\big\}.$$

It is straightforward to check that I with the partial ordering by inclusion in both components is directed and λ-complete. If we now define M_i as the cokernel of the map

$$f \restriction R^{(X')} : R^{(X')} \to R^{(Y')}$$

for every $i = (X', Y') \in I$, it is easy to check that $(M_i \mid i \in I)$ together with the natural maps forms a λ-continuous direct system with M as its direct limit. □

For every directed set I, there is an *associated filter* $\mathfrak{F}_I$ on $(\mathfrak{P}(I), \subseteq)$; namely the one with a basis consisting of the upper sets $\uparrow i = \{j \in I \mid j \geq i\}$ for all $i \in I$. That is

$$\mathfrak{F}_I = \{X \subseteq I \mid (\exists i \in I)(\uparrow i \subseteq X)\}.$$

Recall that a filter $\mathfrak{F}$ on a power set is called λ-*complete* if any intersection of less than λ elements from $\mathfrak{F}$ is again in $\mathfrak{F}$.

Lemma 1.2. *Let $(I, \leq)$ be a λ-complete directed set. Then any subset $J \subseteq I$ such that $|J| < \lambda$ has an upper bound in I. In particular, the associated filter $\mathfrak{F}_I$ is λ-complete, and it is a principal filter if and only if $(I, \leq)$ has a (unique) maximal element.*

Proof. We can well-order J; that is $J = \{j_\alpha \mid \alpha < \tau\}$ for some $\tau < \lambda$. Then we construct by induction a chain $(k_\alpha \mid \alpha < \tau)$ in I such that $k_0 = j_0$ and k_α is a common upper bound for j_α and $\sup_{\beta<\alpha} k_\beta$. Then $\sup_{\beta<\tau} k_\beta$ is clearly an upper bound for J. The rest is also easy. □

1.2. **Filtrations and cotorsion pairs.** Given a module M and an ordinal number σ, an ascending chain $\mathcal{F} = (M_\alpha \mid \alpha \leq \sigma)$ of submodules of M is called a *filtration of* M if $M_0 = 0$, $M_\sigma = M$ and $\mathcal{F}$ is *continuous*—that is, $\bigcup_{\alpha<\beta} M_\alpha = M_\beta$ for each limit ordinal $\beta \leq \sigma$.

Furthermore, let a class $\mathcal{C} \subseteq$ Mod-R be given. Then $\mathcal{F}$ is said to be a $\mathcal{C}$-*filtration* if it has the extra property that each its consecutive factor $M_{\alpha+1}/M_\alpha$, $\alpha < \sigma$, is isomorphic to a module from $\mathcal{C}$. A module M is called $\mathcal{C}$-*filtered* if it admits (at least one) $\mathcal{C}$-filtration.

Let us turn our attention to cotorsion pairs now. By a *cotorsion pair* in Mod-R, we mean a pair $(\mathcal{A}, \mathcal{B})$ of classes of right R-modules such that $\mathcal{A} = \operatorname{Ker}\operatorname{Ext}^1_R(-, \mathcal{B})$ and $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{A}, -)$. We say that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *hereditary* provided that $\mathcal{A}$ is closed under kernels of epimorphisms or, equivalently, $\mathcal{B}$ is closed under cokernels of monomorphisms.

If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair, then the class $\mathcal{A}$ is always closed under arbitrary direct sums and contains all projective modules. Dually, the class $\mathcal{B}$ is closed under direct products and it contains all injective modules. Also, every class of modules $\mathcal{C}$ determines two distinguished cotorsion pairs—the cotorsion pair *generated* by $\mathcal{C}$, that is the one with the right-hand class $\mathcal{B}$ equal to $\operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{C}, -)$, and dually the cotorsion pair *cogenerated*[1] by $\mathcal{C}$—the one with the

[1]It may cause some confusion that the meaning of the terms *generated* and *cogenerated* is sometimes swapped in the literature. Our terminology follows the monograph [19].

left-hand class $\mathcal{A}$ equal to $\operatorname{Ker}\operatorname{Ext}^1_R(-,\mathcal{C})$. We say that $(\mathcal{A},\mathcal{B})$ is of *finite* or *countable type* if it is generated by a set of strongly finitely or strongly countably presented modules, respectively.

We say that a cotorsion pair $(\mathcal{A},\mathcal{B})$ is *complete* if for every module $M \in \text{Mod-}R$, there is a short exact sequence $0 \to B \to A \to M \to 0$ such that $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The map $A \to M$ is then called a *special $\mathcal{A}$-precover* of M. It is well-known that this condition is equivalent to the dual one saying that $\mathcal{B}$ provides for *special $\mathcal{B}$-preenvelopes*; thus, for every $M \in \text{Mod-}R$ there is in this case also a short exact sequence $0 \to M \to B' \to A' \to 0$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$.

Finally, a cotorsion pair is said to be *projective* in the sense of [10] if it is hereditary, complete, and $\mathcal{A} \cap \mathcal{B}$ is precisely the class of all projective modules. It is an easy exercise to prove that $(\mathcal{A},\mathcal{B})$ is projective if and only if it is complete and $\mathcal{B}$ contains all projective modules and has the "two out of three" property—that is: all three modules in a short exact sequence are in $\mathcal{B}$ provided that two of them are in $\mathcal{B}$. To conclude the discussion of terminology concerning cotorsion pairs, we recall that projective cotorsion pairs over self-injective artin algebras are (with a slightly different but equivalent definition) called *thick* in [30].

1.3. **Definable classes and coherent functors.** We will also need the notion of a definable class of modules. First recall that a covariant additive functor from Mod-R to the category of abelian groups is called *coherent* if it commutes with arbitrary products and direct limits. The following important characterization was obtained by Crawley-Boevey:

Lemma 1.3. [13, §2.1, Lemma 1] *A functor $F : \text{Mod-}R \to \text{Ab}$ is coherent if and only if it is isomorphic to* $\operatorname{Coker}\operatorname{Hom}_R(f,-)$ *for some homomorphism $f : X \to Y$ between finitely presented modules X and Y.*

A class $\mathcal{C} \subseteq \text{Mod-}R$ is called *definable* if it satisfies one of the following three equivalent conditions:

(1) $\mathcal{C}$ is closed under taking arbitrary products, direct limits, and pure submodules;
(2) $\mathcal{C}$ is defined by vanishing of some set of coherent functors;
(3) $\mathcal{C}$ is defined in the first order language of R-modules by satisfying some implications $\varphi(\bar{x}) \to \psi(\bar{x})$ where $\varphi(\bar{x})$ and $\psi(\bar{x})$ are primitive positive formulas.

Primitive positive formulas (pp-formulas for short) are first-order language formulas of the form $(\exists\bar{y})(\bar{x}A = \bar{y}B)$ for some matrices A, B over R. For this paper, the most important consequence of (3) is that definable classes are closed under taking elementarily equivalent modules since they are definable in the first-order language. This in particular implies the well-known fact that a definable class is determined by the pure-injective modules it contains since any module is elementarily equivalent to its pure-injective hull. For equivalence between the three definitions and more details, we refer to [37], [13, §2.3], and [45, Section 1].

1.4. **Inverse limits and the Mittag-Leffler condition.** The computation of Ext groups can sometimes be reduced to the computation of the derived functors of inverse limit. We will recall this here only for countable inverse systems. For more details on the topic see [44, §3.5]. Let

$$\cdots \to H_{n+1} \xrightarrow{h_n} H_n \to \cdots \to H_2 \xrightarrow{h_1} H_1 \xrightarrow{h_0} H_0$$

be a countable inverse system of abelian groups—a tower in the terminology of [44]. Then its inverse limit $\varprojlim H_n$ and the *first derived functor of the inverse limit*, $\varprojlim^1 H_n$, can be computed using the exact sequence

$$0 \to \varprojlim H_n \to \prod H_n \xrightarrow{\Delta} \prod H_n \to \varprojlim{}^1 H_n \to 0$$

where $\Delta((x_n)_{n<\omega}) = (x_n - h_n(x_{n+1}))_{n<\omega}$. The first derived functor is closely related to the fact that inverse limit is not exact—it is only left exact. Using the exact sequence above and the snake lemma, one easily observes that, given a countable inverse system of short exact sequences $0 \to H_n \to K_n \to L_n \to 0$, there is a canonical long exact sequence

$$0 \to \varprojlim H_n \to \varprojlim K_n \to \varprojlim L_n \to \varprojlim{}^1 H_n \to \varprojlim{}^1 K_n \to \varprojlim{}^1 L_n \to 0$$

In particular, $\varprojlim^1$ is right exact on countable inverse systems.

In practice, one is often interested whether or not $\varprojlim^1 H_n = 0$. To decide this can sometimes be tedious, but there is a useful tool—the notion of Mittag-Leffler inverse systems. Given a countable inverse system of abelian groups $(H_n, h_n \mid n < \omega)$ as above, we say that it is *Mittag-Leffler* if for each n the descending chain

$$H_n \supseteq h_n(H_{n+1}) \supseteq \cdots \supseteq h_n h_{n+1} \cdots h_{k-1}(H_k) \supseteq \cdots$$

is stationary. This occurs, for example, if all the maps h_n are onto. The following important result gives a connection to $\varprojlim^1$:

Proposition 1.4. *Let $(H_n, h_n \mid n < \omega)$ be a countable inverse system of abelian groups. Then the following hold:*

(1) [44, Proposition 3.5.7] *If (H_n, h_n) is Mittag-Leffler, then $\varprojlim^1 H_n = 0$.*
(2) [2, Theorem 1.3] *(H_n, h_n) is Mittag-Leffler if and only if $\varprojlim^1 H_n^{(\omega)} = 0$.*

We will also use a related notion of T-nilpotency. We say that $(H_n, h_n)_{n<\omega}$ is *T-nilpotent* if for each n there exists $k > n$ such that the composition $H_k \to H_n$ is zero.

2. Filter-closed classes and factorization systems

We start with analyzing properties of modules lying in $\operatorname{Ker}\operatorname{Ext}^1_R(-, \mathcal{G})$ for a class $\mathcal{G}$ closed under arbitrary direct products and unions of well-ordered chains. We will always assume in this case that $\mathcal{G}$ is closed under isomorphic images and that $0 \in \mathcal{G}$, since the trivial module could be viewed as a product of an empty system. As an application to keep in mind, such classes occur as right-hand classes of cotorsion pairs satisfying the hypotheses of TCMC.

Definition 2.1. Let $\mathfrak{F}$ be a filter on the power set $\mathfrak{P}(X)$ for some set X, and let $\{M_x \mid x \in X\}$ be a set of modules. Set $M = \prod_{x \in X} M_x$. Then the *$\mathfrak{F}$-product* $\Sigma_{\mathfrak{F}} M$ is the submodule of M such that

$$\Sigma_{\mathfrak{F}} M = \{m \in M \mid z(m) \in \mathfrak{F}\}$$

where for an element $m = (m_x \mid x \in X) \in M$, we denote by $z(m)$ its zero set $\{x \in X \mid m_x = 0\}$.

The module $M/\Sigma_{\mathfrak{F}} M$ is then called an *$\mathfrak{F}$-reduced product*. Note that for $a, b \in M$, we have an equality $\bar{a} = \bar{b}$ in the $\mathfrak{F}$-reduced product if and only if a and b agree on a set of indices that is in the filter $\mathfrak{F}$.

In the case that $M_x = M_y$ for every pair of elements $x, y \in X$, we speak of an *$\mathfrak{F}$-power* and an *$\mathfrak{F}$-reduced power* (of the module M_x) instead of an $\mathfrak{F}$-product and an $\mathfrak{F}$-reduced product, respectively.

Finally, a nonempty class of modules $\mathcal{G}$ is called *filter-closed*, if it is closed under arbitrary $\mathfrak{F}$-products (for any set X and an arbitrary filter $\mathfrak{F}$ on $\mathfrak{P}(X)$).

Lemma 2.2. *Let $\mathcal{G}$ be a class of modules closed under arbitrary direct products and unions of well-ordered chains. Then $\mathcal{G}$ is filter-closed.*

Proof. It is just a matter of straightforward induction to prove that the closure under unions of well-ordered chains implies closure under arbitrary directed unions—see for instance [1, Corollary 1.7] which is easily adapted for unions. Moreover, any $\mathfrak{F}$-product is just the directed union of products of the modules with indices from the complementary sets to those belonging to $\mathfrak{F}$. □

In the next few paragraphs, we will show that filter-closedness of $\mathcal{G}$ forces existence of certain factoring systems inside modules from $\operatorname{Ker}\operatorname{Ext}^1_R(-, \mathcal{G})$. Let us note that the following lemma presents the crucial technical step in proving the Countable Telescope Conjecture.

Lemma 2.3. *Let $\mathcal{G}$ be a filter-closed class of modules. Let λ be an* uncountable *regular cardinal and $(M, f_i \mid i \in I)$ be a direct limit of a λ-continuous direct system $(M_i, f_{ji} \mid i \leq j)$ indexed by a set I and consisting of $< \lambda$-generated modules.*

Assume that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$. Then there is a λ-closed cofinal subset $J \subseteq I$ such that every homomorphism from M_j to B factors through f_j whenever $j \in J$ and $B \in \mathcal{G}$.

Proof. Suppose that the claim of the lemma is not true. Then the set

$$S = \{i \in I \mid (\exists B_i \in \mathcal{G})(\exists g_i \in \operatorname{Hom}_R(M_i, B_i))(g_i \text{ does not factor through } f_i)\} \qquad (*)$$

must intersect every λ-closed cofinal subset of I (so S is a generalized stationary set, in an obvious sense). For each $i \in S$, choose some $B_i \in \mathcal{G}$ and $g_i : M_i \to B_i$ whose existence is claimed in $(*)$. For the indices $i \in I \setminus S$, let B_i be an arbitrary module from $\mathcal{G}$ and $g_i : M_i \to B_i$ be the zero map. Put $B = \prod_{i \in I} B_i$.

Now, define a homomorphism $h_{ji} : M_i \to B_j$ for each pair $i, j \in I$ in the following way: $h_{ji} = g_j \circ f_{ji}$ if $i \leq j$ and $h_{ji} = 0$ otherwise. This family of maps gives rise to a canonical homomorphism $h : \bigoplus_{k \in I} M_k \to B$. More precisely, if we denote by $\pi_j : B \to B_j$ the projection to the j-th component and by $\nu_i : M_i \to \bigoplus_{k \in I} M_k$ the canonical inclusion of the i-th component, h is (unique) such that $\pi_j \circ h \circ \nu_i = h_{ji}$. Note that for every $i, j \in I$ such that $i \leq j$, the set $\{k \in I \mid h_{ki} = h_{kj} \circ f_{ji}\}$ is in the associated filter $\mathfrak{F}_I$ since it contains $\uparrow j$. Hence, if we denote by φ the canonical pure epimorphism $\bigoplus_{i \in I} M_i \to M = \varinjlim_{i \in I} M_i$ (that is such that $\varphi \circ \nu_i = f_i$ for all $i \in I$), there is a well-defined homomorphism u from M to the $\mathfrak{F}_I$-reduced product $B/\Sigma_{\mathfrak{F}_I} B$ making the following diagram commutative (ρ denotes the canonical projection):

$$\begin{array}{ccccc} B & \xrightarrow{\rho} & B/\Sigma_{\mathfrak{F}_I} B & \longrightarrow & 0 \\ {\scriptstyle h}\uparrow & & {\scriptstyle u}\uparrow & & \\ \bigoplus_{i \in I} M_i & \xrightarrow{\varphi} & M & \longrightarrow & 0. \end{array}$$

We have $\Sigma_{\mathfrak{F}_I} B \in \mathcal{G}$ since $\mathcal{G}$ is filter-closed. Hence, using the assumption that $\operatorname{Ext}^1_R(M, \Sigma_{\mathfrak{F}_I} B) = 0$, we can factorize u through ρ to get some $g \in \operatorname{Hom}_R(M, B)$ such that $u = \rho \circ g$. Since the M_i are all $< \lambda$-generated and $\mathfrak{F}_I$ is λ-complete by Lemma 1.2, we obtain (for every $i \in I$) that "$h \circ \nu_i$ coincides with $g \circ \varphi \circ \nu_i = g \circ f_i$ on a set from the filter", that is:

$$\{k \in I \mid \pi_k \circ g \circ f_i = \pi_k \circ h \circ \nu_i\} \in \mathfrak{F}_I. \qquad (**)$$

Let us define J as follows:

$$J = \{i \in I \mid (\forall k \geq i)(\pi_k \circ g \circ f_i = g_k \circ f_{ki})\}.$$

Then clearly, g_i factors through f_i for every $i \in J$ (just by applying the definition of J for $k = i$). Hence certainly $J \cap S = \varnothing$.

To obtain a contradiction and finish the proof of the lemma, it is now enough to show that J is λ-closed cofinal. The fact that J is λ-closed follows easily by λ-continuity of the direct system $(M_i, f_{ji} \mid i \leq j)$. So we are left to prove that J is cofinal in I. But by $(**)$ and the definition of $\mathfrak{F}_I$, we can find for every $i \in I$ an element $s(i) \in I$ such that $s(i) \geq i$ and

$$(\forall k \geq s(i))(\pi_k \circ g \circ f_i = \pi_k \circ h \circ \nu_i). \qquad (\Delta)$$

Recall that $\pi_k \circ h \circ \nu_i = h_{ki} = g_k \circ f_{ki}$. Now, if we fix any $i' \in I$, we can define $j_0 = i'$, $j_{n+1} = s(j_n)$ for all $n \geq 0$, and $j = \sup_{n<\omega} j_n$. Then clearly $j \geq i'$, and it is easy to check that $j \in J$ using the $\aleph_1$-continuity of the direct system $(M_i, f_{ji} \mid i \leq j)$. □

An important consequence follows by applying Lemma 2.3 to the case when the class $\mathcal{G}$ cogenerates every module. This is for instance always the case when $\mathcal{G}$ is a right-hand class of a cotorsion pair, since then all injective modules are inside $\mathcal{G}$.

Proposition 2.4. *Let $\mathcal{G}$ be a cogenerating filter-closed class of modules. Then for any uncountable regular cardinal λ and any module M such that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$, there is a family $\mathcal{C}_\lambda$ of $< \lambda$-presented submodules of M such that*

(1) *$\mathcal{C}_\lambda$ is closed under unions of well-ordered ascending chains of length $< \lambda$,*
(2) *every subset $X \subseteq M$ such that $|X| < \lambda$ is contained in some $N \in \mathcal{C}_\lambda$, and*
(3) *$\operatorname{Ext}^1_R(M/N, \mathcal{G}) = 0$ for every $N \in \mathcal{C}_\lambda$.*

Proof. By Lemma 1.1, there is a λ-continuous direct system $(M_i, f_{ji} \mid i \leq j)$ of $< \lambda$-presented modules indexed by a set I such that M together with some maps $f_i : M_i \to M$ forms its direct limit. Now, the data $\mathcal{G}$, λ, $(M, f_i \mid i \in I)$, $(M_i, f_{ji} \mid i \leq j)$ and I fits exactly to Lemma 2.3. Hence, there is a λ-closed cofinal subset $J \subseteq I$ such that for every $j \in J$, every homomorphism from M_j to a module in $\mathcal{G}$ factors through f_j. But the fact that $\mathcal{G}$ is a cogenerating class implies that f_j is injective. Thus, we can view the modules M_j for $j \in J$ as submodules of M, and the maps f_j and f_{ji} as inclusions. Let us define

$$\mathcal{D} = \{M_j \mid j \in J\}$$

and let $\overline{\mathcal{D}}$ be the closure of $\mathcal{D}$ under unions of well-ordered chains of length $< \lambda$. Observe, that $(\mathcal{D}, \subseteq)$ is a directed poset since J is a cofinal subset of the directed set I. Using Lemma 1.2, we easily deduce that $\overline{\mathcal{D}}$ is directed, too. Now, we can view the modules in $\overline{\mathcal{D}}$ together with inclusions between them as a λ-continuous direct system indexed by $\overline{\mathcal{D}}$ itself. Hence, we can apply Lemma 2.3 for the second time to get a λ-closed cofinal subset $\mathcal{C}_\lambda$ of $\overline{\mathcal{D}}$ such that every homomorphism from a module $N \in \mathcal{C}_\lambda$ to a module in $\mathcal{G}$ extends to M.

The latter property together with the fact that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$ immediately implies (3). The property (1) is just another way to say that $\mathcal{C}_\lambda$ is λ-closed in $\overline{\mathcal{D}}$. For (2), first notice that $\bigcup \mathcal{C}_\lambda = M$ since $\mathcal{C}_\lambda$ is cofinal in $\overline{\mathcal{D}}$. Hence, if $X \subseteq M$ is a subset of cardinality $< \lambda$, there is a subset $\mathcal{M} \subseteq \mathcal{C}_\lambda$ of cardinality $< \lambda$ such that every $x \in X$ is contained in some $N' \in \mathcal{M}$. Finally, Lemma 1.2 provides us with an upper bound $N \in \mathcal{C}_\lambda$ for $\mathcal{M}$, and clearly $X \subseteq N$. □

In Lemma 2.3, the assumption of λ being uncountable is essential. We can, nevertheless, obtain a weaker but important result using the same technique for the choice $\lambda = \omega$ and $(I, \leq) = (\omega, \leq)$. Lemma 2.5 actually says that, for $B \in \mathcal{G}$, the inverse system of groups $(\operatorname{Hom}_R(M_i, B), \operatorname{Hom}_R(f_{ji}, B) \mid i \leq j < \omega)$ is Mittag-Leffler, and the stationary indices determined by s are common over all $B \in \mathcal{G}$. In this terminology, a proof of the lemma is mostly contained in the proof of [8, Theorems 2.5 and 3.7].

We give a different proof here and we do this for two main reasons: First, the statement about common stationary indices has an important interpretation in the first-order theory of modules and is missing in [8]. Second, we show that the Mittag-Leffler property is a part of a common framework which works for both countable and uncountable systems.

Lemma 2.5. *Let $\mathcal{G}$ be a class of modules closed under countable direct sums. Let $(M, f_i \mid i < \omega)$ be a direct limit of a countable direct system $(M_i, f_{ji} \mid i \leq j < \omega)$ consisting of finitely generated modules.*

Assume that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$. Then there is a strictly increasing function $s : \omega \to \omega$ such that for each $B \in \mathcal{G}$, $i < \omega$ and $c : M_i \to B$ the following holds: If c factors through $f_{s(i)i}$, then it factors through f_{ni} for all $n \geq s(i)$.

Proof. We will show that it is possible to construct the values $s(i)$ by induction on i. Suppose by way of contradiction that there is some $i < \omega$ for which we cannot define $s(i)$. This can only happen if for each $j \geq i$, there is a homomorphism $g_j : M_j \to B_j$ such that $B_j \in \mathcal{G}$, and $g_j \circ f_{ji}$ does not factor through f_{ni} for some $n > j$. For $j < i$ let g_j be zero maps and $B_j \in \mathcal{G}$ be arbitrary. Put $B = \prod_{j<\omega} B_j$.

Now, we follow the proof of Lemma 2.3 (with ω in place of I and λ) starting with the second paragraph and ending just after the definition of $(**)$. Note that the corresponding notion of $\aleph_0$-completeness is void, $\mathfrak{F}_\omega$ is the Fréchet filter on ω, and the $\mathfrak{F}_\omega$-product $\Sigma_{\mathfrak{F}_\omega} B$ is just the direct sum $\bigoplus_{j<\omega} B_j$.

By the same argument as for (Δ) in the proof of Lemma 2.3 and with the same notation as there, there is some $s' \geq i$ such that

$$(\forall k \geq s')(\pi_k \circ g \circ f_i = \pi_k \circ h \circ \nu_i)$$

holds and $\pi_k \circ h \circ \nu_i = h_{ki} = g_k \circ f_{ki}$ for each $k \geq s'$. But this contradicts the fact implied by the choice of g_k that $g_k \circ f_{ki}$ does not factor through f_i. □

Let us remark that we have actually proved a little more than we stated in Lemma 2.5—we have constructed $s : \omega \to \omega$ such that if $c : M_i \to B$ factors through $f_{s(i)i}$, then it factors through $f_i : M_i \to M$. The motivation for the seemingly more complicated statement of the lemma should become clear in the following paragraphs.

If the modules M_i in the direct system from the lemma above are finitely presented instead of finitely generated, we have a statement about factorization through maps between finitely presented modules. Which in other words means that some coherent functors vanish and the Mittag-Leffler property is preserved within the smallest definable class containing $\mathcal{G}$. This is made precise by the following lemma.

Lemma 2.6. *Let $\mathcal{G}$ be a class of modules closed under countable direct sums and $\mathcal{D}$ be the smallest definable class containing $\mathcal{G}$. Let $(M, f_i \mid i < \omega)$ be a direct limit of a direct system $(M_i, f_{ji} \mid i \leq j < \omega)$ consisting of finitely presented modules.*

Assume that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$. Then there is a strictly increasing function $s : \omega \to \omega$ such that for each $D \in \mathcal{D}$, $i < \omega$ and $c : M_i \to D$ the following holds: If c factors through $f_{s(i)i}$, then it factors through f_{ni} for all $n \geq s(i)$.

Proof. By restating the conclusion of Lemma 2.5, we obtain that $\operatorname{Im}\operatorname{Hom}_R(f_{s(i)i}, D) = \operatorname{Im}\operatorname{Hom}_R(f_{ni}, D)$ for each $D \in \mathcal{G}$ and $i \le s(i) \le n < \omega$. It is also straightforward to check that $F = \operatorname{Im}\operatorname{Hom}_R(f_{s(i)i}, -)/\operatorname{Im}\operatorname{Hom}_R(f_{ni}, -)$ is a coherent functor. Hence we have $\operatorname{Im}\operatorname{Hom}_R(f_{s(i)i}, D) = \operatorname{Im}\operatorname{Hom}_R(f_{ni}, D)$ also for each $D \in \mathcal{D}$ and the claim follows. □

Note also that instead of vanishing of the coherent functors in the proof above, we can equivalently consider that certain implications between pp-formulas are satisfied [13, §2.1], thus reformulating the proof in a more model theoretic way.

Now, we can prove a crucial statement similar to [8, Theorem 2.5]:

Proposition 2.7. *Let $\mathcal{G}$ be a class of modules closed under countable direct sums, and let M be a countably presented module such that $\operatorname{Ext}^1_R(M, \mathcal{G}) = 0$. Then $\operatorname{Ext}^1_R(M, D) = 0$ for every D isomorphic to a pure submodule of a product of modules from $\mathcal{G}$.*

Proof. Let D be a pure submodule of $\prod_k B_k$ for some $B_k \in \mathcal{G}$. Since M is countably presented, it can be considered as a direct limit of a countable chain of finitely presented modules $M_i, i < \omega$, as in the assumptions of Lemma 2.6. Hence $(\operatorname{Hom}_R(M_i, D), \operatorname{Hom}_R(f_{ji}, D) \mid i \le j < \omega)$ is Mittag-Leffler since any definable class is closed under taking products and pure submodules.

Then we continue as in the proof of [8, Theorem 2.5]. Since $\operatorname{Ext}^1_R\big(M, \prod_k B_k\big) = 0$ by assumption, we have the exact sequence

$$\operatorname{Hom}_R\big(M, \prod_k B_k\big) \xrightarrow{h} \operatorname{Hom}_R\big(M, (\prod_k B_k)/D\big) \to \operatorname{Ext}^1_R(M, D) \to 0,$$

and so it suffices to show that h is an epimorphism. This easily follows from Proposition 1.4 applied on the inverse system $(\operatorname{Hom}_R(M_i, D), \operatorname{Hom}_R(f_{ji}, D) \mid i \le j < \omega)$. Indeed, we see that $\varprojlim^1_i \operatorname{Hom}_R(M_i, D) = 0$ and obtain the exact sequence

$$\varprojlim_i \operatorname{Hom}_R\big(M_i, \prod_k B_k\big) \to \varprojlim_i \operatorname{Hom}_R\big(M_i, (\prod_k B_k)/D\big) \to 0.$$

It remains to use the basic fact that contravariant Hom-functors take colimits to limits. □

3. Countable type

In this section, we prove the main result of our paper—the Countable Telescope Conjecture for Module Categories. But before doing this, we introduce a fairly simplified version of Shelah's Singular Compactness Theorem. It is based on [15, Theorem IV.3.7]. In the terminology there, systems witnessing strong λ-"freeness" correspond to the λ-dense systems defined below.

A reader acquainted with the full-fledged compactness theorem for filtrations of modules proved in [15, XII.1.14 and IV.3.7] or [14] may well skip Lemma 3.2. We state and prove the lemma for the sake of completeness, and also because we are using only a fragment of the full compactness theorem, and it makes the proof of the Countable Telescope Conjecture more transparent.

Definition 3.1. Let M be a module and λ be a regular uncountable cardinal. Then a set $\mathcal{C}_\lambda$ of $< \lambda$-generated submodules of M is called a *λ-dense system in M* if

(1) $0 \in \mathcal{C}_\lambda$,
(2) $\mathcal{C}_\lambda$ is closed under unions of well-ordered ascending chains of length $< \lambda$, and
(3) every subset $X \subseteq M$ such that $|X| < \lambda$ is contained in some $N \in \mathcal{C}_\lambda$.

Lemma 3.2 (Simplified Shelah's Singular Compactness Theorem)**.** *Let κ be a singular cardinal, M a κ-generated module, and let μ be a cardinal such that $\operatorname{cf}\kappa \le \mu < \kappa$. Suppose we are given a λ-dense system, $\mathcal{C}_\lambda$, in M for each regular λ such that $\mu < \lambda < \kappa$. Then there is a filtration $(M_\alpha \mid \alpha \le \operatorname{cf}\kappa)$ of M and a continuous strictly increasing chain of cardinals $(\kappa_\alpha \mid \alpha < \operatorname{cf}\kappa)$ cofinal in κ such that $M_\alpha \in \mathcal{C}_{\kappa_\alpha^+}$ for each $\alpha < \operatorname{cf}\kappa$.*

Proof. We will start with choosing the chain $(\kappa_\alpha \mid \alpha < \operatorname{cf}\kappa)$. In fact, we can choose any such chain provided that $\mu \le \kappa_0$, just to make sure that $\mathcal{C}_{\kappa_\alpha^+}$ is always available. Let us fix one such chain $(\kappa_\alpha \mid \alpha < \operatorname{cf}\kappa)$.

Next, let $(X_\alpha \mid \alpha < \operatorname{cf}\kappa)$ be an ascending chain of subsets of M such that $\bigcup_{\alpha<\operatorname{cf}\kappa} X_\alpha$ generates M and $|X_\alpha| = \kappa_\alpha$ for each $\alpha < \operatorname{cf}\kappa$. Then, we can by induction construct a (not necessarily continuous) chain $(N^0_\alpha \mid \alpha < \operatorname{cf}\kappa)$ of submodules of M such that $N^0_\alpha \in \mathcal{C}_{\kappa_\alpha^+}$ and

$X_\alpha \cup \bigcup_{\beta<\alpha} N^0_\beta \subseteq N^0_\alpha$ for every $\alpha < \operatorname{cf}\kappa$. Since N_α is κ_α-generated, we can fix for each α a generating set Y^0_α of N^0_α together with some enumeration $Y^0_\alpha = \{y^0_{\alpha,\gamma} \mid \gamma < \kappa_\alpha\}$. Next, we proceed by induction on $n < \omega$ and construct for each $n > 0$ chain of modules $(N^n_\alpha \mid \alpha < \operatorname{cf}\kappa)$ and sets $Y^n_\alpha = \{y^n_{\alpha,\gamma} \mid \gamma < \kappa_\alpha\}$ such that

(1) $(N^n_\alpha \mid \alpha < \operatorname{cf}\kappa)$ is a (not necessarily continuous) chain of submodules of M,
(2) $N^n_\alpha \in \mathcal{C}_{\kappa^+_\alpha}$ and $N^n_\alpha \supseteq \{y^{n-1}_{\zeta,\gamma} \mid \alpha \le \zeta < \operatorname{cf}\kappa \;\&\; \gamma < \kappa_\alpha\} \cup \bigcup_{\beta<\alpha} N^n_\beta$, and
(3) $Y^n_\alpha = \{y^n_{\alpha,\gamma} \mid \gamma < \kappa_\alpha\}$ is a fixed enumeration of some set of generators of N^n_α, for each $\alpha < \operatorname{cf}\kappa$.

For each $n < \omega$, we clearly can construct such a chain and sets by induction on α. Note in particular that we have always $N^{n-1}_\alpha \subseteq N^n_\alpha$ since $Y^{n-1}_\alpha = \{y^{n-1}_{\alpha,\gamma} \mid \gamma < \kappa_\alpha\} \subseteq N^n_\alpha$ by (2). Hence, if we define $M_\alpha = \bigcup_{n<\omega} N^n_\alpha$, we clearly have $M_\alpha \in \mathcal{C}_{\kappa^+_\alpha}$ for each $\alpha < \operatorname{cf}\kappa$. Also, $\bigcup_{\alpha<\operatorname{cf}\kappa} M_\alpha = M$ since $X_\alpha \subseteq N^0_\alpha \subseteq M_\alpha$ for each α. We claim that the chain $(M_\alpha \mid \alpha < \operatorname{cf}\kappa)$ is continuous. To see this, fix for this moment a limit ordinal $\alpha < \operatorname{cf}\kappa$. Then clearly $M_\alpha \supseteq \bigcup_{\beta<\alpha} M_\beta$. On the other hand, for a given $n > 0$ and $\beta < \alpha$, we have $\{y^{n-1}_{\alpha,\gamma} \mid \gamma < \kappa_\beta\} \subseteq N^n_\beta$ by (2). Therefore, $Y^{n-1}_\alpha \subseteq \bigcup_{\beta<\alpha} N^n_\beta$ and also $N^{n-1}_\alpha \subseteq \bigcup_{\beta<\alpha} N^n_\beta$ by (3). Hence $M_\alpha \subseteq \bigcup_{\beta<\alpha} M_\beta$ and the claim is proved. Now, if we change M_0 for the zero module and put $M_{\operatorname{cf}\kappa} = M$, $(M_\alpha \mid \alpha \le \operatorname{cf}\kappa)$ becomes a filtration with the desired properties. □

While Lemma 3.2 or Shelah's Singular Compactness Theorem give us some information about the structure of a module with enough dense systems for a singular number of generators, we can prove a rather straightforward lemma which takes care of regular cardinals.

Lemma 3.3. *Let κ be a regular uncountable cardinal, M be a κ-generated module and $\mathcal{C}_\kappa$ be a κ-dense system in M. Then there is a filtration $(M_\alpha \mid \alpha \le \kappa)$ of M such that $M_\alpha \in \mathcal{C}_\kappa$ for each $\alpha < \kappa$.*

Proof. Let us fix an enumeration $\{m_\gamma \mid \gamma < \kappa\}$ of generators of M. We will construct the filtration by induction. Put $M_0 = 0$ and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha \le \kappa$. For $\alpha = \beta + 1$, we can find $M_\alpha \in \mathcal{C}_\kappa$ such that $M_\beta \cup \{m_\beta\} \subseteq M_\alpha$, using (3) from Definition 3.1. □

Before stating and proving the main result, we need a technical lemma about filtrations which has been studied in [17, 41, 43], and whose origins can be traced back to an ingenious idea of P. Hill [22].

Lemma 3.4. [43, Theorem 6]. *Let $\mathcal{S}$ be a set of countably presented modules and M be a module possessing an $\mathcal{S}$-filtration $(M_\alpha \mid \alpha \le \sigma)$. Then there is a family $\mathcal{F}$ of submodules of M such that:*

(1) *$M_\alpha \in \mathcal{F}$ for all $\alpha \le \sigma$.*
(2) *$\mathcal{F}$ is closed under arbitrary sums and intersections.*
(3) *For each $N, P \in \mathcal{F}$ such that $N \subseteq P$, the module P/N is $\mathcal{S}$-filtered.*
(4) *For each $N \in \mathcal{F}$ and a countable subset $X \subseteq M$, there is $P \in \mathcal{F}$ such that $N \cup X \subseteq P$ and P/N is countably presented.*

Now, we are in a position to prove the Countable Telescope Conjecture.

Theorem 3.5 (Countable Telescope Conjecture)**.** *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair of R-modules such that $\mathcal{B}$ is closed under unions of well-ordered chains. Then*

(1) *$\mathfrak{C}$ is generated by a set of strongly countably presented modules,*
(2) *$\mathfrak{C}$ is complete, and*
(3) *$\mathcal{B}$ is a definable class.*

Proof. (1). First, we claim that $\mathfrak{C}$ is generated by a representative set $\mathcal{S}$ of the class of all countably presented modules from $\mathcal{A}$. To do this, in view of Eklof's Lemma ([19, Lemma 3.1.2] or [16, Lemma 1]), it is enough to prove that every module $M \in \mathcal{A}$ has an $\mathcal{S}$-filtration $(M_\alpha \mid \alpha \le \sigma)$.

We will prove this by induction on the minimal cardinal κ such that M is κ-presented. If κ is finite or countable, then we are done since M itself is isomorphic to a module from $\mathcal{S}$. Assume that κ is uncountable. By our assumption and Lemma 2.2, the class $\mathcal{B}$ is filter-closed and cogenerating. Hence, we can fix for each regular uncountable $\lambda \le \kappa$ a family $\mathcal{C}_\lambda$ of $< \lambda$-presented modules given by Proposition 2.4 used with $\mathcal{G} = \mathcal{B}$. Note that we can without loss of generality assume that $\mathcal{C}_\lambda$ is a λ-dense system, since we always can add the zero module to $\mathcal{C}_\lambda$ without changing its properties. Then, we can use Lemma 3.3 if κ is regular, and Lemma 3.2 if κ is singular to obtain a filtration $(L_\beta \mid \beta \leq \tau)$ of M such that for each $\beta < \tau$

(i) L_β is $< \kappa$-presented, and
(ii) $M/L_\beta \in \mathcal{A}$.

We also have $L_{\beta+1}/L_\beta \in \mathcal{A}$ since it is a kernel of the projection $M/L_\beta \to M/L_{\beta+1}$ and $\mathfrak{C}$ is hereditary. Thus, each of the modules $L_{\beta+1}/L_\beta$ has an $\mathcal{S}$-filtration by the inductive hypothesis, so we can refine the filtration $(L_\beta \mid \beta \leq \tau)$ to an $\mathcal{S}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$ of M and the claim is proved.

Let us note that for the induction step at singular cardinals κ, we can alternatively use the full version of Shelah's Singular Compactness Theorem, considering $\mathcal{S}$-filtered modules as "free" (cf. [15, XII.1.14 and IV.3.7] or [14]).

It is still left to show that all modules in $\mathcal{S}$ are actually strongly countably presented. Note that it is enough to prove that every countably generated module $M \in \mathcal{A}$ is countably presented. If we prove this, we can take for every module $N \in \mathcal{S}$ a presentation $0 \to K \to R^{(\omega)} \to N \to 0$ with K a countably generated module. Since $\mathfrak{C}$ is hereditary, we have $K \in \mathcal{A}$. Now, if K is countably presented, it must be isomorphic to a module from $\mathcal{S}$ again, and we can proceed by induction to construct a free resolution of N consisting of countably generated free modules.

So fix $M \in \mathcal{A}$ countably generated. Then M is $\mathcal{S}$-filtered by the arguments above. Hence, we can consider the family $\mathcal{F}$ given by Lemma 3.4 for M. To finish our proof, we use (4) from this lemma with $N = 0$ and X a countable set of generators of M as parameters.

(2). This follows from (1) by [19, Theorem 3.2.1].

(3). Note that $\mathcal{B}$ is always closed under arbitrary direct products. It is closed under infinite direct sums too since these are precisely $\mathfrak{F}$-products corresponding to Fréchet filters $\mathfrak{F}$. Then $\mathcal{B}$ is closed under pure submodules by (1) and Proposition 2.7. Further, $\mathcal{B}$ is closed under pure epimorphic images and, therefore, also under arbitrary direct limits since $\mathfrak{C}$ is hereditary. Hence $\mathcal{B}$ is definable. □

Remark. We can actually prove a little more than we state in Theorem 3.5. Notice that the proof of (1) and (2) works also for any hereditary cotorsion pair cogenerated (as a cotorsion pair) by some cogenerating (in the module category) filter-closed class $\mathcal{G}$.

To conclude this section, we will discuss the relation of Theorem 3.5 to tilting theory. In fact, it turns out that the countable type and definability of tilting classes is a rather easy consequence of Theorem 3.5. This allows us to give a more direct argumentation for most of the proof of the fact that all tilting classes are of finite type [8, 9].

Recall that $\mathfrak{T} = (\mathcal{A}, \mathcal{B})$ is called a *tilting cotorsion pair* if $\mathfrak{T}$ is hereditary, $\mathcal{A}$ consists of modules of finite projective dimension, and $\mathcal{B}$ is closed under direct sums. In this case, $\mathcal{B}$ is said to be a *tilting class.*

Theorem 3.6. *Let R be a ring and $\mathfrak{T} = (\mathcal{A}, \mathcal{B})$ be a tilting cotorsion pair. Then $\mathfrak{T}$ is generated by a set of strongly countably presented modules and $\mathcal{B}$ is definable.*

Proof. Notice that since $\mathcal{A}$ is closed under direct sums, there is $n < \omega$ such that projective dimension of any module from $\mathcal{A}$ is at most n. We will prove the theorem by induction on this n.

If the $n = 0$, then $\mathcal{B} = \text{Mod-}R$ and the statement follows trivially. Let $n > 0$. Then it is easy to see that the class $\mathcal{D} = \operatorname{Ker}\operatorname{Ext}^2_R(\mathcal{A}, -)$ is tilting and in the corresponding tilting cotorsion pair $(\mathcal{C}, \mathcal{D})$, all modules in $\mathcal{C}$ have projective dimension $< n$ (cf. [4, Lemma 4.8]). Thus $\mathcal{D}$ is definable by the inductive hypothesis. In particular, it is closed under pure submodules. By a simple dimension shifting argument, one observes that $\mathcal{B}$ is closed under pure-epimorphic images. Since, by our assumption, $\mathcal{B}$ is closed under direct sums, it follows that $\mathcal{B}$ is closed under arbitrary direct limits. Thus we may apply Theorem 3.5 to $\mathfrak{T}$ to finish the proof. □

4. Definability

In this section, we will give a description of which coherent functors define the class $\mathcal{B}$ of a cotorsion pair $(\mathcal{A}, \mathcal{B})$ satisfying the hypotheses of TCMC. Our aim is twofold: First, vanishing of a coherent functor on a module M translates to the fact that a certain implication between pp-formulas is satisfied in M, [13, §2.1]. So there is a clear model-theoretic motivation. Second, proving that the cotorsion pair is of finite type amounts to showing that $\mathcal{B}$ is defined by a family of coherent functors of the form $\operatorname{Coker}\operatorname{Hom}_R(f,-)$ where $f : X \to Y$ is an inclusion of $X \in \text{mod-}R$ into a finitely generated projective module Y. The projectivity of Y is essential here: it implies that $Y \in \mathcal{A}$ which in turn means that the functor $\operatorname{Coker}\operatorname{Hom}_R(f,-)$ vanishes on all modules from $\mathcal{B}$ if and only if $Y/X \in \mathcal{A}$. Compare this with Remark (*ii*) at the end of the section.

Even though the finite type question still remains open, we will describe a family of coherent functors defining $\mathcal{B}$ in Theorem 4.9—this can be viewed as a counterpart of [29, Theorem A (3)] for module categories. We will also characterize the countably presented modules from the class $\mathcal{A}$ in Theorem 4.8. In both tasks, the key role is played by the ideal $\mathfrak{I}$ of the category mod-R consisting of the morphisms which, when considered in Mod-R, factor through some module from $\mathcal{A}$.

For the whole section, let R be a *right coherent* ring; that is, finitely (and also countably) presented modules are precisely the strongly finitely (countably) presented ones, respectively. We will deal with countable direct systems of finitely generated modules of the form:

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \to \cdots \to C_n \xrightarrow{f_n} C_{n+1} \to \cdots .$$

Here, we write for simplicity f_n instead of $f_{n+1,n}$. We start with recalling some important preliminary results whose proofs are essentially in [8] and [2]:

Lemma 4.1. *Let $(C_n, f_n)_{n<\omega}$ be a countable direct system of R-modules. Let M be a module such that $\operatorname{Ext}^1_R(\varinjlim C_n, M) = 0$. Then $\varprojlim^1 \operatorname{Hom}_R(C_n, M) = 0$.*

Proof. The proof here is in fact a part of the proof of [8, Theorem 5.1]. If we apply the functor $\operatorname{Hom}_R(-, M)$ to the canonical presentation

$$0 \to \bigoplus C_n \xrightarrow{\phi} \bigoplus C_n \to \varinjlim C_n \to 0$$

of the countable direct limit $\varinjlim C_n$, we get exactly the first three terms of the exact sequence defining the first derived functor of inverse limit of the system $(H_n \mid n < \omega)$, where $H_n = \operatorname{Hom}_R(C_n, M)$:

$$0 \to \varprojlim H_n \to \prod H_n \xrightarrow{\Delta} \prod H_n \to \varprojlim{}^1 H_n \to 0$$

Since $\operatorname{Ext}^1_R(\varinjlim C_n, M) = 0$, the map $\Delta = \operatorname{Hom}_R(\phi, M)$ is surjective. Hence $\varprojlim^1 H_n = 0$. □

Corollary 4.2. *Let $(C_n, f_n)_{n<\omega}$ be a countable direct system of finitely generated R-modules. Let M be a module such that $\operatorname{Ext}^1_R(\varinjlim C_n, M^{(\omega)}) = 0$. Then the inverse system $(\operatorname{Hom}_R(C_n, M), \operatorname{Hom}_R(f_n, M))_{n<\omega}$ is Mittag-Leffler.*

Proof. This follows either immediately from Lemma 2.5 for $\mathcal{G} = \{N \mid N \cong M^{(\omega)}\}$, or from Proposition 1.4. Note that in both cases we use the fact that all modules C_n are finitely generated. □

The following lemma gives us information about a syzygy of a countable direct limit of finitely presented modules and it will be useful for computation.

Lemma 4.3. *Let $(C_n, f_n)_{n<\omega}$ be a countable direct system of finitely presented modules. Then there exists a countable direct system*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & D_2 & \xrightarrow{i_2} & P_2 & \xrightarrow{p_2} & C_2 & \longrightarrow & 0 \\
 & & {\scriptstyle g_1}\uparrow & & {\scriptstyle s_1}\uparrow & & {\scriptstyle f_1}\uparrow & & \\
0 & \longrightarrow & D_1 & \xrightarrow{i_1} & P_1 & \xrightarrow{p_1} & C_1 & \longrightarrow & 0 \\
 & & {\scriptstyle g_0}\uparrow & & {\scriptstyle s_0}\uparrow & & {\scriptstyle f_0}\uparrow & & \\
0 & \longrightarrow & D_0 & \xrightarrow{i_0} & P_0 & \xrightarrow{p_0} & C_0 & \longrightarrow & 0
\end{array}$$

of short exact sequences of finitely presented modules such that P_n is projective and s_n is split mono for each $n < \omega$. In particular, $\varinjlim P_n$ is projective.

Proof. We will construct the short exact sequences by induction on n. For $n = 0$, let $0 \to D_0 \overset{i_0}{\to} P_0 \overset{p_0}{\to} C_0 \to 0$ be a short exact sequence with P_0 projective finitely generated. Then D_0 is finitely generated, hence finitely presented since we are working over a right coherent ring. If $0 \to D_n \overset{i_n}{\to} P_n \overset{p_n}{\to} C_n \to 0$ has already been constructed, let $q : Q \to C_{n+1}$ be an epimorphism such that Q is a finitely generated projective module. Now define $P_{n+1} = P_n \oplus Q$, $s_n : P_n \to P_{n+1}$ as the canonical inclusion, and $p_{n+1} = (f_n p_n, q)$. Then $D_{n+1} = \operatorname{Ker} p_{n+1}$ is finitely presented and g_n is determined by the commutative diagram above. The last assertion is clear. $\square$

Next, we will need a generalized version of Auslander's well-known lemma. It says that $\operatorname{Ext}^1_R(\varinjlim C_i, M) \cong \varprojlim \operatorname{Ext}^1_R(C_i, M)$ whenever M is a pure-injective module. Note that for a countable direct system $(C_n, f_n)_{n<\omega}$, the pure-injectivity of M implies that $\varprojlim^1 \operatorname{Hom}_R(C_n, M) = 0$. To see this, we will again use the fact that after applying $\operatorname{Hom}_R(-, M)$ on the canonical pure-exact sequence

$$0 \to \bigoplus C_i \overset{\phi}{\to} \bigoplus C_i \to \varinjlim C_i \to 0, \qquad (\dagger)$$

we get first three terms of the exact sequence

$$0 \to \varprojlim H_n \to \prod H_n \overset{\Delta}{\to} \prod H_n \to \varprojlim{}^1 H_n \to 0$$

where $H_n = \operatorname{Hom}_R(C_n, M)$. But if M is pure-injective, then applying $\operatorname{Hom}_R(-, M)$ on (†) yields an exact sequence and consequently $\varprojlim^1 \operatorname{Hom}_R(C_i, M) = 0$. It turns out that the latter condition is sufficient for $\operatorname{Ext}^1_R(-, M)$ to turn a direct limit into an inverse limit over a right coherent ring:

Lemma 4.4. *Let $(C_n, f_n)_{n<\omega}$ be a countable direct system and let M be a module such that $\varprojlim^1 \operatorname{Hom}_R(C_i, M) = 0$. Then $\operatorname{Ext}^1_R(\varinjlim C_i, M) \cong \varprojlim \operatorname{Ext}^1_R(C_i, M)$.*

Proof. Consider the direct system of short exact sequences $0 \to D_n \overset{i_n}{\to} P_n \overset{p_n}{\to} C_n \to 0$ given by Lemma 4.3. After applying $\operatorname{Hom}_R(-, M)$, we get an inverse system of exact sequences

$$0 \to \operatorname{Hom}_R(C_n, M) \overset{p_n^*}{\to} \operatorname{Hom}_R(P_n, M) \overset{i_n^*}{\to} \operatorname{Hom}_R(D_n, M) \overset{\delta_n}{\to} \operatorname{Ext}^1_R(C_n, M) \to 0.$$

By assumption, the following short sequence is exact:

$$0 \to \varprojlim \operatorname{Hom}_R(C_n, M) \to \varprojlim \operatorname{Hom}_R(P_n, M) \to \varprojlim \operatorname{Im} i_n^* \to 0.$$

On the other hand, it follows from Proposition 1.4 that $\varprojlim^1 \operatorname{Hom}_R(P_n, M) = 0$ since the countable inverse system $(\operatorname{Hom}_R(P_n, M), \operatorname{Hom}_R(s_n, M))_{n<\omega}$ has all the maps (split) epic. Moreover, $\varprojlim^1 \operatorname{Im} i_n^* = 0$ since $\varprojlim^1$ is right exact on countable inverse systems. Hence, the following sequence is also exact:

$$0 \to \varprojlim \operatorname{Im} i_n^* \to \varprojlim \operatorname{Hom}_R(D_n, M) \to \varprojlim \operatorname{Ext}^1_R(C_n, M) \to 0.$$

Putting everything together, we have obtained the following diagram with canonical maps and exact rows:

$$\begin{array}{ccccccc}
\varprojlim \operatorname{Hom}_R(P_n, M) & \longrightarrow & \varprojlim \operatorname{Hom}_R(D_n, M) & \longrightarrow & \varprojlim \operatorname{Ext}^1_R(C_n, M) & \longrightarrow & 0 \\
\cong \big\uparrow & & \cong \big\uparrow & & & & \\
\operatorname{Hom}(\varinjlim P_n, M) & \longrightarrow & \operatorname{Hom}(\varinjlim D_n, M) & \longrightarrow & \operatorname{Ext}^1_R(\varinjlim C_n, M) & \longrightarrow & 0
\end{array}$$

It follows that $\operatorname{Ext}^1_R(\varinjlim C_n, M) \cong \varprojlim \operatorname{Ext}^1_R(C_n, M)$. $\square$

Now, we will focus on T-nilpotent inverse systems. It is clear that every T-nilpotent countable inverse system is Mittag-Leffler. It turns out that the converse is true precisely when the inverse limit of the system vanishes. This is made precise by the following lemma:

Lemma 4.5. *Let $(H_n, h_n)_{n<\omega}$ be a countable inverse system of abelian groups. Then the following are equivalent:*

(1) *$(H_n, h_n)_{n<\omega}$ is T-nilpotent,*

(2) $(H_n, h_n)_{n<\omega}$ *is Mittag-Leffler and* $\varprojlim H_n = 0$.

Proof. (1) $\implies$ (2) follows easily from the definitions. Let us prove (2) $\implies$ (1). For each $m < \omega$, let $s(m) > m$ be minimal such that the chain

$$H_m \supseteq h_m(H_{m+1}) \supseteq \cdots \supseteq h_m h_{m+1} \cdots h_{n-1}(H_n) \supseteq \cdots$$

is constant for $n \geq s(m)$ and let $\rho_m : \varprojlim H_n \to H_m$ be the limit map for each m. It follows easily that $s(m) \leq s(m')$ for $m < m'$. We will prove by induction that $\operatorname{Im} \rho_m = \operatorname{Im} h_m h_{m+1} \cdots h_{s(m)-1}$. Together with the assumption that $\varprojlim H_n = 0$, this will imply the T-nilpotency. Let us fix $x_m \in \operatorname{Im} h_m h_{m+1} \cdots h_{s(m)-1}$. All we need to do is to construct by induction a sequence of elements $(x_n)_{m<n<\omega}$ such that $x_n \in \operatorname{Im} h_n h_{n+1} \cdots h_{s(n)-1} \subseteq H_n$ and $x_{n-1} = h_{n-1}(x_n)$ for each $n > m$. Suppose we have already constructed x_{n-1} for some n. Then, by the chain condition, there is $y \in H_{s(n)}$ such that $h_{n-1} h_n \cdots h_{s(n)-1}(y) = x_{n-1}$. We can put $x_n = h_n \cdots h_{s(n)-1}(y)$. □

We are in a position now to give a connection between vanishing of Ext^i_R and the chain conditions mentioned above (the Mittag-Leffler condition and T-nilpotency). We state the connection in the following key lemma:

Lemma 4.6. *Let $(C_n, f_n)_{n<\omega}$ be a countable direct system of finitely presented modules and let M be an arbitrary module. Consider the following conditions:*

(1) $\operatorname{Ext}^1_R(\varinjlim C_n, M^{(\omega)}) = \operatorname{Ext}^2_R(\varinjlim C_n, M^{(\omega)}) = 0$.

(2) *The inverse system* $(\operatorname{Hom}_R(C_n, M), \operatorname{Hom}_R(f_n, M))_{n<\omega}$ *is Mittag-Leffler and* $(\operatorname{Ext}^1_R(C_n, M), \operatorname{Ext}^1_R(f_n, M))_{n<\omega}$ *is T-nilpotent.*

(3) $\operatorname{Ext}^1_R(\varinjlim C_n, M^{(\omega)}) = 0$.

Then (1) implies (2) and (2) implies (3).

Proof. (1) $\implies$ (2). Assume $\operatorname{Ext}^1_R(\varinjlim C_n, M^{(\omega)}) = \operatorname{Ext}^2_R(\varinjlim C_n, M^{(\omega)}) = 0$. Then the inverse system $(\operatorname{Hom}_R(C_n, M), \operatorname{Hom}_R(f_n, M))_{n<\omega}$ is Mittag-Leffler by Corollary 4.2. By Proposition 1.4 we have $\varprojlim^1 \operatorname{Hom}_R(C_n, M) = 0$, and subsequently it follows by Lemma 4.4 that

$$\varprojlim \operatorname{Ext}^1_R(C_n, M) \cong \operatorname{Ext}^1_R(\varinjlim C_n, M) = 0$$

Next, let $0 \to D_n \to P_n \to C_n \to 0$ be the countable direct system given by Lemma 4.3. Since

$$\operatorname{Ext}^1_R(\varinjlim D_n, M^{(\omega)}) = \operatorname{Ext}^2_R(\varinjlim C_n, M^{(\omega)}) = 0$$

by dimension shifting, the inverse system $(\operatorname{Hom}_R(D_n, M))_{n<\omega}$ is also Mittag-Leffler by Corollary 4.2. Then $(\operatorname{Ext}^1_R(C_n, M))_{n<\omega}$ is Mittag-Leffler as well, since an epimorfic image of a Mittag-Leffler inverse system is Mittag-Leffler again, [20, Proposition 13.2.1]. Thus, $(\operatorname{Ext}^1_R(C_n, M))_{n<\omega}$ is T-nilpotent by Lemma 4.5.

(2) $\implies$ (3). Clearly, condition (2) implies that $(\operatorname{Hom}_R(C_n, M^{(\omega)}))_{n<\omega}$ is Mittag-Leffler and $(\operatorname{Ext}^1_R(C_n, M^{(\omega)}))_{n<\omega}$ is T-nilpotent. Hence

$$\operatorname{Ext}^1_R(\varinjlim C_n, M^{(\omega)}) = \varprojlim \operatorname{Ext}^1_R(C_n, M^{(\omega)}) = 0$$

by Lemmas 4.4 and 4.5. □

With the previous lemma in mind, a natural question arises when $\operatorname{Ext}^1_R(f, M)$ is a zero map for a homomorphism $f : X \to Y$ between finitely presented modules. It is possible to characterize such maps f when $\operatorname{Ext}^1_R(f, M) = 0$ as M runs over all modules in the right-hand class of a complete cotorsion pair. We state this precisely in Lemma 4.7. In view of [30], the lemma can be viewed as a module-theoretic counterpart of [29, Lemmas 3.4 (3) and 3.8].

Lemma 4.7. *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair in* Mod-R *and let $f : X \to Y$ be a homomorphism between R-modules. Then the following are equivalent:*

(1) $\operatorname{Ext}^1_R(f, B) = 0$ *for every* $B \in \mathcal{B}$,

(2) f *factors through some module in* $\mathcal{A}$.

Proof. (1) $\implies$ (2). Let $0 \to B \to A \to Y \to 0$ be a special $\mathcal{A}$-precover of Y and consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & Q & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & B & \longrightarrow & A & \longrightarrow & Y & \longrightarrow & 0
\end{array}$$

Then the upper row splits by assumption and f factors through A.

(2) $\implies$ (1). This is easy, since the assumption that f factors through some $A \in \mathcal{A}$ implies that $\mathrm{Ext}^1_R(f, B)$ factors through $\mathrm{Ext}^1_R(A, B) = 0$ for each $B \in \mathcal{B}$. $\square$

Now, we can characterize countably presented modules in the left-hand class of a cotorsion pair satisfying the hypotheses of TCMC. Actually, we state the theorem more generally, for cotorsion pairs satisfying somewhat weaker conditions. Recall that by Theorem 3.5, every cotorsion pair satisfying the hypotheses of TCMC is complete.

Theorem 4.8. *Let R be a right coherent ring and $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair with $\mathcal{B}$ closed under (countable) direct sums. Denote by $\mathfrak{I}$ the ideal of all morphisms in* mod-R *which factor through some module from $\mathcal{A}$. Then the following are equivalent for a countably presented module M:*

(1) *$M \in \mathcal{A}$,*
(2) *M is a direct limit of a countable system $(C_n, f_n)_{n<\omega}$ of finitely presented modules such that $f_n \in \mathfrak{I}$ for every n and $(\mathrm{Hom}_R(C_n, B), \mathrm{Hom}_R(f_n, B))_{n<\omega}$ is Mittag-Leffler for each $B \in \mathcal{B}$.*

If, in addition, $\mathcal{A}$ is closed under (countable) direct limits, then these conditions are further equivalent to:

(3) *M is a direct limit of a countable system $(C_n, f_n)_{n<\omega}$ of finitely presented modules such that $f_n \in \mathfrak{I}$ for every n.*

Proof. (1) $\implies$ (2). Let us fix (any) countable system $(D_n, g_n)_{n<\omega}$ of finitely presented modules such that $M = \varinjlim D_n$. Assume $M \in \mathcal{A}$ and $B \in \mathcal{B}$. Then $B^{(\omega)} \in \mathcal{B}$ and $\mathrm{Ext}^1_R(\varinjlim D_n, B^{(\omega)}) = \mathrm{Ext}^2_R(\varinjlim D_n, B^{(\omega)}) = 0$ by assumption. So the inverse system $(\mathrm{Hom}_R(D_n, B), \mathrm{Hom}_R(g_n, B))_{n<\omega}$ is Mittag-Leffler and the system $(\mathrm{Ext}^1_R(D_n, B), \mathrm{Ext}^1_R(g_n, B))_{n<\omega}$ is T-nilpotent for each $B \in \mathcal{B}$ by Lemma 4.6.

Now, we will by induction construct a strictly increasing sequence $n_0 < n_1 < \cdots$ of natural numbers such that the compositions

$$f_i = g_{n_{i+1}-1} \dots g_{n_i+1} g_{n_i} : D_{n_i} \to D_{n_{i+1}}$$

satisfy $\mathrm{Ext}^1_R(f_i, B) = 0$ for each $i < \omega$ and $B \in \mathcal{B}$. Let us start with $n_0 = 0$. For the inductive step, assume that n_i has already been constructed. If there is some $l > n_i$ such that $\mathrm{Ext}^1_R(g_{l-1} \dots g_{n_i+1} g_{n_i}, B) = 0$ for each $B \in \mathcal{B}$, we are done since we can put $n_{i+1} = l$. If this was not the case, there would be some $B_l \in \mathcal{B}$ for each $l > n_i$ such that $\mathrm{Ext}^1_R(g_{l-1} \dots g_{n_i+1} g_{n_i}, B_l) \neq 0$. But this would imply that $(\mathrm{Ext}^1_R(D_n, \bigoplus_{l>n_i} B_l))_{n<\omega}$ is not T-nilpotent, a contradiction.

Finally, we can just put $C_i = D_{n_i}$ and observe using Lemma 4.7 that $f_i \in \mathfrak{I}$ for each $i < \omega$.

(2) $\implies$ (1). This follows directly from Lemma 4.6, since the inverse system $(\mathrm{Ext}^1_R(C_n, B), \mathrm{Ext}^1_R(f_n, B))_{n<\omega}$ is clearly T-nilpotent for each $B \in \mathcal{B}$ (see Lemma 4.7).

(2) $\implies$ (3) is obvious.

(3) $\implies$ (1). For each n, write f_n as a composition of the form $C_n \xrightarrow{u_n} A_n \xrightarrow{v_n} C_{n+1}$ with $A_n \in \mathcal{A}$. In this way, we get a direct system

$$C_0 \xrightarrow{u_0} A_0 \xrightarrow{v_0} C_1 \xrightarrow{u_1} A_1 \xrightarrow{v_1} C_2 \xrightarrow{u_2} \cdots .$$

Now, $\varinjlim_{n<\omega} C_n = \varinjlim_{n<\omega} A_n$. Hence $M \in \mathcal{A}$ since $\mathcal{A}$ is closed under countable direct limits. $\square$

The preceding theorem allows us to characterize modules in the right-hand class of a cotorsion pair satisfying the assumptions of TCMC. Again, we state the following theorem for more general cotorsion pairs than those in question for TCMC. Note that for projective cotorsion pairs over self-injective artin algebras, the following statement is a consequence of [30, Corollary 7.7] and [29, Theorem A].

Theorem 4.9. *Let R be a right coherent ring and $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* Mod-R *with $\mathcal{B}$ closed under unions of well-ordered chains. Denote by $\mathfrak{I}$ the ideal of all morphisms in* mod-R *which factor through some module from $\mathcal{A}$. Then the following are equivalent:*

(1) *$B \in \mathcal{B}$,*
(2) $\mathrm{Ext}^1_R(f, B) = 0$ *for each $f \in \mathfrak{I}$.*

Proof. (1) $\implies$ (2). This is clear, since in this case, for each $f \in \mathfrak{I}$, the map $\operatorname{Ext}^1_R(f, B)$ factors through $\operatorname{Ext}^1_R(A, B) = 0$ for some $A \in \mathcal{A}$.

(2) $\implies$ (1). Recall that the cotorsion pair is of countable type and complete by Theorem 3.5. Moreover, every countably presented module in $\mathcal{A}$ can be expressed as a direct limit of a direct system $(C_n, f_n)_{n<\omega}$ with all the morphisms f_n in $\mathfrak{I}$ by Theorem 4.8.

Let us define a class of modules $\mathcal{C}$ as

$$\mathcal{C} = \{M \in \text{Mod-}R \mid \operatorname{Ext}^1_R(f, M) = 0 \text{ for each } f \in \mathfrak{I}\}$$

By definition $\mathcal{B} \subseteq \mathcal{C}$.

Note that since every $f \in \mathfrak{I}$ is a morphism between strongly finitely presented modules, say $f : X \to Y$, and it is not difficult to see that the functors $\operatorname{Ext}^1_R(X, -)$ and $\operatorname{Ext}^1_R(Y, -)$ are coherent in this case, so is the functor $F_f = \operatorname{Im} \operatorname{Ext}^1_R(f, -)$. Hence $\mathcal{C}$ is a definable class as it is defined by vanishing of the functors F_f where f runs through a representative set of morphisms from $\mathfrak{I}$. In particular, this means that showing $\mathcal{C} \subseteq \mathcal{B}$ reduces just to showing that every *pure-injective* module $M \in \mathcal{C}$ is already in $\mathcal{B}$, since definable classes are determined by the pure-injective modules they contain.

To this end, assume that $M \in \mathcal{C}$ is pure-injective and $A \in \mathcal{A}$ is countably presented. Then $A = \varinjlim C_n$ where $(C_n, f_n)_{n<\omega}$ is a direct system such that $f_n \in \mathfrak{I}$ for each n. In particular, $\operatorname{Ext}^1_R(f_n, M) = 0$ by assumption and

$$\operatorname{Ext}^1_R(A, M) = \operatorname{Ext}^1_R(\varinjlim C_n, M) \cong \varprojlim \operatorname{Ext}^1_R(C_n, M) = 0$$

by Auslander's lemma. Finally, since $(\mathcal{A}, \mathcal{B})$ is of countable type and A was arbitrary, it follows that $M \in \mathcal{B}$. □

Remark. (i) Countable type of the cotorsion pair considered in Theorem 4.9 together with Lemma 3.4 imply that when defining $\mathfrak{I}$, we may assume that the modules from $\mathcal{A}$ through which the maps $f \in \mathfrak{I}$ are required to factorize are all countably presented.

(ii) To determine which implication of pp-formulas corresponds to the coherent functor F_f from the proof of Theorem 4.9, we build the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i_X} & F_X & \xrightarrow{p_X} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & L & \xrightarrow{i_Y} & F_Y & \xrightarrow{p_Y} & Y & \longrightarrow & 0 \end{array}$$

with F_X, F_Y finitely generated free, K, L finitely presented, s a split embedding and i, i_X, i_Y inclusions. Now, an equivalent statement to $F_f(M) = 0$ is that every homomorphism from K into M which extends to L must extend to F_X as well, and this can be routinely translated to an implication between two pp-formulas to be satisfied in M. If we denote by H the pushout of i and i_X, and by h the pushout map $L \to H$, then the latter actually means that $\operatorname{Coker} \operatorname{Hom}_R(h, M) = 0$. Thus, $\operatorname{Coker} \operatorname{Hom}_R(h, -)$ is a coherent functor which may be equivalently used instead of F_f when defining $\mathcal{B}$.

5. Direct limits and pure-epimorphic images

In the cases when TCMC holds true, the class $\mathcal{A}$ of any cotorsion pair $(\mathcal{A}, \mathcal{B})$ meeting its assumptions must be closed under pure-epimorphic images. Indeed, in this setting, we have $\mathcal{A} = \varinjlim(\mathcal{A} \cap \text{mod-}R)$ and the latter class is closed under pure-epimorphic images by the well-known result of Lenzing (cf. [32] or [19, Lemma 1.2.9]). In this section, we prove that the hypotheses of TCMC do always imply that $\mathcal{A}$ is closed under pure-epimorphic images. As a consequence, we prove that every complete cotorsion pair with both classes closed under arbitrary direct limits is cogenerated by a single pure-injective module—this can be viewed as a module-theoretic counterpart of [29, Theorem C].

Note that the first part—to make sure that $\mathcal{A}$ is closed under pure-epimorphic images—is the crucial one. For projective cotorsion pairs over self-injective algebras which satisfy the hypotheses of TCMC, this property follows by analysis of the proofs in [29] and [30]. But when proving this in a more general setting, one obstacle appears. Namely, complete cotorsion pairs provide us with approximations (special precovers and preenvelopes) which are not functorial in general. Therefore, implementing the rather simple underlying idea—expressing each module in

$\mathcal{A}$ in terms of direct limits of $\mathcal{A}$-precovers of finitely presented modules and proving that this transfers to pure-epimorphic images—requires several technical steps. In particular, we need special indexing sets for our direct systems which we call *inverse trees*.

We start with a preparatory lemma. Recall that for an ordinal number α, we denote by $|\alpha|$ the cardinality of α when viewed as the set of all smaller ordinals.

Definition 5.1. A direct system $(M_i, f_{ji} \mid i, j \in I \ \&\ i \leq j)$ of R-modules is said to be *continuous* if $(M_k, f_{kj} \mid j \in J)$ is the direct limit of the system $(M_i, f_{ji} \mid i, j \in J \ \&\ i \leq j)$ whenever J is a directed subposet of I and k is a supremum of J in I.

Lemma 5.2. *Let κ be an infinite cardinal and M be a κ-presented module. Then M is a direct limit of a continuous well-ordered system $(M_\alpha, f_{\beta\alpha} \mid \alpha \leq \beta < \kappa)$ such that for all $\alpha < \kappa$, M_α is $|\alpha|$-presented.*

Proof. We can start as in Lemma 1.1. Let

$$\bigoplus_{\beta<\kappa} x_\beta R \xrightarrow{g} \bigoplus_{\gamma<\kappa} y_\gamma R \to M \to 0$$

be a free presentation of M. For each $\alpha < \kappa$, let X_α be the subset of all ordinals $\beta < \alpha$ such that $f(x_\beta) \in \bigoplus_{\gamma<\alpha} y_\gamma R$. If we define M_α as the cokernel of the restriction $\bigoplus_{\beta \in X_\alpha} x_\beta R \to \bigoplus_{\gamma<\alpha} y_\gamma R$ of g, it is easy to see that the direct system $(M_\alpha \mid \alpha < \kappa)$ together with the natural maps has the properties we require. □

For a set X, we will denote by X^* the set of all finite strings over X, that is, all functions $u : n \to X$ for $n < \omega$. We will denote strings by letters $u, v, w, \dots$ and write them as sequences of elements of X, which we will denote by Greek letters for a reason which will be clear soon. For example, we write $u = \alpha_0\alpha_1 \dots \alpha_{n-1}$. When u, v are strings, we denote by uv their *concatenation*, we define the *length* of a string u in the usual way and denote it by $\ell(u)$, and we identify strings of length 1 with elements in X. The empty string is denoted by $\varnothing$. Note that the set X^* together with the concatenation operation is nothing else than the free monoid over X.

Definition 5.3. Let κ be an infinite cardinal and κ^* be the free monoid over κ. Let us equip $\kappa^* \setminus \{\varnothing\}$ with a partial order in the following way: If $u = \alpha_0\alpha_1 \dots \alpha_{n-1}$ and $v = \beta_0\beta_1 \dots \beta_{m-1}$, we put $u \leq v$ if

(1) $n \geq m$,
(2) $\alpha_0\alpha_1 \dots \alpha_{m-2} = \beta_0\beta_1 \dots \beta_{m-2}$, and
(3) $\alpha_{m-1} \leq \beta_{m-1}$ as ordinal numbers.

Then an *inverse tree* over κ is the subposet of $(\kappa^* \setminus \{\varnothing\}, \leq)$ defined as

$$I_\kappa = \big\{\alpha_0\alpha_1 \dots \alpha_{n-1} \mid (\forall i \leq n-2)\big(\alpha_i \text{ is infinite, non-limit } \&\ \alpha_{i+1} < |\alpha_i|\big)\big\}.$$

For convenience, given a non-empty string $u = \alpha_0\alpha_1 \dots \alpha_{n-1} \in \kappa^*$, we define the *tail* of u, denoted by $t(u)$, to be the last symbol α_{n-1} of u, and the *rank* of u, $\mathrm{rk}(u)$, to be the cardinal number $|\alpha_{n-1}|$. Notice that in this terminology, the tail of a string $u \in I_\kappa$ is allowed to be a limit or finite ordinal.

Having defined inverse trees, we can start collecting basic properties of the partial ordering:

Lemma 5.4. *Let $(I_\kappa, \leq)$ be an inverse tree, and let v and $u = \beta_0 \dots \beta_{m-2}\beta_{m-1}$ be two elements of I_κ such that $v < u$. Then there is $w \in I_\kappa$ such that $v \leq w < u$ and one of the following cases holds true:*

(1) *There is an ordinal $\gamma < \beta_{m-1}$ such that $w = \beta_0\beta_1 \dots \beta_{m-2}\gamma$.*
(2) *There is an ordinal $\gamma < |\beta_{m-1}|$ such that $w = \beta_0\beta_1 \dots \beta_{m-2}\beta_{m-1}\gamma$.*

Proof. This follows easily from the definition. Notice that (2) can only hold if $\beta_{m-1} = t(u)$ is infinite and non-limit. □

As an immediate corollary, we will see that the properties of $u \in I_\kappa$ with respect to the ordering depend very much on the tail (and rank) of u:

Corollary 5.5. *Let $u = \alpha_0 \dots \alpha_{n-2}\alpha_{n-1} \in I_\kappa$. Then the following hold in $(I_\kappa, \leq)$:*

(1) *If $t(u) = 0$, then u is a minimal element.*
(2) *If $t(u)$ is non-zero finite, then u has a unique immediate predecessor.*

(3) *If* $t(u)$ *is an infinite non-limit ordinal, then* $u = \sup\{u\gamma \mid \gamma < \mathrm{rk}(u)\}$.
(4) *If* $t(u)$ *is a limit ordinal, then* $u = \sup\{\alpha_0 \dots \alpha_{n-2}\gamma \mid \gamma < t(u)\}$.

We have seen that an element $u \in I_\kappa$ can be expressed as a supremum of a chain of strictly smaller elements if and only if $\mathrm{rk}(u)$ is infinite. If so, this chain depends on whether $t(u)$ is a limit ordinal or not. We will prove in the next lemma that as far as we are concerned with continuous direct systems indexed with I_κ, this expression of u as a supremum is essentially unique.

Lemma 5.6. *Let* $u \in I_\kappa$ *be of infinite rank and* C *be the chain as in Corollary 5.5* (3) *or* (4) *such that* $u = \sup C$ *in* I_κ. *Let* $J \subseteq I_\kappa$ *be a directed subposet of* I_κ *such that* $u = \sup J$ *in* I_κ *and* $u \notin J$. *Then* $C \cap J$ *is cofinal in* J.

Proof. Choose some $j \in J$ of the least possible length. Since J is directed, u is the supremum of the upper set $\uparrow j = \{i \in J \mid i \geq j\}$, too. By the definition of the ordering and the fact that j has been taken of the least possible length, we see that each $i \in (\uparrow j)$ is of the form $\beta_0\beta_1 \dots \beta_{m-2}\gamma_i$ where $\beta_0, \beta_1, \dots, \beta_{m-2}$ are fixed and $\gamma_i < |\beta_{m-2}|$. Thus $u = \beta_0\beta_1 \dots \beta_{m-2}$ provided that $\sup\{\gamma_i \mid i \in (\uparrow j)\} = |\beta_{m-2}|$ (case (3)), and $u = \beta_0\beta_1 \dots \beta_{m-2}\beta_{m-1}$ if $\beta_{m-1} = \sup\{\gamma_i \mid i \in (\uparrow j)\} < |\beta_{m-2}|$ (case (4)). Hence, $\uparrow j \subseteq C \cap J$ by assumption, and $C \cap J$ is cofinal in J since $\uparrow j$ is. □

So far, we have studied elements strictly smaller than a given $u \in I_\kappa$. But, we will also need to look "upwards":

Lemma 5.7. *Let* $(I_\kappa, \leq)$ *be an inverse tree. Then*
(1) *For each* $u \in I_\kappa$, *the upper set* $\uparrow u = \{w \in I_\kappa \mid w \geq u\}$ *is well-ordered.*
(2) $(I_\kappa, \leq)$ *is directed.*
(3) *Every non-empty bounded subset* $X \subseteq I_\kappa$ *has a supremum in* I_κ.

Proof. (1). It follows from the definition that $\uparrow u$ is a totally ordered subset of I_κ. If $X \subseteq (\uparrow u)$ is nonempty, then the longest string $u \in X$ with the minimum tail $t(u)$ is the least element in X. Hence, $\uparrow u$ is well-ordered.

(2). Let $u = \alpha_1 \dots \alpha_{n-1}$, $v = \beta_1 \dots \beta_{m-1}$ be elements in I_κ. Then $\max\{\alpha_1, \beta_1\}$, viewed as a string of length 1, is greater than both u and v.

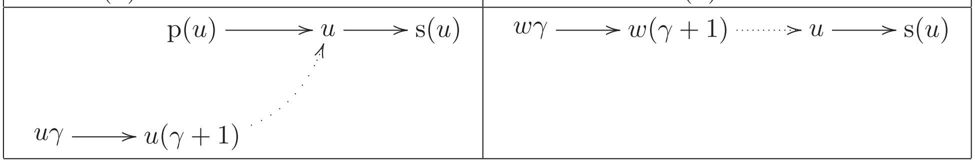
(3). Suppose $X \subseteq I_\kappa$ is non-empty and has an upper bound $u \in I_\kappa$. In other words, $u \in Y$ for $Y = \bigcap_{w \in X}(\uparrow w)$. But since for any $v \in X$ clearly $Y \subseteq (\uparrow v)$, there must be the least element in Y, which is by definition the supremum of X. □

In view of the preceding lemma, we can introduce the following definition:

Definition 5.8. Let $(I_\kappa, \leq)$ be an inverse tree and $u = \alpha_0 \dots \alpha_{n-2}\alpha_{n-1} \in I_\kappa$. Then the *successor* of u in I_κ is defined as $\mathrm{s}(u) = \alpha_0 \dots \alpha_{n-2}\beta$ where $\beta = \alpha + 1$ is the ordinal successor of α. Similarly, if $t(u) = \alpha_{n-1}$ is non-limit and non-zero, we define the *predecessor* of u as $\mathrm{p}(u) = \alpha_0 \dots \alpha_{n-2}\gamma$ where $\gamma = \alpha - 1$ is the ordinal predecessor of α.

Note that by Lemma 5.7, $\mathrm{s}(u)$ is the unique immediate successor of u in $(I_\kappa, \leq)$. On the other hand, even if $\mathrm{p}(u)$ is defined, there still may be other elements in I_κ less than u that are incomparable with $\mathrm{p}(u)$—see Lemma 5.4. We can summarize our observations in a figure showing "neighbourhoods" of elements $u \in I_\kappa$ depending on $t(u)$, where $w \in \kappa^*$ is the string obtained from u by removing its last symbol:

$t(u)$ infinite and non-limit	$t(u)$ limit



This picture also shows the motivation for calling $(I_\kappa, \leq)$ an inverse tree. From each $u \in I_\kappa$, there is exactly one possible way towards greater elements, while when traveling in I_κ down the ordering, there are many branches. The rank zero elements of I_κ can be viewed as leaves. Just the root is missing—it is easy to see that I_κ has no maximal element.

Next, we will turn our attention back to modules. We shall see that each infinitely presented module is the direct limit of a special direct system indexed by an inverse tree.

Lemma 5.9. *Let κ be an infinite cardinal and M be a κ-presented module. Then M is the direct limit of a continuous direct system $(M_u, f_{vu} \mid u, v \in I_\kappa \ \& \ u \leq v)$ indexed by the inverse tree I_κ and such that M_u is $\mathrm{rk}(u)$-presented for each $u \in I_\kappa$.*

Proof. We will construct the direct system by induction on $\ell(u)$ using Lemma 5.2. If $\ell(u) = 1$, then u can be viewed as an ordinal number $< \kappa$ and we just use the modules M_u and morphisms f_{vu} obtained for M by Lemma 5.2.

Suppose we have defined M_u and f_{vu} for all $u, v \in I_\kappa$ with $\ell(u), \ell(v) \leq n$. Let $v \in I_\kappa$ be arbitrary with $\ell(v) = n$ and such that $t(v)$ is infinite and non-limit. Then by using Lemma 5.2 for M_v, we obtain a well-ordered continuous system $(M^v_\alpha, f^v_{\beta\alpha} \mid \alpha \leq \beta < \mathrm{rk}(v))$, and we set $M_{v\alpha} = M^v_\alpha$ and $f_{v\beta,v\alpha} = f^v_{\beta\alpha}$ for all $\alpha \leq \beta < \mathrm{rk}(v)$. Finally, the morphisms $f_{v,v\alpha}$, $\alpha < \mathrm{rk}(v)$, will be defined as the colimit maps $M^v_\alpha \to M_v$, and the rest of the morphisms $f_{u,v\alpha}$ just by taking the appropriate compositions.

The correctness of this construction is ensured by the properties of I_κ proved above, and the fact that $(M_u \mid u \in I_\kappa)$ is continuous is taken care of by Lemma 5.6. □

The crucial fact about inverse trees is that, under the assumptions of TCMC, they allow us to construct for each module a continuous direct system of special precovers:

Lemma 5.10. *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair with both classes closed under direct limits, κ be an infinite cardinal, and M be a κ-presented module. Then there is a continuous direct system of short exact sequences $0 \to B_u \xrightarrow{\iota_u} A_u \xrightarrow{\pi_u} M_u \to 0$ indexed by I_κ such that $B_u \in \mathcal{B}$, $A_u \in \mathcal{A}$, M_u is $\mathrm{rk}(u)$-presented for each $u \in I_\kappa$, and M is the direct limit of the modules M_u.*

Proof. We start with the continuous direct system $(M_u, f_{vu} \mid u, v \in I_\kappa \ \& \ u \leq v)$ given by Lemma 5.9 and construct the exact sequences for each $u \in I_\kappa$ by transfinite induction on $t(u)$.

For each $u \in I_\kappa$ of finite rank, we choose a special $\mathcal{A}$-precover,

$$0 \to B_u \xrightarrow{\iota_u} A_u \xrightarrow{\pi_u} M_u \to 0,$$

of M_u, and if $t(u) > 0$, we find appropriate morphisms $g_{u\mathrm{p}(u)} : A_{\mathrm{p}(u)} \to A_u$ and $h_{u\mathrm{p}(u)} : B_{\mathrm{p}(u)} \to B_u$ using the precover property for the map $f_{u\mathrm{p}(u)} \circ \pi_{\mathrm{p}(u)}$.

Suppose that α is a limit ordinal and the sequences $0 \to B_u \xrightarrow{\iota_u} A_u \xrightarrow{\pi_u} M_u \to 0$ and the maps between them have been constructed for all $u \in I_\kappa$ with $t(u) < \alpha$. Then for each $v \in I_\kappa$ with $t(v) = \alpha$, we define the exact sequence $0 \to B_v \xrightarrow{\iota_v} A_v \xrightarrow{\pi_v} M_v \to 0$ as the direct limit of the direct system of already constructed short exact sequences $0 \to B_w \xrightarrow{\iota_w} A_w \xrightarrow{\pi_w} M_w \to 0$ where w runs over the chain given by Corollary 5.5 (4) used for v. By assumption, we get $A_v \in \mathcal{A}$ and $B_v \in \mathcal{B}$.

Finally, suppose that $\alpha = \delta + 1$ for some infinite δ and we have constructed the exact sequences for all $u \in I_\kappa$ such that $t(u) \leq \delta$. Similarly as above, we define for each $v \in I_\kappa$ with $t(v) = \alpha$ the exact sequence $0 \to B_v \xrightarrow{\iota_v} A_v \xrightarrow{\pi_v} M_v \to 0$ as the direct limit of the direct system of short exact sequences $0 \to B_{v\beta} \xrightarrow{\iota_{v\beta}} A_{v\beta} \xrightarrow{\pi_{v\beta}} M_{v\beta} \to 0$ where β runs over all ordinal numbers $< \mathrm{rk}(v)$. The morphisms $g_{v\mathrm{p}(v)} : A_{\mathrm{p}(v)} \to A_v$ and $h_{v\mathrm{p}(v)} : B_{\mathrm{p}(v)} \to B_v$ can be defined again by the precover property and the rest of the morphisms by obvious compositions. This concludes the construction.

The fact that the direct system of the exact sequences just constructed is well-defined and continuous follows from the lemmas above, in particular from Lemmas 5.4 and 5.6. □

Before stating one of the main results in this section, let us recall that a cotorsion pair satisfying the assumptions of TCMC is complete by Theorem 3.5 (2), thus it fits the setting of the following theorem.

Theorem 5.11. *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair with both classes closed under direct limits. Then $\mathcal{A}$ is closed under pure epimorphic images.*

Proof. Let M be a pure epimorphic image of a module from $\mathcal{A}$. We can assume that M is not finitely presented since otherwise M is trivially in $\mathcal{A}$. Hence, Lemma 5.10 gives us a continuous direct system $0 \to B_u \xrightarrow{\iota_u} A_u \xrightarrow{\pi_u} M_u \to 0$ indexed by I_κ for some κ, and the direct limit $0 \to B \xrightarrow{\iota} A \xrightarrow{\pi} M \to 0$ of this system is a special $\mathcal{A}$-precover of M. It follows from our assumption on M that π is a pure epimorphism.

Now, M is also the direct limit of some direct system $(K_i, k_{ji} \mid i \preceq j)$ consisting of finitely presented modules and indexed by some poset $(J, \preceq)$. We claim that although there is no obvious relation between the direct systems $(M_u \mid u \in I_\kappa)$ and $(K_i \mid i \in J)$, the following holds: For each $i \in J$, there is $s(i) \in J$ such that $i \prec s(i)$ and $k_{s(i)i}$ factors through A_u for some $u \in I_\kappa$ of finite rank.

To this end, denote for all $i \in J$ by $k_i : K_i \to M$ the colimit maps and fix an arbitrary $i \in J$. Then k_i can be factorized through π since K_i is finitely presented and π is pure. Moreover, since $A = \varinjlim_{I_\kappa} A_u$, there is $u_1 \in I_\kappa$ such that k_i factors through A_{u_1}. If $\mathrm{rk}(u_1)$ is finite, we put $u = u_1$. If not, A_{u_1} is by Corollary 5.5 the direct limit of a direct system consisting of some modules A_v with $t(v) < t(u_1)$. Hence, k_i further factors through A_{u_2} for some $u_2 \in I_\kappa$ such that $t(u_2) < t(u_1)$. If the rank of u_2 is finite, we put $u = u_2$. Otherwise, we construct in a similar way u_3 such that $t(u_3) < t(u_2)$, and so forth. Since there are no infinite descending sequences of ordinals, we must arrive at some $u = u_n$ of finite rank after finitely many steps.

Hence, there must be $u_i \in I_\kappa$ of finite rank such that k_i factors through $\pi \circ g_{u_i} = f_{u_i} \circ \pi_{u_i}$ where $g_{u_i} : A_{u_i} \to A$ and $f_{u_i} : M_{u_i} \to M$ are the colimit maps. That is, $k_i = f_{u_i} \circ \pi_{u_i} \circ e_i$ for some $e_i : K_i \to A_{u_i}$ and, since M_{u_i} is finitely presented by Lemma 5.10, f_{u_i} further factors as $k_{j_i} \circ d_{u_i}$ for some $d_{u_i} : M_{u_i} \to K_{j_i}$ and $j_i \in J$ such that $j_i \succ i$. Together, we have $k_i = k_{j_i} \circ d_{u_i} \circ \pi_{u_i} \circ e_i$. Thus, using the fact that K_i is finitely presented and well-known properties of direct limits, there must exist some $s(i) \succeq j_i$ such that $k_{s(i)i} = k_{s(i)j_i} \circ d_{u_i} \circ \pi_{u_i} \circ e_i$, and the claim is proved.

Now set $\tilde{J} = J \times \{0,1\}$ and define $(\tilde{J}, \preceq)$ as the poset generated by the relations $(i,0) \preceq (j,0)$ and $(i,0) \preceq (i,1) \preceq (s(i),0)$ where $i,j \in J, i \preceq j$. Further, for such i,j, put $K_{(i,0)} = K_i$, $K_{(i,1)} = A_{u_i}$, $k_{(j,0),(i,0)} = k_{ji}$, $k_{(i,1),(i,0)} = e_i$, and $k_{(s(i),0),(i,1)} = k_{s(i)j_i} \circ d_{u_i} \circ \pi_{u_i}$, using the same notation as above. In this way, defining the remaining morphisms as the appropriate compositions, we obtain the system $(K_x, k_{yx} \mid x,y \in \tilde{J} \ \& \ x \preceq y)$ which is easily seen to be direct, it has M as its direct limit, and $(K_{(i,1)} \mid i \in J)$ forms a cofinal subsystem. Therefore, M is a direct limit of this cofinal subsystem, which clearly consists of modules from $\mathcal{A}$.

□

Now, we can prove the crucial statement regarding cogeneration of cotorsion pairs by a single pure-injective module. To this end, we need the following notion from [37, Section 9.4]: A pure-injective module N is said to be an *elementary cogenerator* if every pure-injective direct summand of a module elementarily equivalent to $N^{\aleph_0}$ is a direct summand of some power of N. Further recall that the *dual module* M^d of a module M is defined as $M^d = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. It is a well-known fact that any module M is an elementary submodel in its double dual M^{dd} as well as in any reduced $\mathfrak{F}$-power $M^I/\Sigma_{\mathfrak{F}} M^I$ provided that $\mathfrak{F}$ is an ultrafilter on $\mathfrak{P}(I)$ (cf. Definition 2.1, these reduced powers are called *ultrapowers*).

Proposition 5.12. *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair with $\mathcal{B}$ closed under direct limits. Then there exists a pure-injective module E such that the class $\mathrm{Ker}\,\mathrm{Ext}^1_R(-,E)$ coincides with the class of all pure-epimorphic images of modules from $\mathcal{A}$. Moreover, E can be taken of the form $\prod_{k \in K} E_k$, with E_k indecomposable for each $k \in K$.*

Proof. First of all, since $\mathcal{B}$ is closed under direct products and direct limits, it is closed under ultrapowers as well. Thence $M \in \mathcal{B}$ implies by Frayne's Theorem that $N \in \mathcal{B}$ provided that N is a pure-injective direct summand of a module elementarily equivalent to M. In particular, $\mathcal{B}$ is closed under taking double dual modules.

If we denote by $(\mathcal{D}, \mathcal{E})$ the cotorsion pair cogenenerated by the class of all pure-injective modules from $\mathcal{B}$, then $\mathcal{D}$ is exactly the class of all pure-epimorphic images of modules from $\mathcal{A}$ (cf. [5, Lemmas 2.1 and 2.2]; here, the completeness of $(\mathcal{A}, \mathcal{B})$ and $\mathcal{B}$ being closed under *double duals* are actually needed).

By [37, Corollary 9.36], for every module M there exists an elementary cogenerator elementarily equivalent to M. Thus, by the first paragraph, we may consider a representative set $\mathcal{S}$ consisting of elementary cogenerators in $\mathcal{B}$ such that any module in $\mathcal{B}$ is elementarily equivalent to a module from $\mathcal{S}$. Now define E to be the direct product of all modules from $\mathcal{S}$. To finish the main part of our proof, it is enough to show that any pure-injective module from $\mathcal{B}$ is in $\mathrm{Prod}(E)$, the class of all direct summands of powers of E. This is sufficient since then the left-hand class of the cotorsion pair cogenerated by $\{E\}$ will coincide with $\mathcal{D}$.

Let, therefore, $M \in \mathcal{B}$ be a pure-injective module and $N \in \mathcal{S}$ be a module elementarily equivalent to M. By [37, Proposition 2.30], M is a pure submodule (hence a direct summand) in a module elementarily equivalent to $N^{\aleph_0}$. Thus M is a direct summand of some power of N by the definition of elementary cogenerator.

To prove the moreover statement, first recall that, by a well-known result of Fischer, $E = PE(\bigoplus_{j\in J} E_j) \oplus F$ where PE stands for pure-injective hull, E_j is indecomposable pure-injective for each $j \in J$, and F has no indecomposable direct summands; it may happen that J is empty or $F = 0$. By [37, Corollary 4.38], F is a direct summand of a direct product, say $\prod_{l\in L} E_l$, of indecomposable pure-injective direct summands of modules elementarily equivalent to E. According to the first paragraph, $E_l \in \mathcal{B}$ for every $l \in L$. It follows that $PE(\bigoplus_{j\in J} E_j) \oplus \prod_{l\in L} E_l$ cogenerates the same cotorsion pair as E does. Further, $PE(\bigoplus_{j\in J} E_j)$ is a direct summand in $\prod_{j\in J} E_j$ and the latter module is in $\mathcal{B}$ since it is elementarily equivalent to $PE(\bigoplus_{j\in J} E_j) \in \mathcal{B}$. (Here, we use the fact that the direct sum is an elementary submodel in its pure-injective hull as well as in the direct product.) Thus, again, $\prod_{k\in J\cup L} E_k$ cogenerates the same cotorsion pair as E did. □

We are in a position to state the main result of this section. It is in fact an immediate consequence of the previous statements.

Theorem 5.13. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair with both classes closed under direct limits. Then $\mathfrak{C}$ is cogenerated by a direct product of indecomposable pure-injective modules.*

Proof. This follows easily by Theorem 5.11 and Proposition 5.12. □

Remark. (1). Note that if R is an artin algebra or, more generally, a semi-primary ring and $(\mathcal{A}, \mathcal{B})$ is a projective cotorsion pair satisfying the hypotheses of TCMC, it follows from [31, Corollary 4.5] that the class $\mathcal{B}$ is also of the form $\operatorname{Ker}\operatorname{Ext}^1_R(-, N)$ for a pure-injective module N.

(2). The distinction between closure under direct limits and closure under pure-epimorphic images is rather subtle. The two notions often coincide, but no example of a (hereditary) cotorsion pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}$ closed under direct limits and *not* closed under pure-epimorphic images is known to the authors as yet.

6. Compactly generated triangulated categories

In this section, we compare the results we have obtained above with the work of Krause on smashing localizations of triangulated categories in [29, 27]. As mentioned before, there is a bijective correspondence between smashing localizing pairs in the stable module category and certain cotorsion pairs in the usual module category which works for self-injective artin algebras [30]. However, as we want to indicate now, there are strong analogues of both settings well beyond where the correspondence from [30] works. First, we will recall some necessary terminology.

Let $\mathcal{T}$ be a *triangulated category* which admits arbitrary (set indexed) coproducts. We will not define this concept here since it is well-known and the definition is rather complicated, but we refer for example to [18, IV], [21] or [25, §3]. We say that an object $C \in \mathcal{T}$ is *compact* if the canonical map $\bigoplus_i \operatorname{Hom}_{\mathcal{T}}(C, X_i) \to \operatorname{Hom}_{\mathcal{T}}(C, \coprod_i X_i)$ is an isomorphism for any family $(X_i)_{i\in I}$ of objects of $\mathcal{T}$. Here, we will denote coproducts in $\mathcal{T}$ by the symbol $\coprod$ to distinguish them from direct sums of abelian groups. Let us denote by $\mathcal{T}_0$ the full subcategory of $\mathcal{T}$ formed by the compact objects. The category $\mathcal{T}$ is then called *compactly generated* if

(1) $\mathcal{T}_0$ is equivalent to a small category.
(2) Whenever $X \in \mathcal{T}$ such that $\operatorname{Hom}_{\mathcal{T}}(C, X) = 0$ for all $C \in \mathcal{T}_0$, then $X = 0$.

As an important example here, let R be a *quasi-Frobenius ring*, that is a ring for which projective and injective modules coincide, and let $\underline{\text{Mod}}$-R be the *stable category*, that is the quotient of Mod-R modulo the projective modules. Then $\underline{\text{Mod}}$-R is triangulated [21] and compactly generated [29, §1.5]. Moreover, compact objects are precisely those isomorphic in $\underline{\text{Mod}}$-R to finitely generated R-modules. Other examples of compactly generated triangulated categories are unbounded derived categories of module categories and the stable homotopy category.

Let $\mathcal{X}$ be a full triangulated subcategory of $\mathcal{T}$. Then $\mathcal{X}$ is called *localizing* if $\mathcal{X}$ is closed under forming coproducts with respect to $\mathcal{T}$. We call $\mathcal{X}$ *strictly localizing* if the inclusion $\mathcal{X} \to \mathcal{T}$ has a right adjoint. Finally, $\mathcal{X}$ is said to be *smashing* if the right adjoint preserves coproducts. Note that being a smashing subcategory is stronger than being strictly localizing, which in turn is stronger than being a localizing subcategory.

A localizing subcategory $\mathcal{X} \subseteq \mathcal{T}$ is *generated* by a class $\mathcal{C}$ of objects in $\mathcal{T}$ if it is the smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{C}$. Notice that $\mathcal{T}$ itself is generated by $\mathcal{T}_0$ as a localizing subcategory (cf. [39, §5] or [35, Theorem 2.1]).

As in [30], we define $(\mathcal{X}, \mathcal{Y})$ to be a *localizing pair* if $\mathcal{X}$ is a strictly localizing subcategory of $\mathcal{T}$ and $\mathcal{Y} = \operatorname{Ker} \operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, -)$. The objects in $\mathcal{Y}$ are then called $\mathcal{X}$*-local.* Note that this definition makes sense also for non-compactly generated triangulated categories and with this in mind, $(\mathcal{X}, \mathcal{Y})$ is a localizing pair in $\mathcal{T}$ if and only if $(\mathcal{Y}, \mathcal{X})$ is a localizing pair in $\mathcal{T}^{op}$. Moreover, the class $\mathcal{X}$ is smashing if and only if the class $\mathcal{Y}$ of all $\mathcal{X}$-local objects is closed under coproducts.

There is a useful analogue of countable direct limits in a triangulated category, called a homotopy colimit. Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \cdots$$

be a sequence of maps in $\mathcal{T}$. A *homotopy colimit* of the sequence, denoted by $\underrightarrow{\operatorname{hocolim}}\, X_i$, is by definition an object X which occurs in the triangle

$$\coprod_{i<\omega} X_i \xrightarrow{\Phi} \coprod_{i<\omega} X_i \to X \to \coprod_{i<\omega} X_i[1] \qquad (\ddagger)$$

where the i-th component of the map Φ is the composite

$$X_i \xrightarrow{\binom{\mathrm{id}}{-\varphi_i}} X_i \sqcup X_{i+1} \xrightarrow{j} \coprod_{i<\omega} X_i$$

and j is the split monomorphism to the coproduct. Note that a homotopy colimit is unique up to a (non-unique) isomorphism. As an easy but important fact, we point up that when applying the functor $\operatorname{Hom}_{\mathcal{T}}(-, Z)$ on $(\ddagger)$ for any $Z \in \mathcal{T}$, we get an exact sequence

$$0 \leftarrow \underleftarrow{\lim}^1 \operatorname{Hom}_{\mathcal{T}}(X_i, Z) \leftarrow \prod \operatorname{Hom}_{\mathcal{T}}(X_i, Z) \xleftarrow{\Phi^*} \prod \operatorname{Hom}_{\mathcal{T}}(X_i, Z) \leftarrow \underleftarrow{\lim} \operatorname{Hom}_{\mathcal{T}}(X_i, Z) \leftarrow 0$$

where $\Phi^* = \operatorname{Hom}_{\mathcal{T}}(\Phi, Z)$ and $\underleftarrow{\lim}^1$ is the first derived functor of inverse limit.

Having recalled the terminology, we also recall the crucial correspondence between cotorsion pairs and localizing pairs shown in [30]:

Theorem 6.1. *Let R be a self-injective artin algebra,* Mod-R *the category of all right R-modules and* $\underline{\mathrm{Mod}}$-R *the stable category. Then the assignment*

$$(\mathcal{A}, \mathcal{B}) \to (\underline{\mathcal{A}}, \underline{\mathcal{B}})$$

gives a bijective correspondence between projective cotorsion pairs in Mod-R *and localizing pairs in* $\underline{\mathrm{Mod}}$-R*. Moreover, the following hold:*

1. $\underline{\mathcal{A}}$ *is smashing in* $\underline{\mathrm{Mod}}$-R *if and only if both $\mathcal{A}$ and $\mathcal{B}$ are closed under direct limits in* Mod-R.
2. $\underline{\mathcal{A}}$ *is generated, as a localizing subcategory in* $\underline{\mathrm{Mod}}$-R*, by a set of compact objects if and only if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair of finite type in* Mod-R.

Proof. This is an immediate consequence of [30, Theorem 7.6 and Corollary 7.7] and [4, Corollary 4.6]. □

We have proved in Theorem 3.5 that any cotorsion pair $(\mathcal{A}, \mathcal{B})$ coming from a smashing localizing pair is of countable type. We show that it is possible to state a similar countable type result for $\underline{\mathrm{Mod}}$-R purely in the language of triangulated categories.

Definition 6.2. Let $\mathcal{T}$ be a compactly generated triangulated category. We call an object $X \in \mathcal{T}$ *countable* if it is isomorphic to the homotopy colimit of a sequence of maps $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \cdots$ between compact objects. Furthermore, let $\mathcal{T}_\omega$ stand for the full subcategory of $\mathcal{T}$ formed by all countable objects.

Note that $\mathcal{T}_\omega$ is skeletally small. Now we can state the following theorem:

Theorem 6.3. *Let R be a self-injective artin algebra and $\mathcal{T} =$ $\underline{\mathrm{Mod}}$-R the stable category of right R-modules. Then every smashing subcategory of $\mathcal{T}$ is generated, as a localizing subcategory of $\mathcal{T}$, by a set of countable objects.*

We postpone the proof until after a few preparatory observations and lemmas. First note that countable objects in $\underline{\mathrm{Mod}}$-R for a self-injective algebra R are precisely those isomorphic in $\underline{\mathrm{Mod}}$-R to countably generated modules from Mod-R, see [39, Lemma 4.3].

Next, we recall a technical statement concerning vanishing of derived functors of inverse limits. We recall that $\varprojlim^k$ stands for the k-th derived functor of inverse limit and, for convenience, we let $\aleph_{-1} = 1$.

Lemma 6.4. [33] *Let R be a ring and I be a directed set whose smallest cofinal subset has cardinality $\aleph_\alpha$, where α is an ordinal number or -1. Put*

$$d = \sup\{k < \omega \mid \varprojlim^k N_i \neq 0 \text{ for some } (N_i)_{i \in I^{op}}\}$$

where $(N_i)_{i\in I^{op}}$ stands for an inverse system of right R-modules indexed by I^{op}. Then $d = \alpha + 1$ if α is finite and $d = \omega$ if α is an infinite ordinal number.

The latter lemma has important consequences for direct limits that are "small enough". Recall that given a class $\mathcal{C}$ of modules, we denote by $\mathrm{Add}\,\mathcal{C}$ the class of all direct summands of arbitrary direct sums of modules in $\mathcal{C}$.

Lemma 6.5. *Let R be a ring and $(M_i)_{i\in I}$ be a direct system of R-modules such that $|I| < \aleph_\omega$. Then there is an exact sequence:*

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to \varinjlim M_i \to 0,$$

where n is a non-negative integer and $X_j \in \mathrm{Add}\,\{M_i \mid i \in I\}$ for all $j = 0, \dots, n$.

Proof. Consider the canonical presentation of $\varinjlim M_i$:

$$\cdots \xrightarrow{\delta_2} \bigoplus_{i_0<i_1<i_2} M_{i_0 i_1 i_2} \xrightarrow{\delta_1} \bigoplus_{i_0<i_1} M_{i_0 i_1} \xrightarrow{\delta_0} \bigoplus_{i_0 \in I} M_{i_0} \to \varinjlim M_i \to 0,$$

where $M_{i_0 i_1 \dots i_k} = M_{i_0}$ for all k-tuples $i_0 < i_1 < \cdots < i_k$ of elements of I. This is an exact sequence and it follows from [23] that

$$\varprojlim^k \mathrm{Hom}_R(M_i, Y) = \mathrm{Ker}\,\mathrm{Hom}_R(\delta_k, Y) / \mathrm{Im}\,\mathrm{Hom}_R(\delta_{k-1}, Y)$$

for any R-module Y and any $k \geq 0$ (we let $\delta_{-1} = 0$ here). If we take the smallest n such that $|I| \leq \aleph_n$ and $Y = \mathrm{Ker}\,\delta_n$, it follows from Lemma 6.4 that the inclusion

$$0 \to \mathrm{Ker}\,\delta_n \to \bigoplus_{i_0<i_1<\cdots<i_{n+1}} M_{i_0 i_1 \dots i_{n+1}}$$

splits since $\varprojlim^{n+2} \mathrm{Hom}_R(M_i, Y) = 0$ in this case. The claim of the lemma follows immediately. □

Corollary 6.6. *Let R be a quasi-Frobenius ring and let $\underline{\mathcal{A}}$ be a localizing subcategory of $\underline{\mathrm{Mod}}$-R. Assume that $(M_i)_{i\in I}$ is a direct system in* Mod-R *such that $|I| < \aleph_\omega$ and M_i is an object of $\underline{\mathcal{A}}$ for each $i \in I$. Then also $\varinjlim M_i$ is an object of $\underline{\mathcal{A}}$.*

Proof. Note that any localizing subcategory is closed under direct summands [11]. Then the claim follows immediately from the preceding lemma when taking into account that triangles in $\underline{\mathrm{Mod}}$-R correspond to short exact sequences in Mod-R and that the canonical functor Mod-$R \to \underline{\mathrm{Mod}}$-$R$ preserves coproducts. □

Now we are in a position to prove the theorem.

Proof of Theorem 6.3. Let $\underline{\mathcal{A}}$ be a smashing subcategory of $\mathcal{T} = \underline{\mathrm{Mod}}$-$R$ and let $(\mathcal{A}, \mathcal{B})$ be the corresponding projective cotorsion pair in Mod-R with $\mathcal{B}$ closed under direct limits given by Theorem 6.1. Then by Theorem 3.5, there is a set $\mathcal{S}$ of countably generated R-modules that generates the cotorsion pair.

Let us denote by $\mathcal{L}$ the localizing subcategory of $\mathcal{T}$ generated by $\mathcal{S}$, viewed as set of (countable) objects of $\mathcal{T}$. We claim that then for each $X \in \mathcal{T}$, there is a triangle $X \overset{w_X}{\to} B_X \to L_X \to X[1]$ in $\mathcal{T}$ such that $B_X \in \underline{\mathcal{B}}$ and $L_X \in \mathcal{L}$.

Let us assume for a moment that we have proved the claim and let $A \in \underline{\mathcal{A}}$. If we consider the shifted triangle $L_A[-1] \overset{f}{\to} A \overset{w_A}{\to} B_A \to L_A$, then clearly $w_A = 0$ and f is split epi. Hence, A is a direct summand of $L_A[-1]$ and consequently, since $\mathcal{L}$ is closed under direct summands by [11], $A \in \mathcal{L}$. Thus, $\underline{\mathcal{A}} = \mathcal{L}$ and the theorem follows.

Therefore, it remains to prove the claim. Let $X \in \mathcal{T}$. If we view X as an R-module, we can construct a special $\mathcal{B}$-preenvelope $0 \to X \to B_X \to L_X \to 0$ following the lines of [19, Theorem 3.2.1]: We construct a well-ordered continuous chain

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_\alpha \subseteq \cdots$$

indexed by ordinal numbers such that $B_0 = X$ and $B_{\alpha+1}$ is a *universal extension* of B_α by modules from $\mathcal{S}$. That is, there is an exact sequence of the form:

$$0 \to B_\alpha \to B_{\alpha+1} \to \bigoplus_{j \in J_\alpha} Y_j \to 0,$$

where Y_j is isomorphic to a module from $\mathcal{S}$ for each $j \in J_\alpha$ and the connecting homomorphisms $\delta_Z : \mathrm{Hom}_R(Z, \bigoplus_{j \in J} Y_j) \to \mathrm{Ext}^1_R(Z, B_\alpha)$ are surjective for all $Z \in \mathcal{S}$. In particular, $\mathrm{Ext}^1_R(Z, -)$ applied on $B_\alpha \subseteq B_\beta$ for any $\alpha < \beta$ gives the zero map. Since all the modules in $\mathcal{S}$ are countably presented, any morphism $\Omega(Z) \to B_{\aleph_1}$ in Mod-R, where $Z \in \mathcal{S}$, factors through the inclusion $B_\alpha \subseteq B_{\aleph_1}$ for some $\alpha < \aleph_1$. It follows that $\mathrm{Ext}^1_R(Z, B_{\aleph_1}) = 0$ for each $Z \in \mathcal{S}$; hence $B_{\aleph_1} \in \mathcal{B}$. Now, if we set $L_\alpha = B_\alpha / X$ for each α, we have a well-ordered continuous chain

$$L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_\alpha \subseteq \cdots$$

such that $L_{\alpha+1}/L_\alpha \cong B_{\alpha+1}/B_\alpha \in \mathrm{Add}\,\mathcal{S}$. It follows from Eklof's Lemma ([19, Lemma 3.1.2] or [16, Lemma 1]) that $L_\alpha \in \mathcal{A}$ for each ordinal α. Hence, $0 \to X \to B_{\aleph_1} \to L_{\aleph_1} \to 0$ is a special $\mathcal{B}$-preenvelope of X.

Now let us focus on the corresponding triangle $X \to B_{\aleph_1} \to L_{\aleph_1} \to X[1]$ in $\mathcal{T}$. Clearly $B_{\aleph_1} \in \underline{\mathcal{B}}$. Moreover, it follows by a straightforward transfinite induction on α that $L_\alpha \in \mathcal{L}$ for each $\alpha \le \aleph_1$. For $\alpha = 0$, obviously $L_0 = 0 \in \mathcal{L}$. To pass from α to $\alpha + 1$, we use the fact that the third term in the triangle $L_\alpha \to L_{\alpha+1} \to \coprod_{j \in J_\alpha} Y_j \to L_\alpha[1]$ is in $\mathrm{Add}\,\mathcal{S}$. Finally, limit steps are taken care of by Corollary 6.6. The claim is proved and so is the theorem. □

Inspired by Theorem 6.3, we can ask the following question:

Question (Countable Telescope Conjecture). Let $\mathcal{T}$ be an arbitrary compactly generated triangulated category. Is every smashing localizing subcategory of $\mathcal{T}$ generated by a set of countable objects?[2]

In this context, it is a natural question if one can characterize the countable objects in a smashing subcategory of a triangulated category. That is, we are looking for a triangulated category analogue of Theorem 4.8. It turns out that there is an analogous statement that holds for any compactly generated triangulated category.

Theorem 6.7. *Let $\mathcal{T}$ be a compactly generated triangulated category and let $\mathcal{X}$ be a smashing subcategory of $\mathcal{T}$. Denote by $\mathfrak{I}$ the ideal of all morphisms between compact objects which factor through some object in $\mathcal{X}$. Then the following are equivalent for a countable object $X \in \mathcal{T}$:*

(1) *$X \in \mathcal{X}$,*
(2) *X is the homotopy colimit of a countable direct system (X_n, φ_n) of compact objects such that $\varphi_n \in \mathfrak{I}$ for every n.*

Proof. (1) $\implies$ (2). Since X is countable, we have $X = \underrightarrow{\mathrm{hocolim}}\, Y_n$ where (Y_n, ψ_n) is a direct system of compact objects (not necessarily from $\mathcal{X}$). Let Z be an $\mathcal{X}$-local object and let $\tilde{Z} = \coprod_{i<\omega} Z_i$, where $Z_i = Z$ for each $i < \omega$. By assumption, $\tilde{Z}$ is also $\mathcal{X}$-local. If we apply $\mathrm{Hom}_{\mathcal{T}}(-, \tilde{Z})$ on the triangle $\coprod_n Y_n \xrightarrow{\Phi} \coprod_n Y_n \to X \to \coprod_n Y_n[1]$, we see that $\mathrm{Hom}_{\mathcal{T}}(\Phi, \tilde{Z})$ is an isomorphism. Hence we get:

$$\varprojlim \mathrm{Hom}_{\mathcal{T}}(Y_n, \tilde{Z}) = 0 = \varprojlim{}^1 \mathrm{Hom}_{\mathcal{T}}(Y_n, \tilde{Z}).$$

Note also that $\mathrm{Hom}_{\mathcal{T}}(Y_n, \tilde{Z})$ is canonically isomorphic to $\mathrm{Hom}_{\mathcal{T}}(Y_n, Z)^{(\omega)}$ for each $n < \omega$ since all the Y_n are compact. Consequently, the inverse system

$$(\mathrm{Hom}_{\mathcal{T}}(Y_n, Z), \mathrm{Hom}_{\mathcal{T}}(\psi_n, Z))_{n<\omega}$$

[2]An affirmative and far more general answer to this question was given by Krause in [28, §7.4] after submission of this paper.

is Mittag-Leffler by Proposition 1.4 and T-nilpotent by Lemma 4.5. Since the class of all $\mathcal{X}$-local objects is closed under coproducts, we infer, as in the proof of Theorem 4.8, that there are some bounds for T-nilpotency common for all $\mathcal{X}$-local objects Z. In other words, there is a cofinal subsystem $(Y_{n_k}, \varphi_k \mid k < \omega)$ of the direct system (Y_n, ψ_n) such that $\mathrm{Hom}_{\mathcal{T}}(\varphi_k, Z) = 0$ for all $k < \omega$ and $\mathcal{X}$-local objects Z. Note that $X \cong \underrightarrow{\mathrm{hocolim}}_k Y_{n_k}$ since the homotopy colimit does not change when passing to a cofinal subsystem, [36, Lemma 1.7.1].

Finally, if φ is a morphism in $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(\varphi, Z) = 0$ whenever Z is $\mathcal{X}$-local, then φ factors through an object in $\mathcal{X}$ by [29, Lemmas 3.4 and 3.8]. Hence, $\varphi_k \in \mathfrak{I}$ for each k and we can just put $X_k = Y_{n_k}$.

(2) $\implies$ (1). If X and (X_n, φ_n) are as in the assumption, then, by Lemma 4.5,

$$\varprojlim \mathrm{Hom}_{\mathcal{T}}(X_n, Z) = 0 = \varprojlim{}^1 \mathrm{Hom}_{\mathcal{T}}(X_n, Z)$$

whenever Z is $\mathcal{X}$-local. Thus, if we consider the triangle $\coprod_n X_n \xrightarrow{\Phi} \coprod_n X_n \to X \to \coprod_n X_n[1]$ defining X, then $\mathrm{Hom}_{\mathcal{T}}(\Phi, Z)$ is an isomorphism. For a similar reason, $\mathrm{Hom}_{\mathcal{T}}(\Phi[1], Z)$ is an isomorphism, and consequently $\mathrm{Hom}_{\mathcal{T}}(X, Z) = 0$ for all $\mathcal{X}$-local objects Z. In other words: $X \in \mathcal{X}$. □

Triangulated category analogues of Theorems 4.9 and 5.13, the remaining main results of this paper, have been proved by Krause in [29]. We include the corresponding statements from [29] here to underline how straightforward the translation is. Let us start with Theorem 4.9—actually, [29, Theorem A] served as an inspiration for it:

Theorem 6.8. [29, Theorem A] *Let $\mathcal{T}$ be a compactly generated triangulated category and let $\mathcal{X}$ be a smashing subcategory of $\mathcal{T}$. Denote by $\mathfrak{I}$ the ideal of all morphisms between compact objects which factor through some object in $\mathcal{X}$. Then the following are equivalent for $Y \in \mathcal{T}$:*

(1) *Y is $\mathcal{X}$-local,*
(2) *$\mathrm{Hom}_{\mathcal{T}}(f, Y) = 0$ for each $f \in \mathfrak{I}$.*

We conclude the paper with an analogue of Theorem 5.13. Let us first recall that one defines pure-injective objects in a compactly generated triangulated category $\mathcal{T}$ as follows (see [29]): Let us call a morphism $X \to Y$ in $\mathcal{T}$ a *pure monomorphism* if the induced map $\mathrm{Hom}_{\mathcal{T}}(C, X) \to \mathrm{Hom}_{\mathcal{T}}(C, Y)$ is a monomorphism for every compact objects C. An object X is then called *pure-injective* if every pure monomorphism $X \to Y$ splits. As for module categories, the isomorphism classes of indecomposable pure-injective objects form a set which we call a spectrum of $\mathcal{T}$. The following has been proved in [29]:

Theorem 6.9. [29, Theorem C] *Let $\mathcal{T}$ be a compactly generated triangulated category and let $\mathcal{X}$ be a smashing subcategory of $\mathcal{T}$. Then $X \in \mathcal{X}$ if and only if $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for each indecomposable pure-injective $\mathcal{X}$-local object Y.*

For stable module categories over self-injective artin algebras, the correspondence via Theorem 6.1 works especially well because of the following result from [29]:

Proposition 6.10. [29, Proposition 1.16] *Let R be a quasi-Frobenius ring and X be a right R-module. Then X is a pure-injective module if and only if X is a pure-injective object in $\underline{\mathrm{Mod}}$-R.*

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KAPLANSKY CLASSES, FINITE CHARACTER, AND $\aleph_1$–PROJECTIVITY

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Abstract. Kaplansky classes emerged in the context of Enochs' solution of the Flat Cover Conjecture. Their connection to abstract model theory goes back to [3]: a class $\mathcal{C}$ of roots of Ext is a Kaplansky class closed under direct limits, iff the pair $(\mathcal{C}, \leq)$ is an abstract elementary class (AEC) in the sense of Shelah. A question was raised whether this AEC has finite character. We give a positive answer in case $\mathcal{C} = {}^{\perp}\mathcal{C}'$ for a class of pure–injective modules C'. This yields a positive answer for all AECs of roots of Ext over any right noetherian right hereditary ring R.

If $(\mathcal{C}, \leq)$ is an AEC of roots of Ext then $\mathcal{C}$ is known to be a covering class. However, Kaplansky classes need not even be precovering in general: We prove that the class $\mathcal{D}$ of all $\aleph_1$–projective modules is a Kaplansky class for any ring R, but it fails to be precovering in case R is not right perfect, the class ${}^{\perp}(\mathcal{D}^{\perp})$ equals the class of all flat modules and consists of modules of projective dimension ≤ 1. Assuming the Singular Cardinal Hypothesis, we prove that $\mathcal{D}$ is not precovering for each countable non–right perfect ring R.

Introduction

A class $\mathcal{A}$ of (right R–) modules is a *Kaplansky class* provided there is a regular infinite cardinal κ such that for each $0 \neq A \in \mathcal{A}$ and $X \subseteq A$ with $|X| \leq \kappa$, there exists a $\leq \kappa$–presented module $A' \in \mathcal{A}$ such that $X \subseteq A' \subseteq A$ and $A/A' \in \mathcal{A}$.

Kaplansky classes naturally occur in algebra, homotopy theory, and model theory. The fact that the class $\mathcal{FL}$ of all flat modules over an arbitrary ring is a Kaplansky class was crucial for proving the Flat Cover Conjecture in [6]. In [11] it was shown that Kaplansky classes are important sources of module approximations. In [14] the notion was extended to Grothendieck categories $\mathcal{G}$, and applied to constructing model category structures in the category of all unbounded chain complexes over $\mathcal{G}$. In these cases the focus was on Kaplansky classes closed under direct limits.

In parallel, deconstructible classes of modules have widely been used as a set–theoretic tool of homological algebra in [7], [8], [9], [15] et al. (see Definition 1.3 below). Recently it has been shown in [12] that deconstructible classes provide an appropriate setting for application of the methods of Hovey [19]; thus a generalization of the main results of [14] to classes not necessarily closed under direct limits was obtained in [12, Theorem 1.1].

There is close relation between Kaplansky classes and deconstructible classes: Let $\mathcal{C}$ be a class of modules closed under transfinite extensions. If $\mathcal{C}$ is deconstructible then $\mathcal{C}$ is a Kaplansky class, and the converse holds when $\mathcal{C}$ is closed under direct limits (cf. Lemma 1.4).

Many classes $\mathcal{C}$ closed under transfinite extensions are the *classes of roots of Ext*, that is, they are of the form $\mathcal{C} = {}^{\perp}\mathcal{C}'$ for a class of modules $\mathcal{C}'$, where

$$ {}^{\perp}\mathcal{C}' = \bigcap_{1 \leq i < \omega} \operatorname{KerExt}^i_R(-, \mathcal{C}') = \{M \mid \operatorname{Ext}^i_R(M, C') = 0 \text{ for all } C' \in \mathcal{C}' \text{ and } i \geq 1\}. $$

Thus, if a class $\mathcal{C}$ of roots of Ext is closed under direct limits, then $\mathcal{C}$ is a Kaplansky class if and only if $\mathcal{C}$ is deconstructible.

The connection to model theory was discovered somewhat later. In [3] the following link between Kaplansky classes and abstract model theory was established: Consider a pair $(\mathcal{C}, \leq)$ where $\mathcal{C}$ is a class of roots of Ext, and $\leq$ is the partial order on $\mathcal{C}$ defined by $C_0 \leq C_1$ if C_0 is a submodule of C_1 such that $C_0, C_1, C_1/C_0 \in \mathcal{C}$. By [3, 1.18], $(\mathcal{C}, \leq)$ is an abstract elementary

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class (AEC) in the sense of Shelah [24] if and only if $\mathcal{C}$ is a Kaplansky class closed under direct limits. Such AECs are called the *AECs of roots of Ext*. By [3, Theorem 1.20(1)], if $\mathcal{C}'$ is a class of pure–injective modules then $({}^{\perp}\mathcal{C}', \leq)$ is an AEC; indeed, all known examples of AECs of roots of Ext are of this form (we refer to Section 1 for unexplained notation).

An important property of abstract elementary classes, called finite character, was introduced in [20] (cf. Definition 1.5). [3, Question 4.1(2)] asks whether AECs of roots of Ext have finite character. A positive answer for the AECs arising from tilting and cotilting modules was obtained in [26, Theorems 1.4 and 2.2]. The main result of our Section 1 gives a positive answer for all known examples of AECs of roots of Ext:

Theorem 0.1. (1) *Let R be any ring and $(\mathcal{C}, \leq)$ be an AEC of roots of Ext such that $\mathcal{C} = {}^{\perp}\mathcal{C}'$ for a class of pure–injective modules $\mathcal{C}'$. Then $(\mathcal{C}, \leq)$ has finite character.*
(2) *Let R be a right noetherian and right hereditary ring. Then each AEC of roots of Ext has finite character.*

If $\mathcal{C}$ is a Kaplansky class closed under direct limits and extensions, then $\mathcal{C}$ is deconstructible. If $\mathcal{C}$ moreover contains all projective modules, then $\mathcal{C}$ is a covering class, [11, 2.9]. As mentioned above, for classes of modules not closed under direct limits, the property of being Kaplansky is weaker than that of being deconstructible. In fact, the gap between the two notions is rather big in general: While a deconstructible class closed under transfinite extensions, direct summands, and containing R, is always precovering [15, 3.2.4], this is not true of Kaplansky classes.

In order to prove this, we will consider the class $\mathcal{D}$ of all $\aleph_1$–projective modules (see Definition 2.2). It is well known that $\mathcal{D} \subseteq \mathcal{FL}$, and that $\mathcal{D}$ is closed under transfinite extensions, direct summands, and contains all projective modules. It has recently been proved in [18] that $\mathcal{D}$ is deconstructible, if and only if R is a right perfect ring. The main results of Section 2 reads as follows:

Theorem 0.2. *Let R be an arbitrary ring.*

(1) *$\mathcal{D}$ is a Kaplansky class.*
(2) *(SCH) If R is countable then ${}^{\perp}(\mathcal{D}^{\perp}) = \mathcal{FL}$.*
(3) *Assume that R is not right perfect, $\mathcal{FL}$ consists of modules of projective dimension ≤ 1, and ${}^{\perp}(\mathcal{D}^{\perp}) = \mathcal{FL}$. Then $\mathcal{D}$ is not precovering.*
(4) *(SCH) Let R be a countable non–right perfect ring. Then $\mathcal{D}$ is not precovering.*

Here, the notation SCH means that the proof uses a set–theoretic assumption called the Singular Cardinal Hypothesis (see Section §2 for more details).

For unexplained terminology, we refer to [8], [10] and [15].

1. Kaplansky classes of roots of Ext and AECs of finite character

The following definition is due to Shelah [24] (see also [2, Chap. 4]):

Definition 1.1. A pair $(\mathcal{C}, \leq)$ is an *abstract elementary class* (or *AEC* for short) if $\mathcal{C}$ is a class of structures (in a fixed vocabulary τ), and $\leq$ is a partial order on $\mathcal{C}$, both $\mathcal{C}$ and $\leq$ are closed under isomorphism, and satisfy

- (A1) If $A \leq B$ then A is a substructure of B.
- (A2) If $(A_i \mid i < \delta)$ is a $\leq$–increasing chain of elements of $\mathcal{C}$ (that is, $A_i \leq A_{i+1}$ for all $i < \delta$, and $A_i = \bigcup_{j<i} A_j$ for all limit ordinals $i < \delta$) then
  (1) $\bigcup_{i<\delta} A_i \in \mathcal{C}$;
  (2) $A_j \leq \bigcup_{i<\delta} A_i$ for each $j < \delta$;
  (3) If $M \in \mathcal{C}$ and $A_i \leq M$ for each $j < \delta$, then $\bigcup_{i<\delta} A_i \leq M$.
- (A3) If $A, B, C \in \mathcal{C}$, $A \leq C$, $B \leq C$ and A is a substructure of B then $A \leq B$.
- (A4) There is a cardinal κ such that if A is a substructure of $B \in \mathcal{C}$ then there is $A' \in \mathcal{C}$ which contains A as a substructure so that $A' \leq B$, and the cardinality of A' is at most $|A| + \kappa$.

If $A \leq B$ then A is called a *strong substructure* of B. An embedding $f : A \to B$ is *strong* if $f(A) \leq B$.

Basic examples of AECs come from first order logic: the class of all models of a first order theory with the relation of being an elementary submodel is an AEC. The theory of AECs thus generalizes classical model theory to a much more abstract setting — we refer to [2] for more details.

In [3], a new kind of AECs were introduced that arise in homological algebra, namely those of the form $({}^{\perp}\mathcal{D}, \leq)$ where $\mathcal{D}$ is a class of modules over a ring R (see Introduction).

Before describing the results from [3] that are relevant here, we pause to recall the notions and basic properties of transfinite extensions and deconstructible classes from [7] and [18]:

Definition 1.2. Let R be a ring and $\mathcal{A}$ a class of modules. A module M is a *transfinite extension* of modules in $\mathcal{A}$ provided there exists a chain of submodules of M, $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$, such that $M_\alpha \subseteq M_{\alpha+1}$ and $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{A}$ for each $\alpha < \lambda$, $M_0 = 0$, $M_\sigma = M$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$.

The well–known Eklof Lemma ([8, XII.1.5]) says that each class of roots of Ext is closed under transfinite extensions, that is, for any class of modules $\mathcal{C}'$, $M \in {}^{\perp}\mathcal{C}'$ whenever M is a transfinite extension of modules in ${}^{\perp}\mathcal{C}'$.

Definition 1.3. ([3], [7]). Let R be a ring. A class of modules $\mathcal{A}$ is *deconstructible* (or *has refinements*) in case there is an infinite cardinal κ such that each module $M \in \mathcal{A}$ is a transfinite extension of modules in $\mathcal{A}^{<\kappa}$, where $\mathcal{A}^{<\kappa}$ denotes the class of all $< \kappa$–presented modules in $\mathcal{A}$.

The following was proved in [18, Lemmas 6.7 and 6.9]:

Lemma 1.4. *Let R be a ring and $\mathcal{A}$ a class of modules.*

(1) *Assume that $\mathcal{A}$ is a deconstructible class closed under transfinite extensions. Then $\mathcal{A}$ is a Kaplansky class.*
(2) *Assume that $\mathcal{A}$ is closed under extensions and direct limits. Then $\mathcal{A}$ is deconstructible, if and only if $\mathcal{A}$ is a Kaplansky class.*

In [18, Example 6.8], an example is given showing that the notion of a Kaplansky class is weaker in general than that of a deconstructible class: If R is a right self–injective von Neumann regular ring and $\mathcal{A}$ is the class of all non–singular modules, then $\mathcal{A}$ is Kaplansky, but not deconstructible. Since in this particular setting, non–singular = $\aleph_1$–projective (see Definition 2.2), Theorem 0.2(1) generalizes [18, Example 6.8] to arbitrary non–right perfect rings R.

In [3, §1] it was proved that for each ring R and each class of modules $\mathcal{C}'$, the pair $({}^{\perp}\mathcal{C}', \leq)$ is an AEC, if and only if the class ${}^{\perp}\mathcal{C}'$ is closed under direct limits and it is deconstructible.

If $\mathcal{C}'$ is a class of pure–injective modules then ${}^{\perp}\mathcal{C}'$ is closed under direct limits and is deconstructible by [9], so $({}^{\perp}\mathcal{C}', \leq)$ is an AEC. All known examples of AECs of roots of Ext arise in this way (cf. [3]). We denote the class of all pure–injective modules by $\mathcal{PI}$.

We will prove that if $\mathcal{C}' \subseteq \mathcal{PI}$ then the AEC $({}^{\perp}\mathcal{C}', \leq)$ always satisfies the following finiteness property introduced by Hyttinen and Kesälä in [20]:

Definition 1.5. An AEC $(\mathcal{C}, \leq)$ has *finite character* if for all $A, B \in \mathcal{C}$ such that A is a substructure of B, we have $A \leq B$ provided that for each finite set $F \subseteq A$ there exist $C \in \mathcal{C}$ and strong embeddings $f : A \to C$ and $g : B \to C$ such that $f \restriction F = g \restriction F$.

For AECs having amalgamation there is another way of expressing finite character due to Kueker [21, §3]. (An AEC $(\mathcal{C}, \leq)$ *has amalgamation* if for all $A, B, C \in \mathcal{C}$ such that $A \leq B$, $A \leq C$, and $A = B \cap C$, there exist $D \in \mathcal{C}$ and a map $f : B \cup C \to D$ such that $f \restriction B$ and $f \restriction C$ are strong embeddings. The existence of pushouts in module categories easily yields amalgamation for each AEC of roots of Ext, see [3, Lemma 2.1].)

Lemma 1.6. (Kueker) Let $(\mathcal{C}, \leq)$ be an AEC with amalgamation. Then the following are equivalent:

(1) $(\mathcal{C}, \leq)$ has finite character.
(2) For all $A, B \in \mathcal{C}$, if A is a substructure of B and for each finite set $F \subseteq A$ there is a strong embedding $f : A \to B$ such that $f \restriction F = \mathrm{id}_F$, then $A \leq B$.

Proof. That (1) implies (2) is clear even without amalgamation: one takes $C = B$ and $g = \mathrm{id}_B$.

(2) implies (1): Let $\{F_\alpha \mid \alpha < \lambda\}$ be the set of all finite subsets of A. The premise of (1) implies that for each $\alpha < \lambda$ there exist $C'_\alpha \in \mathcal{C}$ such that $B \leq C'_\alpha$ and a strong embedding $f_\alpha : A \to C'_\alpha$ such that $f_\alpha \restriction F_\alpha = \mathrm{id}_{F_\alpha}$.

By induction of α, we will construct a $\leq$–increasing chain $(C_\alpha \mid \alpha < \lambda)$ of elements of $\mathcal{C}$ such that for each $\alpha < \lambda$, B is a strong submodel of C_α, and for each $\beta \leq \alpha$, there is a strong embedding $f_\beta : A \to C_\alpha$ such that $f_\beta \restriction F_\beta = \mathrm{id}_{F_\beta}$. For $\alpha = 0$ we take $C_0 = C'_0$.

For the non–limit step, we can w.l.o.g. assume that $C_\alpha \cap C'_\alpha = B$, use amalgamation for the strong embeddings $B \leq C_\alpha$ and $B \leq C'_\alpha$, and define $C_{\alpha+1} = D$. If $\alpha < \lambda$ is a limit ordinal, it suffices to take $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$ because $C_\alpha \in \mathcal{C}$ by Axiom (A2)(2), and $B \leq C_\alpha$ by Axiom (A2)(3).

Let $C = \bigcup_{\alpha<\lambda} C_\alpha$. Then (2) applied to the pair (A, C) gives $A \leq C$. Since $B \leq C$, Axiom (A3) yields $A \leq B$. □

Before stating the key proposition of this section, let us briefly recall some notions from approximation theory of modules. A pair $(\mathcal{A}, \mathcal{B})$ of classes of modules is a *cotorsion pair* if $\mathcal{A} = {}^{\perp_1}\mathcal{B} = \mathrm{KerExt}^1_R(-, \mathcal{B})$ and $\mathcal{B} = \mathcal{A}^{\perp_1} = \mathrm{KerExt}^1_R(\mathcal{A}, -)$. We say that the cotorsion pair is *hereditary* provided that $\mathcal{A}$ is closed under taking kernels of epimorphisms, or equivalently, $\mathcal{A} = {}^{\perp}\mathcal{B}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *complete* if $\mathcal{A}$ is a special precovering class, i.e., for every module M there exists an R–epimorphism $\psi : A \to M$ with $A \in \mathcal{A}$ and $\mathrm{Ker}\ \psi \in \mathcal{B}$; such ψ is called a *special $\mathcal{A}$–precover* of M.

More in general, a class of modules $\mathcal{C}$ is a *precovering class* provided that for each module M there exists a R–homomorphism $\psi : C \to M$ with $C \in \mathcal{C}$ such that for each R–homomorphism $\psi' : C' \to M$ with $C' \in \mathcal{C}$ there exists an R–homomorphism $\varphi : C' \to C$ such that $\psi' = \psi\varphi$. The map ψ is a *$\mathcal{C}$–precover* of M.

Proposition 1.7. Let R be a ring and $(\mathcal{C}, \mathcal{C}')$ be a hereditary cotorsion pair such that $(\mathcal{C}, \leq)$ is an AEC. If ${}^{\perp_1}(\mathcal{C} \cap \mathcal{C}') \supseteq {}^{\perp_1}(\mathcal{PI} \cap \mathcal{C}')$ then $(\mathcal{C}, \leq)$ has finite character.

Proof. We will verify condition (2) of Lemma 1.6, so we consider $A, B \in \mathcal{C}$ with $A \subseteq B$ satisfying the premise of that condition. Let $\{b_\alpha \mid \alpha < \lambda\}$ be an R–generating subset of B and denote by S the set of all sequences $(r_\alpha) \in R^{(\lambda)}$ such that $\sum_{\alpha<\lambda} b_\alpha r_\alpha \in A$.

In order to prove that $A \leq B$, we have to show that $\mathrm{Ext}^1_R(B/A, C') = 0$ for each $C' \in \mathcal{C}'$. Note that it is not necessary to check all of $\mathcal{C}'$—it suffices to prove vanishing of Ext just for the pure–injective modules from $\mathcal{C}'$; then, by our assumption, $\mathrm{Ext}^1_R(B/A, C') = 0$ will hold for each $C' \in \mathcal{C} \cap \mathcal{C}'$. Now, if $\mathrm{Ext}^1_R(B/A, E) \neq 0$ for some $E \in \mathcal{C}'$, then there must be $g \in \mathrm{Hom}_R(A, E)$ that cannot be extended to $h \in \mathrm{Hom}_R(B, E)$ (because $\mathrm{Ext}^1_R(B, C') = 0$ for all $C' \in \mathcal{C}'$). Since $\mathcal{C}$ is deconstructible, $(\mathcal{C}, \mathcal{C}')$ is complete by [15, 3.2.1]. Thus there is a special $\mathcal{C}$–precover $\psi : C \to E$ of E. Then clearly $C \in \mathcal{C} \cap \mathcal{C}'$, and as $A \in \mathcal{C}$, we may factorize g through ψ and then extend it to a homomorphism from B to C' which, composed with ψ, yields an extension h of g, a contradiction.

So it remains to prove that for all $C' \in \mathcal{PI} \cap \mathcal{C}'$ and $g \in \mathrm{Hom}_R(A, C')$ there exists an extension $h \in \mathrm{Hom}_R(B, C')$ of g; then $\mathrm{Ext}^1_R(B/A, C') = 0$, and $A \leq B$ as claimed.

We define an (infinite) system $\mathcal{S}$ of R–linear equations in the variables $\{x_\alpha \mid \alpha < \lambda\}$ as follows. The equations will be indexed by the elements of S: for each sequence $\bar{r} = (r_\alpha) \in S$ we define $a_{\bar{r}} = \sum_{\alpha<\lambda} b_\alpha r_\alpha (\in A)$, and let the equation indexed by $\bar{r}$ be $\sum_{\alpha<\lambda} x_\alpha r_\alpha = g(a_{\bar{r}})$.

We claim that each finite subsystem of $\mathcal{S}$ is solvable in C'. Indeed, let $\bar{r}^0 = (r^0_\alpha), \dots, \bar{r}^{n-1} = (r^{n-1}_\alpha)$ be finitely many elements of S. Let $F = \{a_{\bar{r}^i} \mid i < n\}$.

By the premise on A and B, there is a monomorphism $f \in \mathrm{Hom}_R(A, B)$ such that $f \restriction F = \mathrm{id}_F$ and $A' = f(A) \leq B$. So $\mathrm{Ext}^1_R(B/A', C') = 0$, hence there is $h' \in \mathrm{Hom}_R(B, C')$ such that $h' \restriction A' = gf^{-1}$.

Define $d'_\alpha = h'(b_\alpha)$ for each $\alpha < \lambda$. Then for each $i < n$ we have

$$\sum_{\alpha<\lambda} d'_\alpha r^i_\alpha = h'(\sum_{\alpha<\lambda} b_\alpha r^i_\alpha) = h'(a_{\bar{r}^i}) = gf^{-1}(a_{\bar{r}^i}) = g(a_{\bar{r}^i}).$$

So $(d'_\alpha \mid \alpha < \lambda)$ is a solution in C' of the finite subsystem of $\mathcal{S}$ indexed by $\{\bar{r}^0, \dots, \bar{r}^{n-1}\} \subseteq S$.

Since C' is pure–injective, the system $\mathcal{S}$ has a global solution in C', that is, there exist $(d_\alpha \mid \alpha < \lambda)$ in C' such that $\sum_{\alpha<\lambda} d_\alpha r_\alpha = g(a_{\bar{r}})$ for each $\bar{r} = (r_\alpha) \in S$ (see e.g. [15, 1.2.19]).

Define $h \in \mathrm{Hom}_R(B, C')$ by $h(\sum_{\alpha<\lambda} b_\alpha r_\alpha) = \sum_{\alpha<\lambda} d_\alpha r_\alpha$. This is possible since $\sum_{\alpha<\lambda} b_\alpha r^0_\alpha = \sum_{\alpha<\lambda} b_\alpha r^1_\alpha$ implies $\sum_{\alpha<\lambda} b_\alpha r_\alpha = 0 \in A$ where $\bar{r} = \bar{r}^0 - \bar{r}^1 \in S$ and $a_{\bar{r}} = 0$. The equation indexed by $\bar{r}$ yields $\sum_{\alpha<\lambda} d_\alpha r_\alpha = g(a_{\bar{r}}) = 0$, hence $h(\sum_{\alpha<\lambda} b_\alpha r^0_\alpha) = h(\sum_{\alpha<\lambda} b_\alpha r^1_\alpha)$.

Finally, if $a \in A$ then $a = a_{\bar{r}} = \sum_{\alpha<\lambda} b_\alpha r_\alpha$ for some $\bar{r} = (r_\alpha) \in S$, so $h(a) = h(\sum_{\alpha<\lambda} b_\alpha r_\alpha) = \sum_{\alpha<\lambda} d_\alpha r_\alpha = g(a)$. This proves that h extends g. □

Now, we are in a position to prove a more general version of Theorem 0.1(1):

Theorem 1.8. *Let R be a ring and $({}^\perp\mathcal{C}', \leq)$ be an AEC such that either $\mathcal{C}'$ consists of pure–injective modules or ${}^\perp\mathcal{C}'$ is closed under products. Then $({}^\perp\mathcal{C}', \leq)$ has finite character.*

Proof. Let $\mathcal{C} = {}^\perp\mathcal{C}'$. In the first case, since all cosyzygies of pure–injective modules are pure–injective (see e.g. [15, 3.2.10]), possibly enlarging $\mathcal{C}'$, we can assume that $\mathcal{C} = {}^{\perp_1}\mathcal{C}'$. The class $\mathcal{C}$ is thus a left–hand class of a hereditary cotorsion pair satisfying the assumptions of Proposition 1.7.

Let $\mathcal{C}$ be closed under products and assume w.l.o.g. that $(\mathcal{C}, \mathcal{C}')$ is a hereditary cotorsion pair. We show that $\mathcal{C} \cap \mathcal{C}'$ consists of pure–injective modules; this will again allow us to use Proposition 1.7.

By the well–known characterization of pure–injective modules (cf. [15, 1.2.19]), it suffices to show that $\mathrm{Ext}^1_R(M^\lambda/M^{(\lambda)}, M) = 0$ for every $M \in \mathcal{C} \cap \mathcal{C}'$ and all cardinals λ. However

$$M^\lambda/M^{(\lambda)} = \varinjlim_{F \in [\lambda]^{<\omega}} M^{\lambda \setminus F},$$

where $[\lambda]^{<\omega}$ denotes the set of all finite subsets of λ, and $\mathcal{C}$ is closed under direct limits and products, so $M^\lambda/M^{(\lambda)} \in \mathcal{C}$, and $\mathrm{Ext}^1_R(M^\lambda/M^{(\lambda)}, M) = 0$, q.e.d. □

As a corollary, we obtain the two main results from [26]:

Corollary 1.9. *Let R be a ring and let $({}^\perp\mathcal{C}', \leq)$ be an AEC such that either $\mathcal{C} = {}^\perp\mathcal{C}'$ is a cotilting class, or $\mathcal{C}'$ is a tilting class. Then $({}^\perp\mathcal{C}', \leq)$ has finite character.*

Proof. In the first case, $\mathcal{C}$ is closed under direct products, so Theorem 1.8 applies directly.

In the second case, there are an $n < \omega$ and a resolving set $\mathcal{S}$ consisting of strongly finitely presented modules of projective dimension $\leq n$ such that $\mathcal{C}' = \mathcal{S}^{\perp_1}$ (see [5] or [15, 5.2.10]). Moreover, $\mathcal{C} = {}^\perp\mathcal{E}$ where $\mathcal{E}$ is the class of all pure–injective modules in $\mathcal{C}'$ (see [1, §2] or [15, 4.5.8]), so Theorem 1.8 applies again. □

Whether Theorem 1.8 covers the general case, that is, whether *each* AEC of roots of Ext has finite character, remains an open problem.

There is, however, a case where the answer is positive, namely the case of Theorem 0.1(2):

Corollary 1.10. *Let R be a right noetherian right hereditary ring and let $({}^\perp\mathcal{C}', \leq)$ be any AEC of roots of Ext. Then $({}^\perp\mathcal{C}', \leq)$ has finite character.*

Proof. By assumption $\mathcal{C}'$ consists of modules of injective dimension ≤ 1, so we have ${}^\perp\mathcal{C}' = {}^\perp\mathcal{E}$ where $\mathcal{E}$ is the class of pure–injective envelopes of all modules in $\mathcal{C}'$ by [3, 1.10], and Theorem 1.8 applies. □

2. The Kaplansky class of all $\aleph_1$–projective modules

We start with refining the notion of a Kaplansky class. Given an infinite cardinal κ and a class $\mathcal{A}$ of modules, we say that $\mathcal{A}$ is a *κ–Kaplansky class* provided that for each $0 \neq A \in \mathcal{A}$ and $X \subseteq A$ with $|X| \leq \kappa$, there exists a $\leq \kappa$–presented $C \in \mathcal{A}$ such that $X \subseteq C \subseteq A$ and $A/C \in \mathcal{A}$. (So $\mathcal{A}$ is Kaplansky class in case there is an infinite regular cardinal κ such that $\mathcal{A}$ is κ–Kaplansky.)

Lemma 2.1. *Let R be a ring and $\lambda = |R| + \aleph_0$. Let A be a module and κ be a cardinal such that $\kappa^\lambda = \kappa$. Then for each $X \subseteq A$ with $|X| \leq \kappa$ there exists a $\leq \kappa$–presented submodule C of A containing X, with the following property:*

Each system of R–linear equations with $\leq \lambda$ variables and $\leq \lambda$ parameters from C has a solution in C provided that it has a solution in A.

Proof. The module C will be defined as the union of a well–ordered chain $(C_i \mid i < \lambda^+)$ of $\leq \kappa$–presented submodules of A. Put $C_0 = \langle X \rangle$. Having constructed C_i for some $i < \lambda^+$, we define C_{i+1} in such a way that it contains a solution of each system of R–linear equations with $\leq \lambda$ variables and $\leq \lambda$ parameters from C_i that has a solution in A. Since the length of the solution vector is $\leq \lambda$ and the number of such systems of equations is $\leq \kappa^\lambda \cdot 2^\lambda = \kappa$, C_{i+1} can be chosen $\leq \kappa$–presented. If $i < \lambda^+$ is limit, we put $C_i = \bigcup_{j<i} C_j$.

Now it is easy to see, that C defined in this way has the desired property. Indeed, if a system of R–linear equations uses only $\leq \lambda$ parameters from C, then these are already elements of some C_i and the solution we are looking for can be found in C_{i+1}. □

We recall the definitions of an $\aleph_1$–projective module and a $\mathcal{Q}$–Mittag–Leffler module from [18]:

Definition 2.2. Let R be a ring and M be a module.

1. M is $\aleph_1$*–projective* provided that there is a system $\mathcal{S}$ consisting of countably generated projective submodules of M such that
   1. $\mathcal{S}$ is closed under unions of countable well–ordered ascending chains,
   2. every countable subset of M is contained in an element of $\mathcal{S}$.
2. Let $\mathcal{Q}$ be a class of left R–modules. Denote by $\mathcal{D}_\mathcal{Q}$ the class of all *flat* $\mathcal{Q}$*–Mittag–Leffler* modules, that is, the flat modules M such that the canonical morphism
$$\rho\colon M \otimes_R \prod_{i\in I} Q_i \to \prod_{i\in I} M \otimes_R Q_i$$
is monic for each family of left R–modules $(Q_i \mid i \in I)$ from $\mathcal{Q}$.

The remarkable fact is that $\aleph_1$–projective modules coincide with the flat $\mathcal{Q}$–Mittag–Leffler modules when $\mathcal{Q}$ is the class of *all* left R–modules (see [18, Theorem 2.9]).

Now we can prove a generalization of Theorem 0.2(1):

Theorem 2.3. *Let $\mathcal{D}_\mathcal{Q}$ denote the class of all flat $\mathcal{Q}$–Mittag–Leffler modules. Then $\mathcal{D}_\mathcal{Q}$ is a κ–Kaplansky class provided that $\kappa^\lambda = \kappa$, where $\lambda = |R| + \aleph_0$.*

In particular, the class of all $\aleph_1$–projective modules is a Kaplansky class.

Proof. Let us put $\mathcal{A} = \mathcal{D}_\mathcal{Q}$. For given $A \in \mathcal{A}$ and $X \subseteq A$ with $|X| \leq \kappa$, Lemma 2.1 provides us with a module C.

Clearly $C \in \mathcal{A}$, since C is pure in A and $\mathcal{A}$ is closed under pure submodules by [18, Lemma 4.1]. So it remains to prove that $A/C \in \mathcal{A}$.

First (by [18, Theorem 2.5]), it is enough to show that every $\leq \lambda$–generated pure submodule of A/C is an element of $\mathcal{A}$. So let B/C be such a submodule and let $\{b_i + C \mid i < \mu\}$ be its generating subset ($\mu \leq \lambda$). Now consider the system of all R–linear equations $\sum_{i\in I} x_i r_i = c$ where $c \in C, r_i \in R, I$ is a finite subset of μ and $\sum_{i\in I} b_i r_i = c$ holds in A. This system has μ variables, $\leq \lambda$ parameters from C, since $|\sum_{i<\mu} b_i R| \leq \lambda$, and $(b_i \mid i < \mu)$ solves it in A. Thus there must be also a solution $(c_i \mid i < \mu)$ in C. Put $J = \sum_{i<\mu}(b_i - c_i)R$.

By the definition of J, we have $J + C = B$ and $J \cap C = 0$. Moreover, C pure in A and B/C pure in A/C imply that B is pure in A. Then J, being a direct summand of B, is pure in A too, hence $J \cong B/C \in \mathcal{A}$ by [18, Lemma 4.1].

As mentioned above, the class of $\aleph_1$–projective modules coincides with $\mathcal{D}_{\text{Mod}-R}$ and it suffices to choose $\kappa = (2^\lambda)^+$. (2^λ need not be a regular cardinal; however κ is regular and $\kappa^\lambda = \kappa$ holds by a cardinal–arithmetic result due to Hausdorff.) □

Given an infinite cardinal λ, we say that a submodule C of a module A is λ*–pure* if every system of R–linear equations of cardinality $< \lambda$ with parameters from C has a solution in C provided that it has a solution in A. Thus $\aleph_0$–purity coincides with the classical purity. Lemma 2.1 says that for every subset X of A with $|X| \leq \kappa$, there is a λ^+–pure submodule C of A containing X with $|C| \leq \kappa$ provided that $\kappa = \kappa^\lambda$. We can refer to this C as to λ^+*–purification* of X in A.

Corollary 2.4. *$\mathcal{D}_\mathcal{Q}$ is closed under taking cokernels of $(|R| + \aleph_0)^+$–pure embeddings.*

Proof. Contained in the proof of Theorem 2.3.

Now, we recall some notions and results from the set theory. We denote by SCH the so–called *Singular Cardinal Hypothesis.* It says that, whenever κ is a singular cardinal, then $\kappa^{\text{cf}(\kappa)} =$

$\max\{2^{\mathrm{cf}(\kappa)}, \kappa^+\}$. One of its useful consequences is that $2^\kappa = (2^{<\kappa})^+$ provided that κ is a singular cardinal with $\mathrm{cf}(2^{<\kappa}) < \kappa$. From this, it follows that $2^\kappa = \kappa^+$ holds for all strong limit singular cardinals κ, ie. singular cardinals with the property that $2^\mu < \kappa$ whenever $\mu < \kappa$. It is known, that in order to construct a model of ZFC where SCH does not hold, one needs to use larger cardinal than just a measurable one. On the other hand, SCH follows, for instance, from the generalized continuum hypothesis (GCH) which is provably consistent with ZFC.

Given an uncountable regular cardinal κ and E its stationary subset, we denote by $\diamondsuit^*_\kappa E$ the following assertion:

There exists a sequence $(W_\alpha \mid \alpha \in E, W_\alpha \subseteq \mathcal{P}(\alpha))$ such that $(\forall \alpha \in E)\, |W_\alpha| \leq |\alpha|$ and with the property that, for every $X \subseteq \kappa$, there exists a closed unbounded set $C \subseteq \kappa$ such that $(\forall \alpha \in C \cap E)\, X \cap \alpha \in W_\alpha$.

By the result of Gregory from [17] (also see [23]), $\diamondsuit^*_{\kappa^+} E(\lambda)$ holds if $\kappa > \aleph_0$, $2^\kappa = \kappa^+, \kappa^\lambda = \kappa$ and λ is an infinite regular cardinal. Here $E(\lambda) = \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \lambda\}$. On the other hand, it is a well-known result due to Kunen that $\diamondsuit^*_\kappa E$ implies $\diamondsuit_\kappa D$ for all stationary $D \subseteq E$ (for all regular uncountable cardinals κ).

Finally, we recall the definition of the $\beth$ (Beth) function. Given an infinite cardinal λ, we put $\beth_0(\lambda) = \lambda$ and then inductively $\beth_{\alpha+1}(\lambda) = 2^{\beth_\alpha(\lambda)}$ for an ordinal α, and $\beth_\alpha(\lambda) = \bigcup_{\beta<\alpha} \beth_\beta(\lambda)$ whenever α is limit.

For the rest of this section, let $\lambda = |R| + \aleph_0$. Before proving our next result, we need to recall a proposition from [18].

Proposition 2.5. ([18, 5.5]) *Let κ be an infinite regular cardinal and $N = \varinjlim_{i<\omega} F_i$ be the direct limit of a countable direct system of modules. Then there exists a module M with the following properties and such that $M \in \mathcal{D}_\mathcal{Q}$ if $\bigoplus_{i<\omega} F_i \in \mathcal{D}_\mathcal{Q}$.*

(i) M has a strictly increasing filtration $\mathcal{L} = (L_\alpha \mid \alpha \leq \kappa^+)$; so $L_{\kappa^+} = M$.

(ii) If $|F| + \lambda \leq \kappa$, then L_α is a $\leq \kappa$–presented module for each $\alpha < \kappa^+$. In particular, $|M| = \kappa^+$.

(iii) Let $\nu < \mu \leq \kappa^+$ and assume that $\mathrm{cf}(\nu) = \omega$. Then there exists a module $K \subseteq L_\mu/L_\nu$ such that $L_\mu/L_\nu = L_{\nu+1}/L_\nu \oplus K$, and $L_{\nu+1}/L_\nu \cong N$.

(iv) Let $n < \omega$. If $\bigoplus_{i<\omega} F_i \in \mathcal{P}_n$, then $M \in \mathcal{P}_{n+1}$.

The following theorem relies on the Singular Cardinal Hypothesis.

Theorem 2.6. (SCH) *Let B be an R–module and $\lambda = |R| + \aleph_0$. Assume that $\mathrm{Ext}^1_R(\mathcal{D}_\mathcal{Q}, B) = 0$. Then $\mathrm{Ker}\,\mathrm{Ext}^1_R(-, B)$ contains all direct limits of countable direct systems of modules from $\mathcal{D}_\mathcal{Q}$.*

Moreover, if there is $n < \omega$ such that the countable direct system consists of modules from $\mathcal{P}_n$, then the hypothesis can be weakened to $\mathrm{Ext}^1_R(\mathcal{D}_\mathcal{Q} \cap \mathcal{P}_{n+1}, B) = 0$.

Proof. Put $\mathcal{A} = \mathrm{Ker}\,\mathrm{Ext}^1_R(-, B)$. We shall work along the lines of the proof of [18, Theorem 6.10] with $\mathcal{A}' = \mathcal{D}_\mathcal{Q}$. So assume, by the way of contradiction, that there is a module $N \notin \mathcal{A}$ which is the direct limit of a countable direct system of modules F_i $(i < \omega)$ from $\mathcal{D}_\mathcal{Q}$. Put $F = \bigoplus_{i<\omega} F_i$ and define $\kappa = \beth_{\lambda^+}(|E(B)| + |F|)$; notice that κ is a strong limit singular cardinal with cofinality λ^+. Let $M \in \mathcal{D}_\mathcal{Q}$ be a module obtained by Proposition 2.5 for this κ and N.

For this M, we construct a new filtration $\mathcal{M} = (M_\alpha \in \mathcal{D}_\mathcal{Q} \mid \alpha \leq \kappa^+)$ with the property that $|M_\alpha| \leq \kappa$, for $\alpha < \kappa^+$, and $M/M_\alpha \in \mathcal{D}_\mathcal{Q}$ provided that $\mathrm{cf}(\alpha) > \lambda$ or α is non–limit. If the moreover clause applies, we require additionally that $M_\alpha, M/M_\alpha \in \mathcal{P}_{n+1}$ for all $\alpha \leq \kappa^+$.

So let us start with $M_0 = 0$ and assume that M_β has been constructed for all $\beta < \alpha \leq \kappa^+$. If α is limit, we put $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. If $\alpha = \gamma + 1$ for γ with $\mathrm{cf}(\gamma) > \lambda$, obtain M_α as a module C from Lemma 2.1 with $X = M_\gamma + L_\alpha$ and $A = M$. Otherwise said, M_α is a λ^+–purification of $M_\gamma + L_\alpha$ in M. Notice that $\kappa = \kappa^\lambda$ (even in ZFC) because $\mathrm{cf}(\kappa) = \lambda^+$ and κ is strong limit. In the last remaining case, there are two possibilities:

(i) Every $f \in \mathrm{Hom}_R(M_\gamma, B)$ can be extended to an element of $\mathrm{Hom}_R(K, B)$ whenever $M_\gamma \subseteq K \subseteq M$ and $|K| \leq \kappa$. Then proceed as in the previous step.

(ii) There is an $f \in \mathrm{Hom}_R(M_\gamma, B)$ which does not extend to some submodule $K \supset M_\gamma$ of M with $|K| \leq \kappa$. Then take $X = K + L_\alpha$ instead of $M_\gamma + L_\alpha$.

By Corollary 2.4, we have $M/M_\alpha \in \mathcal{D}_\mathcal{Q}$ whenever α is non–limit. If $\mathrm{cf}(\alpha) > \lambda$, then M_α is also a λ^+–pure submodule in M since the $\leq \lambda$ parameters in the system of equations have to be contained in some M_β for $\beta < \alpha$ non–limit. Thus $M/M_\alpha \in \mathcal{D}_\mathcal{Q}$ again. Regardless of the cofinality of α, we have $M_\alpha \in \mathcal{D}_\mathcal{Q}$ since all the modules M_α are pure in M. If there is no $n < \omega$ such that $F \in \mathcal{P}_n$, then the filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \kappa^+)$ has the desired properties and we can skip the following paragraph.

If $F \in \mathcal{P}_n$ for some $n < \omega$, we know that $M \in \mathcal{P}_{n+1}$ by Proposition 2.5(*iv*). Then there is a filtration $(P_\alpha \mid \alpha \leq \kappa^+)$ with $|P_\alpha| \leq \kappa$ for $\alpha < \kappa^+$, and such that its consecutive factors belong to $\mathcal{P}_{n+1}$ (cf. [15, 4.1.11]). Further, the set $S = \{\alpha < \kappa^+ \mid P_\alpha = M_\alpha\}$ is closed and unbounded in κ^+. Since $E(\lambda^+)$ is stationary in κ^+, we can define a continuous non–decreasing function $f : \kappa^+ \to S$ such that $f(\alpha + 1) \in E(\lambda^+)$ for all $\alpha < \kappa^+$. If we put $f(\kappa^+) = \kappa^+$ and $f(0) = 0$, then $(M_{f(\alpha)} \mid \alpha \leq \kappa^+)$ is a filtration of M with the same properties as $(M_\alpha \mid \alpha \leq \kappa^+)$. Additionaly, for all $\alpha \leq \kappa^+$, we have $M_{f(\alpha)}, M/M_{f(\alpha)} \in \mathcal{P}_{n+1}$ by Eklof Lemma. For the sake of convenience, we renumber the modules M_α by putting $M_\alpha := M_{f(\alpha)}$; then $\mathcal{M} = (M_\alpha \mid \alpha \leq \kappa^+)$ is a filtration of M with the desired properties.

We show that we can find a subfiltration of $\mathcal{M}$ with consecutive factors in $\mathcal{A}$. The only possible obstacle could be that the set $D = \{\alpha < \kappa^+ \mid M_{\alpha+1}/M_\alpha \notin \mathcal{A}\}$ is stationary. If so, by the construction of $\mathcal{M}$, we have that $D = \bigcup\{E(\mu) \cap D \mid \mu \text{ is an infinite regular cardinal}, \mu \leq \lambda\}$. It follows that there is some $\mu \leq \lambda$ such that $E(\mu) \cap D$ is stationary (since a union of $\leq \kappa$ non–stationary subsets of κ^+ cannot be stationary). We have that $\kappa^\mu = \kappa$ and SCH implies $2^\kappa = \kappa^+$. Thus we have $\diamondsuit_{\kappa^+}(E(\mu) \cap D)$ by the set–theoretic results mentioned above. It follows immediately from the definition of $\diamondsuit$ that $\diamondsuit_{\kappa^+} D$ holds too. So we may use [15, Lemma 10.1.1] (see also [22, Lemma 1.2]) to conclude that $M \notin \mathcal{A}$ which is a contradiction with $M \in \mathcal{D}_\mathcal{Q}$.

Now since we know that D is not a stationary subset in κ^+, we can choose a closed and unbounded subset U of κ^+ such that $E \cap U = \varnothing$, and we can take the subfiltration $\mathcal{M}' = (M'_\alpha \mid \alpha < \kappa^+)$ of $\mathcal{M}$ indexed by the elements of U. The step (ii) of our construction has guaranteed that, in $\mathcal{M}$, whenever there is $\alpha < \beta < \kappa^+$ with $M_\beta/M_\alpha \notin \mathcal{A}$ then it was the case that $M_{\alpha+1}/M_\alpha \notin \mathcal{A}$ too. It means that all consecutive factors in $\mathcal{M}'$ have to belong to $\mathcal{A}$.

Finally, $C = \{\alpha < \kappa^+ \mid M'_\alpha = L_\alpha\}$ is closed and unbounded in κ^+. Thus there are $\nu, \mu \in C$ such that $\nu < \mu$ and $\mathrm{cf}(\nu) = \omega$. We know that $M'_\mu/M'_\nu \in \mathcal{A}$ by Eklof Lemma. At the same time however, $L_\mu/L_\nu = M'_\mu/M'_\nu$ contains N as a direct summand by Proposition 2.5(*iii*). So we arrived at the desired contradiction with $N \notin \mathcal{A}$ which completes our proof. □

Remark 2.7. By an unexpected result of Shelah from [25, pg. 309], for cardinals $\kappa \geq \beth_\omega(\aleph_0)$, $2^\kappa = \kappa^+$ holds if and only if $\diamondsuit_{\kappa^+}$. It says in fact that the diamond principle loses its combinatorial strength above the first singular strong limit cardinal; the implication $\diamondsuit_{\kappa^+} \Rightarrow 2^\kappa = \kappa^+$ is trivial. For our case, it means that SCH is not a uselessly strong assumption. In fact, we cannot weaken it much further if we intend to use [15, Lemma 10.1.1] as the key step in the proof of Theorem 2.6.

The following implies Theorem 0.2(2):

Theorem 2.8. *(SCH) Let R be a countable ring. Then a module B is cotorsion if and only if $\mathrm{Ext}^1_R(\mathcal{D}, B) = 0$. So ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp) = \mathcal{FL}$ for any class of left R–modules $\mathcal{Q}$.*

Proof. The only–if part is trivial, so let us turn our attention to the if one. Recall that over a countable ring, the class of all flat modules is $\aleph_1$–deconstructible. In other words, in order to show that B is a cotorsion module, it suffices to prove that $\mathrm{Ext}^1_R(N, B) = 0$ for every countable flat module N. Since any flat module is a direct limit of free ones, which are always in $\mathcal{D}$, Theorem 2.6 applies. □

Next, we prove Theorem 0.2(3):

Theorem 2.9. *Let R be a ring such that ${}^\perp(\mathcal{D}^\perp) = \mathcal{FL}$ and $\mathcal{FL}$ consists of modules of projective dimension ≤ 1. Then $\mathcal{D}$ is a precovering class if and only if R is right perfect.*

Proof. If R is a right perfect ring, then $\mathcal{D} = \mathcal{FL}$ is covering by a classic result of Bass. For the rest of the proof, we assume that R is not right perfect. So there exists a flat module N which is not $\aleph_1$–projective. Let us denote by H the cotorsion hull of N. Then H is also flat, and $H \notin \mathcal{D}$ because $\mathcal{D}$ is closed under pure submodules. Assume there exists a $\mathcal{D}$–precover $f : D \to H$. Since $\mathcal{D}$ contains a projective generator, f is surjective.

By assumption, for each non–cotorsion submodule B of D there exists $U_B \in \mathcal{D}$ such that $\mathrm{Ext}^1_R(U_B, B) \neq 0$. Let $U = \bigoplus_B U_B$ and let

$$0 \longrightarrow K \xrightarrow{\ \theta\ } F \longrightarrow U \longrightarrow 0$$

be a projective presentation of U. By assumption, K is projective. We are going to define a module $G \in \mathcal{D}$ and an epimorphism $g : G \to H$ which cannot be factorized through f; this will contradict the assumption that f is a $\mathcal{D}$–precover of H.

G will be constructed as the last term of a continuous chain $(G_\alpha \mid \alpha \leq \tau)$ of modules from $\mathcal{D}$. The ordinal τ will be chosen $\leq \mu^+$, where $\mu = |D|$. Together with the chain $(G_\alpha \mid \alpha \leq \tau)$, we will also construct a chain $(g_\alpha \mid \alpha \leq \tau)$ of epimorphisms $g_\alpha : G_\alpha \to H$.

We start with $G_0 = R^{(H)}$, and let $g_0 : G_0 \to H$ be any epimorphism. For the induction step, assume that G_β and $g_\beta : G_\beta \to H$ are defined for all $\beta < \alpha \leq \mu^+$. Then G_α and g_α are defined as the unions of the respective objects constructed in previous steps provided that α is limit. The case of α non–limit is divided into two subcases:

(i) $\alpha = \gamma + 1$ is an odd ordinal. This step is almost identical to the one in the proof of [12, Theorem 5.8]. Denote by I_γ the set of all homomorphisms h from G_γ to D such that $\mathrm{Im}(fh) = H$ and $\mathrm{Im}(h)$ is not cotorsion. For each $h \in I_\gamma$ there exists a $\phi_h \in \mathrm{Hom}_R(K, \mathrm{Im}(h))$ that cannot be factorized through θ. Moreover, since K is projective, there is $\psi_h \in \mathrm{Hom}_R(K, G_\gamma)$ such that $\phi_h = h\psi_h$. Denote by Θ the inclusion of $K^{(I_\gamma)}$ into $F^{(I_\gamma)}$, and define $\Psi \in \mathrm{Hom}_R(K^{(I_\gamma)}, G_\gamma)$ so that the h-th component of Ψ is ψ_h, for each $h \in I_\gamma$. The module G_α is defined by the pushout of Θ and Ψ:

$$\begin{array}{ccc} K^{(I_\gamma)} & \xrightarrow{\ \Theta\ } & F^{(I_\gamma)} \\ {\scriptstyle\Psi}\big\downarrow & & {\scriptstyle\Omega}\big\downarrow \\ G_\gamma & \xrightarrow{\ \subseteq\ } & G_\alpha. \end{array}$$

Since $U^{(I_\gamma)}$ is flat and H is cotorsion, g_γ extends to some $g_\alpha : G_\alpha \to H$.

(ii) $\alpha = \gamma+1$ is an even ordinal. In this step, we will deal with cotorsion submodules of D. Denote by J_γ the set of all cotorsion submodules C of D which are images of some $h \in \mathrm{Hom}_R(G_\gamma, D)$ and such that $\mathrm{Im}(f \restriction C) = H$, that is, $C + \mathrm{Ker}(f) = D$. Note that for each $C \in J_\gamma$, $C \cap \mathrm{Ker}(f)$ is not a cotorsion module. Otherwise $C = (C \cap \mathrm{Ker}(f)) \oplus C'$ where $C' \cong D/\mathrm{Ker}(f) \cong H$, hence $D = \mathrm{Ker}(f) \oplus C'$, and $H \in \mathcal{D}$, a contradiction. Since $\mathrm{Ext}^1_R(U, C) = 0$, it follows that for each $C \in J_\gamma$, there exists a homomorphism $\psi_C : U \to H$ which cannot be factorized through $f \restriction C$. Define $G_\alpha = G_\gamma \oplus U^{(J_\gamma)}$ and extend g_γ to g_α using the maps ψ_C.

In both cases, we have that $G_{\gamma+1}/G_\gamma \in \mathcal{D}$, so every G_α constructed is in $\mathcal{D}$. Note that for $\gamma < \mu^+$, either $G_\gamma \neq G_{\gamma+1}$ or $G_{\gamma+1} \neq G_{\gamma+2}$ (otherwise g_γ cannot be factorized through f, so f is not a $\mathcal{D}$–precover of H).

So $G = G_{\mu^+}$ is defined, and $g = g_{\mu^+}$ can be factorized through f, that is, there is $h : G \to D$ such that $fh = g$. Put $C = \mathrm{Im}(h)$. Then there is $\gamma < \mu^+$ such that $\mathrm{Im}(h \restriction G_\alpha) = \mathrm{Im}(h \restriction G_\gamma)$ whenever $\gamma < \alpha \leq \mu^+$ (because $\mu = |D|$). We can assume that γ is odd provided that C is cotorsion, and even otherwise.

If γ is even, we use the same argument as in the proof of [12, Theorem 5.8]. Put $h^0 = h \restriction G_\gamma$ and $h^1 = h \restriction G_{\gamma+1}$. Then $h^1\Omega$ extends $h^0\Psi$ to a homomorphism $F^{(I_\gamma)} \to C$. Denote by ι_{h^0} and ι'_{h^0} the h^0-th canonical embedding of K into $K^{(I_\gamma)}$ and of F into $F^{(I_\gamma)}$, respectively. Then $h^1\Omega\iota'_{h^0}$ extends $h^0\Psi\iota_{h^0} = h^0\psi_{h^0} = \phi_{h^0}$ to a homomorphism $F \to C$, in contradiction with the definition of ϕ_{h^0}.

If γ is odd, $C \in J_\gamma$ and the factorization $h \restriction G_{\gamma+1}$ of $g_{\gamma+1}$ through f yields a factorization of ψ_C through $f \restriction C$, a contradiction.

So $g \in \mathrm{Hom}_R(G, H)$ cannot be factorized through f, a contradiction. □

We finish by a discussion of the two assumptions of Theorem 2.9.

The rings R for which $\mathcal{FL}$ consists of modules of projective dimension ≤ 1 include all right hereditary rings (trivially), all countable rings (because a countably presented flat module has projective dimension ≤ 1 by a classic result of Jensen and Osofsky), all commutative noetherian rings of Krull dimension ≤ 1 (by a classic result of Gruson and Jensen), and all almost perfect domains (by a recent result of Fuchs and Lee, [13, Corollary 6.4]).

In particular, assertion (4) of Theorem 0.2 follows from its parts (2) and (3).

As for the condition ${}^{\perp}(\mathcal{D}^{\perp}) = \mathcal{FL}$, we first recall the following recent results:

Lemma 2.10. *Let R be a ring.*

(i) Assume that the class $\mathcal{D}$ is closed under direct products. Then ${}^{\perp}(\mathcal{D}^{\perp}) = \mathcal{FL}$.

(ii) The class $\mathcal{D}$ is closed under direct products, iff R is left coherent and for each $n \geq 1$, intersections of arbitrary families of finitely generated left R–submodules of R^n are finitely generated. In particular, this holds when R is left noetherian.

(iii) Let R be a von Neumann regular ring. Then $\mathcal{D}$ is the class of all modules M such that each finitely generated submodule of M is projective. If R is left or right self–injective then $\mathcal{D}$ is closed under direct products.

Proof. (i) is proved in [4], (ii) in [18, Theorem 4.7], and (iii) follows from [18, Propositions 3.4 and 4.11]. □

There is an immediate corollary of Theorem 2.9 and Lemma 2.10(i):

Corollary 2.11. *Let R be a non–right perfect ring such that $\mathcal{D}$ is closed under direct products and $\mathcal{FL}$ consists of modules of projective dimension ≤ 1. Then $\mathcal{D}$ is not a precovering class. In particular, this holds when R is 1–Gorenstein.*

Notice that unlike Theorem 0.2(4), Corollary 2.11 does not cover all countable non–right perfect rings:

Example 2.12. Let R be any countable von Neumann regular ring such that R is not right perfect (For example, let $R = \varinjlim_{n<\omega} M_{2^n}(K)$ where K is a countable field, $M_m(K)$ denotes the full $m \times m$ matrix ring over K, and the direct system maps are induced by the 2–block diagonal ring embeddings of $M_{2^n}(K)$ into $M_{2^{n+1}}(K)$ for each $n < \omega$.) We will show that $\mathcal{D}$ is not closed under direct products.

Since R is countable and von Neumann regular, each finitely generated left ideal is principal, generated by an idempotent, and each infinitely generated left ideal is generated by a countable set of orthogonal idempotents, see [16, §2]. Let $\{e_i \mid i < \omega\}$ be an orthogonal set of non–zero idempotents generating an infinitely generated maximal left ideal $I = \bigoplus_{i<\omega} Re_i$ of R (such left ideal exists because R is not right perfect).

Consider the decreasing chain of left ideals

$$R(1-e_0) \supsetneq R(1-e_0-e_2) \supsetneq \cdots \supsetneq R(1-\sum_{j<i} e_{2j}) \supsetneq R(1-\sum_{j\leq i} e_{2j}) \supsetneq \cdots .$$

Assume $\mathcal{D}$ is closed under direct products. Then, by Lemmma 2.10(ii), the intersection of this chain is finitely generated, so $\bigcap_{i<\omega} R(1-\sum_{j<i} e_{2j}) = Rf$ for an idempotent $f \in R$. Note that for each $r \in R$, we have $r.f = r$ iff $r.e_{2i} = 0$ for all $i < \omega$. It follows that $\bigoplus_{i<\omega} Re_{2i} \cap Rf = 0$, and $\bigoplus_{i<\omega} Re_{2i+1} \subseteq Rf$. So $I \subsetneq \bigoplus_{i<\omega} Re_{2i} \oplus Rf$, whence $\bigoplus_{i<\omega} Re_{2i} \oplus Rf = R$, in contradiction with R being finitely generated.

Finally, we prove that in the particular case of von Neumann regular rings, the condition '$\mathcal{FL}$ consists of modules of projective dimension ≤ 1' in Corollary 2.11 can actually be dropped.

Proposition 2.13. *Let R be a von Neumann regular ring which is not right perfect. Assume $\mathcal{D}$ is closed under direct products (for example, assume R is left or right self–injective). Then $\mathcal{D}$ is not a precovering class.*

Proof. Since R is von Neumann regular, each module is flat, and each cotorsion module is injective. By Lemma 2.10(iii), $\mathcal{D}$ coincides with the class of all modules M such that each finitely generated submodule of M is projective. Since R is non–singular, $\mathcal{D} \subseteq \mathcal{N}$ where $\mathcal{N}$ denotes the class of all non–singular modules. Notice that $\mathcal{N}$ is closed under injective hulls.

By assumption, R is not right perfect, so there is a simple non–projective module S. Clearly S is singular, so $E(S) \notin \mathcal{N}$ where $E(S)$ denotes the injective hull of S.

Assume there is a $\mathcal{D}$–precover $f : D \to E(S)$ of $E(S)$. Since $\mathcal{D}$ contains all projective modules, f is surjective. Moreover f extends to $g : E(D) \to E(S)$, and $E(D) \in \mathcal{N}$. Take an arbitrary $D' \in \mathcal{D}$ and consider the exact sequence

$$0 \longrightarrow K \longrightarrow E(D) \xrightarrow{g} E(S) \longrightarrow 0.$$

Then $\mathrm{Ext}^1_R(D', E(D)) = 0$, $\mathrm{Hom}_R(D', g)$ is surjective because f is $\mathcal{D}$–precover of $E(S)$, and $\mathrm{Ext}^1_R(D', K) = 0$. So $K \in \mathcal{D}^\perp$ is cotorsion by Lemma 2.10(i). Then the exact sequence above splits and $E(S) \in \mathcal{N}$, a contradiction. □

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