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DOCTORAL THESIS

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Helical symmetry and the non-existence of asymptotically flat periodic solutions in general relativity

Institute of Theoretical Physics

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Název práce: Helikální symetrie a neexistence asymptoticky plochých periodických řešení v obecné teorii relativity

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Abstrakt. V současnosti není známé žádné přesné helikálně symetrické řešení v obecné teorii relativity. Jsou však důvody očekávat, že tato řešení, existují-li, nemohou být asymptoticky plochá. V předkládané dizertační práci vyšetřujeme obecnější otázku, zda existují periodická asymptoticky plochá řešení Einsteinových rovnic. Navazujeme na práci Gibbonse a Stewarta [12], kteří ukázali, že neexistují vakuová periodická asymptoticky plochá řešení analytická v okolí světlupodobného nekonečna $\mathcal{I}$. Diskutujeme nutné korekce Gibbonsova a Stewartova důkazu a zobecňujeme jejich výsledek pro soustavu Einsteinových-Maxwellových rovnic, rovnic Einsteinových-Klein-Gordonových a Einsteinových-konformně-skalárních. Ukazujeme tedy, že neexistují asymptoticky ploché periodické prostoročasy analytické v okolí $\mathcal{I}$, kde zdrojem gravitace je elektromagnetické, Kleinovo-Gordonovo nebo konformní skalární pole. Pro potřeby důkazu odvozujeme přislušné konformní polní rovnice, vztah pro Bondiho hmotnost skalárních polí, diskutujeme problém dědičnosti symetrie. Součásti dizertace je teoretický úvod do problematiky helikální symetrie a matematických metod použitých v důkazu (spinory, konformní techniky), články [2] a [3] publikované v časopise Classical and Quantum Gravity, a dokumentace k programu pro práci s Newmanovým-Penroseovým formalizmem. Program byl napsán pro potřeby dizertační práce v softwaru Mathematica.

Klíčová slova: helikální symetrie, periodická řešení, spinory, asymptotické vlastnosti, konformní rovnice pole

Title: Helical symmetry and the non-existence of asymptotically flat periodic solutions in general relativity Author: Martin Scholtz

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Abstract. No exact helically symmetric solution in general relativity is known today. There are reasons, however, to expect that such solutions, if they exist, cannot be asymptotically flat. In the thesis presented we investigate a more general question whether there exist periodic asymptotically flat solutions of Einstein's equations. We follow the work of Gibbons and Stewart [12] who have shown that there are no periodic vacuum asymptotically flat solutions analytic near null infinity $\mathcal{I}$. We discuss necessary corrections of Gibbons and Stewart proof and generalize their results for the system of Einstein-Maxwell, Einstein-Klein-Gordon and Einstein-conformal-scalar field equations. Thus, we show that there are no asymptotically flat periodic space-times analytic near $\mathcal{I}$ if as the source of gravity we take electromagnetic, Klein-Gordon or conformally invariant scalar field. The auxilliary results consist of corresponding conformal field equations, the Bondi mass and the Bondi massloss formula for scalar fields. We also discuss the problem of inheritance of the symmetry. The thesis includes theoretical introduction to the notion of helical symmetry, mathematical methods used in the proof (spinors, conformal techniques), papers [2] and [3] published in Classical and Quantum Gravity journal, and documentation to the program for working with the Newman-Penrose formalism. The program was developed under software Mathematica for the purposes of the thesis.

Keywords: helical symmetry, periodic solutions, spinors, asymptotic properties, conformal field equations

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Part I

Introduction

Chapter 1

Introduction

1.1 Why to study General Relativity?

General theory of relativity (GR) is the only satisfactory theory of gravitation known today. Despite the enormous effort, no significant progress in unifying quantum theory and general relativity has been made. Theories proposed as alternatives to GR were either disproved by observations, or they were not comparable with observations at all. Even today, string theory or loop quantum gravity are far from giving some clear experimental predictions. One of few results of quantum gravity which we can perhaps trust is Hawking's prediciton of black holes' radiation. This prediciton, however, is formulated in the framework of quantum field theory on a fixed curved background, which cannot be regarded as a full quantum theory of gravity.

For many years people thought that hypothetic theory of quantum gravity will differ from general relativity only under extreme conditions: shortly after Big Bang, near singularities of black holes, etc. It seems reasonable that whenever quantum effects are negligible, general relativity should be an excellent approximation. For this reason, research in the area of classical general relativity is not as popular as in quantum gravity, especially string theory. Various important properties of GR have been understood during Golden age of GR (1960–1975). A lot of new mathematical methods have been introduced and they led to new insights, like algebraic classification of Riemann tensor and its relation to gravitational radiation, topological methods and their role in the analysis of singularities and global structure of space-times, etc. Later also cosmology became the part of "serious physics". As the experimental techniques and technologies improved, predictions of GR were being justified with higher and higher accuracy. And although gravitational radiation has never been detected directly, discovery of binary pulsars provided a new "laboratory" for testing GR. In fact, decay of orbits of such pulsars is in excellent agreement with simulations based on GR. For a brief discussion of these issues see Chapter 3.

Thus, GR seems to be perfectly suitable for describing phenomena not influenced by quantum effects. Studying classical GR means, roughly speaking, solving Einstein's equations in more and more realistic situations. Probably, since Einstein's equations are complicated and non-linear, we will never exhaust their richness and there will always be what to study in GR.

A kind of continuing renaissance of interest in GR is due to several facts. First, there are some deep fundamental theoretical issues like "cosmic censorship hypothesis" which remain open. Second, experimental devices have undergone significant improvements during last years and raised a new hope to detect gravitational radiation by terrestrial detectors. Increasing computational abilities connected with more sophisticated numerical algorithms provide new theoretical data which can be compared with new observations. Prediction of gravitational radiation is the most challenging prediction of GR and its detection is the biggest challenge for contemporary experimentalists. This thesis is a contribution to the theoretical background related to gravitational radiation.

But there are antoher reasons why GR is worth to study. Since 1998 observations suggest that the Universe is not only expanding but it is expanding with an increasing rate, the expansion is accelerating. This is in contradiction with Einstein's equations unless we admit the cosmological constant to be nonzero. Thus, in words of Paul Davies, Einstein's "biggest mistake" turns out to be his biggest triumph. Models and solutions of Einstein's equations with cosmological constant (and there are many of them) have been studied without direct physical motivation, but after discovery of acceleration of Universe they became as relevant as the other ones. The origin of cosmological constant remains unclear and is usually referred to as the problem "dark energy".

Moreover, there are suggestions that general relativity does not give correct predicitons in some situations when we expect the Newtonian gravity to hold. Theories like MOND (MOdified Newtonian Dynamics) are trying to explain effects of dark matter as a result of modified Newton's laws. Beside this, several theories have been formulated in which the Lorentz invariance is violated. For example, in so-called deformed special relativity (or doubly special relativity), the invariance of the Planck energy is postulated. However, there is no direct experimental evidence for violation of the Lorentz invariance today.

All the theories mentioned are highly speculative and only time will show whether any of them has a physical relevance. According to the modest opinion of the author of this thesis, GR will appear to be the valid description of all situations where quantum effects do not play an important rôle. Today, there are many theorists propagating the idea that in order to find quantum theory of gravity, also the rules of quantum mechanics have to be modified. Perhaps the most influential contemporary mathematical physicist who considers quantum theory incomplete is Roger Penrose. He is also one of the main authors of mathematical methods used in this thesis.

In this context, it is necessary to test our theories again and again, to make new predictions and compare them with observations. If we will be able to obtain new predictions in GR and these will be confirmed (as the author believes), our confidence in this theory will increase. If, however, GR is not appropriate for the description of gravity, we would like to find its limits as soon as possible.

1.2 The goals of the thesis

In this thesis we study some general properties of GR – the existence of asymptotically flat, periodic solutions of Einstein's equations. The question whether periodic solutions in GR exist is interesting by itself, but it is also related to the problem of gravitational radiation and its detection, mentioned in the previous section. The most promising sources of gravitational radiation are binary systems consisting of massive black holes or neutron stars. Can such a system be in equilibrium or does it have to loose the energy by radiation? Can a solution describing a binary system be periodic? Or, more generaly, can an isolated system evolve in a periodic way? Imagine a sufficiently compact system of gravitational waves (a geon) – will it exist being periodic all the time? Will periodic solutions exist if gravitational field interacts with some other physical fields like electromagnetic or scalar field with zero rest mass?

There are several physical arguments to be given later that the answer is no: isolated systems cannot move periodically. Can these arguments be justified by rigorous mathematical proofs? There are efficient mathematical methods to analyze the properties of isolated systems. These methods, invented by Roger Penrose and others, include conformal techniques, spinors and the Newman-Penrose formalism. In 1984 Gibbons and Stewart [12] used these techniques to show that asymptotically flat solutions of Einstein's equations cannot be periodic. More precisely, they have shown that any periodic asymptotically flat *vacuum* solution which is analytic near infinity is necessarily stationary. This proof is very general because it does not assume anything about the configuration or motion of the sources. But since the solution is assumed to be vacuum near infinity, these sources are not allowed to produce some other type of radiation. Thus, it was desirable to generalize the proof to the presence of non-gravitational fields, especially to the electromagnetic and the scalar fields. This was the original aim of the thesis.

It appeared, however, that the proof of Gibbons and Stewart suffers from several drawbacks. What they have actually shown is that if the geometrical quantities describing the gravitational field are periodic at null infinity $\mathcal{I}$, then they are in fact time-independent. In their paper [12], periodicity in time is defined as the periodicity in coordinate v defined in such a way that

$$K = \partial_v$$

is a null vector. Gibbons and Stewart were able to prove that quantities periodic in v on $\mathcal{I}$ are $v-$independent which means that K is the Killing vector of the metric. From this they conclude that the space-time is stationary.

The problem is that vector K is *everywhere null* by construction while the stationarity requires a time-like Killing vector. It turns out that the coordinates and the tetrad used by Gibbons and Stewart are too restrictive and impossible to be constructed even in the Minkowski space-time! Indeed, in double null coordinates, the conformally rescaled Minkowski metric reads

$$\mathrm{d}s^2 = \mathrm{d}u\,\mathrm{d}v - \frac{1}{4}(u-v)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right).$$

Since the metric coefficients depend on v, $K = \partial_v$ is not a Killing vector, i.e. the Minkowski space-time is not stationary according to the definition of Gibbons and Stewart! In addition, the conformal gauge they use is incompatible with periodicity. All these issues are discussed in detail in paper [2] attached to the thesis.

Thus, in order to justify the results of Gibbons and Stewart, it is necessary to repeat their proof in a coordinate system which allows us to deduce the stationarity in usual sense, that is, to show that if the quantities are periodic in time near infinity, then there exists a time-like Killing vector. In addition, it is of interest to generalize the results for cases when other physical fields are present. It could well happen that a physical field is time-dependent and periodic, but its energy-momentum tensor, which is the source for gravity, is not time-dependent so that gravitational field is stationary. Or, does a physical field have to be stationary as well?

The goals of the thesis can be summarized as follows:

- Repeat the procedure of Gibbons and Stewart to prove the non-existence of asymptotically flat vacuum solutions of Einstein's equations which are periodic near infinity, but in less restrictive coordinates and conformal gauge and show the existence of the time-like Killing vector; proving that vacuum periodic asymptotically flat solutions must be stationary;
- Generalize the proof for the presence of the electromagnetic and scalar fields;
- Discuss the inheritance of the stationarity by electromagnetic and scalar fields.

1.3 Organization of the thesis

The purpose of the thesis is twofold. Beside the presentation of the results of the research we intend to provide also a brief pedagogical introduction to the topic presented and the techniques used. In Chapter 2 we explain the notion of helical symmetry and discuss selected helically symmetric solutions in the Newtonian gravity and electrodynamics. Chapter 3 deals primarily with the definition of helical symmetry in the framework of General Relativity. In this chapter we also motivate the study of periodic solutions in asymptotically flat space-times.

In Chapter 4 we introduce the concept of asymptotic flatness and discuss asymptotic properties of the Minkowski space-time. Then we give geometrical definition of asymptotic flatness as proposed by Penrose and briefly mention some related issues. Chapter 5 is dedicated to spinorial techniques, in Chapter 6 we explain how the usual field equations can be rewritten in terms of spinors and how the spinorial equations can be translated into the language of the Newman-Penrose formalism. The author believes that the material presented in these chapters can serve as a useful introductory text for the students interested in the spinor formalism. Or it can be used as a reminder for the readers who are familiar with this formalism but who do not use it regularly in their work. In addition, we occasionally illustrate how several spinorial operations can be done in the Mathematica program which was written for the purposes of the thesis. Finally, sometimes we present the details of some derivations which are hard to find in the literature.

In chapters 7 and 8 we attach papers [2] and [3] in the form they were published in Classical and Quantum Gravity. This is the main original part of the thesis and all results obtained are discussed there in detail.

In the thesis we use the Newman-Penrose formalism heavily. The Newman-Penrose equations are scalar equations arising as the projections of spinorial field equations onto the spin basis. Especially in the case of conformal Einstein-Klein-Gordon equations, resulting Newman-Penrose equations are extremely long and it is impossible to derive them "by hand". For this reason we have developed programs for working with the Newman-Penrose formalism using Mathematica software. These programs are attached on the CD accompanying the thesis and their functionality is fully described in the Appendix A.

1.4 Results

In papers [2] and [3] included in chapters 7 and 8, the solutions to the problems indicated are given. The proof of Gibbons and Stewart has been corrected and generalized to the presence of non-gravitational fields. In Chapter 7 (paper [2]) we discuss the differences between our approach and the approach of Gibbons and Stewart and show the non-existence of asymptotically flat analytic electrovacuum solutions of Einstein's equations periodic in time. In order to perform the proof we derive the conformal Einstein-Maxwell equations which are essentially the same as those of Friedrich [9]. We also prove that once the asymptotically flat electrovacuum space-time is stationary, then the electromagnetic field is necessarily stationary too. This is related to the question of inheritance: in asymptotically flat stationary space-time, the electromagnetic field necessarily inherits the stationarity.

In Chapter 8 (paper [3]) we use the same coordinate system as in [2] to analyze space-times with the scalar field sources. We consider two kinds of the scalar field, the massless Klein-Gordon field and conformally invariant scalar field. Again, we derive the conformal Einstein-scalar-field equations for both kinds of the scalar field. These equations are believed to be new, although some of them appear in [16].

Theorems about the non-existence of asymptotically flat periodic solutions form the nucleus of this thesis. On the other hand, they also represent a partial result in understanding the helical symmetry in General Relativity. Helical symmetry and the theorems presented are the content of introductory part of the thesis. For this reason we do not enter the discussion about them here and we concentrate on the other, auxiliary results obtained in the thesis.

As we explain in Chapter 8, the assumption of periodicity implies the constancy of the Bondi mass. Expression for the Bondi mass of an electrovacuum space-time has been known since long time ago and we have used it in paper [2]. For the scalar fields, both Klein-Gordon and conformal-scalar-field, no clear expression of the Bondi mass and its loss can be found. Thus, as a part of the proof we present also the derivation of the Bondi mass for these fields. The derivation is based on the twistor equation. There are two surprising features of these relations. First, in electromagnetic case only gravitational quantities enter the expression of the Bondi mass, namely the Ψ_2-component of the Weyl spinor and the news function. We have shown, however, that in the case of the Klein-Gordon field the relevant expression is

$$M_B \;\propto\; \int \mathrm{d}S\,\left[\Psi_2^{(0)} \;+\; \frac{1}{3}\,\partial_u\left(\phi^{(0)}\,\bar{\phi}^{(0)}\right) \;+\; \sigma^{(0)}\,\dot{\bar{\sigma}}^{(0)}\right],$$

where $\phi^{(0)}$ is the leading term in the expansion of the scalar field itself and variable u is to be interpreted as the time coordinate. Thus, the Bondi mass depends explicitly on the radiative part of the scalar field. Massloss formula for the Klein-Gordon field reads

$$\dot{M}_B(u) \;\propto\; -\int \mathrm{d}S\; \left[\dot{\sigma}^{(0)}\,\dot{\bar{\sigma}}^{(0)} \;+\; 2\,\dot{\phi}^{(0)}\,\dot{\bar{\phi}}^{(0)}\right].$$

Although this expression is derived using asymptotic expansions of all quantities near $\mathcal{I}$ and the twistor equation, we present also simple physical argument why to expect this form of the massloss formula. Notice that the decrease of the Bondi mass is manifestly positive.

In the case of conformal-scalar-field, the Bondi mass is given by usual formula, i.e. formula not involving $\phi^{(0)}$-terms. There is another surprise, however, for the massloss formula

$$\dot{M}_B(u) \;\propto\; -\int \mathrm{d}S\; \left[\dot{\sigma}^{(0)}\,\dot{\bar{\sigma}}^{(0)} \;+\; 2\,\left(\dot{\phi}^{(0)}\right)^2 \;-\; \phi^{(0)}\,\ddot{\phi}^{(0)}\right]$$

is now "indefinite", so the Bondi mass can increase as well as decrease. This is a consequence of the fact that energy-momentum tensor of the field under consideration does not satisfy the weak energy condition $T_{ab}\,n^a\,l^b \geq 0$ for arbitrary future null vectors l^a and n^b. This is a potential problem even for our proof which relies on the positivity of the Bondi mass. Fortunately, for the periodic space-times one can integrate the massloss formula by parts and obtain positive overal decrease during one period, which is sufficient for our proof.

As in paper [2], we again discuss the issue of the inheritance of symmetry. For the Klein-Gordon field we have shown that in stationary asymptotically flat space-time the scalar fields inherits the stationarity. We were not able to arrive at any conclusion in the case of the conformal-scalar-field. The main difficulty appears to be indefinitness of its Bondi mass. We present just two simple arguments indicating that one can expect global positivity of the energy of conformal-scalar-field and the inheritance of the stationarity of gravitational field in asymptotically flat space-times.

Results presented in this thesis and explained in detail in papers attached can be summarized as follows.

- **Theorems**
  - In a weakly asymptotically simple space-time, all time-periodic solutions of the Einstein-Maxwell equations analytic near null infinity are stationary, i.e. they possess a Killing vector which is time-like near infinity.
  - The same theorems hold for the system of Einstein-Klein-Gordon equations and Einstein-conformal-scalar-field equations.
- **Corollaries**
  - In the case of electromagnetic and Klein-Gordon field, in any asymptotically flat stationary space-time also the non-gravitational field (i.e. EM or KG) is stationary.
- **Auxiliary results**

– Conformal Einstein-Maxwell, Einstein-Klein-Gordon and Einstein-conformal-scalar field equations.
– The Bondi mass and Bondi massloss formulae for the Klein-Gordon and conformal-scalar fields.

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I am also grateful to my father who always wanted me to study what I like and what I am interested in. Finally, I would like to thank to my girlfriend Alena Malá for her unlimited moral and personal support, more precious than anything else.

Chapter 2

Helical symmetry

The notion of symmetry belongs to the most important concepts in physics. An intuitive feeling that symmetry is the property of ordered, regular systems has found its mathematical formulation: the symmetry is a transformation which leaves the system invariant. Originally, such a transformation was meant to be a translation (crystal lattice), rotation (sphere), or another kind of space transformation. By Noether's theorem, each symmetry implies a corresponding conservation law, e.g. rotational symmetry implies the conservation of angular momentum. Later it was discovered that physical laws described by mathematical formulae can have deeper, hidden symmetries, which may not be connected with spatial transformations. Although such transformations have less intuitive meaning, they also imply some conservation laws. A nice example is the gauge symmetry in quantum field theory, where the invariance of the action of, say, scalar field under a global gauge transformation implies the conservation of the charge. In addition, imposing invariance of the action under the local gauge transformation, one can construct the action of coupled scalar and electromagnetic fields. Symmetries (and their breaking) play an essential role in our understanding of physical laws.

Beside this fundamental status, symmetries have also "practical“ importance. In classical machanics we can use known symmetries of the system to simplify the equations of motion, or even to solve them. Another nice example comes from general relativity through Birkhoff's theorem: imposing spherical symmetry leads to unique solution of vacuum Einstein's equations which is necessarily asymptotically flat.

Results presented in this thesis are motivated by the study of *helical symmetry* in electrodynamics and general relativity. In the context of spatial geometry, the helical symmetry is invariance under continuous rotation combined with translation in the direction of axis of rotation, see figure 2.1. Helical symmetry can be found in Nature, especially in biology, on both microscopic and macroscopic scale, figures 2.2 and 2.3. But helically symmetric structures occur also in technology: springs, spiral staircases, or drill bits (figure 2.4).

Helical symmetry in the context of field theories (in flat space-time) means the invariance under a *spatial rotation* combined with a *time translation*. To be more precise, let $x^\mu = (t, r, \phi, z)$ be standard cylindrical coordinates in Minkowski space-time, so that vector field ∂_t is generator of time translations and ∂_ϕ is generator of rotation about the $z-$axis. Orbits (integral curves) of

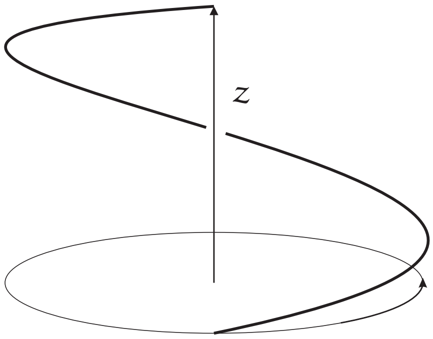

Figure 2.1: Helical symmetry is the invariance under rotation about the axis z connected with translation along the same axis.

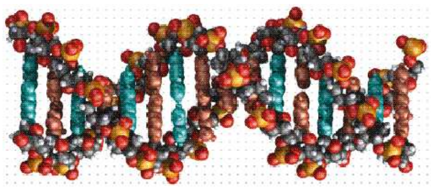

Figure 2.2: A segment of DNA showing two spiraling chains forming a helically symmetric structure.

Figure 2.3: Snail shell exhibits a kind of helical symmetry.

Figure 2.4: Helical symmetry in everyday life: a drill bit.

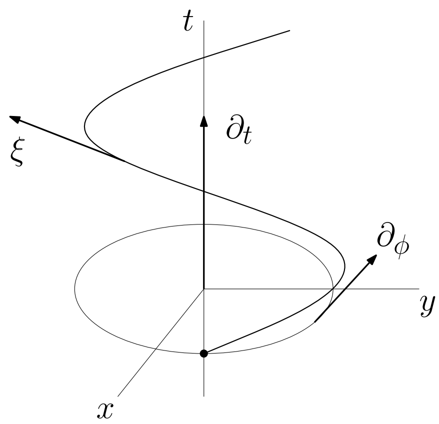

Figure 2.5: Helical vector field $\xi = \partial_t + \omega\partial_\phi$ describes periodic motion of the particle in the plane $z =$constant with constant angular velocity ω.

∂_t are straight time-like lines, while orbits of ∂_ϕ are closed space-like curves. Then, the vector field

$$\xi \; = \; \partial_t \; + \; \omega \, \partial_\phi,$$

where ω is constant, is called the *helical vector field*; it generates transformations consisting of translations in $t-$direction and rotations about axis z, figure 2.5. The orbit of ξ is helix and represents the world line of a particle orbiting along a circle in the plane $z =$constant with constant angular velocity ω.A more detailed definition and analysis will be given in chapter

A typical example of a helically symmetric system is a binary system, i.e. the system of two bodies or particles orbiting about their common centre of mass with the same angular velocity, figure 2.6. Such system can consist of two stars (neutron stars, black holes), but it can be also the system of two charged particles acting on each other via electromagnetic interaction. Another helically symmetric configuration consists of n particles of the same mass, uniformly

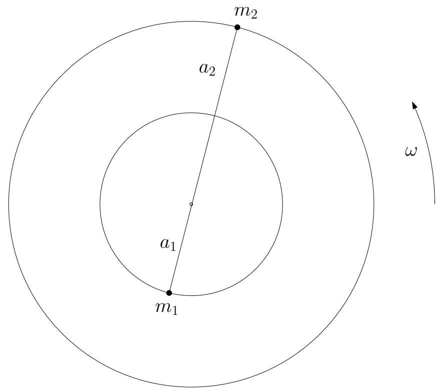

Figure 2.6: Equilibrium configuration of two particles of masses m_1 and m_2 orbiting about the centre of mass with angular velocity ω. This system possesses helical symmetry.

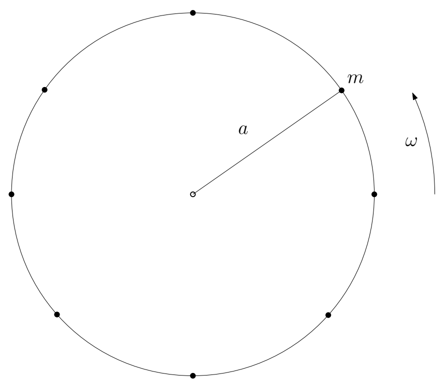

Figure 2.7: Equilibrium configuration of n particles of mass m.

distributed along the circle of radius a and orbiting about the centre, figure 2.7. In the rest of this chapter we discuss these helically symmetric configurations in several theories: Newtonian gravity, electrodynamics, scalar gravity and general relativity.

2.1 Newtonian gravity

Helically symmetric solution in the Newtonian gravity is in fact very familiar and the existence of equilibrium configuration of two point masses orbiting about the centre of mass is well-known. In 1922 it was shown by Lichtenstein

[14] that helically symmetric solution exists also for two fluid bodies of finite sizes.

To show that a helically symmetric solution for two point masses indeed exists is rather trivial. Despite this, however, we list relevant equations here, as they will be helpful later, when we will try to understand properties of a helically symmetric solution in linearized Einstein's theory. The idea of our approach is the following. Consider a point particle with mass m_1, the *source* particle. Assume, in addition, that this particle is moving along the circle of radius a_1 with constant angular velocity ω. At this stage we do not ask *why* the particle orbits, we simply assume that it *does*. Next we compute gravitational field produced by this particle and imagine that we insert second, *test* particle into this field. We derive the equations of motion of the test particle and find conditions of equilibrium. The adjectives "source" and "test" serve merely to distinguish the particles. In fact, to obtain an equlibrium configuration it is necessary to assume that both particles are the sources of gravitational field. At first stage, however, we consider just the motion of the test particle in the field of the source particle. Once we find conditions for the test particle to move along the circle, we can switch the roles of both particles. By the symmetry of the problem, conditions of equlibrium are supposed to be satisfied again.

Thus, we assume that the position vector $\boldsymbol{r}$ of the source particle is, in Cartesian coordinates $x_i = (x, y, z)$,

$$\boldsymbol{r}(t) \;=\; (a_1\,\cos(\omega t + \phi_0),\; a_1\,\sin(\omega t + \phi_0),\; 0)\,. \tag{2.1}$$

In other words, the particle is orbiting about the origin of the coordinate system in the plane $z = 0$ with angular velocity ω. Initial position of the particle at time $t = 0$ is determined by the initial phase ϕ_0. According to Newton's law of gravitation, gravitational field $\boldsymbol{K}$ produced by source particle is given by

$$\boldsymbol{K}(t, \boldsymbol{x}) \;=\; -\,G\,m_1\,\frac{\boldsymbol{x} - \boldsymbol{r}(t)}{|\boldsymbol{x} - \boldsymbol{r}(t)|^3}\,.$$

Equations of motion can be derived from Newton's law of force

$$\ddot{\boldsymbol{x}} \;=\; \boldsymbol{K}(t, \boldsymbol{x})$$

or, equivalently, from the Lagrange equations of the second kind. By the symmetry of the problem it is useful to introduce cylindrical coordinates, so the Lagrange formalism is more appropriate. Cylindrical coordinates $q_a = (r, \phi, z)$ are introduced by usual relations

$$x \;=\; r\,\cos\phi, \quad y \;=\; r\,\sin\phi, \quad z \;=\; z.$$

Components of $\boldsymbol{K}$ with respect to cylindrical coordinates are defined by

$$K_a \;=\; \frac{\partial x_i}{\partial q_a}\,K_i,$$

and they read

$$\begin{aligned}
K_r &= -\,G\,m_1\,\frac{r - a_1\,\cos\theta}{R^3}, \\
K_\phi &= G\,m_1\,\frac{r\,\sin\theta}{R^3}, \\
K_z &= -\,G\,m_1\,\frac{z}{R^3},
\end{aligned} \tag{2.2}$$

where we introduced notation

$$\begin{aligned} R &= |\boldsymbol{x} - \boldsymbol{r}(t)| = \sqrt{a_1^2 + z^2 + r^2 - 2a_1 r \cos\theta}, \\ \theta &= \omega\, t - \phi + \phi_0. \end{aligned} \tag{2.3}$$

Equations of motion of the test particle in the field produced by the source particle are

$$\begin{aligned} \ddot{r} &= -\frac{G\, m_1}{R^3}\,(r - a_1 \cos\theta) + r\,\dot{\phi}^2, \\ \ddot{\phi} &= \frac{G\, m_1}{R^3}\,\frac{\sin\theta}{r} - \frac{2}{r}\,\dot{r}\,\dot{\phi}, \\ \ddot{z} &= -\frac{G\, m_1}{R^3}\, z. \end{aligned} \tag{2.4}$$

Now we want to find conditions for an equilibrium. We restrict ourselves to the initial time $t = 0$ and set the initial phase of the source particle to be $\phi_0 = 0$. Position vector of the source particle is therefore (in Cartesian coordinates)

$$\boldsymbol{r}(0) = (a_1,\ 0,\ 0).$$

The test particle is assumed to be at position

$$\boldsymbol{x} = (-a_2,\ 0,\ 0)$$

with velocity

$$\dot{\boldsymbol{x}} = (0,\ -\omega\, a_2,\ 0).$$

Thus, the initial conditions for the test particle are

$$r = a_2,\ \phi = \pi,\ z = 0, \quad \dot{r} = 0,\ \dot{\phi} = \omega,\ \dot{z} = 0.$$

Parameters θ and R then simplify to

$$\theta = -\pi, \quad R = a_1 + a_2.$$

Situation is sketched in figure 2.8. Equations of motion then reduce to

$$\begin{aligned} \ddot{r} &= -\frac{G\, m_1}{(a_1 + a_2)^2} + a_2\,\omega^2, \\ \ddot{\phi} &= 0, \qquad \ddot{z} = 0. \end{aligned} \tag{2.5}$$

The first observation is that $\ddot{\phi} = 0$, as the consequence of the radial character of the force in Newton's theory. If the system is to be in equilibrium, the quantity $\ddot{r}$ must vanish. This can be achieved by setting

$$\omega = \sqrt{\frac{G\, m_1}{a_2\,(a_1 + a_2)^2}}. \tag{2.6}$$

For ω given by relation (2.6) the system of two point particles will be in equilibrium, because r remains constant ($\dot{r} = 0, \ddot{r} = 0$), and the angular velocity as well ($\dot{\phi} = \omega, \ddot{\phi} = 0$). Honestly, we derived the conditions of equlibrium only at time $t = 0$. However, since the choice of instant $t = 0$ is arbitrary, our equations hold for any initial phase ϕ_0 and corresponding position of the test

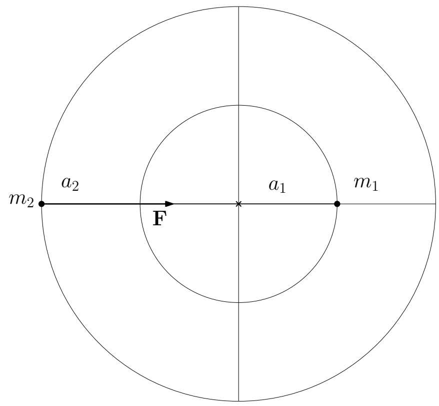

Figure 2.8: Equilibrium configuration in Newton's theory. The force is purely radial, so $\ddot{\phi} = 0$. By appropriate choice of parameter ω we can achieve also $\ddot{r} = 0$.

particle, namely $\phi = \phi_0 + \pi$. Consequently, if the conditions of equilibrium hold at time $t = 0$, they will hold also at all later times.

Thus, we arrived at expected conclusion: helically symmetric solutions describing two particles orbiting about their centre of mass exist. If the masses of particles m_1, m_2 and radii a_1, a_2 of circular trajectories are chosen arbitrarily, there exists a unique angular velocity ω given by (2.6), for which the system of two particles will be in equilibrium. Let us have a look at analogous problem in the framework of classical electrodynamics.

2.2 Electrodynamics

Consider now a system of two charged particles of masses m_1, m_2 and charges e_1, e_2 (with opposite sign), interacting in accordance with classical Maxwell's theory of electromagnetic field. Newton's law of gravitation was historically the first mathematical/physical law describing a fundamental force of Nature. It is natural that after discovery and succeses of Newton's laws people tried to understand the other known forces in a similar manner. Perhaps the first result obtained in this way is the Coulomb law of electrostatic force. Two particles at rest in an inertial frame act on each other by electrostatic force of magnitude

$$F \; = \; \frac{1}{4\pi\varepsilon_0} \, \frac{e_1 \, e_2}{r^2},$$

whith e_1, e_2 being the charges and r the distance between the particles. Analogy with Newton's law of gravitational force is obvious: both electrostatic and gravitational force is described by the same mathematical formula. (Un)fortunately, analogy between Newtonian gravity and electromagnetism is essentially exhausted by Coulomb's law. As we know, the effort spent in understanding electromagnetism led to new breakthroughs, especially to the notion of

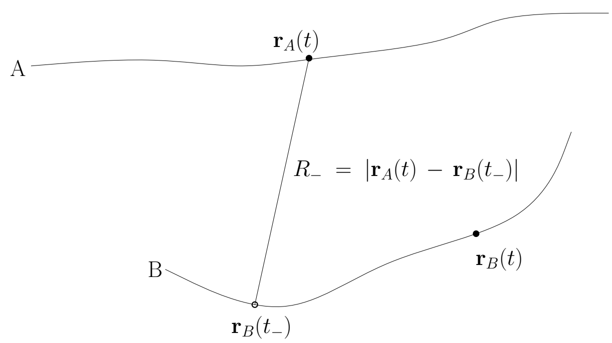

Figure 2.9: Particle A in the position $\boldsymbol{r}_A(t)$ feels the force from particle B when it was at $\boldsymbol{r}_B(t_-)$ in retarded time $t_- = R_-/c$.

the *field* and the discovery of Maxwell's equations (which are valid without a change even today). On the other hand, Newton's theory was replaced by Einstein's general theory of relativity. It is a beautiful fact that from the modern point of view the analogy between electromagnetic and gravitational fields goes further beyond the formal similarity of Coulomb's and Newton's law.

One of the most important qualitative differences between the Newtonian gravity and Maxwell's electrodynamics is the finite speed of propagation. In the standard concept of electrodynamics, a charged particle produces an electromagnetic field which "travels" at the speed of light c and propagates to the space. If there are any other charged particles, the field acts on them by Lorentz force. In this picture, particles are not interacting with each other directly, only through the fields they produce.

There is, however, a different picture, the concept of *action at distance*, which does not use the notion of the field. Rather it resembles original Newton's theory with one modification: particles act on each other directly, but the finite speed of signal is taken into account. In other words, particle A at time t feels the force of the particle B "produced" by it at retarded time $t_- = t - R_-/c$, where R_- is retarded distance, see figure 2.9. Formulation of electrodynamics as action-at-distance theory can be found e.g in paper [24] by Wheeler and Feynman. This theory explains the self-force felt by an accelerating particle as a consequence of the interaction of the particle with so-called absorber–particles which will absorb the radiation produced by source particle sometimes in the future. For details see the Wheeler-Feynman absorber theory [23].

The effect of retardation apparently makes the existence of helically symmetric solutions in electrodynamics impossible. A charged particle moving along circular trajectory is accelerated and therefore emits an electromagnetic radiation (*bremsstrahlung*). This radiation decreases the energy of the particle and therefore the particle must spiral into the centre of mass, see figure 2.10. From the action-at-distance point of view, test particle feels the force from the source particle at *retarded* position, so that the force is not radial but it has non-vanishing tangential component, see figure 2.11. Naively we could expect

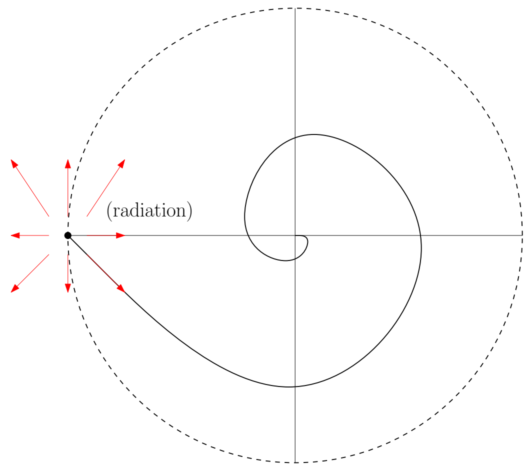

Figure 2.10: Accelerated particle is radiating and loosing energy. Instead of equilibrium configuration we have particle spiraling to the centre.

that resulting force will have the direction of line connecting test particle and source particle at a retarded time. However, such force would lead to *positive* angular acceleration, as it is obvious from figure 2.11. On the other hand, angular acceleration caused by bremsstrahlung should be negative. Detailed calculation, following soon, shows that the force will actually have direction sketched on 2.12 and will cause expected angular deceleration.

Notice the important difference between field and action-at-distance picture. In the former, system cannot stay in equilibrium because each particle emits radiation. This radiation is caused by motion with acceleration. In the latter picture, system cannot stay in equilibrium because of mutual interaction: electromagnetic force is not radial.

Following these arguments we expect that equilibrium configuration of two charged particles cannot exist. In fact, because of bremsstrahlung, classical electrodynamics cannot explain the stability of atoms and it was necessary to develop quantum mechanics. Although there are no doubts about the validity of quantum mechanics, there *is* a way how to achieve an equilibrium configuration. We explain this from the field point of view. The system of two charged particles will emit (outgoing) radiation and both particles will spiral to the centre of mass because of energy loss. Now imagine that there is some *ingoing* radiation coming from infinity and falling on the system of two particles, so there is also energy gain. If the amount of the incoming radiation will exactly compensate the energy loss, the system can stay in equilibrium, see figure 2.13. Usually we assume that radiation is produced by the particle and then propagates from the source to infinity. In this case we talk about *retarded solution* of Maxwell's equations. Ingoing radiation is equivalent to time-reversed propagation: radiation comes from infinity and ends up at the source. Such solution is called *advanced*. It is well-known that Maxwell's theory admits both retarded

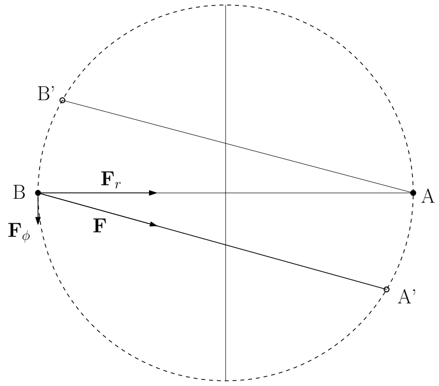

Figure 2.11: Particle B feels the force from particle A, but when it was in retarded position A' and vice versa. Force acting on each particle is therefore not radial and the system is not in equilibrium.

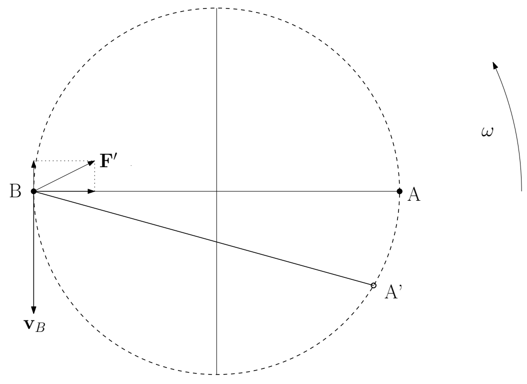

Figure 2.12: Actual direction of the force by particle A corresponds to expected negative angular acceleration (point masses are orbitting in counterclockwise direction).

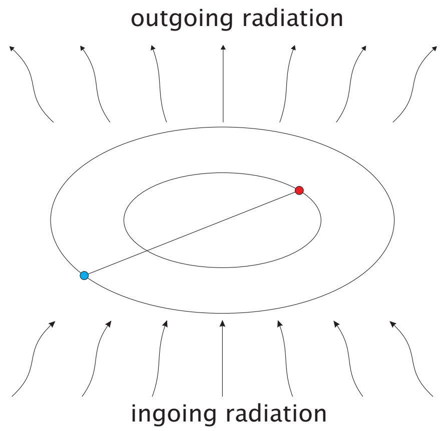

Figure 2.13: If the energy loss caused by outgoing radiation is compensated by ingoing radiation, system can stay in equilibrium

and advanced solutions. The reason why advanced solutions are typically not realized is of statistical nature. To obtain wave going from infinity to one particular point of the space we need extremely special and thus highly improbable initial conditions. In this text, however, we do not ask wether such solution is physically realistic. We merely ask whether such solution exists. After all, it may well describe situations in which the effects of retardation on the system are small.

Action-at-distance point of view suggests how to achieve equilibrium. In the figure 2.12 we can see that particle A at retarded position A' causes the force $\boldsymbol{F}'$ acting on particle B, which has non-zero tangential component. By the symmetry of configuration, the force $\boldsymbol{F}''$ from the advanced position A'' must be of the same magnitude, but with opposite tangential component, see figure 2.14. Then the force

$$\boldsymbol{F} \;=\; \frac{1}{2}\,(\boldsymbol{F}' \;+\; \boldsymbol{F}'')\,,$$

i.e. the force corresponding to the sum of half retarded and half advanced solution, is purely radial, and the system *can* stay in equilibrium for appropriate choice of angular velocity ω.

Solution of Maxwell's equations

We proceed analogously to the Newtonian case. For simplicity we choose units in which $c = 1$. First we assume that there is a source particle orbiting in the plane $z = 0$ with angular velocity ω at radius a_1, so that its position vector (in Cartesian coordinates) is given by (2.1) and its velocity reads

$$\dot{\boldsymbol{r}}(t) \;=\; (-\,a_1\,\omega\,\sin(\omega\,t+\phi_0),\; a_1\,\omega\,\cos(\omega\,t+\phi_0),\; 0). \tag{2.7}$$

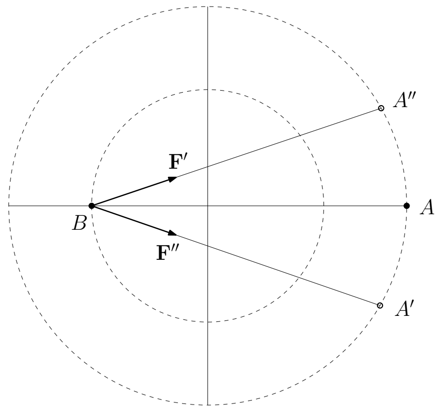

Figure 2.14: Equilibrium configuration can be achieved, if both retarded and advanced fields of the source particle are taken into account.

Cartesian coordinates will be denoted by

$$X^\alpha = (t,\ x,\ y,\ z) = (t,\ \boldsymbol{x})$$

and labelled by Greek letters $\alpha, \beta, \ldots$ Later we will switch to cylindrical coordinates

$$x^\mu = (t,\ r,\ \phi,\ z)$$

labelled by Greek letters $\mu, \nu, \ldots$ Position four-vector of the particle will be denoted by

$$r^\alpha = (t,\ \boldsymbol{r}(t)).$$

Four-current corresponding to the source particle is

$$j^\alpha = (\rho,\ \rho\,\dot{\boldsymbol{r}}) = \rho\,\frac{\mathrm{d}r^\alpha}{\mathrm{d}t},$$

where $\rho = \rho(X)$ is the charge density. For the point charge we have

$$\rho(X) = e_1\,\delta(\boldsymbol{x} - \boldsymbol{r}(t)).$$

Notice that charge density defined in this way is not a four-scalar, but j^α is four-vector.

Electromagnetic field produced by the source particle can be computed from the four-potential A^α satisfying (in the Lorenz gauge) inhomogeneous wave equation

$$\Box\, A^\alpha = j^\alpha.$$

Advanced(+) and retarded(-) solution is given by

$$A^\alpha_\pm(X) = \frac{1}{4\,\pi}\int \frac{j^\alpha(t \pm |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{x}'. \tag{2.8}$$

Evaluation of the integral leads to the familiar advanced/retarded *Liénard-Wiechert* potential

$$A^{\alpha}_{\pm} \;=\; \frac{e_1}{4\,\pi}\,\frac{1}{\mathcal{R}_{\pm}\,\pm\,\dot{\boldsymbol{r}}\cdot\boldsymbol{\mathcal{R}}_{\pm}}\,\frac{\mathrm{d}r^{\alpha}}{\mathrm{d}t},$$

where

$$\boldsymbol{\mathcal{R}}_{\pm} \;=\; \boldsymbol{x}\,-\,\boldsymbol{r}(t_{\pm}), \qquad \mathcal{R}_{\pm} \;=\; |\boldsymbol{\mathcal{R}}_{\pm}|, \tag{2.9}$$

and $t_{\pm} = t \pm \mathcal{R}_{\pm}$ is advanced/retarded time. Now we impose relations for $\boldsymbol{r}(t)$ given by (2.1) and $\dot{\boldsymbol{r}}$ given by (2.7) and transform all quantities to cylindrical coordinates x^{μ}. The result is

$$A^{\mu}(x) \quad=\quad \frac{e_1}{4\,\pi}\,\frac{1}{\rho_{\pm}}\,\frac{\mathrm{d}r^{\alpha}}{\mathrm{d}t} \tag{2.10}$$

where $\rho_{\pm}$ is implicitly determined by equations

$$\begin{aligned}
\rho_{\pm} \;&=\; \mathcal{R}_{\pm}\,\mp\,a_1\,\omega\,r\,\sin\theta_{\pm},\\
\theta_{\pm} \;&=\; \omega\,t_{\pm}\,-\,\phi\,+\,\phi_0, \quad t_{\pm} \;=\; t\,\pm\,\mathcal{R}_{\pm},\\
\mathcal{R}_{\pm} \;&=\; \sqrt{a_1^2\,+\,r^2\,+\,z^2\,-\,2\,a_1\,r\,\cos\theta_{\pm}}\;.
\end{aligned} \tag{2.11}$$

Explicitly, covariant components of the four-potential in cylindrical coordinates read

$$\begin{aligned}
A^{\pm}_0 \;&=\; \frac{e_1}{4\,\pi}\,\frac{1}{\rho_{\pm}},\\
A^{\pm}_1 \;&=\; \frac{e_1}{4\,\pi}\,\frac{\omega\,a_1\,\sin\theta_{\pm}}{\rho_{\pm}},\\
A^{\pm}_2 \;&=\; -\,\frac{e_1}{4\,\pi}\,\frac{\omega\,a_1\,r\,\cos\theta_{\pm}}{\rho_{\pm}},\\
A^{\pm}_3 \;&=\; 0.
\end{aligned} \tag{2.12}$$

Thus, we have computed advanced/retarded potential of the field produced by the source particle; components $A^{\pm}_{\mu}$ are functions of coordinates x^{μ}. Corresponding advanced/retarded electromagnetic field can be computed by taking the curl of four-potential,

$$F^{\pm}_{\mu\nu} \;=\; \partial_{\mu}A^{\pm}_{\nu}\,-\,\partial_{\nu}A^{\pm}_{\mu}.$$

Since functions $\rho_{\pm}$ and $\theta_{\pm}$ are given only implicitly, we have to use equations (2.11) to find their derivatives with respect to space-time coordinates. Straighforward but lengthy calculations yields

$\boldsymbol{\partial_t}$

$$\begin{aligned}
\frac{\partial\rho_{\pm}}{\partial t} \;&=\; \frac{\omega\,a\,r}{\rho_{\pm}}\left(1\,-\,a\,\omega^2\,r\,\cos\theta_{\pm}\right)\sin\theta_{\pm}\,\mp\,a\,\omega^2\,r\,\cos\theta_{\pm},\\
\frac{\partial\theta_{\pm}}{\partial t} \;&=\; \omega\,\pm\,\frac{a\,\omega^2\,r}{\rho_{\pm}}\,\sin\theta_{\pm},
\end{aligned}$$

$$
\begin{aligned}
& \boldsymbol{\partial_r} \\
\frac{\partial \rho_\pm}{\partial r} &= \frac{r \,-\, a\,\cos\theta_\pm}{\rho_\pm}\,\bigl(1 \,-\, a\,\omega^2\,r\,\cos\theta_\pm\bigr) \,\mp\, a\,\omega\,\sin\theta_\pm, \\
\frac{\partial \theta_\pm}{\partial r} &= \pm\,\frac{\omega(r - a\cos\theta_\pm)}{\rho_\pm}, \\
& \boldsymbol{\partial_\phi} \\
\frac{\partial \rho_\pm}{\partial \phi} &= -\,\frac{a\,r}{\rho_\pm}\,\bigl(1 \,-\, a\,\omega^2\,r\,\cos\theta_\pm\bigr)\,\sin\theta_\pm \,\pm\, a\,\omega\,r\,\cos\theta_\pm, \\
\frac{\partial \theta_\pm}{\partial \phi} &= -\,1 \,\mp\, \frac{a\,\omega\,r}{\rho_\pm}\,\sin\theta_\pm, \\
& \boldsymbol{\partial_z} \\
\frac{\partial \rho_\pm}{\partial z} &= \frac{z}{\rho_\pm}\,\bigl(1 \,-\, a\,\omega^2\,r\,\cos\theta_\pm\bigr)\,, \\
\frac{\partial \theta_\pm}{\partial z} &= \pm\,\frac{\omega\,z}{\rho_\pm}.
\end{aligned}
\tag{2.13}
$$

Explicit form of components $F^\pm_{\mu\nu}$ is quite long and not necessary for our purposes, see [1] for some expressions also in NP formalism.

Conditions of equilibrium

In this section we investigate the possibility of equlibrium configuration of two point charges. As in the Newtonian case we analyze the situation at time $t = 0$, assuming that initial phase of the source particle is $\phi_0 = 0$. First we show that taking purely retarded or purely advanced solution it is impossible to achieve equilibrium. For this demonstration we use purely retarded solution, but the discussion can be easily done for the advanced one.

Retarded solution

Thus, we set

$$A_\mu \;=\; A^-_\mu$$

and electromagnetic field is given by

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu \,-\, \partial_\nu A_\mu.$$

Initial position of the source particle is (in cylindrical coordinates)

$$r^\mu \;=\; (0,\; a_1,\; 0,\; 0)$$

and its four-velocity

$$\frac{\mathrm{d}r^\mu}{\mathrm{d}s} \;=\; \gamma_1\,(1,\; 0,\; \omega,\; 0),$$

where

$$\gamma_1 \;=\; \frac{1}{\sqrt{1 \,-\, a_1^2\,\omega^2}}.$$

Position of the test particle and its velocity read

$$x^\mu \;=\; (0,\; a_2,\; \pi,\; 0), \quad u^\mu \;=\; \gamma_2(1,\; 0,\; \omega,\; 0), \tag{2.14}$$

where

$$\gamma_2 \;=\; \frac{1}{\sqrt{1 \,-\, a_2^2\,\omega^2}}.$$

Situation is sketched in figure 2.15.

Let us briefly examine the meaning of parameters $\mathcal{R}_\pm, \theta_\pm$ and $\rho_\pm$, when the position of test particle is given by (2.14). Since we are now interested in purely retarded solution, let us denote

$$\mathcal{R} \;=\; \mathcal{R}_-, \quad \theta \;=\; \theta_-, \quad \rho \;=\; \rho_-.$$

Equations (2.11) then simplify to

$$\begin{aligned} t_- \;&=\; -\,\mathcal{R}, \\ \mathcal{R} \;&=\; \sqrt{a_1^2 \,+\, a_2^2 \,-\, 2\,a_1\,a_2\,\cos\theta}, \\ \theta \;&=\; -\,\omega\,\mathcal{R} \,-\, \pi, \\ \rho \;&=\; \mathcal{R} \,+\, a_1\,a_2\,\omega\,\sin\theta. \end{aligned} \tag{2.15}$$

Quantity $\boldsymbol{\mathcal{R}}$ was defined as a retarded distance, i.e. the distance between the test particle and the source particle at retarded time. Now, looking at picture 2.15 and using cosine formula we can write

$$\mathcal{R}^2 \;=\; a_1^2 \,+\, a_2^2 \,-\, 2\,a_1\,a_2\,\cos(\pi - \vartheta),$$

where ϑ is called *deficit angle* and it is a measure of retardation. If the difference between time and retarded is zero, so is the angle ϑ and no retardation effects appear. This can happen if the speed of interaction is infinitely high, or if the speed of the bodies is negligible. In contrary, if the retardation becomes important (velocities of the bodies are high), deficit angle ϑ grows. From the last equation and definition (2.15) we see that our function θ is in fact related to deficit angle via

$$\theta \;=\; \pi \,-\, \vartheta.$$

Equations of motion of the test particle in the field $F_{\mu\nu}$ produced by source particle are

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}s} \;=\; \frac{e_2}{m_2}\,F^\mu{}_\nu\,u^\nu.$$

The complication is that we want coordinates r, ϕ and z to be parametrized by the time coordinate t rather than by proper time s. Since

$$\frac{\mathrm{d}}{\mathrm{d}s} \;=\; \frac{\mathrm{d}t}{\mathrm{d}s}\,\frac{\mathrm{d}}{\mathrm{d}t} \;=\; \gamma_2\,\frac{\mathrm{d}}{\mathrm{d}t},$$

on the left hand side of the equation of motion we have (for initial conditions (2.14))

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}s} \;=\; \gamma_2^2\,\frac{\mathrm{d}^2x^\mu}{\mathrm{d}t^2} \,+\, \gamma_2^4\,a_2^2\,\omega\,\ddot{\phi}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}t}.$$

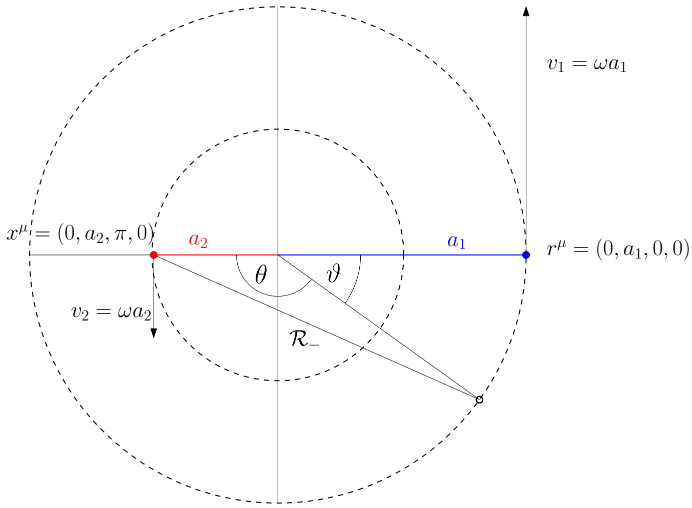

Figure 2.15: Initial conditions for the source and test particle. Only retarded solution is taken into account, so the equilibrium is impossible: force has non-radial component (not shown in the figure).

Finally, the equations of motion are

$$\ddot{x}^\mu \; + \; \gamma_2^2\, a_2^2\, \omega\, \ddot{\phi}\, \dot{x}^\mu \quad = \quad \frac{e_2}{m_2\, \gamma_2}\, F^\mu{}_\nu\, \dot{x}^\nu. \tag{2.16}$$

Resulting equations of motion are lengthy and complicated. If we want to understand basic properties of such configuration, it is useful to expand all quantities into the series in ω. For $\omega = 0$ we expect that the force will be purely radial and its magnitude will be given by Coulomb's law. To see what happens in higher orders, we can use expansions assuming small $\omega \ll 1$

$$\begin{aligned}
\theta \quad &= \quad -\pi \; - \; (a_1 + a_2)\, \omega \; + \; \frac{1}{2}\, a_1\, a_2 (a_1 + a_2)\, \omega^3 \\
&\quad - \; \frac{1}{24}\, a_1\, a_2\, (a_1 + a_2)\, (a_1^2 + 11\, a_1\, a_2 + a_2^2)\, \omega^5 \; + \; \mathcal{O}\left(\omega^7\right), \\
\rho \quad &= \quad (a_1 + a_2) \; + \; \frac{1}{2}\, a_1\, a_2 (a_1 + a_2)\, \omega^2 \\
&\quad - \; \frac{1}{8}\, a_1\, a_2\, (a_1 + a_2)\, (a_1^2 + 3\, a_1\, a_2 + a_2^2)\, \omega^4 \; + \; \mathcal{O}\left(\omega^6\right).
\end{aligned} \tag{2.17}$$

In this order, equations of motion acquire the following form:

$$\begin{aligned}
\ddot{r} \quad &= \quad \frac{e_1\, e_2}{4\, \pi\, m_2}\, \frac{1}{(a_1 + a_2)^2} \; - \; \frac{e_1\, e_2}{8\, \pi\, m_2}\, \frac{a_1^2 \, + \, a_2^2}{(a_1 + a_2)^2}\, \omega^2 \; + \; \mathcal{O}\left(\omega^4\right), \\
\ddot{\phi} \quad &= \quad \frac{e_1\, e_2}{6\, \pi\, m_2}\, \frac{a_1}{a_2}\, \omega^3 \; + \; \mathcal{O}\left(\omega^4\right).
\end{aligned} \tag{2.18}$$

Since the charges of particles have opposite signs, we have

$$e_1\, e_2 \; < \; 0.$$

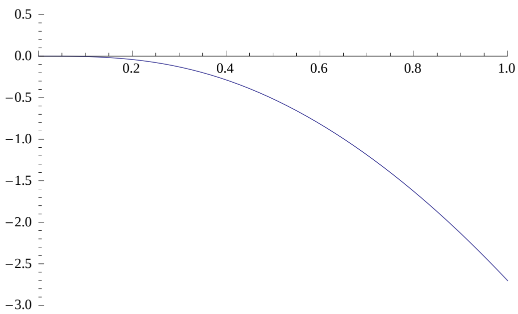

Figure 2.16: $\ddot{\phi}$ as a function of ω for $a_1 = a_2 = 1$.

Then the leading term in $\ddot{r}$ is obviously acceleration caused by Coulomb's force

$$F_C = -\frac{|e_1\, e_2|}{4\,\pi\, m_2}\,\frac{1}{R_0^2},$$

where $R_0 = a_1 + a_2$ is the distance of particles at $t = 0$.

By contrast to the Newtonian case, the angular acceleration is not identically zero. In addition, we can see that leading term of $\ddot{\phi}$ is negative, so the force will cause angular deceleration of the test particle, as sketched in figure 2.12. The question is whether we can find such a combination of parameters a_1, a_2 and ω, that $\ddot{\phi}$ will vanish. In order to find the answer, we have to take full expression for $\ddot{\phi}$ (not shown here) and numerically investigate value of $\ddot{\phi}$ for different values of a_1, a_2 and ω. Here we choose fixed a_1 and a_2 and plot angular acceleration as a function of ω, see figures 2.16 and 2.17[1]. Quantity $\ddot{\phi}$ is zero for $\omega = 0$, but we are not interested in this solution. For bigger values of ω the acceleration acquires bigger negative values, so $\ddot{\phi}$ does not vanish for any positive angular velocity.

Clearly if only retarded solution is taken into account, the equilibrium of two charged particles is impossible and corresponding helically symmetric solution does not exist.

Retarded+advanced solution

Let us now take the solution to be

$$A_\mu = \left(A_\mu^+ + A_\mu^-\right)/2.$$

By the symmetry of configuration we have

$$\mathcal{R}_+ = \mathcal{R}_- = \mathcal{R}.$$

Using relations (2.11) we find

$$\rho_\pm = \rho_- = \rho, \quad \cos\theta_+ = \cos\theta_- = \cos\theta, \quad \sin\theta_+ = -\sin\theta_- = -\sin\theta.$$

[1] In these pictures, ω must be $\omega \in (0, 1/\max\{a_1, a_2\})$, as the velocities of both particles must be subluminal, i.e. $\omega\, a_1 < 1$

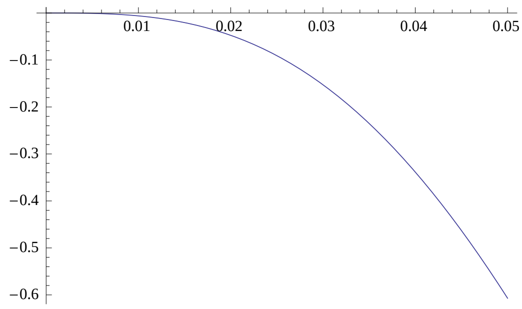

Figure 2.17: $\ddot{\phi}$ as a function of ω for $a_1 = 20$, $a_2 = 1$.

Equations of motion (2.16) then immediately give, using (2.13),

$$\ddot{\phi} \; = \; 0.$$

Thus, resulting force is purely radial as we expected.

Chapter 3

Helical symmetry in General Relativity

In previous chapter we mentioned that a typical example of helically symmetric system is a binary system, i.e. system of two objects orbiting about their common centre of mass. In astrophysics, objects of particular interest are binary systems consisting of neutron stars or black holes. Such binary systems allow us to test general relativity in the case of strong gravitational fields. If one of the constituents is pulsar its period is very precisely measurable. If the pulsar is part of a binary system, the period of pulsar is periodically increasing and decreasing as a consequence of redshift. By measuring these fluctuations it is possible to determine physical properties, e.g. the mass of the second component of binary system.

If the binary system is in equilibrium, i.e. its components are orbiting about the centre along closed trajectories periodically, it possesses a helical symmetry according to our previous definition. We have shown that in Newtonian gravity there indeed exists helically symmetric solution. In general relativity an argument similar to that in electrodynamics applies: accelerated objects emit gravitational radiation. This will cause the loss of energy and consequently the loss of the velocity: both stars will be continuously approaching the centre and emitting gravitational radiation, figure 3.1. This process is called *inspiral* and the orbits of stars are said to *decay*. During final stage of inspiral, shortly before coalescence, energy carried by gravitational radiation is expected to be high. At the end of inspiral both objects merge into one neutron star or black hole.

In fact, coalescence of binary system is one of the most promising sources of gravitational radiation. For example, the decay of orbits in the case of pulsars causes systematic decrease of pulsar's period. This decrease can be measured and is in perfect accordance with predictions of general relativity and can be regarded as indirect, but very convincing proof of the existence of gravitational waves (until now primarily for object called PSR 1913+16). A more direct proof is desirable, however. Gravitational radiation produced by binary system before its coalescence can be intensive enough to be detected by terrestrial gravitational wave detectors. Thus, much effort has been spent on understanding this.

There is, unfortunately, no hope that complicated problem of binary inspiral

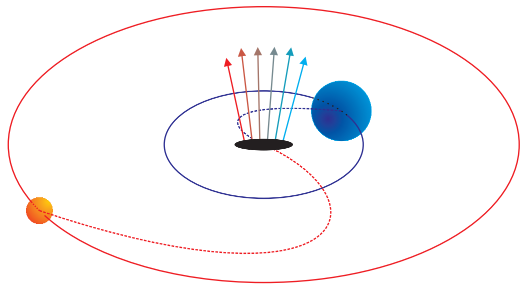

Figure 3.1: Coalescence of binary system: stars are not following circular orbits but spirals. Shortly before final merging process the system emits intensive gravitational radiation (schematic picture).

could be solved analytically and we are devised to numerical simulations – coalescing binary systems exhibit no symmetries and it is impossible to simplify full set of equations. But also in numerical relativity, the two-body problem is very difficult and delicate and was not yet fully solved.

One approach to the simulation of the binary inspiral, initiated by Blackburn and Detweiler [4, 6], is based on an approximate helical symmetry. Numerical simulations typically cannot be evolved for large times so it is difficult to simulate the inspiral from the early stages until the final coalescence. It is therefore necessary to choose appropriate conditions. It is generally expected [13] that initial motion, when the distance of stars is large, can be approximated by the sequence of circular trajectories – quasicircular motion, figure 3.2. The last circular trajectory is called ISCO, innermost stable circular orbit. As the star reaches ISCO, quasicircular motion is followed by a rapid spiral, at the end of which both stars will merge and create a new stationary black hole or a neutron star with larger mass. As a starting point of the simulation we can assume that initial space-time has helical symmetry.

The question, of course, arises, whether the assumption of helical symmetry is compatible with general relativity, i.e. whether helically symmetric solutions of Einstein's equations exist. Argumentation here is very similar to the discussion in electrodynamics. Periodic motion is possible only if there is incoming radiation compensating the energy loss, see figure 2.13. Because of this radiation, helically symmetric binary system cannot be treated as *isolated* system: for all the times there is radiation coming from infinity and falling on the system, and the system is radiating for all times gravitational radiation to infinity. As we shall see later, the fact that system is not isolated can be expressed by statement that corresponding space-time is not asymptotically flat in a precisely defined sense.

We have seen that the existence of helically symmetric solution in electrodynamics is not as trivial as in Newtonian case. Existence of helical symmetry in general relativity is not only interesting fundamental question, but has also practical importance in modelling binary inspiral. Because Einstein's theory is non-linear, we expect that the answer will be more involved. Closer ana-

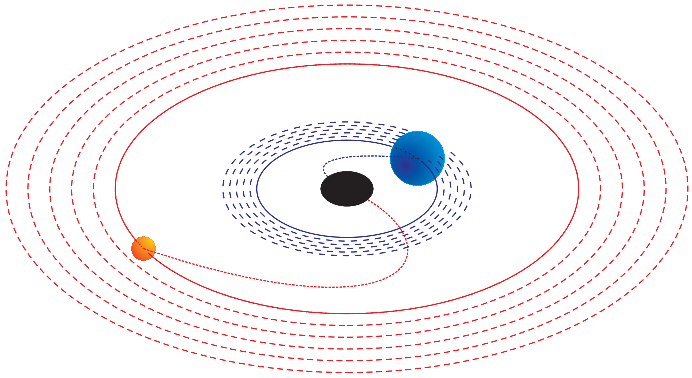

Figure 3.2: Before the final stage stars are moving along quasi-circular trajectories (dashed red and blue circles). This regime can be described by approximate helical symmetry. When the star passes ISCO (full red and blue circles), helical symmetry is not appropriate approximation anymore and rapid inspiral follows.

lyisis, however, shows that in curved space-time of general relativity even the definition of helical symmetry is not fully clear.

3.1 Helical Killing vector in flat space-time

In general, space-time[1] (M, g) has certain symmetry, if vector field ξ generating corresponding transformation is the *Killing vector* of the metric, i.e. if the Lie derivative of the metric with respect to ξ vanishes:

$$\mathcal{L}_\xi \, g \;=\; 0. \tag{3.1}$$

Geometrically, vector field ξ generates one-parameter family of diffeomorphisms $\Phi_t : M \mapsto M$ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_t(P) \;=\; \xi_P$$

at arbitrary point $P \in M$. If the metric g is invariant under diffeomorphism Φ_t for any $t \in \mathbb{R}$, we say that Φ_t is the symmetry of the space-time and vector field ξ is generator of this symmetry; the Lie derivative of g is zero. Equations (3.1) are called *Killing equations*.

We have already seen that the generator of the helical symmetry, called *helical Killing vector*, in Minkowski space-time (M, η) can be written in the form

$$\xi \;=\; \partial_t \;+\; \omega \, \partial_\phi,$$

[1] Manifold M equipped with metric tensor g of signature $(+ - - -)$ and corresponding Levi-Civita connection.

where ∂_t is the generator of translations in time direction and ∂_ϕ is the generator of rotations about $z-$axis. Vector ∂_t is time-like, ∂_ϕ is space-like with closed orbits. Both vectors ∂_t and ∂_ϕ are separately Killing vectors of Minkowski metric, so their linear combination ξ with constant coefficients must be Killing vector as well. The norm of the helical Killing vector expressed in cylindrical coordinates is

$$N \;:=\; \|\xi\|^2 \;=\; 1 \;-\; \omega^2\, r^2.$$

According to the sign of N we can divide the space-time into three regions. Vector ξ is light-like/null ($N = 0$) on the so-called *null cylinder* given by the equation

$$r \;=\; \frac{1}{\omega}.$$

Outside the cylinder, vector ξ is space-like ($N < 0$), inside the cylinder it is time-like ($N > 0$). Quantity $v = \omega\, r$ is in fact the velocity of the observer co-rotating with angular velocity ω at distance r from the $z-$axis. Inside the null cylinder $v < 1$ so that the observer is time-like, while on the cylinder, $v = 1$, the "observer" is rotating with the speed of light. Outside the null cylinder these "observers" would be super-luminal, i.e. non-physical.

Using (2.13) it is straightforward to show that electromagnetic four-potential $A^{\pm}_{\mu}$ given by (2.12) has also helical symmetry in the sense that the Lie derivative of $A^{\pm}_{\mu}$ is zero:

$$\mathcal{L}_\xi A^{\pm}_{\mu} \;=\; \xi^\nu \partial_\nu A^{\pm}_{\mu} \;=\; 0.$$

Here we used the fact that the components of ξ with respect to cylindrical coordinates are constant and therefore the Lie derivative reduces to $\mathcal{L}_\xi \;=\; \xi^\nu \partial_\nu$ (even if it acts on tensors of higher orders).

Let $f = f(t, r, \phi, z)$ be tensor of arbitrary rank (including zero) with helical symmetry, i.e. with vanishing Lie derivative $\mathcal{L}_\xi$:

$$\mathcal{L}_\xi f \;=\; \xi(f) \;=\; \partial_t f \;+\; \omega\, \partial_\phi f.$$

Sometimes it is useful to introduce a *co-rotating frame* by coordinate transformation

$$\hat{t} \;=\; t, \quad \hat{r} \;=\; r, \quad \hat{\phi} \;=\; \phi - \omega\, t, \quad \hat{z} \;=\; z. \tag{3.2}$$

In these coordinates we have

$$\partial_{\hat{t}} \;=\; \partial_t \;+\; \omega\, \partial_{\hat{\phi}},$$

so helically symmetric tensor f must satisfy

$$\frac{\partial f}{\partial \hat{t}} \;=\; 0,$$

which means that f is not a function of $\hat{t}$:

$$f \;=\; f(\hat{r}, \hat{\phi}, \hat{z}) \;=\; f(r, \hat{\phi}, z) \;=\; f(r, \phi - \omega t, z).$$

Helical symmetry therefore reduces the dimension of a problem to three. Advantage of co-rotating coordinates rests in the fact that the four-velocity of orbiting particle in these coordinates has only the zeroth (time) component.

The concept of the helical symmetry on Minkowski background is clear. What is not so clear is how to generalize the definition of the helically symmetric Killing vector to a general curved space-time.

3.2 Curved space-time

Definition of helical symmetry in general curved space-time is neither trivial nor straightforward. Helical symmetry is a special kind of periodicity: particles or stars are moving along closed orbits in a periodic way and the world line of each particle is a helix. Periodicity is, however, not a local property. For we cannot see that system is moving periodically by local temporal and spatial measurements. To see periodicity we have to wait at least one period. We can say that system is periodic, if it is invariant under *discrete* time translation $t \mapsto t + T$, where $T > 0$ is the period.

On the other hand, Killing vectors can be used only for description of *continuous* symmetries; a Killing vector is the infinitesimal generator of finite transformations under which the system is invariant. Obviously, a discrete time periodicity does not have an infinitesimal generator. Helical symmetry is a special case of periodic motion. The world line of a particle in an orbital motion is an infinite continuous curve in space-time, but it may correspond to a periodic motion in a three-dimensional space. In Minkowski space-time, the generators ∂_t and ∂_ϕ guarantee that spatial trajectory of the particle moving along $\partial_t + \omega\partial_\phi$ will be a closed circular curve. An important point is that slices t =constant in a given inertial frame can be identified – they have the same geometry, so it is possible to say that particle is "at the same place" at two different times. In curved space-time, geometry of slices t =constant need not be the same and it may be hard to see whether the particle returned to its initial position after some finite time T.

The notion of helical symmetry naturally extends to *stationary axisymmetric* space-times. The space-time is *stationary* [22, 20], if it possesses a time-like Killing vector e_t. The space-time is *axisymmetric* if there exists a space-like Killing vector e_ϕ with closed orbits. Finally, the space-time is called stationary axisymmetric if it is stationary, axisymmetric *and* both Killing vectors commute:

$$[e_t,\ e_\phi] \ = \ 0.$$

Then also corresponding actions of e_t and e_ϕ commute, i.e. time translations and spatial rotations commute, the corresponding group is Abelian. This condition is always satisfied for asymptotically flat (see below) axisymetric stationary space-times. In stationary axisymmetric space-times we can define helical Killing vector by

$$\xi \ = \ e_t \ + \ \omega\, e_\phi, \ \ \omega \ = \ \text{constant}.$$

Since ξ is an $\mathcal{R}-$linear combination of Killing vectors, it is itself a Killing vector.

First definition of helical symmetry for the case of binary system has been given by Bonazzola et al. [5]. It is essentialy the same as in stationary axisymmetric case, but with a crucial difference: ω is again assumed constant, ξ is a Killing vector, but e_t and e_ϕ are not assumed to be Killing vectors. If the distance between coalescing objects is large enough (about 50 km for neutron stars), the evolution can be approximated by the sequence of equilibrium (helically symmetric) configurations. Since the objects emit gravitational radiation, their orbits will tend to be circular, with zero eccentricity. Thus, according to [5], there exists Killing vector of the form

$$\xi \;=\; e_t \,+\, \omega\, e_\phi, \tag{3.3}$$

where ω is constant angular velocity as measured by a distant inertial observer. Vector e_t is time-like and far from the binary system it coincides with four-velocity of the distant observer. Vector e_ϕ is space-like with closed orbits, and it vanishes on the axis of rotation. Vector field ξ is called the helical Killing vector.

More precise and more general definition of helical symmetry is that of Friedman et al. [8]. This definition tries to resolve problems mentioned in the beginning of this section. Let χ_t be one-parametric family of diffeomorphisms and ξ its generator:

$$\xi_P \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \chi_t(P), \quad P \in M.$$

In figure 3.3, P is point of the particle's world line at time $t = 0$. At time $t > 0$, world line will reach point $P' = \chi_t(P)$. We want to express the idea that after some period T spatial trajectory returns to the initial point. Since it is impossible to define what the "same point" for different slices really means, we only demand the initial point P of the world line and the final point $Q = \chi_T(P)$ to be time-like separated. That is, there exists a time-like curve connecting points P and Q.

It can happen, however, that we choose the initial point P of the world line *under* the horizon of a black hole. Then, of course, the world line will not be a helix, because the particle must fall into singularity. For helical symmetry we thus require that points P and Q must be time-like separated, if they lie outside the space-time region swept out by black hole's horizon.

Definition. Killing vector field ξ (with the flow χ_t) is called *helical*, if there exists smallest $T > 0$ such that P and $\chi_T(P)$ are time-like separated for all $P \in M - \mathcal{T}$, where

$$\mathcal{T} \;=\; \{\chi_t(\mathcal{S}) \mid t \in \mathbb{R}\}$$

is the history of a space-like two-sphere $\mathcal{S}$.

It is obvious that Killing vector of the form (3.3) is helical in the sense of this definition. Now we want to show that every helical Killing vector can be written in the form (3.3).

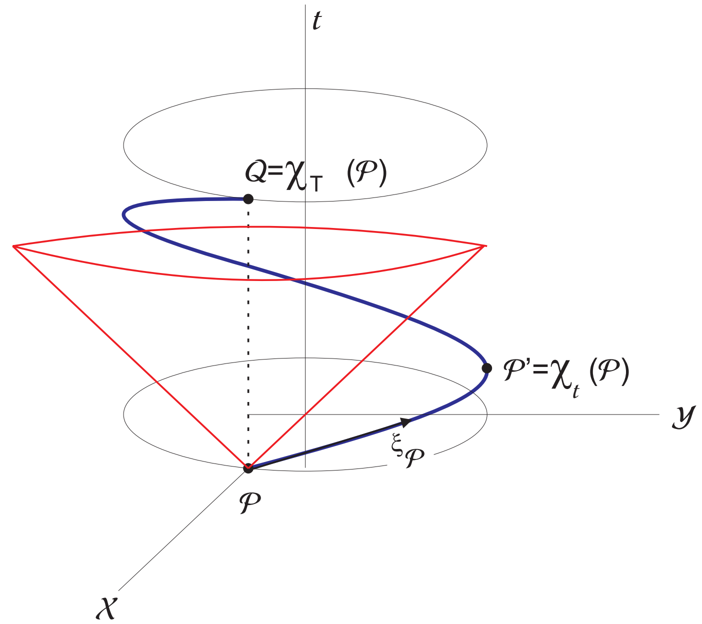

Figure 3.3: World line of the particle starts at time $t = 0$ in the point $P = \chi_0(P)$. At time $T > 0$ the world line reaches point $Q = \chi_T(P)$. If χ_t is to describe helical symmetry, points P and Q must be time-like separated, which is a weaker formulation of requirement that P and Q are the same points in a (3D) space.

Let ξ be a helical Killing vector and $\chi_t : M \mapsto M$ its flow, let Σ be a Cauchy hypersurface[2]. We can use diffeomorphisms χ_t to construct foliation of the space-time by hypersurfaces

$$\Sigma_t \;=\; \{\chi_t(P) \,|\, P \in \Sigma\}\,,$$

see also figure 3.4. This definition implies $\Sigma = \Sigma_0$. Next we can define scalar function ψ to have value t on Σ_t so that Σ_t is given by equation

$$\Sigma_t \;:\; \psi \;=\; t.$$

Now we can define (co-)vector field

$$\psi_a \;=\; \nabla_a \psi.$$

This vector field is orthogonal to space-like hypersurface $\Sigma_t : \psi = t$, so that ψ^a is time-like. Let Ψ_t be the flow of vector field ψ^a.

[2]For precise definitions see Wald [22]. Roughly speaking, Cauchy hypersurface is a space-like hypersurface such that its domain of dependence is full space-time M. It means that each point of the space-time in the future of the hypersruface can be reached by causal (time-like or null) curve starting on the Cauchy hypersurface.

Now choose $t = T$, where T is the period included in the definition of helical Killing vector. According to this definition, point $\chi_T(P)$ must be time-like separated from P which, however, means that it must lie on a time-like curve connecting P and $\chi_T(P)$. In other words, flows of vector fields ξ^a and ψ^a must intersect at point $\chi_T(P)$:

$$\chi_T(P) \;=\; \Psi_T(P),$$

or, equivalently,

$$\Psi_{-T} \circ \chi_T(P) \;=\; P. \tag{3.4}$$

By the flow Ψ_t of vector field ψ^a we can project integral curves χ_t of ξ onto hypersurface Σ. We define curve c by

$$c(t) \;=\; \Psi_{-t} \circ \chi_t \;:\; \mathbb{R}[t] \mapsto \Sigma_0.$$

As t varies from 0 to T, the integral curve is going from point $P \in \Sigma_0$ to point $Q = \chi_T(P)$. For any $t \in (0, T)$, $\chi_t(P)$ is point on the helix. Mapping Ψ_{-t} then drags point $\chi_t(P)$ back to point $P' = \Psi_{-t}(\chi_t(P))$ lying in Σ_0, figure 3.5. Thus, c is indeed curve lying in Σ_0. If χ_t is regarded as world line of the particle, c is its projection onto Σ_0 and therefore represents spatial trajectory of the particle. Because of relation (3.4), curve c is necessarily *closed*.

On Σ_0, we can define vector field e_ϕ by

$$e_\phi|_\Sigma \;=\; \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} c(t)$$

and extend it to entire space-time via Lie-dragging by flow Ψ_t of time-like vector field ψ^a:

$$e_\phi \;=\; (\Psi_t)_*\, e_\phi|_\Sigma.$$

By this procedure we obtain vector field e_ϕ with closed orbits which is always tangent to corresponding hypersurface Σ_t. Helical Killing vector ξ^a can then be written as a sum of part e_t orthogonal to Σ_t (i.e. proportional to ψ^a) and part e_ϕ tangent to it:

$$\xi \;=\; e_t \;+\; e_\phi.$$

Orbits of vector e_ϕ have parametric length T by definition. In order to obtain curves with parametric length 2π we use reparametrization

$$C(t) \;=\; c\left(\frac{2\,\pi\,t}{T}\right),$$

and define e_ϕ to be tangent to this new curve C. Then we obtain Killing vector ξ to be expressed as

$$\xi \;=\; e_t \;+\; \omega\, e_\phi, \tag{3.5}$$

where $\omega = 2\pi/T$.

We have shown that definitions of [5] and [8] are equivalent and they agree with our intuitive understanding of helical symmetry. Decomposition (3.5) is not unique as it depends on particular choice of the foliation of space-time by

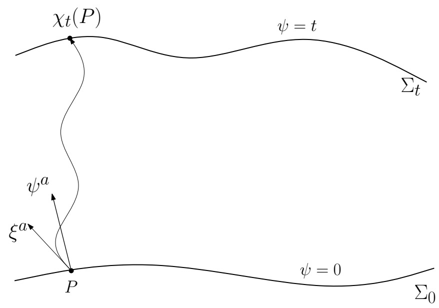

Figure 3.4: A foliation of the space-time by hypersurfaces Σ_t using initial Cauchy hypersurface Σ and flow χ_t of helical Killing vector field ξ. Helix in the picture represents helical integral curve of vector field ξ.

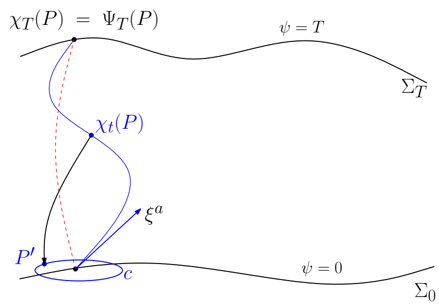

Figure 3.5: Evolution from the Cauchy hypersurface Σ_0 to the hypersurface Σ_T corresponding to a period T. The dashed line connecting P and $\chi_t(P)$ is integral curve of time-like vector field e_0.

hypersurfaces Σ_t. On the other hand, not all space-times allow foliation respecting diffeomorphism χ_t, therefore definition [8] seems to be less restrictive.

It is unknown, however, whether solutions with helical symmetry defined above exist. Moreover, it is not clear whether this definition of helical symmetry is indeed most satisfactory and encapsulates essential properties of helically symmetric systems as we understand them intuitively. These questions remain open.

3.3 Linearized gravity

While the notion of a helical symmetry in general curved space-times is a delicate issue, it has a clear meaning in the linearized Einstein's theory. When the gravitational field is weak enough, it can be treated as a small perturbation of a background flat metric. If all non-linear terms are neglected, gravitational field can be regarded as a field propagating in Minkowski space-time. And in Minkowski space-time, the helical Killing vector is, in cylindrical coordinates, simply

$$\xi \;=\; \partial_t \;+\; \omega\,\partial_\phi.$$

One may hope that constructing helically symmetric solution in linearized gravity one finds hints how to to understand helical symmetry also in curved space-times.

Again, we can raise the question of *existence* of helically symmetric solution in linearized gravity. Situation is essentially the same as in electrodynamics. Body under acceleration produces gravitational radiation, so to obtain equilibrium configuration we have to include both retarded and advanced fields. At the time of writing the thesis, we have found such a solution, but corresponding paper is still in preparation. For this reason, helically symmetric solution of linearized Einstein's equation is not included in the thesis.

3.4 Periodic solutions

In this chapter we discussed the existence of helically symmetric solutions in general relativity. We explained that this question is hard to answer as it is difficult even to define what the helical symmetry in curved space-times is. Space-time is helically symmetric if it possesses helical Killing vector. We have presented definitions following [5] and [8] and shown their equivalence. Problem with definition of helical symmetry is not present in the linearized gravity because of fixed flat background.

Physically, however, we expect that helically symmetric solutions exist if we take both retarded and advanced fields into account, just as in the case of electrodynamics. That is, the energy lost by emitted radiation must be compensated by energy of radiation coming from infinity, figure 3.6. Binary system in equilibrium cannot be regarded as isolated, for the incoming radiation is present for all times.

Since gravity is manifestation of space-time curvature, isolated systems should produce space-times with curvature vanishing at large distances from the source. Such space-times are called *asymptotically flat* and precise definition will be given in the next chapter. Schwarzschild's solution is the simplest non-trivial example of asymptotically flat space-time. Binary system of coalescing black holes is expected to produce asymptotically flat space-time as well. But if the binary system is to be in equilibrium, both incoming and outgoing radiation must be present, and cooresponding space-time is not asymptotically flat in the sense to be defined later.

Helical symmetry is a special case of periodic motion and by physical argumentation, helically symmetric system cannot be asymptotically flat. New questions here arise. First, is it possible to prove rigorously that helical symmetry is incompatible with asymptotical flatness? And can there be other periodic solutions which are asymptotically flat? In other words, can an isolated system be in periodic motion? Can, for example, two black holes orbit about the centre of mass without emitting radiation?

These questions are motivated by study of helical symmetry, but they are interesting in general. Main results of this thesis are theorems about the non-

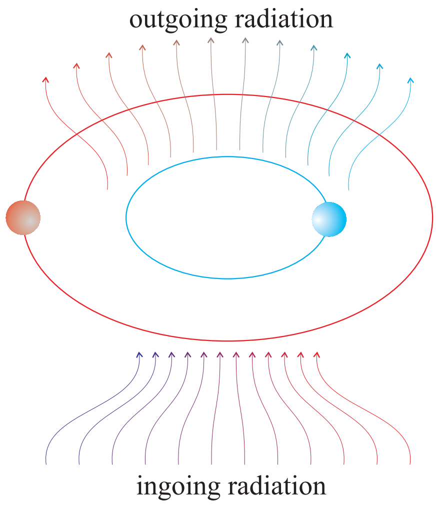

Figure 3.6: Radiated energy must be compensated by radiation coming from infinity.

existence of periodic solutions of Einstein's equations under some assumptions. Before we state them, it is necessary to define the notion of asymptotic flatness and introduce related mathematical tools. This is the subject of following chapters.

Chapter 4

Asymptotic properties of space-times

In this chapter we briefly review conformal techniques developed mainly by Penrose for the study of asymptotically flat space-times. These methods are useful for investigation of properties of radiation (electromagnetic or gravitational) and general properties of asymptotically flat space-times. A lot of problems related to these topics can be much easier analyzed in the spinor or mixed spinor-tensor formalism, which was also introduced to general relativity by Penrose and developed by others. A comprehensive treatment of spinor formalism and asymptotic properties of space-times in spinor and twistor formalism can be found in Penrose and Rindler [18, 19]. Original papers are, for example, Penrose [17] or Newman and Penrose [15]. Brief introduction to the subject with accent on practical calculations can be found in Stewart [21]. An useful "historical" review on the notion of asymptotic flatness and conformal field equations, primarily aimed to applications in numerical relativity, is in Frauendiener [7]. Geometrical aspects of asymptotically flat space-times are in detail given in Geroch [11].

Roughly speaking, asymptotically flat space-times describe isolated gravitating systems. Isolated systems represent idealized models on which one can illustrate basic properties of the theory under consideration. In Newtonian gravity, gravitational field is described by potential Φ satisfying Poisson's equation

$$\Delta\Phi = 4\pi G\rho,$$

where ρ is the mass density of a source. System is *isolated* if function ρ has a compact support or sufficiently rapid decay. As we move from this region to infinity, no additional mass will appear and the field will become weaker and weaker and will vanish at infinity.

There is a simple reason why it is so easy to define isolated system in Newtonian gravity. Total gravitational field is a sum of gravitational fields produced by individual constituents. In general relativity, by contrast, for a given solution of Einstein's equations there is no way how to split resulting field into separate individual parts. Knowing two solutions for two objects, we cannot say what is the solution representing same objects at specific distance. Of course, it is a consequence of non-linearity of Einstein's equations.

Gravitational field in general relativity is described by metric and curvature of the space-time. Intuitively, gravitational field of isolated system must be weak at large distances and it must vanish at infinity, just as in the Newtonian case. We thus expect that the space-time geometry will approach the Minkowski geometry near infinity. Such a space-time is said to be asymptotically flat, i.e. flat at large distances. If we appropriatelly define the "distance coordinate" r, then we expect the metric tensor $g_{\mu\nu}$ to be

$$g_{\mu\nu} \;=\; \eta_{\mu\nu} \;+\; \mathcal{O}\left(\frac{1}{r}\right)$$

for large r, where $\eta_{\mu\nu}$ is the flat metric. This approach has been followed by Bondi, Sachs and others and many useful result have been obtained. In [17], however, Penrose defined asympotic flatness in a geometrical and invariant way and in subsequent papers he was able to recover all results obtained by older approaches. In addition, his approach is not only more elegant and more powerful, but it provides us with a deeper insight into the geometry of asymptotically flat space-times. Moreover, it has had an important influence on numerical relativity.

4.1 Minkowski space-time

Before we define the notion of asymptotic flatness in the sense of Penrose, it is useful to investigate Minkowski space-time. Minkowski space-time is flat, so it must also be asymptotically flat, whatever "asymptotically flat" means precisely. In this section we extract important properties of Minkowski space-time related to its asymptotic flatness. These properties will then serve as an inspiration for general definition.

We would like to analyze geometry of the space-time near infinity, i.e. for large r, where r is the distance from the source in some appropriate sense. In a general space-time it can be hard to find meaningful coordinate r, but this is no problem in Minkowski space-time: r can be chosen to be radial spherical coordinate. Flat metric in standard spherical coordinates reads

$$\mathrm{d}\tilde{s}^2 \;=\; \mathrm{d}t^2 \;-\; \mathrm{d}r^2 \;-\; r^2\left(\mathrm{d}\theta^2 \;+\; \sin^2\theta\,\mathrm{d}\phi^2\right).$$

Line element $\mathrm{d}\tilde{s}^2$ carries tilde for reasons to be clear soon. Quantities with tilde will be called *physical quantities*, e.g. $\mathrm{d}\tilde{s}^2$ is *physical metric*. We also employ notation

$$\mathrm{d}\Sigma^2 \;=\; \left(\mathrm{d}\theta^2 \;+\; \sin^2\theta\,\mathrm{d}\phi^2\right)$$

for the line element on the unit sphere.

First we replace coordinates t and r by so-called *null coordinates*

$$v \;=\; t \;+\; r, \quad u \;=\; t \;-\; r.$$

Epithet *null* reflects the fact that u (retarded null coordinate) and v (advanced null coordinate) are constant along future and past null cones centred in $r = 0$, see figure 4.1. Using null coordinates we can write the physical metric as

$$\mathrm{d}\tilde{s}^2 \;=\; \mathrm{d}u\,\mathrm{d}v \;-\; \frac{1}{4}\,(v-u)^2\,\mathrm{d}\Sigma^2.$$

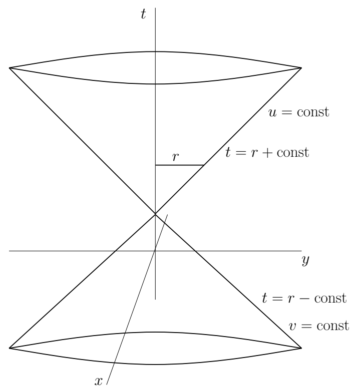

Figure 4.1: Null coordinate $u = t - r$ is constant along future null cones emerging from the origin $r = 0$.

Coordinates u and v satisfy condition $r = (v - u)/2 \geq 0$:

$$u, v \ \in \ (-\infty, \infty)\,, \quad v \ \geq \ u.$$

Now, infinity is the region with $r \to \infty$. Obvious difficulty is that infinity described in such a way is not really well-defined because the limit can be done with various coordinates fixed. Moreover, metric components diverge for $r \to \infty$ because of coefficient $r^2 = (v - u)^2/4$. Idea of Penrose's approach is, basically, to shrink entire space-time into a finite region so that infinity will be represented by points at a finite distance. Infinite range of u and v can be shrunk to a finite interval defining

$$U \ = \ \arctan u, \quad V \ = \ \arctan v,$$

so that

$$U, V \ \in \ \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad V \ \geq \ U.$$

Then we have

$$\mathrm{d}\tilde{s}^2 \ = \ \frac{1}{4\, \cos^2 U\, \cos^2 V} \left(4\, \mathrm{d}U\, \mathrm{d}V \ - \ \sin^2(U - V)\, \mathrm{d}\Sigma^2\right).$$

Thus, infinity is located at finite value $\pm\pi/2$ of U or V, but metric components are still singular there, for the coefficient standing at $\mathrm{d}\Sigma^2$ is not regular function. The crucial point is to introduce new, *unphysical metric* by transformation

$$\mathrm{d}s^2 \ = \ \Omega^2\, \mathrm{d}\tilde{s}^2$$

where

$$\Omega \ = \ 2\, \cos U\, \cos V,$$

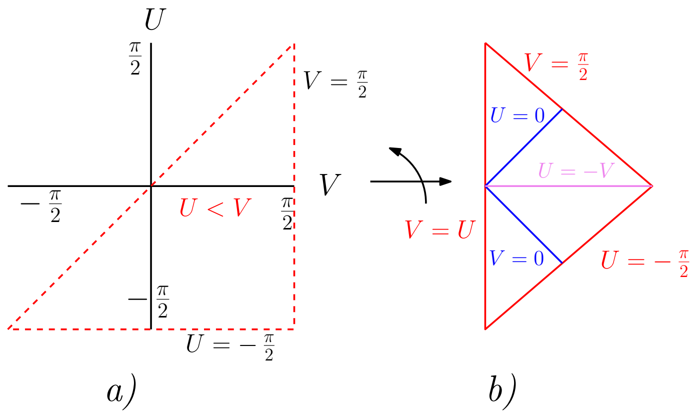

Figure 4.2: With angular coordinates suppressed, unphysical manifold can be visualised in $U - V$ plane. Coordinates U and V are constrained by condition $U \leq V$, so we are interested only in red dashed region in picture *a)*. Rotating the triangle by $+\pi/4$ we obtain triangle in picture *b*).

so that the unphysical metric reads

$$\mathrm{d}s^2 \;=\; 4\,\mathrm{d}U\,\mathrm{d}V \;-\; \sin^2(U-V)\,\mathrm{d}\Sigma^2. \tag{4.1}$$

This new metric is not a metric of Minkowski space-time but metric on some new manifold called *unphysical space-time*. In general, replacement of physical metric tensor $\tilde{g}_{\mu\nu}$ by unphysical metric

$$g_{\mu\nu} \;=\; \Omega^2\,\tilde{g}_{\mu\nu}$$

is a *conformal rescaling* with *conformal factor* Ω. Some general properties of this kind of transformation are discussed later.

Let us examine properties of manifold with metric (4.1). Since $r > 0$, coordinates U and V must satisfy $U - V \leq 0$, see figure 4.2. Region of interest is therefore only triangle given by $U \leq V$ sketched in figure *a)*. For plasticity it is more convenient to rotate this triangle by angle $\pi/4$, see figure *b)*. Picture so obtained is called *conformal* or *Penrose-Carter* diagram. Now we are going to relate geodesics in physical Minkowski space-time to world lines in unphysical space-time.

First, consider lines $r =$ constant in physical space-time. Such lines are geodesics of time-like observers sitting at position $r =$ constant. By definition of U and V we have

$$V \;=\; \arctan(t+r), \qquad U \;=\; \arctan(t-r).$$

Coordinates of the observer at time $t \to -\infty$ are $U = V = -\pi/2$ which corresponds to point i^- in figure 4.3. This point is called, for obvious reason, *past time-like infinity.* Later, at time $t = 0$ the coordinates U and V are given by

$$V = \arctan r, \quad U = -\arctan r,$$

so the point (U, V) lies somewhere on the line $U = -V$, e.g. point O in the figure. Finally, for $t \to \infty$ we have $U = V = \pi/2$, which corresponds to point i^+, so called *future time-like infinity.* Thus, any time-like observer at constant r will move from the past time-like infinity through point O to future time-like infinity. For any r we obtain curve from i^- to i^+ intersecting horizontal line $U = -V$. As r approaches infinity, one approaches point i^0 called *spatial infinity.* For special case $r = 0$ we get the vertical line connecting past and future time-like infinities (vertical red line in the figure).

In Minkowski space-time there exist radial space-like geodesics, i.e. geodesics given by $t =$ constant, which start on vertical line $r = 0$ and end at the space-like infinity i^0, see figure 4.3.

Finally we investigate radial null (light-like) geodesics. In ordinary space-time diagram, null geodesics are represented by lines forming angle $\pi/4$ with vertical time axis, see again figure 4.1. One of essential properties of conformal rescalings is preserving of angles while changing lengths, so even after rescaling of the metric, null geodesics form angle $\pi/4$ with axes. Indeed, future null geodesics are given by equation $t = r +$ constant or, equivalently, by

$$U = \text{constant},$$

and, similarly, past null geodesics are given by

$$V = \text{constant}.$$

In (U, V) plane, null geodesics are vertical or horizontal lines, but our conformal diagram is (U, V) plane rotated by $\pi/4$. All future null geodesics have their endpoints on the line $V = \pi/2$ called *future null infinity* and denoted $\mathcal{I}^+$ (pronounced "scri"). Similarly, all past null geodesics end on surface $U = -\pi/2$ called *past null infinity* $\mathcal{I}^-$. Obviously, the world line of a time-like observer can also reach $\mathcal{I}$ but the observer cannot stay at constant radius r (uniformly accelerated observers serve as a good example).

Thus, all geodesics in physical Minkowski space-time have their images in the unphysical space. These unphysical world lines are not geodesics anymore, with the exception of *null* geodesics. Conformal rescaling preservers not only angles but also null geodesics. Since null geodesics separate time-like and space-like events, we say that they define *causal structure* of the space-time. Conformal rescaling then preserves null cones and therefore also the causal structure of the space-time.

Consider a time-like geodesic $\gamma = \gamma(t)$ in physical space-time, such that $\gamma(0)$ is point in the interior of the space-time. For all $t \in \mathbb{R}$, $\gamma(t)$ is still the point of physical space-time. The unphysical image of the geodesic, however, will reach time-like infinity i^+ for finite value of t. Point i^+ is not part of physical space-time but it belongs to the unphysical space. Points i^0, $i^\pm$ and sets $\mathcal{I}^\pm$ constitute the boundary of Minkowski space-time, but they are not parts of it. We see that the unphysical manifold is larger than the original Minkowski space. We can even see what this unphysical manifold represents.

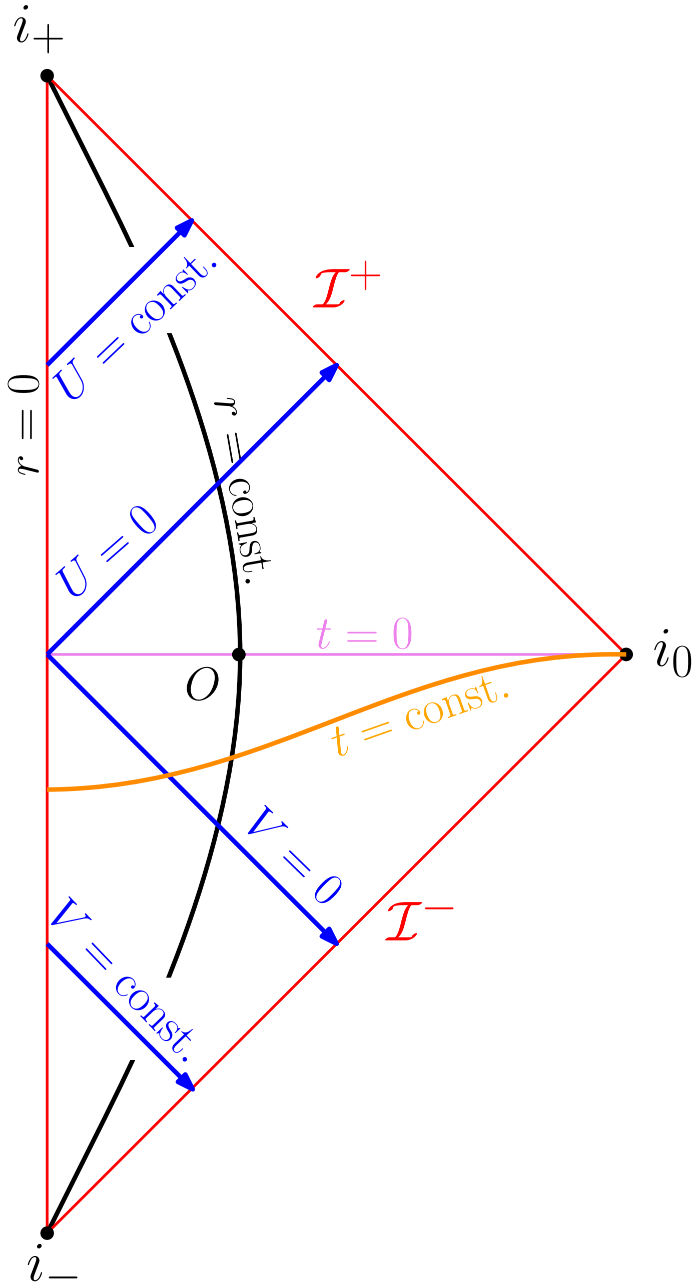

Figure 4.3: Conformal diagram of Minkowski space-time in 2-dimensional schematic picture.

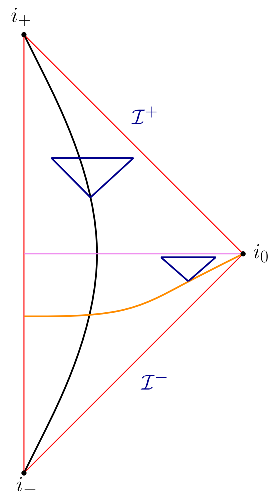

Figure 4.4: Null cones in the conformal diagram.

We have found that in double null coordinates U, V, θ, ϕ the unphysical metric is given by (4.1). We can proceed further and define coordinates

$$T = V + U, \quad R = V - U,$$

so unphysical metric acquires the form

$$\mathrm{d}s^2 = \mathrm{d}T^2 - \mathrm{d}R^2 - \sin^2 R \,\mathrm{d}\Sigma^2.$$

This is the metric of *Einstein's cylinder* $\mathcal{S}^3 \times \mathbb{R}$ representing static closed universe, see figure 4.5.

Now we are almost prepared to define the notion of asymptotic flatness. In order to elucidate still one technical detail, it is useful to dicuss properties of the conformal factor

$$\Omega = 2 \cos U \cos V$$

introduced above. First, notice that Ω vanishes at all points of infitity, i.e. at points lying on the boundary ∂M. Indeed, future null infinity $\mathcal{I}^+$ is given by equation $V = \pi/2$, at past null infinity we have $U = -\pi/2$, so that $\Omega = 0$ there. Second, *gradient* of Ω *does not vanish* at $\partial\Omega$. For the gradient $\mathrm{d}\Omega$ reads

$$\mathrm{d}\Omega = -2\,(\sin U \cos V \,\mathrm{d}U + \sin V \cos U \mathrm{d}V)\,,$$

at $\mathcal{I}^+$ it has the only nonvanishing component

$$\mathrm{d}\Omega|_{\mathcal{I}^+} = -2 \cos U \,\mathrm{d}V.$$

Gradient $\mathrm{d}V$ corresponds to vector orthogonal to hypersurfaces $V = \text{constant}$. Thus, $\mathrm{d}V$ is orthogonal to $\mathcal{I}^+ : V = \pi/2$. Similar statement holds for $\mathcal{I}^-$, figure 4.6.

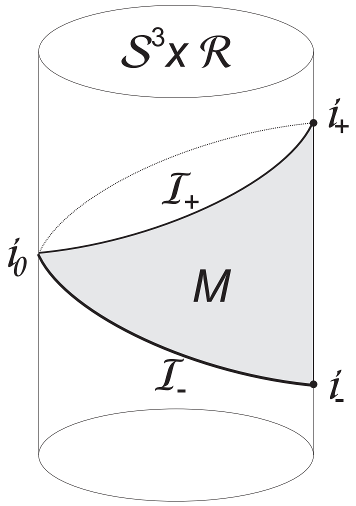

Figure 4.5: Minkowski space-time embedded into Einstein's cylinder $\mathcal{S}^3 \times \mathbb{R}$. Infinity of Minkowski space is represented by the boundary $\partial M = \mathcal{I}^+ \cup \mathcal{I}^- \cup i^0 \cup i^+ \cup i^-$. Adopted from [22].

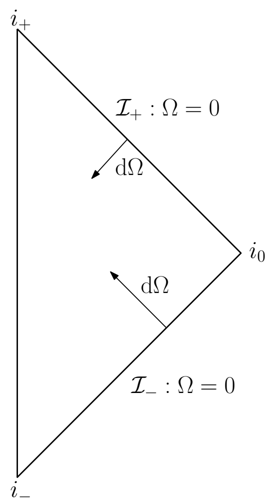

Figure 4.6: Conformal factor Ω vanishes at the boundary ∂M of M. Its gradient $\mathrm{d}\Omega$ is non-zero there and orthogonal to $\mathcal{I}^\pm$.

Let us summarize what we have done. We started with physical metric of Minkowski space-time and rewrote it in coordinates V, U, θ, ϕ instead of original spherical coordinates t, r, θ, ϕ. Coordinates U resp. V are constant along future resp. past null geodesics but they have finite range of values, namely they lie in interval $(-\pi/2, \pi/2)$. In these coordinates even the points at infinity have finite coordinates but the metric is singular at infinity. In order to regularize it we rescaled the metric by appropriate conformal factor. New, unphysical metric, is regular at infinity, but it cannot be regarded as a metric on the original Minkowski space-time. In this case, unphysical manifold is Einstein's cylinder into which Minkowski space-time is embedded. Points at infinity do not belong to Minkowski space-time but form its boundary. This boundary has non-trivial structure and can be divided into future/past null infinity $\mathcal{I}^{\pm}$, future/past time-like infinity $i^{\pm}$ and space-like infinity i^0.

4.2 Asymptotic flatness

We have seen how Minkowski space-time can be compactified and how points at infinity can be attached to it. Inspired by this, we can define the prototype of asymptotically flat space-time. Here we follow the definition presented in [21], but in slightly modified notation. Space-time $(\tilde{M}, \tilde{g})$, where $\tilde{M}$ is smooth manifold and $\tilde{g}$ smooth metric tensor on $\tilde{M}$, is said to be *asymptotically simple* if there exists unphysical space-time (M, g), diffeomorphism $\psi : \tilde{M} \mapsto M$ and function $\Omega : M \mapsto \mathbb{R}$ such that:

1. $\psi(\tilde{M})$ is an open submanifold of M,
2. $\Omega \;=\; 0$ on boundary $I = \partial\psi(\tilde{M})$,
3. $\mathrm{d}\Omega \neq 0$ on I,
4. metric g is conformally rescaled physical metric $\tilde{g}$, i.e. $g = \Omega^2\, \psi_* \tilde{g}$,
5. for any geodesic $\tilde{\gamma}$ in M, geodesic $\gamma \;=\; \psi_* \tilde{\gamma}$ acquires future and past endpoint on I.

If physical Ricci tensor vanishes near I, space-time is *asymptotically flat*, which expresses that there is no matter at infinity. In fact, this requirement can be weakened: there can be matter near infinity, but components of Ricci tensor must fall off sufficiently fast.

Relation between $\tilde{M}$ and M is sketched in figure 4.7. According to requirement 1, physical space-time $\tilde{M}$ can be embedded by diffeomorphism ψ into larger unphysical manifold M. Image of $\tilde{M}$ is denoted by $\psi(\tilde{M})$, but since $\tilde{M}$ and $\psi(\tilde{M})$ are diffeomorphic, we do not have to distinguish between them. Thus, for simplicity, by $\tilde{M}$ we mean either $\tilde{M}$ or its image $\psi(\tilde{M})$.

In M, space-time $\tilde{M}$ has boundary denoted by I. Physical space-time together with its boundary is called *asymptote*. Requirement 2 states that the boundary I is given by equation $\Omega = 0$, where Ω is a real function defined on the unphysical manifold. Function Ω can be pulled-back by diffeomorphism ψ to obtain function

$$\tilde{\Omega} \;=\; \psi^* \Omega \;=\; \Omega \circ \psi^{-1} : \tilde{M} \mapsto \mathbb{R}$$

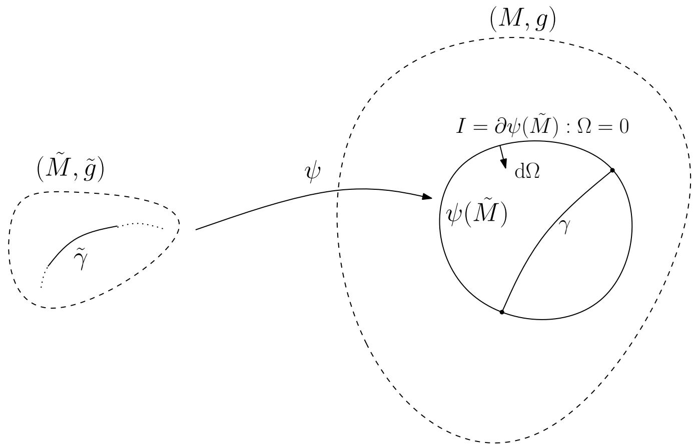

Figure 4.7: Asymptotically simple space-time $\tilde{M}$ embedded into unphysical manifold M via diffeomorphism ψ. Surface $\Omega = 0$ constitutes the boundary of $\tilde{M}$ in M.

on $\tilde{M}$. But again, we will not distinguish between $\tilde{\Omega}$ and Ω. By requirement 3, gradient of Ω is strictly non-vanishing on the boundary I. Requirements 2 and 3 together ensure that conformal factor Ω has asymptotic behaviour

$$\Omega \;=\; \mathcal{O}\left(\frac{1}{r}\right), \quad r \to \infty,$$

where r is suitable "radial distance".

By definition, unphysical manifold M is equipped with its own metric g. To compare g and physical metric $\tilde{g}$ we must first push-forward metric $\tilde{g}$ using diffeomorphism ψ. Again, for the sake of simplicity, by $\tilde{g}$ we mean either $\tilde{g}$ or $\psi_* \tilde{g}$. Requirement 4 can be then written simply as

$$g \;=\; \Omega^2\, \tilde{g},$$

so that unphysical metric is equal to physical metric conformally rescaled by factor Ω.

Finally, any geodesic $\tilde{\gamma}$ can be lifted up to curve $\gamma = \psi \circ \tilde{\gamma}$ in the unphysical space-time. As mentioned above, γ will be geodesic only if $\tilde{\gamma}$ is *null*. If geodesic $\tilde{\gamma}$ can be maximally extended, intuitively, it should reach both past and future infinity. According to the last requirement 5, curves γ have endpoint on the boundary I, so that they reach infinity for finite values of their parameters.

The last requirement can be satisfied only if each geodesic in the physical space-time can be maximally extended. We have seen on the example of Minkowski spacetime, that requirement 5 is fulfilled. We cannot expect the same for Schwarzschild space-time, however, as there exist closed null geodesics. On the other hand, we would like to include Schwarzschild space-time into our definition of asymptotic flatness. Since we are interested in asymptotic properties, it is enough if our definition will be concerned with the neighbourhood of infinity.

Space-time $(\tilde{M}, \tilde{g})$ is *weakly asymptotically simple*, if there exists asymptotically simple space-time $(\tilde{N}, \tilde{h})$ with asymptote (M, g) and diffeomorphism $\psi : N \mapsto M$, and neighbourhood $V \subset M$ of $\partial\psi(\tilde{M})$ of infinity, such that $\tilde{V} = \psi^{-1}(V)$ is isometric to some open subspace $\tilde{U} \subset \tilde{M}$.

This definition deserves a brief explanation (which seems to be not present in the literature). We have a physical space-time $(\tilde{M}, \tilde{g})$ which is not asymptotically simple, but some neighbourhood of its infinity can be embedded into a larger manifold. Thus we demand that there exists asymptotically simple space-time $(\tilde{N}, \tilde{h})$, which can be embedded into manifold (M, g) by diffeomorphism ψ. In M, space-time $\tilde{N}$ has well-defined boundary $I = \psi(\tilde{N})$. Consider some open neighbourhood $V \subset N$ of I, such that $I \cap V = 0$, i.e. the boundary itself is not included in V. Subset V can be dragged back to $\tilde{N}$ defining $\tilde{V} = \psi^{-1}(V)$. Subset $\tilde{V}$ is then neighbourhood of infinity of $\tilde{N}$. If there exists open subset $\tilde{U} \subset \tilde{M}$, such that $\tilde{U}$ and $\tilde{V}$ are isometric, space-time $\tilde{M}$ is weakly asymptotically simple. By this procedure we excluded interior of $\tilde{M}$, where we can expect problems with geodesics.

From now, by *asymptotically flat space-time* we mean weakly asymptotically flat space-time, whose Ricci tensor vanishes on the boundary I. It can be nonzero near I, but must approach zero sufficiently fast. It can be shown that boundary I has the same structure as in the case of Minkowski space-time, i.e. it consists of $\mathcal{I}^{\pm}, i^{\pm}$ and i^0. Each point of I in the conformal diagram 4.3 is a sphere $\mathcal{S}^2$. Space-like and time-like infinities in the conformal diagram are represented by single points, while null infinity is represented by lines. In our considerations we restrict to null infinity $\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^-$. In vacuum, sets $\mathcal{I}^{\pm}$ have topology $\mathbb{R} \times \mathcal{S}^2$.

Thus, we have explained how asymptotic flatness can be defined geometrically. The definition is manifestly coordinate-independent and the notion of "infinity" has clear meaning: it is a hypersurface in the unphysical manifold and it is given by equation $\Omega = 0$. As mentioned in the beginning of this chapter, in older approaches one had to define suitable radial coordinate r and then construct expansions of geometrical quantitities near infinity. In the Penrose approach, one can analyze asymptotic properties of the space-times by local techniques, because in the unphysical manifold, the "physical infinity" is located at a finite distance. Moreover, conformal factor Ω can serve as one of the coordinates and the radial distance $\tilde{r}$ (in physical space-time) can be chosen in such a way, that

$$\Omega \sim \frac{1}{\tilde{r}}$$

near infinity. So, instead of expansions in $\tilde{r}$ for $\tilde{r} \to \infty$ one can use expansions in Ω for $\Omega \to 0$, which is more convenient. Geometric formulation of asymptotic flatness allows us to raise questions concerning geometry and topology of the infinity. Understanding these properties of infinity provides us with the clearer picture of asymptotic properties of the space-times, like the peeling property, relation between ADM and the Bondi mass, and a lot of others.

In the last section we briefly discuss another advantage of this approach: regularity of the unphysical quantities at infinity.

4.3 Conformal field equations

In previous section we introduced the geometric definition of asymptotic flatness. As we explained, asymptotically flat space-times model isolated systems. Study of isolated systems, i.e. study of asymptotic properties of space-times, is useful in problems of gravitational (or other) radiation. To understand asymptotic properties of radiation we do not have to care about the sources and about the mechanism of radiation. Even the basic example, binary system, is too difficult to be understood in detail. Nevertheless, the space-time describing binary system is expected to be asymptotically flat and so we can study general properties of radiation produced by the system at large distances. This model seems to be appropriate, if we are interested in possibility of gravitational wave detection.

In the conformal picture, large distances from source correspond to points in the neighbourhood of $\mathcal{I}$. Physical metric, however, is singular on $\mathcal{I}$, recall physical Minkowski metric containing r^2. By contrast, unphysical, conformally rescaled metric is regular on $\mathcal{I}$. If we want to study gravitational field near $\mathcal{I}$, we have to translate all physical quantities and equations of interest to unphysical ones. By this procedure we obtain set of regular equations in the unphysical space-time. In the following we briefly explain, how these equations can be derived and what are unknown variables. A lot of fundamental theorems about conformal field equations and existence of their solutions have been found by Friedrich. For a comprehensive review see Friedrich [10] or references in Frauendiener[7] and Bičák, Scholtz, Tod [2, 3].

Chapter 5

Spinors

Before we discuss the existence of periodic solutions, which is the main part of this thesis, it is necessary to introduce spinorial techniques in general relativity. Spinor formalism connected with Newman-Penrose formalism is powerful tool for analysis of radiation, asymptotic properties of fields, but also black holes. The Newman-Penrose formalism, which is best formulated and understood in terms of spinors, is useful for the study of algebraically special fields, perturbations of black holes, etc. In this chapter we will not go into details, we just introduce basic terms and notation, spinor decomposition of tensors, and covariant derivative. The Newman-Penrose null tetrad is also introduced.

5.1 Abstract index notation

Abstract index notation developed by Penrose allows us to exploit advantages of standard index notation but without referring to a particular basis. In standard index notation tensors are represented by their components with respect to given coordinates or vector basis (tetrad). For example, second rank mixed tensor t is represented by components

$$t^{\mu}{}_{\nu}$$

with respect to coordinates x^{μ}. Advantage of this notation is that usual tensor operations like contraction or (anti)symmetrization can be written in compact form, e.g. the trace of tensor t is

$$\operatorname{Tr} t \;=\; t^{\mu}{}_{\mu}.$$

If we regard tensor t on vector space L as linear mapping

$$t : L^{*} \times L \mapsto \mathbb{R},$$

and if e_{μ} and e^{μ} are basis vectors and covectors, the trace of tensor t is

$$\operatorname{Tr} t \;=\; t(e^{\mu}, e_{\mu}).$$

We can see that index notation is shorter than the usual geometrical one.

It has also drawbacks, however. Geometrical definition of a tensor as a linear mapping is coordinate independent. The tensor itself is geometrical object which maps vectors and covectors to real numbers. Its properties do not depend on the choice of coordinate system and it can be defined globally on entire manifold. By contrast, coordinates in general cannot be defined globally and the manifold must be covered by several coordinate charts. Thus, the set of components $t^\mu{}_\nu$ is not well-defined globally. If we arrive at any statemenent about tensor t, we must verify that this statement holds in arbitrary coordinate system and that the components transform correctly on the overlaps of different coordinate charts.

Penrose's abstract notation is an original solution to this problem. Let L be a vector space and L^* its dual space. Vector $v \in L$ is linear mapping

$$v \;:\; L^* \mapsto \mathbb{R},$$

while covector $\omega \in L^*$ is linear mapping

$$\omega \;:\; L \mapsto \mathbb{R}.$$

Acting of covector on vector in index notation is written as a scalar product

$$\omega(v) \;=\; \omega_\mu \, v^\mu.$$

Now admit v and ω to be (co)vector *fields* on manifold. Then the left hand side of the last equation is globally well-defined but the right hand side is valid only in given coordinate chart. We would like to use the notation on the right hand side, but in such a way that it is defined globally.

The idea is to label tensors by indices, but these indices will not mean the components of tensors. We define the set of labels (called *abstract indices*) as infinite set of symbols

$$Q \;=\; \{a, b, c, \ldots\}\,.$$

Here a is not an index which acquires values $0, 1, 2, 3$ as in index notation. It is merely a symbol. Now we introduce isomorphic copies of vector space L

$$L^a, L^b, L^c, \ldots$$

These vector spaces are *different* but isomorphic. Elements of $L^q, q \in Q$ will carry the same label as vector space, e.g.

$$v^a \in L^a, \;\; v^a \notin L^b, \;\; v^b \in L^b.$$

Similarly we introduce isomorphic copies of dual space L^* and label them with lower indices:

$$L_a, L_b, L_c, \ldots,$$

so that L_a is dual of L^a, etc. Elements of $L_q, q \in Q$ will again carry corresponding label, e.g.

$$\omega_a \in L_a, \omega_b \notin L_a.$$

Thus, ω_a is not a component of covector ω with respect to some basis, ω_a is the covector itself. Index a just indicates vector space whose element ω_a is.

Covector ω_a is a linear mapping

$$\omega_a \;:\; L^a \mapsto \mathbb{R},$$

i.e. ω_a can act only on elements of L^a. So, if $v^a \in L^a$ is vector, we can insert it into covector ω_a:

$$\omega_a(v^a) \in \mathbb{R}.$$

Omitting brackets we can write

$$\omega_a v^a \in \mathbb{R}.$$

This expression is globally well-defined because it is the same expression as in the usual geometrical approach. But since (co)vectors are labelled by indices, it also resembles usual index notation. It is formally the same but has a different meaning.

Notice that there is no ambiguity in this approach. Expression $\omega_a v^b$ cannot be understood as covector ω_a acting on vector v^b, because v^b is not the element of correct vector space L^a. Only covector ω_b can act on v^b. So, expression $\omega_a v^b$ must be treated as tensor product of two vectors from two different vector spaces, which is well-defined operation. In a similar manner all expressions in the standard index notation can be converted to abstract index notation, while staying formally unchanged.

From now we employ abstract index notation. Indices $a, b, c, \dots$ will always mean labels, not components of tensors. On the other hand, indices $\mu, \nu, \alpha, \beta, \dots$ will denote components of tensors with respect to some coordinates x^μ. Components with respect to a tetrad will be denoted by bold indices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \dots$.

5.2 Spin basis and spin dyad

In this section we introduce two-component spinors. Let S be complex two-dimensional vector space with symplectic form

$$[\cdot,\ \cdot] \,:\, S \times S \mapsto \mathbb{C}.$$

Symplectic form is by definition bilinear, antisymetric and non-degenerate form. Elements of S will be called *spinors* and symplectic form defines inner product on the space of spinors. In S we can choose basis spinors o and ι normalized by condition

$$[o, \iota] \;=\; 1.$$

Pair (o, ι) is called *spin basis*. Arbitrary spinor $\tau \in S$ is then a linear combination of basis spinors,

$$\tau \;=\; \tau^{\mathbf{0}}\, o \;+\; \tau^{\mathbf{1}}\, \iota,$$

where $\tau^{\mathbf{0}}$ and $\tau^{\mathbf{1}}$ are components of τ with respect to basis (o, ι). Components of the symplectic form with respect to a given spin basis read

$$\epsilon_{\mathbf{00}} \;=\; [o, o] \;=\; 0, \quad \epsilon_{\mathbf{01}} \;=\; -\,\epsilon_{\mathbf{10}} \;=\; [o, \iota] \;=\; 1, \quad \epsilon_{\mathbf{11}} \;=\; [o, \iota] \;=\; 0.$$

Symplectic form has a role similar to that of the metric tensor in Riemann geometry. Namely, it defines canonical isomorphism of spaces S and S^*, where

S^* means dual space. Employing Penrose abstract index notation, we define isomorphic copies of S labelled by indices $A, B, C, \ldots$:

$$S^A,\ S^B,\ S^C, \ldots .$$

Corresponding dual spaces are

$$S_A,\ S_B,\ S_C, \ldots\ .$$

Since S is complex space, we have to define also complex conjugated spaces

$$S^{A'},\ S^{B'},\ S^{C'}, \ldots$$

and complex conjugated dual spaces

$$S_{A'},\ S_{B'},\ S_{C'}, \ldots$$

Thus, τ^A is element of S^A, $\tau_{C'}$ is element of $S_{C'}$, etc. Higher valence spinors are elements of tensor products of spinor spaces, e.g. $\tau_A{}^B$ is element of $S_A \otimes S^B = S_A{}^B$. If $\tau_A \in S_A$, we can define complex conjugated spinor

$$\bar{\tau}_{A'} \ \in \ S_{A'}.$$

Symplectic form ϵ_{AB} is element of $S_{[AB]}$ where square brackets denote antisymmetrization. Since S_A is two dimensional, space $S_{[AB]}$ is one dimensional, i.e. every antisymmetric two-form is proportional to ϵ_{AB}. Correspondence between spinors and dual spinors is realized by relations

$$\tau_A \ = \ \epsilon_{BA}\, \tau^B, \quad \tau^A \ = \ \epsilon^{AB}\, \tau_B,$$

where $\epsilon^{AB} \in S^{[AB]}$ is defined by

$$\epsilon_{AB}\, \epsilon^{AB} \ = \ \dim S \ = \ 2.$$

For given spin basis, the symplectic form can be decomposed as follows:

$$\epsilon_{AB} \ = \ o_A\, \iota_B \ - \ o_B\, \iota_A.$$

Next we define *spin dyad* as an ordered pair of basis spinors:

$$\epsilon_{\boldsymbol{A}}{}^A \ = \ (o^A, \iota^A), \quad \boldsymbol{A} \ = \ 0, 1. \tag{5.1}$$

Dual dyad $\epsilon^{\boldsymbol{A}}{}_A$ is defined by normalization condition

$$\epsilon^{\boldsymbol{A}}{}_A\, \epsilon_{\boldsymbol{A}}{}^B \ = \ \epsilon_A{}^B,$$

which can be achieved by the choice

$$\epsilon^{\boldsymbol{A}}{}_A \ = \ (-\,\iota_A, o_A).$$

Arbitrary spinor can be then expressed as a linear combination of elements of the dyad, e.g.

$$\tau^A \ = \ \tau^{\boldsymbol{A}}\, \epsilon_{\boldsymbol{A}}{}^A, \quad \tau_A \ = \ \tau_{\boldsymbol{A}}\, \epsilon^{\boldsymbol{A}}{}_A.$$

5.3 Spinors and tensors

In this section we consider transformations of spin basis (o^A, ι^A). General linear transformation of spinor τ^A reads

$$\tau^A \;\mapsto\; l^A{}_B\,\tau^B,$$

where $l^A{}_B$ is arbitrary 2×2 complex matrix. Dual spinor τ_A then transforms according to formula

$$\tau_A \;\mapsto\; -\,l_A{}^B\,\tau_B.$$

The only additional structure given in the spinor space is symplectic form represented by spinor ϵ_{AB}. It is thus natural to restrict to transformations preserving this symplectic form or, equivalently, transformations preserving inner product. Under general linear transformation symplectic form transforms as

$$\epsilon_{AB} \;\mapsto\; l_A{}^C\,l_B{}^D\,\epsilon_{CD}.$$

Transformations preserving ϵ_{AB} are called *symplectic* or *spin transformations* and they are defined by condition

$$l_A{}^C\,l_B{}^D\,\epsilon_{CD} \;=\; \epsilon_{AB}.$$

This equation immediately implies

$$\left|\det l_A{}^B\right| \;=\; 1.$$

Matrix $l_A{}^B$ with this property is called *spin matrix.*

There is an obvious analogy between spin tranformation in the space of spinors and Lorentz transformations in tangent vector space of the space-time. Let $\mathcal{T}$ be a tangent space at some space-time point P. Then, on $\mathcal{T}$ we have metric tensor g_{ab} which defines inner (scalar) product on $\mathcal{T}$. Arbitrary transformation of tangent space can be written in the form

$$k^a \;\mapsto\; \Lambda^a{}_b\,k^b, \quad k^a \in \mathcal{T}^a.$$

Next we define the Lorentz transformation as transformation preserving inner product induced by metric tensor, i.e. transformation satisfying

$$\Lambda^c{}_a\,g_{cd}\,\Lambda^d{}_b \;=\; g_{ab},$$

which also implies

$$\left|\det \Lambda^a{}_b\right| \;=\; 1.$$

Thus, metric tensor defines an inner product on the tangent space and it is preserved by (unimodular) Lorentz transformations. Symplectic form defines an inner product on the spin space and it is preserved by (unimodular) spin transformations. Basic difference between symplectic form and metric tensor is in their symmetry: the former is antisymmetric, the latter is symmetric. Let us see, how far we can proceed with this analogy.

Spinor τ is said to be *Hermitian*, if it is equal to its complex conjugate, i.e. if $\bar\tau = \tau$. Consider $\tau^A \in S^A$. Then its complex conjugate is spinor $\tau^{A'} \in S^{A'}$. Resulting complex conjugated spinor belongs to different vector space, so it cannot be even compared with the original one. It is therefore senseless to talk about Hermitian univalent spinors. To get Hermitian spinor, we need at least bivalent spinor

$$\tau^{AA'} \;\in\; S^{AA'},$$

so that its complex conjugate is

$$\overline{\tau^{AA'}} \;=\; \bar{\tau}^{A'A} \;=\; \bar{\tau}^{AA'} \;\in\; S^{AA'}.$$

This is again element of the same vector space $S^{AA'}$ and thus both spinors can be compared. If they are equal, $\tau^{AA'}$ is Hermitian. We can see that on matrix level, definition of Hermitian spinor agrees with definition of Hermitian matrix, i.e. spinor is Hermitian, if its matrix is invariant under transposition and complex conjugation. Hermitian 2×2 matrix X must have real diagonal elements, so they do not change under complex conjugation, and complex conjugated non-diagonal elements:

$$A \;=\; \begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$. Thus, space of Hermitian 2×2 matrices is 4-dimensional, and so is the space of Hermitian spinors. It has the same dimensionality as the tangent vector space. Moreover, we know that Lorentz transformations form 6-parametric Lorentz group. What is dimension of group of spin transformations? Spin matrix $l_A{}^B$ has 4 complex elements, i.e. 8 real components. Condition of unimodularity decreases degrees of freedom by two (one complex equation for the determinant). Therefore, group of spin transformation is also 6-dimensional. This suggests that spinor space $S^{AA'}$ of Hermitian spinors can be identified with tangent vector space, and the space of spin transformations can be identified with Lorentz group.

Identification of spinors and tensors can be realized in the following way. Let $x^\mu = (t, x, y, z)$ be arbitrary space-time vector, let $\sigma_\mu^{AA'}$ be standard Pauli matrices

$$\sigma_0^{AA'} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1^{AA'} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.2}$$

$$\sigma_2^{AA'} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3^{AA'} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.3}$$

Then we can define spinor

$$x^{AA'} \;=\; x^\mu\,\sigma_\mu^{AA'} \;=\; \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix},$$

which is a *spinor equivalent* of world vector x^a. Thus, each pair of primed and unprimed spinor indices (e.g. AA') is equivalent to one ordinary tensor index (e.g. a).

We have seen the analogy between symplectic form and metric tensor. Now we know that spinor equivalent of g_{ab} must be a spinor $g_{ABA'B'}$ constructed from ϵ_{AB}. Since g_{ab} must be real, while ϵ_{AB} is not, the obvious choice is

$$g_{ABA'B'} \;=\; \epsilon_{AB}\,\epsilon_{A'B'}.$$

Direct calculation shows

$$g_{ABA'B'}\,\sigma_a^{AA'}\,\sigma_b^{BB'} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which justifies our identification.

In Penrose's abstract index notation, indices at tensorial or spinorial objects do not represent the components of the objects, they merely indicate vector space where a particular object belongs to. In this formalism, one useful convention can be introduced. Let us identify symbols

$$a \text{ and } AA', \quad b \text{ and } BB', \quad c \text{ and } CC', \; \ldots.$$

Then, spinor $x^{AA'}$ is an element of $S^{AA'}$, but according to convention just introduced we can write

$$x^{AA'} \;=\; x^a \;\in\; S^a \;=\; S^{AA'}.$$

Symbols a and AA' are identified, so they represent two different labels for the same vector space $S^{AA'} = S^a$. This is consistent with the fact that $x^{AA'}$ is the spinor equivalent of x^a. Details can be found in [18]. In this convention, we can write simply

$$g_{ab} \;=\; \epsilon_{AB}\,\epsilon_{A'B'}.$$

Here, on both sides of the equality there are spinors, elements of $S_{AB}\otimes S_{A'B'} = S_{ab}$, but we use different labels on the left and right hand side.

5.4 Newman-Penrose null tetrad

Consider space $S^{AA'} = S^a$. Its elements are spinors equivalent to some (possibly complex) space-time vectors. Real space-time vectors are those represented by Hermitian spinors. Let (o^A, ι^A) be given spin basis of the original spinor space S^A. Basis (tetrad) of S^a is then naturally

$$e^a_{\boldsymbol{a}} \;=\; \epsilon^A{}_{\boldsymbol{A}}\,\epsilon^{A'}{}_{\boldsymbol{A'}},$$

where $\epsilon^A{}_{\boldsymbol{A}}$ is the spinor dyad. Explicitly,

$$e^a_{\mathbf{0}} \;=\; o^A\,\bar{o}^{A'}, \quad e^a_{\mathbf{1}} \;=\; \iota^A\,\bar{\iota}^{A'}, \quad e^a_{\mathbf{2}} \;=\; o^A\,\bar{\iota}^{A'}, \quad e^a_{\mathbf{3}} \;=\; \iota^A\,\bar{o}^{A'}.$$

In the notation introduced by Newman and Penrose [15], these vectors are denoted as

$$l^a \;=\; e^a_{\mathbf{0}}, \quad n^a \;=\; e^a_{\mathbf{1}}, \quad m^a \;=\; e^a_{\mathbf{2}}, \quad \bar{m}^a \;=\; e^a_{\mathbf{3}},$$

where it is explicitly emphasized that $e^a_{\mathbf{2}}$ and $e^a_{\mathbf{3}}$ are mutually complex conjugate. Notice that l^a and n^a are real, while m^a is necessarily complex. Vectors $e^a_{\boldsymbol{a}}$ form the basis of *complexified* tangent space and arbitrary complex space-time vector can be written in the form

$$k^a \;=\; k^{\boldsymbol{a}}\,e^a_{\boldsymbol{a}} \;=\; k^{\mathbf{0}}\,l^a \;+\; k^{\mathbf{1}}\,n^a \;+\; k^{\mathbf{2}}\,m^a \;+\; k^{\mathbf{3}}\,\bar{m}^a,$$

where tetrad components $k^{\boldsymbol{a}}$ are complex numbers. For *real* vectors k^a, however, relations

$$k^{\mathbf{0}},\ k^{\mathbf{1}} \in \mathbb{R}, \quad k^{\mathbf{2}} \ =\ \overline{k^{\mathbf{3}}} \in \mathbb{C},$$

must hold. Striking feature of the tetrad induced by the spin basis is that all its constituents are null vectors. Indeed, by direct calculation one can show the following:

$$l^a\, l_a \ =\ n^a\, n_a \ =\ m^a\, m_a \ =\ 0, \quad l^a\, n_a \ =\ -\, m^a\, \bar{m}_a \ =\ 1.$$

For this reason, tetrad $e^a_{\boldsymbol{a}}$ is called the Newman-Penrose *null tetrad*. Components of the metric tensor with respect to the null tetrad read

$$g_{\boldsymbol{ab}} \ =\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{5.4}$$

Finally, it is possible to show that the spinor equivalent of the volume form ϵ_{abcd} is

$$\epsilon_{abcd} \ =\ i\, (\epsilon_{AB}\, \epsilon_{CD}\, \epsilon_{A'C'}\, \epsilon_{B'D'} \ -\ \epsilon_{A'B'}\, \epsilon_{C'D'}\, \epsilon_{AC}\, \epsilon_{BD})\,. \tag{5.5}$$

5.5 Decomposition of spinors

One of the most important properties of spinor-vector identification is that any real null space-time vector k^a can be written in the form

$$k^a \ =\ \kappa^A\, \bar{\kappa}^{A'},$$

where κ^A is some complex univalent spinor. Any vector k^a of this form is necessarily null as follows form the antisymmetry of symplectic form ϵ_{AB}:

$$\epsilon_{AB}\, \kappa^A\, \kappa^B \ =\ 0.$$

It can be easily shown that also the converse is true: for any reall null vector k^a there exists κ^A. Notice, however, the non-uniqueness of κ^A, for the transformation $\kappa^A \mapsto e^{i\theta}\kappa^A$ leaves k^a invariant.

Another important property follows from the fact already mentioned: space of anti-symmetric bivalent spinors is one-dimensional, because any such spinor must be proportional to the symplectic form. Factor of proportionality is related to the trace of the spinor. Let $\tau_{AB} \in S_{[AB]}$ be an antisymmetric spinor. With respect to an arbitrary basis, it must be represented by antisymmetric 2×2 matrix, which has only one independent non-zero component:

$$\tau_{\boldsymbol{AB}} \ =\ \begin{pmatrix} 0 & \tau_{\mathbf{01}} \\ -\tau_{\mathbf{01}} & 0 \end{pmatrix} \ =\ \tau_{\mathbf{01}}\, \epsilon_{\boldsymbol{AB}}.$$

Thus, in arbitrary basis, matrix of τ_{AB} is proportional to the Levi-Civita symbol. In the abstract index notation we can write

$$\tau_{AB} \ =\ \lambda\, \epsilon_{AB}.$$

Factor λ can be obtained by contraction of the equation with spinor ϵ^{AB}:

$$\epsilon^{AB}\,\tau_{AB} \;=\; 2\,\lambda,$$

so that

$$\lambda \;=\; \frac{1}{2}\,\tau_A{}^A.$$

Consider now bivalent spinor τ_{AB} which is neither symmetric nor antisymmetric. It still can be, however, decomposed into symmetric and antisymmetric parts:

$$\tau_{AB} \;=\; \tau_{(AB)} \;+\; \tau_{[AB]} \;=\; \tau_{(AB)} \;+\; \frac{1}{2}\,\tau_X{}^X\,\epsilon_{AB},$$

where we have used the fact that antisymmetric part $\tau_{[AB]}$ must be proportional to ϵ_{AB}. Here we can see that arbitrary spinor τ_{AB} can be decomposed into

- traceless symmetric part $\tau_{(AB)}$;
- antisymmetric part proportional to ϵ_{AB} carrying the trace of τ_{AB}.

For this reason, it is often said that in spinor formalism only symmetric spinors matter: each spinor can be decomposed into irreducible symmetric part and part proportional to symplectic form.

Let us see another important example. Electromagnetic field is described by Faraday's two-form F_{ab}. Its spinor equivalent $F_{ABA'B'}$ can be decomposed using the same rules applied separately on indices AB and $A'B'$:

$$\begin{aligned} F_{ab} \;&=\; F_{(AB)(A'B')} \;+\; \frac{1}{4}\,\epsilon_{AB}\,\epsilon_{A'B'}\,F_X{}^X{}_{X'}{}^{X'} \\ &+\; \frac{1}{2}\,\epsilon_{AB}F_X{}^X{}_{(A'B')} \;+\; \frac{1}{2}\,\epsilon_{A'B'}\,F_{(AB)X'}{}^{X'}. \end{aligned}$$

First two terms on the right-hand side form totally symmetric part of F_{ab} (first term is manifestly symmetric, while second term changes the sign twice under $a \leftrightarrow b$, i.e. does not change the sign). Remaining terms form totally antisymmetric part of F_{ab}. Since F_{ab} in the case of electromagnetism is *antisymmetric*, the symmetric part must be zero. Defining

$$\phi_{AB} \;=\; \frac{1}{2}\,F_{(AB)X'}{}^{X'},$$

we can thus write electromagnetic tensor as

$$F_{ab} \;=\; \phi_{AB}\,\epsilon_{A'B,} \;+\; \bar{\phi}_{A'B'}\,\epsilon_{AB}.$$

We can see that electromagnetic field can be equivalently described by symmetric(!) bivalent spinor ϕ_{AB}.

Generally, by decomposition of any tensor into its spinorial symmetric and antisymmetric parts, we obtain equivalent representation of tensor in terms of some spinors. If the tensor, however, has some algebraic symmetry, the number of spinors is reduced, so that the representation of the tensor is simpler in spinor terms. Usually, spinor parts of tensor are simpler objects and they also satisfy simpler equations.

5.6 Covariant derivative

The notion of tensorial covariant derivative naturally extends to spinor objects. Spinor equivalent of ∇_a is, using Penrose's abstract index notation, $\nabla_{AA'}$. Notice that in the spinor formalism, covariant derivative has two indices, one primed and one unprimed. That is because covariant derivative measures the rate of change of tensor(spinor) field in the direction of some vector field. Arbitrary spinor τ^A does not define a direction and only null vector $\tau^A\bar{\tau}^{A'}$ can be constructed from τ^A. Covariant derivative can be defined only with respect to two-valent spinor $\tau^{AA'}$, which is simply a spinor equivalent of some vector τ^a.

On the other hand, covariant derivative *can* be applied to an arbitrary univalent spinors. Formally, spinor covariant derivative can be introduced axiomatically in a way similar to that of defining tensor covariant derivative, see e.g. [18]. Here we adopt the definition introduced in [21] with one modification – using abstract indices. Covariant derivative $\nabla_{AA'}$ maps arbitrary spinor space S^β to spinor space $S^\alpha_{AA'}$. Here β is a multiindex containing both upper and lower indices indicating the valence of spinor on which the derivative acts. For example, if the derivative acts on spinor τ^B, then $\beta = B$ and $\nabla_{AA'}$ maps space S^B to space $S^B{}_{AA'}$. Complex conjugation is $\beta' = B'$ in our case.

Let θ^β, ϕ^β and ψ^β be spinors of the same valence, elements of S^β, let f be any scalar function. *Spinor covariant derivative* is mapping

$$\nabla_{AA'} \; : \; S^\beta \; \mapsto \; S^\beta{}_{AA'}$$

satisfying following properties:

1. $\nabla_{AA'}\left(\theta^\beta \; + \; \phi^\beta\right) \; = \; \nabla_{AA'}\theta^\beta \; + \; \nabla_{AA'}\phi^\beta$ (additivity)
2. $\nabla_{AA'}\left(\theta^\beta\,\phi^\gamma\right) \; = \; \theta^\beta\,\nabla_{AA'}\phi^\gamma \; + \; \phi^\gamma\,\nabla_{AA'}\theta^\beta$ (Leibniz rule)
3. $\psi^\beta \; = \; \nabla_{AA'}\theta^\beta$ implies $\bar{\psi}^{\beta'} \; = \; \nabla_{AA'}\bar{\theta}^{\beta'}$ (reality of $\nabla_{AA'}$)
4. $\nabla_{AA'}\epsilon_{XY} \; = \; 0$ (compatibility with metric)
5. $\nabla_{AA'}$ commutes with any index substitution not involving A and A'
6. $\nabla_{AA'}\nabla_{BB'}f \; = \; \nabla_{BB'}\nabla_{AA'}f$ (vanishing of torsion)
7. for any derivative D of degree zero on spinor fields[1] there exists spinor field $\tau^{AA'}$ such that $D\theta^\beta = \tau^{AA'}\nabla_{AA'}\theta^\beta$.

In previous sections we defined the Newman-Penrose null tetrad consisting of vectors l^a, n^a, m^a and $\bar{m}^a$ defined by

$$l^a \; = \; o^A\,\bar{o}^{A'}, \quad n^a \; = \; \iota^A\,\bar{\iota}^{A'}, \quad m^a \; = \; o^A\,\bar{\iota}^{A'},$$

[1]Here, by derivative we mean linear mapping obeying Leibniz rule. Derivative of degree zero in addition does not change the valence of spinor (unlike inner product of degree -1 or exterior derivative of degree +1)

where o^A and ι^A form the spin basis. In the Newman-Penrose formalism we denote the covariant derivative with respect to vectors of null tetrad by special symbols:

$$
\begin{aligned}
D &= l^a\,\nabla_a \;=\; o^A\bar{o}^{A'}\,\nabla_{AA'}, \\
\Delta &= n^a\,\nabla_a \;=\; \iota^A\bar{\iota}^{A'}\,\nabla_{AA'}, \\
\delta &= m^a\,\nabla_a \;=\; o^A\bar{\iota}^{A'}\,\nabla_{AA'}, \\
\bar{\delta} &= \bar{m}^a\,\nabla_a \;=\; \iota^A\bar{o}^{A'}\,\nabla_{AA'}.
\end{aligned}
\tag{5.6}
$$

In what follows we use this notation heavily. We also often write NP instead of Newman-Penrose for the sake of brevity.

Chapter 6

Field equations in the NP formalism

Now we are in position to introduce spinorial equivalents of basic field equations: equations of scalar field, electromagnetic field and gravitational field. Abstract spinor equations we obtain in this way can be made concrete by projecting them onto fixed spin basis. By this procedure we arrive at the set of scalar equations. Choosing suitable coordinate system, these equations become differential equations for unknown field quantities. Abstract scalar equations are written in Newman-Penrose formalism whose elements will be introduced in this chapter.

While the Newman-Penrose equations for electromagnetic and gravitational fields are known from the very beginning of the invention of Newman-Penrose formalism, equations for scalar field and especially conformal equations of scalar field are believed to be new in this work and in the paper [3]. In our work, we consider two types of scalar fields: the Klein-Gordon field and the conformally invariant scalar field.

6.1 Spin coefficients

Spin coefficients are essentially the Ricci rotation coefficients and, hence, they describe the affine connection. In this sense, discussion about them belongs to the discussion about gravitational field. In what follows, however, we consider several physical fields on a general curved space-time and thus the connection plays an important role in the formulation of their equations. For this reason we define the spin coefficients here and return to the full treatment of gravitational field later.

In usual tensor calculus, the connection allows one to define the notion of parallel transport and covariant derivative. Similarly, defining covariant derivative one can deduce the connection. For the moment, we suppress abstract index notation and use usual language of differential geometry. Let e_a be basis vector fields, let V be an arbitrary vector field. We define *connection form* ${\omega_a}^b$ by

$$\nabla_V \, e_a \;=\; {\omega_a}^b(V)\, e_b.$$

Components of the connection form are given by

$$\omega_a{}^b{}_c \;=\; \langle e^b,\, \nabla_c\, e_a \rangle \,.$$

We can see that components of $\omega_a{}^b$, called the *Ricci rotation coefficients*, are given by scalar products of basis covectors and vectors $\nabla_c e_a$. Notice that in the Penrose index notation, $\nabla_c e_a$ is second rank tensor, while in usual notation it is a vector – the derivative of vector e_a in the direction of vector e_c.

This definition can be naturally translated to spinor language (from now we again use abstract index notation). Let $\epsilon_{\boldsymbol{B}}{}^B$ be spin dyad and $\epsilon^{\boldsymbol{B}}{}_B$ its dual, cf. (5.1). Bold index $\boldsymbol{B}$ takes values 0 and 1 and it labels spinors of dyad, index B is an abstract one. Spinor connection coefficients $\Upsilon_{a\boldsymbol{B}}{}^{\boldsymbol{C}}$ are defined by

$$\nabla_a \epsilon_{\boldsymbol{B}}{}^B \;=\; \Upsilon_{a\boldsymbol{B}}{}^{\boldsymbol{C}}\, \epsilon_{\boldsymbol{C}}{}^B \,.$$

Using dual dyad we can express connection coefficients as

$$\Upsilon_{a\boldsymbol{B}\boldsymbol{C}} \;=\; \epsilon_{\boldsymbol{C}B}\, \nabla_a\, \epsilon_{\boldsymbol{B}}{}^B \,.$$

Again, connection coefficients are expressed by inner products of basis spinors and covariant derivative of basis spinors. Not all of these coeffiecients are independent, because of antisymmetry in $\boldsymbol{BC}$. In the Newman-Penrose formalism, we use special symbols for each independent connection coefficient:

$$\begin{aligned} \kappa &= m^a D l_a = o^A D o_A, & \tau &= m^a \Delta l_a = o^A \Delta o_A, \\ \sigma &= m^a \delta l_a = o^A \delta o_A, & \rho &= m^a \bar\delta l_a = o^A \bar\delta o_A, \\[1ex] \varepsilon &= \tfrac{1}{2}\left[n^a D l_a - \bar m^a D m_a\right] = \iota^A D o_A, & \beta &= \tfrac{1}{2}\left[n^a \delta l_a - \bar m^a \delta m_a\right] = \iota^A \delta o_A, \\ \gamma &= \tfrac{1}{2}\left[n^a \Delta l_a - \bar m^a \Delta m_a\right] = \iota^A \Delta o_A, & \alpha &= \tfrac{1}{2}\left[n^a \bar\delta l_a - \bar m^a \bar\delta m_a\right] = \iota^A \bar\delta o_A, \\[1ex] \pi &= n^a D \bar m_a = \iota^A D \iota_A, & \nu &= n^a \Delta \bar m_a = \iota^A \Delta \iota_A, \\ \lambda &= n^a \bar\delta \bar m_a = \iota^A \bar\delta \iota_A, & \mu &= n^a \delta \bar m_a = \iota^A \delta \iota_A. \end{aligned} \tag{6.1}$$

Recall the notation (5.6) introduced in the previous chapter. We can see that the spinorial expression of connection coefficients is remarkably simpler than the tensorial one.

Quantities just introduced are called the *Newman-Penrose spin coefficients*. Their precise geometrical meaning depends on the choice of the spin basis or, equivalently, the NP null tetrad. In standard choice, see e.g. [21], coefficients ρ and σ are called *optical scalars* and they describe the behaviour of the congruence of null geodesics. Real part of coefficient ρ describes the expansion, its imaginary part describes the twist. Coefficient σ is so-called *shear* describing the deformation of the congruence without changing its "volume".

Let us see, how abstract spinor equations can be projected onto null tetrad using spin coefficients. As an example we find the derivative of basis spinor o^A in the direction of vector l^a, i.e. we find the expression for quantity Do^A. Because D does not change the rank of spinor, Do^A is again univalent spinor and thus can be written as a linear combination of basis spinors:

$$Do^A \;=\; x\, o^A \;+\; y\, \iota^A .$$

Unknown components x and y can be found by taking inner products of the last equation with basis spinors. Contractions with o^A and ι^A yield

$$o_A\, Do^A \;=\; y, \quad \iota_A\, Do^A \;=\; -\, x,$$

where we remeber $o_A\iota^A = -o^A\iota_A = -1$, while $o_A o^A = \iota_A\iota^A = 0$. Comparing x and y with (6.1) we find

$$x \;=\; \varepsilon, \quad y \;=\; -\,\kappa,$$

so that

$$Do^A \;=\; \varepsilon\, o^A \;-\; \kappa\, \iota^A.$$

Similarly, we can derive relation

$$D\iota^A \;=\; \pi\, o^A \;-\; \varepsilon\, \iota^A.$$

Consider now spinor τ^A. In the spin basis it can be written as

$$\tau^A \;=\; \tau^0\, o^A \;+\; \tau^1\, \iota^A.$$

Covariant derivative of τ^A with respect to basis vector $l^a = o^A\bar{o}^{A'}$ can be found by applying D on τ^A and using the Leibniz rule. The result is

$$\begin{aligned} D\,\tau^A \;&=\; \left(D\tau^0 \;+\; \epsilon\,\tau^0 \;+\; \pi\,\tau^1\right)\, o^A \\ &+\; \left(D\tau^1 \;-\; \kappa\,\tau^0 \;-\; \epsilon\,\tau^1\right)\, \iota^A. \end{aligned} \tag{6.2}$$

Thus, spinorial equations are usualy simpler than tensorial ones, but projections of spinor objects onto null tetrad in NP formalism can be quite clumsy. We have seen that even the simplest example, the directional derivative of univalent spinor, yields quite long expression in the NP formalism. Projecting of equations is, however, straightforward procedure and can be done by computer. In Appendix A we present scripts written in Mathematica to calculate desired projections in the NP formalism.

6.2 Klein-Gordon field

Now we can leave the description of connection and concentrate to equations of non-gravitational fields considered in papers [2, 3]. First, consider usual Klein-Gordon equation of free scalar field. If we denote the scalar field by ϕ, Klein-Gordon equation reads

$$\left(\Box \;+\; m^2\right)\phi \;=\; 0,$$

where m is the mass of particles described by field ϕ. These particles appear after quantizing the field. Since we are not concerned with the quantum theory, we say, for simplicity, that m is the mass of field ϕ.

Klein-Gordon (KG) equation is a scalar equation and thus it is same in the spinor and tensor formalism. In spinor formalism we just use replacement

$$\Box \;=\; \nabla_a\,\nabla^a \mapsto \Box \;=\; \nabla_{AA'}\,\nabla^{AA'}.$$

D'Alembert's operator can be expressed in the NP formalism as

$$\begin{aligned}\Box \;&=\; D\Delta \;+\; \Delta D \;-\; \bar{\delta}\delta \;-\; \delta\bar{\delta}\\ &+\; (\mu+\bar{\mu}-\gamma-\bar{\gamma})D \;+\; (\varepsilon+\bar{\varepsilon}-\rho-\bar{\rho})\Delta\\ &+\; (\alpha-\bar{\beta}+\bar{\tau}-\pi)\delta \;+\; (\bar{\alpha}-\beta+\tau-\bar{\pi})\bar{\delta}.\end{aligned}$$

Nevertheless, using spinor techniques, even this equation can be decomposed into the set of simpler scalar equations. To do this, we define vector field $\varphi_{AA'}$ as a gradient of ϕ:

$$\varphi_{AA'} \;=\; \nabla_{AA'}\phi.$$

Now consider expression

$$\nabla_{AA'}\varphi_B^{A'}.$$

This expression can be decomposed by standard method described in section 5.5 into symmetric and antisymmetric parts,

$$\nabla_{AA'}\varphi_B^{A'} \;=\; \nabla_{A'(A}\varphi_{B)}^{A'} \;+\; \frac{1}{2}\,\epsilon_{AB}\,\Box\phi.$$

The second term on the right-hand side can be simplified using KG equation. The first term actually vanishes. Indeed, the commutator of covariant derivatives annihilates scalar functions as the torsion is assumed to be zero. Contracting equation

$$2\,\nabla_{[a}\nabla_{b]}\phi \;=\; 0$$

by $\epsilon^{A'B'}$ and symmetrizing it in AB we arrive at

$$\nabla_{A'(A}\varphi_{B)}^{A'} \;=\; 0.$$

Spinorial equation satisfied by KG field is therefore

$$\nabla_{AA'}\varphi_B^{A'} \;=\; -\frac{m^2}{2}\,\epsilon_{AB}. \tag{6.3}$$

Next we project this equation onto the null tetrad and write resulting equations in the NP formalism. Components of $\varphi_{AA'}$ will be denoted by

$$\varphi_0 = \varphi_{00'} = D\phi, \quad \varphi_2 = \varphi_{11'} = \Delta\phi, \quad \varphi_1 = \varphi_{01'} = \delta\phi, \quad \varphi_{\bar{1}} = \varphi_{10'} = \bar{\delta}\phi.$$

Projections of the wave equation in spinor form then read

$$\begin{aligned}D\varphi_1 - \delta\varphi_0 \;&=\; (\bar{\pi}-\bar{\alpha}-\beta)\varphi_0 + (\bar{\rho}+\varepsilon-\bar{\varepsilon})\varphi_1 + \sigma\varphi_{\bar{1}} - \kappa\varphi_2,\\ D\varphi_2 - \delta\varphi_{\bar{1}} \;&=\; -\mu\varphi_0 + \pi\varphi_1 + (\bar{\pi}-\bar{\alpha}+\beta)\varphi_{\bar{1}} + (\bar{\rho}-\varepsilon-\bar{\varepsilon})\varphi_2 - \tfrac{m^2}{2},\\ \Delta\varphi_0 - \bar{\delta}\varphi_1 \;&=\; (\gamma+\bar{\gamma}-\bar{\mu})\varphi_0 + (\bar{\beta}-\alpha-\bar{\tau})\varphi_1 - \tau\varphi_{\bar{1}} + \rho\varphi_2 + \tfrac{m^2}{2},\\ \Delta\varphi_{\bar{1}} - \bar{\delta}\varphi_2 \;&=\; \nu\varphi_0 - \lambda\varphi_1 + (\bar{\gamma}-\gamma-\bar{\mu})\varphi_{\bar{1}} + (\alpha+\bar{\beta}-\bar{\tau})\varphi_2.\end{aligned} \tag{6.4}$$

6.3 Conformally invariant scalar field

Since the proof to be presented in the thesis will take place in unphysical, conformally rescaled space-time (in the sense of chapter 4), we will have to investigate how field equations transform under conformal transformations. We will see that d'Alembert's operator has complicated and inelegant transformation properties. KG equation considered above is not conformally invariant, unlike e.g. Maxwell's equations without sources. Fortunately, d'Alembertian transforms in the same way as scalar curvature R. Combining d'Alembertian and scalar curvature we arrive at equation

$$\left(\Box + \frac{R}{6}\right)\phi = 0.$$

This equation is conformally invariant and field ϕ satisfying it will be called *conformally invariant scalar field* or briefly *conformal scalar field.* Form of this equation is the same in both physical and unphysical space-time. We will see, however, that R is in fact zero in the physical space-time. Thus, conformally invariant equation in the physical space-time reduces to

$$\Box\phi = 0,$$

which is equation of massless KG field. Equation of conformal scalar field can be obtained by setting $m = 0$ in the KG equation, and therefore also projections (6.4)–(6.4) are valid in physical space-time, if we set $m = 0$. More detailed account on both kinds of scalar fields is given in [3] contained in the second part oh the thesis.

6.4 Electromagnetic field

Electromagnetic field is described by antisymmetric tensor F_{ab}. In section 5.5 we explained that its spinor equivalent can be written as

$$F_{ab} = \phi_{AB}\,\epsilon_{A'B'} + \bar{\phi}_{A'B'}\,\epsilon_{AB}, \tag{6.5}$$

where ϕ_{AB} is symmetric spinor. Components of ϕ_{AB} with respect to the spin basis are

$$\phi_0 = \phi_{AB}\,o^A\,o^B, \quad \phi_1 = \phi_{AB}\,o^A\,\iota^B, \quad \phi_2 = \phi_{AB}\,o^A\,o^B.$$

Maxwell's equations without sources in tensor form read

$$\nabla_{[c}F_{ab]} = 0, \qquad \nabla_a F^a_{\ b} = 0. \tag{6.6}$$

First of these equations states that F_{ab} is (locally) exact form, i.e. there exists potential A_a such that

$$F_{ab} = \nabla_a A_b - \nabla_b A_a.$$

Using standard rules for decomposition of spinors into symmetric and antisymmetric parts we arrive at relation

$$F_{ab} = \epsilon_{AB}\,\nabla_{X(A'}A^X_{B')} + \epsilon_{A'B'}\,\nabla_{X'(A}A^{X'}_{B)}.$$

Comparing this with decomposition (6.5) we find that spinor ϕ_{AB} and 4-potential A_a are related by

$$\phi_{AB} = \nabla_{X'(A} A^{X'}_{B)},$$

which is manifestly symmetric spinor.

Now consider the second pair of Maxwell's equations, i.e. the second equation in (6.6). In order to obtain its spinor equivalent we define dual tensor ${}^*F_{ab}$ by

$${}^*F_{ab} = \frac{1}{2}\,\epsilon_{abcd}\,F^{cd},$$

where spinor equivalent of alternating tensor ϵ_{abcd} is given by (5.5). By quick calculation we find

$${}^*F_{ab} = -\,i\,\phi_{AB}\,\epsilon_{A'B'} + i\,\bar{\phi}_{A'B'}\,\epsilon_{AB}.$$

We can see that dualization of antisymmetric tensor F_{ab} reduces to multiplying its individual spinor parts by $\pm i$. *Self-dual part* is multiplied by i, *antiself-dual part* is multiplied by $-i$:

$$\bar{\phi}_{A'B'}\,\epsilon_{AB} \;\mapsto\; i\,\bar{\phi}_{A'B'}\,\epsilon_{AB}, \qquad \phi_{AB}\,\epsilon_{A'B'} \;\mapsto\; -\,i\,\phi_{AB}\,\epsilon_{A'B'}.$$

Let us find equation satisfied by dual tensor ${}^*F_{ab}$. Applying divergence yields

$$\nabla^{a\,*}F_{ab} = \frac{1}{2}\,\epsilon_{abcd}\,\nabla^a F^{cd} = \frac{1}{2}\,\epsilon_{abcd}\,\nabla^{[a}F^{cd]} = 0$$

by the first Maxwell equation. Thus, both Maxwell equations can be written in formally the same form

$$\nabla^{a\,*}F_{ab} = 0, \quad \nabla^a F_{ab} = 0.$$

Antiself-dual part of F_{ab} can be expressed using dual tensor as

$$\phi_{AB}\,\epsilon_{A'B'} = \frac{1}{2}\left(F_{ab} + i\,{}^*F_{ab}\right).$$

Applying operator $\nabla^a = \nabla^{AA'}$ gives

$$\nabla^A_{A'}\,\phi_{AB} = 0 \tag{6.7}$$

which is the desired spinor equivalent of Maxwell's equations without sources. Notice that both Maxwell's equations have been used.

Finally, to write Maxwell's equations in the NP formalism we have to project spinor equation (6.7) onto spin basis. One can do it by hand or using Mathematica and script described in Appendix A. Since equation has two free indices and each of them can be contracted with two basis spinors o and ι (or their complex conjugates) we arrive at four scalar equations:

$$\begin{aligned}
D\phi_1 - \bar{\delta}\phi_0 &= (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2,\\
D\phi_2 - \bar{\delta}\phi_1 &= -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\varepsilon)\phi_2,\\
\Delta\phi_0 - \delta\phi_1 &= (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2,\\
\Delta\phi_1 - \delta\phi_2 &= \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2.
\end{aligned} \tag{6.8}$$

6.5 Gravitational field

Gravitational field is the most complicated case for obvious reasons. First, field equations themselves are complicated. Geometry of the space-time is described by the Riemann tensor which is of order four, so its spinor decomposition leads to more spinorial parts than it was in the case of electromagnetism. Next, field equations are non-linear. Maxwell's equations or scalar field equations are equations for fields propagating on curved background and the curvature of the space-time enters those equations through spin coefficients. But the spin coeffiecients are not fixed given functions, but they are derived from the metric tensor which is the unknown of Einstein's equations.

In the standard formulation of general relativity, unknown functions are components of metric tensor with respect to suitable coordinate system. Once the metric tensor is found, one can calculate the connection and the Riemann tensor. By definition, Riemann tensor automatically satisfies several differential identities, namely those of Ricci and Bianchi. In the NP formalism, however, we choose the tetrad in such a way that components of metric tensor are constant and given by matrix (5.4). Thus, components of metric tensor are not unknown variables: they are fixed. What are the unknown variables then? And what equations do they satisfy?

Decomposition of Riemann tensor

Curvature of the space-time is described by the Riemann tensor R_{abcd} (assuming that metric tensor is given and the connection is compatible with metric). This tensor possesses following algebraic symmetries:

$$R_{abcd} = R_{[ab][cd]}, \quad R_{abcd} = R_{cdab}.$$

As in the case of electromagnetic field, section 5.5, we can decompose spinor equivalent of the Riemann tensor into symmetric and antisymmetric parts and simplify them using symmetries of the Riemann tensor. We do not present the details, which can be found in a very understandable form in [22]. Using the rules of spinor algebra one can arrive at the following decomposition:

$$\begin{aligned} R_{abcd} = \; & \Psi_{ABCD}\, \epsilon_{A'B'}\, \epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\, \epsilon_{AB}\, \epsilon_{CD} \\ & + \Phi_{ABC'D'}\, \epsilon_{A'B'}\, \epsilon_{CD} + \Phi_{CDA'B'}\, \epsilon_{C'D'}\, \epsilon_{AB} \\ & + \Lambda\, [(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC})\epsilon_{A'B'}\epsilon_{C'D'} \\ & \quad + (\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{A'D'}\epsilon_{B'C'})\epsilon_{AB}\epsilon_{CD}] \end{aligned} \tag{6.9}$$

Although this expression looks unnecessarily complicated, it provides us with some insight into the structure of Riemann tensor. Its first part,

$$C_{abcd} = \Psi_{ABCD}\, \epsilon_{A'B'}\, \epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\, \epsilon_{AB}\, \epsilon_{CD}$$

is the spinor equivalent of the *Weyl tensor*. The Weyl tensor can be equivalently represented by totally symmetric complex spinor Ψ_{ABCD} which will be called the *Weyl spinor*. Its components are in the NP formalism denoted by

$$\begin{aligned} \Psi_0 &= \Psi_{ABCD}\, o^A\, o^B\, o^C\, o^D, \\ \Psi_1 &= \Psi_{ABCD}\, o^A\, o^B\, o^C\, \iota^D, \end{aligned}$$

$$
\begin{aligned}
\Psi_2 &= \Psi_{ABCD}\, o^A\, o^B\, \iota^C\, \iota^D, \\
\Psi_3 &= \Psi_{ABCD}\, o^A\, \iota^B\, \iota^C\, \iota^D, \\
\Psi_4 &= \Psi_{ABCD}\, \iota^A\, \iota^B\, \iota^C\, \iota^D.
\end{aligned} \tag{6.10}
$$

In vacuum, the Riemann tensor is equal to the Weyl tensor. In this sense the Weyl tensor represents "pure" gravitational field or gravitational radiation propagating in empty space. It describes tidal forces felt by a body moving along geodesics. Geometrically, the Weyl tensor describes the distortion of congurence of geodesics without changing its volume. All contractions of Weyl tensor vanish. This fact is made manifest in spinor form by the total symmetry of the Weyl spinor.

The second part of the Riemann tensor is related to the trace-free Ricci tensor. Spinor $\Phi_{ABA'B'}$ is real and it is called the *Ricci spinor*. It is symmetric in both pair of indices:

$$\Phi_{ABA'B'} = \Phi_{(AB)(A'B')}.$$

The NP components of the Ricci spinor are defined by

$$
\begin{aligned}
\Phi_{00} &= \Phi_{ABA'B'}\, o^A\, o^B\, \bar{o}^{A'}\, \bar{o}^{B'}, \\
\Phi_{01} &= \Phi_{ABA'B'}\, o^A\, o^B\, \bar{o}^{A'}\, \bar{\iota}^{B'}, \\
\Phi_{02} &= \Phi_{ABA'B'}\, o^A\, o^B\, \bar{\iota}^{A'}\, \bar{\iota}^{B'}, \\
\Phi_{10} &= \Phi_{ABA'B'}\, o^A\, \iota^B\, \bar{o}^{A'}\, \bar{o}^{B'}, \\
\Phi_{11} &= \Phi_{ABA'B'}\, o^A\, \iota^B\, \bar{o}^{A'}\, \bar{\iota}^{B'}, \\
\Phi_{12} &= \Phi_{ABA'B'}\, o^A\, \iota^B\, \bar{\iota}^{A'}\, \bar{\iota}^{B'}, \\
\Phi_{21} &= \Phi_{ABA'B'}\, \iota^A\, \iota^B\, \bar{o}^{A'}\, \bar{\iota}^{B'}, \\
\Phi_{22} &= \Phi_{ABA'B'}\, \iota^A\, \iota^B\, \bar{\iota}^{A'}\, \bar{\iota}^{B'},
\end{aligned} \tag{6.11}
$$

Because the Ricci spinor is real, the following relation holds:

$$\overline{\Phi_{mn}} = \Phi_{nm}, \quad m, n = 0, 1, 2.$$

Contracting R_{abcd} with g^{ac} and g^{bd} it is easy to show that

$$-2\,\Phi_{ABA'B'} = R_{ab} - \frac{1}{4}\, g_{ab}\, R, \quad \Lambda = \frac{1}{24} R,$$

where R_{ab} is the Ricci tensor and R is the scalar curvature. Tensor $R_{ab} - R g_{ab}/4$ is the trace-free part of the Ricci tensor, which is again consistent with symmetries of $\Phi_{ABA'B'}$. Real scalar Λ is proportional to the scalar curvature. Ricci tensor measures the change of volume element caused by the curvature of the space-time in the following sense. Let η and g be the flat and curved metric tensors and let ω_η and ω_g be the volume 4-forms associated with η and g. In the neighbourhood of any point P of the manifold we can introduce the Riemann normal coordinates x^μ. Then, the components of the metric g satisfy

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, x^\alpha\, x^\beta + \mathcal{O}\left(|x|^3\right).$$

The volume form ω_g is related to the volume form of flat metric via

$$\omega_g = \left(1 - \frac{1}{6} R_{\mu\nu}\, x^\mu\, x^\nu + \mathcal{O}\left(|x|^3\right)\right) \omega_\eta.$$

We can see that the Ricci tensor describes the deviation of the volume element from the flat volume form.

Ricci tensor is related to the energy-momentum density of the matter via Einstein's equations. Direct calculation shows that spinor equivalent of Einstein's tensor is

$$G_{ab} \;=\; R_{ab} \;-\; \frac{1}{2}\, R\, g_{ab} \;=\; -\,2\,\Phi_{ABA'B'} \;-\; 6\,\Lambda\,\epsilon_{AB}\,\epsilon_{A'B'}.$$

According to convention used in this text, Einstein's equations read

$$G_{ab} \;=\; -\,8\pi\, T_{ab},$$

where T_{ab} is energy-momentum tensor with $T_{ABA'B'}$ being its spinor equivalent. Substituting corresponding spinor expression for G_{ab} we find

$$\Phi_{ABA'B'} \;+\; 3\,\Lambda\,\epsilon_{AB}\,\epsilon_{A'B'} \;=\; 4\,\pi\, T_{ABA'B'}.$$

Notice, first, that the Weyl spinor does not enter Einstein's equations at all. Second, Einstein's equations can be decomposed further. Symmetrization in AB, $A'B'$ and contraction with $\epsilon^{AB}\epsilon^{A'B'}$, respectively, leads to the following pair of spinor equations:

$$\begin{aligned} \Phi_{ABA'B'} &= 4\,\pi\, T_{(AB)(A'B')}, \\ \Lambda &= \frac{1}{3}\, T_{XY}{}^{XY}. \end{aligned} \tag{6.12}$$

Thus, we have decomposed the Riemann tensor into three parts:

- radiative part described by the Weyl spinor Ψ_{ABCD} not determined by Einstein's equations;
- trace-free part of Ricci tensor described by thi Ricci spinor $\Phi_{ABA'B'}$; it is related to trace-free part of energy momentum tensor through Einstein's equations;
- scalar curvature Λ given by the trace of energy-momentum tensor.

We see that Einstein's equations are neither differential equations for the Weyl spinor nor for the Ricci spinor. They are merely algebraic relations between the Ricci spinor and the scalar curvature and energy-momentum tensor. Moreover, they do not contain spin-coefficients which must be part of the set of unknown variables. We must therefore look for the field equations somewhere else. The point is that while in coordinate formulation of general relativity differential identities satisfied by the Riemann tensor (or its parts) are direct consequences of its definition, in the NP formalism they form differential equations for Riemann tensor. We start the discussion with the simplest case of Ricci identities.

Commutation relations

In general relativity we assume that the torsion tensor vanishes. Consequently, the commutator of covariant derivatives vanishes when acting on scalar function f:

$$(\nabla_a \nabla_b \; - \; \nabla_b \nabla_a) f \; = \; 0.$$

It is possible to rewrite this relation using spinor dyad, but for our purposes it is not necessary. Using Mathematica script explained in appendix A, we can obtain the NP projections by simple code:

```
expr = Nabla[a, Nabla[b, f]] - Nabla[b, Nabla[a, f]];
Penrose[expr l[a]n[b]]
Penrose[expr l[a]m[b]]
Penrose[expr n[a]m[b]]
Penrose[expr m[a]m̄[b]]
```

Here we projected expression $\nabla_a \nabla_b f - \nabla_b \nabla_a f$ onto the null tetrad (by antisymmetry in ab, contractions with two same vectors yield zero). Resulting equations are called *commutation relations* in the NP formalism and they express vanishing of the torsion. They have also practical importance, because they can be used to simplify many calculations. On the other hand, if we first introduce coordinate system and define operators D, Δ and δ in coordinates, we can use commutation relations to calculate spin-coefficients. The NP projections of commutation relations are following:

$$\begin{aligned}
\Delta D - D\Delta &= (\gamma+\bar{\gamma})D + (\varepsilon+\bar{\varepsilon})\Delta - (\pi+\bar{\tau})\delta - (\tau+\bar{\pi})\bar{\delta}, \\
D\delta - \delta D &= (\bar{\pi}-\beta-\bar{\alpha})D - \kappa\Delta + (\bar{\rho}+\varepsilon-\bar{\varepsilon})\delta + \sigma\bar{\delta}, \\
\Delta\delta - \delta\Delta &= \bar{\nu}D + (\beta-\tau+\bar{\alpha})\Delta + (\gamma-\bar{\gamma}-\mu)\delta - \bar{\lambda}\bar{\delta}, \\
\bar{\delta}\delta - \delta\bar{\delta} &= (\bar{\mu}-\mu)D + (\bar{\rho}-\rho)\Delta + (\alpha-\bar{\beta})\delta + (\beta-\bar{\alpha})\bar{\delta}.
\end{aligned} \tag{6.13}$$

Ricci identities

Commutation relations discussed above are special case of more general Ricci identities. When the commutator $2\nabla_{[a}\nabla_{b]}$ acts on higher order tensors, it does not vanish anymore. It is more or less the definition of the Riemann tensor, that commutator acting on covector field is given by

$$(\nabla_a \nabla_b \; - \; \nabla_b \nabla_a)\omega_c \;\; = \;\; - R_{abc}{}^{d}\, \omega_d \quad \text{(Ricci identity)}. \tag{6.14}$$

If, on the other hand, the Riemann tensor is known, this relation can be regarded as differential equation for the connection. To see this explicitly, we can project it onto null tetrad. For this, we subsequently choose ω_c to be

$$\omega_c \; = \; l_c, n_c, m_c, \bar{m}_c$$

and contract equation (6.14) with all vectors of the null tetrad. Again, Mathematica is of great help. Script NPformalism includes the definition of antiselfdual part of the Riemann tensor, see (A.1). Here we show only one projection, when $\omega_c = l_c$ and the equation is contracted with $l^a n^b m^c$.

Since NPformalism defines only antiself-dual part of Riemann tensor, to find correspondning contraction we have to use the formula

$$R_{abcd}\, l^a\, n^b\, m^c\, l^d \;=\; {}^{-}R_{abcd}\, l^a\, n^b\, m^c\, l^d \;+\; \overline{{}^{-}R_{abcd}\, l^a\, n^b\, \bar{m}^c\, l^d}.$$

In other words, to find contraction of *self-dual* part of the Riemann tensor we compute the contraction of antiself-dual part with complex conjugated basis, and complex conjugate the result. Corresponding Mathematica code reads

```
(* right hand side *)
ASD = − Penrose[Riemann[A, B, C, D, AA, BB, CC, DD]
        lu[A, AA] nu[B, BB] mu[C, CC] lu[D, DD]];
SD = − Penrose[Riemann[A, B, C, D, AA, BB, CC, DD]
        lu[A, AA] nu[B, BB] mu̅[C, CC] lu[D, DD]];
rhs = ASD − OverBar[SD];
(* left hand side *)
lhs = Penrose[ (Nablad[A, AA, Nablad[B, BB, ld[C, CC]]]
      − Nablad[B, BB, Nablad[A, AA, ld[C, CC]]]) lu[A, AA] nu[B, BB] mu[C, CC]];
lhs == rhs
```

As a result, Mathematica returns the equation

$$[(\nabla_a\nabla_b - \nabla_b\nabla_a)l_c]\, l^a\, n^b\, m^c \;=\; -\,R_{abcd}\, l^a\, n^b\, m^c\, l^d$$

in the NP formalism:

$$D\tau \;-\; \Delta\kappa \;\;=\;\; (\tau+\bar{\pi})\rho + (\pi+\bar{\tau})\sigma + (\varepsilon-\bar{\varepsilon})\tau - (3\gamma+\bar{\gamma})\kappa \;+\; \Psi_1 \;+\; \Phi_{01}.$$

Obviously, this equation is the first-order partial differential equation for spin-coefficients τ and κ. Full set of Ricci identities is not listed here as they are contained in papers [2] (Appendix A, page 18) and [3] (Appendix A, page 19), included in the thesis.

Bianchi identities

So far, we have derived the NP equations for spin-coefficients which are called the Ricci identities. What we still need are differential equations for the components of Weyl and Ricci spinor and scalar curvature. These equations are provided by Bianchi identities.

Bianchi identity is well-known differential identity obeyed by Riemann tensor,

$$\nabla_{[e}R_{ab]cd} \;=\; 0, \qquad \text{(Bianchi identity)}$$

and it resembles the first of Maxwell's equations (6.6). Indeed, the procedure of deriving corresponding spinor equations is similar to derivation of Maxwell's equations in spinor form. But because of the more complicated decomposition of the Riemann tensor, also the procedure is complicated and not necessary for our purposes. Very clear and systematic derivation can be found in [18]. It appears, however, that the Bianchi identity is equivalent to spinor equations

$$\begin{aligned}\nabla^{A}_{A'}\Psi_{ABCD} \;&=\; \nabla^{B'}_{(C}\Phi_{AB)A'B'},\\ \nabla^{BB'}\Phi_{ABA'B'} \;&=\; -\,3\,\nabla_{AA'}\Lambda.\end{aligned} \tag{6.15}$$

The first of these equations is not suitable for direct projection onto spin basis, because it contains symmetrization in three indices. Term $\nabla^{A'}_{(C}\Phi_{AB)A'B'}$ actually represents $3! = 6$ terms and each of them becomes even more complicated after projection. Therefore, we would like to remove the symmetrization. This is somewhat tricky and hard to find in the literature and so we present the procedure here.

Consider, for simplicity, spinor ϕ_{ABC} symmetric in BC:

$$\phi_{ABC} \;=\; \phi_{A(BC)}.$$

Using this symmetry of ϕ_{ABC} we can write

$$\phi_{(ABC)} \;=\; \frac{1}{3}\left[\phi_{(AC)B} \;+\; \phi_{(AB)C} \;+\; \phi_{(BC)A}\right].$$

Now we use the standard decomposition of spinor into symmetric and antisymmetric parts but express the symmetric part:

$$\phi_{(AB)C} \;=\; \phi_{ABC} \;-\; \frac{1}{2}\,\epsilon_{AB}\,\phi_X{}^X{}_C.$$

In the first step we arrive at

$$\phi_{(ABC)} \;=\; \frac{1}{3}\left[2\phi_{(AB)C} \;+\; \phi_{ACB} \;-\; \epsilon_{A(B|}\,\phi_X{}^X{}_{|C)} \;-\; \frac{1}{2}\,\epsilon_{BC}\,\phi_X{}^X{}_A\right].$$

Next we substitute for $\phi_{(AB)C}$ again and find

$$\phi_{(ABC)} \;=\; \phi_{ABC} \;-\; \frac{1}{3}\,\epsilon_{A(B|}\,\phi_X{}^X{}_{|C)} \;-\; \frac{1}{3}\,\epsilon_{AB}\,\phi_X{}^X{}_C \;-\; \frac{1}{6}\,\epsilon_{BC}\,\phi_X{}^X{}_A.$$

This "beautiful" relation can already be applied to our case. For if we replace ϕ_{ABC} by

$$\phi_{ABC} \;\mapsto\; \nabla^{B'}_A\,\Phi_{BCA'B'},$$

obviously having assumed symmetry in BC, and use the second Bianchi identity (6.15), we obtain

$$\nabla^{B'}_{(A}\Phi_{BC)A'B'} \;=\; \nabla^{B'}_A\Phi_{BCA'B'} \;-\; \frac{3}{2}\,\epsilon_{AB}\nabla_{CA'}\Lambda \;+\; \epsilon_{C(A}\nabla_{B)A'}\Lambda.$$

Bianchi identities in spinor form thus read

$$\begin{aligned} \nabla^D_{A'}\Psi_{ABCD} \;&=\; \nabla^{B'}_A\Phi_{BCA'B'} \;-\; \frac{3}{2}\,\epsilon_{AB}\nabla_{CA'}\Lambda \;+\; \epsilon_{C(A}\nabla_{B)A'}\Lambda, \\ \nabla^{BB'}\Phi_{ABA'B'} \;&=\; -\,3\,\nabla_{BB'}\Lambda. \end{aligned} \tag{6.16}$$

Advantage of this form is that now the equations do not containt symmetrization in three indices. It still contains symmetrization in two indices but, first, these terms contain gradient of Λ which is much simpler object than four-valent Ricci spinor, and second, these terms are proportional to ϵ_{AB} which is nonzero only when projected onto $o^A\iota^B$.

Bianchi identities in the NP formalism can again be obtained using script NPformalism for Mathematica. Complete set of equations is listed in papers [2] (Appendix A, pages 18–19) and [3] (Appendix A, pages 19–20) attached to the thesis.

This completes the list of equations of gravitational field in the NP formalism. Einstein's field equations are just algebraic relations among energy-momentum tensor and the Ricci spinor and scalar curvature. In the NP formalism, connection is described by twelwe spin-coefficients. The Ricci identities are non-linear first order equations for spin coefficients. The NP components of the Weyl and Ricci tensor are determined by Bianchi identities, also first order non-linear equations.

6.6 Zero-rest-mass equations

We conclude this chapter by brief recapitulation. We have seen that electromagnetic field in spinor formalism can be described by bivalent symmetric spinor ϕ_{AB}. In the absence of sources, Maxwell's equations are equivalent to equation

$$\nabla^A_{A'} \phi_{AB} \;=\; 0.$$

Pure gravitational field is described by totally symmetric four-valent Weyl tensor. In previous section we have seen that it satisfies Bianchi identities (6.15). In the absence of matter (source of gravitational field), Ricci spinor and scalar curvature vanish so that Bianchi identities reduce to

$$\nabla^{A'}_A \Psi_{ABCD} \;=\; 0,$$

which is formally the same as equation for electromagnetic field, but now the spinor has four indices. Number of indices is in fact related to the spin of the field. Quanta of electromagnetic field are massless photons of spin 1. Hypothetic quanta of gravitational field are massless gravitons of spin 2. Thus, massless field of arbitrary spin s is described by totally symmetric spinor with $2s$ indices satisfying so-called *zero-rest-mass (ZRM) equation*

$$\nabla^{A'}_A \, \phi_{AB..C} \;=\; 0.$$

In the absence of sources, spinor divergence of the spinor vanishes. It is not obvious, however, what is the equation for a scalar field which has spin zero, as it is not possible to perform spinor divergence of the scalar. In [18] it is shown that natural limit of ZRM equation for $s \to 0$ is the massless Klein-Gordon equation

$$\Box \phi \;=\; 0.$$

In the section 6.2 we derived spinor version of Klein-Gordon equation (6.3) which for massless field reduces to

$$\nabla^A_{A'} \varphi_{AB'} \;=\; 0.$$

This equation can also be regarded as a ZRM equation. Spinors with primed indices carry positive helicity (projection of spin onto the direction of momentum) while unprimed indices carry negative helicity. Because the field $\varphi_{AA'}$ has one primed and one unprimed index, total spin carried by field $\varphi_{AA'}$ is zero, as it should be for the scalar field. In this sense Klein-Gordon equation is ZRM equation for spin zero.

Consider now the Dirac field with spin 1/2. It can be shown that the spinor equivalent of massive Dirac equation is pair of spinor equations

$$\nabla_{AA'}\phi^{A'} \;=\; -\frac{i\,m}{\sqrt{2}}\,\psi_A, \quad \nabla_{AA'}\psi^{A} \;=\; \frac{i\,m}{\sqrt{2}}\,\phi_{A'}.$$

Spinors $\phi_{A'}$ and ψ^A together form Dirac four-spinor. For massless field, however, these equations decouple and spinors $\phi_{A'}$ and ψ^A satisfy two separate equations,

$$\nabla^{A}_{A'}\psi_A \;=\; 0, \quad \nabla^{A'}_{A}\phi_{A'} \;=\; 0.$$

Obviously, these equations are ZRM equations for spin 1/2, one for positive helicity and one for the negative. These are Weyl's equations for neutrino with zero rest mass.

Thus, in spinor formalism equations for massless fields without sources have all the same form, the form of ZRM equation. This unified character description of ZRM equations is hidden in the tensor formalism. This observation (and many other properties of spinors) inspired Roger Penrose to that spinors are essenetially more primitive objects than tensors; he tried to derive the notion of space-time itself from more basic mathematical structures. These considerations led him eventually to the theory of twistors as an alternative framework for unifying quantum theory and general relativity.

Part II

Results and papers

Chapter 7

On asymptotically flat solutions of Einsteins equations periodic in time: I. Vacuum and electrovacuum solutions

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On asymptotically flat solutions of Einstein's equations periodic in time: I. Vacuum and electrovacuum solutions

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Abstract
By an argument similar to that of Gibbons and Stewart (1984 Absence of asymptotically flat solutions of Einstein's equations which are periodic and empty near infinity *Classical General Relativity (London, 1983)* ed W Bonnor, J N Islam and M A H Callum (Cambridge: Cambridge University Press) pp 77–94), but in a different coordinate system and less restrictive gauge, we show that any weakly asymptotically simple, analytic vacuum or electrovacuum solutions of the Einstein equations which are periodic in time are necessarily stationary.

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1. Introduction

The inspiral and coalescence of binary black holes or neutron stars appears to be the most promising source for the detectors of gravitational waves, so that there has been much effort going into the development of numerical codes and analytic approximation methods to find the corresponding solutions of Einstein's equations. One of the recent approaches assumes the existence of a helical Killing vector k (see e.g. [25]). The field is assumed stationary in a rotating frame where k generates time translations but k becomes null at the light cylinder and is spacelike outside. k has the form $k = \partial_t + \omega\partial_\phi$, where ∂_t is timelike and ∂_ϕ is spacelike with circular orbits with parameter length 2π (except where $\partial_\phi = 0$); $\omega =$ constant. The spacetime is not stationary but it is still periodic where k is spacelike. Requiring the helical symmetry for a binary system implies equal amounts of outgoing and incoming radiation so that the spacetime, containing energy radiated at all times, is not expected to be asymptotically flat. A corresponding solution in Maxwell's theory for two opposite point charges moving on circular orbits was considered a long time ago by Schild [19]. The properties of the field were analysed recently in the Newman–Penrose formalism in [1]. The rather complicated periodicity properties of the solution became apparent as well as its asymptotic behaviour: at $\mathcal{I}^-$ the advanced fields exhibit the standard Bondi-type expansion and peeling, whereas the retarded fields do decay with $r \to \infty$ but in an oscillatory manner like $(\sin r)/r$. Hence for the retarded plus advanced solution, no radiation field is asymptotically defined. Naturally, one would like to go beyond the linearized theory. There are special exact, time-dependent, solutions known, for example, as Szekeres's dust solution, which has in general no Killing vector, which can be matched to an exterior Schwarzschild metric [3]. One can construct oscillating spherical shells of dust particles moving with the same angular momentum, but in every tangential direction, or oscillating Einstein clusters which are matched to the Schwarzschild spacetime outside [8]. Can there be periodic solutions representing 'bound states' of gravitational or electromagnetic waves so that the radiation field at infinity vanishes and the Bondi mass remains constant?

There have been various attempts to prove that while solutions of the vacuum Einstein equations can be genuinely periodic in a suitable time coordinate (so *not* time independent), these solutions cannot be asymptotically flat. These started with [15, 16], with a summary in English in [17, 24] and more recently was considered in [10]. The method in [15] considers vacuum metrics which are everywhere nonsingular, weak and asymptotically flat and which can be expanded in a series in some parameter, with the flat metric as the first term in the series. Each term in the series is assumed to be periodic in a fixed Minkowski time coordinate and to satisfy the de Donder gauge condition. The second and third terms, call them v_{ab} and w_{ab} respectively, are expanded as Fourier series in the background time coordinate and the Einstein equations then imply that v_{ab} satisfies the source-free wave equation, and w_{ab} satisfies a wave equation whose source is a quadratic expression in v_{ab}. Assuming that the solution for v_{ab} is everywhere regular, the author shows that there cannot be an asymptotically flat solution for w_{ab} unless v_{ab} vanishes. Therefore, the spacetime is flat. In [16], a similar calculation when v_{ab} is regular only outside a certain radius leads to the conclusion that v_{ab} must be time independent in order to have asymptotically flat w_{ab}, and the spacetime is stationary. In [24] it was observed by integrating the Einstein pseudotensor and matter energy–momentum tensor over a four-dimensional volume that 'the mean value of power radiated by a periodic, asymptotically Minkowskian gravitational field is equal to zero'. The question of existence of periodic fields was left open. In [10] the authors used the spin-coefficient formalism (see e.g. [14, 22]) to study the system of conformal Einstein equations of Friedrich [5]. A coordinate system is based on two families of null hypersurfaces, incoming from past null infinity $\mathcal{I}^-$ and labelled by constant v and outgoing near $\mathcal{I}^-$ and labelled by constant u. The authors make a definition of periodicity which enables them to prove that, at $\mathcal{I}^-$, the u-derivatives of all orders of all components of the metric are independent of v. They conclude that if the metric is analytic in these coordinates, then it necessarily has a Killing vector, which in these coordinates is ∂_v, at least in a neighbourhood of $\mathcal{I}^-$. Thus any analytic metric, periodic in their sense, has such a Killing vector. While certainly correct, there is a problem with this conclusion in that, by construction, the Killing vector is null wherever it is defined, and reduces at $\mathcal{I}^-$ to a constant translation along the generators. These are strong conditions and in fact no Killing vector in flat space has these properties (any null Killing vector is necessarily a null translation, and a null translation is zero along one generator of $\mathcal{I}$)[4]. Thus flat space is not periodic according to the definition of [10] and nor is any of the familiar stationary, asymptotically flat solutions, for example the Schwarzschild solution.

[4] For example the null translation $\partial_t + \partial_z$ becomes $2\cos^2(\theta/2)\,\partial_v$ on $\mathcal{I}^-$, which vanishes at $\theta = \pi$.

For convenience, we follow [10] in working at $\mathcal{I}^-$ rather than $\mathcal{I}^+$, though this is trivial to switch, but we shall make a weaker definition of 'periodic in time' which will permit metrics stationary near $\mathcal{I}^-$ and indeed will allow only these for analytic, asymptotically flat vacuum or electrovac metrics. We follow the method of [10] for both the vacuum and electrovac field equations, deferring other cases to a second paper, but in a different coordinate and tetrad system. Our coordinate and tetrad system is similar to the one used at $\mathcal{I}$ in [14], and to prove the existence of a symmetry at the event horizon in [11] and at a compact Cauchy horizon in [12]. We also differ from [10] in the choice of conformal gauge. In [10] the unphysical Ricci scalar is set to zero by a choice of conformal factor obtained by solving a wave equation. However, the solution of the characteristic IVP for this wave equation as posed in [10] will not in general be periodic, so that the rescaled, unphysical metric would not in general share the periodicity of the physical metric—in fact, in the particular case of the Reissner–Nordström solution this gauge choice is compatible with periodicity only for zero mass, as we show in appendix C. Thus we assume that there is at least one conformal factor which is periodic and then modify this choice in the course of the calculation in order to simplify the spin coefficients. From this point on, our method is then essentially the same as in [10], though a little more complicated, and we arrive at the same conclusion, but now with a Killing vector which is timelike in the interior, at least near to $\mathcal{I}^-$. The condition of timelike periodicity which we impose is as follows: a spacetime is timelike periodic if there is a discrete isometry taking any point of the physical spacetime to a point in its chronological future. To define timelike periodicity at $\mathcal{I}^-$ for an asymptotically flat spacetime, we require this isometry to extend to an isometry of a neighbourhood of $\mathcal{I}^-$ which preserves the generators of $\mathcal{I}^-$. In particular, we require the existence of at least one Ω which conformally compactifies the spacetime and preserves the periodicity. The isometry has to be a supertranslation [22],

$$v \to v + a(\theta, \phi), \tag{1}$$

on $\mathcal{I}^-$, in terms of the usual coordinates (v, θ, ϕ) on $\mathcal{I}^-$ and we shall assume that $a \neq 0$. (We could imagine allowing a to vanish on some generators of $\mathcal{I}^-$, since as noted above periodicity along a null translation in flat space would appear like this at $\mathcal{I}^-$, but this would be null-periodicity rather than timelike periodicity.) We could assume further that a is actually a positive constant but this turns out not to be necessary, as we shall find that, for analytic spacetimes, this assumption of periodicity necessarily leads to a spacetime metric with a Killing vector which, in coordinates to be defined, is ∂_v and is timelike near $\mathcal{I}^-$. Our result is

Theorem 1.1. *A weakly asymptotically simple, vacuum or electrovac, time-periodic spacetime which is analytic in a neighbourhood of $\mathcal{I}^-$ in the coordinates introduced below necessarily has a Killing vector which is timelike in the interior and extends to a translation on $\mathcal{I}^-$.*

Thus there are no *non-trivial* time-periodic solutions satisfying these conditions, in the sense that they would necessarily be actually time independent if time periodic. In a later paper, we shall prove the corresponding result for the Einstein equations coupled to either a massless scalar field with the usual energy–momentum tensor, or a solution of the conformally invariant wave equation with the energy–momentum tensor from p 125 of [18] (sometimes called the 'new improved energy–momentum tensor').

The method of proof requires the assumption of analyticity. It was shown in [6] that there are vacuum solutions analytic near $\mathcal{I}^-$. However, one would like either to drop the assumption of analyticity, for example following the lead of [7] or [9] with a similar problem, or to prove that it follows from the assumptions of periodicity and asymptotic-flatness. It remains to be seen in what circumstances this can be done since, as noted above, there are non-analytic solutions with matter in periodic motion and matched to a (static) Schwarzschild exterior.

While this work is primarily motivated by an interest in the possibility or impossibility of helical motions, it is worth noting the connection with the question of the inheritance of symmetry. Recall that, for a solution of Einstein's field equations with matter, the matter is said to inherit the symmetry of the metric if any isometry of the metric is necessarily a symmetry of the matter. There are explicit solutions of the Einstein–Maxwell equations known for which an isometry of the metric is *not* a symmetry of the Maxwell field [13] but these solutions are not asymptotically flat. In [21] some other references may be found for explicit solutions with Maxwell fields which do not share the symmetry of the metric. The same will be true for some Robinson–Trautman solutions with the null electromagnetic field which may depend on time though the metric is static (see [21], section 28.2). These solutions will very likely have wire singularities extending to infinity. From theorem 1.1 noninheritance cannot happen with asymptotically flat, analytic solutions.

Corollary 1.2. *In any weakly asymptotically simple, stationary electrovac spacetime which is analytic in a neighbourhood of $\mathcal{I}^-$ in the coordinates introduced below, the Maxwell field is also stationary.*

One can raise the question of inheritance also for Einstein-scalar field solutions but the answer is rather different: for a massive (complex) Klein–Gordon field, there do exist solutions, the so-called boson stars, for which the metric is spherically symmetric, asymptotically flat and static but the scalar field has a phase linear in time (see e.g. [2]); however, these solutions are not analytic at infinity and, by a scaling argument, such solutions do not exist with massless scalar fields. In a later paper, we shall obtain this result as a corollary of the result corresponding to theorem 1.1. In that subsequent work, we start from the conformal Einstein field equations with a general energy–momentum tensor as a source and specialize them to scalar field cases.

In section 2, we analyse the conformal Einstein–Maxwell equations. We first rewrite Maxwell's equations in the unphysical spacetime, and then translate the physical Bianchi identities and obtain differential equations for the unphysical Weyl spinor and Ricci spinor. In appendix A, we summarize a number of quantities, their relations and behaviour under conformal transformations in the Newman–Penrose formalism [14]; these are extensively used in the main text and in appendices B and C. In particular, all conformal equations for the gravitational and electromagnetic field analysed in terms of spinors in section 2 are projected on the spin basis (i.e. the null tetrad) and written down in the Newman–Penrose formalism in appendix B. In section 3, a suitable coordinate system and a convenient Newman–Penrose null tetrad which gives special values to some of the Newman–Penrose spin coefficients are introduced in the neighbourhood of $\mathcal{I}^-$. As noted above, these differ from those used by the authors of [10]. At the end of section 3, we explain in detail in what our choice of the coordinate system and the null tetrad differs from that of [10]. In appendix C, we demonstrate that in contrast to [10] our choice of gauge admits simple static (i.e. 'periodic') spacetimes like flat space and the Reissner–Nordström metric. In section 4, we follow [10] (although in a different conformal gauge) and study the problem in the NP formalism in the unphysical spacetime, with data on $\mathcal{I}^-$. Assuming periodicity along $\mathcal{I}^-$ we first discover that the only possibility is the independence of all geometric quantities of an affine parameter v along $\mathcal{I}^-$. By induction we then prove that all derivatives of all geometric quantities, including the physical metric components, in the direction into the physical spacetime must also be v-independent. The proof of theorem 1.1 and corollary 1.2 then follows from the assumed analyticity.

This paper arose from a collaboration after PT posted his work [23] on the gr-qc arXiv and JB informed him that he and his PhD student MS were already engaged in tackling the same problem [20].

2. The conformal Einstein–Maxwell equations

We first introduce conformal equations for the gravitational and electromagnetic field in the formalism of two-component spinors. In appendix B, these equations are written down explicitly after the projection on a spin basis, in the form employed in the Newman–Penrose formalism. In the physical spacetime, Maxwell's equations without sources are simply[5] (see e.g. [22])

$$\tilde{\nabla}^{AA'}\tilde{\phi}_{AB} = 0. \tag{2}$$

They are conformally invariant if under conformal rescaling the Maxwell spinor ϕ_{AB} transforms with conformal weight 1,

$$\tilde{\phi}_{AB} = \Omega\phi_{AB}, \tag{3}$$

when the convention used in this paper for conformal rescaling is $\tilde{\epsilon}_{AB} = \Omega^{-1}\epsilon_{AB}$.

From the transformation of the derivative operator (see (A.12)), in the unphysical spacetime, equation (2) becomes

$$\nabla^{AA'}\phi_{AB} = 0. \tag{4}$$

The situation is more complicated in the case of the gravitational field. The physical Bianchi identities read

$$\tilde{\nabla}^{D}_{C'}\tilde{\Psi}_{ABCD} = \tilde{\nabla}^{D'}_{(C}\tilde{\Phi}_{AB)C'D'}, \tag{5}$$

where $\tilde{\Psi}_{ABCD}$ and $\tilde{\Phi}_{ABC'D'}$ are the Weyl and the Ricci spinor, respectively. Using the rules for the conformal transformation of these spinors (equations (A.15) and (A.17)), we find

$$\Omega^2\nabla^{D}_{C'}\psi_{ABCD} = \Omega\nabla^{D'}_{(C}\Phi_{AB)C'D'} + \left(\nabla^{D'}_{(A}\Omega\right)\Phi_{BC)C'D'} + \nabla^{D'}_{(C}\nabla_{A(C'}\nabla_{D')B)}\Omega, \tag{6}$$

where $\psi_{ABCD} = \Omega^{-1}\Psi_{ABCD}$. These equations are the physical Bianchi identities written in terms of the quantities in the unphysical spacetime. We simplify them by employing Einstein's equations in the physical spacetime,

$$\tilde{\Phi}_{ABA'B'} = k\tilde{\phi}_{AB}\bar{\tilde{\phi}}_{A'B'}. \tag{7}$$

Here we used the fact that the physical scalar curvature vanishes for the electromagnetic field; we put the constant factor k on the rhs of (7) equal to 1 following the convention of [14], unlike, e.g., [18]. From equations (A.15), (7) and (3), we obtain

$$\nabla_{A(A'}\nabla_{B')B}\Omega = \Omega^3\phi_{AB}\bar{\phi}_{A'B'} - \Omega\Phi_{ABA'B'}. \tag{8}$$

Applying $\nabla^{D'}_{C}$, symmetrizing and using Maxwell's equations (4), we can express the term containing the third derivative of Ω appearing in (6) as follows:

$$\begin{aligned}&\nabla^{D'}_{(C}\nabla_{A(C'}\nabla_{D')B)}\Omega \\ &\quad = 3\Omega^2\bar{\phi}_{C'D'}\phi_{(AB}\nabla^{D'}_{C)}\Omega + \Omega^3\bar{\phi}_{C'D'}\nabla^{D'}_{(C}\phi_{AB)} - \Omega\nabla^{D'}_{(C}\Phi_{AB)C'D'} - \left(\nabla^{D'}_{(C}\Omega\right)\Phi_{AB)C'D'}.\end{aligned}$$

Inserting this result into (6), we arrive at the conformal Bianchi identities for the Einstein–Maxwell field expressed in terms of the quantities in the unphysical spacetime:

$$\nabla^{D}_{A'}\psi_{ABCD} = 3\bar{\phi}_{A'B'}\phi_{(AB}\,\nabla^{B'}_{C)}\Omega + \Omega\bar{\phi}_{A'B'}\,\nabla^{B'}_{(C}\phi_{AB)}. \tag{9}$$

Projecting these equations onto the spin basis, we obtain the set of the equations which are explicitly written down (using the NP formalism) in appendix B, see (B.5*a*)–(B.5*h*). Equations (9) are differential equations for the unphysical Weyl spinor. To obtain the equations

[5] Spinor indices are labelled by $A, A', B, B', \ldots$ and have values 0, 1. The metric has signature -2.

for the Ricci spinor, we use the Bianchi identities valid for quantities in the unphysical spacetime:

$$\nabla^{D}_{C'}\Psi_{ABCD} = \nabla^{D'}_{(C}\Phi_{AB)C'D'}. \tag{10}$$

Combining the last two equations, we get

$$\nabla^{B'}_{(C}\Phi_{AB)A'B'} = \psi_{ABCD}\,\nabla^{D}_{A'}\Omega + 3\,\Omega\bar{\phi}_{A'B'}\phi_{(AB}\,\nabla^{B'}_{C)}\,\Omega + \Omega^2\bar{\phi}_{A'B'}\nabla^{B'}_{(C}\phi_{AB)}. \tag{11}$$

In the following, we shall also need the expression for quantities $\nabla_{AA'}\nabla_{BB'}\Omega$. Let us decompose $\nabla_{AA'}\nabla_{BB'}\Omega$ into its symmetric and antisymmetric parts

$$\nabla_{AA'}\nabla_{BB'}\Omega = \nabla_{A(A'}\nabla_{B')B}\Omega + \tfrac{1}{2}\epsilon_{A'B'}\nabla_{AC'}\nabla^{C'}_{B}\,\Omega. \tag{12}$$

The first term on the rhs is given in (8); the second term can be decomposed again:

$$\nabla_{AC'}\nabla^{C'}_{B}\,\Omega = \nabla_{C'(A}\nabla^{C'}_{B)}\Omega + \tfrac{1}{2}\epsilon_{AB}\;\Box\,\Omega. \tag{13}$$

Since the operator $\nabla_{C'(A}\nabla^{C'}_{B)}$ is just the commutator $\nabla_{[a}\nabla_{b]}$ contracted by $\epsilon^{A'B'}$, it annihilates scalar quantities. Using equations (8), (12) and (13), we obtain

$$\nabla_{AA'}\nabla_{BB'}\Omega = \Omega^3\phi_{AB}\bar{\phi}_{A'B'} - \Omega\,\Phi_{ABA'B'} + \tfrac{1}{4}\epsilon_{A'B'}\epsilon_{AB}\;\Box\,\Omega. \tag{14}$$

It will be convenient to introduce the quantity

$$F = \tfrac{1}{2}\Omega^{-1}(\nabla_{AA'}\Omega)(\nabla^{AA'}\Omega), \tag{15}$$

which can be seen to be smooth in the unphysical spacetime from the rule for the conformal transformation of the scalar curvature (A.15) in the form

$$\Box\,\Omega = 4\,\Omega\,\Lambda - 4\,\Omega^{-1}\,\tilde{\Lambda} + 4\,F, \tag{16}$$

since the physical scalar curvature $\tilde{\Lambda} = 0$ for the electromagnetic field. From equation (14), we now obtain the following expressions for the second derivatives of Ω:

$$\nabla_{AA'}\nabla_{BB'}\Omega = \Omega^3\phi_{AB}\bar{\phi}_{A'B'} - \Omega\,\Phi_{ABA'B'} + \epsilon_{A'B'}\epsilon_{AB}\,(F + \Omega\,\Lambda). \tag{17}$$

Finally we wish to derive expressions for $\nabla_{AA'}F$. Directly from the definition of the unphysical Riemann tensor and from the decomposition (A.3), we have

$$(\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'})\,\nabla^{BB'}\Omega = -2\Phi_{ABA'B'}\nabla^{BB'}\Omega + 6\,\Lambda\,\nabla_{AA'}\,\Omega. \tag{18}$$

Employing Maxwell's equations (4) and the contracted Bianchi identities (A.21), we find that equations (14) and (18) imply

$$\nabla_{AA'}F = \Omega^2\phi^{B}_{A}\bar{\phi}^{B'}_{A'}\,\nabla_{BB'}\Omega - \Phi_{ABA'B'}\nabla^{BB'}\Omega + \Lambda\,\nabla_{AA'}\Omega. \tag{19}$$

3. Coordinates, tetrad and conformal gauge

We assume that we have an analytic, time-periodic solution of the Einstein–Maxwell equations and an analytic, time-periodic conformal factor so that the unphysical metric with $\mathcal{I}^-$ also has these properties. We construct a convenient coordinate system and a Newman–Penrose null tetrad in the neighbourhood of $\mathcal{I}^-$ (see figures 1 and 2). We stay in the unphysical spacetime in order to include $\mathcal{I}^-$. Let $\mathcal{S} \subset \mathcal{I}^-$ be a particular spacelike 2-sphere. We can introduce arbitrary coordinates x^I, $I = 2, 3$ on $\mathcal{S}$ and propagate them along $\mathcal{I}^-$ by the condition

$$\nabla_{\dot{\gamma}}\,x^I = 0, \tag{20}$$

where $\gamma = \gamma(v)$ is an affinely parametrized null generator of $\mathcal{I}^-$. We may set $v = 0$ on $\mathcal{S}$. The triple (v, x^2, x^3) represents suitable coordinates on $\mathcal{I}^-$. In order to go into the interior of spacetime, we introduce the family of null hypersurfaces $\mathcal{N}_v$ orthogonal to $\mathcal{I}^-$ and intersecting

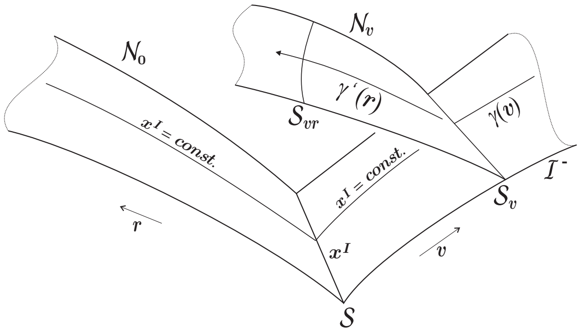

Figure 1. Construction of the coordinate system.

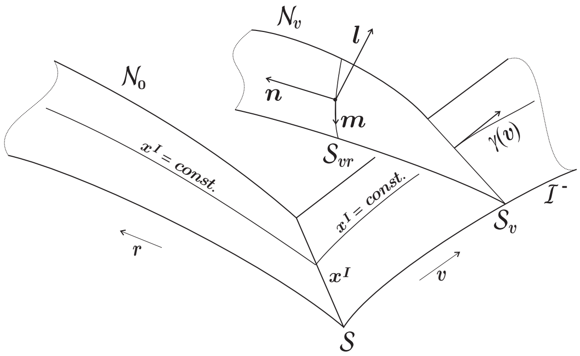

Figure 2. NP null tetrad.

$\mathcal{I}^-$ in the spacelike cuts $\mathcal{S}_v$ of constant v. Let $\gamma' = \gamma'(r)$ be the null generators of the surface $\mathcal{N}_v$ labelled by x^I. Here, r is the affine parameter which can be chosen so that $r = 0$ on $\mathcal{I}^-$ and $g(\mathrm{d}v, \mathrm{d}r) = 1$ at $\mathcal{I}^-$. We propagate the coordinates v and x^I onto $\mathcal{N}_v$ by the conditions

$$\nabla_{\dot{\gamma}'} x^I = 0, \qquad \nabla_{\dot{\gamma}'} v = 0. \tag{21}$$

We thus have established a coordinate chart

$$x^\mu = (v, r, x^2, x^3), \qquad \mu = 0, 1, 2, 3, \tag{22}$$

in the neighbourhood of past null infinity[6].

Next we construct a suitable Newman–Penrose null tetrad. $\mathcal{N}_v$ are null hypersurfaces $v =$ constant; therefore, the gradient of v is both tangent and normal to $\mathcal{N}_v$; we denote it by

$$n_a = \nabla_a v. \tag{23}$$

Since n^a is tangent to γ' along which only r varies,

$$n = \frac{\partial}{\partial r}. \tag{24}$$

[6] Components of tensors with respect to the basis induced by these coordinates will be labelled by Greek letters $\mu, \nu, \ldots$. Components with respect to an arbitrary tetrad will be labelled by Latin letters $a, b, \ldots$ from the beginning of the alphabet. Indices labelled by capital letters $I, J, \ldots$ have values 2, 3.

On each cut $\mathcal{S}_{vr}$: $v, r =$ constant there exists exactly one null direction normal to $\mathcal{S}_{vr}$ not proportional to n^a. We choose the vector field l^a to be parallel to this direction and normalize it by $n_a l^a = 1$. It can be written in the form

$$l = \frac{\partial}{\partial v} - H\frac{\partial}{\partial r} + C^I\frac{\partial}{\partial x^I}. \tag{25}$$

On $\mathcal{I}^-$ l is tangent to the generators $\gamma(v)$, so functions H and C^I vanish on $\mathcal{I}^-$. The conformal gauge can be chosen so that

$$\frac{\partial \Omega}{\partial r} = 1 \quad \text{on } \mathcal{I}^-. \tag{26}$$

Let us now turn to the 2-spheres $\mathcal{S}_{vr}$ on which ∂_I are basis vectors. Since $\mathcal{S}_{vr}$ is a spacelike sphere, we choose, following the standard procedure, a complex vector m and its complex conjugate $\bar{m}$, such that

$$m^a\, m_a = 0, \qquad m^a \bar{m}_a = -1, \tag{27}$$

m has the form

$$m = P^I\frac{\partial}{\partial x^I}, \tag{28}$$

where P^2, P^3 are complex functions. The coordinates x^I can be chosen to be the standard spherical coordinates, $x^I = (\theta, \phi)$. Then the appropriate choice of the null vector m at $\mathcal{I}^-$ is (see e.g. [22])

$$m = \frac{1}{\sqrt{2}}\left(\partial_\theta + \frac{\mathrm{i}}{\sin\theta}\,\partial_\phi\right), \qquad P^I = \frac{1}{\sqrt{2}}\left(1, \frac{\mathrm{i}}{\sin\theta}\right). \tag{29}$$

The vectors $m, \bar{m}$ are orthogonal to l and n. The contravariant components of the tetrad read

$$\begin{aligned} l^\mu &= (1, -H, C^2, C^3),\\ n^\mu &= (0, 1, 0, 0),\\ m^\mu &= (0, 0, P^2, P^3). \end{aligned} \tag{30}$$

The contravariant components of the metric tensor are given, regarding the relation $g^{\mu\nu} = 2l^{(\mu}n^{\nu)} - 2m^{(\mu}\bar{m}^{\nu)}$, by the matrix

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & -2H & C^2 & C^3\\ 0 & C^2 & -2P^2\,\bar{P}^2 & -P^2\,\bar{P}^3 - P^3\,\bar{P}^2\\ 0 & C^3 & -P^2\,\bar{P}^3 - P^3\,\bar{P}^2 & -2P^3\,\bar{P}^3 \end{pmatrix}. \tag{31}$$

Using (30) and the inverse of (31) we find the covariant components of the tetrad vectors:

$$\begin{aligned} l_\mu &= (H, 1, 0, 0),\\ n_\mu &= (1, 0, 0, 0),\\ m_\mu &= (\omega, 0, R_2, R_3), \end{aligned} \tag{32}$$

where

$$\begin{aligned} R_2 &= \frac{P^3}{P^2\,\bar{P}^3 - P^3\,\bar{P}^2}, \qquad R_3 = \frac{P^2}{P^3\,\bar{P}^2 - P^2\,\bar{P}^3},\\ \omega &= -C^I\,R_I. \end{aligned} \tag{33}$$

The covariant components of the metric are

$$g_{\mu\nu} = \begin{pmatrix} 2H - 2\omega\bar{\omega} & 1 & -\omega\bar{R}_2 - \bar{\omega}R_2 & -\omega\bar{R}_3 - \bar{\omega}R_3\\ 1 & 0 & 0 & 0\\ -\omega\bar{R}_2 - \bar{\omega}R_2 & 0 & -2R_2\bar{R}_2 & -R_3\bar{R}_2 - R_2\bar{R}_3\\ -\omega\bar{R}_3 - \bar{\omega}R_3 & 0 & -R_3\bar{R}_2 - R_2\bar{R}_3 & -2R_3\bar{R}_3 \end{pmatrix}. \tag{34}$$

The vectors l, n, m and $\bar{m}$ constitute the NP tetrad. However, it is not unique since there is a rotation gauge freedom $m \to e^{i\chi}m$ which will be used later. Following the standard notation of the NP formalism (e.g. [14, 22]), we define the operators

$$D = l^a\nabla_a, \qquad \Delta = n^a\nabla_a, \qquad \delta = m^a\nabla_a. \tag{35}$$

We shall also employ the spin basis (o^A, ι^A) associated with the null tetrad

$$l^a = o^A\bar{o}^{A'}, \qquad n^a = \iota^A\bar{\iota}^{A'}, \qquad m^a = o^A\bar{\iota}^{A'}, \tag{36}$$

normalized by $o_A\,\iota^A = 1$. Note that this coordinate and tetrad system has some more gauge freedom associated with it. In particular, we may make another choice $\hat{\Omega}$ with $\hat{\Omega} = \Theta\Omega$, where Θ is also periodic and takes the value 1 at $\mathcal{I}^-$. Thus,

$$\tilde{g}_{ab} = \hat{\Omega}^{-2}\hat{g}_{ab} = \Omega^{-2}g_{ab}$$

and so $\hat{g}_{ab} = \Theta^2 g_{ab}$. We assume that $\Theta = 1 + f(v, r, x^I)$, with $f = O(r)$. This will change the definition of the affine parameter r, to $\hat{r}$ say, and then we must accompany the change of the conformal factor with a null rotation of the tetrad so that $\hat{\delta}$ is tangent to the sphere $\mathcal{S}_{v\hat{r}}$; thus,

$$\begin{aligned}
\hat{n}_a &= n_a,\\
\hat{m}_a &= \Theta(m_a + Zn_a),\\
\hat{l}_a &= \Theta^2(l_a + Z\bar{m}_a + \bar{Z}m_a + Z\bar{Z}n_a),
\end{aligned} \tag{37}$$

where Z, which parametrizes the null rotation, is fixed by requiring $\hat{\delta}\hat{r} = 0$. The associated operators change according to

$$\begin{aligned}
\hat{\Delta} &= \Theta^{-2}\Delta,\\
\hat{\delta} &= \Theta^{-1}(\delta + Z\Delta),\\
\hat{D} &= D + Z\bar{\delta} + \bar{Z}\delta + Z\bar{Z}\Delta.
\end{aligned} \tag{38}$$

With the coordinate v common to both systems, we define $\hat{r}$ as the affine parameter with

$$\hat{\Delta}\hat{r} = \Theta^{-2}\Delta\hat{r} = 1.$$

This can be integrated to give

$$\hat{r} = \int_0^r \Theta^2\,\mathrm{d}r = r + O(r^2), \tag{39}$$

and we need

$$0 = \hat{\delta}\hat{r} = \Theta^{-1}(\delta\hat{r} + Z\Delta\hat{r}),$$

so that

$$Z = -\Theta^{-2}\delta\hat{r},$$

which can be calculated from (39). Note that $Z = O(r^2)$. We shall need to exploit this gauge freedom below. Next we examine what special values some of the spin coefficients take due to the above choice of the null tetrad (we calculate for the unhatted system, but the same relations hold in the hatted systems). Acting by the commutators (A.2) on the coordinate v, we find

$$\gamma + \bar{\gamma} = \bar{\alpha} + \beta - \bar{\pi} = \nu = \mu - \bar{\mu} = 0. \tag{40}$$

Furthermore, the commutators $[\delta, \Delta]r$ and $[\bar{\delta}, \delta]r$ give

$$\tau - \bar{\alpha} - \beta = \rho - \bar{\rho} = 0. \tag{41}$$

Applying the remaining commutators on the variables v, r and x^I leads to the 'frame equations', i.e. the equations for the metric functions H, C^I and P^I:

$$\Delta H = -(\varepsilon + \bar{\varepsilon}), \tag{42a}$$

$$\delta H = -\kappa, \tag{42b}$$

$$\Delta C^I = -2\pi P^I - 2\bar{\pi}\bar{P}^I, \tag{42c}$$

$$\bar{\delta} P^I - \delta \bar{P}^I = (\alpha - \bar{\beta}) P^I - (\bar{\alpha} - \beta)\bar{P}^I, \tag{42d}$$

$$\Delta P^I = -(\mu - \gamma + \bar{\gamma}) P^I - \bar{\lambda}\bar{P}^I, \tag{42e}$$

$$\delta C^I - D P^I = -(\rho + \varepsilon - \bar{\varepsilon}) P^I - \sigma \bar{P}^I. \tag{42f}$$

Since the generators $\gamma(v)$ of $\mathcal{I}^-$ are affinely parametrized null geodesics, $Dl^a = 0$ on $\mathcal{I}^-$. Comparing this with the general relation

$$Dl^a = (\varepsilon + \bar{\varepsilon}) l^a - \bar{\kappa}\, m^a - \kappa \bar{m}^a, \tag{43}$$

we see that

$$\varepsilon + \bar{\varepsilon} = \kappa = 0 \quad \text{on } \mathcal{I}^-. \tag{44}$$

Next we wish to show that the freedom in choosing the basis $(m, \bar{m})$ of the space tangential to $\mathcal{S}_{vr}$ allows us to set $\gamma = 0$. From the definition of γ (equation A.1), we have $\gamma - \bar{\gamma} = m^a \Delta \bar{m}_a$. Under the rotation through χ,

$$m^a \to \mathrm{e}^{\mathrm{i}\chi} m^a, \tag{45}$$

the quantity $\gamma - \bar{\gamma}$ transforms according to

$$\gamma - \bar{\gamma} \to \gamma - \bar{\gamma} + \mathrm{i}\Delta\chi, \tag{46}$$

so by solving the equation

$$\Delta\chi = \mathrm{i}(\gamma - \bar{\gamma}), \tag{47}$$

and regarding (40) we can set

$$\gamma = 0. \tag{48}$$

Because the Δ-operator is the derivative with respect to the coordinate r, further rotation (45) with an r-independent function χ does not violate equality (48). The quantity $\varepsilon - \bar{\varepsilon}$ under the rotation (45) transforms according to

$$\varepsilon - \bar{\varepsilon} \to \varepsilon - \bar{\varepsilon} + \mathrm{i}D\chi. \tag{49}$$

Solving the equation

$$D\chi = \mathrm{i}(\varepsilon - \bar{\varepsilon}) \tag{50}$$

on $\mathcal{I}^-$, where $r = 0$, we set $\varepsilon = \bar{\varepsilon}$ which, together with (44), implies

$$\varepsilon = 0 \quad \text{on } \mathcal{I}^-. \tag{51}$$

To end this section, we exploit the gauge freedom (37) and (38) to achieve a further simplification. From the commutator $[\hat{\delta}, \hat{\Delta}]$ (see (A.2) with the values of the spin coefficients fixed above), we calculate

$$\hat{\mu} = \Theta^{-2}(\mu + \Theta^{-1}\Delta\Theta),$$

so that we can set $\hat{\mu} = 0$ by choosing

$$\Theta = \exp\left(-\int_0^r \mu\, \mathrm{d}r\right).$$

Having done this, we omit the hats.

In order to elucidate the differences between our choice of the coordinate system and the null tetrad and those used by Gibbons and Stewart, we conclude this section by giving the details of their construction. Instead of the affine parameter r they use coordinate u, defined as follows. Let $\mathcal{S}_0'$ be a spacelike cut on $\mathcal{I}^-$ and $\mathcal{N}'$, $\mathcal{S}_0' \subset \mathcal{N}'$, the null hypersurface such that the null generators of $\mathcal{N}'$ are orthogonal to $\mathcal{S}_0'$. Now, the real function u on $\mathcal{N}'$ is defined in such a way, that $u = 0$ on $\mathcal{S}_0'$, and on spacelike two-surfaces $\mathcal{S}_u$ $u =$ constant. The cut S_u defines another null hypersurface $\mathcal{N}_u$ with null generators orthogonal to $\mathcal{S}_u$. The coordinate u is obtained by setting $u =$ constant on $\mathcal{N}_u$. Similarly, the family of null hypersurfaces $\mathcal{N}_v'$ orthogonal to spacelike cuts $\mathcal{S}_v'$ on $\mathcal{I}^-$, with v being the affine parameter along the null generators of $\mathcal{I}^-$, is constructed. Coordinates x^I are chosen freely on $\mathcal{S}_0'$ and propagated into the spacetime along $\mathcal{N}'$ and $\mathcal{N}_u$. The functions $x^\mu = (u, v, x^2, x^3)$ constitute a coordinate system in the neighbourhood of $\mathcal{I}^-$ but note that in these coordinates the vector field ∂_v is null, which is not in our coordinates.

The NP tetrad used in [10] consists of vectors l, tangent to $\mathcal{N}_u$, n, tangent to $\mathcal{N}_v'$, and $m, \bar{m}$ spanning the tangent space of $\mathcal{S}_0'$ and propagated into the spacetime. Coordinate expressions of their tetrad read (this should be compared with our expressions (24), (25) and (28))

$$l = Q\,\partial_v, \qquad n = \partial_u + C^I\,\partial_I, \qquad m = P^I\,\partial_I, \tag{52}$$

where Q, C^I and P^I are metric functions. In this tetrad, the following equation holds:

$$\Delta n^a = -(\gamma + \bar{\gamma}) n^a.$$

Therefore, the null generators of $\mathcal{N}_v'$ are geodesics, but u is not an affine parameter.

The periodicity of the spacetime is defined as the periodicity of all geometrical quantities in the variable v. It is shown in [10] that $K = \partial_v$ is the Killing vector of the metric and concluded that the spacetime is stationary. However, K is *null* everywhere by construction as it is tangent to the null generators of $\mathcal{N}_v$, while the stationarity requires the timelike Killing vector. Thus, it is impossible to conclude that the spacetime is stationary from the fact that K is the Killing vector. As was mentioned in the introduction, even the Minkowski spacetime does not posses the Killing vector which is everywhere null and tangent to $\mathcal{I}^-$.

In the following, we use the coordinates and the tetrad introduced in the beginning of this section. We show that $K = \partial_v$ is the Killing vector null on $\mathcal{I}^-$ but *timelike* in its neighbourhood.

4. Proof of the theorem

Having chosen coordinates and tetrad and fixed special values of some of the NP coefficients we now analyse all geometric quantities assuming analyticity in the chosen coordinates and periodicity on $\mathcal{I}^-$ in v. Following [10] we introduce the notation

$$\begin{aligned} &S_0 = D\Omega, \qquad S_1 = \delta\Omega, \qquad S_2 = \Delta\Omega, \\ &F = \frac{1}{\Omega}(S_0 S_2 - S_1 \bar{S}_1), \qquad \psi_n = \frac{\Psi_n}{\Omega}, \qquad n = 0, 1, 2, 3, 4, \end{aligned} \tag{53}$$

where Ψ_n are the NP components of the Weyl spinor (see equation (A.10)). In the case of asymptotically flat spacetime, they vanish on $\mathcal{I}^-$, so assuming smoothness, the ψ_n are regular there. Tangential derivatives of the conformal factor vanish on $\mathcal{I}^-$, i.e. $S_0 = S_1 = 0$, and so, again by smoothness, the quantity F is regular on $\mathcal{I}^-$. The remaining component of $\nabla\Omega$ is S_2 which is 1 on $\mathcal{I}^-$ (cf (26)), so that its tangential derivatives also vanish on $\mathcal{I}^-$. Equations (17) and (19) are explicitly written down in the NP formalism in appendix B as (B.2*a*)–(B.4*c*). Equations (B.2*d*)–(B.2*j*) show that on $\mathcal{I}^-$

$$\sigma = 0, \tag{54a}$$

$$F = 0, \tag{54b}$$

$$\rho = 0, \tag{54c}$$

$$\bar{\pi} = 0 = \beta + \bar{\alpha} = \tau, \tag{54d}$$

$$\Delta S_0 = 0, \tag{54e}$$

$$\Delta S_2 = 0, \tag{54f}$$

$$\Delta S_1 = 0. \tag{54g}$$

Since $F = 0$ on $\mathcal{I}^-$, also the tangential derivatives DF and δF vanish there. From equations (B.4*a*) and (B.4*b*) we thus obtain

$$\Phi_{00} = \Phi_{01} = 0 \quad \text{on } \mathcal{I}^-. \tag{55}$$

The metric functions P^I on $\mathcal{I}^-$ are given by (29). Inserting this expression into the frame equation (42*d*) and using relation (54*d*) we find

$$\alpha = -\beta = -\tfrac{1}{2\sqrt{2}} \cot\theta \quad \text{on } \mathcal{I}^-. \tag{56}$$

The Ricci identity (A.20*q*) now shows that

$$\Lambda + \Phi_{11} = \tfrac{1}{2} \quad \text{on } \mathcal{I}^-. \tag{57}$$

In order to discover the behaviour of the other relevant quantities we shall take into account the properties of the Bondi mass. In a general asymptotically flat electrovacuum spacetime, the total mass energy at $\mathcal{I}^+$ is defined by the formula (see e.g. [4])

$$M_B = -\frac{1}{2\sqrt{\pi}} \int \mathrm{d}S \big(\tilde{\Psi}_2^0 + \tilde{\sigma}^0 \dot{\bar{\tilde{\sigma}}}^0 \big). \tag{58}$$

By the superscript 0 we denote the leading term in the asymptotic expansion of a quantity; the superscripts $1, 2, \ldots$ then denote higher order terms, for example, $\tilde{\sigma} = \tilde{\sigma}^0\,\Omega^2 + \tilde{\sigma}^1\Omega^3 + \mathcal{O}(\Omega^4)$. The rate of decrease of the Bondi mass is given by

$$\dot{M}_B = -\frac{1}{2\sqrt{\pi}} \int \mathrm{d}S \left(\dot{\tilde{\sigma}}^0 \dot{\bar{\tilde{\sigma}}}^0 + \tilde{\phi}_2^0 \bar{\tilde{\phi}}_2^0 \right). \tag{59}$$

The quantities σ and $\phi_i, i = 0, 1, 2$, are defined in (A.1) and (A.26). Following the 'conversion table' between $\mathcal{I}^+$ and $\mathcal{I}^-$ (see (A.14)), we analogously define the Bondi mass at $\mathcal{I}^-$ by

$$M_B = -\frac{1}{2\sqrt{\pi}} \int \mathrm{d}S \big(\tilde{\Psi}_2^0 + \tilde{\lambda}^0 \dot{\bar{\tilde{\lambda}}}^0 \big). \tag{60}$$

Since radiation comes into the physical spacetime through $\mathcal{I}^-$ but cannot exit through it, the total mass energy at $\mathcal{I}^-$ cannot decrease. Its rate of change in (advanced) time v along $\mathcal{I}^-$ is given by

$$\dot{M}_B = \frac{1}{2\sqrt{\pi}} \int \mathrm{d}S \big(\dot{\tilde{\lambda}}^0 \dot{\bar{\tilde{\lambda}}}^0 + \tilde{\phi}_0^0 \bar{\tilde{\phi}}_0^0 \big). \tag{61}$$

Now we assume periodicity. But a non-decreasing periodic function must be a constant. Hence, our assumption of periodicity of the mass energy at $\mathcal{I}^-$ requires

$$\dot{\tilde{\lambda}}^0 = 0, \qquad \tilde{\phi}_0^0 = 0. \tag{62}$$

The leading term in the asymptotic expansion of $\tilde{\Psi}_0$ is then $\tilde{\Psi}_0^0 = \ddot{\bar{\lambda}}^0 = 0$. Regarding equations (A.16) and (A.18) and putting $\tilde{\Psi}_0^0 = 0$, we can write the asymptotic expansion of Ψ_0 near $\mathcal{I}^-$ as

$$\Psi_0 = \Psi_0^1 \Omega^2 + \mathcal{O}(\Omega^3), \tag{63}$$

or (cf equation (53))

$$\psi_0 = \mathcal{O}(\Omega). \tag{64}$$

Equation (63) implies

$$\Delta \Psi_0 = 0 \quad \text{on } \mathcal{I}^-. \tag{65}$$

Similarly, equation (A.27), where we put $\tilde{\phi}_0^0 = 0$, implies $\phi_0 \in \mathcal{O}(\Omega)$ and

$$\Delta \phi_0 = \phi_0^1 \, S_2 \quad \text{on } \mathcal{I}^-. \tag{66}$$

The geometrical quantities consist of the tetrad components, which give the metric functions, the spin coefficients and the components of the Weyl and the Ricci tensor on $\mathcal{I}^-$. Because of our assumption of the periodicity of gravitational field, the geometrical quantities are all assumed to be periodic in the variable v on $\mathcal{I}^-$. We do not assume the periodicity of the electromagnetic field since this field may not have the same symmetries as the gravitational field (this is the issue of inheritance which we shall return to). We have shown that the following spin coefficients vanish on $\mathcal{I}^-$ (and thus do not depend on v):

$$\mu, \rho, \sigma, \kappa, \varepsilon, \nu, \gamma, \pi, \tau. \tag{67}$$

The spin coefficients α and β are v-independent because of (56). Now we wish to show that also the last spin coefficient λ is independent of v. The Bianchi identity (A.23*a*) together with (65) and (55) shows that

$$D\Phi_{02} = 0 \quad \text{on } \mathcal{I}^-. \tag{68}$$

If we now apply D to the Ricci identity (A.20*g*), we get

$$D^2 \lambda = 0 \quad \text{on } \mathcal{I}^-. \tag{69}$$

The general solution of this equation on $\mathcal{I}^-$ is

$$\lambda = \lambda^{(0)} + v\, \lambda^{(1)}, \tag{70}$$

where $\lambda^{(0)}$ and $\lambda^{(1)}$ are functions independent of v. Since λ is assumed to be periodic and a polynomial in v can be periodic only if it is constant, we get $\lambda = \lambda^{(0)}$ and

$$D\lambda = 0 \quad \text{on } \mathcal{I}^- \tag{71}$$

(we borrow this style of argument from [10] where it is used extensively). The Ricci identity (A.20*g*) then implies

$$\Phi_{02} = 0 \quad \text{on } \mathcal{I}^-. \tag{72}$$

The Ricci identity (A.20*h*) on $\mathcal{I}^-$ becomes

$$\Lambda = 0, \tag{73}$$

and then by (57) $\Phi_{11} = 1/2$ there. Now from (A.22*c*) and D on (A.20*k*), $D\Phi_{12}$ and $D\Phi_{22}$ vanish at $\mathcal{I}^-$. We collect these results and some similar ones as a lemma.

Lemma 4.1. *The following are zero on* $\mathcal{I}^-$*:*

$H, C^A, \rho, \sigma, \pi, \kappa, \epsilon, S_0, S_1, F, \psi_0, \Phi_{00}, \Phi_{01}, \Phi_{02}, \phi_0, \Lambda,$
$DP^A, D\alpha, D\beta, DS_2, D\lambda, D\Phi_{11}, D\Phi_{12}, D\Phi_{22}, D\psi_1, D\psi_2, D\psi_3, D\psi_4, D\phi_1, D\phi_2,$
$D\Delta S_0, D\Delta S_1, D\Delta S_2.$

Proof. The first line is done already, as is the second line up to $D\psi_1$, which comes from (B.5*a*). From D applied to (B.5*b*)–(B.5*d*), we obtain $D^2\psi_i = 0$ whence by periodicity $D\psi_i = 0$ at $\mathcal{I}^-$, in order for $i = 2, 3, 4$. The same procedure applied to (A.29*a*), (A.29*b*) takes care of $D\phi_1$, $D\phi_2$. Then the third line follows from D applied to (B.2*h*)–(B.2*j*).

Now we turn to the proof of the theorem. We set up an induction with the following inductive hypothesis. Suppose inductively that $\partial_v \Delta^j Q = 0$ at $\mathcal{I}^-$ for $0 \leqslant j \leqslant k$ with Q one of

$$H, C^I, P^I, \epsilon, \pi, \lambda, \beta, \alpha, \rho, \sigma, \kappa, F, \psi_i, \Phi_{ij}, \phi_i, \Lambda \tag{74}$$

and for $0 \leqslant j \leqslant k+1$ with $Q = S_i$.

This is easily seen by the lemma to hold for $k = 0$, so we need to deduce it for $j = k+1$ from its truth for $j \leqslant k$. In this calculation, we use the fact that $\partial_v = D$ at $\mathcal{I}^-$ and make extensive use of the commutators (A.2). Under the inductive hypothesis, the inductive step follows

- for H, C^I, P^I from (42*a*), (42*c*) and (42*e*);
- for $\epsilon, \pi, \lambda, \beta, \alpha, \rho, \sigma, \kappa$, respectively, from (A.20*f*), (A.20*i*), (A.20*j*), (A.20*l*), (A.20*o*), (A.20*n*), (A.20*m*) and (A.20*c*);
- for F from (B.4*c*);
- for ϕ_0 and ϕ_1 directly from (A.29*c*) and (A.29*d*), respectively; for ϕ_2, from (A.29*b*), we deduce at $\mathcal{I}^-$
$$D^2\Delta^{k+1}\phi_2 = 0,$$
and then periodicity implies
$$D\Delta^{k+1}\phi_2 = 0;$$
- for $\psi_i, i = 0, 1, 2, 3$, from (B.5*e*)–(B.5*h*); for ψ_4, under the inductive hypothesis, we deduce at $\mathcal{I}^-$
$$D^2\Delta^{k+1}\psi_4 = 0$$
from (B.5*d*) and then periodicity implies
$$D\Delta^{k+1}\psi_4 = 0;$$
- for $\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{12}$ from (A.22*b*), (A.23*b*), (A.22*d*) and (A.23*d*), respectively, all with $\Psi_n = \Omega\psi_n$; then for Λ, Φ_{11} and Φ_{22} we use (A.20*h*), (A.24*c*) and (A.20*k*).

This completes the inductive step for the first set of quantities Q. For $Q = S_i$, we use $D\Delta^{k+1}$ applied to (B.2*h*)–(B.2*j*). Thus r-derivatives of all orders of the quantities in (74), which includes the metric functions H, C^I and P^I, are independent of v. Now analyticity in r forces these functions to be independent of v. Therefore, by (30), the metric components are all independent of v and so $K := \partial/\partial v$ is a Killing vector of the unphysical metric. However, for any j,

$$\partial_v \Delta^j \Omega = \partial_v \Delta^{j-1} S_2,$$

at $\mathcal{I}^-$ and the rhs vanishes for all j. Thus, by analyticity in r, Ω is also independent of v and so K is a Killing vector of the physical metric too. The norm-squared of the Killing vector is

$$g(K, K) = 2(H - \omega\bar{\omega}).$$

This is $O(r^2)$ at $\mathcal{I}^-$ but there

$$\Delta^2 g(K, K) = 2\Delta^2 H = -2\Delta(\varepsilon + \bar{\varepsilon}) = 2$$

so that K is null at $\mathcal{I}^-$ but timelike just inside: the metric is stationary. □

This completes the proof of the theorem. Note that we have shown that under the assumption of periodicity, both fields are necessarily time independent. A slightly different question is whether a stationary asymptotically flat gravitational field might be produced by an electromagnetic field which is not itself stationary. The content of corollary 1.2 is that the answer is no.

Proof of corollary 1.2. Starting from the assumption that the metric admits ∂_v as a Killing vector, we want to show that this is also a symmetry of the Maxwell field. We have

$$\tilde{\Phi}_{ij} = \Omega^2 \phi_i \bar{\phi}_j, \tag{75}$$

and $\partial_v \tilde{\Phi}_{ij} = 0$ so that, for some χ possibly depending on v, we have

$$\phi_i = \mathrm{e}^{\mathrm{i}\chi} \varphi_i, \tag{76}$$

where φ_i is v-independent. From the Maxwell equation (A.29*a*), with $\phi_0 = 0$ on $\mathcal{I}^-$, we find $\phi_1 D\chi = 0$ on $\mathcal{I}^-$ so that $D\chi = 0$ unless $\phi_1 = 0$ there. If $\phi_1 = 0$ there, (A.29*b*) gives $D\chi = 0$ unless $\phi_2 = 0$, so we can conclude that $D\phi_i = 0$ on $\mathcal{I}^-$. Now we set up an induction to show that $D\Delta^n \phi_i = 0$ on $\mathcal{I}^-$ for all $n \in \mathbb{N}$ and $i = 0, 1, 2$. The inductive hypothesis will be

$$(\forall k \leqslant n) \quad (\forall i \in \{0, 1, 2\}) \quad (D\Delta^k \phi_i = 0 \text{ on } \mathcal{I}^-). \tag{77}$$

Then by $D\Delta^n$ on (A.29*c*) and (A.29*d*), we obtain this for $k = n + 1$ and $i = 0, 1$. For $i = 2$, $D\Delta^{n+1}$ on (A.29*b*) gives

$$D^2 \Delta^{n+1} \phi_2 = 0 \quad \text{on } \mathcal{I}^-, \tag{78}$$

which integrates to give $\Delta^{n+1}\phi_2 = av + b$. This would contribute a v-dependent term to $\tilde{\Phi}_{22}$ at $\mathcal{O}(\Omega^{2n+4})$, a contradiction unless $a = 0$. Then $D\Delta^{n+1}\phi_2 = 0$ on $\mathcal{I}^-$, which completes the induction.

By assumption, the Maxwell field is analytic and so has a convergent power series in r near to $\mathcal{I}^-$ and we have shown that all coefficients are v-independent. Since the spinor dyad is Lie-dragged by the Killing vector, this proves that the Maxwell field is too: in this situation the Maxwell field inherits the symmetry. □

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Appendix A. The Newman–Penrose formalism and conformal transformations in Einstein–Maxwell spacetimes

A.1. Gravitational field

In the NP formalism, the spin coefficients are the Ricci rotation coefficients with respect to a null tetrad $\{l, n, m\}$ with the corresponding spin basis o_A, ι_A; they encode the connection. The

12 independent complex coefficients are defined by (see e.g. [14, 22] for details)

$$\begin{aligned}
&\kappa = m^a D l_a = o^A D o_A, && \tau = m^a \Delta l_a = o^A \Delta o_A,\\
&\sigma = m^a \delta l_a = o^A \delta o_A, && \rho = m^a \bar\delta l_a = o^A \bar\delta o_A,\\
&\varepsilon = \tfrac{1}{2}[n^a D l_a - \bar m^a D m_a] = \iota^A D o_A, && \beta = \tfrac{1}{2}[n^a \delta l_a - \bar m^a \delta m_a] = \iota^A \delta o_A,\\
&\gamma = \tfrac{1}{2}[n^a \Delta l_a - \bar m^a \Delta m_a] = \iota^A \Delta o_A, && \alpha = \tfrac{1}{2}[n^a \bar\delta l_a - \bar m^a \bar\delta m_a] = \iota^A \bar\delta o_A,\\
&\pi = n^a D \bar m_a = \iota^A D \iota_A, && \nu = n^a \Delta \bar m_a = \iota^A \Delta \iota_A,\\
&\lambda = n^a \bar\delta \bar m_a = \iota^A \bar\delta \iota_A, && \mu = n^a \delta \bar m_a = \iota^A \delta \iota_A,
\end{aligned} \tag{A.1}$$

where $D = \nabla_l, \Delta = \nabla_n, \delta = \nabla_m$. Acting on a scalar, the operators D, Δ, δ obey the commutation relations:

$$\begin{aligned}
D\delta - \delta D &= (\bar\pi - \bar\alpha - \beta) D - \kappa\Delta + (\bar\rho - \bar\varepsilon + \varepsilon)\delta + \sigma\bar\delta,\\
\Delta D - D\Delta &= (\gamma + \bar\gamma) D + (\varepsilon + \bar\varepsilon)\Delta - (\bar\tau + \pi)\delta - (\tau + \bar\pi)\bar\delta,\\
\Delta\delta - \delta\Delta &= \bar\nu D + (\bar\alpha + \beta - \tau)\Delta + (\gamma - \bar\gamma - \mu)\delta - \bar\lambda\bar\delta,\\
\delta\bar\delta - \bar\delta\delta &= (\mu - \bar\mu) D + (\rho - \bar\rho)\Delta + (\bar\alpha - \beta)\bar\delta - (\alpha - \bar\beta)\delta.
\end{aligned} \tag{A.2}$$

The Riemann tensor can be decomposed as follows:

$$\begin{aligned}
R_{abcd} = C_{abcd} &+ \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \bar\Phi_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'}\\
&+ \Lambda\,(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})\,\epsilon_{A'B'}\epsilon_{C'D'}\\
&+ \Lambda\,(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{B'C'}\epsilon_{A'D'})\,\epsilon_{AB}\epsilon_{CD}.
\end{aligned} \tag{A.3}$$

The first part is the Weyl tensor whose spinor equivalent is the totally symmetric Weyl spinor Ψ_{ABCD}:

$$C_{abcd} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar\Psi_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}. \tag{A.4}$$

The scalar Λ is related to the scalar curvature R by

$$\Lambda = \tfrac{1}{24} R. \tag{A.5}$$

The symmetric Ricci spinor $\Phi_{ABC'D'}$ is equivalent to the trace-free part of the Ricci tensor:

$$R_{ab} = -2\Phi_{ABA'B'} + 6\Lambda\epsilon_{AB}\epsilon_{A'B'}. \tag{A.6}$$

The spinor equivalent of the Einstein tensor is

$$G_{ab} = -2\Phi_{ABA'B'} - 6\Lambda\epsilon_{AB}\epsilon_{A'B'}, \tag{A.7}$$

and the spinor equivalent of Einstein's equations is

$$\Phi_{ABA'B'} = -\,3\,\Lambda\epsilon_{AB}\epsilon_{A'B'} + 4\pi\, T_{ABA'B'}. \tag{A.8}$$

Taking the symmetric part or contracting them with $\epsilon^{AB}\epsilon^{A'B'}$, respectively, we obtain two equations, equivalent to (A.8):

$$\begin{aligned}
\Phi_{ABA'B'} &= 4\pi\, T_{(AB)(A'B')},\\
3\Lambda &= \pi\, T_A{}^A{}_{A'}{}^{A'}.
\end{aligned} \tag{A.9}$$

The five complex components of the Weyl spinor are

$$\begin{aligned}
\Psi_0 &= C_{abcd} l^a m^b l^c m^d = \Psi_{ABCD}\, o^A o^B o^C o^D,\\
\Psi_1 &= C_{abcd} l^a n^b l^c m^d = \Psi_{ABCD}\, o^A o^B o^C \iota^D,\\
\Psi_2 &= C_{abcd} l^a m^b \bar m^c n^d = \Psi_{ABCD}\, o^A o^B \iota^C \iota^D,\\
\Psi_3 &= C_{abcd} l^a n^b \bar m^c n^d = \Psi_{ABCD}\, o^A \iota^B \iota^C \iota^D,\\
\Psi_4 &= C_{abcd} \bar m^a n^b \bar m^c n^d = \Psi_{ABCD}\, \iota^A \iota^B \iota^C \iota^D.
\end{aligned} \tag{A.10}$$

The traceless Ricci tensor has the following components (three real and three complex):

$$\begin{aligned}
\Phi_{00} &= -\tfrac{1}{2}R_{ab}l^a l^b = \Phi_{ABA'B'}o^A o^B \bar{o}^{A'}\bar{o}^{B'},\\
\Phi_{01} &= -\tfrac{1}{2}R_{ab}l^a m^b = \Phi_{ABA'B'}o^A o^B \bar{o}^{A'}\bar{\iota}^{B'},\\
\Phi_{02} &= -\tfrac{1}{2}R_{ab}m^a m^b = \Phi_{ABA'B'}o^A o^B \bar{\iota}^{A'}\bar{\iota}^{B'},\\
\Phi_{11} &= -\tfrac{1}{4}R_{ab}(l^a n^b + m^a\bar{m}^b) = \Phi_{ABA'B'}o^A \iota^B \bar{o}^{A'}\bar{\iota}^{B'},\\
\Phi_{12} &= -\tfrac{1}{2}R_{ab}n^a m^b = \Phi_{ABA'B'}o^A \iota^B \bar{\iota}^{A'}\bar{\iota}^{B'},\\
\Phi_{22} &= -\tfrac{1}{2}R_{ab}n^a n^b = \Phi_{ABA'B'}\iota^A \iota^B \bar{\iota}^{A'}\bar{\iota}^{B'}.
\end{aligned} \tag{A.11}$$

The three remaining components can be obtained via the condition $\Phi_{ij} = \bar{\Phi}_{ji}$. Under the conformal rescaling $g_{ab} = \Omega^2\tilde{g}_{ab}$, the covariant derivative acting on a two-component spinor transforms as

$$\tilde{\nabla}_{AA'}\xi_B = \nabla_{AA'}\xi_B + \Omega^{-1}\,\xi_A\nabla_{BA'}\Omega. \tag{A.12}$$

The NP quantities also transform. To find relations between the physical and unphysical quantities, we have to transform the null tetrad. We wish to keep $n_a = \tilde{n}_a = \partial_a v$ so the correct choice is

$$\begin{array}{llll}
o^A = \tilde{o}^A, & \iota^A = \Omega^{-1}\,\tilde{\iota}^A, & o_A = \Omega\,\tilde{o}_A, & \iota_A = \tilde{\iota}_A,\\
l^a = \tilde{l}^a, & n^a = \Omega^{-2}\tilde{n}^a, & m^a = \Omega^{-1}\,\tilde{m}^a, & \bar{m}^a = \Omega^{-1}\,\bar{\tilde{m}}^a\\
l_a = \Omega^2\tilde{l}_a, & n_a = \tilde{n}_a, & m_a = \Omega\,\tilde{m}_a, & \bar{m}_a = \Omega\,\bar{\tilde{m}}_a,
\end{array} \tag{A.13}$$

from which the transformation of the spin coefficients can be found.

The geometrical meaning of the spin coefficients depends on the choice of the null tetrad. With our choices, the vector l is pointing into $\mathcal{I}^+$, while n is tangent to $\mathcal{I}^+$. On $\mathcal{I}^-$ the role of these vectors is interchanged, n is pointing from $\mathcal{I}^-$ and l is tangent to it. To convert quantities from $\mathcal{I}^+$ to $\mathcal{I}^-$, we have only to switch the spinors o^A and ι^A (and adjust some signs). The correspondence between the quantities on $\mathcal{I}^+$ and $\mathcal{I}^-$ is given in the following table:

$$\begin{array}{ll}
\kappa \leftrightarrow \nu, & \tau \leftrightarrow \pi,\\
\sigma \leftrightarrow \lambda, & \rho \leftrightarrow \mu,\\
\varepsilon \leftrightarrow \gamma, & \alpha \leftrightarrow \beta,\\
\Psi_n \leftrightarrow \Psi_{4-n}, & \Phi_{ij} \leftrightarrow \Phi_{(2-i)(2-j)}.
\end{array} \tag{A.14}$$

The scalar curvature and the Ricci spinor transform according to the formulae

$$\begin{aligned}
\tilde{R} &= \Omega^2 R - 6\Omega\Box\Omega + 12g^{ab}\,(\nabla_a\Omega)\,(\nabla_b\Omega)\,,\\
\tilde{\Phi}_{ABA'B'} &= \Phi_{ABA'B'} + \Omega^{-1}\nabla_{A(A'}\nabla_{B')B}\Omega,
\end{aligned} \tag{A.15}$$

the NP components of the Weyl spinor as

$$\tilde{\Psi}_n = \Omega^n\Psi_n. \tag{A.16}$$

The Weyl spinor is conformally invariant with weight zero:

$$\Psi_{ABCD} = \tilde{\Psi}_{ABCD}. \tag{A.17}$$

Because the physical Weyl spinor vanishes on $\mathcal{I}^-$, so does the unphysical one, and assuming smoothness is therefore $\mathcal{O}(\Omega)$. Then we get

$$\tilde{\Psi}_n \in \mathcal{O}(\Omega^{n+1}). \tag{A.18}$$

The Ricci identities can be written in the spinor form as follows:

$$\begin{aligned}
\nabla_{A'(A}\nabla^{A'}_{B)}\xi_C &= \Psi_{ABCD}\xi^D - 2\Lambda\xi_{(A}\epsilon_{B)C},\\
\nabla_{A(A'}\nabla^{A}_{B')}\xi_C &= \Phi_{CDA'B'}\xi^D.
\end{aligned} \tag{A.19}$$

Substituting the basis spinors o_A and ι_A for ξ_A and projecting the last equations onto the spin basis we obtain the Ricci identities in the NP-formalism:

$$D\rho - \bar{\delta}\kappa = \rho^2 + (\epsilon + \bar{\epsilon})\rho - \kappa\left(3\alpha + \bar{\beta} - \pi\right) - \tau\bar{\kappa} + \sigma\bar{\sigma} + \Phi_{00}, \tag{A.20a}$$

$$D\sigma - \delta\kappa = (\rho + \bar{\rho} + 3\varepsilon - \bar{\varepsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0, \tag{A.20b}$$

$$D\tau - \Delta\kappa = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + (\varepsilon - \bar{\varepsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01}, \tag{A.20c}$$

$$D\alpha - \bar{\delta}\varepsilon = (\rho + \bar{\varepsilon} - 2\varepsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\varepsilon - \kappa\lambda - \bar{\kappa}\gamma + (\varepsilon + \rho)\pi + \Phi_{10}, \tag{A.20d}$$

$$D\beta - \delta\varepsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\varepsilon})\beta - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\varepsilon + \Psi_1, \tag{A.20e}$$

$$D\gamma - \Delta\varepsilon = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\varepsilon + \bar{\varepsilon})\gamma - (\gamma + \bar{\gamma})\varepsilon + \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11}, \tag{A.20f}$$

$$D\lambda - \bar{\delta}\pi = (\rho - 3\varepsilon + \bar{\varepsilon})\lambda + \bar{\sigma}\mu + (\pi + \alpha - \bar{\beta})\pi - \nu\bar{\kappa} + \Phi_{20}, \tag{A.20g}$$

$$D\mu - \delta\pi = (\bar{\rho} - \varepsilon - \bar{\varepsilon})\mu + \sigma\lambda + (\bar{\pi} - \bar{\alpha} + \beta)\pi - \nu\kappa + \Psi_2 + 2\Lambda, \tag{A.20h}$$

$$D\nu - \Delta\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi - (3\varepsilon + \bar{\varepsilon})\nu + \Psi_3 + \Phi_{21}, \tag{A.20i}$$

$$\Delta\lambda - \bar{\delta}\nu = -(\mu + \bar{\mu} + 3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4, \tag{A.20j}$$

$$\Delta\mu - \delta\nu = -(\mu + \gamma + \bar{\gamma})\mu - \lambda\bar{\lambda} + \bar{\nu}\pi + (\bar{\alpha} + 3\beta - \tau)\nu - \Phi_{22}, \tag{A.20k}$$

$$\Delta\beta - \delta\gamma = (\bar{\alpha} + \beta - \tau)\gamma - \mu\tau + \sigma\nu + \varepsilon\bar{\nu} + (\gamma - \bar{\gamma} - \mu)\beta - \alpha\bar{\lambda} - \Phi_{12}, \tag{A.20l}$$

$$\Delta\sigma - \delta\tau = -(\mu - 3\gamma + \bar{\gamma})\sigma - \bar{\lambda}\rho - (\tau + \beta - \bar{\alpha})\tau + \kappa\bar{\nu} - \Phi_{02}, \tag{A.20m}$$

$$\Delta\rho - \bar{\delta}\tau = (\gamma + \bar{\gamma} - \bar{\mu})\rho - \sigma\lambda + (\bar{\beta} - \alpha - \bar{\tau})\tau + \nu\kappa - \Psi_2 - 2\Lambda, \tag{A.20n}$$

$$\Delta\alpha - \bar{\delta}\gamma = (\rho + \varepsilon)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3, \tag{A.20o}$$

$$\delta\rho - \bar{\delta}\sigma = (\bar{\alpha} + \beta)\rho - (3\alpha - \bar{\beta})\sigma + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01}, \tag{A.20p}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + (\rho - \bar{\rho})\gamma + (\mu - \bar{\mu})\varepsilon - \Psi_2 + \Lambda + \Phi_{11}, \tag{A.20q}$$

$$\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21}. \tag{A.20r}$$

The spinor form of the Bianchi identities is

$$\nabla^D_{B'}\Psi_{ABCD} = \nabla^{A'}_A\Phi_{BCA'B'} + \epsilon_{C(A}\nabla_{B)B'}\Lambda - \tfrac{3}{2}\epsilon_{AB}\nabla_{CB'}\Lambda. \tag{A.21}$$

Projecting these equations onto the spin basis leads to the Bianchi identities in the NP formalism:

$$\begin{aligned} D\Psi_1 - \bar{\delta}\Psi_0 - D\Phi_{01} + \delta\Phi_{00} = {} & (\pi - 4\alpha)\Psi_0 + 2(2\rho + \varepsilon)\Psi_1 - 3\kappa\Psi_2 + 2\kappa\Phi_{11} \\ & - (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} - 2\sigma\Phi_{10} - 2(\bar{\rho} + \varepsilon)\Phi_{01} + \bar{\kappa}\Phi_{02}, \end{aligned} \tag{A.22a}$$

$$\begin{aligned} D\Psi_2 - \bar{\delta}\Psi_1 + \Delta\Phi_{00} - \bar{\delta}\Phi_{01} + 2D\Lambda = {} & -\lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 \\ & + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} + (2\gamma + 2\bar{\gamma} - \bar{\mu})\Phi_{00} - 2(\alpha + \bar{\tau})\Phi_{01} - 2\tau\Phi_{10}, \end{aligned} \tag{A.22b}$$

$$\begin{aligned} D\Psi_3 - \bar{\delta}\Psi_2 - D\Phi_{21} + \delta\Phi_{20} - 2\bar{\delta}\Lambda = {} & -2\lambda\Psi_1 + 3\pi\Psi_2 + 2(\rho - \varepsilon)\Psi_3 - \kappa\Psi_4 \\ & + 2\mu\Phi_{10} - 2\pi\Phi_{11} - (2\beta + \bar{\pi} - 2\bar{\alpha})\Phi_{20} - 2(\bar{\rho} - \varepsilon)\Phi_{21} + \bar{\kappa}\Phi_{22}, \end{aligned} \tag{A.22c}$$

$$\begin{aligned} D\Psi_4 - \bar{\delta}\Psi_3 + \Delta\Phi_{20} - \bar{\delta}\Phi_{21} = {} & -3\lambda\Psi_2 + 2(\alpha + 2\pi)\Psi_3 + (\rho - 4\varepsilon)\Psi_4 + 2\nu\Phi_{10} \\ & - 2\lambda\Phi_{11} - (2\gamma - 2\bar{\gamma} + \bar{\mu})\Phi_{20} - 2(\bar{\tau} - \alpha)\Phi_{21} + \bar{\sigma}\Phi_{22}, \end{aligned} \tag{A.22d}$$

$$\Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2$$
$$+ (\bar\rho + 2\varepsilon - 2\bar\varepsilon)\Phi_{02} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12} - \bar\lambda\Phi_{00} + 2(\bar\pi - \beta)\Phi_{01}, \qquad (A.23a)$$

$$\Delta\Psi_1 - \delta\Psi_2 - \Delta\Phi_{01} + \bar\delta\Phi_{02} - 2\delta\Lambda = \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3$$
$$- \bar\nu\Phi_{00} + 2(\bar\mu - \gamma)\Phi_{01} + (2\alpha + \bar\tau - 2\bar\beta)\Phi_{02} + 2\tau\Phi_{11} - 2\rho\Phi_{12}, \qquad (A.23b)$$

$$\Delta\Psi_2 - \delta\Psi_3 + D\Phi_{22} - \delta\Phi_{21} + 2\Delta\Lambda = 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4$$
$$- 2\mu\Phi_{11} - \bar\lambda\Phi_{20} + 2\pi\Phi_{12} + 2(\beta + \bar\pi)\Phi_{21} + (\bar\rho - 2\varepsilon - 2\bar\varepsilon)\Phi_{22}, \qquad (A.23c)$$

$$\Delta\Psi_3 - \delta\Psi_4 - \Delta\Phi_{21} + \bar\delta\Phi_{22} = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4 - 2\nu\Phi_{11}$$
$$- \bar\nu\Phi_{20} + 2\lambda\Phi_{12} + 2(\gamma + \bar\mu)\Phi_{21} + (\bar\tau - 2\bar\beta - 2\alpha)\Phi_{22}, \qquad (A.23d)$$

$$D\Phi_{11} - \delta\Phi_{10} + \Delta\Phi_{00} - \bar\delta\Phi_{01} + 3D\Lambda = (2\gamma + 2\bar\gamma - \mu - \bar\mu)\Phi_{00} + (\pi - 2\alpha - 2\bar\tau)\Phi_{01}$$
$$+ (\bar\pi - 2\bar\alpha - 2\tau)\Phi_{10} + 2(\rho + \bar\rho)\Phi_{11} + \bar\sigma\Phi_{02} + \sigma\Phi_{20} - \bar\kappa\Phi_{12} - \kappa\Phi_{21}, \qquad (A.24a)$$

$$D\Phi_{12} - \delta\Phi_{11} + \Delta\Phi_{01} - \bar\delta\Phi_{02} + 3\delta\Lambda = (2\gamma - \mu - 2\bar\mu)\Phi_{01} + \bar\nu\Phi_{00} - \bar\lambda\Phi_{10} + 2(\bar\pi - \tau)\Phi_{11}$$
$$+ (\pi + 2\bar\beta - 2\alpha - \bar\tau)\Phi_{02} + (2\rho + \bar\rho - 2\bar\varepsilon)\Phi_{12} + \sigma\Phi_{21} - \kappa\Phi_{22}, \qquad (A.24b)$$

$$D\Phi_{22} - \delta\Phi_{21} + \Delta\Phi_{11} - \bar\delta\Phi_{12} + 3\Delta\Lambda = \nu\Phi_{01} + \bar\nu\Phi_{10} - 2(\mu + \bar\mu)\Phi_{11} - \lambda\Phi_{02} - \bar\lambda\Phi_{20}$$
$$+ (2\pi - \bar\tau + 2\bar\beta)\Phi_{12} + (2\beta - \tau + 2\bar\pi)\Phi_{21} + (\rho + \bar\rho - 2\varepsilon - 2\bar\varepsilon)\Phi_{22}. \qquad (A.24c)$$

A.2. Electromagnetic field

For the description of an electromagnetic field, we use the electromagnetic field tensor F_{ab} and its spinor equivalent ϕ_{AB}:

$$F_{ab} = \phi_{AB}\epsilon_{A'B'} + \bar\phi_{A'B'}\epsilon_{AB}. \qquad (A.25)$$

The NP components of the Maxwell spinor are defined by

$$\begin{aligned}\phi_0 &= F_{ab}\, l^a\, m^b = \phi_{AB}\, o^A\, o^B,\\ \phi_1 &= \tfrac{1}{2}F_{ab}[l^a n^b - m^a \bar m^b] = \phi_{AB}\, o^A\iota^B,\\ \phi_2 &= F_{ab}\bar m^a\, n^b = \phi_{AB}\iota^A\iota^B.\end{aligned} \qquad (A.26)$$

The conformal transformation of these quantities is given by

$$\tilde\phi_{AB} = \Omega\phi_{AB}, \qquad \tilde\phi_i = \Omega^{i+1}\phi_i. \qquad (A.27)$$

Maxwell's equations without sources are equivalent to the (conformally invariant) spin-1 zero-rest-mass equation

$$\nabla^A_{A'}\phi_{AB} = 0. \qquad (A.28)$$

Projecting this onto the spin basis we obtain Maxwell's equations in the NP formalism:

$$D\phi_1 - \bar\delta\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2, \qquad (A.29a)$$

$$D\phi_2 - \bar\delta\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\varepsilon)\phi_2, \qquad (A.29b)$$

$$\Delta\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2, \qquad (A.29c)$$

$$\Delta\phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2. \qquad (A.29d)$$

Appendix B. Conformal field equations

B.1. Einstein–Maxwell fields

The projections of equation (17),

$$\nabla_{AA'}\nabla_{BB'}\Omega = \Omega^3\phi_{AB}\,\bar{\phi}_{A'B'} - \Omega\,\Phi_{ABA'B'} + (F+\Omega\Lambda)\epsilon_{A'B'}\epsilon_{AB}, \tag{B.1}$$

onto the null tetrad imply the following system of equations:

$$DS_0 + (\varepsilon+\bar{\varepsilon})S_0 + \bar{\kappa}S_1 + \kappa\bar{S}_1 = \Omega^3\phi_0\bar{\phi}_0 - \Omega\Phi_{00}, \tag{B.2a}$$

$$DS_1 - \bar{\pi}S_0 + (\bar{\varepsilon}-\varepsilon)S_1 + \kappa S_2 = \Omega^3\phi_0\bar{\phi}_1 - \Omega\Phi_{01}, \tag{B.2b}$$

$$\delta S_0 - (\bar{\alpha}+\beta)S_0 + \bar{\rho}S_1 + \sigma\bar{S}_1 = \Omega^3\phi_0\bar{\phi}_1 - \Omega\Phi_{01}, \tag{B.2c}$$

$$\delta S_1 - \bar{\lambda}S_0 + (\bar{\alpha}-\beta)S_1 + \sigma S_2 = \Omega^3\phi_0\bar{\phi}_2 - \Omega\Phi_{02}, \tag{B.2d}$$

$$DS_2 - F - \Omega\,\Lambda - \pi S_1 - \bar{\pi}\bar{S}_1 + (\varepsilon+\bar{\varepsilon})S_2 = \Omega^3\phi_1\bar{\phi}_1 - \Omega\Phi_{11}, \tag{B.2e}$$

$$\delta\bar{S}_1 + F\ + \Omega\,\Lambda - \mu S_0 + (\beta-\bar{\alpha})\bar{S}_1 + \bar{\rho}S_2 = \Omega^3\phi_1\bar{\phi}_1 - \Omega\Phi_{11}, \tag{B.2f}$$

$$\delta S_2 - \mu S_1 - \bar{\lambda}\bar{S}_1 + (\bar{\alpha}+\beta)S_2 = \Omega^3\phi_1\bar{\phi}_2 - \Omega\Phi_{12}, \tag{B.2g}$$

$$\Delta S_0 - F - \Omega\,\Lambda - (\gamma+\bar{\gamma})S_0 + \bar{\tau}S_1 + \tau\bar{S}_1 = \Omega^3\phi_1\bar{\phi}_1 - \Omega\Phi_{11}, \tag{B.2h}$$

$$\Delta S_1 - \bar{\nu}S_0 + (\bar{\gamma}-\gamma)S_1 + \tau S_2 = \Omega^3\phi_1\bar{\phi}_2 - \Omega\Phi_{12}, \tag{B.2i}$$

$$\Delta S_2 - \nu S_1 - \bar{\nu}\bar{S}_1 + (\gamma+\bar{\gamma})S_2 = \Omega^3\phi_2\bar{\phi}_2 - \Omega\Phi_{22}. \tag{B.2j}$$

The projections of equation (19),

$$\nabla_{AA'}F = \Omega^2\phi_A^B\bar{\phi}_{A'}^{B'}\nabla_{BB'}\Omega - \Phi_{ABA'B'}\nabla^{BB'}\Omega + \Lambda\,\nabla_{AA'}\Omega, \tag{B.3}$$

give

$$\begin{aligned} DF = {} & -S_0\Phi_{11} + S_1\Phi_{10} + \bar{S}_1\Phi_{01} - S_2\Phi_{00} \\ & + \Omega^2[S_0\phi_1\bar{\phi}_1 - S_1\phi_1\bar{\phi}_0 - \bar{S}_1\phi_0\bar{\phi}_1 + S_2\phi_0\bar{\phi}_0] + \Lambda S_0, \end{aligned} \tag{B.4a}$$

$$\begin{aligned} \delta F = {} & -S_0\Phi_{12} + S_1\Phi_{11} + \bar{S}_1\Phi_{02} - S_2\Phi_{01} \\ & + \Omega^2[S_0\phi_1\bar{\phi}_2 - S_1\phi_1\bar{\phi}_1 - \bar{S}_1\phi_0\bar{\phi}_2 + S_2\phi_0\bar{\phi}_1] + \Lambda S_1, \end{aligned} \tag{B.4b}$$

$$\begin{aligned} \Delta F = {} & -S_0\Phi_{22} + S_1\Phi_{21} + \bar{S}_1\Phi_{12} - S_2\Phi_{11} \\ & + \Omega^2[S_0\phi_2\bar{\phi}_2 - S_1\phi_2\bar{\phi}_1 - \bar{S}_1\phi_1\bar{\phi}_2 + S_2\phi_1\bar{\phi}_1] + \Lambda S_2. \end{aligned} \tag{B.4c}$$

The conformal Bianchi identities (9) for the Einstein–Maxwell field projected onto the spin basis imply the following system:

$$\begin{aligned} D\psi_1 - \bar{\delta}\psi_0 = {} & (\pi - 4\alpha)\psi_0 + 2(\varepsilon+2\rho)\psi_1 - 3\kappa\psi_2 - 3S_1\phi_0\bar{\phi}_0 + 3S_0\phi_0\bar{\phi}_1 \\ & + \Omega[2\sigma\phi_1\bar{\phi}_0 - 2\beta\phi_0\bar{\phi}_0 + 2\varepsilon\phi_0\bar{\phi}_1 - 2\kappa\phi_1\bar{\phi}_1 + \bar{\phi}_0\delta\phi_0 - \bar{\phi}_1 D\phi_0], \end{aligned} \tag{B.5a}$$

$$\begin{aligned} D\psi_2 - \bar{\delta}\psi_1 = {} & -\lambda\psi_0 + 2(\pi-\alpha)\psi_1 + 2\rho\psi_2 - 2\kappa\psi_3 - S_2\phi_0\bar{\phi}_0 - 2S_1\phi_1\bar{\phi}_0 \\ & + 2S_0\phi_1\bar{\phi}_1 + \bar{S}_1\phi_0\bar{\phi}_1 + \tfrac{2}{3}\,\Omega[\bar{\phi}_0\delta\phi_1 - \bar{\phi}_1 D\phi_1 - (\gamma+\mu)\phi_0\bar{\phi}_0 + \tau\phi_1\bar{\phi}_0 \\ & + (\alpha+\pi)\phi_0\bar{\phi}_1 + \sigma\phi_2\bar{\phi}_0 - \rho\phi_1\bar{\phi}_1 - \kappa\phi_2\bar{\phi}_1] + \tfrac{1}{3}\,\Omega[\bar{\phi}_0\Delta\phi_0 - \bar{\phi}_1\bar{\delta}\phi_0], \end{aligned} \tag{B.5b}$$

$$
\begin{aligned}
D\psi_3 - \bar{\delta}\psi_2 = {}& -2\lambda\psi_1 + 3\pi\psi_2 + 2(\rho-\varepsilon)\psi_3 - \kappa\psi_4 - 2S_2\phi_1\bar{\phi}_0 - S_1\phi_2\bar{\phi}_0 + S_0\phi_2\bar{\phi}_1 \\
& + 2\bar{S}_1\phi_1\bar{\phi}_1 + \tfrac{2}{3}\,\Omega[-\nu\phi_0\bar{\phi}_0 - \mu\phi_1\bar{\phi}_0 + \lambda\phi_0\bar{\phi}_1 + (\beta+\tau)\phi_2\bar{\phi}_0 + \pi\phi_1\bar{\phi}_1 \\
& - (\varepsilon+\rho)\phi_2\bar{\phi}_1 + \bar{\phi}_0\Delta\phi_1 - \bar{\phi}_1\bar{\delta}\phi_1] + \tfrac{1}{3}\,\Omega[\bar{\phi}_0\delta\phi_2 - \bar{\phi}_1 D\phi_2], \qquad \text{(B.5}c\text{)}
\end{aligned}
$$

$$
\begin{aligned}
D\psi_4 - \bar{\delta}\psi_3 = {}& -3\lambda\psi_2 + 2(\alpha+2\pi)\psi_3 + (\rho-4\varepsilon)\psi_4 - 3S_2\phi_2\bar{\phi}_0 + 3\bar{S}_1\phi_2\bar{\phi}_1 \\
& + \Omega[\bar{\phi}_0\Delta\phi_2 - \bar{\phi}_1\bar{\delta}\phi_2 - 2\nu\phi_1\bar{\phi}_0 + 2\gamma\phi_2\bar{\phi}_0 + 2\lambda\phi_1\bar{\phi}_1 - 2\alpha\phi_2\bar{\phi}_1], \qquad \text{(B.5}d\text{)}
\end{aligned}
$$

$$
\begin{aligned}
\delta\psi_1 - \Delta\psi_0 = {}& (\mu-4\gamma)\psi_0 + 2(\beta+2\tau)\psi_1 - 3\sigma\psi_2 - 3S_1\phi_0\bar{\phi}_1 + 3S_0\phi_0\bar{\phi}_2 \\
& + \Omega[-2\beta\phi_0\bar{\phi}_1 + 2\sigma\phi_1\bar{\phi}_1 + 2\varepsilon\phi_0\bar{\phi}_2 - 2\kappa\phi_1\bar{\phi}_2 - \bar{\phi}_2 D\phi_0 + \bar{\phi}_1\delta\phi_0], \qquad \text{(B.5}e\text{)}
\end{aligned}
$$

$$
\begin{aligned}
\delta\psi_2 - \Delta\psi_1 = {}& -\nu\psi_0 + 2(\mu-\gamma)\psi_1 + 3\tau\psi_2 - 2\sigma\psi_3 - S_2\phi_0\bar{\phi}_1 - 2S_1\phi_1\bar{\phi}_1 + 2S_0\phi_1\bar{\phi}_1 \\
& + \bar{S}_1\phi_0\bar{\phi}_2 + \tfrac{2}{3}\,\Omega[-(\gamma+\mu)\phi_0\bar{\phi}_1 + \tau\phi_1\bar{\phi}_1 + \sigma\phi_2\bar{\phi}_1 + (\pi+\alpha)\phi_0\bar{\phi}_2 \\
& - \rho\phi_1\bar{\phi}_2 - \kappa\phi_2\bar{\phi}_2 + \bar{\phi}_1\delta\phi_1 - \bar{\phi}_2 D\phi_1] + \tfrac{1}{3}\,\Omega[\bar{\phi}_1\Delta\phi_0 - \bar{\phi}_2\bar{\delta}\phi_0], \qquad \text{(B.5}f\text{)}
\end{aligned}
$$

$$
\begin{aligned}
\delta\psi_3 - \Delta\psi_2 = {}& -2\nu\psi_1 + 3\mu\psi_2 + 2(\tau-\beta)\psi_3 - \sigma\psi_4 - 2S_2\phi_1\bar{\phi}_1 - S_1\phi_2\bar{\phi}_1 + S_0\phi_2\bar{\phi}_2 \\
& + 2\bar{S}_1\phi_1\bar{\phi}_2 + \tfrac{2}{3}\,\Omega[-\nu\phi_0\bar{\phi}_1 - \mu\phi_1\bar{\phi}_1 + (\beta+\tau)\phi_2\bar{\phi}_1 + \lambda\phi_0\bar{\phi}_2 + \pi\phi_1\bar{\phi}_2 \\
& - (\varepsilon+\rho)\phi_2\bar{\phi}_2 + \bar{\phi}_1\Delta\phi_1 - \bar{\phi}_2\bar{\delta}\phi_1] + \tfrac{1}{3}\,\Omega[\bar{\phi}_1\delta\phi_2 - \bar{\phi}_2 D\phi_2], \qquad \text{(B.5}g\text{)}
\end{aligned}
$$

$$
\begin{aligned}
\delta\psi_4 - \Delta\psi_3 = {}& -3\nu\psi_2 + 2(2\gamma+2\mu)\psi_3 + (\tau-4\beta)\psi_4 - 3S_2\phi_2\bar{\phi}_1 + 3\bar{S}_1\phi_1\bar{\phi}_2 \\
& + \Omega[-2\nu\phi_1\bar{\phi}_1 + 2\gamma\phi_2\bar{\phi}_1 + 2\lambda\phi_1\bar{\phi}_2 - 2\alpha\phi_2\bar{\phi}_2 + \bar{\phi}_1\Delta\phi_2 - \bar{\phi}_2\bar{\delta}\phi_2]. \qquad \text{(B.5}h\text{)}
\end{aligned}
$$

Appendix C. Reissner–Nordström spacetime

To justify our choice of gauge and show that the choice made by [10] is too restrictive, we shall show here how a simple spacetime, namely the Reissner–Nordström solution, appears in our gauge. The physical metric is

$$
\mathrm{d}\tilde{s}^2 = \left(1 - \frac{2m}{\tilde{r}} + \frac{Q^2}{\tilde{r}^2}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{\tilde{r}} + \frac{Q^2}{\tilde{r}^2}\right)^{-1}\mathrm{d}\tilde{r}^2 - \tilde{r}^2\,\mathrm{d}\Sigma^2, \qquad \text{(C.1)}
$$

where Q is the charge and $\mathrm{d}\Sigma^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. In the standard conformal compactification of the Reissner–Nordström spacetime one introduces the 'tortoise coordinate' r^* and the advanced time v by

$$
\begin{aligned}
\mathrm{d}\tilde{r} &= \left(1 - \frac{2m}{\tilde{r}} + \frac{Q^2}{\tilde{r}^2}\right)\mathrm{d}r^*, \\
v &= t + r^*.
\end{aligned} \qquad \text{(C.2)}
$$

In these coordinates the physical metric acquires the form

$$
\mathrm{d}\tilde{s}^2 = \left(1 - \frac{2m}{\tilde{r}} + \frac{Q^2}{\tilde{r}^2}\right)(\mathrm{d}v^2 - 2\,\mathrm{d}v\,\mathrm{d}r^*) - \tilde{r}^2\,\mathrm{d}\Sigma^2. \qquad \text{(C.3)}
$$

We compactify it by defining the coordinate

$$
r = \tilde{r}^{-1} \qquad \text{(C.4)}
$$

and the conformal factor

$$
\Omega = r. \qquad \text{(C.5)}
$$

The unphysical metric then reads

$$\mathrm{d}s^2 = r^2(1 - 2\,m\,r + Q^2\,r^2)\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r - \mathrm{d}\Sigma^2. \tag{C.6}$$

Comparing this with (31)–(34), we find the metric functions to be

$$\begin{aligned} H &= \frac{1}{2}\,r^2 - m\,r^3 + \frac{1}{2}\,Q^2\,r^4, \\ C^I &= 0, \\ P^2 &= \frac{1}{\sqrt{2}}, \\ P^3 &= \frac{\mathrm{i}}{\sqrt{2}\,\sin\theta}. \end{aligned} \tag{C.7}$$

From the metric the other geometrical quantities follow. The spin coefficients are all zero, except for

$$\varepsilon = -\tfrac{1}{2}\,r + \tfrac{3}{2}\,m\,r^2 - Q^2\,r^3, \qquad \alpha = -\beta = -\tfrac{1}{2\sqrt{2}}\cot\theta. \tag{C.8}$$

The non-zero components of the Weyl and Ricci tensor read

$$\begin{aligned} &\psi_2 = m - Q^2\,r, \\ &\Phi_{11} = \tfrac{1}{2} - \tfrac{3}{2}\,m\,r + \tfrac{3}{2}\,Q^2\,r^2, \\ &\Lambda = \tfrac{1}{2}\,m\,r - \tfrac{1}{2}\,Q^2\,r^2. \end{aligned} \tag{C.9}$$

The electromagnetic 4-potential and corresponding electromagnetic tensor in these coordinates are

$$A_\mu = (Q\,r, 0, 0, 0), \qquad F_{\mu\nu} = -Q\epsilon_{\mu\nu 2 3}. \tag{C.10}$$

The only non-vanishing NP component of $F_{\mu\nu}$ is

$$\phi_1 = Q, \tag{C.11}$$

as one would expect. All these results are in accordance with results obtained in the text. On the other hand, the gauge condition $\Lambda = 0$ everywhere, imposed in [10], leads to a periodic unphysical metric only if $m = 0$, i.e. flat spacetime. This can be seen as follows: we need to rescale the metric (C.6) say to

$$\hat{g}_{ab} = \Theta^{-2} g_{ab}$$

so that, by (A.15),

$$\hat{\Lambda} = \Theta^{-2}\big(\Theta\Lambda + \tfrac{1}{4}\,\Box\,\Theta\big) = 0,$$

where the boundary conditions on Θ are that $\Theta = 1$ on $r = 0$ and, say, $v = 0$ (in order to preserve the conditions that $\rho = 0$ on $r = 0$, $\mu = 0$ on $v = 0$ and $\Theta = 1$ on $v = r = 0$). With the metric (C.6), this wave equation on Θ becomes

$$2\partial_v\partial_r\Theta = \partial_r(A\partial_r\Theta) - L^2\Theta - 2(mr - Qr^2)\Theta, \tag{C.12}$$

with $A = r^2(1 - 2mr + Q^2r^2)$ and

$$L^2\Theta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Theta}{\partial\theta^2}.$$

Now from (C.12) evaluated at $r = 0$, we calculate $\partial_v\partial_r\Theta = 0$ so that $\partial_r\Theta$ is constant on $\mathcal{I}^-$, but it vanishes at $v = 0$ so it is zero for all v. Then from (C.12) again at $\mathcal{I}^-$,

$$\partial_v\partial_r^2\Theta = -m.$$

Thus Θ cannot be periodic in v unless $m = 0$, in which case the physical metric is flat.

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Chapter 8

On asymptotically flat solutions of Einsteins equations periodic in time: II. Spacetimes with scalar-field sources

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On asymptotically flat solutions of Einstein's equations periodic in time: II. Spacetimes with scalar-field sources

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Abstract

We extend the work in our earlier paper (Bičák J *et al* 2010 *Class. Quantum Grav.* **27** 055007) to show that time-periodic, asymptotically flat solutions of the Einstein equations analytic at $\mathcal{I}$, whose source is one of a range of scalar-field models, are necessarily stationary. We also show that, for some of these scalar-field sources, in stationary, asymptotically flat solutions analytic at $\mathcal{I}$, the scalar field necessarily inherits the symmetry. To prove these results we investigate miscellaneous properties of massless and conformal scalar fields coupled to gravity, in particular Bondi mass and its loss.

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1. Introduction

In this paper, we continue the study begun in [4] (paper I) of asymptotically flat solutions of Einstein's equations that are periodic in time. In [4], we showed that such spacetimes, if either vacuum or electrovacuum and analytic at $\mathcal{I}$, are necessarily stationary near $\mathcal{I}$. Here we extend this result to spacetimes whose source is one of a range of scalar-field models.

In [4], we also considered the problem of inheritance of symmetry. This is the question of whether, if a spacetime which is a solution of Einstein's equations with some matter source has a symmetry, the matter source necessarily has the same symmetry. For asymptotically flat electrovacuum spacetimes which are analytic near $\mathcal{I}$ we showed that the symmetry is necessarily inherited. For scalar-field sources, we now obtain the same result in some cases but not in others.

The scalar fields we consider fall into two broad classes. The first class includes the complex, massless Klein–Gordon (KG) field which satisfies wave equation (1) and has an energy–momentum tensor as in (2). Here we prove

Theorem 5.2. *A weakly asymptotically simple time-periodic solution of the Einstein-massless-KG field equations which is analytic in a neighbourhood of $\mathcal{I}^-$ necessarily has a Killing vector which is time-like in the interior and extends to a translation on $\mathcal{I}^-$.*

It is also possible to include a potential for the scalar field, as in subsection (2.3), and therefore to include a mass-term, and the above result will continue to hold subject to a weak condition on the potential. Now one knows, for example from [5], that there exist boson-star solutions of the Einstein-massive-KG system for which the metric is static, spherically symmetric and asymptotically flat, while the complex scalar field takes the form $f(r)\,\mathrm{e}^{\mathrm{i}\omega t}$: these solutions are genuinely periodic in time but not stationary, and the source does not inherit all the symmetries of the metric. However, it is easy to see that these solutions are not analytic near $\mathcal{I}$, which is why they do not violate our result.

The other class of scalar fields contains what we shall call *the conformal scalar field*, that is, it satisfies the conformally invariant wave equation (35). For simplicity, we shall take the field to be real, though the formalism allows a complex field. In the real case, there is a conserved energy–momentum tensor for such a source due originally to [14] (see also [7] and [15]), given in (36). This leads to a form of the Einstein equations (37) studied in [11], where it was shown that there is a well-posed initial-value problem. With suitable data on hyperboloidal surfaces extending to $\mathcal{I}$, there exist asymptotically flat solutions even with a regular point at i^+ [11]. Explicit static spherically symmetric solutions were earlier given in [2, 3]. Starting from the assumption of an asymptotically flat solution of these Einstein equations, we may proceed as before, with the corresponding result.

Theorem 5.3. *A weakly asymptotically simple time-periodic solution of the Einstein-conformal-scalar-field equations which is analytic in a neighbourhood of $\mathcal{I}^-$ necessarily has a Killing vector which is time-like in the interior and extends to a translation on $\mathcal{I}^-$.*

Turning to the question of inheritance, for the first class of fields we show (theorem 6.1) that the only way a stationary symmetry can fail to be inherited in the class of spacetimes under consideration is if the (necessarily complex) scalar field has the form $f(x^i)\,\mathrm{e}^{\mathrm{i}\omega t}$ in terms of the comoving space-coordinates x^i and time t. This is periodic and the previous result can be applied to deduce that the symmetry is in fact inherited. For a complex conformal scalar field, the same argument can be used to show that there are no non-inheriting fields of this form but this is only a partial result as we cannot characterize the non-inheriting fields in the same way.

The plan of the paper is as follows: in section 2 we review the Einstein-massless-KG equations and show how to formulate the *conformal Einstein-massless-KG equations*, by which we mean the equations formulated for an unphysical, rescaled metric which correspond to the physical Einstein-massless-KG equations. This enables the equations to be extended to $\mathcal{I}$. In section 3, we do the same thing for the Einstein-conformal-scalar equations, using the conserved energy–momentum tensor proposed in [14] (see also [7]). This energy–momentum tensor does not satisfy the dominant energy condition, but we give some arguments why it might nonetheless lead to positive total energy. In section 4, we give expressions for the Bondi mass and Bondi mass-loss for both classes of scalar-field sources. The Bondi mass-loss for the conformal scalar field is not manifestly positive (at $\mathcal{I}^+$) but in the periodic case the average over a period is. In section 5, we recall the coordinate and null-tetrad system used in [4] and prove theorems 5.2 and 5.3 to show that in this setting, periodic solutions are actually stationary. The proof is much as in [4]: one shows inductively that all radial derivatives of all metric components at $\mathcal{I}^-$ are v-independent so $K = \partial/\partial v$ is a Killing vector. In section 6, we discuss inheritance and prove theorem 6.1 to show that stationarity is necessarily inherited in an analytic, weakly asymptotically flat, Einstein-massless-KG solution.

In order to be able to follow clearly the arguments in the main text, in appendix A we review all Newman–Penrose equations for a general source: that is, the commutation relations of the NP operators, and the Ricci and Bianchi identities. In appendix B the conformally rescaled scalar wave equations and conformal Bianchi identities for the massless scalar field are written down in an unphysical space in manifestly regular form. The regular conformal Bianchi identities for conformally invariant scalar fields follow in fact from the conformal Bianchi identities for any matter field for which the Ricci spinor behaves as $\mathcal{O}(\Omega^2)$ at $\mathcal{I}$. The projections of the Bianchi identities are given in section B.4 of appendix B. The asymptotic form of the solutions of the Einstein-massless-scalar-field equations at future null infinity $\mathcal{I}^+$ is discussed in appendix C; this is used in section 4 in the derivation of the Bondi mass and the mass-loss formula for both massless KG field and conformal-scalar field. Finally, in appendix D we give some examples of exact solutions of the Einstein-conformal-scalar equations and discuss the possible presence of singularities.

2. The massless KG field

2.1. Basic relations

First we investigate the complex scalar field which satisfies the massless KG equation in the physical spacetime:

$$\tilde{\Box}\tilde{\phi} = 0. \tag{1}$$

We shall consistently use the tilde to indicate quantities in the physical spacetime, untilded quantities referring to the rescaled, unphysical spacetime. The energy–momentum tensor whose conservation is implied by this equation is

$$\tilde{T}_{ab} = \frac{1}{4\pi}[2(\tilde{\nabla}_{(a}\tilde{\phi})(\tilde{\nabla}_{b)}\bar{\tilde{\phi}}) - \tilde{g}_{ab}\tilde{g}^{cd}(\tilde{\nabla}_c\tilde{\phi})(\tilde{\nabla}_d\bar{\tilde{\phi}})]. \tag{2}$$

First we have to determine the conformal behaviour of the scalar field. Since wave equation (1) is not conformally invariant, there is *a priori* no preferred choice. However, since near $\mathcal{I}^-$ ($\tilde{r} \to \infty$) the radiative part of the field behaves as

$$\tilde{\phi} \sim \frac{1}{\tilde{r}}, \tag{3}$$

and we wish to have a non-vanishing regular unphysical field on $\mathcal{I}^-$, we define

$$\tilde{\phi} = \Omega\,\phi. \tag{4}$$

In the following, we employ the notation

$$\tilde{\varphi}_{AA'} = \nabla_{AA'}\tilde{\phi}, \qquad \varphi_{AA'} = \nabla_{AA'}\phi, \qquad s_{AA'} = \nabla_{AA'}\Omega, \tag{5}$$

and using the NP formalism[4] we denote the components of the fields s_a and φ_a by the special symbols:

$$\begin{aligned} &D\Omega = s_{00'} = S_0, \quad \delta\Omega = s_{01'} = S_1, \quad \bar{\delta}\Omega = s_{10'} = S_{\bar{1}} = \bar{S}_1, \quad \Delta\Omega = s_{11'} = S_2,\\ &D\phi = \varphi_{00'} = \varphi_0, \quad \delta\phi = \varphi_{01'} = \varphi_1, \quad \bar{\delta}\phi = \varphi_{10'} = \varphi_{\bar{1}}, \quad \Delta\phi = \varphi_{11'} = \varphi_2, \end{aligned} \tag{6}$$

and correspondingly with tildes in the physical spacetime.

[4] The explicit expressions for the NP tetrad, the corresponding spin basis, the NP operators, etc. in the coordinate system (v, r, θ, ϕ) used in the following are introduced in section 3 and appendix A of paper I and repeated in section 5.

In this notation, the spinor form of Einstein's equations in the physical spacetime is

$$\begin{aligned} \tilde{\Phi}_{ABA'B'} &= 2\,\tilde{\varphi}_{(A(A'}\,\tilde{\bar{\varphi}}_{B')B)},\\ 6\tilde{\Lambda} &= -\tilde{\varphi}_c\,\tilde{\bar{\varphi}}^c; \end{aligned} \tag{7}$$

the components of the Ricci spinor with respect to the spin basis are

$$\begin{aligned} \tilde{\Phi}_{00} &= 2\,\tilde{\varphi}_0\,\tilde{\bar{\varphi}}_0,\\ \tilde{\Phi}_{01} &= 2\,\tilde{\varphi}_{(0}\,\tilde{\bar{\varphi}}_{1)},\\ \tilde{\Phi}_{02} &= 2\,\tilde{\varphi}_1\,\tilde{\bar{\varphi}}_1,\\ \tilde{\Phi}_{11} &= \tilde{\varphi}_{(0}\,\tilde{\bar{\varphi}}_{2)} + \tilde{\varphi}_{(1}\,\tilde{\bar{\varphi}}_{\bar{1})},\\ \tilde{\Phi}_{12} &= 2\,\tilde{\varphi}_{(1}\,\tilde{\bar{\varphi}}_{2)},\\ \tilde{\Phi}_{22} &= 2\,\tilde{\varphi}_2\,\tilde{\bar{\varphi}}_2, \end{aligned} \tag{8}$$

and the scalar curvature is

$$\tilde{\Lambda} = \tfrac{1}{3}[-\,\tilde{\varphi}_{(0}\,\tilde{\bar{\varphi}}_{2)} + \tilde{\varphi}_{(1}\,\tilde{\bar{\varphi}}_{\bar{1})}]. \tag{9}$$

2.2. The conformal Einstein-massless-KG equations

In this subsection, we find a system of equations regular at $\mathcal{I}$ for all unphysical quantities. This system, by analogy with Friedrich's 'conformal Einstein equations' [9], we shall call the 'conformal Einstein-massless-KG equations'.

In [4] we derived the physical Bianchi identities expressed in terms of the unphysical quantities as

$$\Omega^2\,\nabla^D_{A'}\psi_{ABCD} = \Omega\,\nabla^{B'}_{(C}\,\Phi_{AB)A'B'} + s^{B'}_{(C}\,\Phi_{AB)A'B'} + \nabla^{B'}_{(C}\,\nabla_{A(A'}\,s_{B')B)}, \tag{10}$$

where $s_{AA'}$ is given by (5), $\Phi_{ABA'B'}$ is the Ricci spinor and $\psi_{ABCD} = \Omega^{-1}\Psi_{ABCD}$ is the rescaled Weyl spinor (see equation (6) in paper I). Using the rule for the conformal transformation of the Ricci spinor,

$$\nabla_{A(A'}s_{B')B} = \Omega\,\tilde{\Phi}_{ABA'B'} - \Omega\,\Phi_{ABA'B'}, \tag{11}$$

we find

$$\nabla^D_{A'}\,\psi_{ABCD} = \Omega^{-2}\,s^{B'}_{(C}\,\tilde{\Phi}_{AB)A'B'} + \Omega^{-1}\,\nabla^{B'}_{(C}\,\tilde{\Phi}_{AB)A'B'}. \tag{12}$$

The right-hand side of this equation is not manifestly regular on $\mathcal{I}$, while the left-hand side is regular by assumption of asymptotic flatness.

Next we express the physical Ricci spinor via the unphysical quantities,

$$\tilde{\Phi}_{ABA'B'} = 2\,\Omega^2\,\varphi_{(A(A'}\bar{\varphi}_{B')B)} + 2\,\phi\,\bar{\phi}\,s_{(A(A'}\,s_{B')B)} + 2\,\Omega\,\bar{\phi}\,\varphi_{(A(A'}\,s_{B')B)} + 2\,\Omega\,\phi\,\bar{\varphi}_{(A(A'}\,s_{B')B)}, \tag{13}$$

and insert this expression into (12).

In order to simplify the resulting equations, we introduce the following notation: let X_a, Y_a and Z_a be the arbitrary vector fields and define

$$(XYZ) = X^{B'}_{(C}\,Y_{A(A'}\,Z_{B')B)}. \tag{14}$$

The expression (XYZ) is obviously symmetric in YZ. It is straightforward to derive the relation

$$(XYZ) + (ZXY) + (YZX) = 0, \tag{15}$$

with the special case $(XXX) = 0$.

After inserting the Ricci spinor (13) into the Bianchi identities (12), we arrive at

$$\nabla^D_{A'}\psi_{ABCD} = 2\Omega^{-1}(2\,\bar\phi(s\varphi s) + 2\,\phi\,(s\bar\varphi s) + \phi\,(\bar\varphi s\,s) + \bar\phi(\varphi s\,s) + \phi\bar\phi(\nabla s\,s)) \\ + 6\,(s\varphi\bar\varphi) + 2\,(\bar\varphi\varphi\,s) + 2\,(\varphi\bar\varphi\,s) + 2\,\bar\phi\,(\nabla\varphi\,s) + 2\,\phi\,(\nabla\bar\varphi\,s) + 2\,\Omega\,(\nabla\varphi\,\bar\varphi). \quad (16)$$

This can be simplified using identity (15):

$$\nabla^D_{A'}\psi_{ABCD} = 2\phi\bar\phi\Omega^{-1}(\nabla s\,s) + 2\,\Omega\,(\nabla\varphi\bar\varphi) + 4(s\varphi\bar\varphi) + 2\bar\phi(\nabla\varphi s) + 2\phi(\nabla\bar\varphi s), \quad (17)$$

where, e.g., $(\nabla\varphi s) = \nabla^{B'}_{(C}(\varphi_{A(A'}s_{B')B)})$, so ∇ acts on both φ and s. The last equation is still formally singular on $\mathcal{I}^-$ because of the factor Ω^{-1}, but using (11) we finally obtain

$$\nabla^D_{A'}\psi_{ABCD} = 2\,\phi\,\bar\phi\,s^{B'}_{(C}\Phi_{AB)A'B'} + 4\,(s\,\varphi\,\bar\varphi) + 2\,\phi\,(\nabla\,s\,\bar\varphi) + 2\,\bar\phi\,(\nabla\,s\,\varphi) \\ + 4\,\Omega\big[\tfrac{1}{2}\,(\nabla\,\varphi\,\bar\varphi) - \phi\,\bar\phi^2\,(s\,\varphi\,s) - \bar\phi\,\phi^2\,(s\,\bar\varphi\,s)\big] - 4\,\Omega^2\,\phi\,\bar\phi\,(s\,\varphi\,\bar\varphi), \quad (18)$$

which is manifestly smooth at $\mathcal{I}$.

Next we wish to derive equations for the conformal factor. The commutator of covariant derivatives annihilates scalars, so contracting $\nabla_{[a}\nabla_{b]}\Omega = 0$ with $\epsilon^{A'B'}$ gives the relation

$$\nabla_{A'(A}\,s^{A'}_{B)} = 0. \quad (19)$$

By decomposing $\nabla_{AA'}s_{BB'}$ into its symmetric and antisymmetric parts and using the above equation, we obtain

$$\nabla_{AA'}\,s_{BB'} = \nabla_{(A(A'}\,s_{B')B)} + \tfrac{1}{4}\,\epsilon_{AB}\,\epsilon_{A'B'}\,\Box\,\Omega. \quad (20)$$

The first term on the rhs is given by (11). We now define the quantity (cf equation (15) in paper I)

$$F = \tfrac{1}{2}\,\Omega^{-1}\,g^{ab}\,s_a\,s_b, \quad (21)$$

which is regular on $\mathcal{I}$. The rule for the conformal transformation of the scalar curvature can be written in the form (equation (16) in paper I)

$$\Box\,\Omega = 4\Omega\,\Lambda - 4\Omega^{-1}\,\tilde\Lambda + 4F. \quad (22)$$

We thus have found an expression for the second derivatives of the conformal factor Ω:

$$\nabla_{AA'}s_{BB'} = \Omega\,\tilde\Phi_{ABA'B'} - \Omega\,\Phi_{ABA'B'} + \epsilon_{AB}\,\epsilon_{A'B'}\,(\Omega\Lambda - \Omega^{-1}\tilde\Lambda + F). \quad (23)$$

The last expression contains a term $\Omega^{-1}\tilde\Lambda$ which again seems to be singular on $\mathcal{I}$. This is not the case, however, since by (7), (4) and (21) we have

$$\tilde\Lambda = -\tfrac{1}{6}\,\Omega^3[\Omega\varphi_c\,\bar\varphi^c + \phi\bar\varphi_c\,s^c + \bar\phi\,\varphi_c\,s^c + 2\,\phi\,\bar\phi\,F]. \quad (24)$$

The physical scalar curvature is therefore manifestly at least $\mathcal{O}(\Omega^3)$.

The projections of the last equation are written down explicitly in appendix B, equations (B.6)–(B.15). Now we wish to derive equations governing the quantity F. The contracted Ricci identities read

$$\nabla_a\,\Box\,\Omega - \nabla_b\nabla_a\,s^b = R^d_a\,s_d. \quad (25)$$

Using the spinor decomposition of the Ricci tensor

$$R_{ab} = -2\Phi_{ABA'B'} + 6\Lambda\,\epsilon_{AB}\,\epsilon_{A'B'}, \quad (26)$$

and expression (23), we find after some arrangements

$$\nabla_{AA'}F = \tfrac{1}{3}\,\Omega^1\,\nabla^{BB'}\,\tilde\Phi_{ABA'B'} + \tfrac{1}{3}\,s^{BB'}\,\tilde\Phi_{ABA'B'} - s^{BB'}\,\Phi_{ABA'B'} \\ + \Lambda\,s_{AA'} - \Omega^{-2}\,\tilde\Lambda\,s_{AA'} + \Omega^{-1}\,\nabla_{AA'}\,\tilde\Lambda. \quad (27)$$

The first term on the rhs can be rewritten as

$$\nabla^{BB'}\tilde\Phi_{ABA'B'} = \Omega^{-2}\,\tilde\nabla^{BB'}\,\tilde\Phi_{ABA'B'} + 2\,\Omega^{-1}\,\tilde\Phi_{ABA'B'}\,s^{BB'}. \quad (28)$$

Now we employ the contracted physical Bianchi identities $\tilde{\nabla}^{AA'}\tilde{\Phi}_{ABA'B'} = -3\tilde{\nabla}_{BB'}\tilde{\Lambda}$ and obtain

$$\nabla_{AA'}F = s^{BB'}\,\tilde{\Phi}_{ABA'B'} - s^{BB'}\,\Phi_{ABA'B'} + (\Lambda - \Omega^{-2}\,\tilde{\Lambda})\,s_{AA'}. \tag{29}$$

The projections of this equation can be found in appendix B, equations (B.17)–(B.19).

Finally we derive conformal equations for the field $\varphi_{AA'}$. The expression $(\Box + 4\Lambda)\phi$ is conformally invariant with the conformal weight 3, so

$$\Box\,\phi = -4(\Lambda - \Omega^{-2}\,\tilde{\Lambda})\,\phi, \tag{30}$$

where we used wave equation (1) in the physical spacetime. The symmetric part of $\nabla_A^{A'}\varphi_{BA'}$ is zero and we find

$$\nabla_A^{A'}\varphi_{BA'} = -\tfrac{1}{2}\epsilon_{AB}\,\Box\,\phi. \tag{31}$$

Combining the last two equations we arrive at

$$\nabla_A^{A'}\varphi_{BA'} = 2\,(\Lambda - \Omega^{-2}\tilde{\Lambda})\,\phi\,\epsilon_{AB}. \tag{32}$$

To summarize, in the unphysical spacetime we have the following variables: $\{\Omega, \phi, s_a, \varphi_a, F, \psi_{ABCD}, \Phi_{ABA'B'}, \Lambda\}$. The evolution of these quantities is given by equations (18), (23), (29) and (32), together with the contracted Bianchi identities

$$\nabla^{AA'}\Phi_{ABA'B'} = -3\nabla_{BB'}\Lambda. \tag{33}$$

2.3. *Potentials*

The massless KG equation (1) can be generalized to include self-interactions of the scalar field by adding a potential term to the energy–momentum tensor (2),

$$\tilde{T}_{ab} \;\mapsto\; \tilde{T}_{ab} + \frac{1}{4\pi}\tilde{g}_{ab}V(\tilde{\phi}, \bar{\tilde{\phi}}),$$

so that the field equation acquires the form

$$\tilde{\Box}\,\tilde{\phi} + \frac{\partial V}{\partial\bar{\tilde{\phi}}} = 0.$$

Since the potential term in the energy–momentum tensor is proportional to the metric, it will contribute to the scalar curvature $\tilde{\Lambda}$, but not to the trace-free Ricci spinor. The new form of Einstein's equations (7) is therefore

$$\tilde{\Phi}_{ABA'B'} = 2\tilde{\varphi}_{(A(A'}\,\bar{\tilde{\varphi}}_{B')B)}, \qquad 6\tilde{\Lambda} = -\tilde{\varphi}_c\,\bar{\tilde{\varphi}}^c + 2V. \tag{34}$$

For our proof we require $\phi = \Omega^{-1}\tilde{\phi}$ and $\Omega^{-3}\tilde{\Lambda}$ to be regular on $\mathcal{I}^-$, cf (4), (29) and (24). From (34) we can see that this will be satisfied, if $\Omega^{-3}V$ is regular on $\mathcal{I}^-$. In this case the proof works without change. An example is massless ϕ^4-theory, where $V = (\tilde{\phi}\bar{\tilde{\phi}})^2 = \mathcal{O}(\Omega^4)$.

If there is a mass term $m^2\tilde{\phi}\bar{\tilde{\phi}}$ in V, the asymptotic behaviour of the unphysical field changes to

$$\phi = \mathcal{O}(\mathrm{e}^{-m\tilde{r}}),$$

so

$$\Omega^{-3}m^2\tilde{\phi}\bar{\tilde{\phi}} \;\sim\; m^2\tilde{r}\,\mathrm{e}^{-2m\tilde{r}},$$

which is regular. The field ϕ is now not analytic at $\mathcal{I}^-$, so our argument does not apply to this case, which is the class including the boson stars of [5]. Note, however, that in general the asymptotic behaviour of massive fields at $\mathcal{I}$ is a subtle question which appears to be carefully analysed only at the level of linearized theory [17].

3. The conformal-scalar field

3.1. Basic relations

Consider now the conformal-scalar field, by which we mean a scalar field satisfying the equation

$$\left(\widetilde{\Box} + \tfrac{1}{6}\,\tilde{R}\right)\tilde{\phi} \equiv (\widetilde{\Box} + 4\,\tilde{\Lambda})\tilde{\phi} = 0. \tag{35}$$

This is conformally invariant if ϕ transforms as (4), i.e. $\tilde{\phi} = \Omega\,\phi$. For simplicity we assume the field ϕ to be real, but the procedure is easily generalized to complex ϕ. The energy–momentum tensor conserved due to equation (35) is (see [7, 14] or [15], Volume II, p 125)

$$\widetilde{T}_{ab} = \frac{1}{4\pi}[2\,\tilde{\varphi}_{A(A'}\,\tilde{\varphi}_{B')B} - \tilde{\phi}\,\widetilde{\nabla}_{A(A'}\,\tilde{\varphi}_{B')B} + \tilde{\phi}^2\,\tilde{\Phi}_{ABA'B'}]. \tag{36}$$

Furthermore, this energy–momentum tensor also has good conformal behaviour rescaling as

$$\widetilde{T}_{ab} = \Omega^2 T_{ab},$$

but it will not satisfy any of the usual energy conditions. We shall return to this point. We take Einstein's equations to be, as usual,

$$\tilde{\Phi}_{ab} + 3\tilde{\Lambda}\tilde{g}_{ab} = 4\pi\,\widetilde{T}_{ab};$$

then we can solve to find

$$\begin{aligned} &\tilde{\Phi}_{ABA'B'} = (1 - \tilde{\phi}^2)^{-1}[2\,\tilde{\varphi}_{(A(A'}\,\tilde{\varphi}_{B')B)} - \tilde{\phi}\,\tilde{\nabla}_{(A(A'}\,\tilde{\varphi}_{B')B)}],\\ &\tilde{\Lambda} = 0. \end{aligned} \tag{37}$$

These equations are singular when $\tilde{\phi}^2 = 1$ but there are known solutions which avoid this singularity [2, 3] (for explicit examples, see appendix D) and it is known that there is a well-posed initial-value problem [11] which with suitable data extends to $\mathcal{I}^+$. We shall therefore assume that we have an asymptotically flat solution, periodic in time, with $\tilde{\phi}$ tending to zero at infinity, so that $\tilde{\phi}^2 < 1$ everywhere.

In the absence of the dominant energy condition, it is not clear that any version of the positive mass theorem holds but there is some reason to expect a positive global energy. To see this, integrate the energy density over an asymptotically flat maximal space-like hypersurface Σ (assuming for the moment that one exists) with normal N^a. Note from (36) and (37) that

$$\widetilde{T}_{ab} = \frac{1}{4\pi}\,\frac{1}{1-\tilde{\phi}^2}[2\,\tilde{\varphi}_{A(A'}\,\tilde{\varphi}_{B')B} - \tilde{\phi}\,\widetilde{\nabla}_{A(A'}\,\tilde{\varphi}_{B')B}]. \tag{38}$$

Then a measure of total energy at Σ is

$$\begin{aligned} E := &\int \widetilde{T}_{ab}N^a N^b\,\mathrm{d}\Sigma\\ = &\frac{1}{4\pi}\int\frac{1}{1-\tilde{\phi}^2}\left[(N^a\tilde{\varphi}_a)^2 - \frac{1}{2}\tilde{g}^{ab}\tilde{\varphi}_a\tilde{\varphi}_b - \tilde{\phi}\,N^a\,N^b\,\widetilde{\nabla}_a\,\tilde{\varphi}_b\right]\mathrm{d}\Sigma. \end{aligned} \tag{39}$$

Now,

$$N^a N^b \widetilde{\nabla}_a\tilde{\varphi}_b = (h^{ab} + \tilde{g}^{ab})\widetilde{\nabla}_a\tilde{\varphi}_b = h^{ij}\widetilde{\nabla}_i\tilde{\varphi}_j = h^{ij}\,D_i\tilde{\varphi}_j + (N^a\tilde{\varphi}_a)K,$$

where D_i is the derivative operator associated with the three-dimensional metric h_{ij} induced on Σ, K is the trace of the extrinsic curvature which vanishes for a maximal surface, and we have used $\widetilde{\Box}\tilde{\phi} = 0$.

Note also

$$\tilde{g}^{ab}\tilde{\varphi}_a\tilde{\varphi}_b = (N^a\tilde{\varphi}_a)^2 - h^{ij}\tilde{\varphi}_i\tilde{\varphi}_j,$$

and integrate by parts in (39) to find

$$E = \frac{1}{4\pi}\int \mathrm{d}\Sigma\ (1-\tilde{\phi}^2)^{-1}\left[\frac{3}{2}(N^a\tilde{\varphi}_a)^2 + \frac{1}{2}\frac{3+\tilde{\phi}^2}{1-\tilde{\phi}^2}h^{ij}\tilde{\varphi}_i\tilde{\varphi}_j\right],$$

which is manifestly non-negative. Thus, on a maximal surface the global energy is positive without local positivity. Positive energy also holds for hyperplanes in Minkowski space with T_{ab} as in (36) and we shall see something similar below, namely that, while the Bondi mass-loss is not necessarily positive at any particular cut, nonetheless the mass-loss integrated over a period in a periodic spacetime is non-negative.

The Ricci spinor written in terms of unphysical quantities reads

$$(1-\Omega^2\phi^2)\tilde{\Phi}_{ABA'B'} = 2\,\Omega^2\,\varphi_{(A(A'}\,\varphi_{B')B)} - \Omega^2\,\phi\,\nabla_{(A(A'}\,\varphi_{B')B)} - \Omega\,\phi^2\,\nabla_{(A(A'}\,s_{B')B)}. \tag{40}$$

Let us define the 'rescaled Ricci spinor'

$$\phi_{ABA'B'} = \Omega^{-2}\,\tilde{\Phi}_{ABA'B'}, \tag{41}$$

which should be distinguished from the unphysical Ricci spinor. Substituting the rule for the conformal transformation of the Ricci spinor (11) into (40) we arrive at the following simple expression for the rescaled Ricci spinor:

$$\phi_{ABA'B'} = 2\,\varphi_{(A(A'}\,\varphi_{B')B)} - \phi\,\nabla_{(A(A'}\varphi_{B')B)} + \phi^2\,\phi_{ABA'B'}. \tag{42}$$

This spinor is regular on $\mathcal{I}^-$. Note that we do not write the tilde over $\phi_{ABA'B'}$ (as we wrote over $\tilde{\Phi}_{ABA'B'}$ in (7)), since we expect that the physical Ricci spinor has already been expressed in terms of the unphysical quantities and the following relations become simpler. The components of $\phi_{ABA'B'}$ with respect to the spin basis are

$$\begin{aligned}
\phi_{00} &= 2\varphi_0^2 - \phi[D\varphi_0 - (\varepsilon+\bar{\varepsilon})\varphi_0 + \bar{\kappa}\varphi_1 + \kappa\varphi_{\bar{1}}] + \phi^2\,\Phi_{00},\\
\phi_{01} &= 2\varphi_0\varphi_1 - \tfrac{1}{2}\phi[D\varphi_1 + \delta\varphi_0 - (\bar{\alpha}+\beta+\bar{\pi})\varphi_0 + \kappa\varphi_2 + (\bar{\rho}-\varepsilon+\bar{\varepsilon})\varphi_1 + \sigma\varphi_{\bar{1}}] + \phi^2\,\Phi_{01},\\
\phi_{02} &= 2\varphi_1^2 - \phi[\delta\varphi_1 - \bar{\lambda}\varphi_0 + \sigma\varphi_2 + (\bar{\alpha}-\beta)\varphi_1] + \phi^2\,\Phi_{02},\\
\phi_{12} &= 2\varphi_1\varphi_2 - \tfrac{1}{2}\phi[\Delta\varphi_1 + \delta\varphi_2 - \bar{\nu}\varphi_0 + (\beta+\tau+\bar{\alpha})\varphi_2 + (\bar{\gamma}-\gamma-\mu)\varphi_1 - \bar{\lambda}\varphi_{\bar{1}}] + \phi^2\,\Phi_{12},\\
\phi_{22} &= 2\varphi_2^2 - \phi[\Delta\varphi_2 + (\gamma+\bar{\gamma})\varphi_2 - \nu\varphi_1 - \bar{\nu}\varphi_{\bar{1}}] + \phi^2\,\Phi_{22},\\
\phi_{11} &= \varphi_0\varphi_2 + \varphi_1\varphi_{\bar{1}} - \tfrac{1}{4}\phi[D\varphi_2 + \Delta\varphi_0 + \delta\varphi_{\bar{1}} + \bar{\delta}\varphi_1 - (\gamma+\bar{\gamma}+\mu+\bar{\mu})\varphi_0]\\
&\quad - \tfrac{1}{4}\phi[(\rho+\bar{\rho}+\varepsilon+\bar{\varepsilon})\varphi_2 + (\bar{\tau}-\alpha+\bar{\beta}-\pi)\varphi_1 + (\tau-\bar{\alpha}+\beta-\bar{\pi})\varphi_{\bar{1}}] + \phi^2\,\Phi_{11}.
\end{aligned} \tag{43}$$

3.2. The conformal Einstein-conformal-scalar equations

Now, as in subsection 2.2, we obtain a system of conformal Einstein equations, regular in the unphysical, rescaled spacetime and equivalent to the Einstein-conformal-scalar equations.

In order to derive the conformal Bianchi identities for the conformal-scalar field we return to the general physical Bianchi identities (12). Using the rescaled Ricci spinor instead of $\tilde{\Phi}_{ABA'B'}$ the Bianchi identities become

$$\nabla^D_{A'}\,\psi_{ABCD} = 3\,s^{B'}_{(C}\,\phi_{AB)A'B'} + \Omega\,\nabla^{B'}_{(C}\phi_{AB)A'B'}. \tag{44}$$

Projections of these equations on the spin basis can be found in appendix B, equations (B.51)–(B.58).

Next we turn to the contracted Bianchi identities in the physical spacetime

$$\tilde{\nabla}^{BB'}\,\tilde{\Phi}_{ABA'B'} = -3\tilde{\nabla}_{AA'}\,\tilde{\Lambda}, \tag{45}$$

where $\tilde{\Lambda} = 0$ by (37). Following the rules for the conformal transformation of the covariant derivative we find that the left-hand side transforms like

$$\tilde{\nabla}^{BB'}\,\tilde{\Phi}_{ABA'B'} = \Omega^2\,\nabla^{BB'}\,\tilde{\Phi}_{ABA'B'} - 2\,\Omega\,s^{BB'}\,\tilde{\Phi}_{ABA'B'} \tag{46}$$

or, using (41),

$$\tilde{\nabla}^{BB'}\tilde{\Phi}_{ABA'B'} = \Omega^4\,\nabla^{BB'}\phi_{ABA'B'}, \tag{47}$$

and thus the contracted Bianchi identities have a simple form, just as in the physical spacetime:

$$\nabla^{BB'}\phi_{ABA'B'} = 0. \tag{48}$$

Projections of these equations on the spin basis can be obtained from the Bianchi identities (A.8*a*)–(A.8*c*) by deleting terms containing Λ and replacing $\Phi_{mn} \mapsto \phi_{mn}$.

4. Bondi mass

One of the necessary ingredients in this work is to find restrictions which the assumption of periodicity imposes on the Bondi mass. As long as the Bondi mass $M_B(u)$ on $\mathcal{I}^+$ is a non-increasing function of the retarded time u, it can be periodic only if it is constant. In [4] we used the well-known formula for the Bondi mass of an electrovacuum spacetime[5],

$$M_B(u) = -\frac{1}{2\sqrt{\pi}}\oint \mathrm{d}S\big(\Psi_2^0 + \sigma^0\,\dot{\bar{\sigma}}^0\big), \tag{49}$$

when its time decrease is given by the 'mass-loss' formula

$$\dot{M}_B(u) = -\frac{1}{2\sqrt{\pi}}\oint \mathrm{d}S\big(\dot{\sigma}^0\,\dot{\bar{\sigma}}^0 + \phi_2^0\,\bar{\phi}_2^0\big). \tag{50}$$

This expression is manifestly non-positive. To achieve periodicity of the Bondi mass we thus had to set $\dot{\sigma}^0 = 0$ and $\phi_2^0 = 0$. The loss of the Bondi mass due to the gravitational radiation is described by the news function $-\dot{\bar{\sigma}}^0$ and the electromagnetic contribution by the quantity ϕ_2^0. Periodicity thus requires the absence of both gravitational and electromagnetic radiation.

In order to repeat this reasoning in the case of spacetimes with scalar fields, we need the appropriate formula for Bondi mass-loss. The gravitational contribution will again be expressed by the news function and there will be a contribution from the matter. The energy flux due to the matter is described by the energy–momentum tensor T_{ab} (omitting tildes for clarity, in this subsection only). If we write, using the NP formalism,

$$T_{ab} = A\,l_a\,l_b + B\,n_{(a}l_{b)} + C\,n_a\,n_b + \cdots, \tag{51}$$

then the component $A = T_{ab}n^a n^b$ is the energy radiated out of $\mathcal{I}^+$ (recall that n^a is tangential to $\mathcal{I}^+$, and l^a points into the spacetime towards $\mathcal{I}^-$). In terms of the Ricci spinor NP component, we get

$$T_{ab}\,n^a\,n^b \propto \Phi_{22}. \tag{52}$$

For the complex scalar field ϕ, $\Phi_{22} \propto \dot{\phi}\,\dot{\bar{\phi}}$, where dot means the derivative with respect to u.

For the scalar field we thus expect

$$\dot{M}_B(u) = -\,\frac{1}{2\,\sqrt{\pi}}\oint[\dot{\sigma}^0\,\dot{\bar{\sigma}}^0 + k\,\dot{\phi}^0\,\dot{\bar{\phi}}^0]\,\mathrm{d}S, \tag{53}$$

where k is a positive constant factor and ϕ^0 is the radiative part of the scalar field, i.e. $\phi = \phi^0\,r^{-1} + \mathcal{O}(r^{-2})$. We shall now calculate the Bondi mass for the scalar field which will imply the exact formula for the mass-loss.

[5] In fact, in [4] we constructed the proof—and the same will be done here—at $\mathcal{I}^-$ where the Bondi mass is non-decreasing but it is straightforward to get one from the other. Since it is more common to work at $\mathcal{I}^+$, in this section we discuss the Bondi mass there.

4.1. Massless KG field

To compute the Bondi mass we use a method based on the asymptotic twistor equation as described in [16]. More details can be found in [12] or [15]. In this approach we have to find the asymptotic solution of the Einstein-massless-KG equations in the neighbourhood of $\mathcal{I}^+$ (in the physical spacetime). We give enough of this for our present purposes in appendix C.

The Bondi mass is then given by the coefficient $\mu^{(2)}$, which is the $\mathcal{O}(\Omega^2)$ term in the expansion of the spin coefficient μ. This term is given by (C.5) and reads

$$\mu^{(2)} = -\eth\eth\bar\sigma^{(0)} - \Psi_2^{(0)} - 2\Lambda^{(0)} - \sigma^{(0)}\dot{\bar\sigma}^{(0)}.$$

Since the term $\eth\eth\bar\sigma^{(0)}$ vanishes on integration, we find the Bondi mass to be (with the normalization used in paper I)

$$M_B(u) = -\frac{1}{2\sqrt{\pi}}\oint \mathrm{d}S\big[\Psi_2^{(0)} + 2\Lambda^{(0)} + \sigma^{(0)}\,\dot{\bar\sigma}^{(0)}\big]. \tag{54}$$

Using the expansion of Λ given by (C.6) leads to the final expression

$$M_B(u) = -\frac{1}{2\sqrt{\pi}}\oint \mathrm{d}S\left[\Psi_2^{(0)} + \frac{1}{3}\,\partial_u(\phi^{(0)}\,\bar\phi^{(0)}) + \sigma^{(0)}\,\dot{\bar\sigma}^{(0)}\right]. \tag{55}$$

To find the time derivative of the Bondi mass we use the leading term in the Bianchi identity (A.7*c*):

$$\dot\Psi_2^{(0)} + 2\dot\Lambda^{(0)} = \eth\Psi_3^{(0)} + \Phi_{22}^{(0)} + \sigma^{(0)}\,\Psi_4^{(0)}. \tag{56}$$

The term $\eth\Psi_3^{(0)}$ vanishes on integration. By (C.6) we have $\Psi_4^{(0)} = -\ddot{\bar\sigma}^{(0)}$. The leading term of Φ_{22} is found from (8) to be $\Phi_{22}^{(0)} = 2\dot\phi^{(0)}\dot{\bar\phi}^{(0)}$, and the mass-loss formula thus acquires the form

$$\dot M_B(u) = -\frac{1}{2\sqrt{\pi}}\oint \mathrm{d}S[\dot\sigma^{(0)}\,\dot{\bar\sigma}^{(0)} + 2\,\dot\phi^{(0)}\,\dot{\bar\phi}^{(0)}]. \tag{57}$$

This expression is manifestly non-positive. If we demand the spacetime to be periodic, the Bondi mass must be constant, i.e.

$$\dot{\bar\sigma}^{(0)} = \dot\phi^{(0)} = 0.$$

4.2. Conformal-scalar field

The same calculation can be repeated with minor changes in the case of the conformal-scalar field. Now we obtain the following expressions for the Bondi mass and its 'loss':

$$\begin{aligned} M_B(u) &= -\frac{1}{2\sqrt{\pi}}\oint \mathrm{d}S\big[\Psi_2^{(0)} + \sigma^{(0)}\,\dot{\bar\sigma}^{(0)}\big],\\ \dot M_B(u) &= -\frac{1}{2\sqrt{\pi}}\oint \mathrm{d}S[\dot\sigma^{(0)}\dot{\bar\sigma}^{(0)} + 2\,(\dot\phi^{(0)})^2 - \phi^{(0)}\,\ddot\phi^{(0)}]. \end{aligned} \tag{58}$$

Now the formula for the rate of change of the Bondi mass is not manifestly non-positive, so it can apparently increase as well as decrease. This seems to be a consequence of the fact that the energy–momentum tensor (36) does not obey the energy condition $T_{ab}l^an^b \geqslant 0$ for the arbitrary future null vectors l^a and n^a.

However, if the Bondi mass is supposed to be periodic, its overall change ΔM_B during the one period T is non-positive. Indeed,

$$\Delta M_B = -\frac{1}{2\sqrt{\pi}}\int_u^{u+T}\mathrm{d}u\oint \mathrm{d}S[\dot\sigma^{(0)}\dot{\bar\sigma}^{(0)} + 3\,\dot\phi^{(0)}\,\dot\phi^{(0)}] + \frac{1}{2\sqrt{\pi}}\oint \mathrm{d}S\,[\phi^{(0)}\,\dot\phi^{(0)}]_u^{u+T}, \tag{59}$$

where we have integrated the term containing $\ddot{\phi}^{(0)}$ by parts. The second term in (59) vanishes because of periodicity and we arrive at a manifestly non-positive expression for the loss of mass during one period. Such an expression can be periodic only if it is constant, so we again obtain the condition

$$\dot{\sigma}^{(0)} = \dot{\phi}^{(0)} = 0.$$

5. Periodic solutions are necessarily stationary: proof of the theorems

5.1. The massless KG field

In this section we prove that all periodic, asymptotically flat Einstein-massless-KG spacetimes, analytic near $\mathcal{I}^-$ in the coordinates we shall introduce, are necessarily stationary. First we set up a coordinate system, choose the null tetrad and fix the conformal gauge as in paper I, and the justification for the assertions below is given there. The coordinates are denoted as $x^\mu = (v, r, \theta, \phi)$. Here v is the affine parameter along the generators of $\mathcal{I}^-$ and has the meaning of the advanced time. The coordinate r is an affine parameter along the null geodesics ingoing from $\mathcal{I}^-$ with the property $\Omega = r + \mathcal{O}(r^2)$, and (θ, ϕ) are the standard spherical coordinates on the unit sphere. The NP operators D, Δ and δ representing derivatives in the directions of the vectors l, n and m (constituting the null tetrad) can be expressed in the coordinates x^μ in the following way:

$$\begin{aligned} D &= \partial_v - H\partial_r + C^I\partial_I, \\ \Delta &= \partial_r, \\ \delta &= P^I\partial_I. \end{aligned} \tag{60}$$

The metric functions H, C^I and P^I are governed by the frame equations

$$\Delta H = -(\varepsilon + \bar{\varepsilon}), \tag{61}$$

$$\delta H = -\kappa, \tag{62}$$

$$\Delta C^I = -2\pi P^I - 2\bar{\pi}\bar{P}^I, \tag{63}$$

$$\bar{\delta} P^I - \delta\bar{P}^I = (\alpha - \bar{\beta})\, P^I - (\bar{\alpha} - \beta)\bar{P}^I, \tag{64}$$

$$\Delta P^I = -(\mu - \gamma + \bar{\gamma})P^I - \bar{\lambda}\bar{P}^I, \tag{65}$$

$$\delta C^I - D\, P^I = -(\rho + \varepsilon - \bar{\varepsilon})P^I - \sigma\bar{P}^I, \tag{66}$$

which can be understood as determining the nonzero spin coefficients. We choose

$$P^2 = \frac{1}{\sqrt{2}}, \qquad P^3 = \frac{\mathrm{i}}{\sqrt{2}\,\sin\theta} \qquad \text{on} \quad \mathcal{I}^-. \tag{67}$$

The metric functions H and C^I vanish on $\mathcal{I}^-$ by construction, so the operator D reduces to ∂_v there, and we have

$$H = C^I = 0, \qquad D P^I = 0 \qquad \text{on} \quad \mathcal{I}^-. \tag{68}$$

As a consequence of the choice of the coordinates and the tetrad, we have

$$\begin{aligned} &\rho - \bar{\rho} = \mu - \bar{\mu} = \nu = \pi - \alpha - \bar{\beta} = \bar{\tau} - \beta - \bar{\alpha} = 0, \quad \text{everywhere}, \\ &\alpha = -\beta = -\frac{1}{2\sqrt{2}}\cot\theta, \qquad \kappa = 0, \qquad \text{on} \quad \mathcal{I}^-. \end{aligned} \tag{69}$$

Exploiting the tetrad gauge freedom corresponding to the rotation of $(m, \bar{m})$ we achieve

$$\gamma = 0 \quad \text{everywhere}, \qquad \varepsilon = 0 \qquad \text{on} \quad \mathcal{I}^-. \tag{70}$$

Using the conformal gauge freedom we set

$$\mu = 0 \quad \text{everywhere}. \tag{71}$$

Recall from (24) that the physical scalar curvature is $\mathcal{O}(\Omega^3)$. Equations (B.6)–(B.15) for the conformal factor then reveal that on $\mathcal{I}^-$

$$\begin{aligned} &F = \rho = \sigma = \pi = \bar{\tau} = 0, \\ &\Delta S_0 = \Delta S_1 = \Delta S_2 = DS_2 = 0. \end{aligned} \tag{72}$$

Equations (B.17) and (B.18) for derivatives of F imply

$$\Phi_{00} = \Phi_{01} = 0 \qquad \text{on} \quad \mathcal{I}^-. \tag{73}$$

We saw in the previous section that the periodicity of the solution requires the constancy of the Bondi mass. This is expressed by the relations

$$\left.\begin{aligned} &\psi_0 = \Delta\Psi_0 = 0, \\ &\varphi_0 = D\phi = 0, \end{aligned}\right. \qquad \text{on} \quad \mathcal{I}^-. \tag{74}$$

These equations also imply $D\varphi_1 = D\varphi_{\bar{1}} = 0$ on $\mathcal{I}^-$, as can be seen from (B.2) and (A.1).

First we prove that, assuming periodicity, all NP quantities are time-independent on $\mathcal{I}^-$, i.e. independent of v. This follows immediately from the choices made above for all spin coefficients except for λ. The Ricci identity (A.5*g*) and Bianchi identity (A.7*a*) show

$$D\lambda = \Phi_{20}, \qquad D\Phi_{02} = 0 \qquad \text{on} \quad \mathcal{I}^-, \tag{75}$$

and therefore

$$D^2\lambda = 0. \tag{76}$$

By the same argument as in [10] and [4], we conclude

$$D\lambda = 0 \qquad \text{on} \quad \mathcal{I}^-, \tag{77}$$

since equation (76) has a polynomial solution in v, but λ can be periodic only if it is constant. Equation (75) then gives

$$\Phi_{20} = 0 \qquad \text{on} \;\; \mathcal{I}^-. \tag{78}$$

The conformal Bianchi identities (B.24), (B.26), (B.28) and (B.30) on $\mathcal{I}^-$ simplify to

$$\begin{aligned} &D\psi_1 = 0, \\ &D\psi_2 - \bar{\delta}\psi_1 = -2\alpha\,\psi_1, \\ &D\psi_3 - \bar{\delta}\psi_2 = -2\lambda\,\psi_1 + \tfrac{1}{3}(\phi D\bar{\varphi}_{\bar{1}} + \bar{\phi}D\varphi_{\bar{1}}), \\ &D\psi_4 - \bar{\delta}\psi_3 = -3\lambda\,\psi_2 + 2\alpha(\psi_3 + \bar{\phi}\varphi_{\bar{1}} + \phi\bar{\varphi}_{\bar{1}}) - 4\varphi_{\bar{1}}\,\bar{\varphi}_{\bar{1}} + \bar{\phi}\,\bar{\delta}\varphi_{\bar{1}} + \phi\,\bar{\delta}\bar{\varphi}_{\bar{1}}. \end{aligned} \tag{79}$$

Applying D to these equations, we immediately see that

$$D^2\psi_n = 0 \qquad \text{on} \quad \mathcal{I}^- \tag{80}$$

for all n. By periodicity

$$D\psi_n = 0 \qquad \text{on} \quad \mathcal{I}^-, \tag{81}$$

so all components of the Weyl spinor are v-independent on $\mathcal{I}^-$. Because $\psi_n = \Omega^{-1}\Psi_n$, we have

$$D\Delta\Psi_n = 0 \qquad \text{on} \quad \mathcal{I}^-. \tag{82}$$

Finally, we investigate the behaviour of the remaining components of the Ricci tensor, i.e. Φ_{11}, Φ_{12}, Φ_{22} and Λ. The Ricci identity (A.5*h*) immediately shows

$$\Lambda = 0 \qquad \text{on} \quad \mathcal{I}^{-}. \tag{83}$$

Since μ is identically zero not only on $\mathcal{I}^{-}$, but also in its neighbourhood, the Ricci identity (A.5*k*) on $\mathcal{I}^{-}$ reduces to

$$\Phi_{22} = -\lambda\bar{\lambda} \qquad \text{on} \quad \mathcal{I}^{-}, \tag{84}$$

and therefore

$$D\Phi_{22} = 0 \qquad \text{on} \quad \mathcal{I}^{-}. \tag{85}$$

Applying D on the Ricci identity (A.5*r*) leads to

$$D\Phi_{21} = 0 \qquad \text{on} \quad \mathcal{I}^{-}. \tag{86}$$

The spin coefficients α and β on $\mathcal{I}^{-}$ are given by (69). Inserting these into the Ricci identity (A.5*q*) we find

$$\Phi_{11} = \tfrac{1}{2} \qquad \text{on} \quad \mathcal{I}^{-}, \tag{87}$$

so Φ_{11} is obviously v-independent on $\mathcal{I}^{-}$.

We have already shown that φ_1 and $\varphi_{\bar{1}}$ are v-independent on $\mathcal{I}^{-}$. Equation (B.3) implies

$$D\varphi_2 - \delta\varphi_{\bar{1}} = (\beta - \bar{\alpha})\varphi_{\bar{1}} \qquad \text{on} \quad \mathcal{I}^{-}. \tag{88}$$

Applying D and assuming periodicity of the scalar field we conclude

$$D\varphi_2 = 0 \qquad \text{on} \quad \mathcal{I}^{-}. \tag{89}$$

Projections (B.4) and (B.5) of the wave equation and commutator (A.3) applied to ϕ reveal, after differentiating with D, that $D\Delta Q = 0$ on $\mathcal{I}^{-}$, with

$$Q \in \{\varphi_0, \varphi_{\bar{1}}, \varphi_1\}.$$

To show the same for $\Delta\varphi_2$ we apply $D\Delta$ to (B.3) and obtain

$$D^2\Delta\varphi_2 + 2\phi D\Delta\Lambda = \varphi_1 D\Delta\pi - \varphi_2 D\Delta(\varepsilon + \bar{\varepsilon} - \rho). \tag{90}$$

From Ricci identities (A.5*f*), (A.5*i*), (A.5*l*)–(A.5*o*), we find that $D\Delta Q = 0$ on $\mathcal{I}^{-}$ for

$$Q \in \{\rho, \pi, \alpha, \sigma, \varepsilon, \beta\}.$$

Applying $D\Delta$ to (A.5*h*) shows

$$D\Delta\Lambda = 0 \qquad \text{on} \quad \mathcal{I}^{-},$$

and thus (90) implies $D^2\Delta\varphi_2 = 0$ on $\mathcal{I}^{-}$, so by periodicity

$$D\Delta\varphi_2 = 0 \qquad \text{on} \quad \mathcal{I}^{-}. \tag{91}$$

Thus, we have proved the lemma

Lemma 5.1. *The following quantities vanish on* $\mathcal{I}^{-}$*:*

$$\begin{aligned} &H, C^I, \rho, \sigma, \pi, \tau, \kappa, \varepsilon, S_0, S_1, F, \psi_0, \Phi_{00}, \Phi_{01}, \Phi_{02}, \Lambda, \varphi_0,\\ &DP^I, D\alpha, D\beta, D\lambda, DS_2, \Delta S_0, \Delta S_1, \Delta S_2,\\ &D\varphi_1, D\varphi_{\bar{1}}, D\varphi_2, D\Delta\varphi_0, D\Delta\varphi_1, D\Delta\varphi_{\bar{1}}, D\Delta\varphi_2,\\ &D\psi_1, D\psi_2, D\psi_3, D\psi_4, D\Phi_{11}, D\Phi_{12}, D\Phi_{22}. \end{aligned} \tag{92}$$

Now we set up an induction, with inductive hypothesis.

Suppose that $D\Delta^j Q = 0$ on $\mathcal{I}^-$ for $0 \leqslant j \leqslant k$ with

$$Q \in \{H, C^I, P^I, \varepsilon, \rho, \sigma, \lambda, \pi, \tau, \kappa, \alpha, \beta, F, \psi_n, \Phi_{mn}, \Lambda\},$$

and for $0 \leqslant j \leqslant k+1$ with $Q \in \{S_m, \varphi_m\}$.

This holds for $k = 0$ by lemma 5.1, so we need to deduce it for $j = k+1$ from its validity for $j \leqslant k$. Here we closely follow the procedure we used in [4]. Applying $D\Delta^k$ on (61) we find

$$D\Delta^{k+1} H = -D\Delta^k(\varepsilon + \bar{\varepsilon}),$$

where the rhs vanishes on $\mathcal{I}^-$ by the inductive hypothesis. By a similar argument we can deduce $D\Delta^{k+1} Q = 0$ on $\mathcal{I}^-$:

- for H, C^I and P^I from (61), (63) and (65);
- for $\kappa, \varepsilon, \pi, \tau, \lambda, \beta, \sigma, \rho$ and α from (A.5c), (A.5f), (A.5i)–(A.5o);
- for F from (B.19), taking $\tilde{\Phi}_{mn}$ from (8) and $\tilde{\Lambda}$ from (9) and (24);
- for $\Phi_{00}, \Phi_{20}, \Phi_{01}$ and Φ_{21} from (A.6b), (A.6d), (A.7b) and (A.7d);
- for Λ, Φ_{22} and Φ_{11} from (A.5h), (A.5k) and (A.5q);
- for ψ_0, ψ_1, ψ_2 and ψ_3 from (B.25), (B.27), (B.29) and (B.31).

Now, all quantities except for ψ_4 are proved to satisfy $D\Delta^{k+1} Q = 0$ on $\mathcal{I}^-$. Applying $D\Delta^{k+1}$ on (B.7), (B.10), (B.14), (B.4), (B.5) and (A.3) shows $D\Delta^{k+2} Q = 0$ on $\mathcal{I}^-$ for $Q \in \{S_0, S_1, S_2, \varphi_0, \varphi_{\bar{1}}, \varphi_1\}$.

Ricci identities (A.5f), (A.5i), (A.5l)–(A.5o) and (A.5h) in this order imply $D\Delta^{k+2} Q = 0$ on $\mathcal{I}^-$ for

$$Q \in \{\varepsilon, \pi, \beta, \sigma, \rho, \alpha, \Lambda\}.$$

Applying $D\Delta^{k+2}$ on (B.3) implies $D^2\Delta^{k+2}\varphi_2 = 0$ on $\mathcal{I}^-$ and using periodicity we obtain $D\Delta^{k+2}\varphi_2 = 0$ on $\mathcal{I}^-$.

Finally, acting by $D\Delta^{k+1}$ on (B.30) we find $D^2\Delta^{k+1}\psi_4 = 0$ on $\mathcal{I}^-$, and therefore, by periodicity,

$$D\Delta^{k+1}\psi_4 = 0 \qquad \text{on} \quad \mathcal{I}^-.$$

This completes the induction.

We have thus proved that all variables are v-independent on $\mathcal{I}^-$ and, assuming analyticity in r, in a finite neighbourhood. Since our set of variables also includes the functions H, C^I and P^I constituting the components of the metric tensor, we can conclude that $K = \partial_v$ is a Killing vector of the unphysical metric. However, the conformal factor is v-independent as well, so the Lie derivative of the physical metric is

$$\mathcal{L}_K \tilde{g}_{ab} = -2\Omega^{-3} g_{ab} \mathcal{L}_K \Omega = 0,$$

i.e. K is also a Killing vector of the physical metric.

The norm of K is given by the component g_{vv} in the coordinates x^μ. The full form of the metric tensor $g_{\mu\nu}$ can be found in paper I, equation (34). The norm of K is then

$$g(K, K) = g_{vv} = 2H - 2\omega\bar{\omega},$$

where $\omega = -C^I R_I$ and R_I are the $\mathcal{O}(1)$ functions (see (33) in paper I). Frame equations (61) and (63) imply $H, C^I = \mathcal{O}(r^2)$, and the Ricci identity (A.5f) with (87) shows $\varepsilon = -1/2$ on $\mathcal{I}^-$. From these relations we find the norm of the Killing vector to be

$$g(K, K) = 2r^2 + \mathcal{O}(r^3).$$

We can see that K is null on $\mathcal{I}^-$ and time-like in its neighbourhood. Our results are summarized in the following theorem.

Theorem 5.2. *A weakly asymptotically simple time-periodic solution of the Einstein-massless-KG field equations which is analytic in a neighbourhood of $\mathcal{I}^-$ in the coordinates introduced above necessarily has a Killing vector which is time-like in the interior and extends to a translation on $\mathcal{I}^-$.*

5.2. The conformal-scalar field

The proof for the conformal-scalar field is essentially the same as in the case of massless KG field, the only difference lying in the Bianchi identities. These are not so complicated as in the previous case and we present them in their full form in appendix B, equations (B.51)–(B.58). Our results are summarized in the following theorem.

Theorem 5.3. *A weakly asymptotically simple time-periodic solution of the Einstein-conformal-scalar equations which is analytic in a neighbourhood of $\mathcal{I}^-$ in the coordinates introduced above necessarily has a Killing vector which is time-like in the interior and extends to a translation on $\mathcal{I}^-$.*

We briefly outline the main steps of the proof. We use the same coordinate system and tetrad, defined by (60) and (61)–(66), so all consequences of the choice of the gauge remain unchanged. Projections of wave equation (B.2)–(B.5) and equations for the conformal factor (B.6)–(B.19) differ only in the presence of the physical scalar curvature $\tilde{\Lambda}$, which in this case is zero. The form of the Ricci identities does not depend on the type of the matter field. On the whole, equations (60)–(78) hold without change.

Now it is straightforward to see from the conformal Bianchi identities (B.51)–(B.54) that the Weyl scalars ψ_n are v-independent on $\mathcal{I}^-$. Next we return to equations (83)–(91), which are again valid. So lemma (5.1) holds.

To finalize the proof we need to repeat the induction. In the previous case, in the inductive hypothesis we assumed that each quantity Q satisfies $D\Delta^j Q = 0$ on $\mathcal{I}^-$, where $0 \leqslant j \leqslant k$, and in addition, $D\Delta^{k+1} Q = 0$ on $\mathcal{I}^-$ for $Q \in \{S_a, \varphi_a\}$. This was necessary, since the Bianchi identities contained derivatives of these fields. In the inductive step we were able to prove $D\Delta^{k+1} Q = 0$ on $\mathcal{I}^-$ for ψ_n's and for all other quantities. Moreover, we proved $D\Delta^{k+2} Q = 0$ on $\mathcal{I}^-$ for $Q \in \{S_a, \varphi_a\}$.

In this case, the Bianchi identities actually contain the second derivatives of the fields S_a and φ_a, i.e. the third derivatives of Ω and ϕ. This is not a problem, however, as all third derivatives are multiplied by Ω. Therefore, terms with problematic $D\Delta^{k+2}$-derivatives vanish on $\mathcal{I}^-$, and the induction can be repeated without change.

6. Inheritance

In the previous section, we proved that if both gravitational and scalar fields are periodic near infinity, the spacetime is stationary there and the scalar field does not depend on time. However, there are examples known in which the gravitational field and its matter source do not share the same symmetries (see paper I for a longer discussion). The question therefore is, whether a stationary gravitational field can be produced by a time-dependent source. In paper I we showed that this is not the case with an electromagnetic field—once the spacetime is stationary, the electromagnetic field must be too. Let us briefly recall the idea of the proof.

In the electromagnetic case, the components of the physical Ricci spinor have the simple form $\tilde{\phi}_{mn} = \tilde{\phi}_m \bar{\tilde{\phi}}_n$. It is clear that if the Ricci spinor is to be stationary, the electromagnetic field can depend on time only through the phase of $\tilde{\phi}_m$, i.e. $\tilde{\phi}_m = \tilde{\varphi}_m\, \mathrm{e}^{\mathrm{i}\chi}$, where $\chi = \chi(v, r, x^I)$, but the modulus $\tilde{\varphi}$ is time-independent. Now, by the Bondi mass-loss formula $\phi_0 = 0$ on $\mathcal{I}^-$.

Using Maxwell's equations we deduced that ϕ_m are time-independent on $\mathcal{I}^-$, and by induction also in its neighbourhood. Therefore, if an asymptotically flat electrovacuum spacetime is stationary, the electromagnetic field has to inherit stationarity.

The situation is more complicated in the case of the massless KG field, for now it is not obvious what kind of time dependence of the scalar field $\tilde{\phi}$ is compatible with the stationarity of the spacetime. We first find this time dependence and then the result follows, using the Bondi mass-loss formula and induction.

Theorem 6.1. *In a stationary, analytic, weakly asymptotically simple solution of the Einstein-massless-KG equations with the stationarity Killing vector $K = \partial/\partial v$, the physical massless-KG field must take the form $\tilde{\phi} = \mathrm{e}^{\mathrm{i}\omega v}\tilde{\phi}_0$, where $\partial_v\tilde{\phi}_0 = 0$. If the metric is analytic in a neighbourhood of $\mathcal{I}^-$ in the coordinates introduced above then $\tilde{\phi}$ is in fact time-independent.*

For the first part, we use the coordinate system introduced above and assume the stationarity of the spacetime. Hence, $K = \partial_v$ is a Killing vector of the metric and the Lie derivative $\mathcal{L}_K$ reduces to a simple partial derivative with respect to v, which is also denoted by a dot. Since $\tilde{\Lambda}$ is stationary, Einstein's equations (7) imply

$$\partial_v(\tilde{\varphi}_c\tilde{\bar{\varphi}}^c) = 0.$$

The Lie derivative of the energy–momentum tensor (2) is then

$$4\pi\mathcal{L}_K\tilde{T}_{ab} = \tilde{\psi}_a\tilde{\bar{\varphi}}_b + \tilde{\varphi}_a\tilde{\bar{\psi}}_b + \tilde{\varphi}_b\tilde{\bar{\psi}}_a + \tilde{\psi}_b\tilde{\bar{\varphi}}_a,$$

where $\tilde{\psi} = \dot{\tilde{\phi}}$ and $\tilde{\psi}_a = \nabla_a\tilde{\psi}$. Let us decompose the fields $\tilde{\phi}$ and $\tilde{\psi}$ into real and imaginary parts:

$$\tilde{\phi} = X + \mathrm{i}Y, \qquad \tilde{\psi} = U + \mathrm{i}W, \tag{93}$$

with X, Y, U and W being the real functions. In this notation we have

$$2\pi\dot{\tilde{T}}_{ab} = X_aU_b + Y_aW_b + X_bU_a + Y_bW_a = 0, \tag{94}$$

where $X_a = \nabla_aX$, etc.

We first consider the case when the gradient fields X_a and Y_a are proportional in some finite region so that, by analyticity, they are proportional everywhere. Thus X and Y are functionally dependent. If either were constant then that constant would be zero, since $\tilde{\phi} = 0$ at infinity. Thus we may suppose that Y is a function of X and then

$$\tilde{\varphi}_a = (1 + \mathrm{i}Y')X_a, \qquad \tilde{\Box}\tilde{\phi} = (1 + \mathrm{i}Y')\tilde{\Box}X + \mathrm{i}Y''\tilde{g}^{ab}X_aX_b,$$

where the prime indicates derivative w.r.t. X.

If $Y'' = 0$ then $Y = aX + b$ for constants a, b, but once again b must vanish by asymptotic flatness so $\tilde{\phi} = (1 + \mathrm{i}a)X$ and, after rescaling $\tilde{\phi}$ by a constant, we may assume $Y = 0$ whence also $W = 0$. Now (94) becomes $X_{(a}U_{b)} = 0$ from which necessarily $U_a = 0$ and $\dot{\tilde{\phi}} =$ constant, but then asymptotic flatness forces $\dot{\tilde{\phi}} = 0$.

If $Y'' \neq 0$ then

$$\tilde{\Box}X = 0 = \tilde{g}^{ab}X_aX_b.$$

Now

$$4\pi\tilde{T}_{ab} = 2(1 + Y'^2)X_aX_b, \tag{95}$$

and one may impose on this expression the vanishing of $\mathcal{L}_K\tilde{T}_{ab}$. Introduce

$$h := \mathcal{L}_KX,$$

then this is

$$\mathcal{L}_K(4\pi\tilde{T}_{ab}) = 4Y'Y''hX_aX_b + 2(1+Y'^2)(h_aX_b + X_ah_b) = 0.$$

For nonzero h, this is only possible if h_a is proportional to X_a, so that h is a function of X and this condition becomes

$$4Y'Y''h + 4h'(1+Y'^2) = 0,$$

which can be integrated to give $h^2(1+Y'^2) = C$, a constant. Now from (95)

$$4\pi\tilde{T}_{ab}K^aK^b = 2h^2(1+Y'^2) = 2C,$$

but this expression must vanish at infinity for asymptotic flatness, so $C = 0$ and so $h = 0$, and the scalar field inherits the symmetry of the metric.

When X_a and Y_a are not proportional (except possibly on a set of measure zero), we return to (94) and choose a vector field Z^a with $Z^aX_a = 0$ but $Z^aY_a \neq 0$. Contracting (94) with such Z^a we find that W_b is a linear combination of X_b and Y_b, from which we deduce

$$W = f(X, Y).$$

Similarly, contracting (94) with a different Z^a satisfying $Z^aX_a \neq 0$ and $Z^aY_a = 0$ we arrive at

$$U = g(X, Y).$$

Inserting this back into (94) we obtain

$$g_XX_aX_b + 2(f_X + g_Y)X_{(a}Y_{b)} + f_YY_aY_b = 0,$$

where the subscript on f or g indicates the corresponding partial derivative. Since X_a and Y_a are assumed to be linearly independent, all terms in the last equation must vanish separately. We thus have three differential equations for f and g. The general solution is

$$W = f = \omega X + \beta, \qquad U = g = -\omega Y + \gamma,$$

with constants ω, β and γ. Regarding (93), for the field $\tilde{\psi}$ we have

$$\tilde{\psi} = \mathrm{i}\omega(X + \mathrm{i}Y) + (\gamma + \mathrm{i}\beta).$$

Since $\tilde{\psi} = \dot{\tilde{\phi}}$, we can solve the last equation to find

$$\tilde{\phi} = \tilde{\phi}_0\,\mathrm{e}^{\mathrm{i}\omega v} + \text{const}. \tag{96}$$

However, the constant second term must be set to zero, as the field itself must vanish at infinity.

We have shown that the most general non-stationary scalar field compatible with stationarity of the spacetime is of the form

$$\tilde{\phi}(v, r, x^I) = \tilde{\phi}_0(r, x^I)\,\mathrm{e}^{\mathrm{i}\omega v}. \tag{97}$$

(It is not difficult to show that the same result is obtained with a potential term $V(\tilde{\phi}, \bar{\tilde{\phi}})$ added as in subsection 2.3, with the extra condition that necessarily V must also have the form $V = F(\tilde{\phi}\bar{\tilde{\phi}})$.) We shall next show that nonzero ω leads to the vanishing of $\tilde{\phi}$. The stationarity of the spacetime implies the constancy of the Bondi mass, so φ_0 is again zero on $\mathcal{I}^-$. Now, if $\omega = 0$, the field ϕ is v-independent everywhere and is therefore stationary. On the other hand, if $\omega \neq 0$, then expanding ϕ_0 in the variable r and using (96), we find

$$\mathrm{i}\omega\phi_0^{(0)} = 0 \qquad \text{on} \quad \mathcal{I}^-,$$

so that $\phi_0^{(0)} = 0$. Continuing by induction, suppose that $\phi_0^{(j)} = 0$ for $0 \leqslant j \leqslant k$. Acting with Δ^k on (B.3) leads to (recall that ρ and ε vanish on $\mathcal{I}^-$)

$$\mathrm{i}\omega\Delta^{(k+1)}\phi = 0 \qquad \text{on} \quad \mathcal{I}^-$$

and since the constant ω is assumed to be nonzero, it follows immediately that

$$\phi^{(k+1)} = 0.$$

Hence, by induction and analyticity, the field ϕ vanishes in a neighbourhood of $\mathcal{I}^-$. This completes the proof of theorem 6.1 and of inheritance for the massless KG field.

Let us now turn to the conformal-scalar field. Again, we demand the stationarity of the metric, and therefore also the stationarity of the energy–momentum tensor, but not the stationarity of the scalar field. Unfortunately, the complicated form of the energy–momentum tensor (36) does not allow us to find the most general time dependence of $\tilde{\phi}$ compatible with the stationarity of the metric, and thus we cannot proceed as before. In addition, we cannot deduce any concrete condition on $\mathcal{I}^-$, as we do not have a negative semi-definite mass-loss formula. Because of these complications we will only show that the scalar field inherits the symmetry in a simpler case. Let us consider a complex conformal-scalar field with the energy–momentum tensor,

$$\tilde{T}^{\mathbb{C}}_{ab} = \frac{1}{4\pi}\left[2\,\tilde{\varphi}_{(a}\,\bar{\tilde{\varphi}}_{b)} - \frac{1}{2}\,\tilde{g}_{ab}\,\tilde{\varphi}_c\,\bar{\tilde{\varphi}}^c - \frac{1}{2}\,\tilde{\phi}\,\tilde{\nabla}_a\bar{\tilde{\varphi}}_b - \frac{1}{2}\,\bar{\tilde{\phi}}\,\tilde{\nabla}_a\tilde{\varphi}_b + \tilde{\phi}\,\bar{\tilde{\phi}}\,\tilde{\Phi}_{ab}\right]. \tag{98}$$

The Bondi mass-loss formula (59) now takes the form

$$\dot{M}_B = -\frac{1}{2\sqrt{\pi}}\oint \mathrm{d}S\left[\dot{\sigma}^{(0)}\,\dot{\bar{\sigma}}^{(0)} + 2\,\dot{\phi}^{(0)}\,\dot{\bar{\phi}}^{(0)} - \frac{1}{2}\,\phi^{(0)}\,\ddot{\bar{\phi}}^{(0)} - \frac{1}{2}\,\ddot{\phi}^{(0)}\,\bar{\phi}^{(0)}\right]. \tag{99}$$

Although we cannot exclude the existence of some more general time dependence of $\tilde{\phi}$, for the field of the form (97) the energy–momentum tensor (98) is stationary. In this case we can integrate by parts in (99) to find as in (59)

$$\Delta M_B = -\frac{1}{2\sqrt{\pi}}\int_v^{v+2\pi/\omega}\mathrm{d}v\oint \mathrm{d}S[\dot{\sigma}^{(0)}\,\dot{\bar{\sigma}}^{(0)} + 3\,\dot{\phi}^{(0)}\,\dot{\bar{\phi}}^{(0)}]. \tag{100}$$

Since we assume the stationarity of the spacetime, $\dot{\sigma}^{(0)} = 0$. The constancy of the Bondi mass then implies $\dot{\phi}^{(0)} = 0$, i.e.

$$D\phi \equiv \varphi_0 = 0 \qquad \text{on} \quad \mathcal{I}^-. \tag{101}$$

Now we can proceed as in the case of the massless scalar field. By (101) we have $\omega = 0$ or $\phi^{(0)} = 0$. If $\omega = 0$, the field is time-independent everywhere. If $\phi^{(0)} = 0$, or equivalently, $\phi = 0$ on $\mathcal{I}^-$, we prove by induction that $\phi = 0$ everywhere.

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Appendix A. The general Newman–Penrose equations

A.1. Commutation relations

The operators D, Δ, δ and $\bar{\delta}$ satisfy the commutation relations

$$D\delta - \delta D = (\bar{\pi} - \bar{\alpha} - \beta)D - \kappa\Delta + (\bar{\rho} - \bar{\varepsilon} + \varepsilon)\delta + \sigma\bar{\delta}, \tag{A.1}$$

$$\Delta D - D\Delta = (\gamma + \bar{\gamma})D + (\varepsilon + \bar{\varepsilon})\Delta - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})\bar{\delta}, \tag{A.2}$$

$$\Delta\delta - \delta\Delta = \bar{\nu}D + (\bar{\alpha} + \beta - \tau)\Delta + (\gamma - \bar{\gamma} - \mu)\delta - \bar{\lambda}\bar{\delta}, \tag{A.3}$$

$$\delta\bar{\delta} - \bar{\delta}\delta = (\mu - \bar{\mu})D + (\rho - \bar{\rho})\Delta + (\bar{\alpha} - \beta)\bar{\delta} - (\alpha - \bar{\beta})\delta. \tag{A.4}$$

A.2. Ricci identities

$$D\rho - \bar{\delta}\kappa = \rho^2 + (\epsilon + \bar{\epsilon})\rho - \kappa(3\alpha + \bar{\beta} - \pi) - \tau\bar{\kappa} + \sigma\bar{\sigma} + \Phi_{00}, \tag{A.5a}$$

$$D\sigma - \delta\kappa = (\rho + \bar{\rho} + 3\varepsilon - \bar{\varepsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0, \tag{A.5b}$$

$$D\tau - \Delta\kappa = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + (\varepsilon - \bar{\varepsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01}, \tag{A.5c}$$

$$D\alpha - \bar{\delta}\varepsilon = (\rho + \bar{\varepsilon} - 2\varepsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\varepsilon - \kappa\lambda - \bar{\kappa}\gamma + (\varepsilon + \rho)\pi + \Phi_{10}, \tag{A.5d}$$

$$D\beta - \delta\varepsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\varepsilon})\beta - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\varepsilon + \Psi_1, \tag{A.5e}$$

$$D\gamma - \Delta\varepsilon = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\varepsilon + \bar{\varepsilon})\gamma - (\gamma + \bar{\gamma})\varepsilon + \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11}, \tag{A.5f}$$

$$D\lambda - \bar{\delta}\pi = (\rho - 3\varepsilon + \bar{\varepsilon})\lambda + \bar{\sigma}\mu + (\pi + \alpha - \bar{\beta})\pi - \nu\bar{\kappa} + \Phi_{20}, \tag{A.5g}$$

$$D\mu - \delta\pi = (\bar{\rho} - \varepsilon - \bar{\varepsilon})\mu + \sigma\lambda + (\bar{\pi} - \bar{\alpha} + \beta)\pi - \nu\kappa + \Psi_2 + 2\Lambda, \tag{A.5h}$$

$$D\nu - \Delta\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi - (3\varepsilon + \bar{\varepsilon})\nu + \Psi_3 + \Phi_{21}, \tag{A.5i}$$

$$\Delta\lambda - \bar{\delta}\nu = -(\mu + \bar{\mu} + 3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4, \tag{A.5j}$$

$$\Delta\mu - \delta\nu = -(\mu + \gamma + \bar{\gamma})\mu - \lambda\bar{\lambda} + \bar{\nu}\pi + (\bar{\alpha} + 3\beta - \tau)\nu - \Phi_{22}, \tag{A.5k}$$

$$\Delta\beta - \delta\gamma = (\bar{\alpha} + \beta - \tau)\gamma - \mu\tau + \sigma\nu + \varepsilon\bar{\nu} + (\gamma - \bar{\gamma} - \mu)\beta - \alpha\bar{\lambda} - \Phi_{12}, \tag{A.5l}$$

$$\Delta\sigma - \delta\tau = -(\mu - 3\gamma + \bar{\gamma})\sigma - \bar{\lambda}\rho - (\tau + \beta - \bar{\alpha})\tau + \kappa\bar{\nu} - \Phi_{02}, \tag{A.5m}$$

$$\Delta\rho - \bar{\delta}\tau = (\gamma + \bar{\gamma} - \bar{\mu})\rho - \sigma\lambda + (\bar{\beta} - \alpha - \bar{\tau})\tau + \nu\kappa - \Psi_2 - 2\Lambda, \tag{A.5n}$$

$$\Delta\alpha - \bar{\delta}\gamma = (\rho + \varepsilon)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3, \tag{A.5o}$$

$$\delta\rho - \bar{\delta}\sigma = (\bar{\alpha} + \beta)\rho - (3\alpha - \bar{\beta})\sigma + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01}, \tag{A.5p}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + (\rho - \bar{\rho})\gamma + (\mu - \bar{\mu})\varepsilon - \Psi_2 + \Lambda + \Phi_{11}, \tag{A.5q}$$

$$\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21}. \tag{A.5r}$$

A.3. Bianchi identities

$$\begin{aligned} D\Psi_1 - \bar{\delta}\Psi_0 - D\Phi_{01} + \delta\Phi_{00} = {} & (\pi - 4\alpha)\Psi_0 + 2(2\rho + \varepsilon)\Psi_1 - 3\kappa\Psi_2 + 2\kappa\Phi_{11} \\ & - (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} - 2\sigma\Phi_{10} - 2(\bar{\rho} + \varepsilon)\Phi_{01} + \bar{\kappa}\Phi_{02}, \end{aligned} \tag{A.6a}$$

$$\begin{aligned} D\Psi_2 - \bar{\delta}\Psi_1 + \Delta\Phi_{00} - \bar{\delta}\Phi_{01} + 2D\Lambda = {} & -\lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 \\ & + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} + (2\gamma + 2\bar{\gamma} - \bar{\mu})\Phi_{00} - 2(\alpha + \bar{\tau})\Phi_{01} - 2\tau\Phi_{10}, \end{aligned} \tag{A.6b}$$

$$D\Psi_3 - \bar{\delta}\Psi_2 - D\Phi_{21} + \delta\Phi_{20} - 2\bar{\delta}\Lambda = -2\lambda\Psi_1 + 3\pi\Psi_2 + 2(\rho - \varepsilon)\Psi_3 - \kappa\Psi_4 + 2\mu\Phi_{10} - 2\pi\Phi_{11} - (2\beta + \bar{\pi} - 2\bar{\alpha})\Phi_{20} - 2(\bar{\rho} - \varepsilon)\Phi_{21} + \bar{\kappa}\Phi_{22}, \tag{A.6c}$$

$$D\Psi_4 - \bar{\delta}\Psi_3 + \Delta\Phi_{20} - \bar{\delta}\Phi_{21} = -3\lambda\Psi_2 + 2(\alpha + 2\pi)\Psi_3 + (\rho - 4\varepsilon)\Psi_4 + 2\nu\Phi_{10} - 2\lambda\Phi_{11} - (2\gamma - 2\bar{\gamma} + \bar{\mu})\Phi_{20} - 2(\bar{\tau} - \alpha)\Phi_{21} + \bar{\sigma}\Phi_{22}, \tag{A.6d}$$

$$\Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 + (\bar{\rho} + 2\varepsilon - 2\bar{\varepsilon})\Phi_{02} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00} + 2(\bar{\pi} - \beta)\Phi_{01}, \tag{A.7a}$$

$$\Delta\Psi_1 - \delta\Psi_2 - \Delta\Phi_{01} + \bar{\delta}\Phi_{02} - 2\delta\Lambda = \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 - \bar{\nu}\Phi_{00} + 2(\bar{\mu} - \gamma)\Phi_{01} + (2\alpha + \bar{\tau} - 2\bar{\beta})\Phi_{02} + 2\tau\Phi_{11} - 2\rho\Phi_{12}, \tag{A.7b}$$

$$\Delta\Psi_2 - \delta\Psi_3 + D\Phi_{22} - \delta\Phi_{21} + 2\Delta\Lambda = 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4 - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + 2\pi\Phi_{12} + 2(\beta + \bar{\pi})\Phi_{21} + (\bar{\rho} - 2\varepsilon - 2\bar{\varepsilon})\Phi_{22}, \tag{A.7c}$$

$$\Delta\Psi_3 - \delta\Psi_4 - \Delta\Phi_{21} + \bar{\delta}\Phi_{22} = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4 - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + 2(\gamma + \bar{\mu})\Phi_{21} + (\bar{\tau} - 2\bar{\beta} - 2\alpha)\Phi_{22}, \tag{A.7d}$$

$$D\Phi_{11} - \delta\Phi_{10} + \Delta\Phi_{00} - \bar{\delta}\Phi_{01} + 3D\Lambda = (2\gamma + 2\bar{\gamma} - \mu - \bar{\mu})\Phi_{00} + (\pi - 2\alpha - 2\bar{\tau})\Phi_{01} + (\bar{\pi} - 2\bar{\alpha} - 2\tau)\Phi_{10} + 2(\rho + \bar{\rho})\Phi_{11} + \bar{\sigma}\Phi_{02} + \sigma\Phi_{20} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21}, \tag{A.8a}$$

$$D\Phi_{12} - \delta\Phi_{11} + \Delta\Phi_{01} - \bar{\delta}\Phi_{02} + 3\delta\Lambda = (2\gamma - \mu - 2\bar{\mu})\Phi_{01} + \bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{10} + 2(\bar{\pi} - \tau)\Phi_{11} + (\pi + 2\bar{\beta} - 2\alpha - \bar{\tau})\Phi_{02} + (2\rho + \bar{\rho} - 2\bar{\varepsilon})\Phi_{12} + \sigma\Phi_{21} - \kappa\Phi_{22}, \tag{A.8b}$$

$$D\Phi_{22} - \delta\Phi_{21} + \Delta\Phi_{11} - \bar{\delta}\Phi_{12} + 3\Delta\Lambda = \nu\Phi_{01} + \bar{\nu}\Phi_{10} - 2(\mu + \bar{\mu})\Phi_{11} - \lambda\Phi_{02} - \bar{\lambda}\Phi_{20} + (2\pi - \bar{\tau} + 2\bar{\beta})\Phi_{12} + (2\beta - \tau + 2\bar{\pi})\Phi_{21} + (\rho + \bar{\rho} - 2\varepsilon - 2\bar{\varepsilon})\Phi_{22}. \tag{A.8c}$$

Appendix B. The conformal field equations

B.1. The conformally rescaled wave equation

The wave equation $\tilde{\Box}\tilde{\phi} = 0$ in the physical spacetime is not conformally invariant. If $\tilde{\phi}$ is the solution of the physical wave equation, then the unphysical scalar field ϕ satisfies equation (32),

$$\nabla_A^{A'}\varphi_{BA'} = 2\,(\Lambda - \Omega^{-2}\tilde{\Lambda})\,\phi\,\epsilon_{AB}. \tag{B.1}$$

The projections of this equation are as follows:

$$D\varphi_1 - \delta\varphi_0 = (\bar{\pi} - \bar{\alpha} - \beta)\varphi_0 + (\bar{\rho} + \varepsilon - \bar{\varepsilon})\varphi_1 + \sigma\varphi_{\bar{1}} - \kappa\varphi_2, \tag{B.2}$$

$$D\varphi_2 - \delta\varphi_{\bar{1}} = -\mu\varphi_0 + \pi\varphi_1 + (\bar{\pi} - \bar{\alpha} + \beta)\varphi_{\bar{1}} + (\bar{\rho} - \varepsilon - \bar{\varepsilon})\varphi_2 - 2\phi(\Lambda - \Omega^{-2}\tilde{\Lambda}), \tag{B.3}$$

$$\Delta\varphi_0 - \bar{\delta}\varphi_1 = (\gamma + \bar{\gamma} - \bar{\mu})\varphi_0 + (\bar{\beta} - \alpha - \bar{\tau})\varphi_1 - \tau\varphi_{\bar{1}} + \rho\varphi_2 - 2\phi(\Lambda - \Omega^{-2}\tilde{\Lambda}), \tag{B.4}$$

$$\Delta\varphi_{\bar{1}} - \bar{\delta}\varphi_2 = \nu\varphi_0 - \lambda\varphi_1 + (\bar{\gamma} - \gamma - \bar{\mu})\varphi_{\bar{1}} + (\alpha + \bar{\beta} - \bar{\tau})\varphi_2. \tag{B.5}$$

Appropriate equations for the conformal-scalar field can be obtained from (B.2)–(B.5) by setting $\tilde{\Lambda} = 0$.

B.2. Equations for the conformal factor

The projections of equation (23),

$\nabla_{AA'} s_{BB'} = \Omega\, \tilde{\Phi}_{ABA'B'} - \Omega\, \Phi_{ABA'B'} + \epsilon_{AB}\, \epsilon_{A'B'}\, (\Omega\Lambda - \Omega^{-1}\tilde{\Lambda} + F),$

are as follows[6]:

$$D S_0 - (\varepsilon + \bar{\varepsilon})\, S_0 + \bar{\kappa}\, S_1 + \kappa\, \bar{S}_1 = \Omega\, \tilde{\Phi}_{00} - \Omega\, \Phi_{00}, \tag{B.6}$$

$$\Delta S_0 - (\gamma + \bar{\gamma})\, S_0 + \bar{\tau}\, S_1 + \tau\, \bar{S}_1 = \Omega\, \tilde{\Phi}_{11} - \Omega\, \Phi_{11} + \Omega\, \Lambda - \Omega^{-1}\, \tilde{\Lambda} + F, \tag{B.7}$$

$$\delta S_0 - (\bar{\alpha} + \beta)\, S_0 + \bar{\rho}\, S_1 + \sigma\, \bar{S}_1 = \Omega\, \tilde{\Phi}_{01} - \Omega\, \Phi_{01}, \tag{B.8}$$

$$D S_1 - \bar{\pi}\, S_0 + (\bar{\varepsilon} - \varepsilon)\, S_1 + \kappa\, S_2 = \Omega\, \tilde{\Phi}_{01} - \Omega\, \Phi_{01}, \tag{B.9}$$

$$\Delta S_1 - \bar{\nu}\, S_0 + (\bar{\gamma} - \gamma)\, S_1 + \tau\, S_2 = \Omega\, \tilde{\Phi}_{12} - \Omega\, \Phi_{12} \tag{B.10}$$

$$\delta S_1 - \bar{\lambda}\, S_0 + (\bar{\alpha} - \beta)\, S_1 + \sigma\, S_2 = \Omega\, \tilde{\Phi}_{02} - \Omega\, \Phi_{02}, \tag{B.11}$$

$$\bar{\delta} S_1 - \bar{\mu}\, S_0 + (\bar{\beta} - \alpha)\, S_1 + \rho\, S_2 = \Omega\, \tilde{\Phi}_{11} - \Omega\, \Phi_{11} - \Omega\, \Lambda + \Omega^{-1}\, \tilde{\Lambda} - F, \tag{B.12}$$

$$D S_2 - \pi\, S_1 - \bar{\pi}\, \bar{S}_1 + (\varepsilon + \bar{\varepsilon})\, S_2 = \Omega\, \tilde{\Phi}_{11} - \Omega\, \Phi_{11} + \Omega\, \Lambda - \Omega^{-1}\, \tilde{\Lambda} + F, \tag{B.13}$$

$$\Delta S_2 - \nu\, S_1 - \bar{\nu}\, \bar{S}_1 + (\gamma + \bar{\gamma})\, S_2 = \Omega\, \tilde{\Phi}_{22} - \Omega\, \Phi_{22}, \tag{B.14}$$

$$\delta S_2 - \mu\, S_1 - \bar{\lambda}\, \bar{S}_1 + (\bar{\alpha} + \beta)\, S_2 = \Omega\, \tilde{\Phi}_{12} - \Omega\, \Phi_{12}. \tag{B.15}$$

Equation (29) for the derivatives of $F = (1/2)\Omega^{-1} s_c s^c$,

$$\nabla_{AA'} F = s^{BB'}\, \tilde{\Phi}_{ABA'B'} - s^{BB'}\, \Phi_{ABA'B'} + (\Lambda - \Omega^{-2}\, \tilde{\Lambda})\, s_{AA'}, \tag{B.16}$$

has the following projections:

$$\begin{aligned} DF = {} & S_2\, \tilde{\Phi}_{00} - S_1\, \tilde{\Phi}_{10} + S_0\, \tilde{\Phi}_{11} - \bar{S}_1\, \tilde{\Phi}_{01} \\ & - S_2\, \Phi_{00} + S_1\, \Phi_{10} - S_0\, \Phi_{11} + \bar{S}_1\, \Phi_{01} + (\Lambda - \Omega^{-1}\tilde{\Lambda}) S_0, \end{aligned} \tag{B.17}$$

$$\begin{aligned} \delta F = {} & S_2\, \tilde{\Phi}_{01} - S_1\, \tilde{\Phi}_{11} + S_0\, \tilde{\Phi}_{12} - \bar{S}_1\, \tilde{\Phi}_{02} \\ & - S_2\, \Phi_{01} + S_1\, \Phi_{11} - S_0\, \Phi_{12} + \bar{S}_1\, \Phi_{02} + (\Lambda - \Omega^{-1}\tilde{\Lambda}) S_1, \end{aligned} \tag{B.18}$$

$$\begin{aligned} \Delta F = {} & S_2\, \tilde{\Phi}_{11} - S_1\, \tilde{\Phi}_{21} + S_0\, \tilde{\Phi}_{22} - \bar{S}_1\, \tilde{\Phi}_{12} \\ & - S_2\, \Phi_{11} + S_1\, \Phi_{21} - S_0\, \Phi_{22} + \bar{S}_1\, \Phi_{12} + (\Lambda - \Omega^{-1}\tilde{\Lambda}) S_2. \end{aligned} \tag{B.19}$$

B.3. Conformal Bianchi identities for the scalar field

Let us write the conformal Bianchi identities (18) for the scalar field in the form

$$X_{ABCA'} = Y_{ABCA'}, \tag{B.20}$$

where (for notation see (14))

$$X_{ABCA'} = \nabla^{D}_{A'} \psi_{ABCD} - 2\, \phi\, \bar{\phi}\, s^{B'}_{(C} \Phi_{AB)A'B'},$$

$$Y_{ABCA'} = 4\, (s\, \varphi\, \bar{\varphi}) + 2\, \phi\, (\nabla\, s\, \bar{\varphi}) + 2\, \bar{\phi}\, (\nabla\, s\, \varphi) \tag{B.21}$$

$$+\ 4\, \Omega \left[\tfrac{1}{2} (\nabla \varphi \bar{\varphi}) - \phi \bar{\phi}^2 (s\, \varphi\, s) - \bar{\phi}\, \phi^2 (s\, \bar{\varphi}\, s) \right] \tag{B.22}$$

$$-\ 4\, \Omega^2\, \phi\, \bar{\phi}\, (s\, \varphi\, \bar{\varphi}). \tag{B.23}$$

[6] There is a misprint in paper I: in equation (B.2a) the sign of $(\varepsilon + \bar{\varepsilon}) S_0$ should be 'minus' as in (B.6). This does not affect the results of paper I.

Since both sides are totally symmetric in ABC, we denote their contractions with spinors o and ι by the number of ι's in the first index and number of $\bar{\iota}$'s in the second, e.g. $X_{20} = X_{ABCA'}o^A\iota^B\iota^C\bar{o}^{A'}$, $X_{01} = X_{ABCA'}o^Ao^Bo^C\bar{\iota}^{A'}$.

The components of $X_{ABCA'}$ read

$$X_{00} = -D\psi_1 + \bar{\delta}\psi_0 + (\pi - 4\alpha)\psi_0 + 2(\varepsilon + 2\rho)\psi_1 - 3\kappa\psi_2 + \phi\bar{\phi}(-2S_1\Phi_{00} + 2S_0\Phi_{01}), \tag{B.24}$$

$$X_{01} = \Delta\psi_0 - \delta\psi_1 + (\mu - 4\gamma)\psi_0 + 2(\beta + 2\tau)\psi_1 - 3\sigma\psi_2 + \phi\bar{\phi}(2S_0\Phi_{02} - 2S_1\Phi_{01}), \tag{B.25}$$

$$X_{10} = -D\psi_2 + \bar{\delta}\psi_1 - \lambda\psi_0 + 2(\pi - \alpha)\psi_1 + 3\rho\psi_2 - 2\kappa\psi_3 + (2/3)\phi\bar{\phi}(\bar{S}_1\Phi_{01} - S_2\Phi_{00} + 2S_0\Phi_{11} - 2S_1\Phi_{10}), \tag{B.26}$$

$$X_{11} = \Delta\psi_1 - \delta\psi_2 - \nu\psi_0 + 2(\mu - \gamma)\psi_1 + 3\tau\psi_2 - 2\sigma\psi_3 + (2/3)\phi\bar{\phi}(\bar{S}_1\Phi_{02} - S_2\Phi_{01} + 2S_0\Phi_{12} - 2S_1\Phi_{11}), \tag{B.27}$$

$$X_{20} = -D\psi_3 + \bar{\delta}\psi_2 - 2\lambda\psi_1 + 3\pi\psi_2 + 2(\rho - \varepsilon)\psi_3 - \kappa\psi_4 + (2/3)\phi\bar{\phi}(S_0\Phi_{21} - S_1\Phi_{20} + 2\bar{S}_1\Phi_{11} - S_2\Phi_{10}), \tag{B.28}$$

$$X_{21} = \Delta\psi_2 - \delta\psi_3 - 2\nu\psi_1 + 3\mu\psi_2 + 2(\tau - \beta)\psi_3 - \sigma\psi_4 + (2/3)\phi\bar{\phi}(S_0\Phi_{22} - S_1\Phi_{21} + \bar{S}_1\Phi_{12} - S_2\Phi_{11}), \tag{B.29}$$

$$X_{30} = -D\psi_4 + \bar{\delta}\psi_3 - 3\lambda\psi_2 + 2(\alpha + 2\pi)\psi_3 + (\rho - 4\varepsilon)\psi_4 + \phi\bar{\phi}(2\bar{S}_1\Phi_{21} - 2S_2\Phi_{20}), \tag{B.30}$$

$$X_{31} = \Delta\psi_3 - \delta\psi_4 - 3\nu\psi_2 + 2(\gamma + 2\mu)\psi_3 + (\tau - 4\beta)\psi_4 + \phi\bar{\phi}(2\bar{S}_1\Phi_{22} - 2S_2\Phi_{21}). \tag{B.31}$$

We do not present the projections of $Y_{ABCA'}$ in full detail since they are too long. However, the structure of all the terms entering this spinor allows one to reconstruct its components from knowledge of the components of spinors $(s\varphi\bar{\varphi})$ and $(\nabla s\varphi)$, if appropriate interchanges of s, φ and $\bar{\varphi}$ are made. For expressions of type $(s\varphi\bar{\varphi})$ we get

$$2\,(s\,\varphi\,\bar{\varphi})_{00} = 2S_1\varphi_0\bar{\varphi}_0 - S_0(\varphi_0\bar{\varphi}_1 + \varphi_1\bar{\varphi}_0), \tag{B.32}$$

$$2\,(s\,\varphi\,\bar{\varphi})_{01} = -2S_0\varphi_1\bar{\varphi}_1 + S_1(\varphi_0\bar{\varphi}_1 + \varphi_1\bar{\varphi}_0), \tag{B.33}$$

$$6(s\,\varphi\,\bar{\varphi})_{10} = -S_0(\varphi_0\bar{\varphi}_2 + \varphi_2\bar{\varphi}_0 + \varphi_1\bar{\varphi}_{\bar{1}} + \varphi_{\bar{1}}\bar{\varphi}_1) + 2S_1(\varphi_0\bar{\varphi}_{\bar{1}} + \varphi_{\bar{1}}\bar{\varphi}_0) - \bar{S}_1(\varphi_0\bar{\varphi}_1 + \varphi_1\bar{\varphi}_0) + 2S_2\varphi_0\bar{\varphi}_0, \tag{B.34}$$

$$6(s\,\varphi\,\bar{\varphi})_{11} = -2S_0(\varphi_2\bar{\varphi}_1 + \varphi_1\bar{\varphi}_2) + S_1(\varphi_0\bar{\varphi}_2 + \varphi_2\bar{\varphi}_0 + \varphi_1\bar{\varphi}_{\bar{1}} + \varphi_{\bar{1}}\bar{\varphi}_1) - 2\bar{S}_1\varphi_1\bar{\varphi}_1 + S_2(\varphi_0\bar{\varphi}_1 + \varphi_1\bar{\varphi}_0), \tag{B.35}$$

$$6(s\,\varphi\,\bar{\varphi})_{20} = -S_0(\varphi_2\bar{\varphi}_{\bar{1}} + \varphi_{\bar{1}}\bar{\varphi}_2) + 2S_1\varphi_{\bar{1}}\bar{\varphi}_{\bar{1}} - \bar{S}_1(\varphi_0\bar{\varphi}_2 + \varphi_2\bar{\varphi}_0 + \varphi_1\bar{\varphi}_{\bar{1}} + \varphi_{\bar{1}}\bar{\varphi}_1) + 2S_2(\varphi_0\bar{\varphi}_{\bar{1}} + \varphi_{\bar{1}}\bar{\varphi}_0), \tag{B.36}$$

$$6(s\,\varphi\,\bar{\varphi})_{21} = -2S_0\varphi_2\bar{\varphi}_2 + S_1(\varphi_2\bar{\varphi}_{\bar{1}} + \varphi_{\bar{1}}\bar{\varphi}_2) - 2\bar{S}_1(\varphi_2\bar{\varphi}_1 + \varphi_1\bar{\varphi}_2) + S_2(\varphi_0\bar{\varphi}_2 + \varphi_2\bar{\varphi}_0 + \varphi_1\bar{\varphi}_{\bar{1}} + \varphi_{\bar{1}}\bar{\varphi}_1), \tag{B.37}$$

$$2(s\,\varphi\,\bar{\varphi})_{30} = -\bar{S}_1(\varphi_2\bar{\varphi}_{\bar{1}} + \varphi_{\bar{1}}\bar{\varphi}_2) + 2S_2\varphi_{\bar{1}}\bar{\varphi}_{\bar{1}}, \tag{B.38}$$

$$2(s\,\varphi\,\bar\varphi)_{31} = -2\bar S_1\varphi_2\bar\varphi_2 + S_2(\varphi_{\bar 1}\bar\varphi_2 + \varphi_2\bar\varphi_{\bar 1}). \tag{B.39}$$

The expressions of type $(\nabla s\varphi)$ (and their projections) can be slightly simplified by observing that both s_a and φ_a are the gradients of scalar functions, namely Ω and ϕ. Since the commutator $\nabla_{A'(A}\nabla^{A'}_{B)}$ annihilates any scalar quantity, we have

$$\nabla_{A'(A}s^{B'}_{B)} = \nabla_{A'(A}\varphi^{B'}_{B)} = 0, \tag{B.40}$$

and thus

$$(\nabla s\varphi) = \tfrac{1}{2}\,s_{B'(A}\nabla^{B'}_{C}\varphi_{B)A'} + \tfrac{1}{2}\,\varphi_{B'(A}\nabla^{B'}_{C}s_{B)A'} = -(s\nabla\varphi) - (\varphi\nabla s). \tag{B.41}$$

The components of $(s\nabla\varphi)$ are

$$2(s\nabla\varphi)_{00} = S_0[\delta\varphi_0 - (\beta+\bar\alpha)\varphi_0 + \bar\rho\varphi_1 + \sigma\varphi_{\bar 1}] + S_1[-D\varphi_0 + (\varepsilon+\bar\varepsilon)\varphi_0 - \bar\kappa\varphi_1 - \kappa\varphi_{\bar 1}], \tag{B.42}$$

$$2(s\nabla\varphi)_{01} = S_0[\delta\varphi_1 - \bar\lambda\varphi_0 + \sigma\varphi_2 + (\bar\alpha-\beta)\varphi_1] + S_1[-D\varphi_1 + \bar\pi\varphi_0 - \kappa\varphi_2 + (\varepsilon-\bar\varepsilon)\varphi_1], \tag{B.43}$$

$$\begin{aligned}6(s\nabla\varphi)_{10} = {} & S_0[\Delta\varphi_0 + \delta\varphi_{\bar 1} - (\gamma+\bar\gamma+\mu)\varphi_0 + \bar\rho\varphi_2 + \bar\tau\varphi_1 + (\beta+\tau-\bar\alpha)\varphi_{\bar 1}]\\ & + S_1[-D\varphi_{\bar 1} - \bar\delta\varphi_0 + (\pi+\alpha+\bar\beta)\varphi_0 - \bar\kappa\varphi_2 - \bar\sigma\varphi_1 + (\bar\varepsilon-\rho-\varepsilon)\varphi_{\bar 1}]\\ & + \bar S_1[\delta\varphi_0 - (\bar\alpha+\beta)\varphi_0 + \bar\rho\varphi_1 + \sigma\varphi_{\bar 1}] + S_2[-D\varphi_0 + (\varepsilon+\bar\varepsilon)\varphi_0 - \bar\kappa\varphi_1 - \kappa\varphi_{\bar 1}],\end{aligned} \tag{B.44}$$

$$\begin{aligned}6(s\nabla\varphi)_{11} = {} & S_0[\Delta\varphi_1 + \delta\varphi_2 - \bar\nu\varphi_0 + (\beta+\tau+\bar\alpha)\varphi_2 + (\bar\gamma-\gamma-\mu)\varphi_1 - \bar\lambda\varphi_{\bar 1}]\\ & + S_1[-D\varphi_2 - \bar\delta\varphi_1 + \bar\mu\varphi_0 - (\rho+\varepsilon+\bar\varepsilon)\varphi_2 + (\pi+\alpha-\bar\beta)\varphi_1 + \bar\pi\varphi_{\bar 1}]\\ & + \bar S_1[\delta\varphi_1 - \bar\lambda\varphi_0 + \sigma\varphi_2 + (\bar\alpha-\beta)\varphi_1] + S_2[-D\varphi_1 + \bar\pi\varphi_0 - \kappa\varphi_2 + (\varepsilon-\bar\varepsilon)\varphi_1],\end{aligned} \tag{B.45}$$

$$\begin{aligned}6(s\nabla\varphi)_{20} = {} & S_0[\Delta\varphi_{\bar 1} - \nu\varphi_0 + \bar\tau\varphi_2 + (\gamma-\bar\gamma)\varphi_{\bar 1}] + S_1[-\bar\delta\varphi_{\bar 1} + \lambda\varphi_0 - \bar\sigma\varphi_2 + (\bar\beta-\alpha)\varphi_{\bar 1}]\\ & + \bar S_1[\Delta\varphi_0 + \delta\varphi_{\bar 1} - (\gamma+\bar\gamma+\mu)\varphi_0 + \bar\rho\varphi_2 + \bar\tau\varphi_1 + (\beta+\tau-\bar\alpha)\varphi_{\bar 1}]\\ & + S_2[-D\varphi_{\bar 1} - \bar\delta\varphi_0 + (\bar\pi+\alpha+\bar\beta)\varphi_0 - \bar\kappa\varphi_2 - \bar\sigma\varphi_1 + (\bar\varepsilon-\varepsilon-\rho)\varphi_{\bar 1}],\end{aligned} \tag{B.46}$$

$$\begin{aligned}6(s\nabla\varphi)_{21} = {} & S_0[\Delta\varphi_2 + (\gamma+\bar\gamma)\varphi_2 - \nu\varphi_1 - \bar\nu\varphi_{\bar 1}]\\ & + S_1[-\bar\delta\varphi_2 - (\alpha+\bar\beta)\varphi_2 + \lambda\varphi_1 + \bar\mu\varphi_{\bar 1}]\\ & + \bar S_1[\Delta\varphi_1 + \delta\varphi_2 - \bar\nu\varphi_0 + (\beta+\tau+\bar\alpha)\varphi_2 + (\bar\gamma-\gamma-\mu)\varphi_1 - \bar\lambda\varphi_{\bar 1}]\\ & + S_2[-D\varphi_2 - \bar\delta\varphi_1 + \bar\mu\varphi_0 - (\rho+\varepsilon+\bar\varepsilon)\varphi_2 + (\pi+\alpha-\bar\beta)\varphi_1 + \bar\pi\varphi_{\bar 1}],\end{aligned} \tag{B.47}$$

$$2(s\nabla\varphi)_{30} = \bar S_1[\Delta\varphi_{\bar 1} - \nu\varphi_0 + \bar\tau\varphi_2 + (\gamma-\bar\gamma)\varphi_{\bar 1}] + S_2[-\bar\delta\varphi_{\bar 1} + \lambda\varphi_0 - \bar\sigma\varphi_2 + (\bar\beta-\alpha)\varphi_{\bar 1}], \tag{B.48}$$

$$2(s\nabla\varphi)_{31} = \bar S_1[\Delta\varphi_2 + (\gamma+\bar\gamma)\varphi_2 - \nu\varphi_1 - \bar\nu\varphi_{\bar 1}] + S_2[-\bar\delta\varphi_2 - (\alpha+\bar\beta)\varphi_2 + \lambda\varphi_1 + \bar\mu\varphi_{\bar 1}]. \tag{B.49}$$

B.4. Conformal Bianchi identities for the conformal-scalar field

The projections of the Bianchi identities (44)

$$\nabla^{D}_{A'}\,\psi_{ABCD} = 3\,s^{B'}_{(C}\,\phi_{AB)A'B'} + \Omega\,\nabla^{B'}_{(C}\,\phi_{AB)A'B'} \tag{B.50}$$

are as follows:

$$
\begin{aligned}
D\psi_1 - \bar{\delta}\psi_0 = {} & (\pi - 4\alpha)\psi_0 + 2(\varepsilon + 2\rho)\psi_1 - 3\kappa\psi_2 - 3S_1\phi_{00} + 3S_0\phi_{01} \\
& + \Omega[D\phi_{01} - \delta\phi_{00} + (2\beta - 2\bar{\alpha} - \bar{\pi})\phi_{00} - 2(\varepsilon + \bar{\rho})\phi_{01} - 2\sigma\phi_{11} + \bar{\kappa}\phi_{02}],
\end{aligned} \tag{B.51}
$$

$$
\begin{aligned}
D\psi_2 - \bar{\delta}\psi_1 = {} & -\lambda\psi_0 + 2(\pi - \alpha)\psi_1 + 3\rho\psi_2 - 2\kappa\psi_3 \\
& + 2S_0\phi_{11} + \bar{S}_1\phi_{01} - S_2\phi_{00} - S_1\phi_{10} \\
& + \tfrac{1}{3}\Omega[2D\phi_{11} - \Delta\phi_{00} + 2\delta\phi_{10} - \bar{\delta}\phi_{01} \\
& + (2\gamma + 2\bar{\gamma} + 2\mu - \bar{\mu})\phi_{00} - 2(\pi + \alpha + \bar{\tau})\phi_{01} - 2(\bar{\pi} + \tau - 2\bar{\alpha})\phi_{10} \\
& + 2(\rho - 2\bar{\rho})\phi_{11} + 2\bar{\kappa}\phi_{12} + 2\kappa\phi_{21} + \bar{\sigma}\phi_{02} - 2\sigma\phi_{20}],
\end{aligned} \tag{B.52}
$$

$$
\begin{aligned}
D\psi_3 - \bar{\delta}\psi_2 = {} & -2\lambda\psi_1 + 2\pi\psi_2 + 2(\rho - \varepsilon)\psi_3 - \kappa\psi_4 \\
& + S_0\phi_{21} - S_1\phi_{20} - 2S_2\phi_{10} + 2\bar{S}_1\phi_{11} \\
& + \tfrac{1}{3}\Omega[2\Delta\phi_{10} - D\phi_{21} - 2\bar{\delta}\phi_{11} + \delta\phi_{20} \\
& + 2\nu\phi_{00} - 2\lambda\phi_{01} + 2(\mu - \bar{\mu} + 2\bar{\gamma})\phi_{10} - 2(\pi + 2\bar{\tau})\phi_{11} \\
& + 2\bar{\sigma}\phi_{12} + 2(\rho - \bar{\rho} + \varepsilon)\phi_{21} + (2\bar{\alpha} - 2\beta - 2\tau - \bar{\pi})\phi_{20} + \bar{\kappa}\phi_{22}],
\end{aligned} \tag{B.53}
$$

$$
\begin{aligned}
D\psi_4 - \bar{\delta}\psi_3 = {} & -3\lambda\psi_2 + 2(\alpha + 2\pi)\psi_3 + (\rho - 4\varepsilon)\psi_4 + 3\bar{S}_1\phi_{21} - 3S_2\phi_{20} + \Omega[\Delta\phi_{20} - \bar{\delta}\phi_{21} \\
& + 2\nu\phi_{10} - 2\lambda\phi_{11} + 2(\alpha - \bar{\tau})\phi_{21} + (2\bar{\gamma} - 2\gamma - \bar{\mu})\phi_{20} + \bar{\sigma}\phi_{22}],
\end{aligned} \tag{B.54}
$$

$$
\begin{aligned}
\Delta\psi_0 - \delta\psi_1 = {} & (4\gamma - \mu)\psi_0 - 2(\beta + 2\tau)\psi_1 + 3\sigma\psi_2 + 3S_1\phi_{01} - 3S_0\phi_{02} + \Omega[-D\phi_{02} + \delta\phi_{01} \\
& - \bar{\lambda}\phi_{00} + 2(\bar{\pi} - \beta)\phi_{01} + 2\sigma\phi_{11} - 2\kappa\phi_{12} + (2\varepsilon - 2\bar{\varepsilon} + \bar{\rho})\phi_{02}],
\end{aligned} \tag{B.55}
$$

$$
\begin{aligned}
\Delta\psi_1 - \delta\psi_2 = {} & \nu\psi_0 + 2(\gamma - \mu)\psi_1 - 3\tau\psi_2 + 2\sigma\psi_3 \\
& + S_2\phi_{01} + 2S_1\phi_{11} - 2S_0\phi_{12} - \bar{S}_1\phi_{02} \\
& + \tfrac{1}{3}\Omega[\Delta\phi_{01} - \bar{\delta}\phi_{02} + 2\delta\phi_{11} - 2D\phi_{12} \\
& - \bar{\nu}\phi_{00} + 2(\bar{\mu} - \mu - \gamma)\phi_{01} - 2\bar{\lambda}\phi_{10} + 2(\tau + 2\bar{\pi})\phi_{11} \\
& + 2(\bar{\rho} - \rho - 2\bar{\varepsilon})\phi_{12} + 2\sigma\phi_{21} + (2\pi + 2\alpha - 2\bar{\beta} + \bar{\tau})\phi_{02} - 2\kappa\phi_{22}],
\end{aligned} \tag{B.56}
$$

$$
\begin{aligned}
\Delta\psi_2 - \delta\psi_3 = {} & 2\nu\psi_1 - 3\mu\psi_2 + 2(\beta - \tau)\psi_3 + \sigma\psi_4 \\
& + 2S_2\phi_{11} + S_1\phi_{21} - S_0\phi_{22} - 2\bar{S}_1\phi_{12} \\
& + \tfrac{1}{3}\Omega[2\Delta\phi_{11} + \delta\phi_{21} - 2\bar{\delta}\phi_{12} - D\phi_{22} \\
& - 2\nu\phi_{01} - 2\bar{\nu}\phi_{10} + 2(2\bar{\mu} - \mu)\phi_{11} + 2(\pi + \bar{\tau} - 2\bar{\beta})\phi_{12} \\
& + 2(\beta + \tau + \bar{\pi})\phi_{21} + 2\lambda\phi_{02} - \bar{\lambda}\phi_{20} + (\bar{\rho} - 2\rho - 2\varepsilon - 3\bar{\varepsilon})\phi_{22}],
\end{aligned} \tag{B.57}
$$

$$
\begin{aligned}
\Delta\psi_3 - \delta\psi_4 = {} & 3\nu\psi_2 - 2(\gamma + 2\mu)\psi_3 + (4\beta - \tau)\psi_4 + 3S_2\phi_{21} - 3\bar{S}_1\phi_{22} + \Omega[\Delta\phi_{21} - \bar{\delta}\phi_{22} \\
& - 2\nu\phi_{11} + 2\lambda\phi_{12} + 2(\gamma + \bar{\mu})\phi_{21} - \bar{\nu}\phi_{20} + (\bar{\tau} - 2\alpha - 2\bar{\beta})\phi_{22}].
\end{aligned} \tag{B.58}
$$

Appendix C.

C.1. The asymptotic solution of the Einstein-massless-scalar-field equations

Although we want the results at $\mathcal{I}^-$, we follow the usual convention and find the asymptotic solution of the field equations in the physical spacetime first in the neighbourhood of $\mathcal{I}^+$. The results can easily be translated to $\mathcal{I}^-$. For the solution, we closely follow the procedure presented in [16] for the vacuum spacetimes. The coordinates, tetrad and conformal transformations of the spin basis are identical to those used therein, and in this appendix, since we do not consider unphysical quantities, we omit the tildes from physical quantities.

We define

$$\varphi_0 = D\phi, \qquad \varphi_1 = \delta\phi, \qquad \varphi_{\bar{1}} = \bar{\delta}\phi, \qquad \varphi_2 = \Delta\phi. \tag{C.1}$$

The components of the Ricci spinor and the scalar curvature are given by (8) and (9). The asymptotic behaviour of these quantities is as follows:

$$\begin{aligned}&\Phi_{00}, \Phi_{01}, \Phi_{02} = \mathcal{O}(\Omega^4),\\ &\Phi_{11}, \Phi_{12}, \Lambda = \mathcal{O}(\Omega^3),\\ &\Phi_{22} = \mathcal{O}(\Omega^2).\end{aligned} \tag{C.2}$$

Assuming analyticity we can expand any quantity $X = \mathcal{O}(\Omega^k)$ in a series of the form

$$X = \sum_{i=0}^{\infty} X^{(i)}(u, \theta, \phi)\, \Omega^{i+k}. \tag{C.3}$$

Using the field equations, i.e. the Ricci and Bianchi identities and the frame equations, we arrive at the following asymptotic solution for the spin coefficients (setting $\Phi_{mn} = 0$ and $\Lambda = 0$ we recover expansions valid for the vacuum case which can be found, e.g. in [16], section 3.10):

$$\begin{aligned}\sigma &= \sigma^{(0)}\,\Omega^2 + \mathcal{O}(\Omega^4),\\ \rho &= -\Omega + \rho^{(2)}\,\Omega^3 + \mathcal{O}(\Omega^4),\\ \alpha &= a\,\Omega + \alpha^{(1)}\,\Omega^2 + \mathcal{O}(\Omega^3),\\ \beta &= -a\,\Omega - a\,\sigma^{(0)}\,\Omega^2 + \mathcal{O}(\Omega^3),\\ \pi &= \eth\bar{\sigma}^{(0)}\,\Omega^2 + \mathcal{O}(\Omega^3),\\ \lambda &= \dot{\bar{\sigma}}^{(0)}\,\Omega + \lambda^{(2)}\,\Omega^2 + \mathcal{O}(\Omega^3),\\ \gamma &= \gamma^{(2)}\,\Omega^2 + \mathcal{O}(\Omega^3),\\ \mu &= -\tfrac{1}{2}\,\Omega + \mu^{(2)}\,\Omega^2 + \mathcal{O}(\Omega^3),\\ \nu &= \mathcal{O}(\Omega),\end{aligned} \tag{C.4}$$

where

$$\begin{aligned}\rho^{(2)} &= -\left[\sigma^{(0)}\,\bar{\sigma}^{(0)} + \Phi_{00}^{(0)}\right],\\ a &= -\,(2\sqrt{2})^{-1}\,\cot\theta,\\ \alpha^{(1)} &= \eth\bar{\sigma}^{(0)} + a\,\bar{\sigma}^{(0)},\\ \gamma^{(2)} &= a\eth\bar{\sigma}^{(0)} - a\bar{\eth}\sigma^{(0)} - \tfrac{1}{2}\,(\Psi_2^{(0)} + \Phi_{11}^{(0)} - \Lambda^{(0)}),\\ \lambda^{(2)} &= \tfrac{1}{2}\,\bar{\sigma}^{(0)} - \bar{\eth}\eth\bar{\sigma}^{(0)},\\ \mu^{(2)} &= -\,\eth\eth\bar{\sigma}^{(0)} - \Psi_2^{(0)} - 2\,\Lambda^{(0)} - \sigma^{(0)}\,\dot{\bar{\sigma}}^{(0)},\end{aligned} \tag{C.5}$$

For the relevant Weyl scalars and Ricci tensor components we have

$$\begin{aligned}\Psi_2 &= \Psi_2^{(0)}\,\Omega^3 + \mathcal{O}(\Omega^4),\\ \Psi_4 &= -\,\ddot{\bar{\sigma}}^{(0)}\,\Omega + \mathcal{O}(\Omega^2),\\ \Phi_{11} &= -\tfrac{1}{2}\partial_u(\phi^{(0)}\,\bar{\phi}^{(0)})\,\Omega^3 + \mathcal{O}(\Omega^4),\\ \Lambda &= \tfrac{1}{6}\,\partial_u(\phi^{(0)}\,\bar{\phi}^{(0)})\,\Omega^3 + \mathcal{O}(\Omega^4).\end{aligned} \tag{C.6}$$

Appendix D. Selected solutions to the Einstein-conformal-scalar equations

We first briefly survey some explicit stationary solutions to the Einstein-conformal-scalar equations which satisfy the requirements of our theorem. To explore the field equation (37) further, we also present two families of time-dependent solutions. Some are singular when $\phi^2 = 1$, some are not and in some ϕ^2 never takes the value 1.

D.1. Stationary solutions

Over 50 years ago Buchdahl [6] demonstrated how from any given static *vacuum* solution a one-parameter family of pairs of solutions of Einstein's equations with the massless scalar field can be constructed. Later Bekenstein [2] showed how from any Einstein-scalar field solution the corresponding Einstein-*conformal*-scalar field solution can be found. In particular, considering any static vacuum solution in the form

$$\mathrm{d}s^2 = W^2\mathrm{d}t^2 - W^{-2}\, h_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j, \tag{D.1}$$

the two Einstein-conformal-scalar solutions are

$$\begin{aligned} \mathrm{d}s^2 &= \frac{1}{4}\,(W^\beta \pm W^{-\beta})^2[W^{2\alpha}\,\mathrm{d}t^2 - W^{-2\alpha} h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j], \\ \phi &= \sqrt{\frac{3}{4\pi}}\frac{1 \mp W^{2\beta}}{1 \pm W^{2\beta}}, \end{aligned} \tag{D.2}$$

where $\alpha = (1 - 3\beta^2)^{1/2}$, and $\beta \in \left\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ is a free parameter. Upper and lower signs, respectively, in (D.2) correspond to two types of solutions A and B. If the solution (D.1) is asymptotically flat, so it is the type A solution. Hence, many solutions satisfying our assumptions are available.

A special spherically symmetric solution—after choosing a suitable radial coordinate—reads

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 - \frac{m}{\bar{r}}\right)^2 \mathrm{d}t^2 - \left(1 - \frac{m}{\bar{r}}\right)^{-2} \mathrm{d}\bar{r}^2 - \bar{r}^2\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \\ \phi &= \sqrt{\frac{3}{4\pi}}\,\frac{m}{\bar{r} - m}. \end{aligned} \tag{D.3}$$

The geometry is identical to that of an extreme Reissner–Nordström black hole, so it can be analytically continued to $\bar{r} < m$. However, ϕ and $(\nabla_a\phi)(\nabla^a\phi)$ diverge at the 'horizon' $\bar{r} = m$. Nevertheless, this infinite scalar field does not imply an infinite barrier for test scalar charges and the solutions are often regarded as 'black holes with scalar charge' [3]. In any case, both geometry and scalar fields are analytic at $\bar{r} \to \infty$ satisfying our requirements.

Bekenstein's work inspired a number of more recent papers: for example, Einstein-conformal-scalar-field solutions were analysed in arbitrary dimensions [18], self-interacting scalar fields were considered [8] and transversable wormholes from massless conformally coupled and other scalar fields non-minimally coupled to gravity were constructed [1].

D.2. FLRW metric

In this section, we present simple homogenous isotropic solutions of the Einstein-conformal-scalar equations. We shall take the metric in the standard form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1 - k\,r^2} + r^2\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{D.4}$$

where $k \in \{-1, 0, 1\}$. The energy–momentum tensor is given by (38). Since this tensor is traceless, the scalar curvature must vanish:

$$R = \frac{6}{a^2}[k + \dot{a}^2 + \ddot{a}a] = 0.$$

Solutions to this equation are

$$\begin{aligned} a(t) &= \sqrt{c_1 - k(t + c_2)^2} \qquad \text{for} \quad k \neq 0, \\ a(t) &= c_1\sqrt{2t + c_2} \qquad \text{for} \quad k = 0. \end{aligned} \tag{D.5}$$

The conformally invariant scalar field in the physical spacetime satisfies d'Alembert's equation $\Box\phi = 0$. Because the spacetime is assumed to be homogeneous and isotropic, we suppose that the field does not depend on the spatial coordinates. D'Alembert's equation then reduces to

$$\Box\phi(t) = \ddot{\phi} + \frac{3\,\dot{a}\,\dot{\phi}}{a} = 0. \tag{D.6}$$

For given $a(t)$, the solution can be found explicitly:

$$\phi(t) = C_3 + C_4 \int \frac{\mathrm{d}t}{a^3(t)}.$$

In the following short discussion we consider three cases for the three possible values of k. We solve the wave equation, find the components of the energy–momentum tensor and see that these components do not exhibit the singularity formally present in (38).

(1) $k = 0$. Imposing the initial condition $a(0) = 0$ leads to

$$a(t) = \sqrt{2\,C\,t}, \tag{D.7}$$

where C is an arbitrary positive constant. The general solution of (D.4) is

$$\phi(t) = \alpha + \frac{\beta}{\sqrt{t}}.$$

Einstein's equations then imply

$$\alpha = \pm 1,$$

and β is nonzero but arbitrary. Note that $\phi^2 = 1$ at $t = \beta^2/4$ if $\alpha\beta/|\beta| = -1$, but ϕ^2 is never 1 if $\alpha\beta/|\beta| = +1$. The components of the energy–momentum tensor are

$$T_{ab} = \frac{1}{16\pi}\,\mathrm{diag}\left(\frac{3}{2\,t^2}, \frac{C}{t}, \frac{C\,r^2}{t}, \frac{C\,r^2\,\sin^2\theta}{t}\right). \tag{D.8}$$

Obviously, T_{ab} is regular unless $t = 0$. This is the expected initial curvature singularity (for example, the Kretschmann invariant $R_{abcd}\,R^{abcd} = 3/(2t^4)$ diverges for $t = 0$). As noted above, the term $(1 - \phi^2)$ may or may not vanish depending on the constants of integration but even when it does there is no singularity in T_{ab} despite the form of (38).

(2) $k = -1$. Again, we demand $a(0) = 0$, so $a(t)$ is of the form

$$a(t) = \sqrt{t(t + C)}.$$

The general solution of the wave equation is

$$\phi(t) = \alpha + \beta\frac{2t + C}{2a(t)}, \tag{D.9}$$

and Einstein's equations give

$$\alpha = \cosh\chi, \qquad \beta = \sinh\chi, \tag{D.10}$$

where χ is an arbitrary constant. Now $(1-\phi^2)$ will vanish at some $t>0$ for $\chi<0$ but not for $\chi>0$. The components of the energy–momentum tensor are

$$T_{ab} = \frac{C^2}{32\pi a^2(t)} \operatorname{diag}\left(\frac{3}{a^2(t)}, \frac{1}{1+r^2}, r^2, r^2\sin^2\theta\right), \tag{D.11}$$

and they are again singular only for $t=0$, and not at $\phi^2=1$.

(3) $k=1$. Now we impose the conditions $a(0)=0$ and $\dot{a}(T)=0$, so that

$$a(t) = \sqrt{t(2T-t)}.$$

The solution of the wave equation is

$$\phi(t) = \alpha + \beta\,\frac{T-t}{a(t)}, \tag{D.12}$$

and Einstein's equations imply

$$\alpha = \cos\chi, \qquad \beta = \sin\chi. \tag{D.13}$$

In this case, ϕ^2 always takes the value 1 for some time, but the components of the energy–momentum tensor are

$$T_{ab} = \frac{T^2}{8\pi\, a^2(t)} \operatorname{diag}\left(\frac{3}{a^2(t)}, \frac{1}{1-r^2}, r^2, r^2\,\sin^2\theta\right), \tag{D.14}$$

and are nonsingular at $\phi^2=1$.

In [2], cosmological solutions were also considered (both conformal scalar field and incoherent radiation); however, singularities in T_{ab} for scalar field were not discussed.

D.3. pp-waves

We can find *pp*-wave solutions with this source: consider the *pp*-wave with the metric given by

$$\mathrm{d}s^2 = 2\,H(u,x,y)\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2. \tag{D.15}$$

For simplicity, we assume that the scalar field $\phi=\phi(u,x,y)$ does not depend on v. The wave equation is then

$$\Box\phi = -\phi_{xx} - \phi_{yy} = 0, \tag{D.16}$$

with subscripts denoting corresponding derivatives. We can take the general real solution to be

$$\phi(u,x,y) = f(u, x+\mathrm{i}y)/2 + f(u, x-\mathrm{i}y)/2, \tag{D.17}$$

where f is an arbitrary real function of two variables. Let us denote

$$K_{ab} = R_{ab} + 8\pi\,T_{ab}, \tag{D.18}$$

so that Einstein's equation are $K_{ab}=0$. One of these equations is

$$K_{01} = \frac{\phi_x^2+\phi_y^2}{1-\phi^2} = 0, \tag{D.19}$$

from which we find

$$\phi = f(u). \tag{D.20}$$

Then the only remaining nonzero component of K_{ab} is

$$K_{00} = H_{xx} + H_{yy} + \frac{2}{1-f^2}\big(f\,f_{uu} - 2f_u^2\big). \tag{D.21}$$

Solving the equation $K_{00} = 0$ with respect to H we arrive at

$$H = C(u, x + \mathrm{i}y) + C(u, x - \mathrm{i}y) + \frac{x^2 + y^2}{2} \frac{\left(2f_u^2 - f f_{uu}\right)}{1 - f^2}. \tag{D.22}$$

Here, C and f are the arbitrary real functions. As we can now see, the metric function H is singular if ever $f \equiv \phi = \pm 1$ and this, if it occurs, will be a curvature singularity.

In [13], a large class of solutions of the Einstein-conformal-scalar equations for colliding plane waves was found by employing the Bekenstein transformation [2].

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Appendix A

NP formalism in Mathematica

Spinor formalism provides an efficient way how to treat problems discussed in this thesis. However, the Newman-Penrose equations, which are projections of spinor equations onto null tetrad or spin basis, are usually very long. Examples of simple equations were described in section 6.1. Obviously, procedure of projecting the equation is straightforward but for more complicated equations tedious. Thus, the most efficient way how to find the NP equations for given spinor equations is to calculate them using computer and software which is able to perform analytic computations. For all calculations in this thesis software Mathematica has been used. In this appendix we present short guide on the script in Mathematica programmed for the purposes of the thesis. Source code of the script is contained in the file NP-formalism.nb attached on CD accompanying the thesis. The file itself contains brief instructions.

A.1 Conventions

In the script we try to follow standard NP notation as far as possible. Basis spinors o^A and ι^A are represented by symbols

$$o^A \mapsto \mathsf{ou[A]} \quad \iota^A \mapsto \iota\mathsf{u[A]}$$

$$o_A \mapsto \mathsf{od[A]} \quad \iota_A \mapsto \iota\mathsf{d[A]}$$

Letters u (up) and d (down) indicate the position of index. Symbol ιu[A] can be typed by pressing

```
ESC i ESC u[A]
```

Complex conjugated spinors must be typed with bar. Program does not distinguish between primed and unprimed indices, but it uses the convention that primed index A' is written as AA. So, the correspondence between usual notation and notation we used is following:

$$\bar{o}^{A'} \mapsto \overline{\mathsf{ou}}\mathsf{[AA]} \quad \bar{\iota}^{A'} \mapsto \overline{\iota\mathsf{u}}\mathsf{[AA]}$$

$$\bar{o}_{A'} \mapsto \overline{\mathsf{od}}\mathsf{[AA]} \quad \bar{\iota}_{A'} \mapsto \overline{\iota\mathsf{d}}\mathsf{[AA]}$$

Symbol $\overline{\mathsf{ou}}$[AA] can by typed pressing `ou CTRL+& _ [AA]`

Program can work with both spinor and tensor structures. Beside spin basis it defines also null tetrad indexed by small Latin letters:

$$l_a, l^a \mapsto \mathsf{l[a]} \qquad n_a, n^a \mapsto \mathsf{n[a]}$$

$$m_a, m^a \mapsto \mathsf{m[a]} \quad \bar{m}_a, \bar{m}^a \mapsto \overline{\mathsf{m}}\mathsf{[a]}$$

Notice that the program does not distinguish between upper and lower tensor indices. That is because contraction can be made only through indices in different positions, but the result is symmetric, i.e. for any contraction we have

$$t_a\, s^a \;=\; t^a\, s_a.$$

In the case of spinors, however, we must be careful because there the contractions are antisymmetric.

NPformalism defines also spinor equivalents of vectors of null tetrad. For example,

$$l^a \;=\; l^{AA'} \;=\; o^A\, \bar{o}^{A'}.$$

If we project spinor equivalents of tensors, we have to contract them with spinor equivalents of the null tetrad. Since it is annoying to write $o^A \bar{o}^{A'}$, our program employs following notation:

$$\begin{array}{llll} l^{AA'} \mapsto \mathsf{lu[A,AA]} & & n^{AA'} \mapsto \mathsf{nu[A,AA]} \\ m^{AA'} \mapsto \mathsf{mu[A,AA]} & & \bar{m}^{AA'} \mapsto \overline{\mathsf{mu}}\mathsf{[A,AA]} \end{array}$$

Beside spin basis and null tetrad one often needs to use their special combinations, namely Levi-Civita symbol

$$\epsilon_{AB} \;=\; o_A\, \iota_B \;-\; o_B\, \iota_A$$

and metric tensor

$$g_{ab} \;=\; l_a\, n_b \;+\; l_b\, n_a \;-\; m_a\, \bar{m}_b \;-\; \bar{m}_a\, m_b.$$

Thus, the program defines following symbols:

$$\epsilon_{AB} \mapsto \mathsf{Ed[A,B]} \qquad \epsilon_{A'B'} \mapsto \mathsf{Eccd[AA,BB]}$$

$$\epsilon^{AB} \mapsto \mathsf{Eu[A,B]} \qquad \epsilon^{A'B'} \mapsto \mathsf{Eccu[AA,BB]}$$

$$\epsilon_A{}^B \mapsto \mathsf{Edu[A,B]} \qquad \epsilon_{A'}{}^{B'} \mapsto \mathsf{Eccdu[AA,BB]}$$

$$\epsilon^A{}_B \mapsto \mathsf{Eud[A,B]} \qquad \epsilon^{A'}{}_{B'} \mapsto \mathsf{Eccud[AA,BB]}$$

$$g_{ab}, g^{ab} \mapsto \mathsf{g[a,b]}$$

Covariant derivative $\nabla_a = \nabla_{AA'}$ is represented by several versions of symbol Nabla:

$$\nabla_{AA'}x \;\mapsto\; \mathsf{Nablad[A,AA,x]} \qquad \nabla^{AA'}x \;\mapsto\; \mathsf{Nablau[A,AA,x]}$$

$$\nabla_A^{A'}x \;\mapsto\; \mathsf{Nabladu[A,AA,x]} \qquad \nabla_{A'}^{A}x \;\mapsto\; \mathsf{Nablaud[A,AA,x]}$$

$$\nabla_a x, \nabla^a x \;\mapsto\; \mathsf{Nabla[a,x]}$$

The NP operators D, Δ, δ and $\bar{\delta}$ are naturaly represented by symbols $\mathsf{DD}, \Delta, \delta, \bar{\delta}$ (operator D is reserved for the differentiation and cannot be used as a symbol):

$$Dx \;\mapsto\; \mathsf{DD[x]} \quad \Delta x \;\mapsto\; \Delta\mathsf{[x]} \quad \delta x \;\mapsto\; \delta\mathsf{[x]} \quad \bar{\delta}x \;\mapsto\; \bar{\delta}\mathsf{[x]}$$

In Mathematica, Greek letters can be typed using ESC, e.g. letter Δ can be typed by pressing

```
ESC D ESC
```

Program has built-in definitions of the spin coefficients and their complex conjugates. They are denoted by the symbols

$$\alpha, \beta, \gamma, \varepsilon, \kappa, \lambda, \mu, \nu, \mathsf{pi}, \rho, \sigma, \tau.$$

The only exception is π coefficient, because the symbol π is reserved for Ludolph's constant. Thus, for example, Mathematica evaluates expression $\mathsf{DD}[\pi]$ as a derivative of constant which is zero. For this reason we used symbol pi instead of more natural π.

The other built-in NP quantities are the Weyl and Ricci spinor. The Weyl spinor is totally symmetric four-valent spinor Ψ_{ABCD} and in the program it is represented by symbol $\psi\mathsf{[A,B,C,D]}$. Ricci spinor is also four-valent, but its first two indices are unprimed and the other two are primed. Riemann tensor related NP quantities are therefore

$$\Psi_{ABCD} \;\mapsto\; \psi\mathsf{[A,B,C,D]}, \qquad \Phi_{ABA'B'} \;\mapsto\; \Phi\mathsf{[A,B,AA,BB]}$$

$$\Psi_0 \;\mapsto\; \Psi\mathsf{[0]} \qquad \dots \qquad \Psi_4 \;\mapsto\; \Psi\mathsf{[4]}$$

$$\Phi_{00} \;\mapsto\; \Phi\mathsf{[0,0]} \qquad \dots \qquad \Phi_{22} \;\mapsto\; \Phi\mathsf{[2,2]}$$

Symbol $\psi\mathsf{[A,B,C,D]}$ is in fact Weyl spinor expanded into its components with respect to spin basis in terms of NP quantities $\Psi_0, \dots \Psi_4$, i.e.

$$\Psi_{ABCD} \;=\; \Psi_0\, \iota_A\, \iota_B\, \iota_C\, \iota_D \;-\; 4\, \Psi_1\, o_{(A}\, \iota_B\, \iota_C\, \iota_{D)} \;+\; \cdots.$$

Similarly, symbol $\Phi\mathsf{[A,B,AA,BB]}$ is Ricci spinor expanded in terms of NP quantities:

$$\Phi_{ABA'B'} \;=\; \Phi_{00}\, \iota_A\, \iota_B\, \bar{\iota}_{A'}\, \bar{\iota}_{B'} \;-\; 2\, \Phi_{01}\, \iota_A\, \iota_B\, \bar{o}_{(A'}\, \bar{\iota}_{B')} \;+\; \cdots.$$

Finally, NPformalism defines the spinor equivalent of Riemann tensor. For simplicity, however, script defines only the antiself- dual part of Riemann tensor, cf. (6.9):

$$\begin{aligned} {}^{-}R_{abcd} \;=\;& \Psi_{ABCD}\, \epsilon_{A'B'}\, \epsilon_{C'D'} \;+\; \Phi_{ABC'D'}\, \epsilon_{A'B'} \epsilon_{CD} \\ &+ \Lambda\, \epsilon_{A'B'}\, \epsilon_{C'D'}\, (\epsilon_{AC}\, \epsilon_{BD} \;+\; \epsilon_{AD}\, \epsilon_{BC})\,. \end{aligned} \tag{A.1}$$

Complete Riemann tensor is then sum of its self-dual and antiself-dual part,

$$R_{abcd} = {}^{+}R_{abcd} + {}^{-}R_{abcd},$$

where ${}^{+}R_{abcd}$ is complex conjugate of ${}^{-}R_{abcd}$. In the script, antiself-dual part of Riemann tensor is represented by symbol

$${}^{-}R_{abcd} \mapsto \mathsf{Riemann[A, B, C, D, AA, BB, CC, DD]}.$$

This review exhausts built-in symbols of the program presented. For example, symbol ϕ_{AB} for electromagnetic spinor is not defined in order to avoid possible collisions, since the letter ϕ is used in many situations. Several additional symbols are defined in a separate file and are briefly described later.

A.2 Working with the NP-formalism

After introducing basic notation and conventions used, we list few examples illustrating properties of the program and its possibilities. Central function of the script is called Penrose and its purpose is to translate expresion into Newman-Penrose formalism. The algorithm si very simple: expression is expanded so that no brackets remain. Then the Leibniz rule for derivatives is being applied until each derivative acts on a single symbol. Finally, the program looks for contractions of spinors or vectors which can be evaluated or simplified using standard rules of spinor and tensor calculus.

For example, spin coefficient ε is defined as

$$\varepsilon = o^A D\iota_A.$$

Function Penrose expects expression like that on the right hand side and tries to re-express it in terms of spin coefficients. Typing

```
Penrose[ ou[A] DD[ιd[A]] ]
```

yields

$$\varepsilon.$$

To see that function Penrose works correctly with spinor and tensor contractions one can try following examples and compare them with calculation made by hand:

Mathematica command (argument of function Penrose)	Standard notation	Result
od[A] ιu[A]	$o_A\, \iota^A$	1
ou[B] ιd[B]	$o^B\, \iota_B$	-1
od[A] ιu[B]	$o_A\, \iota^B$	od[A] ιu[B]
l[d] n[d]	$l_d\, n^d,\ \ l^d\, n_d$	1
g[a, b] g[a, b]	$g^{ab}\, g_{ab}$	4
g[a, b] l[b]	$g_{ab}\, l^b$	l[a]
Ed[A, B] ou[B]	$\epsilon_{AB}\, o^B$	−od[A]
ψ[A, B, C, D] ou[A] ιu[B] ou[C] ou[D]	$\Psi_{ABCD}\, o^A\, \iota^B\, o^C\, o^D$	Ψ[1]
Eu[A, B] od[A] ιd[B]	$\epsilon^{AB}\, o_A\, \iota_B$	1
Φ[A, B, AA, BB] Eccu[AA, BB]	$\Phi_{ABA'B'}\, \epsilon^{A'B'}$	0
ou[A] $\overline{\text{ou}}$[AA] Nablad[A, AA, x]	$o^A\, \bar{o}^{A'}\, \nabla_{AA'} x$	DD[x]
od[A] ou[B] Δ[ιd[A]ιd[B]]	$o^A\, o^B\, \Delta(\iota_A\, \iota_B)$	-2γ

Let us see another example. In section 6.1 we have seen how to express quantity $D\tau^A$ in the NP formalism, if τ^A is an arbitrary spinor. In Mathematica we can proceed as follows. First we define spinor τ^A in terms of its components with respect to spin basis by

$$\tau\mathsf{u}[\mathsf{A}\_] = \tau 0\, \mathsf{ou}[\mathsf{A}] \ + \ \tau 1\, \iota\mathsf{u}[\mathsf{A}].$$

Now Mathematica can calculate contractions of τ^A with basis spinors:

$$\begin{array}{lcl} \mathsf{Penrose}[\ \tau\mathsf{u}[\mathsf{X}]\, \mathsf{od}[\mathsf{X}]\ ] & \to & \tau 1 \\ \mathsf{Penrose}[\ \tau\mathsf{u}[\mathsf{X}]\, \iota\mathsf{d}[\mathsf{X}]\ ] & \to & -\tau 0 \end{array}$$

Applying function Penrose to expression DD[τu[A]] will insert our definition of τ^A into operator DD. Then, by additivity of the derivative and by the Leibniz rule, our program arrives (internally) at expression

$$\mathsf{DD}[\tau 0]\, \mathsf{ou}[\mathsf{A}] \ + \ \tau 0\, \mathsf{DD}[\mathsf{ou}[\mathsf{A}]] \ + \ \mathsf{DD}[\tau 1]\, \iota\mathsf{u}[\mathsf{A}] \ + \ \tau 1\, \mathsf{DD}[\iota\mathsf{u}[\mathsf{A}]].$$

Now relations

$$Do^A \ = \ \varepsilon\, o^A \ - \ \kappa\, \iota^A, \quad D\iota^A \ = \ \pi\, ou^A \ - \ \varepsilon\, \iota^A$$

will be used, so Mathematica gives the result

$$\varepsilon\, \tau 0\, \mathsf{ou}[\mathsf{A}] \ + \ \mathsf{pi}\, \tau 1\, \mathsf{ou}[\mathsf{A}] \ + \ \mathsf{DD}[\tau 0]\, \mathsf{ou}[\mathsf{A}] \ - \ \kappa\, \tau 0\, \iota\mathsf{u}[\mathsf{A}] \ - \ \varepsilon\, \tau 1\, \iota\mathsf{u}[\mathsf{A}] \ + \ \mathsf{DD}[\tau 1]\, \iota\mathsf{u}[\mathsf{A}].$$

This is already final expression but it is worth to give it more "tidy" form. The first possibility is to contract it with basis spinors to find the components with respect to o^A and ι^A, but we must be careful because of signs. More efficient way is to extract factors standing in front of basis spinors using Mathematica function Coefficient. Code suitable for convenient reading coefficients off can look like follows:

```
τu[A_] = τ0 ou[A] + τ1 ιu[A];
expr = Penrose[DD[τu[A]]];
Coefficient[expr, ou[A]]
Coefficient[expr, ιu[A]]
```

This code produces two expressions,

```
ε τ0 + pi τ1 + DD[τ0]
− κ τ0 − ε τ1 + DD[τ1]
```

which implies

$$D\tau^A \;=\; \left(D\tau^0 \,+\, \varepsilon\,\tau^0 \,+\, \pi\,\tau^1\right)\, o^A \;+\; \left(D\tau^1 \,-\, \kappa\,\tau^0 \,-\, \varepsilon\,\tau^1\right)\, \iota^A.$$

Consider yet another example. Let $\phi_{A'}$ be spinor defined by equation

$$\phi_{A'} \;=\; \nabla_{AA'}\tau^A. \tag{A.2}$$

In the NP formalism, the components of spinors are usually indexed according to the number of contractions with spinor ι, i.e. the NP components of $\phi_{A'}$ are defined as

$$\phi_0 \;=\; \phi_{A'}\,\bar{o}^{A'}, \quad \phi_1 \;=\; \phi_{A'}\,\bar{\iota}^{A'}.$$

Let us find components of spinor $\phi_{A'}$ defined by equation (A.2) using Mathematica and function Penrose. First we define spinor

$$\tau_A \;=\; \tau_1\, o_A \;-\; \tau_0\, \iota_A,$$

so that

$$\tau_0 \;=\; \tau_A\, o^A, \quad \tau_1 \;=\; \tau_A\, \iota^A,$$

following convention of the NP formalism. Spinor τ^A is then

$$\tau^A \;=\; \epsilon^{AB}\,\tau_B.$$

Next we define

$$\phi_{A'} \;=\; \nabla_{AA'}\,\tau^A$$

and compute ϕ_0 and ϕ_1. Corresponding Mathematica code reads

```
τd[A_] = τ1 od[A] − τ0 ιd[A];
τu[A_] = Eu[A, B] τd[B];
φd[AA_] = Nablad[A, AA, τu[A] ];
expr1 = Penrose[ φd[AA] ou̅[AA] ]
expr2 = Penrose[ φd[AA] ιu̅[AA] ]
```

Looking at resulting expression (and perhaps using Coefficient) we find

$$\begin{aligned}
\phi_0 &= D\tau_1 \;-\; \bar{\delta}\tau_0 \;+\; (\alpha-\pi)\,\tau_0 \;+\; (\varepsilon-\rho)\,\tau_1,\\
\phi_1 &= \Delta\tau_0 \;-\; \delta\tau_1 \;+\; (\gamma-\mu)\,\tau_0 \;+\; (\beta-\tau)\,\tau_1.
\end{aligned}$$

A.3 Additional features

In the spinor formalism one often uses symmetrization and antisymmetrization. Library offers routine for symmetrization in two and three indices. Syntax is straightforward, e.g.

$$\mathsf{Symmetrize}[\ \tau[\mathsf{A},\mathsf{B}],\ \mathsf{A},\ \mathsf{B}\ ]$$

yields

$$\frac{1}{2}\,(\tau[\mathsf{A},\mathsf{B}]\ +\ \tau[\mathsf{B},\mathsf{A}]).$$

and similarly for three indices. For example, the NP projections of the so-called univalent twistor equation (this equation was used in the paper [3] for the calculation of the Bondi mass)

$$\nabla_{A'(A}\,\tau_{B)}\ =\ 0$$

can be found by taking projections of expression

```
expr = Symmetrize[ Nablad[A, AA, τd[B]], A, B];
Penrose[expr ou[A] ιu[B] ōu[AA] ]
```

Here we define expression $\nabla_{A'(A}\tau_{B)}$ and then contract it with $o^A\,\iota^B\,\bar{o}^{A'}$.

Antisymmetrization makes sense only for two indices, because antisymmetrization for three and more indices gives zero. Antisymmetrization in AB replaces AB by dummy indices XY and multiplies expression under antisymmetrization by

$$\frac{1}{2}\,\epsilon_{AB}\,\epsilon^{XY}.$$

There are four routines for antisymmetrization, depending on position and type of indices:

AntiSymmd[Q, A, B]	lower unprimed indices
AntiSymmccd[Q, AA, BB]	lower primed indices
AntiSymmu[Q, A, B]	upper unprimed indices
AntiSymmccu[Q, AA, BB]	upper primed indices

For example, antisymmetrization of ϵ_{AB} must leave this spinor unchanged:

$$\epsilon_{[AB]}\ =\ \frac{1}{2}\,\epsilon_{AB}\,\epsilon^{XY}\,\epsilon_{XY}\ =\ \epsilon_{AB},$$

because trace of symplectic form is $\epsilon^{XY}\epsilon_{XY}\ =\ 2$. We can verify that our program gives correct result by

$$\mathsf{AntiSymm}[\mathsf{Ed}[\mathsf{A},\mathsf{B}],\mathsf{A},\mathsf{B}]$$

which yields

$$\mathsf{od}[\mathsf{A}]\,\iota\mathsf{d}[\mathsf{B}]\ -\ \iota\mathsf{d}[\mathsf{A}]\,\mathsf{o}[\mathsf{B}],$$

expression equal to ϵ_{AB}.

Additional definitions of objects used in calculations necessary for the thesis are contained in separate file NPextra.nb. Electromagnetic spinor ϕ_{AB} is defined as

$$\phi_{AB} \;=\; \phi_0\,\iota_A\,\iota_B \;-\; 2\,\phi_1\,o_{(A}\,\iota_{B)} \;+\; 2\,\phi_2\,o_A\,o_B.$$

Maxwell's equation are equivalent to spinor equation

$$\nabla^A_{A'}\phi_{AB} \;=\; 0.$$

Its NP projections can be obtained by

expr = Nablaud[A, AA, ϕd[A]]	$\nabla^A_{A'}\phi_{AB}$
Penrose[expr ou[B] $\overline{\text{ou}}$[AA]]	$\left(\nabla^A_{A'}\phi_{AB}\right) o^B\,\bar{o}^{A'}$

which gives equation

$$D\phi_1 \;-\; \bar{\delta}\phi_0 \;=\; (\pi - 2\alpha)\,\phi_0 \;+\; 2\,\rho\,\phi_1 \;-\; \kappa\,\phi_2.$$

As it is explained in the main text, scalar field ϕ is described by its gradient, i.e. vector field

$$\varphi_{AA'} \;=\; \nabla_{AA'}\phi.$$

Since we consider also complex fields, our script defines eight spinor fields:

$$\begin{array}{ll}
\varphi_{AA'} \;\mapsto\; \varphi\mathsf{d[A,AA]} & \overline{\varphi}_{AA'} \;\mapsto\; \overline{\varphi\mathsf{d}}\mathsf{[A,AA]} \\
\varphi_A^{\;A'} \;\mapsto\; \varphi\mathsf{du[A,AA]} & \overline{\varphi}_A^{\;A'} \;\mapsto\; \overline{\varphi\mathsf{du}}\mathsf{[A,AA]} \\
\varphi^A_{\;A'} \;\mapsto\; \varphi\mathsf{ud[A,AA]} & \overline{\varphi}^A_{\;A'} \;\mapsto\; \overline{\varphi\mathsf{ud}}\mathsf{[A,AA]} \\
\varphi^{AA'} \;\mapsto\; \varphi\mathsf{u[A,AA]} & \overline{\varphi}^{AA'} \;\mapsto\; \overline{\varphi\mathsf{u}}\mathsf{[A,AA]}
\end{array}$$

Components of these vector fields are denoted as follows:

$$\begin{array}{ll}
D\phi \;\mapsto\; \varphi[0] & \Delta\phi \;\mapsto\; \varphi[2] \\
\delta\phi \;\mapsto\; \varphi[1] & \bar{\delta}\phi \;\mapsto\; \varphi[\bar{1}] \\
D\overline{\phi} \;\mapsto\; \overline{\varphi}[0] & \Delta\overline{\phi} \;\mapsto\; \overline{\varphi}[2] \\
\delta\overline{\phi} \;\mapsto\; \overline{\varphi}[1] & \overline{\delta\phi} \;\mapsto\; \overline{\varphi}[\bar{1}]
\end{array}$$

Finally we define the last vector field, gradient of conformal factor Ω denoted by

$$s_{AA'} \;=\; \nabla_{AA'}\Omega.$$

Because conformal factor is real, it is sufficient to define only four fields:

$$\begin{array}{ll}
s_{AA'} \;\mapsto\; \mathsf{sd[A,AA]} & s^{AA'} \;\mapsto\; \mathsf{su[A,AA]} \\
s_A^{\;A'} \;\mapsto\; \mathsf{sdu[A,AA]} & s^A_{\;A'} \;\mapsto\; \mathsf{sud[A,AA]}
\end{array}$$

Components of $s_{AA'}$ are denoted in following way:

$$D\Omega \;\mapsto\; \mathsf{S0} \qquad \Delta\Omega \;\mapsto\; \mathsf{S2}$$

$$\delta\Omega \;\mapsto\; \mathsf{S1} \qquad \bar{\delta}\Omega \;\mapsto\; \overline{\mathsf{S1}}$$

As the last example, consider scalar product

$$s^a\,\varphi_a \;=\; (\nabla_{AA'}\Omega)(\nabla^{AA'}\phi).$$

Straightforward command

$$\mathsf{Penrose[su[A,AA]}\,\varphi\mathsf{d[A,AA]]}$$

yields correct expression

$$\mathsf{S2}\,\varphi[0] \;+\; \mathsf{S0}\,\varphi_2 \;-\; \mathsf{S1}\,\varphi[\bar{1}] \;-\; \overline{\mathsf{S1}}\,\varphi[1].$$

A.4 Conclusion

In this appendix we introduced scripts written in Mathematica for projecting spinor/tensor equations onto the spin basis/null tetrad and writing resulting equations in the NP formalism. We explained how any spinor/tensor equation can be converted to the notation used by script and listed a number of examples.

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