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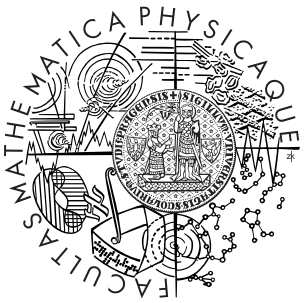


Taras Perevorskyy

## Testy bodových procesů

Katedra pravděpodobnosti a matematické statistiky

Vedoucí práce: RNDr. Tomáš Mrkvička, Ph.D., Jihočeská univerzita, Přírodovědecká fakulta, Ústav matematiky a biomechaniky.

Studijní program: Teorie pravděpodobnosti a náhodné procesy

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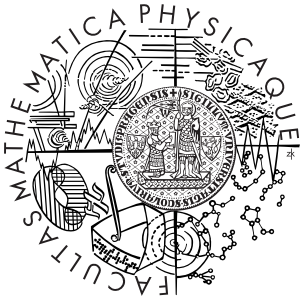
# THESIS




Taras Perevorskyy

## Point processes testing methods

Department of Probability and Mathematical Statistics

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Study programme: Probability theory and random processes

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I wish to thank my supervisor, Tomáš Mrkvička, for his guidance during my work on this thesis. I am very indebted to the company Mineus )( s.r.o. and especially its managing director Mirko Nedeljkovic for generously providing me with the processor time without which this thesis would not be possible to complete. Finally I would like to express my sincerest thanks to miss Eliška Vránová and my beloved mother for their unconditional support and help at all times.

I hereby declare, that I have written this thesis on my own using only the cited sources. I grant my consent for this work to be loaned.

On in Prague Taras Perevorskyy

# Contents

**Abstrakt**

Název práce: *Testy bodových procesů*
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Abstrakt: Hlavním cílem této práce je provést srovnávací analýzu různých testů modelů bodových procesů, které jsou nyní k dispozici. Za tímto účelem byl pro statistický software `R` byl napsán skript ve kterém jsou implementované různé testy modelů bodových procesů. Pomocí tohoto softwaru byla provedena rozsáhlá simulační studie skládající se z více než 500 000 testů. Velká pozornost byla věnována teprve nedávno zveřejněnému envelopes testu. Kromě toho byly testovány další dvě simulační metody a také $J$-test a quadrant-count test. Pro testování jsme zvolili 5 různých typů procesů. Získané výsledky byly pečlivě analyzovány a zdokumentovány.

Klíčová slova: *bodové procesy, envelopes test, simulační testy.*

**Abstract**

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Abstract: The main objective of this thesis is to carry out a comparative analysis of the now available point process model tests. A script file has been written implementing the different point process test methods in the `R` program. With the help of this software, a simulation study has been conducted resulting in over 500 000 tests. A great attention has been paid to the only recently published envelopes test. Except for envelopes test other two simulation tests have been considered along with the $J$-test and the quadrant-count test. As test subjects 5 different process families were chosen. The obtained results have been carefully analyzed and the interesting patterns have been documented.

Keywords: *point processes, simulation tests, envelopes test, model testing.*

# Chapter 1

# Introduction

The modern point process theory provides powerful tools for modelling and analyzing spatial data. The development of the theory started in the early 20th century and its major applications appeared in consequence of the rapid growth of machine computing capabilities. Spatial data is or can always be represented as points in the space of whatever dimension. Such data can be of almost infinite complexity and is easily found in a broad range of disciplines such as forestry and plant ecology (location of trees or plants), geography (locations of geographical objects), astronomy (planetary formations), material science (position of defects in industrial materials) and the list is not even nearly complete. Point process theory offers a robust platform for the statistical treatment and classification of the collected data of such type.

The basic idea behind the main point process application is simple and can roughly be described as follows: the observed data is in one way or another compared to the existing point process class and parameters of the class representative are chosen to create a match with the data. After the matching process is determined, the theory offers us a multitude of probability statements about the interactions among points sometimes revealing data properties of incomprehensible magnitude. Point process model testing methods are therefore of vital importance and so we decided to devote this thesis to this problem.

Our ultimate goal is to compare the strength of the now available tests for point process models and to complete this task we have created a computer software enabling persons interested to carry out such tests in the `R` statistical environment. The software is published under GNU public license in hope that it may be of use and its archive will be made available for download

from `http://home.pf.jcu.cz/~mrkvicka/math/`. The archive contains an R script file which offers calculation of simulation test with deviation measure, simulation test with envelopes, asymptotic normality test as well as a dynamic library for calculating morphological functions in `R` under Windows operating system. Furthermore the package contains results of an extensive simulation study in form of MySql text file dump and the source codes for compiling the morphological functions calculation algorithm for other platforms. For instructions on how to use this software please refer to the readme file enclosed in the package.

The thesis is organized in 6 chapters. Chapter 2 offers a brief introduction to the theory of the point processes and presents some basic definitions and statements aiming to acquaint the reader with the topic. After that in chapter 3 we direct our attention to the common methods of estimating point process statistics and discuss the problem of their asymptotic normality. Chapter 4 describes the now available methods for testing the point process models. Finally Chapter 5 presents and compares the results of the simulation study and tries to make suggestions on the most appropriate testing method. Chapter 6 concludes this thesis.

# Chapter 2

# Brief introduction to the theory of point processes

This chapter defines the basics of the point process theory and provides the necessary base for the further development. We shall start with giving a set of basic and well-known definitions, proceed with defining a point process and finally describe some of its basic classes and their properties focussing on the point process families which we have observed in detail while performing the simulation study.

## 2.1 Definition of a point process

Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $\mathcal{B}^d$ denote a space of all Borel sets on $\mathbb{R}^d$, and $\mathcal{B}_0^d \subseteq \mathcal{B}^d$ be a system of all finite Borel sets. Then a space of locally finite subsets of $\mathbb{R}^d$ can be defined as

$$\mathcal{N} = \left\{\phi \subseteq \mathbb{R}^d : \phi(B) < \infty, \forall B \in \mathcal{B}_0^d\right\}$$

where $\phi(B)$ denotes number of points falling into $\phi \cap B$. On that space we will set up a sigma-algebra in a following manner:

$$\mathfrak{N} = \sigma\left\{\left\{\phi \in \mathcal{N} : \phi(B) = m\right\}, m \in \mathbb{N}_0, B \in \mathcal{B}_0^d\right\}$$

A measurable function $\Phi : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathcal{N}, \mathfrak{N})$ is then called a *point process*. Point processes can be considered either as random sets of discrete points or as random measures counting the number of points. Both approaches prove to be advantageous in different applications and thus the following notations

may be used:

$x \in \Phi$ asserts that the point $x$ belongs to the random sequence $\Phi$;
$\Phi(B) = n$ asserts that there are $n$ points belonging to $\Phi$ in the set $B$.

A realization of a point process $\Phi(B)$ is usually observed in a window which can be practically any measurable set. In applications, windows are usually balls or squares of an appropriate dimension. All our tests have been performed with point processes observed in a unit square.

One of the important qualities of a point process is stationarity. A point process is referred to as *stationary* if for an arbitrary $y \in \mathbb{R}^d$

$$\mathbb{P}(\Phi \in U) = \mathbb{P}(\Phi + y \in U) \text{ for all } U \in \mathcal{N}.$$

In other words, a point process is stationary if its distribution remains unchanged after shifting each point on the same distance $y$. The notation of *isotropy* is fairly equivalent, except that the invariance is required for point rotations. The name 'point process' at first sight implies a dynamic evolution over time, but this is not entirely true. The name random point field would perhaps be more adequate. Nevertheless due to the history of evolution of this theory the term point process has been employed.

In this thesis we will exclusively consider *simple* point patterns, that is patterns which do not possess any two points located at the exact same coordinates.

The *distribution* of a point process is a probability measure $\Pi$ defined on $(\mathcal{N}, \mathfrak{N})$ in the following manner:

$$\Pi(U) = \mathbb{P}(\Phi \in U) = \mathbb{P}(\{\omega \in \Omega : \Phi(\omega) \in U\}),\ U \in \mathfrak{N}$$

In the system of all finite-dimensional distributions of a point process a smaller subsystem exists, which is of particular importance. Probabilities

$$v_B = \mathbb{P}(\Phi(B) = 0) = \mathbb{P}(\Phi \cap B = \emptyset)$$

are called *void probabilities* of the point process. Their importance lies in the fact, that complete distribution of a point process is determined by $v_B$ as $B$ ranges through the compact sets [8].

## 2.2 Point process characteristics

In this section we will discuss the functional characteristics of a point process. We will start with a brief description of the general $n$-th moment measures and then direct our attention to the commonly used characteristics based on the first and second moment measures. Finally, we will give a brief introduction to another type of characteristics which have been introduced only recently – morphological functions. The discussion will be held only for stationary case, but most definitions can be extended to cover non-stationary point patterns as well. Examples of characteristics for the various process classes can be found in the next section.

### 2.2.1 Moment measures

Since the point process is a mapping operating on the probability space, one could be curious to define its $n$-th moment. However, this requires, due to the complexity of the point process structure, a more general and flexible approach. For a point process $\Phi$ we will define two general counting measures: *$n$-th order moment measure* $M^{(n)}(A)$ and *$n$-th order factorial moment measure* $\alpha^{(n)}(A)$ with the following two expressions:

$$M^{(n)}(A) = \mathbb{E} \sum_{X_1,\ldots,X_n \in \Phi} \mathbf{1}_{[(X_1,\ldots,X_n)\in A]},\ A \in (\mathcal{B}^d)^n$$

$$\alpha^{(n)}(A) = \mathbb{E} \sum_{X_1,\ldots,X_n \in \Phi}^{\neq} \mathbf{1}_{[(X_1,\ldots,X_n)\in A]},\ A \in (\mathcal{B}^d)^n$$

where $\sum^{\neq}$ symbol asserts that the summation is performed using only $n$-tuples of mutually different $X_1, \ldots, X_n$. The above-defined measures are not exactly equivalent to the moments of the classical random variable. However, denoting $\mathbb{E}\Phi(B) = M^{(1)}(B)$ one could show, that

$$M^{(n)}(B_1 \times \ldots \times B_n) = \mathbb{E}[\Phi(B_1)\ldots\Phi(B_n)], \quad B_1, \ldots, B_n \in \mathcal{B}^d;$$

$$\alpha^{(n)}(B \times \ldots \times B) = \mathbb{E}[\Phi(B)(\Phi(B)-1)\ldots(\Phi(B)-n+1)], \quad B \in \mathcal{B}^d$$

consult [8] for details. From the definition of $M^{(n)}$ it follows that the first order moment measure $M^{(1)}(B)$ is, in a nutshell, an expected number of

points of a process $\Phi$ which fall into the set $B$. $M^{(1)}(B)$ is commonly referred to as *intensity measure* and denoted $\Lambda(B)$.

In a stationary case, if the intensity measure $\Lambda$ is locally finite, then it is a multiple of the Lebesgue measure. This is obvious taking into consideration the fact that Lebesgue measure is the only finite measure on $(\mathbb{R}^d, \mathcal{B}^d)$ which is translation invariant. Therefore, intensity measure of a stationary point process is referred to simply as intensity. If $\lambda$ – the density of $\Lambda$ with respect to the Lebesgue measure exists, then $\lambda$ is called *intensity function* (or just intensity if the process is stationary) and is in fact a Borel measure. One could also describe $\lambda$ as a mean number of points of $\Phi$ per unit volume.

### 2.2.2 First and second order properties

In order to properly investigate the given point pattern in applications, it is important to have an extensive set of tools which are based on the different process qualities. The most basic of them are the so called first and second order properties of the process. $\Lambda$ – the intensity measure, mentioned in the previous subsection, is one of the first order properties. Other first order properties are based on inter-point distances. We will define those while still assuming the stationarity of the point process $\Phi$. Note that extending the following definitions to cover the non-stationary case is non-trivial. Refer to [6] for more details.

The *empty space function* $F$, which is sometimes also called the spherical contact distribution function, is the distribution function of the distance from the origin (or another fixed point in $\mathbb{R}^d$) to the nearest point in $\Phi$. Formally:

$$F(r) = \mathbb{P}(\Phi \cap b(0,r) \neq \emptyset), \quad r > 0 \tag{2.1}$$

The *nearest-neighbour function* $G$ is

$$G(r) = \frac{1}{\lambda |A|} \, \mathbb{E} \sum_{x \in \Phi \cap A} \mathbf{1}_{[\Phi \setminus x \cap b(x,r) \neq \emptyset]} \tag{2.2}$$

for the arbitrary set $A$ with $0 < |A| < \infty$. Conceptual difference between $F$ and $G$ functions is obvious. While the $F$ function considers distances from an arbitrary point in $\mathbb{R}^d$ to the point of the process, the $G$ function deals with distances between points of the process. Finally the *J-function* is defined by

$$J(r) = \frac{1 - G(r)}{1 - F(r)}, \quad F(r) < 1 \tag{2.3}$$

Both $F$ and $G$ functions use distances to the nearest point of the process, which makes them somewhat shortsighted. Therefore we need a characteristics, which takes into consideration more point data. We will turn to the second order moment measure and define the second order reduced moment measure $\mathcal{K}$. For a stationary point process $\Phi$ with intensity $\lambda$ the measure

$$\mathcal{K}(B) = \frac{1}{|A|} \, \mathbb{E} \sum_{x,y \in \Phi}^{\neq} \frac{\mathbf{1}_{[x \in A, y - x \in B]}}{\lambda^2}, \quad B \subseteq \mathbb{R}^d \tag{2.4}$$

is called the *second order reduced moment measure*, if it does not depend on the choice of $A \subseteq \mathbb{R}^d$ with $A$ being locally finite. $K(r) = \lambda \mathcal{K}(b(o, r))$, where $o \in \Phi$ is then called the *second order reduced moment function* or simply $K$-function. As follows from the definition, one can interpret $\lambda K(r)$ as the mean number of points of the process $\Phi$ in the sphere with radius $r$ and centre at the typical point of the process, which is not itself counted. Other functions than $K$ may be used to describe the second-order behaviour of the point process. Some of them are adopted from statistical physics, where they have been used for a long time. For example the product density $\rho^{(2)}$

$$\rho^2(r) = \lambda^2 \frac{\dfrac{\mathrm{d}K(r)}{\mathrm{d}r}}{d\omega_d r^{d-1}} \tag{2.5}$$

the pair correlation function $g$

$$g(r) = \frac{\rho^{(2)}(r)}{\lambda^2} \tag{2.6}$$

or the $L$ function

$$L(r) = \sqrt[d]{\frac{K(r)}{\omega_d}} \tag{2.7}$$

$\omega_d$ mentioned in the definitions above and also found later in this paper denotes the volume of the unit ball in $\mathbb{R}^d$. For $d \leq 3$ we have: $\omega_0 = 1$, $\omega_1 = 2$, $\omega_2 = \pi$, $\omega_3 = 4\pi/3$.

In applications the preferred choice of the second order property depends on statistical considerations and on convenience, of course.

Important thing to note is that $K(r)$ does not necessarily represent all the important information about a stationary point process. Autors of [2] gave an example of the point process, which is quite different from the Poisson point process, defined later in this thesis, yet has the same $K$-function. A good rule of thumb is to rely on more then one characteristics, possibly of different types. For example, besides the $K$-function quantity one could consider any of the three morphological functions. Those are discussed in the immediately following subsection.

### 2.2.3 Morphological functions

The morphological functions are a relatively young but quite powerful statistics of a point process. Originally they have been used in statistical physics and have been adopted recently by the point process theory. The geometrical idea behind morphological function is simple. Each point pattern is enlarged to a disc of radius $r$, the same for all points. In this way, the pattern of discs is obtained, which for a sufficiently large $r$ are mutually connected. The topology or morphology of this pattern is then analyzed for all reasonable values of $r$. Three geometrical characteristics are used: area, boundary length and Euler characteristics. Integral geometry suggests that determining the aforementioned functions for all positive $r$ will give us valuable information about the analyzed pattern, its clustering or regularity degree, thus about the possible interactions between the points of the process. We will discuss the definition and basic properties of morphological functions for the stationary point process on the plane. It is with some effort possible to extend this definition to non-stationary processes. Readers seeking a way to generalize those definitions to higher dimensions are advised to turn to [1].

Consider a simple stationary point process of intensity $\lambda$ and denote by $\Phi$ the set of all its points. Moreover, let $b(o, r)$ denote the disc of radius $r$ with the center at the origin $o$ of $\mathbb{R}^2$. The symbol $\otimes$ denotes the Minkovski-addition operation also known as dilation [9] and consider the closed random set $\Phi_r$ produced by this operation

$$\Phi_r = \Phi \otimes b(o, r).$$

$\Phi_r$ is the union of all discs of radius $r$ centered at the points of $\Phi$ and thus is as well stationary. So it can be characterized using the intensities of corresponding associated curvature measures. Denote by $A_A(r)$ the area

fraction of $\Phi_r$ by $U_A(r)$ the mean boundary length of $\Phi_r$ and by $N_A(r)$ the specific Euler characteristic per area unit, as hinted by the subscript $A$. In the planar case $N_A$ can be interpreted as the number of connected components minus the number of holes. Thus $N_A(0) = \lambda$. For large $r$, however $N_A(r)$ can be negative and for $r \to \infty$, $\Phi_r \to \mathbb{R}^2$ and $N_A(r) \to 0$.

Now we can define morphological functions as:

$$a(r) = \frac{A_A(r)}{\lambda \pi r^2}, \tag{2.8}$$

$$u(r) = \frac{U_A(r)}{2\lambda \pi r}, \tag{2.9}$$

$$n(r) = \frac{N_A(r)}{\lambda}, \text{ for } r \geq 0 \tag{2.10}$$

all of these functions have value 1 for $r = 0$ but their behaviour differs. $a(r)$ and $u(r)$ decrease monotonically and are bounded by 0 and 1. $n(r)$ on the other hand can take negative values. Furthermore, $a(r)$ is closely related to the empty space distribution function $F(r)$ see Stoyan and Stoyan (1994)

For a specific point pattern morphological functions can be conveniently calculated using a standalone software written by A. Tscheschel using the ideas of Brodatzki and Mecke available for download under GNU license at `http://www.mathe.tu-freiberg.de/inst/stoch/Stoyan/morph2D/`. We have used the core algorithm of this program to enable morphological function calculation in the `R` statistical environment under Windows operating system. The dynamic library accommodating this algorithm and also source codes for compiling `R` compatible packages for other platforms can be found in the aforementioned package.

To enable `R` make use of the library, extract the contents of the package and execute `dyn.load(path/to/Area.dll)` command in the `R` command prompt. Then call the now available function `Morph(P)` on the desired point pattern `P`. Further notices on usage and the description of input and output values are provided in the readme file included in the package.

## 2.3 Point process classes

A point pattern is a structure which can be observed virtually everywhere. The forest trees, the selected type of cells in a unit volume, the birds nests in

a certain area, cosmological structures, simply any type of objects in space. While the variety of patterns can seem overwhelming and chaotic we can still single out three fundamental types of point patterns.

If the points of the point pattern arrange themselves in groups, then they constitute a *cluster* process. On the other hand the points may exhibit repulsion and allocate themselves in a standalone manner. Should that be the case, a point process is informally referred to as *repulsive*. This type of processes further divides in smaller subgroups such as *Cox* or *hard-core* processes. The third case occurs if the observed points in whole do not demonstrate clustering or repulsion but are arranged in a rather random manner, which in the point process theory is referred to as *complete spatial randomness model*.

In this section we are going to describe some of the representative families of the aforementioned types of point patterns, determine their properties and touch the extensive topic of their statistics. We will focus on the Poisson point process, as this process is the corner stone of almost every point process model. Then we will turn to cluster processes, provide the necessary theory for defining the Thomas cluster process and finally present the Strauss process. Both Thomas and Strauss processes as representatives of the cluster and regular processes along with Poisson point process were subject of vast automated model testing with a variety of test methods.

More information about those processes as well as other process types can be found for example in [9] or [6].

### 2.3.1 Poisson point process

Poisson point process describes a complete spatial randomness model, that is a model where there are absolutely no interactions between points. It also serves as a reference process when summary statistics are studied and when more advanced processes are constructed. In our case study we have considered 1000 different simulations of a Poisson process with intensity 200. Such simulation can be easily obtained using the `R` statistical software with the `spatstat` extension by executing the `rpoispp` command. Figure 2.1 provides a visualization of the three selected candidates obtained in such way.

As apparent from the illustrations below, points of the Poisson process as a whole do not exhibit repulsion or clustering and therefore the term "complete spatial randomness model" is clearly very appropriate. We are now

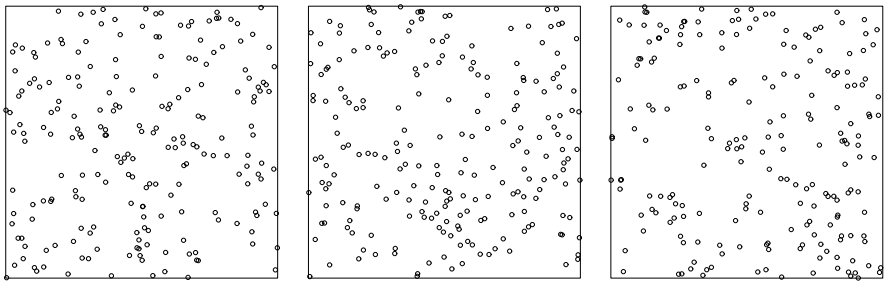

Figure 2.1: Different realizations of Poisson process with intensity 200

going to investigate in detail basic properties and statistics of the Poisson point process. There is, however, just one more preparatory step to make before we get to this matter.

### Binomial process

Before we get to the definition of the Poisson point process, a related process is to be considered. To define it let $\nu$ be a diffusional measure on $\mathbb{R}^d$. That is for every $x \in \mathbb{R}^d,\ \nu(\{x\}) = 0$. Let $B \in \mathcal{B}^d$ be such that $0 < \nu(B) < \infty$. Furthermore let $X_1, \ldots, X_n,\ n \in \mathbb{N}$ be independent, $\nu$-equally distributed random vectors, for which

$$\mathbb{P}(X_i \in A) = \frac{\nu(A)}{\nu(B)}, \qquad A \subseteq B, \quad A \in \mathcal{B}^d.$$

Then $\Phi^{(n)} = \{X_1, \ldots X_n\}$ constitutes a binomial point process with $n$ points on $B$.

The void probabilities of the binomial point process observed through the window $W$ are:

$$\mathbb{P}(\Phi^n(A) = 0) = \left(\frac{\nu(W) - \nu(A)}{\nu(W)}\right)^n$$

### Poisson process definition

A Poisson point process is defined in a similar manner. Let $\Lambda$ be a locally finite diffusional measure. Then a point process $\Phi$ is a *Poisson point process* with intensity measure $\Lambda$ if it has following properties:

1. $\Phi(B)$ has Poisson distribution with parameter $\Lambda(B)$ for every $B \in \mathcal{B}_0^d$
2. $\Phi(B_1) \dots \Phi(B_n)$ are independent for every $n \in \mathbb{N}$ and $B_1 \dots B_n \in \mathcal{B}_0^d$

As usually, we will direct our attention to a stationary case of the Poisson process. With little effort most results can be extended to cover the non-stationary case. See [9] for details.

For any bounded Borel set $B$ the mean number of points belonging to the stationary Poisson point process in $B$ is $\lambda\nu(B)$. In fact the whole distribution of the Poisson process can be determined solely from knowing the value of $\lambda$. The void probabilities are:

$$\mathbb{P}(\Phi(B) = 0) = \mathrm{e}^{-\lambda\nu(B)}$$

Restricting the stationary Poisson point process $\Phi$ to a compact set B and conditioning on $\Phi(B) = n$ actually yields a binomial point process. This can be proven by showing that all finite dimensional distributions of both processes coincide. However it suffices to prove that both processes share the same system of void probabilities. The following lemma shows exactly that.

**Lemma:** Let $\Phi$ be a Poisson point process with intensity measure $\Lambda$. Take $B \in \mathcal{B}^d$ a fixed set for which $\Lambda(B) < \infty$. Then, conditionally on $\Phi(B) = n$, the restriction of $\Phi$ on $B$ is a binomial process on $B$ with $n$ points and intensity measure $\Lambda$.

**Proof:** This follows immediately from a simple transformation. Set $\Phi_B = \Phi \cap B$ − a restriction of $\Phi$ on $B$, then for any Borel set $A \subseteq B$ it is:

$$\mathbb{P}\left(\Phi_B\left(A\right) = 0 | \Phi\left(B\right) = n\right) = \frac{\mathbb{P}\left(\Phi\left(A\right) = 0, | \Phi\left(B \setminus A\right) = n\right)}{\mathbb{P}\left(\Phi\left(B\right) = n\right)}$$

$$= \frac{\mathbb{P}\left(\Phi\left(A\right) = 0\right)\mathbb{P}\left(\Phi\left(B \setminus A\right) = n\right)}{\mathbb{P}\left(\Phi\left(B\right) = n\right)} = \frac{\mathrm{e}^{-\lambda\nu(B)}\,\dfrac{-\lambda\nu(B \setminus A)^n}{n!}}{\dfrac{-\lambda\nu(B)^n}{n!}}$$

$$= \left(\frac{\nu(B \setminus A)}{\nu(B)}\right)^n$$

the resulting probabilities are exactly the void probabilities of the binomial process and therefore the proof is concluded.

### Stationary Poisson process characterization

Stationary Poisson point process can be also characterized using a different set of properties:

1. Simplicity: no two points coincide so that process is a simple point process.
2. Stationarity. As defined above.
3. Independent scattering: $\Phi(B_1), \ldots, \Phi(B_n)$ are independent Poisson random variables for disjoint bounded Borel $B_1, \ldots, B_n$.

This characterization is the primary approach to estimate whether or not the empirical point pattern is indeed a Poisson point process. The most important property is that of independent scattering. It asserts that there is no interaction between points. This property can be tested statistically and often provides a base for further statistical analysis.

### Moments of the stationary Poisson process

From the definition of stationarity it follows that for a Borel set $B$ the first moment of the stationary Poisson process is a constant multiple of the Lebesgue measure and that multiple is given by the process intensity. Thus we have:

$$\Lambda(B) = \mathbb{E}\Phi(B) = \lambda\nu(B).$$

The second order moment measure is evaluated straightforward from the definition. Given two Borel sets $B_1$ and $B_2$ one could create a following disjointed decomposition:

$$B_1 = (B_1 \cap B_2) \cup B_1 \setminus B_2,$$
$$B_2 = (B_1 \cap B_2) \cup B_2 \setminus B_1.$$

Using the fact that $\Phi$ is a random measure and that $\{\Phi(B_i)\}$ are independent for disjoint $B_i$ one could establish:

$$\begin{aligned} M^{(2)}(B_1 \times B_2) &= \mathbb{E}\Phi(B_1)\Phi(B_2) \\ &= \mathbb{E}\Phi(B_1 \setminus B_2)\mathbb{E}\Phi(B_2 \setminus B_1) + \mathbb{E}\Phi(B_1 \cap B_2)\mathbb{E}\Phi(B_2 \setminus B_1) \\ &\quad + \mathbb{E}(B_1 \setminus B_2)\mathbb{E}(B_1 \cap B_2) + \mathbb{E}(\Phi(B_1 \cap B_2))^2 \\ &= \mathbb{E}\Phi(B_1)\mathbb{E}\Phi(B_2) + \mathbb{E}(\Phi(B_1 \cap B_2))^2 - (\mathbb{E}(\Phi(B_1 \cap B_2)))^2 \end{aligned}$$

As $\Phi(B_1 \cap B_2)$ is of Poisson distribution with mean $\lambda\nu(B_1 \cap B_2)$ its second non-centered moment is:

$$\lambda\nu(B_1 \cap B_2)(1 + \lambda\nu(B_1 \cap B_2)).$$

With that being said, the final formula for $M^{(2)}$ is:

$$\begin{aligned} M^{(2)}(B_1 \times B_2) &= \Lambda(B_1)\Lambda(B_2) + \lambda\nu(B_1 \cap B_2) \\ &= \lambda^2\nu(B_1)\nu(B_2) + \lambda\nu(B_1 \cap B_2). \end{aligned}$$

Summarizing: the second-order moment measure $M^{(2)}$ can be expressed in terms of the Lebesgue measure.

All variances and covariances can also be calculated directly from the second-order moment measure. The relevant formulae following immediately from the definitions of variance and covariance are:

$$\operatorname{var}(\Phi(B)) = M^{(2)}(B \times B) - (\Lambda(B))^2,$$

$$\operatorname{cov}(\Phi(B_1), \Phi(B_2)) = M^{(2)}(B_1 \times B_2) - \Lambda(B_1)\Lambda(B_2).$$

To calculate the second order product density recall the counting nature of the both $M^{(2)}$ and $\alpha^{(2)}$

$$\begin{aligned} M^{(2)}(B_1 \times B_2) &= \mathbb{E}\,\#\{(x, y) : x \in \Phi \cap B_1, y \in \Phi \cap B_2\} \\ &= \mathbb{E} \sum_{x,y \in \Phi} \mathbf{1}_{B_1}(x)\mathbf{1}_{B_2}(y), \end{aligned}$$

$$\begin{aligned} \alpha^{(2)}(B_1 \times B_2) &= \mathbb{E}\,\#\{(x, y) : x \in \Phi \cap B_1, y \in \Phi \cap B_2, x \neq y\} \\ &= \mathbb{E} \sum_{x,y \in \Phi}^{\neq} \mathbf{1}_{B_1}(x)\mathbf{1}_{B_2}(y). \end{aligned}$$

Clearly the only difference between $M^{(2)}$ and $\alpha^{(2)}$ is the expectation of the sum:

$$\sum_{x,y \in \Phi:\ x=y} \mathbf{1}_{B_1}(x)\mathbf{1}_{B_2}(y) = \sum_{x \in \Phi} \mathbf{1}_{B_1 \cap B_2}(x).$$

Hence

$$M^{(2)}(B_1 \times B_2) = \alpha^{(2)}(B_1 \times B_2) + \Lambda(B_1 \cap B_2).$$

The above shows that the second-order product density $\rho^{(2)}$ can be calculated as:

$$\rho^{(2)}(x_1, x_2) = \lambda^2.$$

The $n$-th moment product density and factorial measure in case of stationary Poisson point process are then obviously defined as follows:

$\rho^{(n)}(x_1, \ldots, x_n) = \lambda^n,$

$\alpha^{(n)}(x_1, \ldots, x_n) = \lambda^n \nu(B_1) \cdots \nu(B_n).$

The relations between the $\alpha^{(n)}$ and $M^{(n)}$ even for small $n$ are quite complicated and therefore we omit those in this thesis.

#### Morphological functions for Poisson process

In case of stationary Poisson process with intensity $\lambda$ formulae for all three morphological functions can be obtained. [5] We will omit the calculation process and present here only the resulting formulae. For details see [5].

$$a_P(r) = \frac{1}{x}(1 - \mathrm{e}^{-x}),$$

$$u_P(r) = \mathrm{e}^{-x},$$

$$n_P(r) = (1 - x)\,\mathrm{e}^{-x}$$

where $x = \pi r^2 \lambda$

### 2.3.2 Cluster processes

Another fundamental type of point processes are cluster processes. These processes in contrary to Poisson point process are more suitable to model real life data. This section briefly describes only the basics of cluster processes. More material on cluster processes can be found for example in [6], [9] or [8]

#### The Cox process

To properly introduce a Cox process we first need a definition of the random field. $Y$ is called a random field on $S$ when $Y(x)$ is a random variable for all $x \in S$. If $\rho(x) = \mathbb{E}Y(x)$ exits and is locally integrable, than with probability one $Y(x)$ is a locally integrable function.

Cox processes can be seen on as a natural extension of the Poisson point process obtained by considering the intensity function of the Poisson point process as a realization of a random field. Formally, let $\Lambda$ be a random $\sigma$-finite diffusional measure on $(\mathbb{R}^d, \mathcal{B}^d)$. A point process $\Phi$ that conditionally on $\Lambda = \mu$ becomes a Poisson point process with intensity $\mu$ is called *Cox*

*point process with driving measure* $\Lambda$. If there exists the density of $\Lambda$ in form of a random field $Y = \{Y(x) : x \in \mathbb{R}^d\}$, then it is called a *driving function* of the Cox process.

It is clear from the definition that the distribution of the Cox point process conditioned on a given $Y$ is a Poisson point process with intensity $Y$. The intensity measure of the process $\Phi|Y$ is

$$M(B) = \int_B Y(x)\mathrm{d}x \tag{2.11}$$

In the special case when $Y$ is deterministic, $\Phi$ simply becomes a Poisson process with intensity function $\lambda = Y$. Examples of other specific constructions can be found in [6].

Distributional properties of $\Phi$ being a Cox process can be easily obtained by conditioning on $Y$ and referencing the properties of the Poisson process $\Phi|Y$. Therefore given bounded $B \in \mathcal{B}^d$ one could establish for a Cox process: The void probabilities

$$\begin{aligned} P(\Phi(B) = 0) &= \mathbb{E}\mathbf{1}_{[\Phi(B)=0]} = \mathbb{E}\left(\mathbb{E}\mathbf{1}_{[\Phi(B)=0]}|\Lambda\right) \\ &= \mathbb{E}\left[\mathbb{P}(\Phi(B) = 0|\Lambda)\right] = \mathbb{E}\ \mathrm{e}^{-\Lambda(B)} \end{aligned}$$

The intensity function

$$\rho(\xi) = \mathbb{E}Y(\xi) \tag{2.12}$$

the second order factorial moment measure is

$$\alpha^{(2)}(B_1 \times B_2) = \mathbb{E}\Lambda(B_1)\Lambda(B_2) \tag{2.13}$$

and, if $Y$ has finite variance, the pair correlation function is given by

$$g(\xi, \eta) = \mathbb{E}\frac{[Y(\xi)Y(\eta)]}{[\rho(\xi)\rho(\eta)]} \tag{2.14}$$

### Cluster processes

The notion of cluster process can be defined in a very natural manner. Given the parent process $\Phi_p$ replace each point of $\Phi_p$ by a cluster of $\Psi_X$ points. Formally, let $\Psi_X$ be a finite point process for every $X \in \Phi_p$. Then

$$\Phi = \bigcup_{X \in \Phi_p} \Psi_X$$

is called a *cluster point process*.

Now allow $\Phi$ be a cluster process, such that $\Phi_p$ is a Poisson point process and conditionally on $\Phi_p$ let $\Psi_X$ be independent processes. Furthermore, let $\Phi_p$ and $\Phi_X - X$ be independent, points of $\Psi_X - X$ be independent and equally distributed with density $p$ on $\mathbb{R}^d$ and total point counts of $\Psi_X(\mathbb{R}^d)$ be independent random variables. Then $\Phi$ is called a *Neumann-Scott point process*. If $\Psi_X$ additionally are of Poisson distribution with intensity $\lambda_c$ then $\Phi$ is referred to as *Neumann-Scott Poisson process*.

An important example of the Neumann-Scott Poisson process is Thomas process which has $p$ equal to the density of normal distribution in the sphere $b(o, R)$. That is

$$p(x) = \frac{1}{(2\pi\sigma^2)^{d/2}} \mathrm{e}^{\frac{||x||^2}{2\sigma^2}} \tag{2.15}$$

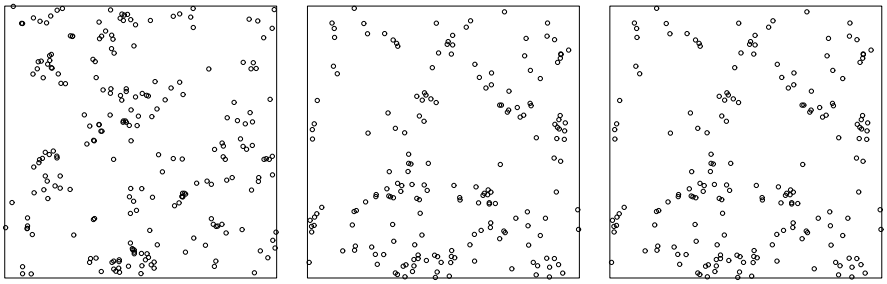

Figure 2.2: Different realizations of Thomas process with parameters $\kappa = 100, \sigma = 0.02, \mu = 2$

A Thomas process depends on 3 parameters: the intensity of Poisson process of cluster centers $\kappa$, the standard deviation of displacement of a point from its cluster centre $\sigma$ and expected numbers of points per cluster $\mu$. Once those three parameters are decided the Thomas process can be easily simulated in `R` equipped with the `spatstat` extension by executing the `rThomas` command and passing the aforementioned parameters to it. Care must be taken while choosing the parameters since for some sets of parameters simulating one realization in `R` can take a considerable amount of time or even make the program unresponsive. Figure 2.2 demonstrates

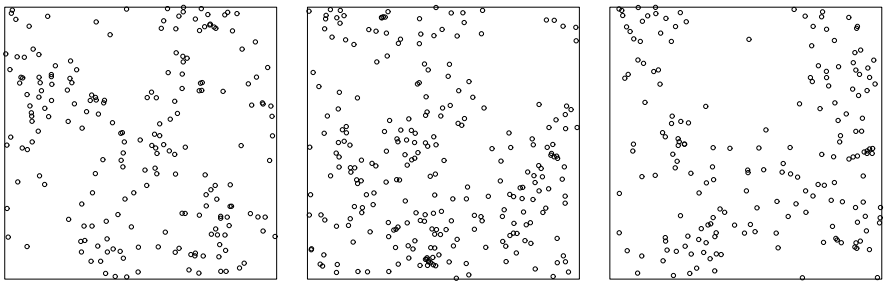

Figure 2.3: Different realizations of Thomas process with parameters $\kappa = 20, \sigma = 0.08, \mu = 10$

three selected processes with $\kappa = 100, \sigma = 0.02$ and $\mu = 2$ and Figure 2.3 presents three simulations of the Thomas process with parameters $\kappa = 20, \sigma = 0.08,$ and $\mu = 10$. Both types have been subjected to the model testing.

Expressions for summary statistics for Thomas point process are by far more complicated than for Poisson process. We only state that the Thomas process is isotropic with the pair correlation function

$$g(x) = 1 + \frac{1}{(4\pi\sigma^2)^{-d/2}\kappa} \, \mathrm{e}^{-\frac{||\xi||^2}{4\sigma^2}} \tag{2.16}$$

The $K$-function can be expressed in terms of cumulative distribution for $\chi^2$-distribution with $d$-degrees of freedom. For $d = 2$ it is

$$K(r) = \pi r^2 + \frac{1 - \mathrm{e}^{-\frac{r^2}{4\sigma^2}}}{\kappa} \tag{2.17}$$

Morphological functions for Thomas process as well as for other cluster processes tend to be smaller as their Poisson process counterparts. If the cluster centres are regularly distributed, which is not the case of Thomas process, and there are large spaces between the clusters then $n(r)$ may be constant in some interval of values of $r$ while $a(r)$ and $u(r)$ continue to decrease.

### Hard core processes

A hardcore process is a process with points which are distant from one another by no less then a certain fixed number. This type of process is the counterpart of the cluster process, though it is obtained basically by the same operation. While to construct the cluster process one could execute the "clustering" operation, where points of the given process will be replaced with sets of points, a hard core process can be constructed by using the replacement sets with one point to keep the distant points or zero points to remove the clustered points. For the sake of clarity such operation carries a distinct name and is referred to as *thinning*.

As apparent from the above construction of the hard core process the thinning operation can not be conducted independently from the process points as it does not necessarily lead to a hard core process. The following proposition establishes that executing the independent thinning operation on a Poisson point process actually yields a new Poisson process.

**Proposition:** Suppose $\Phi$ is a Poisson point process with intensity measure $\Lambda$, $B \in \mathcal{B}^d$ is a fixed set, $p : \mathbb{R}^d \to [0,1]$ is a measurable function and let $\{U(x) : x \in B\}$ be independent random variables identically distributed on $(0,1)$ and independent of $\Phi$. Set $\Phi_{th} = \{X \in B \cap \Phi : U(X) < p(X)\}$ – a thinned process. Then $\Phi_{th}$ is a Poisson point process with intensity measure

$$\Lambda_{th}(A) = \int_A p(y)\Lambda(\mathrm{d}x)$$

for every $A \subseteq B, A \in \mathcal{B}_0^d$.

The proof of this proposition is based on showing that the void probabilities of the process $\Phi_{th}$ coincide with the void probabilities of the Poisson process with intensity measure $\Lambda_{th}$ and is fairly straightforward. Details can be found in [8].

The above proposition makes it clear, that to obtain a hard core process with thinning another process, one must use dependent thinning. For example, if given a Poisson point process $\Phi$ and a fixed positive $r$ one could construct the *Matern I process* using the following rule:

$$\Phi_I = \{X \in \Phi : ||X - Y|| > r \; \forall Y \in \Phi, Y \neq X\}$$

as obvious from the construction, this is a hard core process. Another example of a hard core process is a *Matern II* process which is defined as

$$\Phi_{II} = \{X \in \Phi : U(X) < U(Y) \ \ \forall Y \in \Phi \cap b(X, r) \setminus X\},$$

where $\{U(x) : x \in \mathbb{R}^d\}$ are mutually independent random variables, identically distributed on $(0, 1)$ and independent of $\Phi$

Note that both processes are stationary. This follows immediately from the definition of those processes.

Should one ever find himself in need to construct a hard core process from the scratch and without thinning another process, then following approach can be put to use:

- Choose real $r > 0$ and $B \in \mathcal{B}_0^d$
- Randomly and independently choose $X_1 \in B$
- Randomly and independently choose $X_k$ in $B \setminus \cup_{i=1}^{k-1} b(X_i, r)$
- Keep choosing $X_k$ until $B \subseteq \cup_{i=1}^{k} b(X_i, r)$

The process resulting from this procedure is called a *simple sequential inhibition process* and is clearly a finite point process. In application the union of $b(X_i, r)$ can be a rather complicated set, therefore if using some kind of software to generate such process, the desired number of points has to be specified in advance. For the same reason $X_k$ is generated in $B$ instead of the $B \setminus \cup_{i=1}^{k} b(X_i, r)$ and if the constructed set does not contain the generated point, the point is dismissed and a new point is generated. This procedure continues until the currently generated point is positioned outside the balls surrounding $X_i$.

### Strauss process

One more example of a repulsive process, we would like to present the reader with, is the Strauss process. To begin with, suppose that $\Phi_P$ is a Poisson point process with a finite and non atomic intensity measure $\Lambda$. Denote by $\Pi(U)$ the distribution of $\Phi_P$ that is $\Pi(U) = \mathbb{P}(\Phi_P \in U)$. Additionally, allow $\Phi$ be a point process with density $p$ that is

$$\mathbb{P}(\Phi \in U) = \int_U p(\phi)\Pi(\mathrm{d}\phi)$$

The choice of $\Lambda$ guarantees that process $\Phi$ constructed in this manner is simple and finite. Now allow $p$ be of the shape:

$$p(\phi) = \alpha \prod_{x \in \phi} g(\{x\}) \prod_{\{x,y\} \subseteq \phi} g(\{x, y\}) \tag{2.18}$$

where $\alpha$ is a normalizing constant and $g$ is a non-negative function, such that the right side of the expression 2.18 is integrable.

Process having $p$ as its density function is referred to as *pair-wise interaction process*. If additionally $g(\{x\}) = \beta$ is constant and $g(\{x, y\}) = g(||x - y||)$ then $\Phi$ is a *homogeneous* pair-wise interaction process. An important example of a homogeneous pair-wise interaction point process, is the Strauss process and without further delays here is the definition of this process.
A *Strauss point process* is a point process $\Phi$ with density

$$p(\phi) = \alpha \beta^{\phi(\mathbb{R}^d)} \gamma^{S_R(\phi)}$$

where $S_R(\phi)$ is the total count of all pairs of points which are distant from one another by no more then $R$ and belong to the configuration $\phi$. Formally

$$S_R(\phi) = \sum_{\{x,y\} \in \phi} \mathbf{1}_{[||x-y|| \leq R]}.$$

Constants $\beta > 0, 0 \leq \gamma \leq 1$ and $R > 0$ are parameters of the process. The normalizing constant $\alpha$ is mostly unknown. One could calculate its value for $\gamma = 1$ or $\gamma = 0$.

In the simulation research, we have tested 1000 instances of two different Strauss processes with parameters: $\beta = 200, R = 0.16, \gamma = 0.8$ and $\beta = 200, R = 0.04$ and $\gamma = 0.3$. Figures 2.4 and 2.5 each represent three selected realizations from both process types.

In the `R` statistical environment extended with `spatstat` package the Strauss point process can be simulated using `rStrauss` command. This function requires the user to pass the aforementioned parameters $\beta, \gamma$ and $R$. In comparison to an average generation time of other processes, simulation of

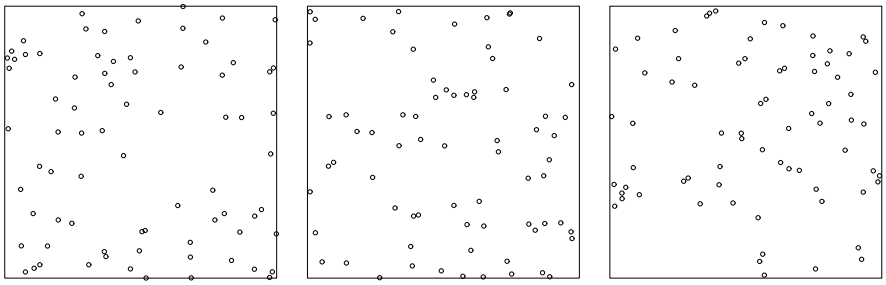

Figure 2.4: Different realizations of Strauss process with parameters $\beta = 200, R = 0.16, \gamma = 0.8$

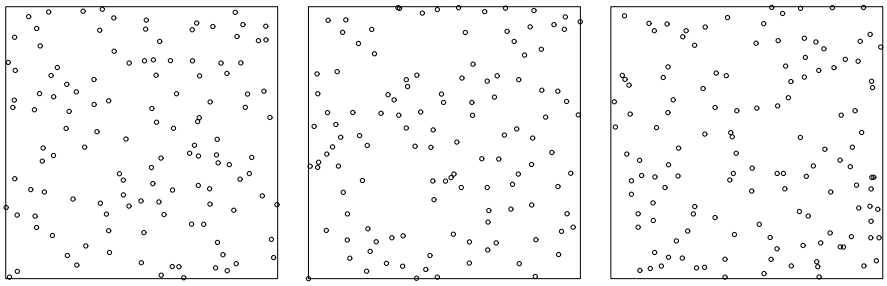

Figure 2.5: Different realizations of Strauss process with parameters $\beta = 200, R = 0.04, \gamma = 0.3$

a Strauss point process can take longer time. As in case with Thomas process, if the parameters are not chosen carefully, there is a great risk, that the program will spend a substantial amount of time trying to deal with the computationally exhaustive algorithm of generation of the process and even may stop responding in some cases.

## 2.4 Concluding this chapter

The objective of this chapter was to introduce the general notion of the point process, define its basic characteristics and qualities and ultimately to present process families which were subjects of the extensive practical test-

ing. As a result, some aspects of the theory presented here may have suffered from the lack of mathematical precision. However, to build the formal point process theory in all its versatility is well beyond the scope of this thesis and thus we will advise the eager reader seeking for more than just a brief introduction to refer to [9], [6] or [8] each providing a comprehensive overview of the modern point process theory while presenting it from a slightly different angle.

# Chapter 3

# Estimates of point process statistics

In this chapter we will briefly describe the most common methods of estimating the statistics of the point pattern. We will start with presenting only the necessary theory and conclude this chapter with the discussion of the asymptotic normality of these estimates. For further information, consult [6].

## 3.1 Nonparametric estimation of summary statistics

Calculating precise values of summary statistics poses quite a challenge as exact formulae are known only for a few point process models. Therefore effective estimates for those functions are much desired for use in applications.

For estimating the intensity one could use a natural unbiased estimator

$$\hat{\lambda} = \frac{n(x)}{|W|} \tag{3.1}$$

where $n(x)$ is the total count of points falling into the window $W$. Moreover, in case of the stationary Poisson point process the expression 3.1 is the maximum likelihood estimate.

To obtain an unbiased estimator of the value of $\mathcal{K}$ of the stationary point process denote $W_\xi = \{\eta + \xi : \eta \in W\}$ the translate of the set $W$ by $\xi \in \mathbb{R}^d$

and suppose that

$$|W \cap W_\xi| > 0 \text{ for all } \xi \in B \in \mathcal{B}^d. \tag{3.2}$$

Then

$$\sum_{\eta,\xi\in\Phi}^{\neq} \frac{\mathbf{1}_{\eta-\xi\in B}}{\lambda(\xi)\lambda(\eta)|W \cap W_{\eta-\xi}|} \tag{3.3}$$

is an unbiased estimate of $\mathcal{K}(\mathcal{B})$. $B$ is usually a square or a ball of an appropriate dimension. If the intensity function is not known, which is often the case, we are left with the only option – to replace the $\lambda(\xi)\lambda(\eta)$ with its estimate $\widehat{\lambda(\xi)\lambda(\eta)}$. This, however, contributes to biasing the estimate of $\mathcal{K}$. The estimate of $L(r)$ obtained from transforming the biased version of the estimate 3.3 using formula 2.7 is of course biased as well. The condition 3.2 ensures that the set $B$ is completely contained in the window $W$. Under some circumstances this condition can be weakened [6].

The quantity

$$\frac{1}{|W \cap W_{\xi-\eta}|} \tag{3.4}$$

is called the *edge correction factor* and is supposed to eliminate potential errors, which may arise in case that some points appear close to the edges of $W$ so that it is not possible to position the testing set $B$ to cover those points. There is a multitude of edge correction factors suggested, each suitable in a particular case.

For estimates of $F$, $G$ and $J$ functions denote by $d(\xi, B)$ the shortest distance from a point $\xi \in \mathbb{R}^d$ to the set $B \subset \mathcal{B}^d$ and let $I \subset W$ be a finite regular grid of points chosen independently from $\Phi$. Also for $r > 0$ let

$$W_{\ominus r} = \{\xi \in W : b(\xi, r) \subseteq W\}$$

Then the unbiased estimate of $F$ is

$$\hat{F}(r) = \sum_{\xi\in\Phi_W} \frac{\mathbf{1}_{d(\xi,\Phi_W)\le r}}{|I \cap W_{\ominus r}|} \tag{3.5}$$

and the estimate of $G$ is

$$\hat{G}(r) = \sum_{\xi \in \Phi_W \cap W_{\ominus r}} \frac{\mathbf{1}_{d(\xi, \Phi_W \setminus \{\xi\}) \leq r}}{\hat{\lambda} |W_{\ominus r}|} \tag{3.6}$$

Now that we have estimates of $F$ and $G$ recalling the definition 2.3 we easily obtain the estimate for the $J$-function:

$$\hat{J}(r) = \frac{1 - \hat{G}(r)}{1 - \hat{F}(r)} \tag{3.7}$$

For the $F$ and $G$ function other more efficient estimates exist in the literature. For example Kaplan-Meier or Chiu-Stoyan estimates can turn useful in some cases. For details see [6] and references within. As one would suspect, the quality of these estimates is very much dependent on how much point process data is available.

Methods for estimating morphological functions are well described in [5]. The general idea behind constructing such estimators is to estimate $A_A$, $U_A$, $N_A$ and the intensity $\lambda$ from the definitions 2.8, 2.9 and 2.10. Please refer to [5] to obtain the precise form of those estimators.

`R` software can calculate estimates of the mentioned functions using different edge correction factors. `Kest()`, `Lest()`, `Fest()`, `Gest()` or `Jest()` functions can be used to obtain estimates of $K$, $L$, $F$, $G$ or $K$ functions respectively.

## 3.2 Asymptotic normality of the summary statistics

For the mathematically correct introduction of the point process model test in the following chapter one more question remains to be answered. Cutting right to the chase, choose a fixed $n$ and consider a family of point processes $\Phi_1, \ldots, \Phi_n$. Denote by $S_i$ the computed estimate of the certain chosen summary statistics for each process in discrete equidistant points $r_1, \ldots, r_m$. Then $S = (S_1, \ldots, S_n)^T$ can be regarded as a realization of the random vector. The question is: can one always find a vector $\mu$ and positive semidefinite matrix $\Sigma$ so that a random vector of distribution $N(\mu, \Sigma)$ is an approximation of $S$?

We were unable to solve this problem in a traditional manner and therefore we decided to test this hypothesis statistically. `R` offers different functions

for testing the multivariate normality, however, those proved to be unusable in our situation. The reason for this is that the offered tests required computation of determinant of $S$ and for $m = 20$ and $n = 50$ the average value of $\det|S|$ was as small as $10^{-106}$ which makes the system computationally singular and prevents the test from execution. While searching for an alternative solution we have discovered the only recently published generalization of Shapiro-Wilk's test for multivariate normality [11]. This test does not require to compute the determinant and thus suits perfectly our purpose. The authors of this test provided an `R` extension as well. Table 3.1 presents the resulting p-values of statistics for $m = 20$, $n = 40$, $r_{max} = 0.1$ and $r_{min} = 0.01$.

| | **Poisson** | **Thomas1** | **Thomas2** | **Strauss1** | **Strauss2** |
|---|---|---|---|---|---|
| **a** | 0.409 | 0.112 | 0.17 | 0.48 | 0.249 |
| **u** | 0.185 | 0.093 | 0.372 | 0.148 | 0.645 |
| **n** | 0.378 | 0.361 | 0.272 | 0.319 | 0.323 |
| **F** | 0.062 | 0.944 | 0.394 | 0.765 | 0.468 |
| **G** | 0.675 | 0.513 | 0.532 | 0.836 | 0.589 |
| **J** | NA | 0 | 0 | 0.333 | 0 |
| **L** | 0.79 | 0.63 | 0.261 | 0.122 | 0.026 |

Table 3.1: p-values from the generalized Shapiro-Wilk's test for summary statistics with parameters $m = 20$, $n = 40$, $r_{max} = 0.1$, $r_{min} = 0.01$.

As apparent from the table 3.1, the hypothesis of asymptotic normality seems to hold for all statistics except the $J$-function. The reason of this issue is obvious from the Figure 3.1. For Poisson process `R` calculates the values of $J$-function only for $r \in [0; 06]$ and for Thomas process only in the interval $[0; 08]$ which is smaller than our chosen $r_{max}$ and thus some of the $S_i(r_j)$ were not taken into account. Table 3.2 contains the results of the generalized Shapiro-Wilk's test for $r_{max} = 0.05$ and the other parameters unchanged.

| | **Poisson** | **Thomas1** | **Thomas2** | **Strauss1** | **Strauss2** |
|---|---|---|---|---|---|
| **J** | 0.234 | 0.634 | 0.415 | 0.176 | 0.935 |

Table 3.2: p-values from the generalized Shapiro-Wilk's test for $J$-function with parameters $m = 20$ $n = 40$, $r_{max} = 0.05$ $r_{min} = 0.01$.

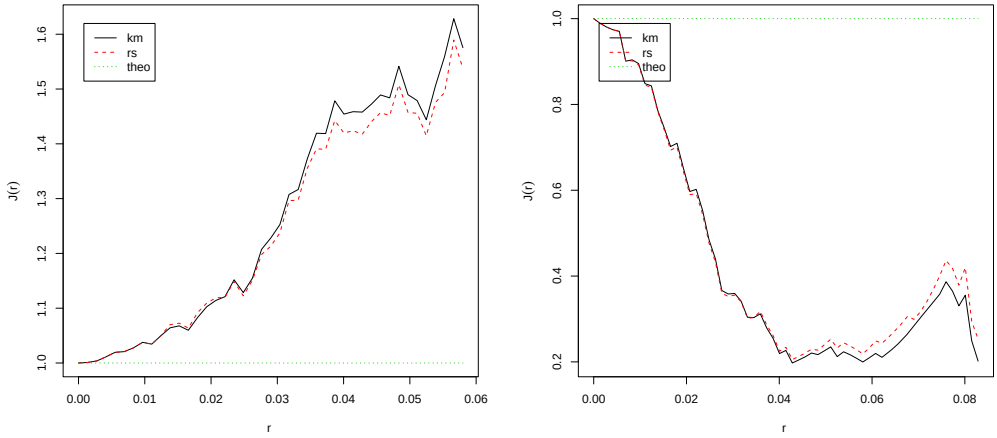


Figure 3.1: $J$-functions for Poisson (left), and Thomas (right) processes

To summarize, the outcome of the asymptotic normality test is heavily dependent on the accurate choice of boundaries of $r$. In case $r$ was chosen correctly, the hypothesis of the asymptotic normality is not to be rejected. To be absolutely sure that this result is not just a coincidence, we have conducted a batch test where we executed the test for 100 processes for each process family and each summary statistics. Tables 3.3 and 3.4 present the number of rejections in each batch.

| | **Poisson** | **Thomas1** | **Thomas2** | **Strauss1** | **Strauss2** |
|---|---|---|---|---|---|
| **a** | 8 | 3 | 38 | 14 | 14 |
| **u** | 18 | 8 | 26 | 22 | 12 |
| **n** | 8 | 12 | 18 | 12 | 14 |
| **F** | 12 | 18 | 6 | 6 | 14 |
| **G** | 14 | 14 | 14 | 6 | 18 |
| **L** | 12 | 12 | 12 | 6 | 5 |

Table 3.3: Number of rejections out of 100 executions of the generalized Shapiro-Wilk's test. Employed parameters: $m = 20$, $n = 40$, $r_{max} = 0.1$, $r_{min} = 0.01$.

Observing the results in tables 3.3 and 3.4, we can conclude that the hypothesis of asymptotic normality holds. The 38 rejections for the case of

| | **Poisson** | **Thomas1** | **Thomas2** | **Strauss1** | **Strauss2** |
|---|---|---|---|---|---|
| **J** | 2 | 4 | 0 | 3 | 2 |

Table 3.4: Number of rejections out of 100 executions of the generalized Shapiro-Wilk's test for $J$-function values. Employed parameters: $m = 20$, $n = 40$, $r_{max} = 0.05$, $r_{min} = 0.01$.

$a$ with Thomas process might look suspicious, however, as in the case of $J$, this seems to be the matter of fine-tuning the model parameters.

One last remark is in order before concluding this chapter. Computation of $K$, $G$, $F$, $L$ and $J$ functions in `R` is according to `R`'s user manual very dependent on the correct choice of the argument vector $r$. The authors of the computation algorithm even advise not to specify this parameter where possible and to leave this matter to the algorithm. In our situation this approach was of course not acceptable and thus some deviations from the correct result are possible.

One could also replicate our result or experiment with different parameter values using the function `normalityTest()` or `batchNormalityTest()` both available in the aforementioned package. Refer to the enclosed readme file or to the source code comments to obtain information on the function input and output values.

# Chapter 4

# Description of tests of point pattern models

In this chapter we will describe tests, which already are available for testing of point process models and then present the reader with a new test method. This chapter will only present the idea of the tests, while discussion about their applicability and performance along with the results of extensive simulation study is being held in the next chapter.

## 4.1 Testing the Poisson hypothesis

We will traditionally start with describing the means for testing the Poisson hypothesis alternatively called CSR (complete spatial randomness) hypothesis. Probably every point pattern investigation routine starts with the CSR assumption and despite of the fact that a multitude of tests have already been developed a vital research of this problem is still being carried on. Every CSR test makes use of one or more summary statistics. According to [4], since every test verifies only particular aspects of the CSR behaviour, it is impossible to decide which test is more reliable if considering only a single criterion. Decision on which test to use in a particular application should be made with regard to the data sampling method and the current alternative hypothesis. A general idea behind the most CSR tests is fairly simple: A summary statistics is estimated from the data and then compared to its theoretical counterpart. In case the difference between the both is to evident, the Poisson null hypothesis is rejected. Formally, if $\hat{S}(r)$ is the estimator of some summary statistics $S(R)$ and $S_P(r)$ is its Poisson counterpart

then reject the CRF hypothesis if

$$\max_{r \leq s} |\hat{S}(r) - S_P(r)| > S_\alpha^m \tag{4.1}$$

or if

$$\int_0^s (\hat{S}(r) - S_P(r))^2 \mathrm{d}r > S_\alpha^i \tag{4.2}$$

where $S_\alpha^m$ and $S_\alpha^i$ are known quantities. $r$ in the expressions 4.2 and 4.1 is usually the inter-point distance and due to the fact that in a bounded observation window $S_r$ cannot be estimated efficiently for large $r$, careful choice of $s$ is of vital importance.

The alternative hypothesis is mostly either that of clustering or regularity and it is advised to choose it prior to data collection. Of course these are not the only alternatives. In applications, a point pattern may exhibit clustering and regularity at the same time for instance if observed under different scales or through different windows. Also non-stationarity is an important example of non-CSR.

### Quadrant counts test

This test is based on the fact, that the stationary Poisson point pattern is uniformly distributed across the observation window. To conduct the test, divide the data observation window into a grid of $k$ equally sized cells of area $S$. Than according to the above, for the CSR hypothesis not to be rejected, a random number of points in each quadrant should follow the Poisson distribution of mean $\lambda S$ and the counts in disjointed quadrants should be independent. It is worth to note, that the dividing grid should be chosen, if possible, before the data mapping process is started. In the opposite case one could simply admit a grid on $W$ after the data mapping process is complete, which could however result in loss of valuable information about the interaction between points.

The quadrant counts test is convenient, as one does not need to conduct a simulation study since classical statistical methods will suffice. The disadvantage of the test is that in applications one can rarely find a point pattern arranging itself that elegantly into quadrants. Furthermore, the test depends on the number of quadrants $k$ which can be chosen as desired.

Greig-Smith test is an improved version of the simple quadrant counts. It does additionally consider counts of points in grouped neighbouring quad-

rants and so it may efficiently detect clustering at different scales. There is more to read about this method in [4] or in [3].

### The $J$-test

The $J$-test is based on the $J$ function as defined with the expression 2.3. For a Poisson process $J(r) \equiv 1$ and so recalling formulas 4.1 and 4.2 we easily obtain the test statistics:

$$\tau_{max} = \max_{r \leq s} |\hat{J}(r) - 1| \tag{4.3}$$

$$\tau_{int} = \int_0^s (J(r) - 1)^2 \mathrm{d}r \tag{4.4}$$

Critical values of $\tau$ for the $J$-test are obtained by simulation. The $J$-function has the particular advantage of measuring both the strength and range of interaction. The author of [4] warns that care should be taken while choosing the $s$ parameter, as it will strongly influence the results. The power of the test is studied in great detail in [10]

### The $L$-test

Because for the Poisson process the $L$-function has a nice and simple linear form

$$L(r) = r, \text{for } r \geq 0 \tag{4.5}$$

it is well suitable for establishing another test. Again using Formula 4.1 we obtain

$$\tau_{max} = \max_{r \leq s} |\hat{L}(r) - r|, \tag{4.6}$$

expression 4.2 can be used as well. Critical value for $\tau$ for the significance level $\alpha = 0.05$ is approximately

$$\tau_{0.05} = 1.45 \frac{\sqrt{\nu(W)}}{n} \tag{4.7}$$

where $\nu(W)$ is the area of the window of observation and $n$ is the number of observed points in that window. This value has been obtained by simulations as no precise analytical value has been yet found as stated by the author of [4].

There are many more methods, some of them being very sophisticated, for testing the CSR hypothesis in the literature (see for example [4] and references within). We allow ourselves to omit those here for the sake of getting to the ultimate matter of this thesis – the simulation based tests.

## 4.2 Simulation based tests

Testing the CRF hypothesis is a first step to make while the point pattern is being investigated. In case this hypothesis is rejected a second step has to be made. The most common approach is to investigate the clustering or regularity of the point pattern. This can be done using the simulation based tests.

While simulation based tests offer wider applicability they are not void of disadvantages. Main inconvenience is that failure to correctly estimate the parameter $\theta$ can lead to the false rejection. Another problem lies in the fact, that such test is in one way or another operating on an imaginative dataset and thus deviation from the correct result may be possible. Furthermore the quality of the simulation algorithm and the simulation software itself will of course contribute to the quality of the result as well.

#### Test with deviation measure

The deviation measure test is probably the most widely used example of the simulation based tests. It is simple to conduct and provides adequate results. Given a particular dataset the test with deviation measure can be carried out in 7 general steps:

1. Choose a summary statistics to base the test upon. Denote that summary statistics by $S = S(r)$ where $r \in \mathbb{R}^+$.

2. Choose $n \in \mathbb{N}$ and choose $n$ equidistant positive real points $r_1, \ldots, r_n$. For the dataset compute the estimates of $S$ in $r_1, \ldots, r_n$ and denote these estimates by $\hat{S}^D_{r_1}, \ldots, \hat{S}^D_{r_n}$.

3. Estimate parameters of the assumed point process model and denote the estimates by $\hat{\theta}$.

4. Produce $N$ independent simulated samples of the model with parameters $\theta = \hat{\theta}$ and for each simulated dataset calculate the estimates of

$S(r)$ in the priorly chosen points $r_1, \ldots, r_n$ denoting these estimates with $\hat{S}^j_{r_1}, \ldots, \hat{S}^j_{r_n}$ for $j \in 1, \ldots, N$.

5. Choose a deviation measure to work with. Possible choices are:

$$\Delta_{max} = \max_{1 \leq i \leq n} |\hat{S}_{r_i} - S_{\hat{\theta}, r_i}| \tag{4.8}$$

or

$$\Delta_{\Sigma} = \sum_{i=1}^{n} (\hat{S}_{r_i} - S_{\hat{\theta}, r_i})^2 \tag{4.9}$$

In both cases $S_{\hat{\theta}, r_i}$ denotes the theoretical value of the summary statistic at $r_i$. Other deviation measures are of course possible.

6. Calculate deviation measure $\Delta^D$ using the $\hat{S}^D_{r_i}$ and deviation measures $\{\Delta_j\}$ using $\hat{S}^j_{r_i}$ for $j \in 1, \ldots, N$ and $i \in 1, \ldots, n$

7. If $\Delta^D$ is greater or equal to the $m$-th maximum of the set $\{\Delta^D, \Delta_1, \cdots, \Delta_N\}$ the hypothesis should be rejected. The p-value of the test is than equal to $\frac{m}{N+1}$.

In applications calculating the value of $S_{\hat{\theta}, r_i}$ can pose a considerable obstacle and so it is mostly replaced with a suitable estimate. In our simulation study we have used the quantity $\frac{1}{n} \sum_j \hat{S}^j_{r_i}$ as the estimated replacement for $S_{\hat{\theta}, r_i}$.

Caution is in order before using the described test. What we are actually testing here is not just a model of the point process but rather the model of the point process *with estimated parameters* and therefore if the whichever chosen estimation procedure fails to return correct model parameters, the reliability of the test will decrease dramatically.

## 4.3 Envelopes test

Envelopes test has been known for a while, but there was no method available to calculate its significance level. Due to this matter this procedure was not considered as primary testing method, but rather as diagnostic plot to find the right direction. Only recently the author of [7] offered a way to compute the p-value of the test and demonstrated the efficiency of this test

on random sets. We will apply his approach for testing point processes. In contrast to the deviation measure test, which tends to ignore a considerable amount of information contained in the point point pattern characteristics, the envelopes test tries to make use of this information and so it is dedicated to produce results of greater precision.

Conducting the envelopes test takes 8 steps.

1. Choose a summary statistic to base the test upon. Denote that summary statistics by $S = S(r)$ where $r \in \mathbb{R}^+$.

2. Choose $n \in \mathbb{N}$ and choose $n$ equidistant positive real points $r_1, \ldots, r_n$. For the dataset compute the estimates of $S$ in $r_1, \ldots, r_n$ and denote these estimates by $\hat{S}^D_{r_1}, \ldots, \hat{S}^D_{r_n}$.

3. Estimate the parameters of the assumed point process model and denote the estimates by $\hat{\theta}$.

4. Produce $N$ independent simulated samples of the model with parameters $\theta = \hat{\theta}$ and for each simulated dataset calculate the estimates of $S(r)$ in the priorly chosen points $r_1, \ldots, r_n$ denoting these estimates with $\hat{S}^j_{r_1}, \ldots, \hat{S}^j_{r_n}$ for $j \in 1, \ldots, N$.

5. Recalling the property of asymptotic normality, as discussed in the previous chapter, approximate the $S_{r_1}, \cdots, S_{r_n}$ with a random vector $X$ with multi normal distribution with mean vector $\mu = (\mu_1, \ldots, \mu_2)$ and the covariance matrix $\Sigma = (\Sigma_{ij})^n_{i,j=1}$.

6. Consider $s > 0$ – a real parameter and construct the envelope boundaries as functions of $s$:

$$LE(s) = (\mu_1 - s\sqrt{\Sigma_{11}}, \ldots, \mu_n - s\sqrt{\Sigma_{nn}})^T, \tag{4.10}$$

$$UE(s) = (\mu_1 + s\sqrt{\Sigma_{11}}, \ldots, \mu_n + s\sqrt{\Sigma_{nn}})^T. \tag{4.11}$$

7. Choose an integer number $K$ denoting the maximum acceptable number of components of $\hat{S}^D_{r_1}, \ldots, \hat{S}^D_{r_n}$ falling outside the envelope constituted by $LE$ and $UE$. Additionally let

$$c_i(s) = \mathbb{I}(\hat{S}^D_{r_i} \notin (LE(s)_i, UE(s)_i)), i = 1, \ldots, n. \tag{4.12}$$

Then the widest envelope will be that with

$$s_{sup} = \sup_s\{c(s) : c(s) = \sum_i \mathbf{1}_{c_i(s)} \leq K\} \tag{4.13}$$

8. The p-value of the test is then the probability that more than $K$ components of the random vector $X$ fall outside the widest envelope $(LE(s_{sup}), UE(s_{sup}))$.

Vector $X$ is usually high dimensional and thus computing the probability from step 8 is often possible only by using the Monte Carlo methods. This also seems to be the only possible approach if implementing this test in a software application. In more detail: for a sufficiently large $m$ a random vector with multi normal distribution is generated as described in step 5. If more than $K$ components of the currently generated vector fall outside the envelope then it is considered as "positive". The ratio between the count of such positive vectors and the total number of drawn vectors is then the Monte Carlo estimate of the p-value of the test.

The implementation of this algorithm is accommodated within the function `envelopesTest()` which can be found in the aforementioned package. See comments above this function in the source code to find out about the input and output values. The Monte Carlo error is approximately $\frac{1}{\sqrt{n}}$ with $n$ being the number of simulated vectors from step 5. Therefore to obtain correct results, one has to set this number to no less than 50 000. In our simulation study this number defaulted to 100 000. It is worth noting that at the time of writing the test with 100 000 normal vector generations took 13-20 minutes to complete on one 2.5 Ghz processor core running in the 32 bit instructions mode. Parallel processing capability as well as the 64 bit processor environment would of course speed up the testing process considerably, but unfortunately the `R` software neither has native multiprocessor support nor does it run in the 64 bit operating mode. There are however ways to achieve some level of pseudo parallel computing in `R` and so every person concerned is welcome to implement such functionality.

Now that there is a method to calculate the p-value of the envelopes test we are able to compare the outcomes from different tests in order to see which test performs better. We shall get to that matter in the following chapter.

# Chapter 5

# Test results and results comparison

In this chapter we will present the results obtained from J-test, quadrant counts test, simulation test with deviation measure with maximum and integral deviation measure as well as simulation test with envelopes. We have performed in total over 500 000 tests. The total processing time on one core would be approximately 220 days therefore the tests results are available in form of an MySql database dump in the aforementioned package. This dump can be eventually imported in the MySql database (or virtually any other SQL based database) for further examination. The first section of this chapter contains the summary from the results organized in tables. The next section discusses the meaning of those results and concludes this thesis.

## 5.1 Test results

For our testing purposes we have chosen 5 different processes; those, along with their parameters, are presented in the table 5.1. The values in the column named alias will be used when referring to a particular process across this chapter.

For each process from table 5.1 we have generated 1000 simulations and conducted the tests using those simulations as a dataset.

For the simulation based tests 14 different summary statistics were used. Those were: $a$, $u$, $n$, $G$, $J$, $L$ separately and then combinations of $(L, G, J, n)$, $(L, G, J, a, u, n)$, $(L, J, n)$, $(J, n)$, $(G, J)$, $(a, u, n)$, $(L, G)$ and finally $(G, J, n)$.

| **Process type** | **Parameters** | **Alias** |
|---|---|---|
| Poisson | $\lambda = 200$ | Po |
| Strauss | $\beta = 200, r = 0.16, \gamma = 0.8$ | St1 |
| Strauss | $\beta = 200, r = 0.04, \gamma = 0.3$ | St2 |
| Thomas | $\kappa = 100, \sigma = 0.02, \mu = 2$ | Th1 |
| Thomas | $\kappa = 20, \sigma = 0.08, \mu = 10$ | Th2 |

Table 5.1: Processes, which were subject of testing with the aforementioned methods.

Tables 5.33, 5.3, 5.4, 5.5 present the significance level (that is a probability of rejection of a true null hypothesis) and average p-value of the different tests where the test subject was the Poisson process.

| | **quadrant counts** | $J$**-test** |
|---|---|---|
| **Avg. p-value** | 0.4979 | 0.3637 |
| **Significance** | 0.039 | 0.0500 |

Table 5.2: Average p-value and significance level for quadrant counts test and $J$-test.

For testing hypotheses other than Poisson process, only simulation tests can be conducted. A set of tables will follow presenting the results obtained from different simulation tests. The tables contain significance level and average p-value of the particular test for a particular process and are organized as follows:

- Deviation measure test with deviation measure 4.8: tables 5.6, 5.7, 5.8, 5.9.
- Deviation measure test with deviation measure 4.9: tables 5.10, 5.11, 5.12, 5.13.
- Envelopes test: tables 5.14, 5.15, 5.16, 5.17.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0 | 0.6677 |
| **u** | 0.0010 | 0.6905 |
| **n** | 0.0020 | 0.6425 |
| **G** | 0.0150 | 0.5696 |
| **J** | 0.0480 | 0.4870 |
| **L** | 0.0340 | 0.5145 |
| **L G J a u n** | 0.0170 | 0.4591 |
| **L G J n** | 0.0150 | 0.4400 |
| **L J n** | 0.0130 | 0.4056 |
| **J n** | 0.0710 | 0.6749 |
| **G J** | 0.0080 | 0.3599 |
| **a u n** | 0.0260 | 0.6234 |
| **L G** | 0.0290 | 0.5133 |
| **G J n** | 0.0090 | 0.4351 |

Table 5.3: Significance and average p-value of the deviation measure test for process `Po`. Deviation measure is defined by the expression 4.9

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0.0040 | 0.6346 |
| **u** | 0.0010 | 0.6571 |
| **n** | 0.0040 | 0.6385 |
| **G** | 0.0220 | 0.5589 |
| **J** | 0.0500 | 0.3637 |
| **L** | 0.0260 | 0.5121 |
| **L G J a u n** | 0.0170 | 0.4591 |
| **L G J n** | 0 | 0.5239 |
| **L J n** | 0 | 0.4674 |
| **J n** | 0.0820 | 0.5526 |
| **G J** | 0 | 0.4199 |
| **a u n** | 0.0350 | 0.5554 |
| **L G** | 0.0140 | 0.5294 |
| **G J n** | 0 | 0.4714 |

Table 5.4: Significance and average p-value of the deviation measure test for process `Po`. Deviation measure is defined by the expression 4.8

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0.0162 | 0.6360 |
| **u** | 0.0404 | 0.7387 |
| **n** | 0.0323 | 0.6951 |
| **G** | 0.0667 | 0.6433 |
| **J** | 0.1567 | 0.4220 |
| **L** | 0.0533 | 0.4889 |
| **L G J a u n** | 0.4433 | 0.1844 |
| **L G J n** | 0.3367 | 0.2519 |
| **L J n** | 0.3167 | 0.2571 |
| **J n** | 0.2633 | 0.2999 |
| **G J** | 0.2133 | 0.3605 |
| **a u n** | 0.0467 | 0.6702 |
| **L G** | 0.0933 | 0.5379 |
| **G J n** | 0.2933 | 0.2828 |

Table 5.5: Significance and average p-value of the envelopes test for process `Po`.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0.0240 | 0.5698 |
| **u** | 0.0240 | 0.5484 |
| **n** | 0.0360 | 0.5177 |
| **G** | 0.0380 | 0.5223 |
| **J** | 0.0430 | 0.4070 |
| **L** | 0.0370 | 0.5646 |
| **L G J a u n** | 0 | 0.5525 |
| **L G J n** | 0 | 0.5100 |
| **L J n** | 0 | 0.4745 |
| **J n** | 0.0761 | 0.5394 |
| **G J** | 0.0010 | 0.4110 |
| **a u n** | 0.0350 | 0.5279 |
| **L G** | 0.0430 | 0.5319 |
| **G J n** | 0.0020 | 0.4589 |

Table 5.6: Significance and average p-value of the deviation measure test with deviation measure 4.8 for process `St1`.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0.0010 | 0.6307 |
| **u** | 0.0010 | 0.6086 |
| **n** | 0.0080 | 0.5450 |
| **G** | 0.0080 | 0.6359 |
| **J** | 0.0361 | 0.3961 |
| **L** | 0.0210 | 0.5923 |
| **L G J a u n** | 0 | 0.5939 |
| **L G J n** | 0 | 0.5439 |
| **L J n** | 0 | 0.4968 |
| **J n** | 0.0872 | 0.5465 |
| **G J** | 0 | 0.4449 |
| **a u n** | 0.0341 | 0.5654 |
| **L G** | 0.0210 | 0.5655 |
| **G J n** | 0 | 0.4903 |

Table 5.7: Significance and average p-value of the deviation measure test with deviation measure 4.8 for process `St2`.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0.0160 | 0.6423 |
| **u** | 0.0150 | 0.6437 |
| **n** | 0.0180 | 0.6016 |
| **G** | 0.0580 | 0.5160 |
| **J** | 0.0520 | 0.3540 |
| **L** | 0.0550 | 0.6727 |
| **L G J a u n** | 0.0100 | 0.5390 |
| **L G J n** | 0.0090 | 0.5171 |
| **L J n** | 0.0090 | 0.4993 |
| **J n** | 0.1010 | 0.5522 |
| **G J** | 0.0030 | 0.4185 |
| **a u n** | 0.0630 | 0.5370 |
| **L G** | 0.0750 | 0.5842 |
| **G J n** | 0.0030 | 0.4506 |

Table 5.8: Significance and average p-value of the deviation measure test with deviation measure 4.8 for process `Th1`.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0.0060 | 0.7146 |
| **u** | 0.0030 | 0.7262 |
| **n** | 0.0030 | 0.6954 |
| **G** | 0.0020 | 0.6506 |
| **J** | 0.0420 | 0.3624 |
| **L** | 0.0250 | 0.7182 |
| **L G J a u n** | 0.0070 | 0.5128 |
| **L G J n** | 0.0060 | 0.5073 |
| **L J n** | 0.0040 | 0.5071 |
| **J n** | 0.0860 | 0.5516 |
| **a u n** | 0.0560 | 0.5892 |
| **L G** | 0.0600 | 0.6017 |
| **G J n** | 0 | 0.5300 |

Table 5.9: Significance and average p-value of the deviation measure test with deviation measure 4.8 for process `Th2`.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0.0110 | 0.6492 |
| **u** | 0.0180 | 0.5839 |
| **n** | 0.0170 | 0.5415 |
| **G** | 0.0390 | 0.5363 |
| **J** | 0.0400 | 0.5147 |
| **L** | 0.0240 | 0.5613 |
| **L G J a u n** | 0.0080 | 0.4699 |
| **L G J n** | 0.0070 | 0.4532 |
| **L J n** | 0.0060 | 0.4244 |
| **J n** | 0.0661 | 0.6626 |
| **G J** | 0.0120 | 0.3512 |
| **a u n** | 0.0350 | 0.5938 |
| **L G** | 0.0250 | 0.5207 |
| **G J n** | 0.0240 | 0.4199 |

Table 5.10: Significance and average p-value of the deviation measure test with deviation measure 4.9 for process `St1`.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0 | 0.6068 |
| **u** | 0 | 0.5961 |
| **n** | 0.0020 | 0.5852 |
| **G** | 0.0010 | 0.6599 |
| **J** | 0.0401 | 0.5171 |
| **L** | 0.0100 | 0.6021 |
| **L G J a u n** | 0.0120 | 0.4603 |
| **L G J n** | 0.0150 | 0.4538 |
| **L J n** | 0.0060 | 0.4276 |
| **J n** | 0.0872 | 0.6675 |
| **G J** | 0.0060 | 0.3966 |
| **a u n** | 0.0341 | 0.5921 |
| **L G** | 0.0090 | 0.5502 |
| **G J n** | 0.0070 | 0.4672 |

Table 5.11: Significance and average p-value of the deviation measure test with deviation measure 4.9 for process `St2`.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0.0050 | 0.6615 |
| **u** | 0.0050 | 0.6799 |
| **n** | 0.0110 | 0.6151 |
| **G** | 0.0480 | 0.5140 |
| **J** | 0.0580 | 0.4761 |
| **L** | 0.0510 | 0.6962 |
| **L G J a u n** | 0.0450 | 0.5296 |
| **L G J n** | 0.0370 | 0.5227 |
| **L J n** | 0.0360 | 0.5025 |
| **J n** | 0.1050 | 0.6673 |
| **G J** | 0.0110 | 0.3616 |
| **a u n** | 0.0580 | 0.6047 |
| **L G** | 0.0700 | 0.6345 |
| **G J n** | 0.0100 | 0.4282 |

Table 5.12: Significance and average p-value of the deviation measure test with deviation measure 4.9 for process `Th1`.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0.0030 | 0.7101 |
| **u** | 0.0030 | 0.7524 |
| **n** | 0.0030 | 0.7110 |
| **G** | 0.0010 | 0.6554 |
| **J** | 0.0420 | 0.4837 |
| **L** | 0.0390 | 0.7069 |
| **L G J a u n** | 0.0360 | 0.5417 |
| **L G J n** | 0.0210 | 0.5334 |
| **L J n** | 0.0170 | 0.5216 |
| **J n** | 0.0810 | 0.6639 |
| **G J** | 0 | 0.4480 |
| **a u n** | 0.0510 | 0.6360 |
| **L G** | 0.0590 | 0.6326 |
| **G J n** | 0.0030 | 0.5145 |

Table 5.13: Significance and average p-value of the deviation measure test with deviation measure 4.9 for process `Th2`.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0.0567 | 0.5600 |
| **u** | 0.0383 | 0.6032 |
| **n** | 0.0267 | 0.5798 |
| **G** | 0.0267 | 0.5990 |
| **J** | 0.2650 | 0.3639 |
| **L** | 0.0367 | 0.5671 |
| **L G J a u n** | 0.4300 | 0.2135 |
| **L G J n** | 0.3880 | 0.2460 |
| **L J n** | 0.3660 | 0.2576 |
| **J n** | 0.3440 | 0.2626 |
| **G J** | 0.3100 | 0.3166 |
| **a u n** | 0.0440 | 0.5341 |
| **L G** | 0.0300 | 0.5834 |
| **G J n** | 0.3620 | 0.2496 |

Table 5.14: Significance and average p-value of the envelopes test for process `St1`.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0 | 0.6445 |
| **u** | 0.0123 | 0.7145 |
| **n** | 0.0088 | 0.6913 |
| **G** | 0.0630 | 0.6993 |
| **J** | 0.1979 | 0.3682 |
| **L** | 0.0140 | 0.5703 |
| **L G J a u n** | 0.4588 | 0.1623 |
| **L G J n** | 0.3345 | 0.2287 |
| **L J n** | 0.3047 | 0.2423 |
| **J n** | 0.2925 | 0.2448 |
| **G J** | 0.2434 | 0.3515 |
| **a u n** | 0.0193 | 0.6999 |
| **L G** | 0.0630 | 0.6214 |
| **G J n** | 0.3275 | 0.2418 |

Table 5.15: Significance and average p-value of the envelopes test for process `St2`.

| Statistics | Significance | Avg. p-value |
|---|---|---|
| **a** | 0.0702 | 0.6518 |
| **u** | 0.0807 | 0.7360 |
| **n** | 0.0632 | 0.6544 |
| **G** | 0.1158 | 0.5473 |
| **J** | 0.3228 | 0.3239 |
| **L** | 0.0667 | 0.7917 |
| **L G J a u n** | 0.4919 | 0.2135 |
| **L G J n** | 0.4432 | 0.2385 |
| **L J n** | 0.4162 | 0.2608 |
| **J n** | 0.4270 | 0.2348 |
| **G J** | 0.3838 | 0.2947 |
| **a u n** | 0.1189 | 0.6713 |
| **L G** | 0.1405 | 0.6268 |
| **G J n** | 0.4432 | 0.2192 |

Table 5.16: Significance and average p-value of the envelopes test for process `Th1`.

| **Statistics** | **Significance** | **Avg. p-value** |
|---|---|---|
| **a** | 0.0300 | 0.7363 |
| **u** | 0.0550 | 0.8157 |
| **n** | 0.0550 | 0.7555 |
| **G** | 0.1250 | 0.6822 |
| **J** | 0.3700 | 0.2990 |
| **L** | 0.0950 | 0.7043 |
| **L G J a u n** | 0.4390 | 0.3189 |
| **L G J n** | 0.4390 | 0.3129 |
| **L J n** | 0.4228 | 0.3337 |
| **J n** | 0.4146 | 0.3208 |
| **G J** | 0.3984 | 0.3165 |
| **a u n** | 0.0650 | 0.7322 |
| **L G** | 0.1707 | 0.6270 |
| **G J n** | 0.4146 | 0.3072 |

Table 5.17: Significance and average p-value of the envelopes test for process `Th2`.

Another important property of any statistical test is its power, i.e. the probability that the test will effectively reject the null hypothesis, whereas the null hypothesis is indeed to be rejected. On the next pages we present the estimated power of the test and its average p-value. The tables are organized as follows:

- Envelopes test. Tables: 5.18, 5.19, 5.20, 5.21, 5.22.
- Deviation measure test with deviation measure 4.8. Tables: 5.23, 5.24, 5.25, 5.26, 5.27,.
- Deviation measure test with deviation measure 4.9. Tables: 5.28, 5.29, 5.30, 5.31, 5.32.
- $J$-test and quadrant counts test Table 5.33.

| **Statistics** | **Est. power** | **Avg. p-value** |
|---|---|---|
| **a** | 0.3314 | 0.4015 |
| **u** | 0.3643 | 0.3878 |
| **n** | 0.3186 | 0.4065 |
| **G** | 0.2871 | 0.4013 |
| **J** | 0.2314 | 0.3699 |
| **L** | 0.0557 | 0.5830 |
| **L G J a u n** | 0.6671 | 0.1049 |
| **L G J n** | 0.5786 | 0.1376 |
| **L J n** | 0.5386 | 0.1470 |
| **J n** | 0.5171 | 0.1873 |
| **G J** | 0.4300 | 0.2168 |
| **a u n** | 0.3714 | 0.3893 |
| **L G** | 0.2714 | 0.3690 |
| **G J n** | 0.5286 | 0.1597 |

Table 5.18: Estimated test power and average p-value of the envelopes test for process `Po` as dataset and `St1` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 1.0000 | 0.0000 |
| **u** | 1.0000 | 0.0000 |
| **n** | 1.0000 | 0.0000 |
| **G** | 1.0000 | 0.0000 |
| **J** | 1.0000 | 0.0001 |
| **L** | 1.0000 | 0.0000 |
| **L G J a u n** | 1.0000 | 0.0000 |
| **L G J n** | 1.0000 | 0.0000 |
| **L J n** | 1.0000 | 0.0000 |
| **J n** | 1.0000 | 0.0000 |
| **G J** | 1.0000 | 0.0000 |
| **a u n** | 1.0000 | 0.0000 |
| **L G** | 1.0000 | 0.0000 |
| **G J n** | 1.0000 | 0.0000 |

Table 5.19: Estimated test power and average p-value of the envelopes test for process `Th1` as dataset and `Po` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 0.8775 | 0.0338 |
| **u** | 0.9700 | 0.0062 |
| **n** | 0.9600 | 0.0105 |
| **G** | 0.6850 | 0.0798 |
| **J** | 0.8275 | 0.0369 |
| **L** | 1.0000 | 0.0000 |
| **L G J a u n** | 1.0000 | 0.0000 |
| **L G J n** | 1.0000 | 0.0000 |
| **L J n** | 1.0000 | 0.0000 |
| **J n** | 0.9725 | 0.0050 |
| **G J** | 0.8800 | 0.0254 |
| **a u n** | 0.9850 | 0.0044 |
| **L G** | 1.0000 | 0.0001 |
| **G J n** | 0.9825 | 0.0045 |

Table 5.20: Estimated test power and average p-value of the envelopes test for process `Th2` as dataset and `Po` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 0.1103 | 0.3812 |
| **u** | 0.0795 | 0.4651 |
| **n** | 0.1744 | 0.3858 |
| **G** | 0.0359 | 0.6144 |
| **J** | 0.3923 | 0.2964 |
| **L** | 0.3103 | 0.1901 |
| **L G J a u n** | 0.5256 | 0.1536 |
| **L G J n** | 0.5128 | 0.1621 |
| **L J n** | 0.5487 | 0.1408 |
| **J n** | 0.4795 | 0.1927 |
| **G J** | 0.3923 | 0.2939 |
| **a u n** | 0.1385 | 0.3669 |
| **L G** | 0.1974 | 0.2796 |
| **G J n** | 0.4538 | 0.2082 |

Table 5.21: Estimated test power and average p-value of the envelopes test for process `St1` as dataset and `Po` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 0.6725 | 0.0744 |
| **u** | 0.5600 | 0.1176 |
| **n** | 0.4525 | 0.1502 |
| **G** | 0.8225 | 0.0369 |
| **J** | 0.9450 | 0.0094 |
| **L** | 0.9650 | 0.0081 |
| **L G J a u n** | 0.9900 | 0.0036 |
| **L G J n** | 0.9825 | 0.0041 |
| **L J n** | 0.9800 | 0.0038 |
| **J n** | 0.9725 | 0.0077 |
| **G J** | 0.9725 | 0.0077 |
| **a u n** | 0.7675 | 0.0508 |
| **L G** | 0.9600 | 0.0095 |
| **G J n** | 0.9725 | 0.0077 |

Table 5.22: Estimated test power and average p-value of the envelopes test for process `St2` as dataset and `Po` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 1.0000 | 0.0100 |
| **u** | 1.0000 | 0.0100 |
| **n** | 1.0000 | 0.0100 |
| **G** | 1.0000 | 0.0100 |
| **J** | 0.0050 | 0.4078 |
| **L** | 0.9960 | 0.0117 |
| **L G J n** | 0.0030 | 0.1027 |
| **L J n** | 0.0090 | 0.0907 |
| **J n** | 0.0430 | 0.6144 |
| **G J** | 0.4234 | 0.0481 |
| **a u n** | 1.0000 | 0.0142 |
| **L G** | 0.9469 | 0.0251 |
| **G J n** | 0.3554 | 0.0504 |

Table 5.23: Estimated test power and average p-value of the deviation measure test with deviation measure 4.8 test for process `Po` as dataset and `St1` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 0.0840 | 0.3784 |
| **u** | 0.0880 | 0.3708 |
| **n** | 0.0750 | 0.3935 |
| **G** | 0.0340 | 0.5078 |
| **J** | 0.0710 | 0.3787 |
| **L** | 0.1180 | 0.2312 |
| **L G J a u n** | 0.0010 | 0.4830 |
| **J n** | 0.1100 | 0.5040 |
| **a u n** | 0.0770 | 0.4143 |
| **L G** | 0.0540 | 0.3232 |

Table 5.24: Estimated test power and average p-value of the deviation measure test with deviation measure 4.8 test for process `St1` as dataset and `Po` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 0.4770 | 0.0812 |
| **u** | 0.5180 | 0.0721 |
| **n** | 0.7900 | 0.0385 |
| **G** | 0.9560 | 0.0177 |
| **J** | 0.0560 | 0.3819 |
| **L** | 0.9570 | 0.0169 |
| **L G J a u n** | 0.0080 | 0.1375 |
| **L G J n** | 0.0070 | 0.1139 |
| **L J n** | 0.0150 | 0.0991 |
| **J n** | 0.1010 | 0.5175 |
| **G J** | 0.0410 | 0.0847 |
| **a u n** | 0.2220 | 0.1503 |
| **L G** | 0.8310 | 0.0362 |
| **G J n** | 0.0500 | 0.0869 |

Table 5.25: Estimated test power and average p-value of the deviation measure test with deviation measure 4.8 test for process `St2` as dataset and `Po` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 1.0000 | 0.0100 |
| **u** | 1.0000 | 0.0100 |
| **n** | 1.0000 | 0.0101 |
| **G** | 1.0000 | 0.0100 |
| **J** | 0.0210 | 0.4328 |
| **L** | 1.0000 | 0.0100 |
| **L G J a u n** | 0.5620 | 0.0453 |
| **L G J n** | 0.7080 | 0.0395 |
| **L J n** | 0.7160 | 0.0391 |
| **J n** | 0.0600 | 0.5951 |
| **G J** | 0.3740 | 0.0507 |
| **a u n** | 1.0000 | 0.0135 |
| **L G** | 1.0000 | 0.0108 |
| **G J n** | 0.3340 | 0.0527 |

Table 5.26: Estimated test power and average p-value of the deviation measure test with deviation measure 4.8 test for process `Th1` as dataset and `Po` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 0.8300 | 0.0392 |
| **u** | 0.8390 | 0.0356 |
| **n** | 0.7850 | 0.0477 |
| **G** | 0.5620 | 0.1035 |
| **J** | 0.0130 | 0.4990 |
| **L** | 0.9980 | 0.0102 |
| **L G J a u n** | 0.7680 | 0.0342 |
| **L G J n** | 0.7620 | 0.0344 |
| **L J n** | 0.7550 | 0.0343 |
| **J n** | 0.0540 | 0.6153 |
| **G J** | 0.0100 | 0.1678 |
| **a u n** | 0.7230 | 0.0699 |
| **L G** | 0.9970 | 0.0104 |
| **G J n** | 0.0090 | 0.2027 |

Table 5.27: Estimated test power and average p-value of the deviation measure test with deviation measure 4.8 test for process `Th2` as dataset and `Po` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 1.0000 | 0.0100 |
| **u** | 1.0000 | 0.0100 |
| **n** | 1.0000 | 0.0100 |
| **G** | 1.0000 | 0.0100 |
| **J** | 0.0100 | 0.4591 |
| **L** | 0.9980 | 0.0112 |
| **L G J a u n** | 0.0741 | 0.0775 |
| **L G J n** | 0.3934 | 0.0521 |
| **L J n** | 0.6667 | 0.0411 |
| **J n** | 0.0380 | 0.7039 |
| **G J** | 1.0000 | 0.0184 |
| **a u n** | 1.0000 | 0.0111 |
| **L G** | 0.9760 | 0.0223 |
| **G J n** | 1.0000 | 0.0185 |

Table 5.28: Estimated test power and average p-value of the deviation measure test with deviation measure 4.9 test for process `Po` as dataset and `St1` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 0.0490 | 0.4219 |
| **u** | 0.0550 | 0.3954 |
| **n** | 0.0730 | 0.3733 |
| **G** | 0.0200 | 0.5224 |
| **J** | 0.0700 | 0.4919 |
| **L** | 0.1830 | 0.1931 |
| **L G J a u n** | 0.0280 | 0.3431 |
| **L G J n** | 0.0320 | 0.2993 |
| **L J n** | 0.0330 | 0.2580 |
| **J n** | 0.0970 | 0.6369 |
| **G J** | 0.0070 | 0.3400 |
| **a u n** | 0.0670 | 0.4720 |
| **L G** | 0.1160 | 0.2617 |
| **G J n** | 0.0120 | 0.4103 |

Table 5.29: Estimated test power and average p-value of the deviation measure test with deviation measure 4.9 test for process `St1` as dataset and `Po` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 0.1110 | 0.1710 |
| **u** | 0.2590 | 0.1155 |
| **n** | 0.6130 | 0.0603 |
| **G** | 0.9060 | 0.0239 |
| **J** | 0.0560 | 0.4623 |
| **L** | 0.7710 | 0.0377 |
| **L G J a u n** | 0.1180 | 0.1661 |
| **L G J n** | 0.1410 | 0.1288 |
| **L J n** | 0.2420 | 0.0970 |
| **J n** | 0.0920 | 0.6426 |
| **G J** | 0.4470 | 0.0600 |
| **a u n** | 0.1020 | 0.2993 |
| **L G** | 0.5370 | 0.0744 |
| **G J n** | 0.4670 | 0.0668 |

Table 5.30: Estimated test power and average p-value of the deviation measure test with deviation measure 4.9 test for process `St2` as dataset and `Po` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 1.0000 | 0.0103 |
| **u** | 1.0000 | 0.0101 |
| **n** | 1.0000 | 0.0100 |
| **G** | 1.0000 | 0.0100 |
| **J** | 0.0270 | 0.2744 |
| **L** | 1.0000 | 0.0100 |
| **L G J a u n** | 0.9870 | 0.0142 |
| **L G J n** | 0.9930 | 0.0142 |
| **L J n** | 0.9990 | 0.0139 |
| **J n** | 0.0470 | 0.6435 |
| **G J** | 0.9900 | 0.0225 |
| **a u n** | 0.9890 | 0.0205 |
| **L G** | 1.0000 | 0.0104 |
| **G J n** | 0.9860 | 0.0227 |

Table 5.31: Estimated test power and average p-value of the deviation measure test with deviation measure 4.9 test for process `Th1` as dataset and `Po` being the null hypothesis.

| Statistics | Est. power | Avg. p-value |
|---|---|---|
| **a** | 0.7600 | 0.0490 |
| **u** | 0.8370 | 0.0341 |
| **n** | 0.8800 | 0.0285 |
| **G** | 0.5860 | 0.0955 |
| **J** | 0.0120 | 0.3014 |
| **L** | 1.0000 | 0.0100 |
| **L G J a u n** | 0.9940 | 0.0124 |
| **L G J n** | 0.9950 | 0.0123 |
| **L J n** | 0.9970 | 0.0124 |
| **J n** | 0.0480 | 0.6585 |
| **G J** | 0.2290 | 0.1214 |
| **a u n** | 0.5900 | 0.1004 |
| **L G** | 1.0000 | 0.0100 |
| **G J n** | 0.1810 | 0.1587 |

Table 5.32: Estimated test power and average p-value of the deviation measure test with deviation measure 4.9 test for process `Th2` as dataset and `Po` being the null hypothesis.

| | quadrant counts | $J$-test |
|---|---|---|
| **Avg. p-value** | 0.3958 | 0.3966 |
| **Power** | 0.487 | 0.0354 |

Table 5.33: Average p-value and significance level for quadrant counts test and $J$-test.

## 5.2 Test comparison

Now that the results have been presented we are in position to make a comparative analysis of the aforementioned tests. In order to do so we have calculated the average significance level and average power for all tests. We are certain that after the excessive amount of tables presented in the previous section the reader would not appreciate any more of those, so we will try and present the most interesting observations in a prosaic form. In the event the reader is curious about comparing significance level or test power for each and every statistics he is left with no choice but to bear with one more table (Table 5.34) at the end of this section.

As for the average significance level, the results show great diversity. The quadrant count test and the $J$ test yielded average significance of 0.5 and 0.36 respectively. For single statistics the average significance level for both deviation measure tests ranges between 0.001 ($(G, J)$ combination) to 0.08 ($(J, n)$ combination) with the majority being smaller than 0.05. In contrast to that the significance level for envelopes test ranges from 0.03 ($n$-function) to a rather surprising 0.45 for $(L, G, J, a, u, n)$ combination. An interesting fact is that for all combinations not involving the $J$ function the significance level is less than 0.07 and for $a, u, n, L$ and $(a, u, n)$ the significance level is lower than 0.05. Such unexpected result is in our opinion highly influenced by the parameters choice. The envelopes test, as apparent from its construction, considers more information provided by the summary statistics than a test with deviation measure but the usable $J$-function values were calculated for a smaller range of the argument $r$ than the testing range. Therefore as outlined in section 3.2 the asymptotic normality requirement is not satisfied and naturally the results are not accurate.

The average test power distribution across different statistics seems to be more homogeneous. For both deviation measure tests the average test power is between 0.035 ($J$- function) to 0.56 for the $L$ function. The tests conducted

with a single statistics have greater power close to 0.41-0.56 than those with combinations. The power of envelopes test was between 0.53 ($G$-function) to 0.81 for the combination of $(L, G, J, a, u, n)$. Such a high value of the test power emerged partially due to 100% rejection of a Poisson hypothesis on Thomas process as apparent from the Table 5.19. But even a quick look at the results shows that the envelopes test is of greater power than the deviation measure test. The quadrant count test and the $J$-test have power of 0.5 and 0.03 respectively.

Morphological functions as summary statistics exhibited the most stable results in all three tests. Average power $a, u$ and $n$ functions is 0.41 to 0.56 while the significance level of the tests is 0.001-0.03. The envelopes test using morphological functions showed higher power while keeping the significance level at the acceptable 0.03. Therefore morphological functions seem to be the best candidates for using in all three tests and especially the envelopes test.

Grouping multiple functions into one summary statistics to our honest surprise did not produce better results. All grouped summary statistics in all three tests tended to raise the significance level and to lower the power.

The most unreliable summary statistics was $(L, G, J, a, u, n)$. While the deviation measure tests produced almost none rejections of the null hypothesis, the envelopes test exhibited exactly the opposite behaviour and favoured the alternative hypothesis most of the time. This diversity can perhaps be contracted to some extent by choosing other parameters set for the test. It is however our belief that even after parameters adjustment the produced results will not be precise.

| Statistics | ES | SS | MS | EP | SP | MP |
|---|---|---|---|---|---|---|
| **a** | 0.032 | 0.004 | 0.010 | 0.566 | 0.426 | 0.487 |
| **u** | 0.039 | 0.005 | 0.009 | 0.567 | 0.410 | 0.448 |
| **n** | 0.031 | 0.007 | 0.014 | 0.548 | 0.466 | 0.483 |
| **G** | 0.066 | 0.021 | 0.026 | 0.532 | 0.517 | 0.522 |
| **J** | 0.248 | 0.046 | 0.045 | 0.622 | 0.036 | 0.035 |
| **L** | 0.043 | 0.032 | 0.033 | 0.588 | 0.548 | 0.566 |
| **L G J a u n** | 0.450 | 0.024 | 0.003 | 0.816 | 0.245 | 0.136 |
| **L G J n** | 0.370 | 0.019 | 0.003 | 0.785 | 0.292 | 0.149 |
| **L J n** | 0.346 | 0.016 | 0.003 | 0.779 | 0.344 | 0.152 |
| **J n** | 0.326 | 0.082 | 0.086 | 0.754 | 0.064 | 0.076 |
| **G J** | 0.285 | 0.007 | 0.001 | 0.697 | 0.378 | 0.096 |
| **a u n** | 0.046 | 0.041 | 0.045 | 0.618 | 0.399 | 0.442 |
| **L G** | 0.075 | 0.038 | 0.043 | 0.634 | 0.499 | 0.535 |
| **G J n** | 0.351 | 0.011 | 0.001 | 0.755 | 0.367 | 0.084 |

Table 5.34: Average significance level (columns labeled with SE SS SM) and power (columns PE PS PM) for the envelopes test, deviation measure test with deviation measure 4.9 and deviation measure test with deviation measure 4.8

# Chapter 6

# Conclusion

There is more to investigate in the fascinating topic of the point process models testing; however, we feel that the capacity of this thesis has been nearly exhausted and the time has come to sum up the above said. The objective of this thesis was to carry out a comparative analysis of the different available methods of point process model testing and it is our hope that the reader will find such analysis of use. Let us outline what has been done as a part of this thesis; two test methods have been implemented for use within the `R` application environment. A library has been provided for calculating morphological functions in `R` and the property of asymptotic normality of summary statistics has been statistically verified. With the help of this software, a comprehensive simulation study has been conducted which involved 5 000 different point process instances and resulting in over 500 000 tests. The outcome of this simulation study has been carefully analyzed and interesting patterns have been documented.

The most important results of this thesis are summed up in the following list:

- Morphological functions are the best candidates for use as summary statistics in the simulation based tests.
- Constructing summary statistics from multiple functions does not improve the test results.
- Envelopes test performs better than a test with deviation measure in case the summary statistics is not based on the $J$-function. The reason for this seems to be the fact that the asymptotic normality requirement does not hold for the chosen interval.

As a result of this comparison we anticipated to answer the question about which test is ultimately better. As apparent from the above list the envelopes test performs better in certain situations. But the results in whole exhibited great versatility and the final verdict on that matter is, in our opinion, difficult to make at this point. Further testing with different parameter sets and perhaps other process families is required. Until such testing has been undertaken we suggest to rely on the outcome of several different tests rather than favour any in particular.

There is of course enough space left for further exploration and improvements. Missing are a set of batch tests employing different parameters or investigation of tests for other process families. A very interesting question which we had no chance to answer is an optimal value for the $K$ parameter in the envelopes test. This parameter controls the maximum acceptable number of the vector components falling outside the envelopes and is therefore of critical importance.

The provided software can also be improved. Part of the code would benefit from more sophisticated parameter processing routines and generalization to different `R` objects. At some points the complexity of the code as well as the count of performed operation can be reduced by using more advanced programming techniques. Also one could suggest more efficient implementation of the test algorithms or make use of the parallel computing techniques. Finally the script file can be converted into an `R` extension for more comfortable use.

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