Univerzita Karlova v Praze

Fakulta sociálních věd

Institut ekonomických studií

Matěj Urban

Optimal Investment Portfolio with Respect to the Term Structure Of the Risk-Return Tradeoff

Diplomová práce

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Autor práce: **Bc. Matěj Urban**

Vedoucí práce: **PhDr. Milan Rippel**

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Abstract

My thesis will focus on optimal investment decisions, especially those that are planned for longer investment horizon. I will review the literature, showing that changes in investment opportunities can alter the risk-return tradeoff over time and that asset return predictability has an important effect on the variance and correlation structure of returns on bonds, stocks and T bills across investment horizons. The main attention will be given to pension funds, which are institutional investors with relatively long investment horizon. I will find the term structure of risk-return tradeoff in the empirical part of this paper. Later on I will add some variables into the model and investigate whether it can improve the results. Finally the optimal investment strategies will be constructed for various levels of risk tolerance and the results will be compared with strategies of Czech pension funds. I am going to use data from Thomson Reuters Datastream, Wharton Research Data Services and additionally from some other sources.

Abstrakt

Tato práce se zabývá optimálním investičním rozhodováním, speciálně v dlouhém časovém horizontu. Nejdříve shrnu předešlou literaturu, která ukazuje, že změny v investičních příležitostech mohou změnit strukturu rizika a očekávaných výnosů v čase a že předvídatelnost návratností aktiv má důležitý dopad na rozptyl a korelaci výnosů u dluhopisů, akcií a pokladničních poukázek s měnícím se investičním horizontem. Hlavní pozornost bude zaměřena na penzijní fondy, což jsou institucionální investoři s relativně dlouhým investičním horizontem. V empirické části této práce ukáži výsledek časové struktury rizika jednotlivých aktiv. Později přidám do modelu další proměnné a prozkoumám, zda mohou zlepšit výsledky modelu. Nakonec vytvořím optimální investiční strategie pro různé míry tolerance k riziku a výsledky porovnám se současnými strategiemi českých penzijních fondů. Pro empirický výzkum použiji data z Thomson Reuters Datastream, Wharton Research Data Services a dalších zdrojů.

Keywords

Term structure of risk-return tradeoff, predictability of asset returns, vector autoregressive model, optimal portfolio, pension funds

Klíčová slova

Časová struktura rizika a očekávaných výnosů, předvídatelnost výnosnosti aktiv, vektorový autoregresní model, optimální portfolio, penzijní fondy

Rozsah práce: 134,045 znaků

Prohlášení

1. Prohlašuji, že jsem předkládanou práci zpracoval samostatně a použil jen uvedené prameny a literaturu.
2. Prohlašuji, že práce nebyla využita k získání jiného titulu.
3. Souhlasím s tím, aby práce byla zpřístupněna pro studijní a výzkumné účely.

V Praze, 10. května 2011

Matěj Urban

Poděkování

Děkuji PhDr. Milanu Rippelovi za ochotu ujmout se vedení této diplomové práce, za konzultace a cenné připomínky. Svým nejbližším děkuji za trpělivost a podporu.

Master Thesis Proposal

Institute of Economic Studies
Faculty of Social Sciences
Charles University in Prague

Author:	**Bc. Matěj Urban**	Supervisor:	**PhDr. Milan Rippel**
E-mail:	mat.urban@seznam.cz	E-mail:	milanrippel@seznam.cz
Phone:	+420606785762	Phone:	+420775127245
Specialization:	FFTaB	Defense Planned:	February 2011

Proposed Topic:

Optimal Investment Portfolio with Respect to the Term Structure of the Risk-Return Tradeoff

Topic Characteristics:

My thesis will focus on optimal investment decisions, especially those that are planned for longer investment horizon. I will review the literature, showing that changes in investment opportunities can alter the risk-return tradeoff over time and that asset return predictability has an important effect on the variance and correlation structure of returns on bonds, stocks and T bills across investment horizons. The main attention will be given to pension funds, which are institutional investors with relatively long investment horizon. I will try to find optimal portfolio allocation with respect to the term structure of the risk-return tradeoff. Later on I will investigate whether it is convenient for the investor to invest into actively managed funds and whether those funds are successful in timing the market. This will address the question if the pension funds should invest into hedge funds, mutual funds etc. or to invest into safer instruments. I am going to use data from the Thomson Reuters Datastream and additionally from some other sources.

Hypotheses:

1. Expected excess returns on bonds and stocks, real interest rates and risk shift over time in predictable ways
2. Optimal investment portfolio changes with different investment horizon
3. Longer the investment horizon, more stocks are present in optimal portfolio
4. Actively managed portfolios (including the fees) do not outperform the market
5. There is almost no persistence in mutual funds performance

Methodology:

Concerning the optimal investment allocation in long investment horizon, I am going to employ the vector autoregressive model. I will divide assets into various asset classes (long term bonds, short term bonds, T bills, stocks, real estate etc.) and try to create the term structure of the risk-return tradeoff. I am going to find optimal investment allocation with respect to the convenient investment horizon of institutional investor (particularly pension fund) and preferred risk. I will focus especially on the pension funds in Czech Republic. Then I am going to compare the resulting efficient frontiers with the frontier of modern portfolio theory. One important implication of time variation in expected returns is that investors may want to engage in market timing. I will employ the four factor Famma&French model to investigate performance and its persistence of actively managed funds. Additionally I am going to use the same method for testing some alternative investments into specific assets.

Outline:

1) **Literature Review**
   - 1.a) Modern Portfolio Theory
   - 1.b) Vector Autoregressive Model
   - 1.c) Other Models of Asset Return Predictability
2) **Model of Risk-Return Tradeoff**
   - 2.a) Investment Allocation According to Modern Portfolio Theory
   - 2.b) Estimation Results of Vector Autoregressive Model
   - 2.c) Optimal Investment with Respect to Investment Horizon
3) **Pension Funds**
   - 3.a) Characteristics of Pension Funds
   - 3.b) Specificity of Czech Pension Funds
   - 3.c) Optimal Investment Strategy for Czech Pension Funds
4) **Alternative Investments**
   - 4.a) Specific Asset Classes (Sin Stocks, Socially Responsible Investment etc.)
   - 4.b) Market Timing
   - 4.c) Testing the Performance of Actively Managed Portfolios
   - 4.d) Discussion of the Results

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Contents:

1. Introduction

The insight of portfolio choice problem has changed after recent research in academic finance. The mean-variance analysis of Markowitz (1952) has provided the basis for financial economists to analyze risk-return problem and diversify portfolio in order to reduce risk. However later studies emphasize that the modern portfolio theory of Markowitz (1952) is useful analytical method for short term investor, but it ignores important factors influencing the portfolio choice of long term investor. Since the work of Merton (1969, 1971 and 1973) and Samuelson (1969) the solution to a portfolio choice problem can be significantly different for long term investor than myopic (short term) investor. In dynamic portfolio theory investors do not care only about risks one period ahead as myopic investors. In reality many investors want to finance a stream of consumption over a long lifetime.

The widespread evidence of predictability of asset returns has important effect on the variance and correlation structure of returns on all assets across investment horizons. Campbell and Viceira (2005) come with the empirical model that is able to work with the complex dynamics of risk and expected returns and which is easily applicable to practice. They model returns and state variables[1] as a vector autoregressive model. They illustrate their approach using quarterly data from U.S. stock, bond and T-bill markets for the post war period. Their results emphasize the relevance of risk horizon effects on asset allocation. Shocks to the forecasting variables are correlated with unexpected returns and therefore optimal portfolio allocations among bills, stocks and bonds changes with the length of investment horizon. The main conclusion in this recent development is that predictability of stock returns lean the optimal portfolio holdings of conservative investors towards stocks and away from bonds and cash.

The purpose of this paper is to find optimal investment decisions, especially those that are planned for longer investment horizon. Typical institutional investor with long investment horizon is the pension fund therefore we will employ the vector autoregressive model to find the optimal portfolio allocation for Czech pension funds. Czech pension fund market is

1 They use as a forecasting state variables dividend yield, nominal yield on T-bill and the yield spread

very specific and regulated sector therefore our findings will be rather theoretically optimal investment strategy in order to maximize the welfare of planholders than advice to current pension plans. We will simply compare the results of optimization problem (without regulations) with real investment strategies of pension plans under regulation.

We will try to solve the vector autoregressive model based on data from the Thomson Reuters DataStream and Wharton Research Data Services. The reasons why we do not rely on the results of previous studies and make our own empirical research are following:

- More recent data are available and therefore the effects of the last financial crisis are included in considerations.
- More variables will be included in the model. We will try to extend the division of asset classes. We will include for example real estate returns and hedge fund returns.
- We will work only with statistically significant variables which is not the case of VAR model made by Campbell and Viceira. We will compare how much this influences the results.
- We will try to apply the model also to European data.
- We will try to find optimal portfolio for a long term investor which will require transforming the risk of excess returns into real returns.

The VAR model of Campbell and Viceira (2005) that we are going to use for the research has some conditions. First, they assume that the variance-covariance structure of the shocks in VAR is constant, thus the short term risk does not change over time. However the empirical evidence suggests that changes in risk are not very persistent, therefore the changes to the model that would count with changing short term risk should not be important. Nonetheless, varying short term risk can be included into the model along models written by other authors[2]. Second, the model is valid only for buy and hold investors who make one-time investment decision and then hold their portfolio until the maturity. This might seem unrealistic, because investors may want to rebalance their portfolio in response to changes in investment opportunities. However Samuelson (1969), Merton (1969, 1971, and 1973) and other economists have shown that for long-horizon

[2] See for example Engle (2002)

investor, not only short-term risk is relevant to the investment decisions with rebalancing strategies. The "intertemporal hedging portfolio" is as important as short term efficient portfolio for optimal asset allocation rebalancing strategies[3].

The more is an investor rebalancing the portfolio, the more is involved in market timing and more he cares about short term risk. The market timing is necessarily connected with more portfolio managers and higher costs of active management. There have been studies that the market timing is too expensive and that it does not bring additional wealth to the planholders. Therefore even though the outcome of my research is relevant for buy and hold investors only, it is conceptually appealing as general recommendation for investment strategies of long horizon investors.

We will model in this paper both, the efficient frontier for myopic portfolio and efficient frontiers for intertemporal hedging portfolio. We will create an optimal portfolio for given levels of risk in section five of this paper. We will also create the global minimum variance portfolio and compare the results with present reality of Czech pension fund sector. In case that the expected real return of pension funds with current investment strategy is lower than the expected real return of global minimum variance portfolio, the change in the regulation of Czech private pension plans would be very appropriate.

The organization of the paper is as follows. Section 2 reviews previous important literature on portfolio choice theory and the term structure of the risk-return tradeoff. Section 3 includes my own empirical research concerning both, the term structure of the risk return tradeoff and the short term mean-variance analysis. Section 4 introduce new asset classes and investigates if it can improve the results of the model. Section 5 shows changing structure of global minimum variance portfolio across investment horizons and suggests some appropriate long term strategies for long term investors. Section 6 is explaining the questions of pension funds and emphasizes the specificity of Czech pension fund market. Finally, section 7 concludes the thesis.

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[3] Brennan, Schwartz and Lagnado (1997) have modelled month to month, year to year optimal rebalancing strategies including both, the intertemporal hedging portfolio and myopic portfolio. This combined portfolio is different than both portfolios individually.

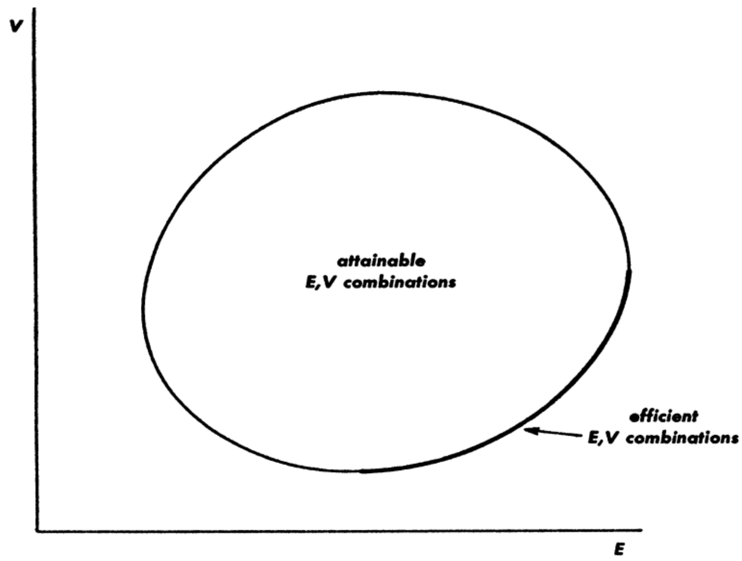
2. Literature Review

This section reviews the important literature that models in this paper are based on. It starts with the pioneering work of Markowitz on portfolio selection that illustrates relations between beliefs and choice of portfolio according to the „expected returns-variance of returns rule”. The model of short term mean-variance analysis in section 3.1 draws from this model presented by Markowitz and his followers and presents the traditional approach of portfolio selection problem. Chapter 2.2 shows the important studies concerning the models based on asset return predictability and serves as an important source for empirical research of the term structure of the risk return tradeoff in sections 3 and 4.

2.1. Modern Portfolio Theory by Markowitz (1952)

Until the work of Markowitz (1952) the economic theory suggested that the selection of securities is based on maximizing the discounted expected future returns. Despite the growing empirical evidence of the behavior of many investors who were diversifying their portfolios, the theory did not capture sufficiently the rule that should be followed by investors. The rule of "only" maximizing discounted future expected returns has been rejected by Markowitz and replaced by the rule that investor considers expected return as desirable and the variance of return as undesirable thing. Markowitz points out that the rule of maximizing discounted future returns does not imply that there is a diversified portfolio which is preferable to all non-diversified portfolios. The terms yield and risk were commonly used in financial writings even before Markowitz, but he replaced the term yield by expected return and risk by the variance of return which enabled to illustrate the risk-return tradeoff. Figure 1 geometrically presents the nature of the efficient surfaces for cases in which the number of available securities is small.

Figure 1: Efficient combinations of variance and expected return



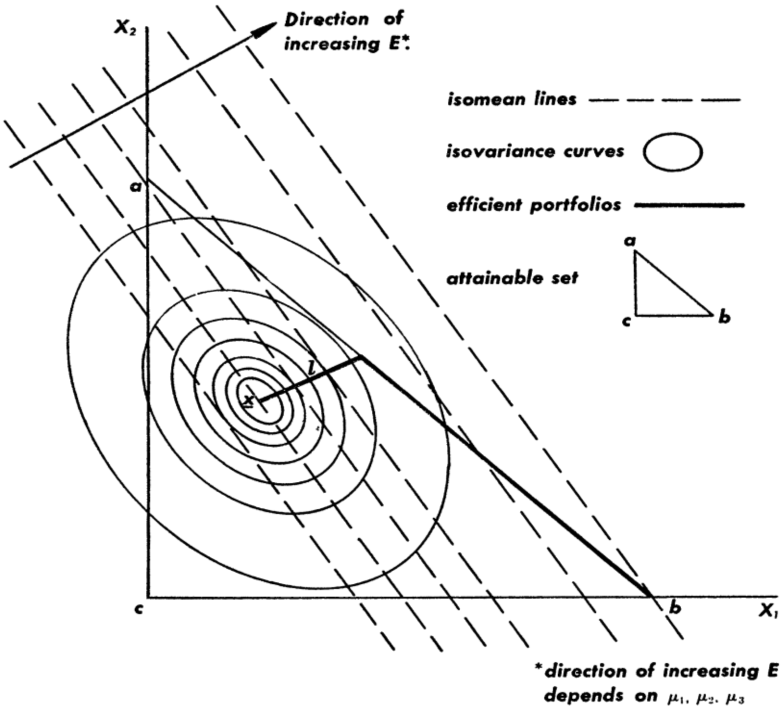
Source: Markowitz (1952)

The author considers the case of three securities where the expected return of portfolio equals the weighted sum of expected returns of all securities and the variance of the portfolio is:

$$V = \sum_{i=1}^{3}\sum_{j=1}^{3} X_i X_j \sigma_{ij},$$

where V is the variance, X_i and X_j are the relative amounts invested into securities and σ_{ij} is the covariance between returns of security i and j. The model reduces to the form where E and V is the function of X_1 and X_2. By using these relations and the constraint prohibiting short sales, we can work with two dimensional geometry. The attainable combinations of X_1, X_2 are represented by the triangle "abc" in Figure 2. The *isomean* curve is the set of all points (portfolios) with a given expected return. An *isovariance* line is defined to be the set of all points with a given variance of return.

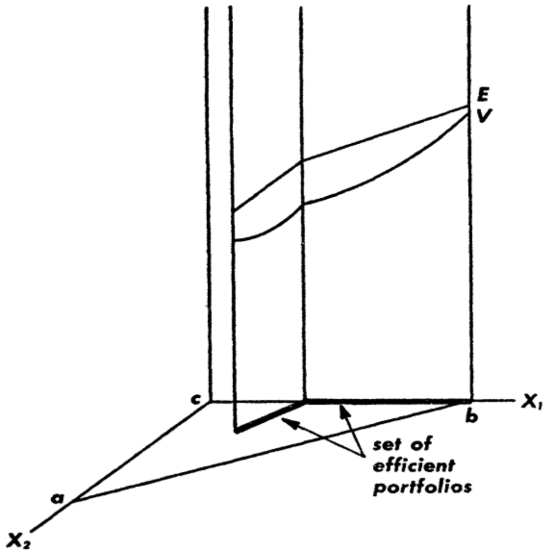
Figure 2: Efficient portfolios



Source: Markowitz (1952)

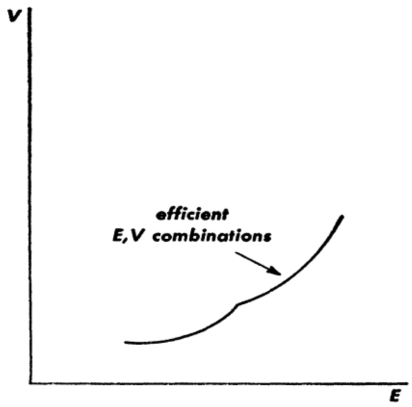
The centre of the ellipses is the point which minimizes V. Variance increases as we move out of this point. Since the point is attainable, it is also efficient. The thick line illustrates the efficient portfolio. The efficient set in N security case is a series of connected line segments. At one end of the efficient set is the point with maximum expected return and at the other end is the point of minimum variance. “A Figure 3 show that the section of the E-plane over the efficient portfolio set is a series of connected line segments. The section of the V-paraboloid over the efficient portfolio set is a series of connected parabola segments. If we plotted V against E for efficient portfolios we would again get a series of connected parabola segments (see Fig. 4). This result obtains for any number of securities.” (Markowitz 1952) The efficient combinations of expected returns and variance of returns is usually called the *efficient frontier*. This model is a good investment guide for the risk averse investor. It tells him how much of the expected return has to be given up in order to achieve lower risk.

Figure 3: Set of efficient portfolios



Source: Markowitz (1952)

Figure 4: Efficient frontier



Source: Markowitz (1952)

The contribution of Markowitz to financial economics was important by introduction of risk measured by variance in returns and also by realization of the importance of covariances between returns. Covariances are as important as variances in returns for total risk of the portfolio. Thus it is important for good diversification to avoid investing in securities with high covariances among themselves.

The Markowitz´s risk-return evaluation brought the “right kind” concept of diversification into financial economics, however the further development has shown that changes in investment opportunities can alter the risk-return tradeoff for investors with long investment horizon.

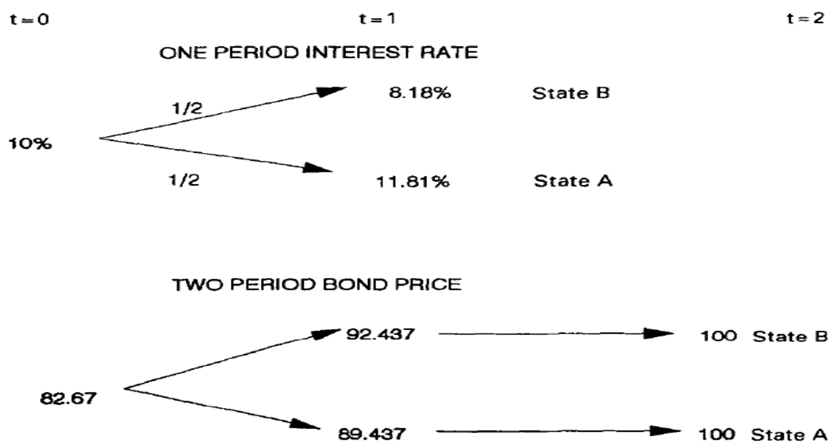
2.2. Models based on asset return predictability

Section 2.1. suggested the solution for portfolio selection problem that is convenient for short horizon investor. However since the work of Merton (1969, 1971, 1973) and Samuelson (1969) the solution to a portfolio choice problem can be changing across investment horizon. The rational risk averse investor with long investment horizon will be concerned about hedging against shifts in the future investment opportunity set. Due to the changes in investment opportunities, the long term investors might want to hedge against the shocks in investment opportunities and create a demand for “strategic asset allocation” that is well explained by the work of Brennan, Schwartz and Lagnado (1997). The tactical asset allocation developed by Markowitz is a single period or myopic strategy. Such a strategy might face some difficulties, because expected rates of return are typically not one period rates of return, but rather estimated rates over long investment period. The other difficulty is that myopic objective function that underlies tactical asset allocation is appropriate only if the investor has a logarithmic utility function[4]. „For general (non-log) utility functions the investor will be concerned about hedging against shifts in the future investment opportunity set (changes in expected returns or covariances) - for an investor with a long horizon, a drop in interest rates may be as important for his future welfare as a substantial reduction in his current wealth. Similar considerations apply to institutional investors such as pension funds, depending on the precise specification of their objective function.”(Brennan, Schwartz, Lagnado 1997) Therefore time varying investment opportunities can change the risk-return tradeoff of stocks, bonds and cash across investment horizons, thus creating a term structure of the risk-return tradeoff. Brennan, Schwartz and Lagnado (1997) explain the optimal control problem on a simple example[5]:

4 See for example Mossin (1968) Brennan, Schwartz, Lagnado (1997) or Campbell, Viceira (2005)
5 The simple model is taken over from Brennan, Schwartz and Lagnado (1997)

Consider an investor with initial wealth ***W*** who is interested in maximizing the expected utility of wealth at the end of a two period horizon. His utility function is of the iso-elastic family: $U(W) = \frac{1}{\gamma}W^{\gamma}$ At time t (t = 0, 1,2) the investor's expected utility will depend on both his current wealth (W) and the investment opportunities, which are assumed to be represented by a vector Y, V(W, Y, t). The investor´s investment opportunities are one-period or two-period bond and expected returns on both bonds are the same[6]. The bond price is obtained from discounting by the relevant period rate.

Figure 5: Binomial model of bond pricing



Source: Brennan, Schwartz, Lagnado (1997)

A myopic investor will not invest anything in the two-period bond, since it is riskier asset with the same expected return as the one-period bond. However an investor with two-period horizon can take an advantage of hedging against change in investment opportunity and take position in two-period bond in the first period. He is compensated by higher reinvestment rate in case of worse payoff in state A. His final wealth in the two states may be written as: W_A = 1.1181W0{1.1 + x(0.0818 - O.l)}

W_B = 1.0818W0{1.1 + x(0.1181 – O.l)}

The investor chooses *x* to maximize: $V = 0.5W_A^{\gamma} + 0.5W_B^{\gamma}$ The optimal values of *x* for different level of risk aversion are shown in table 1.

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6 See binomial model of bond pricing in figure 4, probability of getting into state A is ½ as well as the probability of getting into state B.

Table 1: Optimal allocation to two-period bond

γ	0.9	0.5	0.0	-0.5	-0.9	-2.0
X	-8.9	-1.0	0.0	0.3	0.4	0.7

Source: Brennan, Schwartz, Lagnado (1997)

We can see that for $\gamma > 0$, it is optimal to take a short position in two-period bond, while for $\gamma < 0$ is optimal to take long position. It implies that the risk averse long horizon investor behaves differently than the myopic investor and uses the advantage of hedging against changes in investment opportunities. This simple example explained how the changes in investment opportunities can alter the risk-return tradeoff of assets. Campbell, Viceira (2005) and Campbell, Chan, Viceira (2000) found that asset return predictability has important effects on the variance and correlation structure of returns across investment horizons. They use the vector autoregressive model to illustrate the term structure of risk, using quarterly data from the U.S. Stock, bond and T-bill markets. They use following state variables as return predictors: The short term interest rate, the dividend-price ratio and the yield spread between long-term and short-term bonds. Even though authors have chosen VAR (1) model as a method for regression, the process is stable, therefore we can write any VAR (n) model as VAR (1) model. Thus the order of autoregression does not play a role.

The authors demonstrated on a simple example of VAR (1) model that only when the variance of single period returns is the same at all forecasting horizons and returns are not autocorrelated, there is no horizon effect in risk. However these conditions do not hold when returns are predictable, so horizon matters for risk. This simple model shows that predictability of asset returns has two effects on risk in multiperiod horizon. It increases the conditional variance of future single period returns, because future returns depend on past shocks to the forecasting variable and it evokes autocorrelation in single period returns, because future single period returns react to past shocks of the forecasting variable. The total effect depends on the sign and size of the coefficients of forecasting variables and on the contemporaneous correlation between unexpected returns and the shocks to state variables.

Table (2) reports the estimation results of the VAR system. The first section shows coefficient estimates and the R squared statistic for each equation in the system. The second section reports the covariance structure of the VAR system. The elements on the main diagonal are standard deviations multiplied by 100 and the elements out of the main diagonal are correlation statistics.

Table 2: VAR estimation results, 1952 Q2 – 2002 Q4

A. Slopes (t-statistics in parenthesis)

		Coefficients on lagged variables						R-squared
		(1)	(2)	(3)	(4)	(5)	(6)	
(1)	log real T-bill rate	0.4029	0.0009	0.0002	0.2597	-0.0013	0.3999	0.299
		(5.732)	(0.167)	(0.008)	(3.493)	(-1.211)	(2.254)	
(2)	log stock excess returns	1.0536	0.0268	0.3796	-2.1044	0.0549	-0.1348	0.095
		(0.977)	(0.380)	(1.583)	(-2.431)	(3.021)	(-0.056)	
(3)	log bond excess returns	-0.1088	-0.0568	-0.0686	0.3706	-0.0039	3.1252	0.097
		(-0.374)	(-2.672)	(-0.583)	(0.944)	(-0.769)	(3.169)	
(4)	log nominal yield on T-bills	-0.0023	0.0050	0.0027	0.9539	0.0003	0.0961	0.872
		(-0.074)	(2.315)	(0.217)	(22.262)	(0.772)	(1.084)	
(5)	log dividend yield	-1.2428	-0.0157	-0.3753	1.3617	0.9613	-0.3787	0.951
		(-1.129)	(-0.217)	(-1.489)	(1.518)	(52.792)	(-0.154)	
(6)	log yield spread	0.0041	-0.0015	0.0029	0.0217	-0.0001	0.7596	0.561
		(0.174)	(-0.869)	(0.378)	(0.848)	(-0.352)	(13.160)	

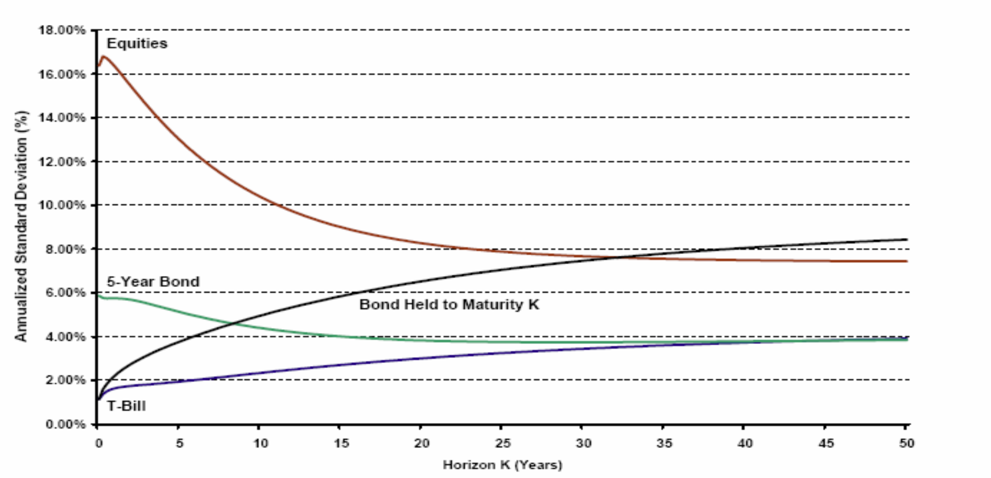
B. Quarterly percent standard deviations of residuals (diagonal) and cross-correlations of residuals (off-diagonal)

		(1)	(2)	(3)	(4)	(5)	(6)
(1)	log real T-bill rate	0.567	0.221	0.357	-0.349	-0.223	0.143
(2)	log stock excess returns	0.221	8.064	0.119	-0.101	-0.983	0.015
(3)	log bond excess returns	0.357	0.119	2.689	-0.755	-0.141	0.147
(4)	log nominal yield on T-bills	-0.349	-0.101	-0.755	0.252	0.132	-0.754
(5)	log dividend yield	-0.223	-0.983	-0.141	0.132	8.175	-0.042
(6)	log yield spread	0.143	0.015	0.147	-0.754	-0.042	0.172

Source: Campbell, Viceira (2005)

We can find that the best predictors of the real bill rate are the lagged real bill rate, the lagged nominal bill rate and the slightly significant yield spread. The best predicting variables of excess stock returns are the lagged nominal short-term interest rate and the dividend yield. Those are the only significant variables. The third row corresponds to the equation for the excess bond return. The only significant predicting variables are the yield spread with a positive coefficient and the excess stock returns with a negative coefficient. The last three estimation results are well described by a persistent univariate AR (1) process. The figure (6) displays the total horizon effects on the annualized risks of equities, bonds and T bills up to 50 years.

Figure 6: Annualized percent standard deviations of real returns

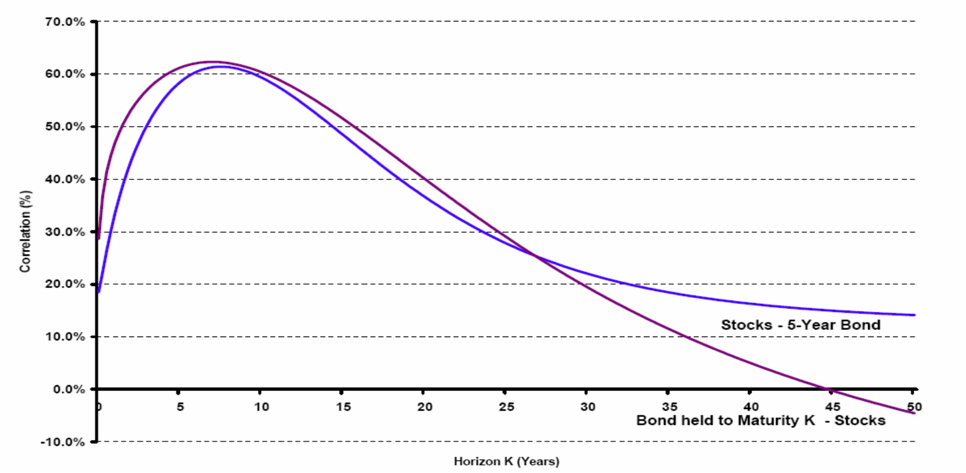


Source: Campbell, Viceira (2005)

We can see that long-horizon returns on stocks are much less volatile than the short-horizon returns. It is driven mainly by the mean reverting behavior in stock returns induced by their predictability from the dividend-price ratio. The positive coefficient of the dividend yield combined with the large negative correlation of shocks to the dividend yield imply that low dividend yields coincide with high current and poor future stock returns. Also the return on the 5 year bond records slight mean-reversion, which is the result of two reverse effects. The mean-reversion in bond returns caused by the nominal T-bill forecast is lowered by the mean-aversion driven by the fact that the yield spread forecasts bond returns positively and its shocks record low positive correlation with unexpected bond returns. The T-bill returns exhibit mean-averting behavior that is caused by the persistent variation in the real interest rate which intensifies the volatility of returns when T-bills are reinvested over long horizons. The bond held to maturity exhibits strong mean-aversion in real returns.

The risk of this bond is the risk of cumulative inflation. Figure (7) illustrates the correlation structure of real returns across investment horizons. The result is also very interesting. The magnitude of the correlation between real returns on stocks and fixed-maturity bonds changes significantly across investment horizons. Similar result is for the correlation between stock returns and variable maturity bond returns.

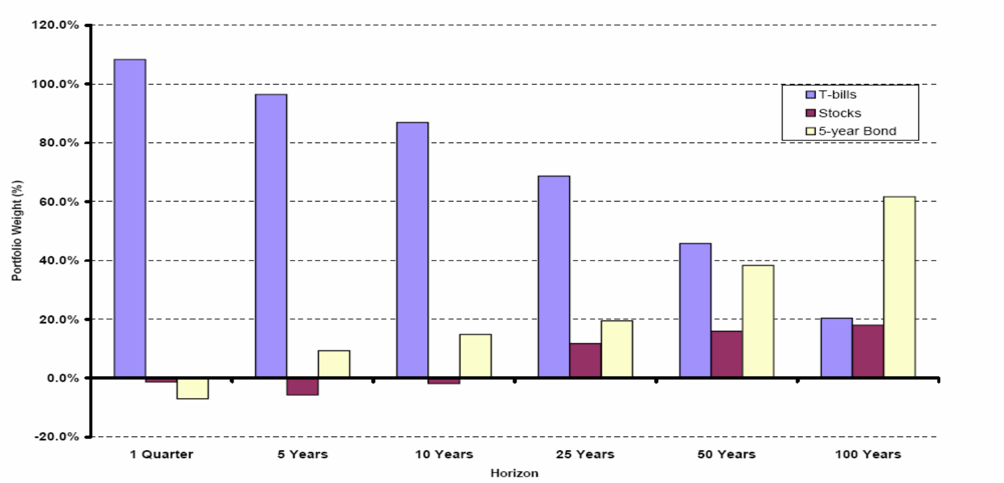
Figure 7: Correlation of real returns implied by quarterly VAR (1) estimates



Source: Campbell, Viceira (2005)

We can see that the highest correlation is in intermediate horizon which is not good for diversification of portfolio that consists of both assets. Changing variance-correlation structure of asset returns across investment horizons has dramatic effect on the structure of global minimum variance portfolio. We can see in figure 7 that the fraction of T-bills in this portfolio declines dramatically from 100 % to almost 20 % for investment horizon 100 years. This result shows that standard practise of considering T-bills as riskless asset does not work as well for long horizons**.**

Figure 8: Composition of global minimum variance portfolio



Source: Campbell, Viceira (2005)

The mean-reversion in stock returns and high volatility per period of T-bills and long term nominal bonds held to maturity suggest altering efficient mean-variance frontiers that investors face at different horizons. It implies different asset allocation recommendations for long horizon investor than those based on short term risk and return.

These results have far reaching consequences for the investment strategy of those institutional investors that have long investment horizon. A typical institutional investor with long investment horizon is the pension fund. The results of the empirical research of this paper will be applicable to investment strategies of Czech pension funds, taking to account that these funds apply rather conservative investment strategies minimizing the risk. The model of Campbell and Viceira has some shortcomings. We will try to solve some of them in the empirical part of this paper.

3. Empirical research

We will first do the traditional mean variance analysis in this section. We will see that there is no horizon effect of risk. Then we will introduce the vector autoregressive model as a result of predictability of asset returns. We will see that there is the term structure of risk-return tradeoff that is not constant over time.

3.1. Short term mean variance analyses: Traditional approach

We are going to show the results of short term mean variance analysis in this section. The outcome of the risk is the unconditional variance and is related to the following section in a way that this would be the outcome of VAR model if no predictability of asset returns are present. The difference between conditional and unconditional variance of returns is described in more detail in section 3.2. We used quarterly data from Thomson Reuters Datastream and Wharton Research Data Service. At first place, we were looking at expected return and variance of 3 different basic asset classes. These are stocks (S&P 500), T-bills and 5 year Treasury bonds. Later we take a look at one year Treasury bond, 30 year

Treasury bond and Real Estate Investment Trust. They are selected because we are going to work with these asset classes later in the vector autoregressive model. The quarterly data start in first quarter of 1960 and end in fourth quarter of 2009. In another table, where Real Estate Investment Trust are added, the data start in first quarter of 1972. The sample statistics are annualized and are in log terms. In table (3), we can see the mean and standard deviation of log T-bill returns, log 5 year bond excess returns, log stock excess returns and other statistics that are used later in VAR model as predictive variables.

Table 3: Mean and standard deviation

	mean	standard deviation
log T-bill	0.0172	0.017285832
log excess 5 year bond	0.012	0.056568542
log excess stock	0.033	0.175606071
log T-bill nominal	0.0552	0.013145341
log yield spread	0.0392	0.024257205
log dividend yield	0.1272	0.024027651

Source: Own calculation

We can notice that stocks are the riskiest assets with highest mean return as expected. T-bills are the safest assets with lowest mean return. Table (4) gives us the mean and standard deviation for different sample period so the mean and standard deviation is slightly different. As already mentioned, we added three more asset classes: log REIT returns, log one year bond excess returns and log 30 year bond excess returns.

Table 4: Mean and standard deviation

	mean	standard deviation
log T-bill	0.017	0.0195
log excess 5 year bond	0.0146	0.061
log excess stock	0.0378	0.179
log excess 1 year bond	0.0066	0.0177
log excess 30 year bond	0.016	0.1285
log excess REIT return	0.0349	0.175

Source: Own calculation

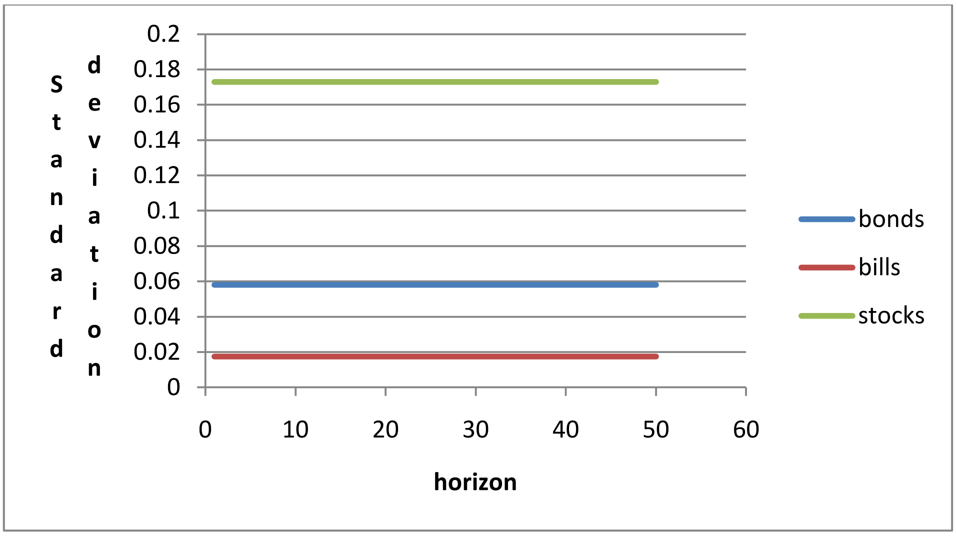
We can notice that log T-bill is still the asset class with lowest expected return. When we compare T-bills with longer maturities Treasury bonds, we can see that the higher the maturity, the higher the expected return. According to the theory it is especially due to liquidity and other risk premiums. The majority of economic researches prove that the short yield is more volatile than long term yield. One might suggest that the standard deviation of T-bill returns should be larger than standard deviation for longer maturities. But we have to keep in mind that we are working with total returns, including the capital gains, thus the short maturity bonds appear as a safer instrument. However we can notice that excess one year bond returns are less volatile than T-bills. Here we have to realize that we are not comparing standard deviation of returns, but standard deviation of one year bond excess returns with standard deviation of T-bill returns. The standard deviation of one year bond returns is 0.028 thus larger than T-bill standard deviation. Return on REIT is smaller than return on S&P 500, but slightly less risky. Since we measure the risk by the standard deviation of returns (or excess returns), we can write the multi period risk by the general formula[7]:

$$\sigma_{re_{t+n}^{(n)}} = \sqrt{n} \cdot \sigma_e ,$$

where $\sigma_{re_{t+n}^{(n)}}$ is the standard deviation of not annualized returns and σ_e is standard deviation of one period returns. This relationship gives a desirable effect that for an investor with long investment horizon, it is better to invest into riskier assets because of the law of large numbers. The Value at Risk will be decreasing faster stocks than for safer asset classes. However this effect has nothing to do with the term structure of risk return tradeoff caused by the predictability of asset returns. When there is no predictability or autocorrelation, then the annualized standard deviation is stable across all investment horizons. Figure (9) and (10) show us that there is no horizon effect in asset returns.

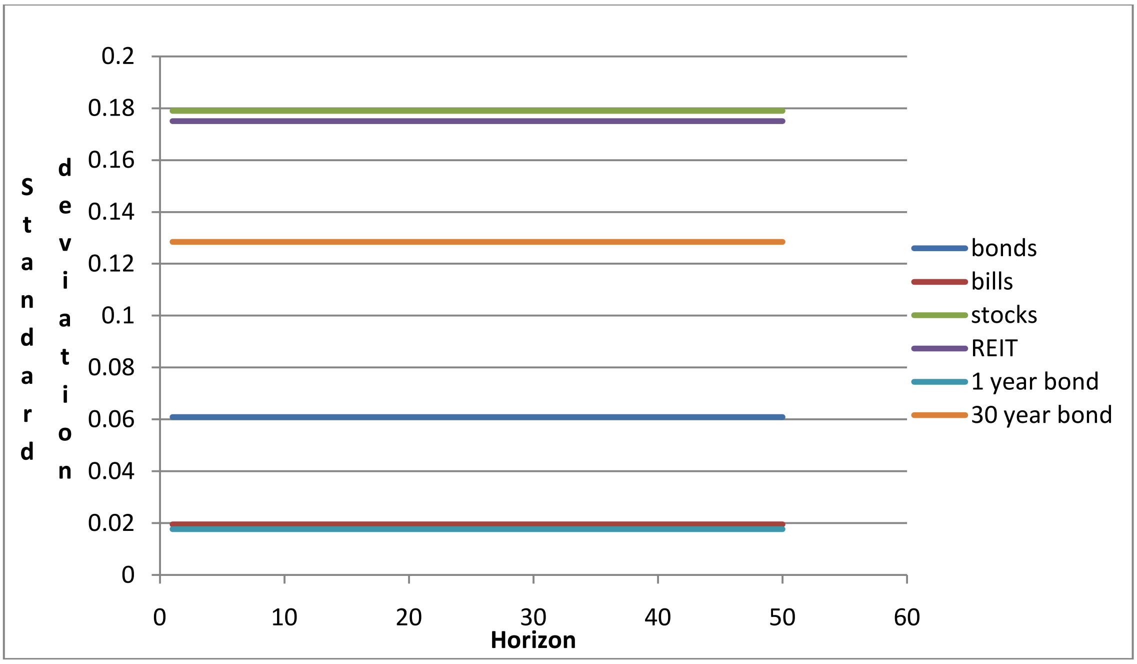
[7] This holds when returns are i.i.d. and are normally distributed.

Figure 9: Risk in standard mean-variance approach



Source: own calculation

Figure 10: Risk in standard mean-variance approach



Source: own calculation

We can see horizontal lines in both figures, representing stable risk across all investment horizons. We will see in following section that it is not the case, when we implement the vector autoregressive model and the predictability of asset returns.

3.2. Vector Autoregressive Model

In this section we are going to introduce the assumptions of the model, elaborate on them when necessary and then we are going to describe how the model works and how the term structure of variances and correlations is calculated.

3.2.1. Assumptions of the model

The investor uses vector autoregressive model of order one to forecast returns. Those returns are predicted by state variables. In first version of the model, the returns that are being predicted are: T bill returns, bond returns and stock returns. The state variables are selected such that they might have some predictive power based on some previous theories. For example according to the expectations hypothesis, the yield spread can predict the future behavior of short and long yields, thus yield spread (or slope of the yield curve) is one of the state variables. Many theories in corporate finance relate the dividend yields to the stock returns therefore we use this as a second state variable in this paper. The last state variable that is chosen for our paper is short nominal yield. It is because central bank targets the short nominal rate based on deviation from inflation target and potential output. It should have some predictive power to real short and long rates.

Just for technical purposes we use the log (continuously compounded) returns instead of gross returns. The data can be transformed any time back to gross returns which is actually the case of section when searching for optimal portfolio allocation. The returns are also measured as excess returns compared to the benchmark. Cash was selected as a benchmark as common practice. We approximate returns on cash by the real return on 3 months T-bills. We use the same assumption as Campbell, Viceira (2005) that short term risk does not change over time. For purposes of this work, we are satisfied with the argument that changes in risk are not very persistent and that this assumption should not have a large effect on results[8].

[8] Chacko and Viceira (1999) include changing risk in a long-term portfolio choice problem, using a continuous time extension.

The concept of this model is valid only to the buy and hold investor, however in practice it is not common and investors rebalance their portfolios. One might conclude that if the investor is rebalancing that he cares only about short term risk, no matter what is the term structure of risk-return tradeoff. However this was proved to be wrong by Samuelson (1969), Brennan, Schwartz and Lagnado (1997) because risk averse investor should search for the intertemporal hedging portfolio to hedge against unexpected changes in investment opportunities. Thus the “Strategic Asset Allocation” should be created, containing both the myopic mean-variance efficient portfolio and the intertemporal hedging portfolio. We suggest to the long term investor to use cash flow matching and decide for every liability to match the assets according to its maturity while using the term structure of risk-return tradeoff. The investor can rebalance the portfolio according to the equation (6) for expected returns.

However the investor should be aware of the fact that while rebalancing, he is losing partially the advantage of the risk term structure. Thus investor should rebalance only in the case when it is worth it, keeping in mind that rebalancing has some costs. In this paper we will consider only the conservative investor that is not involved in market timing and as a justification for that, we show some researches that market timing does not beat the market after the cost deductions. For other investors that tend to rebalance their portfolio very often, the results of this paper should serve only as a concept for construction of “intertemporal hedging portfolio”.

Majority of previous researches on active management has shown that market timing does not outperform the market[9]. Snigaroff (2000) explains active asset management as a zero sum game where “Buyers who want to produce alpha in a zero-sum game have to be better than their competitors“. But this game includes the costs of active management that decreases the average return of funds with active management below the average market performance. French K. R. (2008) compares the fees, expenses and trading costs society pays to invest in U.S. stock market with what would be paid if everyone invest passively: „Averaging over 1980–2006, I find investors spend 0.67% of the aggregate value of the market each year searching for superior returns. Society’s capitalized cost of price

[9] See for example Henriksson (1984)

discovery is at least 10% of the current market cap. Under reasonable assumptions, the typical investor would increase his average annual return by 67 basis points over the 1980–2006 period if he switched to a passive market portfolio.“ French K.R. (2008)
Of course, there are some institutional investors that outperform the market in some years, but the persistence on performance seems to be very poor. We can see that on the work of Carhart, M.M. (1997) or Lakonishok, J. & Shleifer, A. & Vishny, R.W. & Hart, O. & Perry, G.L. (1992). Thus the planholder cannot pick the right fund with good active management with expectations that they will beat the market. It is the reason why we consider buy and hold strategy sufficient for the purposes of this paper.

Another assumption is that the residuals in vector autoregressive model are serially uncorrelated. We also assume that the vector of shocks to asset returns and return forecasting variable is independent and identically distributed random variable and that it is normally distributed with zero mean.

3.2.2. Description of Vector Autoregressive Model

As already mentioned, we selected log returns on T-bills, bonds and stocks as explanatory variables and log dividend yield, log yield spread and log nominal short term yield as state variables. They all follow the vector autoregressive process of order one. However we can easily transform the VAR (1) model into VAR model of any order by simply adding new state variables which represent larger lags. Let us denote Z_{t+1} a column vector consisting of log real return on benchmark asset, log excess returns on other assets and log state variables at time t+1.

$$z_{t+1} = \begin{bmatrix} r_{T,t+1} \\ r_{B,t+1} - r_{T,t+1} \\ r_{E,t+1} - r_{T,t+1} \\ r_{DY,t+1} \\ r_{YS,t+1} \\ r_{NY,t+1} \end{bmatrix} = \begin{bmatrix} r_{T,t+1} \\ x_{t+1} \\ s_{t+1} \end{bmatrix}, \qquad (1)$$

where $r_{T,t+1}$ denotes log real return on cash, $r_{E,t+1}$ denotes log return on equity, $r_{B,t+1}$ is log real return on bonds, $r_{DY,t+1}$ is log dividend yield, $r_{YS,t+1}$ is log yield spread, $r_{NY,t+1}$ is the log nominal yield and for simplification X_{t+1} is the vector of log excess returns and s_{t+1} is a vector of state variables. Any asset return is assumed to follow first order autoregressive process such that:

$$z_{i,t+1} = \phi_0 + \phi_1 z_{1,t} + ... + \phi_i z_{i,t} + ... + v_{i,t+1}$$

Thus each variable depends linearly on its lagged value, on lagged values of all other variables, on a constant and contemporaneous random shock $v_{i,t+1}$. We can represent this equation in matrix form:

$$Z_{t+1} = \Phi_0 + \Phi_1 Z_t + V_{t+1}, \tag{2}$$

Where Φ_0 is a vector of intercepts, Φ_1 is a matrix of slope coefficients and V_{t+1} is a vector of zero mean shocks. To achieve the multivariate stationary condition similarly like in AR (1) where the autoregressive parameter is bounded between -1 and 1, we require the determinant of Φ_1 matrix to be bounded between -1 and 1. The shocks are serially uncorrelated, but we allow shocks for different asset classes and state variables to covary between each other. We assume that the vector of shocks is normally distributed, such that:

$$V_{t+1} \overset{i.i.d.}{\sim} N(0, \Sigma_v), \tag{3}$$

where Σ_v denotes the variance, covariance matrix of contemporaneous shocks. The elements on the main diagonal are the variances of real returns on benchmark, excess returns and state variable and the off diagonal elements represent the covariances. As already mentioned in assumptions of the model, these variances and covariances do not vary over time. The VAR (1) model differs to the traditional of the risk-return tradeoff because it does not expect constant expected returns. Thus the expected returns will differ to the traditional view and also the risk will be measured relative to conditional expectations. The traditional approach is the special case of VAR (1) approach, where returns are not autocorrelated. However when the asset returns are predictable, the returns will be autocorrelated and "the VAR (1) investor will understand that some portion of the unconditional volatility of each asset return is actually predictable time-variation in the return and thus does not count as risk. For this reason the conditional variance is smaller than the unconditional variance." Campbell, Viceira (2005)

We are going to derive what is the variance-covariance structure of the VAR (1) model with extending the investment horizon. As you will see, only under very special conditions, there is no term structure in risk. Similarly as in equation (2), we can write for any time:

$$Z_{t+2} = \Phi_0 + \Phi_1 Z_{t+1} + V_{t+2}$$

$$Z_{t+2} = \Phi_0 + \Phi_1 \Phi_0 + \Phi_1 \Phi_1 Z_t + \Phi_1 V_{t+1} + V_{t+2}$$

....

$$Z_{t+k} = \Phi_0 + \Phi_1 \Phi_0 + ... + \Phi_1^{k-1} \Phi_0 + \Phi_1^k Z_t + \Phi_1^{k-1} V_{t+1} + ... + \Phi_1 V_{t+k-1} + V_{t+k}$$

Because of the properties of logarithmic function, we can write (non annualized) returns of more than one period as the sum of single period returns.

$$Z_{t+k}^{(k)} = Z_{t+1} + Z_{t+2} + ... + Z_{t+k} \, , \tag{4}$$

where the element on the left side of equation denotes k period return. If we want to transform the k period return into annualized version, we can simply divide the k period return by the number of periods. Adding the expression for Z_{t+1}, $Z_{t+2}, \dots, Z_{t+k}$ and expressing vector Z as k period vector of returns and state variables, we get:

$$Z_{t+1} + ... + Z_{t+k} = \left[\sum_{i=0}^{k-1} (k-i) \Phi_1^i \right] \Phi_0 + \left[\sum_{j=1}^{k} \Phi_1^j \right] Z_t + \sum_{q=1}^{k} \left[\sum_{p=0}^{k-q} \Phi_1^p V_{t+q} \right] \tag{5}$$

Now we know how to compute conditional k-period returns. In order to calculate conditional k-period variance-covariance matrix, we need to know the conditional mean. Since the shocks have zero mean, the conditional mean is given by:

$$E_t(Z_{t+1} + ... + Z_{t+k}) = \left[\sum_{i=0}^{k-1} (k-i) \Phi_1^i \right] \Phi_0 + \left[\sum_{j=1}^{k} \Phi_1^j \right] Z_t \tag{6}$$

Hence the conditional variance-covariance matrix is given by:

$$Var_t(Z_{t+1} + .. + Z_{t+k}) = Var_t \left[\sum_{q=1}^{k} \left[\sum_{p=0}^{k-q} \Phi_1^p V_{t+q} \right] \right] \tag{7}$$

We can expand this expression and get:

$$Var_t(Z_{t+1} + .. + Z_{t+k}) = Var_t \left[(\mathrm{I} + \Phi_1 + ... + \Phi_1^{k-1}) V_{t+1} + (\mathrm{I} + .. + \Phi_1^{k-2}) V_{t+2} + .. + (\mathrm{I} + \Phi_1) V_{t+k-1} + V_{t+k} \right]$$

Since we know that shocks are serially uncorrelated, we can split abovementioned equation into the sum of variances of all elements such that:

$$Var_t(Z_{t+1}+..+Z_{t+k}) = Var_t\left[(\mathrm{I}+\Phi_1+...+\Phi_1^{k-1})V_{t+1}\right] + Var_t\left[(\mathrm{I}+\Phi_1+..+\Phi_1^{k-2})V_{t+2}\right] + $$
$$+...+Var_t\left[(\mathrm{I}+\Phi_1)V_{t+k-1}\right]+Var_t\left[V_{t+k}\right]$$

where I is the identity matrix. We know from the nature of the vector of shocks that its variance-covariance matrix is Σ_V. Thus we can get the final equation for the term structure of risk.

Equation (8):

$$Var_t(Z_{t+1}+..+Z_{t+k}) = \Sigma_v + (\mathrm{I}+\Phi_1)\Sigma_v(\mathrm{I}+\Phi_1)^T + (\mathrm{I}+\Phi_1+\Phi_1^2)\Sigma_v(\mathrm{I}+\Phi_1+\Phi_1^2)^T$$
$$+...+(\mathrm{I}+\Phi_1+...+\Phi_1^{k-1})\Sigma_v(\mathrm{I}+\Phi_1+...+\Phi_1^{k-1})^T$$

When we want to annualize the variance of returns, we simply divide the variance by the number of periods. To relate this into traditional unconditional approach, we can see on simple example with only two periods that the conditional and unconditional variance are equal only if the short term risk does not change over time and when returns are not autocorrelated. However when we suppose that asset returns are partially predictable then returns will be autocorrelated and matrix Φ_1 will not disappear from the equation. If the returns are not predictable Φ_1 will become matrix of zeros, thus we can write that the annualized variance of two periods is:

$\frac{1}{2}Var_t(Z_{t+1}+Z_{t+2}) = \frac{1}{2}\Sigma_v + \frac{1}{2}(\mathrm{I}+\Phi_1)\Sigma_v(\mathrm{I}+\Phi_1)^T = \frac{1}{2}\Sigma_v + \frac{1}{2}\Sigma_v = \Sigma_v$, which is the one period variance.

In the following section we are going to make empirical analyses of the term structure of the risk-return tradeoff based on the model presented here. We will see how much the results differ to the traditional approach where the risk is constant over all investment horizons. First we are going to run the regression of VAR (1) model and use the results to approximate the behavior of risk ignoring the significance of variables, just as in research of Campbell and Viceira (2005). Then we will do the same analyses working only with the significant variables and see how it affects the results.

3.2.3. Empirical Research

In this section we are going to use practical application of asset return dynamics, considering three U.S. asset classes and three predicting state variables. The assets are cash, Treasury bonds[10] and equities. The Treasury bonds will be later divided into 3 maturity groups: 1 year, 5 years and 30 years. The forecasting variables are log short term nominal interest rate represented by log yield on 90 day T-bills, log dividend yield that was calculated as value weighted return on stocks including dividends minus value weighted return on stocks excluding dividends. The last state variable used in the model is the log yield spread, which is represented by the difference between log yield on 5 year Treasury bond and the log yield on a 90 day T-bill. The real return on cash was approximated by the real return on 90 day T-bill. The portfolio of S&P 500 was used for log return on equities including dividends.

We used the quarterly data starting in first quarter of 1960 and ending in fourth quarter of 2009. The data were taken from Thomson Reuters Datastream and Wharton Research Data Service. Table (3) in previous section shows the mean and standard deviations of this sample. The sample statistics are annualized and are in log terms. One can approximate the mean gross returns by mean log returns, adding one half of their variance[11].

Our estimates of VAR (1) model of given sample period are presented in table (5) and (6). We used standard OLS method in program R to find estimating coefficients. Table (5) shows the coefficient estimates (represented by matrix Φ_1). This table reports also the t-statistics and R^2 of the model. Table (6) gives the variance-covariance structure of shocks represented by matrix Σ_v. We do not report the intercepts, because we restrict the intercept to be zero. The expected return of an asset according to the VAR model is state dependent. Hence we suppose that the mean of full sample will equal the mean of the traditional approach.

[10] The real returns on bonds include the capital gains.
[11] As suggested by Campbell, Viceira (2005)

Table (5): VAR estimation results- Coefficients on lagged variables

		1	2	3	4	5	6	R-squared adjusted
1	log bond excess returns	-0.10153	0.110627	0.420942	-0.1734	-0.05207	-0.08384	0.028
	t statistics	-1.209	0.397	2.168	-0.34	-2.209	-0.33	
2	log real T-bill rate	0.012734	0.176505	0.18746	0.658222	-0.00225	-0.18108	0.21
	t statistics	0.55	2.296	3.501	4.687	-0.347	-2.584	
3	log yield spread	0.01022	0.001206	0.91664	0.110733	-0.00781	0.015514	0.78
	t statistics	0.593	0.021	23.012	1.06	-1.615	0.298	
4	log nominal yield on T-bill	0.002835	-0.00291	-0.00374	0.940654	0.004267	0.011724	0.903
	t statistics	0.457	-0.142	-0.261	25.013	2.451	0.625	
5	log stock excess returns	0.27246	-0.10397	0.09538	-3.67651	0.06978	2.17997	0.07
	t statistics	1.069	-0.123	0.162	-2.378	0.975	2.825	
6	log dividend yield	-0.00931	-0.02321	0.017268	0.128276	-0.00392	0.910777	0.91
	t statistics	-0.958	-0.72	0.769	2.179	-1.439	31	

Source: Own calculation based on data from WRDS

The first row of table (5) represents the VAR equation for the excess bond return. The lagged stock excess returns and the lagged yield spread are the only significant variables influencing the excess bond return. The yield spread has a positive coefficient which means that steepening of the yield curve forecasts increase of the bond yield next period.[12] On the other hand, the lagged stock excess return has negative coefficient thus predicting future bond excess return negatively. According to Campbell (2001) the low R^2 for equation of bond excess return can be misleading about the magnitude of predictability at lower frequencies[13]. This is because highly persistent return forecasting variables can influence the R^2, such that the implied annual R^2 can be much larger than the quarterly R^2. This is the case of bond excess return forecasting variables. The second row corresponds to the real T-bill rate equation. The lagged real T-bill rate, lagged nominal T-bill rate and the yield spread have positive coefficients and are highly significant. Surprisingly also the lagged dividend yield has predictive power for T-bill rate equation. The sign of dividend yield is negative, thus high dividend yield forecasts a decrease in T-bill rate next period.

[12] It is in line with expectation hypothesis.
[13] Semiannual, annual

The fifth row is the equation for the excess stock return. The lagged nominal yield on T-bill and the dividend yield are the only significant variables in predicting excess stock returns. As expected from economic theory, high dividends signal high profitability and return on stocks is expected to go up. This corresponds to the positive sign of dividend yield variable. The negative sign of lagged nominal short term interest rate shows the inverse relationship between short interest rate and price of the stock[14]. Hence high short term interest rate predicts capital losses next period and it decreases the excess stock return. All the other rows show the estimation results for state variables. The yield spread is predicted mainly by its own lagged values and also by the stock excess return. Nominal short term yield is predicted mainly by the lagged nominal short term yield and the lagged stock excess return. The dividend yield is predicted by the lagged dividend yield and by the lagged nominal yield on T-bill.

Table (6) represents the variance-covariance structure of the VAR system. The variance expresses the short term risk of an investor with investment horizon of one quarter. We can see that unexpected stock excess returns are positively correlated with unexpected bond and T-bill returns. On the other hand, they are negatively correlated with unexpected dividend yield and nominal yield on T-bill. Unexpected excess bond returns are negatively correlated with shocks to the yield spread, short term nominal yield and dividend yield and they are positively correlated with shocks to the real T-bill rate and shocks to the stock excess returns. All other variances and covariances are presented below.

Table (6): VAR estimation results- variance covariance matrix of shocks

		1	2	3	4	5	6
1	log bond excess returns	7.72E-04	9.18E-05	-3.10E-06	-3.71E-05	1.39E-04	-2.37E-05
2	log real T-bill rate	9.18E-05	5.87E-05	7.23E-07	-5.38E-06	7.27E-05	-5.03E-06
3	log yield spread	-3.10E-06	7.23E-07	3.25E-05	-7.52E-06	4.91E-05	-2.98E-06
4	log nominal yield on T-bill	-3.71E-05	-5.38E-06	-7.52E-06	4.21E-06	-1.20E-05	1.59E-06
5	log stock excess returns	1.39E-04	7.27E-05	4.91E-05	-1.20E-05	7.11E-03	-1.88E-04
6	log dividend yield	-2.37E-05	-5.03E-06	-2.98E-06	1.59E-06	-1.88E-04	1.03E-05

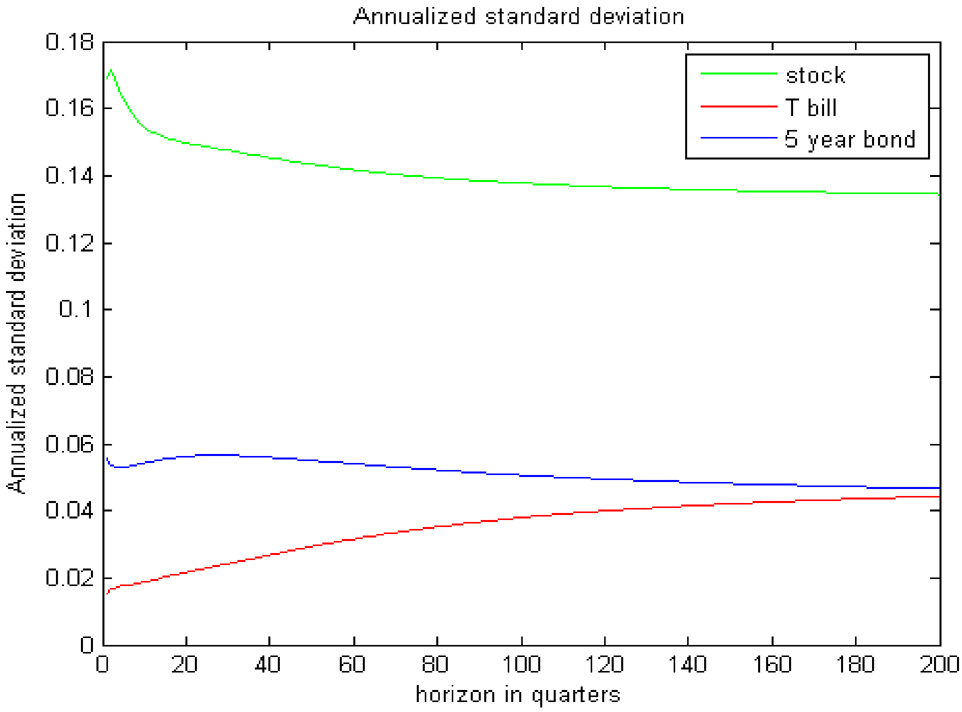
Source: Own computation based on data from WRDS

[14] Thanks to discounting of future cash flows, increasing short term interest rate decreases the price of the stock.

The resulting risk at different investment horizons after the implication of asset return predictability is displayed in Figure (11). The unit of investment horizon is one quarter and risk is measured by annualized standard deviation. The results are calculated in MatLab and the procedure is expressed in equation (8), where matrix Φ_1 is represented by coefficients on lagged variables from table (5) and matrix Σ_v is represented by variance-covariance matrix of shocks from table (6). The VAR estimation results are used regardless of the significance in this case. We can observe slight mean-aversion in 5 year Treasury bond in first five years, followed by mean reversion for longer investment horizons. The volatility starts at about 6% per annum, decreasing to about 4.5% per annum at 50 years horizon. The final mean reversion is the result of more offsetting effects. On one hand real T-bill rate forecasts bond returns positively and its shocks have positive correlation with unexpected bond returns. This causes mean aversion in excess bond returns. On the other hand positive yield spread coefficient combined with negative correlation of shocks with bond returns causes the mean-reversion in bond returns. Also the stock excess return causes mean reversion in excess bond return. The other lagged variables cause mean aversion in excess bond return.

Figure (11) also shows that excess stock returns are less volatile in long horizons than short horizons. This is driven mainly by the predictability of stock returns from the dividend yield. The large negative correlation of shocks to unexpected stock returns and dividend yield, together with positive coefficient of dividend yield in stock excess return equation imply mean reverting behavior of stock returns. The mean reversion in excess stock returns decreases the annualized standard deviation from 17% to less than 14% for longer horizons. However when we compare the results with results of Campbell, Viceira (2005), we see that the mean reversion is not as big. The mean reversion of their paper cuts the annualized standard deviation of excess stock returns from 17% to less than 8% per annum in 25 years horizon. The reason for that might be different sample period or different index representing the stock returns. Campbell and Viceira used sample period from 1952 Q2 till 2002 Q4 and we used sample period since 1960 Q1 till 2009 Q4. Thus our sample period does not include early post WWII period, but includes the latest period including the recent financial crisis. The mean-reversion of stock returns is being reduced by offsetting effect of mean aversive behavior in nominal yield on T-bill. This is caused by the large negative coefficient of lagged short nominal yield in excess stock return equation combined with negative correlation of shocks between excess stock return and nominal yield on T-bill.

Figure (11): Annualized Percent Standard Deviations of Real Returns

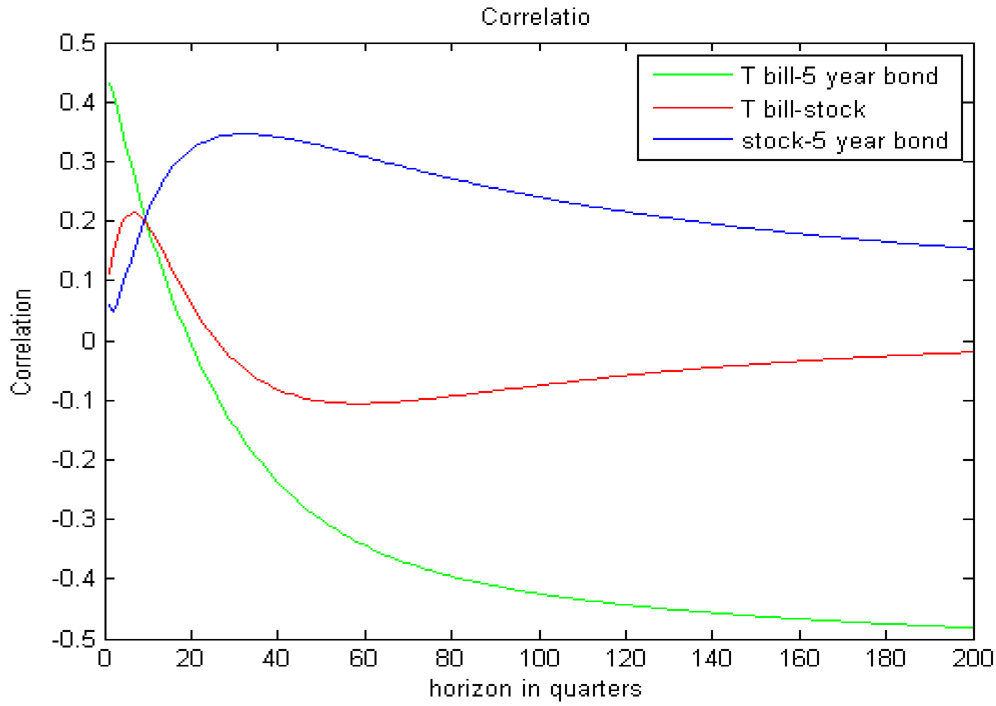


Source: Own calculation

In contrast to mean reversion of stocks, the real T-bill rate exhibits significant mean aversion in risk implied by VAR model. Thus the real return volatility of T-bills increases with the increase of investment horizon. The mean aversion is caused by mean aversive behavior of lagged bond excess return, yield spread and dividend yield. Campbell and Viceira also argue that: "The mean-aversion of T-bill returns is caused by persistent variation in the real interest rate in the postwar period, which amplifies the volatility of returns when Treasury bills are reinvested over long horizons." (Campbell, Viceira 2005)

Figure (12) is showing us that the correlation structure of real returns also differs across investment horizons. Correlation of real returns on stock and 5 year bond is positive at all investment horizons. However it starts initially at small correlation, reaching the top at ten years investment horizon and then it falls back in long horizon. This interesting pattern is the result of the interaction of state variables that dominate at different investment horizons.

Figure (12): Correlation of Real Returns



Source: Own calculation

At intermediate horizon the main variable influencing the correlation is the nominal yield on T-bill. It predicts low returns on both, the stocks and the bonds, however when the T-bill return rises, bond returns falls at once while stock returns fall slowly. Hence the correlation between bonds and stocks is higher at intermediate horizon than the short term correlation. The most important variable at long horizon is the dividend yield. This variable is very persistent and high dividend yields predict high stock returns, but low bond returns. Thus at long horizons, it weakens the correlation between excess bond returns and excess stock returns. The correlation between T-bills and stocks is very small at short horizon of one quarter. It jumps from 0.1 to 0.2 in few quarters and then begins to fall. The lowest correlation is for 15 years horizon and then it slowly rises, converging to zero correlation at very long investment horizons. The most interesting correlation pattern is for T bill and 5 years bond. The correlation starts at 0.45 for very short horizon and decreases continuously to -0.5 for very long investment horizons. It makes these two assets very attractive for diversification of the portfolio for long investment horizons. However as stated above, especially the T-bills do not seem to be so attractive at long horizon because of its mean aversion of real returns.

3.2.4. Empirical Research taking care of significance

In last section we made an empirical research on VAR (1) model by simply taking coefficients from table (5) and (6). In this section we are going to run the same model, but we are going to be statistically correct. We will test the assumptions of ordinary least squares[15] estimates and where necessary, we will use different econometric approach than OLS. We will use for our model only those variables that are significantly different from zero as a forecasting variables in VAR (1) model. We will use the procedure where the least significant variable will be excluded from the equation and the new regression will be processed. We will continue with this method until there are only significant variables present in the VAR model. The remaining (excluded) variables will be set to be zero, thus to have no predicting power. Together with this procedure, we will test the assumptions simultaneously. The tables (7) and (8) show the new VAR estimation results with significant variables.

Table (7): VAR estimation results- Coefficients on lagged variables

		1	2	3	4	5	6	R-squared adjusted
1	log bond excess returns	0	0	0.4086	0	-0.050	0	0.034
	t statistics			2.387		-2.15		
2	log real T-bill rate	0	0.196	0.1816	0.6277	0	-0.1711	0.196
	t statistics		2.931	3.472	4.873		-2.514	
3	log yield spread	0	0	0.9105	0	-0.0086	0	0.774
	t statistics			25.79		-1.794		
4	log nominal yield on T-bill	0	0	0	0.9526	0.0041	0	0.902
	t statistics				42.402	2.445		
5	log stock excess returns	0	0	0	-4.3306	0	2.3323	0.056
	t statistics				-3.421		3.36	
6	log dividend yield	0	0	0	0.1246	0	0.9169	0.927
	t statistics				2.569		34.467	

Source: Own calculation based on data from WRDS

[15] Later OLS

Table (8): VAR estimation results- Variance, Covariance matrix of shocks

		1	2	3	4	5	6
1	log bond excess returns	7.80E-04	9.11E-05	-4.67E-06	-3.74E-05	1.21E-04	-2.29E-05
2	log real T-bill rate	9.11E-05	5.88E-05	7.84E-07	-5.36E-06	7.35E-05	-5.03E-06
3	log yield spread	-4.67E-06	7.84E-07	3.32E-05	-7.49E-06	5.17E-05	-3.10E-06
4	log nominal yield on T-bill	-3.74E-05	-5.36E-06	-7.49E-06	4.22E-06	-1.15E-05	1.57E-06
5	log stock excess returns	1.21E-04	7.35E-05	5.17E-05	-1.15E-05	7.20E-03	-1.92E-04
6	log dividend yield	-2.29E-05	-5.03E-06	-3.10E-06	1.57E-06	-1.92E-04	1.06E-05

Source: Own calculation based on data from WRDS

The bond excess returns are predicted by lagged coefficients of yield spread and stock excess returns in this VAR (1) model. Real T bill returns are predicted by its own lagged values, yield spread, nominal yield on T bill and dividend yield. The excess stock returns are predicted by lagged nominal yield on T bill and lagged dividend yield. It remains to test the assumption of the OLS estimates. We know from econometrics that if the residuals are a sequence of random variables with zero mean and $E\varepsilon_i\varepsilon_j = \sigma^2\delta_{ij}$, $\sigma^2 \in (0,\infty)$, then the OLS estimator is the best linear unbiased estimator. Thus we should check for homoscedasticity and autocorrelation. We will do that by Goldfeld-Quandt test and Durbin-Watson test. If moreover $X^T X = O(n), (X^T X)^{-1} = O(n^{-1})$ and the disturbances are independent, then the OLS estimator is consistent. If further $\lim_{n\to\infty} \frac{1}{n} X^T X = Q$, where Q is the regular matrix, then OLS estimator is asymptotically normal. Therefore we have to check for multicollinearity and if the design matrix is of full rank. We will check the multicollinearity by the condition number.

If moreover $L(\varepsilon_i) = N(0,\sigma^2)$, then the OLS estimator is the best among all unbiased estimators. Hence we will check for the normality of disturbances by Kolmogorov-Smirnov test. The following table shows the results of testing the assumptions.

Table (9): Testing the assumptions

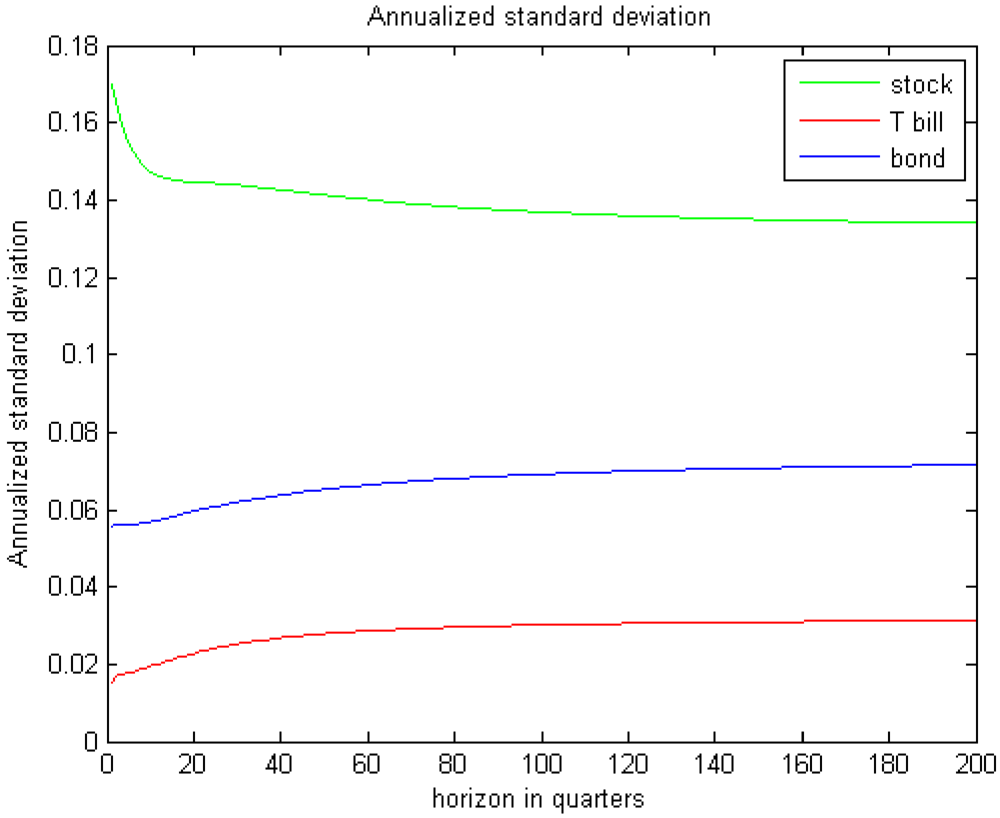
Test	Goldfeld-Quandt	Kolmogorov-Smirnov	condition number
log bond excess returns	p=0.83	p=0.13	5.51
log real T-bill rate	p=0.048	p=0.44	10.42
log yield spread	p=0.99	p=0.32	5.51
log nominal yield on T-bill	p=1	p=0.0003	5.1
log stock excess returns	p=0.11	p=0.12	10.66
log dividend yield	p=1	p=0.028	10.66

Source: Own calculation

We can see that on 5% significance level, there is only the equation for log real T bill rate that rejects the hypotheses of homoscedasticity. However the rejection is very close and the GLS estimates looked similarly. Therefore we decided to keep the OLS estimator for log real T bill rate. The practice in econometrics is that if the condition number is higher than 100, at least one dependent variable should be excluded. This is not our case, but if the condition number is between 10 and 100 or 30 and 100 (depending on tolerance) a special treatment should be applied[16]. If the condition number is smaller than 10 (30), there is nothing to be done and we deny the multicollinearity. We can see that the condition numbers for log real T bill rate, log stock excess returns and log dividend yield are above, but very close to 10. Thus we decided also in this case to accept the data as non collinear and we did not use any transformation or further exclusion of variables. The test for normality fails at 5% significance level for log nominal yield on T-bill and log dividend yield. Thus the OLS estimator is not the best among all unbiased estimators, but is best only among the class of linear unbiased estimators. We could use the box-cox or some other type of transformation to correct this and find better estimator, but since the model follows vector autoregressive process, it would be appropriate to use this transformation for all data. Only log nominal yield on T bill fails to have normally distributed residuals at 1% confidence level so we decided to keep regular OLS estimates and to be satisfied with best linear unbiased estimator for equation of log nominal yield on T bill. Figure (13) displays the term structure of risk for our new VAR (1) model.

[16] The possible special treatment for collinearity is Ridge regression.

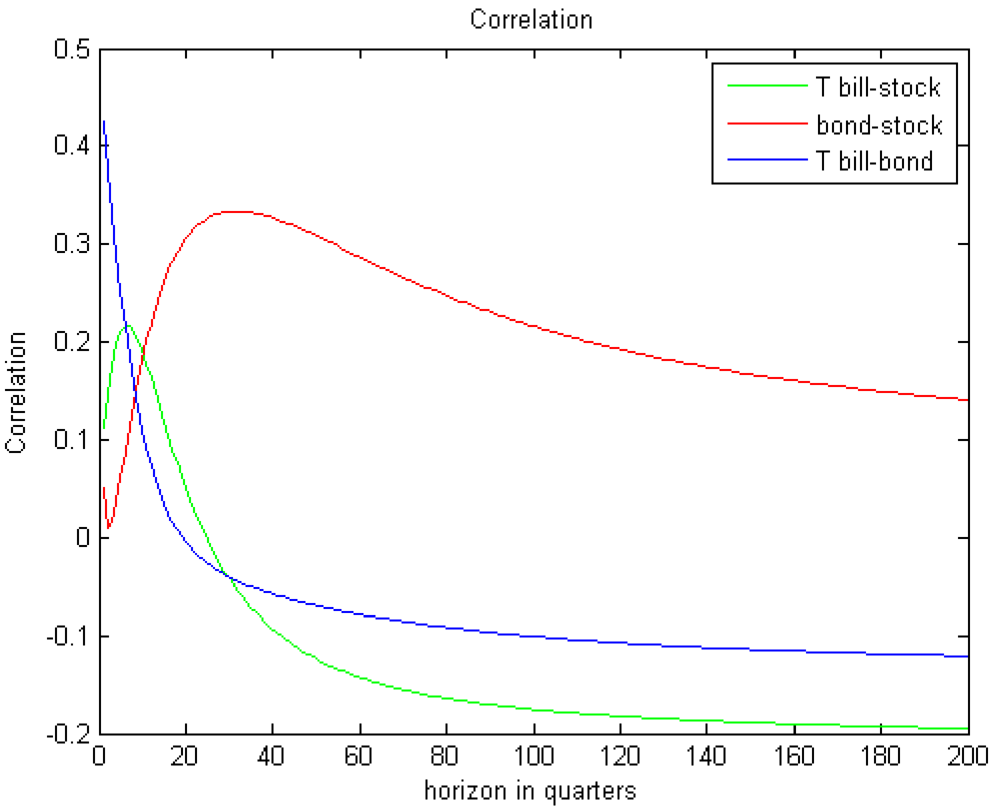
Figure (13): Annualized Percent Standard Deviation



Source: Own calculation

We can see that excluding insignificant variables has almost no effect on the term structure of risk for stock excess return. The mean reversion of stock returns caused by dividend yield variable is still weakened by the per se mean aversion from nominal yield on T bill. However we can observe the changes for the risk of bond excess return and T bill return. The mean aversion of T bill returns follows flatter pattern and therefore the risk of T bills at long horizon is much smaller than in previous model. However the largest change to previous model is in the term structure of bond excess returns. Instead of mean reversion observed in figure (11), the real returns on bond follow the mean aversive pattern. The annualized standard deviation starts at less than 6% for one quarter horizon and then continuously rises, reaching more than 7% in 50 years investment horizon. Figure (14) shows the new correlation structure.

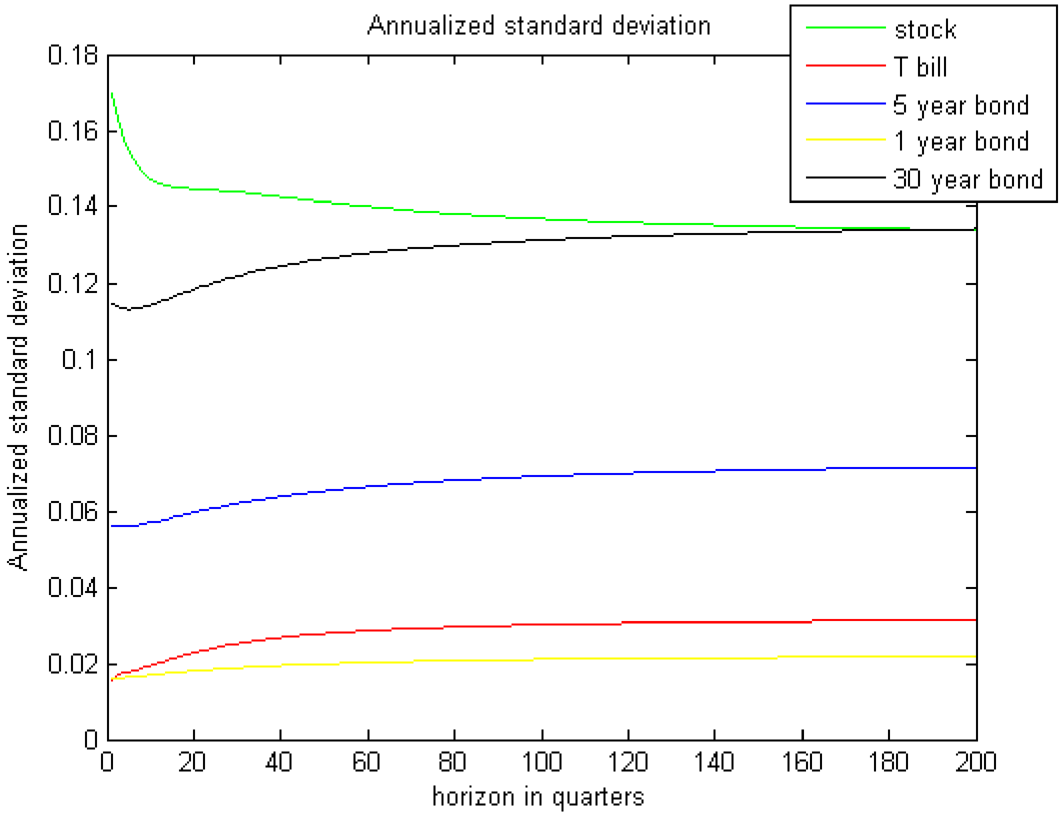
Figure (14): Correlation of Real Returns



Source: Own calculation

We can see almost no change of the term structure of correlation between excess bond and stock returns. The decrease of correlation with increasing investment horizon is not as steep for T bill-bond correlation, it reaches -0.1 at 40 years horizon. The T bill-stock correlation seems to follow the same pattern as previous model up to 15 years horizon, but then it continues to decrease as opposed to the previous model. The resulting correlation is by 0.2 lower at 50 years investment horizon compared to previous model, where we did not exclude insignificant variables. The final conclusion of the changes of correlation term structure is that the new model makes the stock more attractive asset for portfolio composition. The following figures show the term structure of risk for bond excess return for different maturity bonds. We used one shorter maturity- one year bond and one long maturity- 30 years bond.

Figure (15): Annualized Standard Deviation for 3 bonds

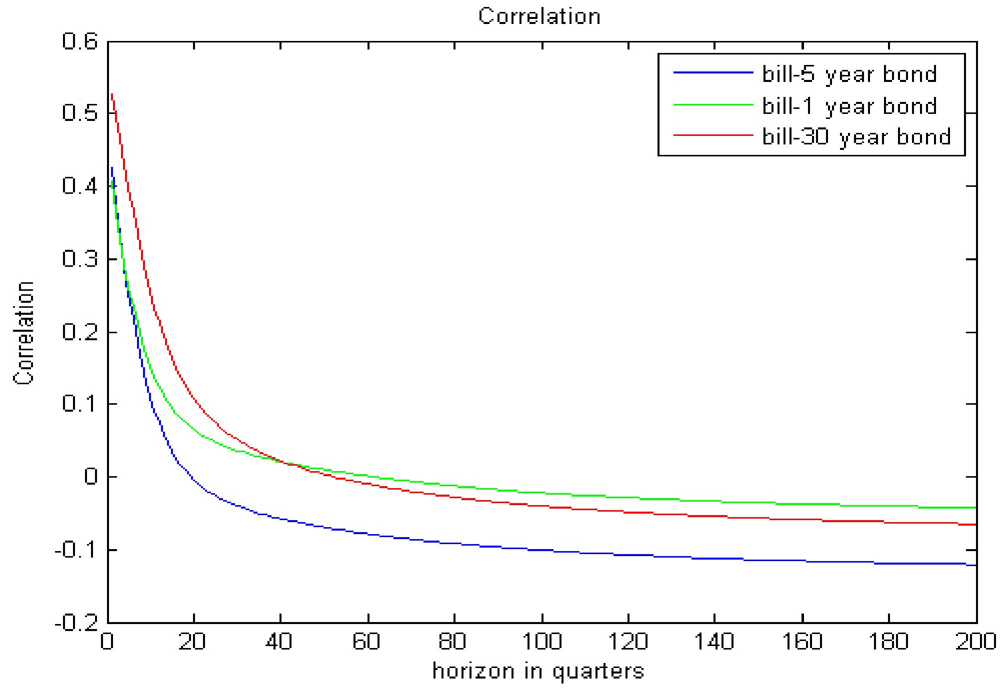


Source: Own calculation

We can see that risk of 30 years bond starts at around 12%, but rises as investment horizon increases. In the long run, it converges to the risk of stocks. From about 30 year horizon, the risk is almost the same, which is favorable for stocks, because they have higher return[17]. Interesting result is that one year bond is risky as T-bill in short horizon, but less risky in longer horizon. Here we have to keep in mind again that in case of one year bond, it is the risk of excess log return, not the gross return. The figure only says that in long horizon, the term premium of one year bond is less volatile than the T-bill return.

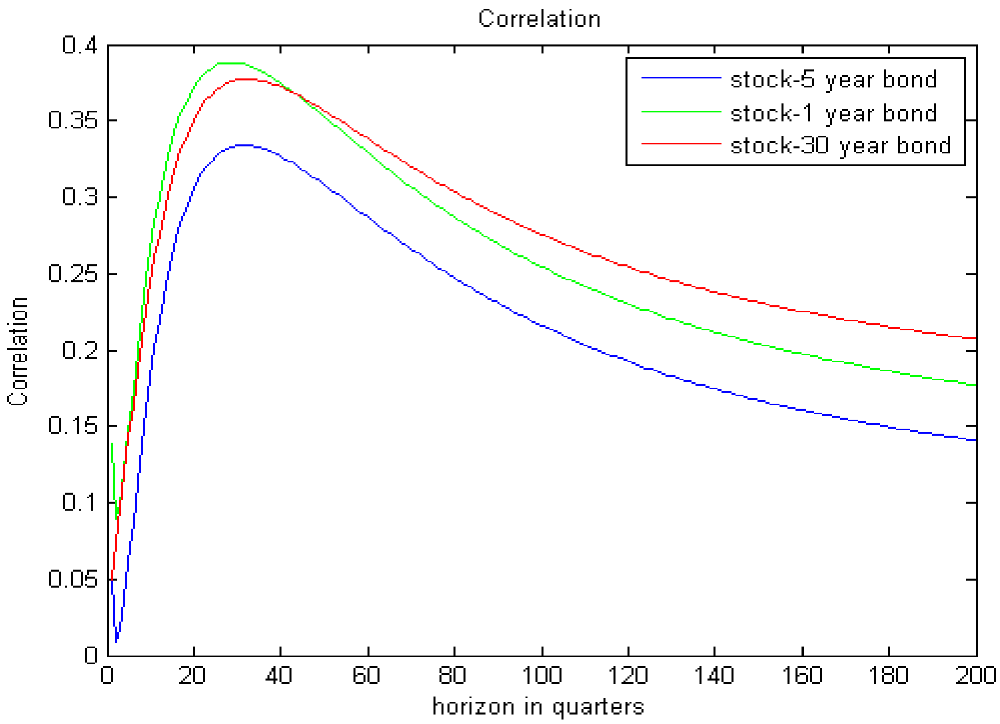
[17] Table 4

Figure (16): Correlation of T bill Returns and 3 Bonds



Source: Own calculation

Figure (17): Correlation of Stock Returns and 3 Bonds



Source: Own calculation

We can see that the correlation of stock excess returns and bill returns with different bond returns behave in the same manner. Correlation with T-bill returns sharply decreases in short horizons and it continues to decrease in long horizons, but in lower speed. Also the correlation with stock excess returns is similar for different maturities of bonds. The highest correlation is in medium horizon.

4. Extended empirical research

In previous section, we investigated the term structure of risk return tradeoff for three different asset classes. We will try to extend the model by some other classes and also try to run the same model for European data. First we try to extend the model by Real Estate Investment Trust (REIT) and then by Hedge fund returns as another asset class. Of course, both the REIT and Hedge funds are already included in stocks as a more general asset class. As a result of that, we will face the problem of endogeneity. However those 2 asset classes represent only a little fraction of S&P 500 thus we will tolerate this problem for the purposes of this analysis. The returns on REIT are selected, because they are representing also the land as another factor of production and Hedge funds are selected, because they might give our model some extra information, since they often try to time the market, thus it can improve the predictability.

4.1. Extension by REIT

We used FTSE/NAREIT index to represent the real estate index series. The data were taken from Thomson Reuters Datastream. The data were available from the first quarter of 1972, but since we created returns out of the index, the first return available is in second quarter of 1972. We had to change the sample period for other asset classes as well, so the sample period starts in Q2 1972 and ends in Q4 2009. Having the new asset class, we can run the same VAR model as in previous section using a shorter sample period. Table (10) and (11) show the VAR estimation results.

Table 10: Extension by REIT- coefficients on lagged variables

		1	2	3	4	5	6	7	R adj
1	log bond excess returns	-0.1362	0.2974	0.288	-0.581	-0.062	0.0399	0.021	0.02
	t statistics	-1.3	0.876	1.14	-0.753	-2.113	0.115	0.657	
2	log real T-bill rate	0.0634	0.0481	0.3116	1.283	-0.008	-0.426	0.0079	0.29
	t statistics	2.228	0.522	4.538	6.112	-0.945	-4.504	0.914	
3	log yield spread	0.0103	-0.013	0.9309	0.189	-0.011	-0.019	0.0039	0.77
	t statistics	0.459	-0.176	17.16	1.144	-1.776	-0.26	0.574	
4	log nominal yield on T-bill	0.0027	0.0052	-0.013	0.901	0.0051	0.031	0.000023	0.89
	t statistics	0.352	0.204	-0.684	15.582	2.311	1.191	0.01	
5	log stock excess returns	0.2488	0.3207	-0.292	-4.602	0.0701	2.484	0.0692	0.04
	t statistics	0.807	0.321	-0.393	-2.025	0.798	2.424	0.742	
6	log dividend yield	-0.0091	-0.0367	0.026	0.151	-0.0018	0.9135	-0.0026	0.95
	t statistics	-0.863	-1.072	1.018	1.93	-0.589	25.853	-0.815	
7	log REIT excess returns	0.6057	0.296	1.007	-1.826	0.3839	0.6904	-0.03	0.25
	t statistics	2.239	0.338	1.544	-0.917	4.987	0.768	0.081	

Source: Own calculation

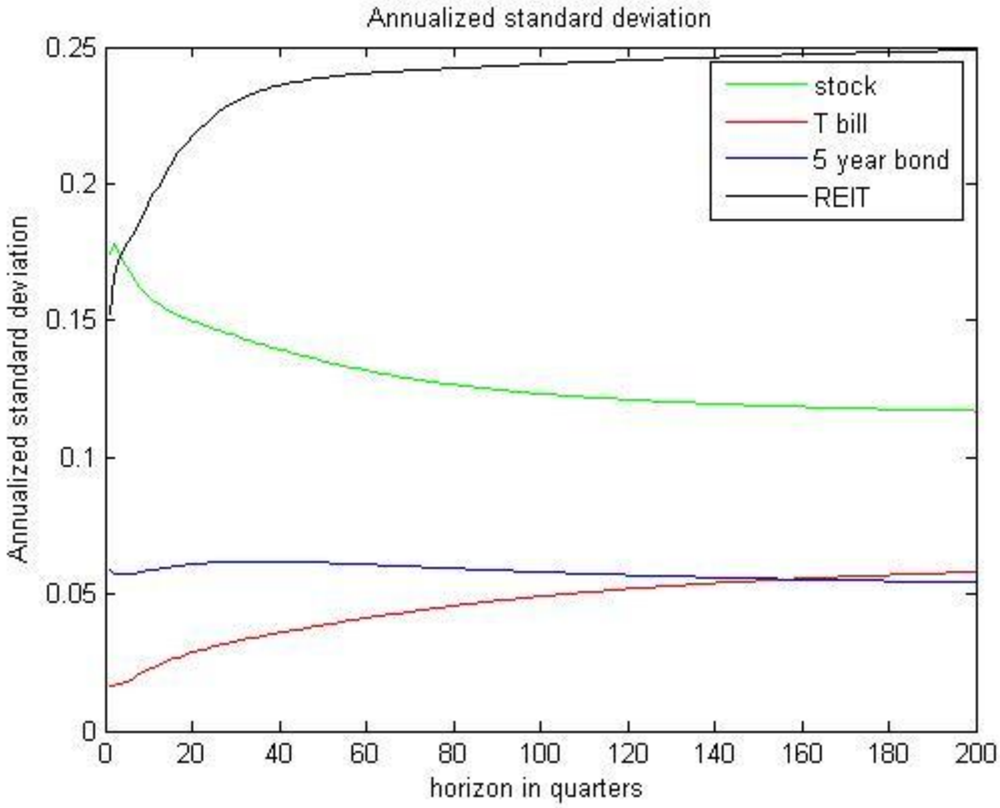
Table 11: Extension by REIT- covariance matrix

		1	2	3	4	5	6	7
1	log bond excess returns	8.75E-04	1.14E-04	-1.10E-05	-4.17E-05	2.25E-05	-2.47E-05	-0.00036
2	log real T-bill rate	1.14E-04	6.47E-05	-1.14E-06	-5.91E-06	8.27E-05	-5.89E-06	-0.00016
3	log yield spread	-1.10E-05	-1.14E-06	4.03E-05	-9.00E-06	5.93E-05	-3.24E-06	2.28E-05
4	log nominal yield on T-bill	-4.17E-05	-5.91E-06	-9.00E-06	4.91E-06	-1.00E-05	1.74E-06	1.98E-05
5	log stock excess returns	2.25E-05	8.27E-05	5.93E-05	-1.00E-05	7.59E-03	-2.17E-04	0.00198
6	log dividend yield	-2.47E-05	-5.89E-06	-3.24E-06	1.74E-06	-2.17E-04	9.02E-06	-3.56E-05
7	log REIT excess reurn	-0.00035	-0.00016	2.28E-05	1.98E-05	0.001977	-3.56E-05	0.0058

Source: Own calculation

Having the coefficients on lagged variable and variance-covariance matrix, we can show the results of the term structure of risk. We do not care about the significance of variables as in section 3.2.3. Figure (18), (19) and (20) show the results of the term structure of standard deviation and correlation for our extended model.

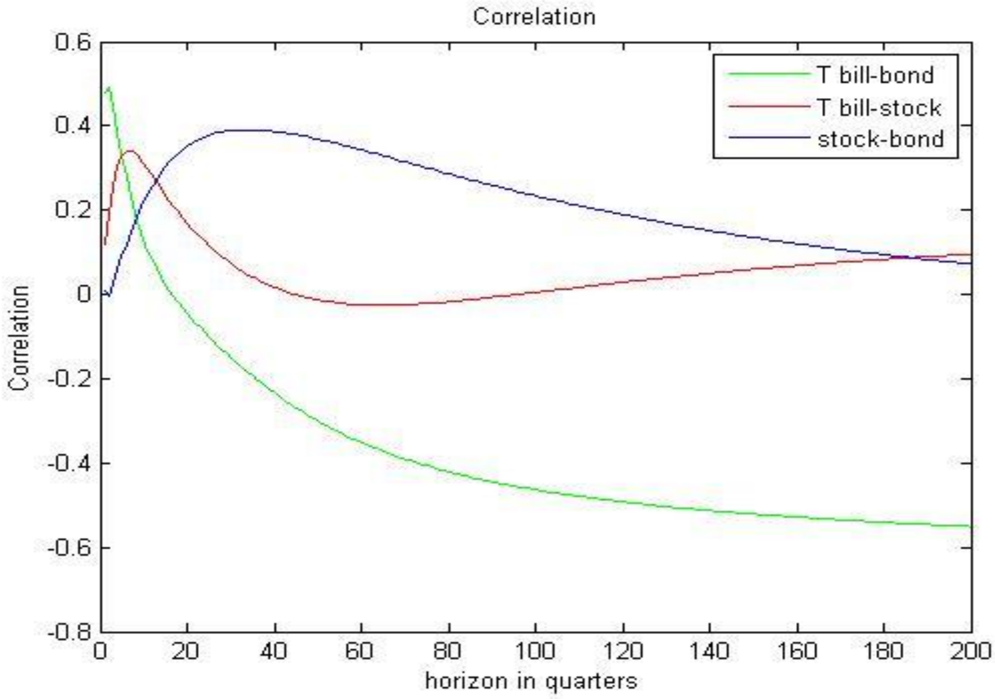
Figure 18: Extension by REIT- Annualized standard deviation



Source: Own calculation

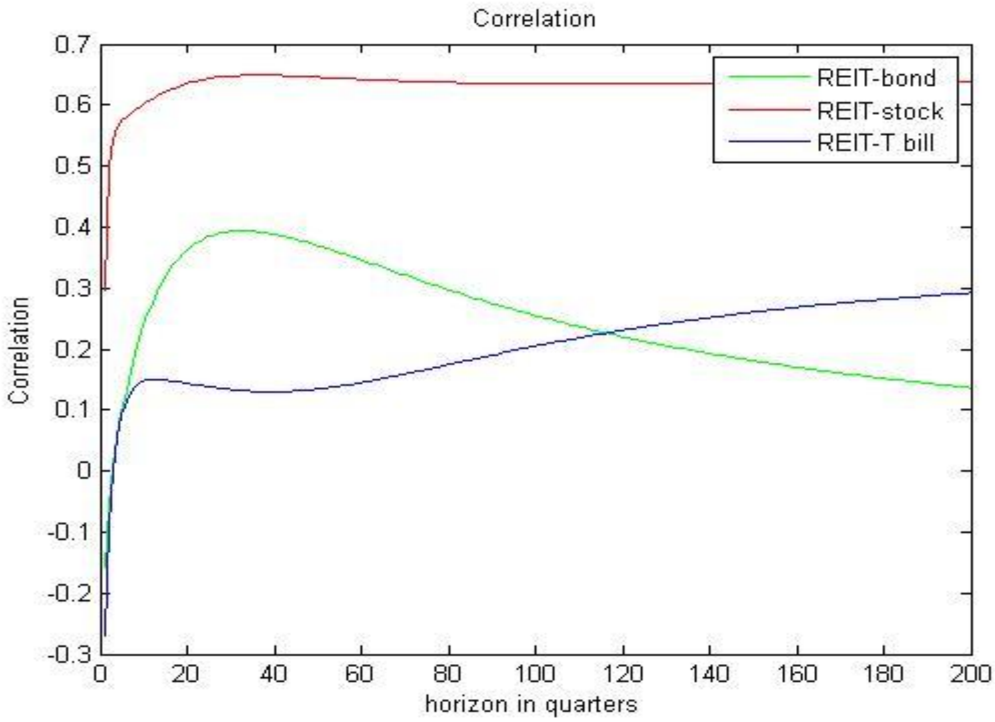
We can see that the risk of log excess bond returns decreased slightly compared to the non extended model. Also notice that the horizon effect of T-bills is much stronger, from 40 year horizon it even grows over the risk of bonds. The risk of REIT starts slightly under the risk of stocks, however we can observe the sharp mean aversion in REIT returns thus at long horizons, the risk of REIT is much larger than risk of stocks. Interesting is that inclusion of REIT helped the predictability of stock returns and therefore we can observe stronger mean reversion than in previous model. The annualized standard deviation of log stock excess returns starts at 17%, but then decreases to around 11% in 50 years horizon. These results make stocks more attractive asset class especially at the expense of Treasury bills.

Figure 19: Extension by REIT- Correlation



Source: Own calculation

Figure 20: Extension by REIT- Correlation of REIT with other asset classes



Source: Own calculation

We can see only minor differences of correlation between stocks and bonds compared to the previous model. The correlation between T-bills and stocks slightly increased. The difference in correlation between T-bills and bonds is also small. It only slightly decreased in long horizons. The correlation between REIT and stock returns starts at 0.3, but then it jumps to 0.6 and stays almost constant across all investment horizons. The correlation between REIT excess returns and T-bill returns starts at -0.3, then it sharply rises, reaching 0.15 in one year horizon. We can see that the correlation remains stable in medium horizon and then it continues rising, reaching 0.3 in 50 years horizon. The correlation between REIT and bond excess returns starts at -0.15 then it rises, reaching the top of 0.4 in medium horizon, but later it starts to decrease to 0.12 approximately in 50 years horizon. This correlation is driven mainly by the yield on T-bill in medium term and by dividend yield in the long run. The relationship is similar as the correlation between stocks and bonds. We will run the same model neglecting the insignificant variables now. The procedure is the same as in section 3.2.4. Table (12) and (13) give us the VAR estimation results when only significant variables are included.

Table 12: Extension by REIT- coefficients on lagged variables (significant)

		1	2	3	4	5	6	7	R-squared adjusted
1	log bond excess returns	0	0	0.391	0	-0.054	0	0	0.03
	t statistics			2.051		-1.939			
2	log real T-bill rate	0.0671	0	0.329	1.329	0	-0.441	0	0.3
	t statistics	2.862		5.572	7.592		-5.185		
3	log yield spread	0	0	0.913	0	-0.01	0	0	0.78
	t statistics			22.253		-1.761			
4	log nominal yield on T-bill	0	0	0	0.95	0.005	0	0	0.9
	t statistics				35.876	2.291			
5	log stock excess returns	0	0	0	-4.709	0	2.385	0	0.06
	t statistics				-3.002		2.978		
6	log dividend yield	0	0	0	0.122	0	0.929	0	0.95
	t statistics				2.238		33.415		
7	log REIT excess return	0.734	0	1.286	0	0.3698	0	0	0.26
	t statistics	3.498		2.604		5.138			

Source: Own calculation

Table 13: Extension by REIT- Covariance matrix (significant)

		1	2	3	4	5	6	7
1	log bond excess returns	8.93E-04	1.16E-04	-1.28E-05	-4.22E-05	1.11E-05	-2.43E-05	-3.52E-04
2	log real T-bill rate	1.16E-04	6.53E-05	-1.00E-06	-5.92E-06	8.27E-05	-5.97E-06	-1.66E-04
3	log yield spread	-1.28E-05	-1.00E-06	4.14E-05	-9.02E-06	6.24E-05	-3.32E-06	1.81E-05
4	log nominal yield on T-bill	-4.22E-05	-5.92E-06	-9.02E-06	4.99E-06	-8.94E-06	1.66E-06	2.08E-05
5	log stock excess returns	1.11E-05	8.27E-05	6.24E-05	-8.94E-06	7.73E-03	-2.23E-04	1.96E-03
6	log dividend yield	-2.43E-05	-5.97E-06	-3.32E-06	1.66E-06	-2.23E-04	9.33E-06	-3.55E-05
7	log REIT excess return	-3.52E-04	-1.66E-04	1.81E-05	2.08E-05	1.96E-03	-3.55E-05	5.88E-03

Source: Own calculation

The log bond excess returns are predicted by the yield spread and log stock excess returns in this model. The log real T-bill rate is predicted by log bond excess returns, log yield spread, long nominal yield on T-bills and log dividend yield. The log stock excess returns are being predicted by log dividend yield and log nominal yield on T-bill. The log REIT excess return is predicted by log bond excess returns, log yield spread and log stock excess returns.

It is interesting that in slopes 2 and 7 of table (12) are only zeros. It means that log real T-bill rate and log REIT excess return are not used as a predictor for any variable. The question is whether the results of the model can change if the REIT returns do not enter the equations for predictability. One might say that the change in results is only due to the different sample period. However we can simply check the equation 8 and we will find out that even when the matrix Φ_1 is not of full rank, the covariance matrix of unexpected shocks enter the equations for finding the multiple period risk. Thus the change in results might be due to shorter sample period or due to including new asset class.

Before turning to the result of the term structure, we have to investigate the assumptions of the model. We have to check for autocorrelation, homoscedasticity, normality of disturbances and multicollinearity as in section 3.2.4. The test for autocorrelation of residuals was done by Durbin-Watson test and there was no autocorrelation found. The homoscedasticity was tested by Goldfeld-Quandt, the normality of residuals by Kolmogorov-Smirnov and multicollinearity by the condition number. Table (14) concludes the results of testing the OLS assumptions.

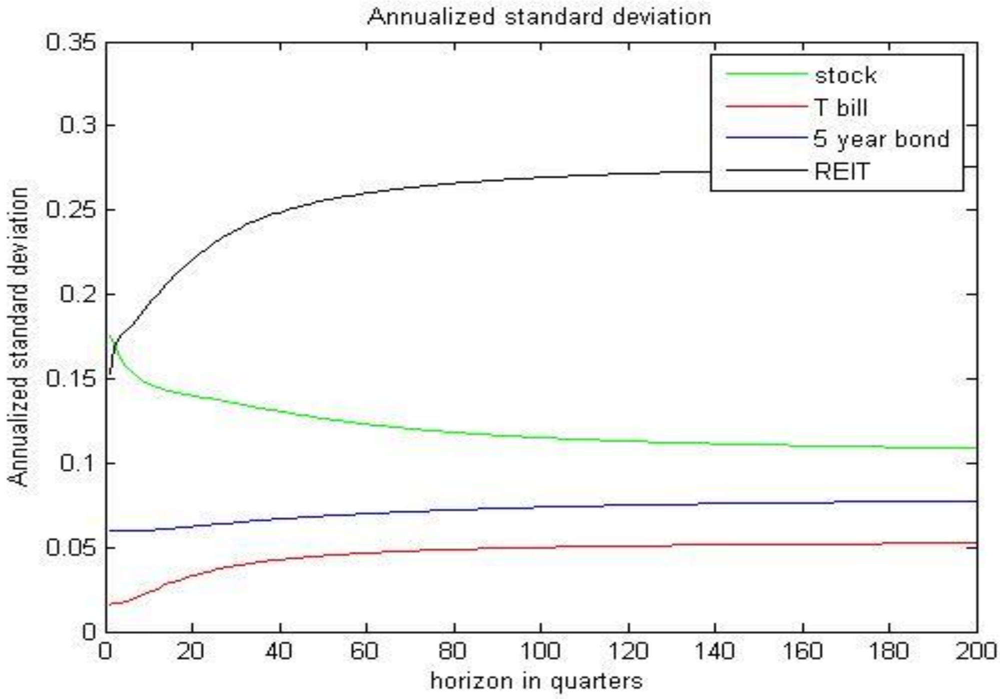
Table 14: Testing the assumptions (extended model)

Test	Goldfeld-Quandt	Kolmogorov-Smirnov	condition number
log bond excess returns	p=0.978	p=0.43	5.19
log real T-bill rate	p=0.027	p=0.34	10.39
log yield spread	p=0.99	p=0.25	5.19
log nominal yield on T-bill	p=1	p=0.0008	4.88
log stock excess returns	p=0.52	p=0.344	11.23
log dividend yield	p=1	p=0.076	11.23
log REIT excess returns	p=0.11	p=0.95	5.18

Source: Own calculation

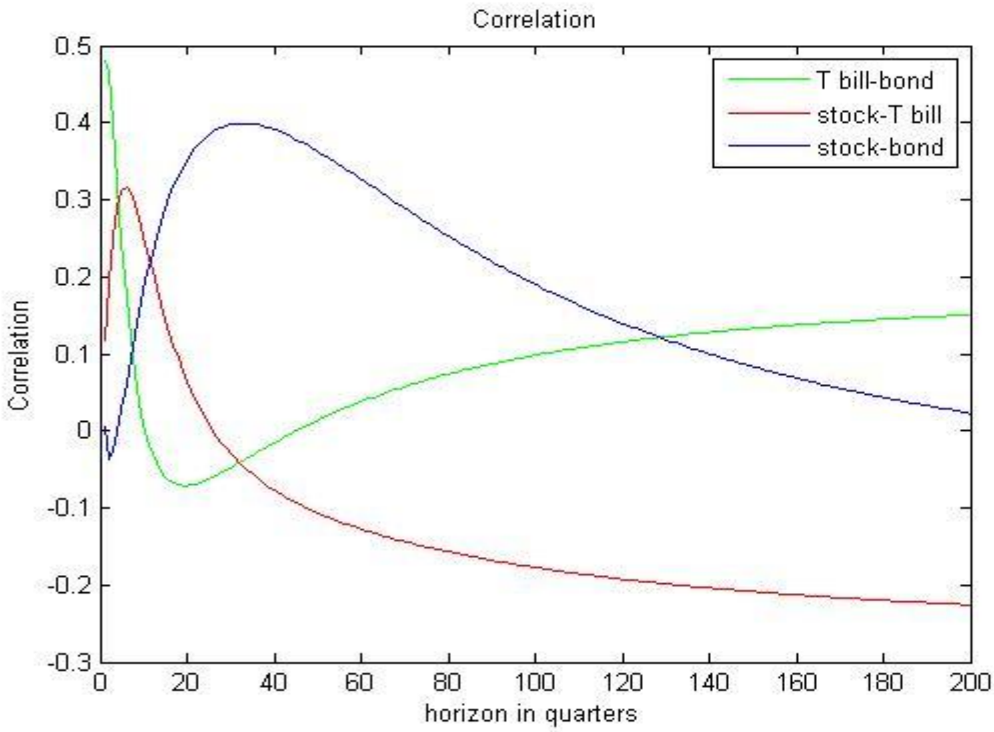
We can see that the results of testing the assumptions are very similar as in section 3.2.4. We only used generalized least squares for log real T-bill rate to correct for heteroscedasticity. We kept OLS estimator for log nominal yield on T-bill. We only have to keep in mind that due to no normality of disturbances, the estimator is only best among the class of linear unbiased estimators. Figure (21), (22) and (23) show the results of the term structure of standard deviation and correlation for our extended model. We can see that the term structure of standard deviation does not change much after elimination of insignificant variables. Only the risk of 5 year bond rises in long horizon to 8%. Generally including REIT as an asset class helped to reduce the risk of stocks and on the other hand increased the risk of bonds and T-bills. This makes the stocks even more attractive asset. The annualized standard deviation of log stock excess returns starts at 17%, but decreases continuously to 10.5% for 50 year horizon. The risk of T-bill returns starts at 2% and rises to 5% in long horizons. The risk of 5 year bond starts at 6% in one quarter horizon, but then it rises to 8% in 50 years horizon. The strong rise of volatility in REIT returns is the result of mean averting behavior caused by the positive predictability of REIT by stock returns and the positive correlation of shocks to the stock returns and unexpected REIT returns. As we can see, the risk starts at 15% and rises to 27.5% in 50 years horizon.

Figure 21: Extension by REIT- Annualized standard deviation (significant)



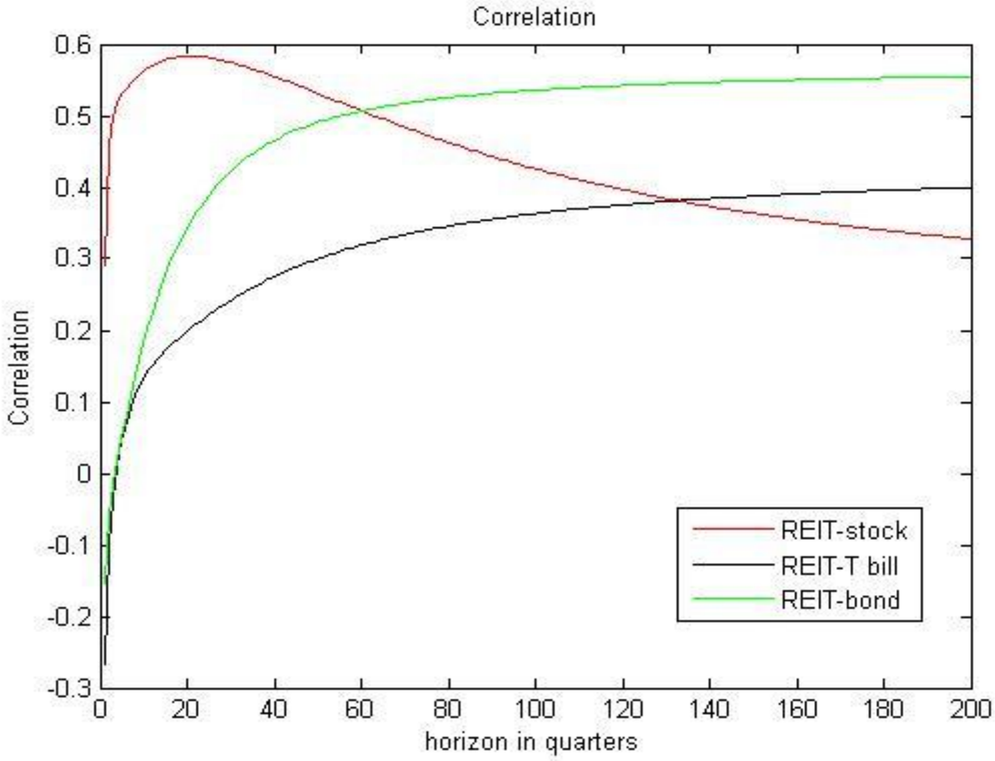
Source: Own calculation

Figure 22: Extension by REIT- Correlation (significant)



Source: Own calculation

Figure 23: Extension by REIT- correlation of REIT with other asset classes (significant)



Source: Own calculation

We can notice that including REIT into the model almost did not change the correlation between bond and stock returns. Only in medium run, it increases more than in previous model, reaching the top of 0.4 in 10 years horizon. There is also almost no change for the correlation between T-bill returns and stock excess returns. The only significant change is in correlation between T-bill and bond returns. It starts at almost 0.5 in one quarter horizon then it decreases sharply to -0.08 in 5 years horizon, but then it starts to rise again reaching 0.15 in 50 years horizon. The correlation between bond returns and REIT returns rises across all investment horizons which is in medium term due to the fact that both REIT and bonds react to the yield spread positively, but the change in bond returns is faster, thus increase in correlation will reveal later. Changes in correlation between REIT and T-bill returns are also driven mainly by the yield spread. The correlation starts at -0.15 but rises to 0.4 in 50 years horizon. The correlation between REIT excess returns and stock returns rises from 0.3 to 0.6 in first five years but then it slowly decreases back to 0.3 in long investment horizons.

So we already investigated the term structure of the risk return tradeoff using the basic VAR model and also the extended version of the model. So far we can conclude that the term structure of risk makes the stocks more attractive as an asset class for long term investor compared to the myopic investor. In next section we will try to extend the model by hedge funds as another asset class.

4.2. Extension by REIT and Hedge funds

In this section we are going to add one more asset class. Thus we extend three original asset classes (bonds, T-bills and stocks) by REIT and hedge fund returns. We calculated the hedge fund returns from DJ CS Hedge Fund Index. The data were taken from Thomson Reuters Datastream and were available from first quarter of 1994 for returns. We had to change the sample period for other assets such that the periods match. Thus the sample period is from Q1 1994 till Q4 2009. Table (15) and (16) conclude the results of vector autoregressive model.

Table 15: Extension by REIT and Hedge funds- coefficients on lagged variables

		1	2	3	4	5	6	7	8
1	log bond excess returns	0	0	0	0	0	0	0	0
	t statistics	0	0	0	0	0	0	0	0
2	log real T-bill rate	0	1.496	0.182	1.399	0	0	-0.088	0.032
	t statistics	0	3.657	1.698	3.872	0	0	-2.931	2.165
3	log yield spread	0	0	0.848	-1.089	0	0	0	0
	t statistics	0	0	9.44	-6.42	0	0	0	0
4	log nominal yield on T-bill	0	0	0	0	0	0	0.08	-0.042
	t statistics	0	0	0	0	0	0	2.68	-3.14
5	log stock excess returns	0	0	0	0	0	2.945	0	0
	t statistics	0	0	0	0	0	1.609	0	0
6	log dividend yield	0	0	0	0	0	0.913	0	0
	t statistics	0	0	0	0	0	23.01	0	0
7	log HEDGE excess return	0	0	0	-0.735	0	0	0.238	0
	t statistics	0	0	0	1.901	0	0	-1.327	0
8	log REIT excess return	0	0	0	0	0.408	0	0	0
	t statistics	0	0	0	0	3.28	0	0	0

Source: Own calculation

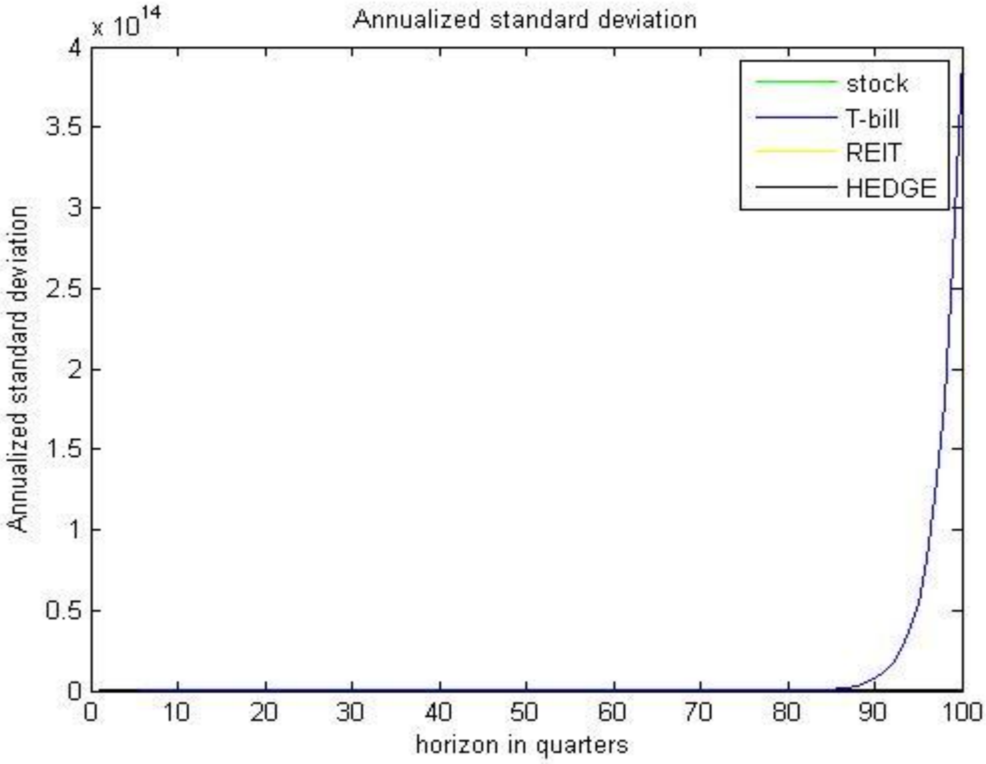
Table 16: Extension by REIT and Hedge funds- Covariance matrix

		1	2	3	4	5	6	7	8
1	log bond excess returns	6.53E-04	-1.94E-07	5.96E-06	1.08E-05	-1.26E-04	-3.15E-06	4.23E-05	6.60E-05
2	log real T-bill rate	-1.94E-07	7.16E-05	-8.28E-05	-7.30E-05	-6.16E-05	1.54E-06	-6.50E-05	-2.70E-04
3	log yield spread	5.96E-06	-8.28E-05	1.30E-04	8.49E-05	2.18E-04	-4.40E-06	1.30E-04	4.23E-04
4	log nominal yield on T-bill	1.08E-05	-7.30E-05	8.49E-05	7.86E-05	7.36E-05	-1.62E-06	6.60E-05	2.47E-04
5	log stock excess returns	-1.26E-04	-6.16E-05	2.18E-04	7.36E-05	8.52E-03	-1.20E-04	2.68E-03	2.71E-03
6	log dividend yield	-3.15E-06	1.54E-06	-4.40E-06	-1.62E-06	-1.20E-04	2.26E-06	-3.98E-05	-3.95E-05
7	log HEDGE excess returns	4.23E-05	-6.50E-05	1.30E-04	6.60E-05	2.68E-03	-3.98E-05	1.65E-03	1.47E-03
8	log REIT excess returns	6.60E-05	-2.70E-04	4.23E-04	2.47E-04	2.71E-03	-3.95E-05	1.47E-03	7.80E-03

Source: Own calculation

Unfortunately due to the small sample period the majority of variables are insignificant. For example log bond excess returns get completely out of the model. T-bill returns are predicted by its own lagged values, by yield spread, nominal yield on T-bill, hedge fund returns and REIT returns. However the outcome of the model is unrealistic as we can see in figure (24). It is especially due to huge autocorrelation in real T-bill rate. The assumptions of OLS estimator are not fulfilled in all cases. The result is that annualized standard deviation of T-bill returns grow exponentially. Hence we have to neglect this result and suggest to try this analysis later when larger sample period will be available.

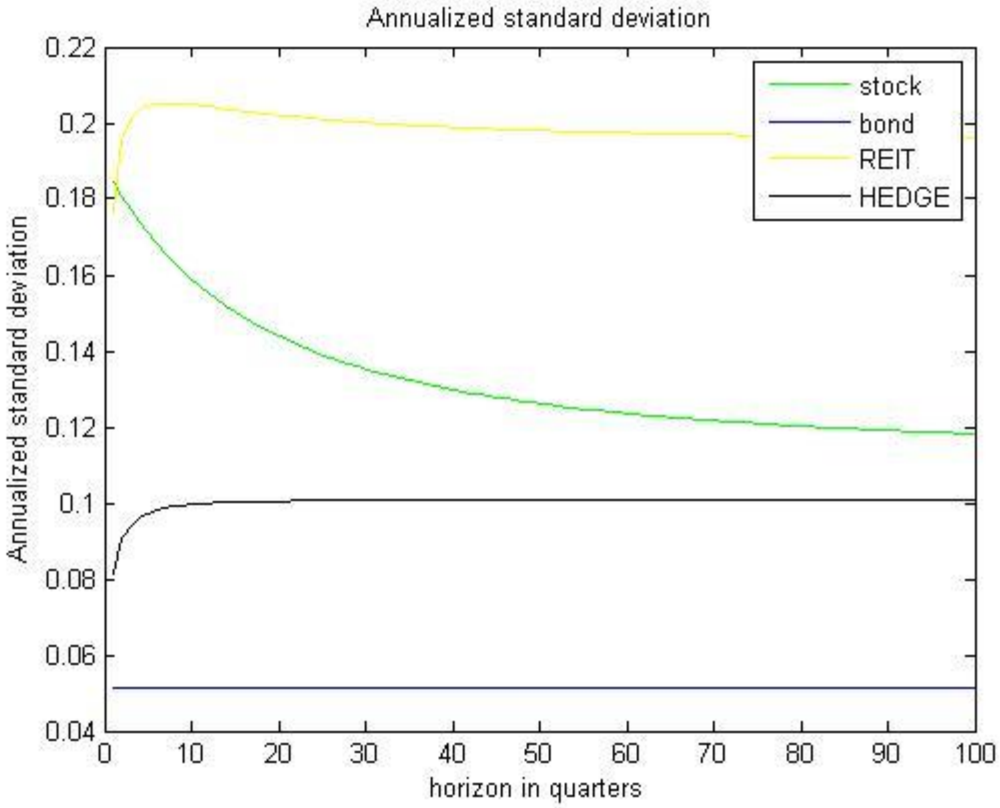
Figure 24: Extension by REIT and Hedge funds- Risk of T-bills



Source: Own calculation

The results of the term structure of risk look more reasonable for other asset classes. We can see that in figure (25). Interesting is that hedge fund returns are less risky than stocks across all investment horizons, even though we can observe mean aversion in annualized standard deviation of hedge fund returns. For our sample period also the average gross return is larger for hedge funds. The annualized gross excess return of hedge funds is 5.4% and annualized gross excess return of stocks is 5.1%. It is especially due to the performance of DJ CS Hedge Fund Index during the financial crisis. If we shorten the sample period to Q1 1994- Q4 2007, then the gross stock excess returns are 7.3% compared to 6.7% for gross excess hedge fund returns. But the performance of DJ CS Hedge Fund Index was much better during the crisis than S&P 500. This would suggest contrary to most of researches that an investor can beat the market by investing into hedge funds which would be successful in searching for superior return. However as we already mentioned the results of this analysis are not sufficient due to small sample period so we should not draw any conclusions.

Figure 25: Extension by REIT and Hedge funds- Annualized standard deviation



Source: Own calculation

4.3. Research on European data

In this section we were trying to do the same model of the term structure of risk return tradeoff as in section 3.2.4., but for European data. The reason for making this analysis is that European market is more appealing for Czech pension funds than the U.S. market. The best way would be to use the Czech capital market and money market, but problem for making this analysis was due to short sample period and also too small stock market. Therefore we tried to model the European market where we do not face any problem with stock market. However since the European Union does not have united fiscal policy, we could only rely on Euro Generic rates for bond and money market. The sample period was too small for 3 months Euro Generic rates, starting from first quarter of 1999. Thus before this date we used 3 months German Generic rate. Unfortunately it was on the account of loosing continuity around this date. The 5 years Euro Generic rate was available from third quarter of 1993 so it is the point where our sample period starts. The end of sample period is in second quarter of 2010. The data for Euro Generic rates and German Generic rates were found on Bloomberg. The data for stock returns and dividend yields at Thomson Reuters Datastream and the rates were derived from FTSE Europe index. Unfortunately the sample period was still too small to get reasonable results. As we can see in table (17) of coefficients on lagged variables, the bond excess returns and dividend yields are not predictable and coefficients for stock excess returns are unreasonably high.

Table 17: Coefficients on lagged variables (European data)

		1	2	3	4	5	6	R-squared adjusted
1	log bond excess returns	0	0	0	0	0	0	
	t statistics							
2	log real T-bill rate	0	0	0	0.796	0	0	0.26
	t statistics				4.89			
3	log yield spread	0	0	0.863	0	0	0	0.77
	t statistics			14.72				
4	log nominal yield on T-bill	0	0	0	0.93	0	0	0.9
	t statistics				25.08			
5	log stock excess returns	-24.06	33.54	22.34	-24.65	0	1.81	0.24
	t statistics	-3.03	4.01	3.36	-4.14		2.28	
6	log dividend yield	0	0	0	0	0	0	
	t statistics							

Source: Own calculation

Unfortunately we have to conclude that we were unsuccessful in making vector autoregressive model for European data and it was impossible to model the term structure of the risk-return tradeoff. We suggest doing this analysis when larger sample period for European data will be available. However for other chapters of this paper, we have to rely on results from U.S. data.

5. Optimal Portfolio Allocation

Knowing the results from previous sections, we can turn into standard mean variance analysis counting with the horizon effect in risk. We will consider many representative horizons and use the comparative analysis of the optimal portfolio choice problem. We will consider a myopic investor with one quarter investment horizon, investor with 5 years, 10 years, 25 years and finally 50 years investment horizon. We will find a global minimum variance portfolio in section 5.1., and optimal allocation with respect to risk tolerance in section 5.2. Because an investor is interested in gross returns rather than log excess returns, we transform both the expected returns and annualized standard deviations and covariances into gross returns. Only exception is finding the optimal portfolio where we treat T-bills as a riskless asset. We always use the term structure of risk for two different possibilities. One based on original VAR model counting with only significant variables and one based on extended version of VAR model counting also only with significant variables.

5.1. Global minimum variance portfolio

We can find the global minimum variance portfolio[18] by first finding the variance-covariance matrix for different horizons. This can be found by equation (8). So recall the equation 8:

$$Var_t(Z_{t+1} + .. + Z_{t+k}) = \Sigma_v + (\mathrm{I} + \Phi_1)\Sigma_v(\mathrm{I} + \Phi_1)^T + (\mathrm{I} + \Phi_1 + \Phi_1^2)\Sigma_v(\mathrm{I} + \Phi_1 + \Phi_1^2)^T + ...$$
$$+ (\mathrm{I} + \Phi_1 + ... + \Phi_1^{k-1})\Sigma_v(\mathrm{I} + \Phi_1 + ... + \Phi_1^{k-1})^T$$

Because we are interested in gross returns and not excess returns, we have to transform the variance of excess returns into variance of returns. This can be done by simple transformation:

$$\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y) + 2 \cdot \mathrm{cov}(X, Y),$$

where X+Y represents total returns, X excess returns and Y Treasury bill returns. Let us denote the resulting variance-covariance matrix by Σ and vector of weights by w. Then the optimization problem for finding minimum variance portfolio looks like this:

[18] Portfolio with the lowest risk

$$\sigma_p^2 = \min_{w^*}(w^T \cdot \Sigma \cdot w)$$

$$s.t. \sum_{i=1}^{n} w_i = 1$$

We do not allow short selling, so another condition is that every weight is equal or larger to zero. Table (18) gives us the global minimum variance portfolio for basic model and table (19) for extended version with REIT returns. They show us the weights of different asset classes, mean portfolio return, variance and standard deviation.

Table 18: Global minimum variance portfolio

horizon	bond w	bill w	stock w	mean return	variance	standard deviation
1 quarter	0.000	1.000	0.000	0.017	0.000	0.015
5 years	0.000	1.000	0.000	0.017	0.000	0.022
10 years	0.020	0.976	0.004	0.018	0.001	0.027
25 years	0.034	0.949	0.017	0.019	0.001	0.029
50 years	0.038	0.920	0.042	0.020	0.001	0.029

Source: Own calculation

Table 19: Global minimum variance portfolio (extended version)

horizon	bond w	bill w	stock w	REIT w	mean return	variance	standard deviation
1 quarter	0.000	0.973	0.000	0.027	0.018	0.000	0.014
5 years	0.041	0.959	0.000	0.000	0.018	0.001	0.035
25 years	0.000	0.933	0.067	0.000	0.021	0.002	0.047
50 years	0.000	0.896	0.104	0.000	0.022	0.002	0.050

Source: Own calculation

We can see that in short horizon the global minimum variance portfolio consists almost entirely from Treasury bills. It is not surprising since T-bills are often being considered as the riskless asset. But T-bills are not inflation indexed thus they are not a real riskless asset. In short term they carry only inflation risk, but at long horizons they face also the reinvestment risk and therefore their risk is increasing across the horizons. The global minimum variance portfolio consists of only around 90% of T-bills in 50 years horizon, shifting the weight towards stocks. Even though this result is interesting, it is not as strong result as in Campbell, Viceira (2005), where they found that in 50 years horizon, 5 year bonds represents 38%, stocks 16% and T-bills only 46% of global minimum variance portfolio. The possible differences of results are described earlier[19].

5.2. Optimal allocation with respect to risk tolerance

In this subsection we are going to try to find optimal portfolio at different investment horizons. In order to do so, we will have to select a demand function for representative investor. This demand function will define the investors willing tradeoff between risk and expected return. This subsection is structured as follows. First we will try to find the highest cut off point (5% significance level) in order to minimize value at risk[20]. Then we will find different optimal portfolios according to horizon and investors risk profile and finally we will find the tangency portfolio, treating Treasury bills as a riskless asset.

5.2.1. Minimizing Value at Risk

Value at Risk can be defined as monetary loss relative to the mean or absolute monetary loss. Here we will try to minimize the value at risk as absolute monetary loss for different investment horizons. It is equivalent to finding the maximum cut off point (quantil) at 5% confidence level. As we foreshadowed in section 3.1, the multiperiod risk does not grow linearly. The multi period risk can be expressed by formula[21]:

$$\sigma_{re_{t+n}^{(n)}} = \sqrt{n} \cdot \sigma_e ,$$

where $\sigma_{re_{t+n}^{(n)}}$ is the standard deviation of not annualized returns and σ_e is standard deviation of one period returns. Table (20) and (21) give us the minimal Value at Risk portfolio for basic model and the extended model respectively.

[19] It might be slightly different methodology, different sample period or usage of different index.
[20] Defined as absolute monetary loss (-cut off point*initial value)
[21] If i.i.d. and normal

Table 20: Minimal Value at Risk portfolio

horizon	bond w	bill w	stock w	mean return	variance	standard deviation	cut off point
1 quarter	0.000	1.000	0.000	0.017	0.000	0.015	-0.008
5 years	0.095	0.823	0.082	0.023	0.001	0.027	0.015
25 years	0.228	0.000	0.772	0.058	0.012	0.112	0.669
50 years	0.000	0.000	1.000	0.066	0.017	0.129	4.752

Source: Own calculation

Table 21: Minimal Value at Risk portfolio (extended version)

horizon	bond w	bill w	stock w	REIT w	mean return	variance	standard deviation	cut off point
1 quarter	0.000	0.973	0.000	0.027	0.018	0.000	0.014	-0.007
5 years	0.211	0.696	0.094	0.000	0.025	0.002	0.040	-0.023
25 years	0.083	0.000	0.917	0.000	0.063	0.013	0.114	0.842
50 years	0.000	0.000	1.000	0.000	0.065	0.012	0.109	6.062

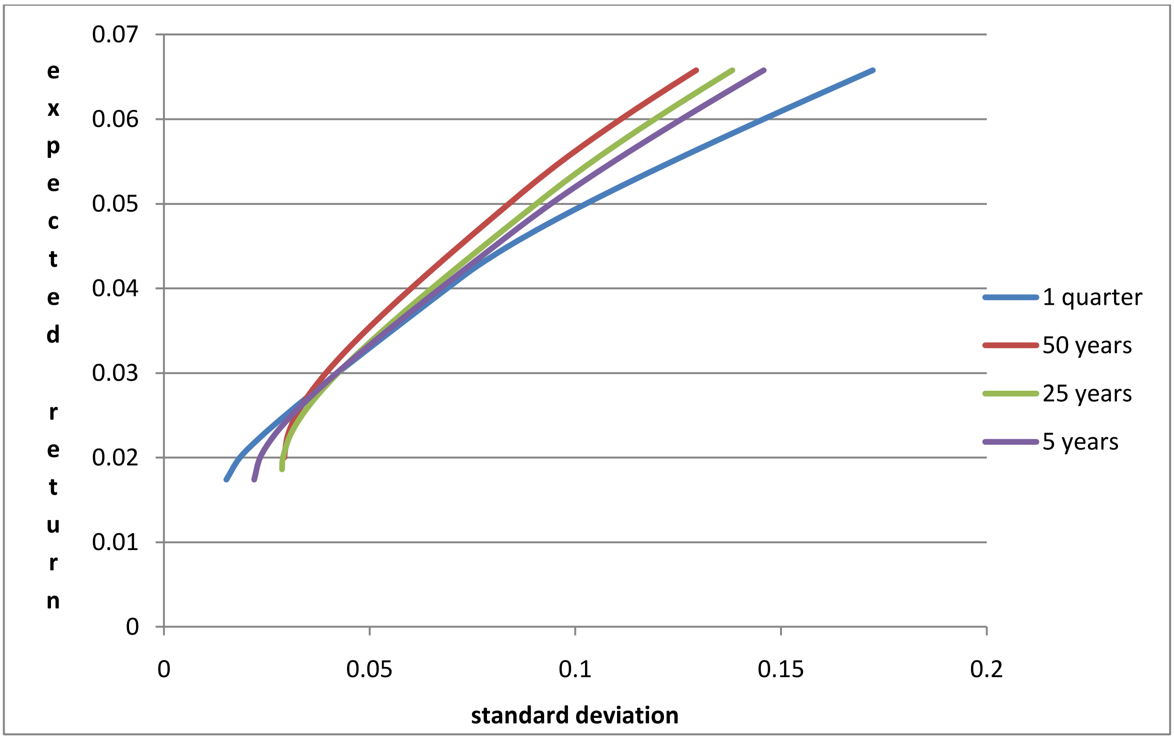
Source: Own calculation

We can see that with increasing investment horizon, the Minimal Value at Risk portfolio leans much more toward stocks and bonds and later only for stocks. This characteristic is typical even without the presence of predictability of asset returns and its implications for the term structure of risk-return tradeoff. However we have to note that this applies only to buy and hold investor for entire period. It is unrealistic that institutional investor can hold its portfolio for 50 years unchanged. But it is conceptually appealing if we divide the portfolio into smaller portfolios and match each to liabilities with different maturities. In practice we can see that for example life cycle funds offer more aggressive (stock oriented) strategy for young people with larger investment horizon and then during the time when people get older, closer to retirement, the strategy changes towards safer short term securities.

5.2.2. Optimal asset allocation

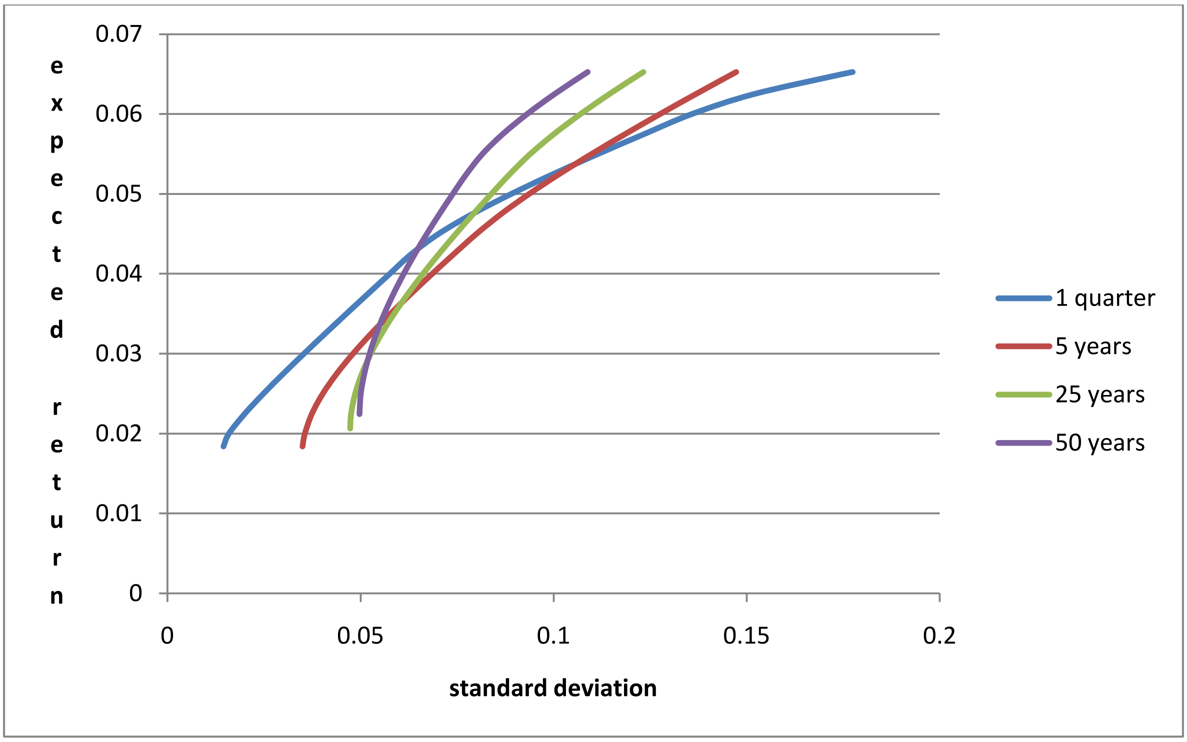
In previous subsection we showed the multiple period risks and the minimization of Value at Risk. Here we return to the annualized portfolio risk and investigate on mean variance analyses how the predictability of asset returns change the optimal portfolio selection with respect to the investment horizon. In order to do so, we have to select the preferred risk-return tradeoff of representative investor. We selected three different investors according to their risk tolerance. All of the investors are risk averse, but they differ in their risk averseness. One representative investor is aggressive, one medium and one moderate. Note that we work with the gross returns and its associated risks, because T-bill is treated as risky in this subsection. Before we introduce the utility functions of representative investors, we will show how the efficient frontiers look like for different investment horizons. Figures (26) and (27) display the efficient frontiers for the basic model and extended model by REIT respectively.

Figure 26: Efficient frontiers



Source: Own calculation

Figure 27: Efficient frontiers (extended version)



Source: Own calculation

As we can see, for both versions of the model, with increasing investment horizon the efficient frontier becomes steeper. It means that an investor with longer investment horizon can achieve higher return in exchange for lower increase in risk than it is for short horizon investor. This effect is much larger for the model that is extended by REIT, especially because of larger mean reversion in stock returns.

In order to find an optimal portfolio, we need to know investor's risk tolerance. Therefore we define three different utility functions for representative investors as mentioned earlier. Let us denote aggressive investor as investor one, the medium investor as investor two and moderate investor as investor three. For every investor and every horizon we need to find the point on efficient frontier, where the slope of efficient frontier is the same as the indifference curve of the representative investor. Assume that the slopes of the indifference curves look like following:

$$\frac{\partial E_1}{\partial \sigma_1} = 0.25 + 2 \cdot \sigma$$

$$\frac{\partial E_2}{\partial \sigma_2} = 0.3 + 6 \cdot \sigma$$

$$\frac{\partial E_3}{\partial \sigma_3} = 0.4 + 10 \cdot \sigma$$

Symbol E represents expected return, symbol σ standard deviation and numbers 1, 2 and 3 different representative investors. Now we have to find for every horizon the appropriate slope of the efficient frontier. We were not finding the slope in a right sense, but we approximated the slope by change of the standard deviation, where expected return is changing by 0.25%. Thus we are looking for points where these relationships approximately hold:

$$\frac{0.0025}{\Delta \sigma} = 0.25 + 2 \cdot \sigma$$

$$\frac{0.0025}{\Delta \sigma} = 0.3 + 6 \cdot \sigma$$

$$\frac{0.0025}{\Delta \sigma} = 0.4 + 10 \cdot \sigma$$

Tables (22) and (23) display the resulting composition of optimal portfolios for three different representative investors and four different investment horizons. Of course with increasing horizon the weight of T-bills usually decreases and weight of stocks increases. The same holds for decreasing risk aversion. In our basic model represented in table (22), the optimal weight for T-bills for moderate myopic investor is 88.4% whereas optimal weight for medium investor with 25 years investment horizon is 61.9%. Even larger differences are for the extended version of the model. Important is that for any investor the horizon effect moves the optimal weights towards stocks and in majority of the cases decreases the weight of T-bills. Not so clear results are for bonds which sometimes increase and sometimes decrease with the investment horizon.

Table 22: Optimal asset allocation

investor	horizon	bond w	bill w	stock w	mean return	variance	standard deviation
moderate	1 quarter	0.055	0.884	0.061	0.021	0.000	0.020
	5 years	0.095	0.823	0.082	0.023	0.001	0.027
	25 years	0.121	0.721	0.158	0.026	0.001	0.036
	50 years	0.119	0.702	0.179	0.028	0.001	0.035
medium	1 quarter	0.101	0.819	0.080	0.023	0.001	0.024
	5 years	0.153	0.707	0.140	0.026	0.001	0.033
	25 years	0.160	0.619	0.221	0.030	0.002	0.042
	50 years	0.172	0.559	0.269	0.033	0.002	0.044
aggressive	1 quarter	0.559	0.164	0.277	0.038	0.004	0.062
	5 years	0.386	0.243	0.371	0.040	0.005	0.067
	25 years	0.316	0.213	0.471	0.044	0.006	0.075
	50 years	0.358	0.057	0.585	0.050	0.007	0.084

Source: Own calculation

Table 23: Optimal asset allocation (extended version)

investor	horizon	bond w	bill w	stock w	REIT w	mean return	variance	standard deviation
moderate	1 quarter	0.0685	0.8263	0.0294	0.0758	0.0225	0.0004	0.0200
	5 years	0.2106	0.6956	0.0937	0.0000	0.0250	0.0016	0.0402
	25 years	0.0933	0.6413	0.2654	0.0000	0.0315	0.0029	0.0542
	50 years	0.1592	0.4699	0.3709	0.0000	0.0375	0.0034	0.0586
medium	1 quarter	0.2046	0.6173	0.0662	0.1118	0.0275	0.0009	0.0300
	5 years	0.3150	0.5185	0.1665	0.0000	0.0300	0.0023	0.0480
	25 years	0.1737	0.4599	0.3664	0.0000	0.0375	0.0039	0.0623
	50 years	0.3321	0.0883	0.5796	0.0000	0.0500	0.0055	0.0739
aggressive	1 quarter	0.5483	0.0000	0.2167	0.2350	0.0440	0.0045	0.0671
	5 years	0.5761	0.0756	0.3484	0.0000	0.0425	0.0055	0.0742
	25 years	0.2786	0.0000	0.7214	0.0000	0.0560	0.0093	0.0963
	50 years	0.2184	0.0000	0.7816	0.0000	0.0580	0.0078	0.0881

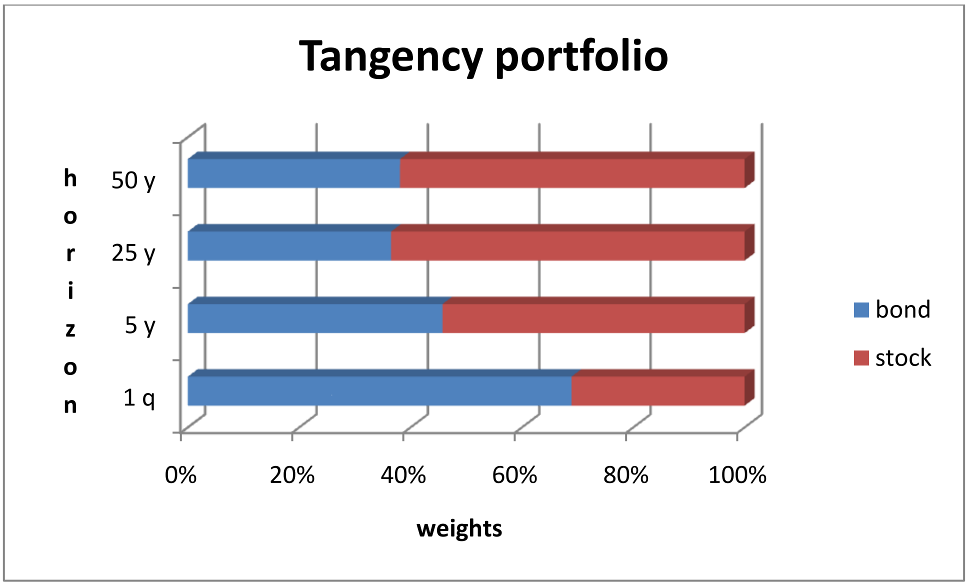
Source: Own calculation

We treated Treasury bills as risky asset in this subsection therefore we were working with gross returns. We could not find the tangency portfolio, because we did not have the risk free rate. We will try to find tangency portfolio in next subsection saying that Treasury bills are the riskless asset.

5.2.3. Tangency portfolio

Following the standard practice from many studies, we treated Treasury bills as riskless asset in this subsection. Having the risk and returns for all investment horizons and the risk free rate, we can construct the security market line and also the tangency portfolio. Figure (28) shows the weights of stocks and bonds in tangency portfolio for different investment horizons. Figure (29) does the same for extended version of the model, thus it includes also weight of Real Estate Investment Trust in the model. Because we treat T-bills as riskless asset, the volatility of T-bills is supposed to be zero. Hence we need to change back the gross returns of other assets into gross excess returns. We have to do the same with the variance-covariance matrix and corresponding correlation.

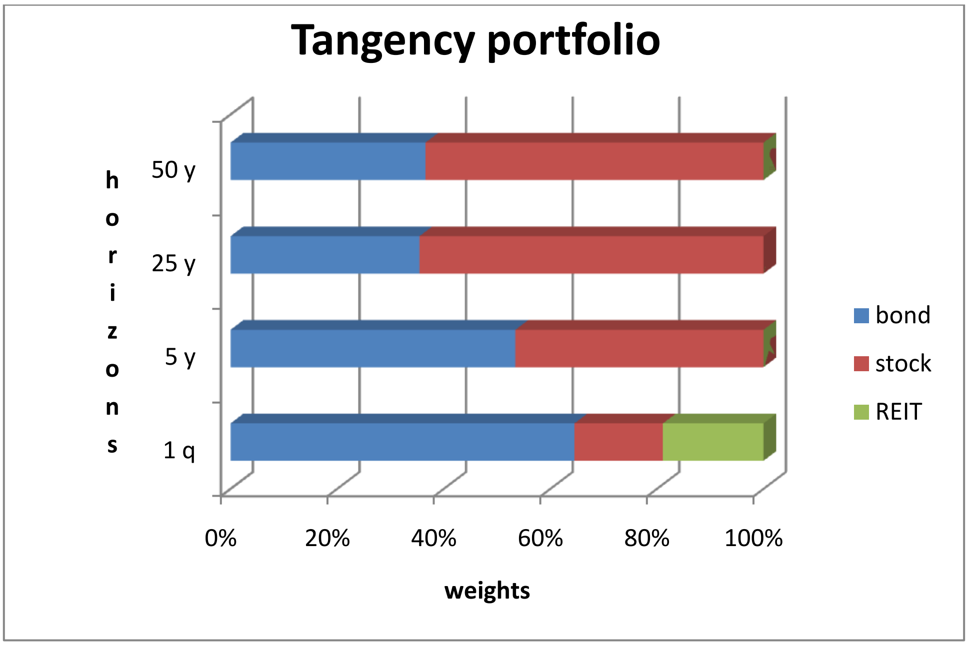
Figure 28: Tangency portfolio



Source: Own calculation

We can see that the horizon effect moves the composition of tangency portfolio from more than 60% of bonds to around 36% of bonds for long term horizons. Surprisingly the weight of bonds slightly increases (for both versions of the model) when the investment horizon increases from 25 to 50 years. The weight of REIT in tangency portfolio is present only for one quarter horizon.

Figure 29: Tangency portfolio (extended version)



Source: Own calculation

We finished the empirical research on the term structure of the risk-return tradeoff by this section. We found that the horizon effect is in favor of stocks hence the investors with longer investment horizons should invest relatively more into stocks than myopic investors. We will turn to the practical application, particularly for Czech pension funds in the following chapter.

6. Pension funds

The results from our empirical research are appealing to investors with long investment horizon. Typical institutional investor with long investment horizon is a pension fund. Our application of the term structure of the risk-return tradeoff will be applied especially to Czech pension funds. But before turning into Czech pension funds problematic, we will briefly describe basic characteristics of pension funds in general.

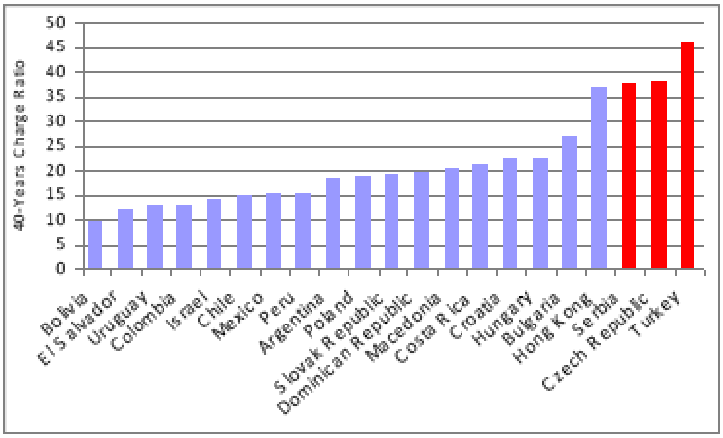
6.1. Characteristics of pension funds

In most of the developed countries the dominant pension system is the public unfunded scheme. This is often called as PAYG (Pay As You Go). The people are paying social security and they receive the pension when retired according to specific formula that includes especially wage of the retiree and number of years working[22]. However in last few decades the pension system relies increasingly on funded pension schemes in many countries. The funded scheme as opposed to PAYG uses the proceeds from contributions to invest these resources that are used in the future for payment of retirement benefits. Thus the liabilities of PAYG scheme are unfunded[23] and liabilities of funded pension plans are backed by the portfolio of assets. The funded pension plans are called the pension funds.

The PAYG is considered by the World Bank as the first pillar of the pension system. The division between compulsory and voluntary pension funds is the main characteristic of the division between second pillar and third pillar of pension system. The funded second pillar is alternative to PAYG system with the opportunity to opt-out the contributions from PAYG into these private or public funded pension funds. The supplementary voluntary pension scheme is considered as the third pillar of pension system. The voluntariness has important effect on the fees and charges imposed upon pension funds. Voluntary systems have higher charge ratios due to marketing costs etc. It is particularly due to the fact that pension funds in voluntary regime are less important players and have fewer assets under management and thus the charges for management are relatively more important than the gains. Despite the fact that it is extremely difficult to compare the charge ratios among countries[24], we can see in figure (30) that countries with voluntary system as Czech Republic, Turkey and Serbia have according to Hernandez, Stewart (2008) relatively high charge ratios. Of course high charge ratios are also caused by other factors as number of providers, age of the funded pension market, the level of active management, contribution and wage rates etc. but we can generally say that there are scale effects in pension funds market.

[22] Therefore PAYG system belongs to the class of Defined Benefit pension schemes
[23] Benefits are being paid from the contribution payments of younger generations.
[24] There is a great national diversity of systems and fee charging methods.

Figure 30: 40 years charge ratio



Source: Hernandez, Stewart (2008)

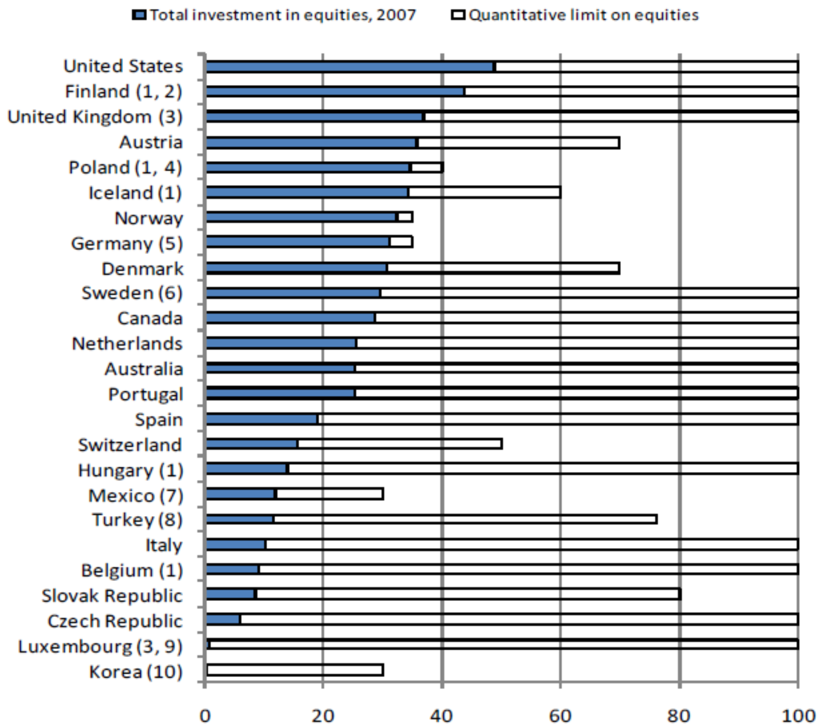
Pension funds can be further distinguished as sponsored or unsponsored. In sponsored funds, the sponsor (employer) contributes to the fund of employee. The two major types of pension funds are *defined benefit* (DB) and *defined contribution* (DC). Defined benefit pension schemes provide a periodic pension at pensionable age as a flat rate or a function of an individual's employment and earnings history. Thus the benefits are predictable and therefore matching the liabilities with appropriate assets is much more important for management of DB than DC. There is a risk of over-commitment to specific level of pensions. In case that the DB scheme gets into trouble with their funds, there is a conflict of interest between the old pensioners and new pensioners in whether to solve the situation by increasing contributions or decreasing benefits. As opposed to that in DC schemes, there are not defined benefits, but instead the predetermined contributions are sent to the individual account. Then together with other bulk of accounts, they are invested in portfolio of securities. The upside and downside risk is moved to the planholder, therefore the management of the DC pension fund is not tide as much with the liabilities. The benefits are paid as lump sum or as an annuity or as a mixture of both. Thus the main advantage of DB scheme is the guaranteed size of pension (theoretically) and the main advantage of DC fund is that it is easier portable to different fund and the investment policy of the management is not as restricted. As a result of that DC funds are in average more involved in market timing and have higher cost of management. The development in last 20 years is the shift from traditional DB schemes to DC schemes. The DB funds are dominant in Italy and France and DC funds are dominant USA, UK and Ireland. The Netherlands for example have mixture of both. A lot of variations of DC scheme exist, one of them is 401 (k) which is the retirement saving plan in the United States of America. It allows the worker to have the savings invested while deferring the income tax on the saved money until withdrawal. The pension funds can be sorted further according to their investment strategy (aggressive vs. moderate) or active management (active vs. passive). Because the planholders have different risk tolerance, pension funds can offer variety of investment strategies. These funds are often called lifestyle funds. The planholders are also of different age, which has effect on their investment strategy. The pension funds that offer more stock oriented strategy in early years and become more bond oriented prior to retirement are called the lifecycle funds.

How do pension funds decide on their investments? First they have to set their investment objective. They choose the benchmark that serves as a lower bound of their performance they need to achieve. For DB plans it is usually the liability structure (defined benefits) that is implied from the contributions of the planholders. For DC plans it might be a bond index or specific return that they promise to the planholders. However if it is not satisfied, it is the risk of planholder. Then they need to specify the risk that they are willing to take in order to track or exceed the performance of benchmark. There are various types of risks that pension funds need to deal with. These are interest rate risk, credit risk, call risk, prepayment risk, yield curve risk, liquidity risk, exchange rate risk, inflation risk etc. They can deal with them in certain extent by immunization of the portfolio, hedging, cash flow matching, by diversification of the portfolio. But after all there will always be risk remaining. Its size is related by market to the expected return and thus their risk tolerance is very important. As we can see from the results of this paper also the investment horizon is very important. For the purposes of this paper we will think of risk only as of the standard deviation from the expected return.

The other thing that pension fund needs to deal with is its constraints. They need to fulfill regulatory requirement that are set by the regulatory authority to decrease investment risk in pension system. There are different regulatory requirements within countries. The most popular are quantitative limits. This is the quantitative investment restriction by asset class.

These restrictions are being used in Denmark, Hungary, Switzerland, Israel, Mexico etc. Other form of restriction is minimum investment return which is used in Switzerland or quantitative risk limit over certain short period of time. This is used in Denmark and Mexico DC plans. As we can see in the figure from OECD working paper (2009), the quantitative limits are usually set very high. It seems to be binding only in Poland, Norway and Germany.

Figure 31: Portfolio limits on OECD pension funds` investment in equities, 2007



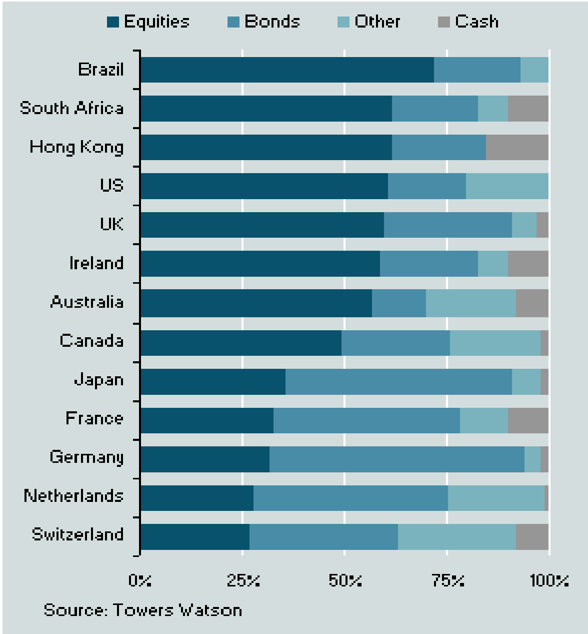
Notes: (1) Investment limit refers to listed equities; (2) Investment limit refers to statutory pension plans; (3) Data refer to the year 2006; (4) Investment limit refers to mandatory personal pension plans; (5) Investment limit refers to *Pensionskassen* (pension institutions); equity investments are probably overstated due to the inclusion of investments in mutual funds that should be broken down and reallocated both to equity and bond investments; (6) Pension foundations are not subject to uniform investment rules and are not, therefore, covered here; (7) Investment limit refers to Basic Fund 5; (8) Data only refer to personal pension plans; (9) Investment limit refers to ASSEP and SEPCAV; and (10) Investment limit refers to corporate DB plans.

Source: OECD Global Pension Statistics.

Some countries, where the contributions to the DC plan are mandatory can also regulate the choice of the pension fund. It might be restricted according to age as in Latin America. In general all these regulatory requirements move the allocation toward more conservative bond oriented rather than expansive equity oriented.

Thus the culture, regulation and the development of pension fund market can have effect on portfolio allocation of pension funds across the countries. Figure (32) shows the basic asset allocation of pension funds by countries.

Figure 32: Pension fund asset allocation, 2009



As we can see no country included in the graph has the problem that the percentage of equities and other shares would be under the global minimum variance portfolio from our empirical research. This holds for all investment horizons. Of course we cannot say whether the portfolio is well diversified or if some pension funds have large exposure to some stocks. We cannot distinguish if the share of equities in entire portfolio is only due to relatively high risk tolerance or whether it is also due to the fact that pension funds already take the term structure of the risk-return tradeoff into account which is in favor of stocks for larger investment horizons. If we suppose that the pension funds take the horizon effect into account and we expect that average investment horizon of the pension fund is twenty five[25] years then pension funds in Switzerland, Netherlands, Germany and France would represent the moderate investors in our extended version of the model. Japan would represent medium risk averse investor and Brazil aggressive investor. The other countries

[25] It is reasonable assumption if we expect that average difference in age of those who start working and those that enter retirement is 40-45 years and also due to expansion of pension funds in last decades, we can expect slightly higher frequency of young people than old people included in the fund.

are somewhere in between medium and aggressive. Of course due to the fact that many pension funds had large equity exposure during the financial crises, it led to the temporal underfunding of the pension funds. We cannot say in general what is the best risk exposure, but we can say that the composition of stocks in global minimum variance portfolio is increasing with the investment horizon. As we will see in next section, the regulations of Czech pension funds may be a striking problem in not allowing them to think as long term investors.

6.2. Czech pension funds

The pension system in Czech Republic consists of first pillar PAYG[26] system and third pillar voluntary supplementary DC pension plans. This paper focuses on investment strategy of the voluntary pension funds and it has relevance for often discussed pension reform in Czech Republic, which would bring the second pillar into Czech pension system[27]. The pension plans would play much larger role in pension system with development of second pillar and therefore the discussion about the changes in regulation of this sector would be appealing. Assets under management of pension funds are gradually increasing even without the reform. It is mainly due to the fact that saving in pension funds has many advantages compared to other financial products. Those are state contributions, tax reliefs for planholder and also the tax reliefs for employers that are motivated to contribute into the pension fund of their employees. However pension fund market is still very small compared to other countries, where the second pillar of pension system is present. Total assets under management of Czech pension funds were 222.662 billons CZK by 30.6.2010 which is around 7% of Czech GDP. This among other factors leads to higher administration fees as shown in figure (30). One of the factors that prevent pension funds from growing faster is their performance. Bad performance of Czech pension funds is connected with the regulations. Tables (24) and (25) conclude the nominal and real gains of Czech pension funds.

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26 Pay as you go: continuously financed state pension system

27 PAYG is considered by World Bank as first pillar of pension system, the pension funds financed by opt out option from mandatory social insurance is considered as second pillar and voluntary supplementary pension funds are considered as third pillar.

Table 24: Performance of Czech pension funds: Nominal returns (%)

Název penzijního fondu	**2000**	**2001**	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**
The Name of the Pension Fund										
AEGON PF	0	0	0	0	0	0	0	4.5	3.50	2.1
Allianz PF	3.8	4.36	3.71	3	3	3	3.11	3.0	3.00	3.00
AXA penzijní fond	4.1	4.25	3.41	3.36	3.1	3.7	2.5	2.2	0.00	2
ČSOB PF Progres	5.62	3.9	4.26	4.3	5.3	5.0	2.3	2.4	0.02	1.00
ČSOB PF Stabilita	4.2	3.2	3	2.3	4.3	4.0	2.8	2.4	0.05	1.37
Generali PF	3.6	4.6	4.1	3	3.0	3.81	3.74	4.1	2.00	2.4
ING penzijní fond	4.4	4.8	4	4	2.5	4.2	3.6	2.5	0.04	0.1
PF České pojišťovny	4.5	3.8	3.2	3.1	3.5	3.8	3.3	2.4	0.20	1.2
PF České spořitelny	4.2	3.8	3.5	2.64	3.74	4.03	3.04	3.1	0.40	1.28
PF Komerční banky	4.89	4.4	4.63	3.4	3.5	4.0	3.0	2.3	0.58	0.24

Source: Asociace penzijních fondů ČR

Table 25: Performance of Czech pension funds: Real returns (%)

Název penzijního fondu	**2000**	**2001**	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**
The Name of the Pension Fund										
AEGON PF								1.7	-2.8	1.1
Allianz PF	-0.1	-0.34	1.91	2.9	0.2	1.1	0.61	0.2	-3.3	2
AXA penzijní fond	0.2	-0.45	1.61	3.26	0.3	1.8	0	-0.6	-6.3	1
ČSOB PF Progres	1.72	-0.8	2.46	4.2	2.5	3.1	-0.2	-0.4	-6.28	0
ČSOB PF Stabilita	0.3	-1.5	1.2	2.2	1.5	2.1	0.3	-0.4	-6.25	0.37
Generali PF	-0.3	-0.1	2.3	2.9	0.2	1.91	1.24	1.3	-4.3	1.4
ING penzijní fond	0.5	0.1	2.2	3.9	-0.3	2.3	1.1	-0.3	-6.26	-0.9
PF České pojišťovny	0.6	-0.9	1.4	3	0.7	1.9	0.8	-0.4	-6.1	0.2
PF České spořitelny	0.3	-0.9	1.7	2.54	0.94	2.13	0.54	0.3	-5.9	0.28
PF Komerční banky	0.99	-0.3	2.83	3.3	0.7	2.1	0.5	-0.5	-5.72	-0.76

Source: Asociace penzijních fondů ČR

Before turning to main cause of relatively poor performance, let us briefly describe some characteristics and regulations of Czech pension funds. Pension fund is privately owned and the majority of Czech pension funds are owned by banks or insurance companies but the bank of deposit needs to be different and this bank cannot own the shares. Since 1.4.2006 the supervision over pension funds is done by Czech National Bank. The minimum of own capital is 50 million CZK. Value of the securities of one issuer cannot exceed 10% of fund`s assets (excluding government bonds of OECD countries and European banks). Total value of tangible and intangible assets cannot exceed 10% of fund`s assets. At least 70% of fund`s property must be placed in assets denominated in the currency in which fund is committed to participants. Pension fund cannot buy shares of another pension funds. The main restriction that is placed on pension funds is the minimum investment return that applies every year[28].

In current Czech law are the client contributions, state contributions and possibly employer contributions invested and continually appreciated by pension fund. Pension funds normally attribute to the clients 85-95% of the profits. At least 5% of the profit is compulsorily transferred to the reserve fund and up to 10% is decided at general meeting. The year to year nominal losses are covered by the reserves or the equity of pension fund therefore the losses are on the account of shareholders. Logically the management of pension fund that acts in interest of shareholder will try to minimize the probability of losses, because they are not fully compensated by possible gains since the majority of gains is attributed to planholders (pensioners). As a result of this regulation, the rational manager behaves as a myopic investor and not as an investor with long investment horizon. The combination of the fact that pension funds are privately owned with short term minimum investment return creates an agency problem between shareholders and planholders. They both have different expected returns and different risks. Management of the pension fund can easily achieve nominal appreciation of their funds when they invest into bonds and T-bills. But they cannot expect the same with stocks. As a result of that Czech pension funds have very high exposure to bonds and very low fraction of portfolio in stocks. Nowadays the average fraction of assets that is invested by Czech pension funds into stocks is around 4.5% which is in clear contradiction to the optimal investment portfolio for long term

[28] Information about regulation of Czech pension funds is taken from Czech National Bank.

investor that is described in dynamic portfolio theory. This implies the often criticized fact that the real returns of Czech pension funds are very low[29]. Tables (26) and (27) show basic balance sheet of Czech pension funds for the fourth quarter of 2010.

Table 26: Asset structure of Czech pension funds

	CZK millions	Assets of pension funds											
Name of the pension fund	**Funds credited to the particip ant**	**Total Assets**	**Total bond**	**%**	**T-bills**	**%**	**Shares+ unit cert.**	**%**	**Cash in bank and term deposits**	**%**	**other**	**%**	**Sum total of for-eign invest-ments**
AEGON PF	3,737	4,234	3,152	74.4	129	3.0	0	0.0	458	10.8	495	11.7	656
Allianz PF	9,539	10,557	9,838	94.7	0	0.0	122	1.2	393	3.8	204	1.9	122
AXA PF	33,245	36,275	28,897	79.7	100	0.3	2	6.9	1,707	4.7	3	8.5	9,871
ČSOB PF Progres	9,271	10,135	9,277	91.5	0	0.0	213	2.1	255	2.5	390	3.8	322
ČSOB PF Stabilita	17,763	19,346	17,842	92.2	0	0.0	542	2.8	451	2.3	511	2.6	920
Generali PF	2,633	2,854	2,496	87.5	0	0.0	151	5.3	100	3.5	107	3.7	776
ING PF	23,908	25,492	21,742	85.3	957	3.8	0	0.0	2,203	8.6	590	2.3	6
PF České pojišťovny	52,125	55,284	49,379	89.3	0	0.0	3	6.3	1,357	2.5	1	2.0	13,068
PF České spořitelny	35,173	37,624	24,794	65.9	0	0.0	3	9.1	8,940	23.8	460	1.2	2,527
PF Komerční banky	28,718	30,602	27,839	91.0	0	0.0	0	0.0	1,865	6.1	898	2.9	0
TOTAL	**216,112**	**232,402**	**195,256**	**84**	**1,186**	**0.5**	**10,410**	**4.5**	**17,729**	**7.6**	**7,822**	**3.4**	**28,268**

Source: APF ČR

[29] Less than 1 % in last decade

Table 27: Liability structure of Czech pension funds

Name of the Pension Fund	Total Liabilities and Equity	Parties means	Reserves	Equity	Other Liabilities
AEGON PF	4,233,434	3,731,665	2	464,535	37,232
Allianz PF	10,556,762	9,538,982	7,920	976,801	33,059
AXA penzijní fond	36,278,659	33,245,247	22,366	2,764,332	246,714
ČSOB PF Progres	10,134,357	9,270,520	329	613,537	249,971
ČSOB PF Stabilita	19,346,185	17,762,848	3,030	1,076,265	504,042
Generali PF	2,854,277	2,632,701	518	204,319	16,739
ING Penzijní fond	25,491,586	23,910,458	24,912	1,369,097	187,118
PF České pojišťovny	55,304,610	52,124,745	23,143	3,026,619	130,103
PF České spořitelny	37,623,796	35,173,368	28,979	2,286,843	134,606
PF Komerční banky	30,601,851	28,718,275	8,119	1,585,836	289,621
Total	**232,425,517**	**216,108,809**	**119,318**	**14,368,184**	**1,829,205**

Source: APF ČR

We can see that as expected by the rational behavior of management representing shareholder, weight of stocks is very low in the portfolio. Shares together with unit certificates represent only 4.5% of portfolio allocation. When we compare this allocation with global minimum variance portfolio from the section with empirical research, we notice that for the extended version of the model, investor with 25 years investment horizon should have higher exposure to stocks if he wants to minimize risk (6.7%)[30]. Of course we do not know the investment horizon of particular pension funds because it depends on age of their participants and rebalancing strategies. However if we look only at buy and hold strategy and we suppose that duration of pension funds' liabilities is somewhere between 20 and 25 years, we have to conclude that weight of stocks is really very low, maybe even under the level that would minimize the long term risk. On the other hand, the weight of bonds is high accounting for 84% from which majority (82%) is government bonds. This is much higher than any of our model shows and it is on account of T-bills[31]. However this might be due to availability of these instruments since Czech money market is small and

[30] Table 24

[31] We have to keep in mind that bonds in empirical model are 5 years bonds and bonds in the balance sheet are all maturities bonds.

investing in foreign instruments is limited because majority of funds' property must be denominated in Czech crowns. T-bills are represented only by 0.5% in the portfolio, which is quite surprising. But since we treated T-bills as cash in our analysis, we can consider cash in bank and term deposits as a substitute for it. Therefore T-bills and cash amount to 8.1% altogether. This would be best represented by aggressive investor with five years investment horizon in our extended model however the level of stocks would have to be at 35% instead of 4.5%. If we take T-bills and bonds together as one asset class then the most representative investor will be the moderate with one quarter or one year investment horizon. This is in line with the expectation that under current regulations, pension funds are forced to act as myopic investors instead of long term investors. Thus the regulations of Czech pension funds leads to suboptimal results. It is especially due to the minimum investment return requirement that is set for one year.

Of course when interpreting the results, we have to keep in mind that our model has many drawbacks. One of the weak points is that we take 5 year bonds as a representative for the whole asset class of bonds. But we could easily do the same analyses for bonds with different maturities since we estimated the term structure of the risk-return tradeoff for bonds with one year and thirty years maturity in figures 15, 16 and 17. Another weakness of the model is mentioned in the beginning of the paper. We expect that short term risk does not change over time. However in this work, we are satisfied with the argument that changes in risk are not very persistent and that this assumption should not have a large effect on results. Another inconsistency is that the empirical analysis is made on U.S. data, but the application is for Czech pension funds that invest mainly into Czech securities. Unfortunately due to absence of data, the same analyses for Czech or European data were not possible to be made. We suggest for further research to do such analyses when more data will be available.

The most important drawback is that our analysis is applicable to buy and hold investor only. But we know that pension funds rebalance their portfolio for two main reasons. One is searching for superior returns by timing the market. But as mentioned earlier, even in presence of short term rebalancing portfolio, it is optimal to keep also the intertemporal hedging portfolio as a hedge against changes in investment opportunities. Thus the models of long term buy and hold portfolio is conceptually appealing. The other reason for rebalancing is the asset liability management that depends on demographic structure of plan's participants. This might be solved by dividing whole portfolio into many small portfolios according to the maturities of liabilities that need to be matched and solve the optimization problem for each portfolio individually. Then the stochastic programs for asset liability management can be used as in paper of Dupačová and Polívka (2004). The term structure of the risk-return tradeoff can be included into these programs later on.

7. Conclusion

We have shown in this paper that portfolio choice problem has changed after recent research in empirical finance. We reviewed the literature showing that contrary to portfolio theory of Markowitz (1952), the long term investor faces different risk than the myopic investor. Campbell and Viceira (2005) came with the empirical model that is able to work with the complex dynamics of risk and expected returns and which is easily applicable to practice. They model returns and state variables as vector autoregressive model. They found that predictability of asset returns has important effect on variance and covariance structure of returns across investment horizons. We run similar model as Campbell and Viceira (2005) based on data from Thomson Reuters Datastream and Wharton Research Data Services. We use slightly different indexes and work with different sample period than other authors, thus the effect of last financial crises is included in coefficient estimates. We further work with vector autoregressive model, where only the coefficients which have significant predictive power are included and where returns on Real Estate Investment Trust are included into the model. This improves the predictability of the model. We found that risk of stocks is decreasing with investment horizon and risk of T-bills is increasing with investment horizon. Also the correlation structure changes significantly across investment horizons. This makes stocks more favorable asset class for long term investor at the account of bonds and T-bills. Unfortunately we could not prove the same with use of European data due to shortage of sample size.

In section 5 we were searching for optimal portfolio allocation where we took the horizon effect from previous sections into account. First we constructed global minimum variance portfolio, where we found that for longer investment horizons the weight of T-bills decreases and weight of stocks increases. For 50 years investment horizon the weight of stocks rises to more than 4% in basic model and more than 10% in extended version of the model. Then we were trying to find optimal asset allocation with respect to risk tolerance. First we minimized the Value at Risk for various investment horizons. We found that with increasing investment horizon the weight of stocks increases dramatically. Then we defined three representative investors according to their risk tolerance and constructed optimal asset allocation for various investment horizons. In the last part of empirical research we treated T-bills as riskless asset and constructed tangency portfolio for different investment horizons. We found again that with increasing horizon the weight of stocks increases and the weight of bonds decreases. The weight of stocks in tangency portfolio exceeded 60% for investment horizons larger than 25 years. Thus the main conclusion of empirical research of this paper is that institutional investors with long investment horizons such as pension funds should include more stocks into their portfolio even if they are very risk averse.

In the last section of this paper we used the results of empirical research for comparison with the practice of Czech pension funds and their asset allocation. We found that Czech pension funds are not only strongly risk averse, but also optimize their asset allocation as short term investors, which is suboptimal. This finding is not as surprising when we realize that regulations of pension funds in Czech Republic force pension funds to act as a myopic investor. We found that the allocation of stocks is maybe even under the theoretically optimal allocation of global minimum variance portfolio. Therefore the suggestion of this paper is to relax the regulation of minimum investment return for one year and make this regulation for longer period. This would enable Czech pension funds to act as a long term investor and use the advantages of the term structure of risk-return tradeoff. Our research applies only to buy and hold investor, but as a theoretical concept may be appealing. Our suggestion for further research using the similar concept of vector autoregressive model is to use the asset return predictability to simulate the short term rebalancing portfolio as the supplement to intertemporal hedging portfolio. It would be interesting to see on out of sample data from last financial crisis whether our vector autoregressive model has good predictive power and whether it would suggest rebalancing the portfolio in the good direction just before the crisis.

8. List of Figures

9. List of Tables

10. Bibliography:

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