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Černé díry v teorii strun

Ústav teoretické fyziky

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Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

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Abstrakt: M. Guica, T. Hartman, W. Song a A. Strominger nedávno ukázali, že entropii extrémní Kerrovy černé díry lze vypočítat pomocí duality mezi kvantovou gravitací na pozadí černé díry a konformní teorií pole poblíž horizontu. V naší diplomové práci jsme nalezli zobecnění této procedury: opouštíme "near horizon" geometrii a definujeme okrajové podmínky a povrchové náboje v plném prostoročase na horizontu černé díry, kde také "žije" duální konformní teorie pole. Pomocí této procedury znovu odvozujeme entropii extrémních černých děr a poté počítáme entropii obecných rotujících černých děr.

Klíčová slova: černé díry, entropie, Kerr/CFT korespondence, konformní teorie pole

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Abstract: Recently it was shown by M. Guica, T. Hartman, W. Song and A. Strominger that the entropy of extreme Kerr black hole can be computed through a duality between the quantum gravity on the background of the Kerr black hole and the conformal field theory near the horizon. I our diploma work we have found a generalization of this procedure: we leave the near horizon geometry and we define our boundary conditions and the surface charges in the full spacetime on the horizon of the black hole, where the dual conformal field theory "lives". We use this procedure to rederive the entropy of extremal black holes and then we compute the entropy of general rotating black holes.

Keywords: black hole, entropy, Kerr/CFT correspondence, conformal field theory

Chapter 1

Introduction

The entropy of black holes is a fascinating phenomena. Since 1973, when Bekenstein [1] proposed that the entropy of a black hole is proportional to its horizon area, there have been many attempts to explain this relation and to find the microscopic degrees of freedom of the black hole.

In the string theory, the black holes can be described as a weakly coupled system of strings and D-branes (for example [2], [3], [4]) and the entropy is explained by the degeneracy of the states of this system. Despite the name of our work, we decided to concentrate on a related, but different approach based on a holographic duality between a quantum gravity and a quantum field theory in fewer dimensions. These dualities work very well with string theory [5]. Our approach does not require connection to string theory, it works both in the classical general relativity and in supergravity theories. In this approach the Bekenstein-Hawking entropy is explained as an entropy of a conformal field theory (CFT) "living" near the horizon, which is dual to quantum gravity on the Kerr background.

The possibility, that the charge algebra of asymptotic symmetries can have a nontrivial central extension, was first realized by Brown and Henneaux [6] in the case of AdS_3. This work inspired many of the black hole applications, because they often contain AdS-like region. For example the first explanation of black hole entropy was done for the BTZ black hole [7] that is embedded in AdS_3. Our starting point is an article on the Kerr/CFT correspondence [8], which was also motivated by the resemblance of the near horizon extreme Kerr geometry to AdS. This article describes the entropy of a realistic (although extremal) black hole. It was followed by many generalizations (for example [9], [10], [11], [12]).

In our work we try to extend the applications of this approach. First we reformulate the Kerr CFT correspondence in context of the whole Kerr spacetime, not just in the near horizon geometry. This can be done by defining the boundary conditions and the conformal field theory on the horizon. Then we use this method to compute the right central charge of the extremal Kerr and we even propose a way how to compute the entropy of a general non-extremal rotating black hole.

This work is organized as follows. In chapter 2 we introduce the concept of the black hole entropy and we discuss the second law of thermodynamics.

In chapter 3 we review the formalism of asymptotic charges and the construction of a holographic conformal field theory. Then we use this formalism to find the asymptotic charges in general relativity.

Chapter 4 reviews the properties of a non-extremal and extremal Kerr solution, because we use the Kerr black hole as a basic example in our work. We review the calculation of the extremal Kerr entropy following [8] in chapter 5.

In chapter 6 we rederive the entropy of the extremal Kerr using our new method based on the definition of particular boundary conditions on the horizon. Then we compare these two approaches and discuss the possible localization of the holographic screen. We generalize the results of the previous two chapters to a much broader class of black holes in chapter 7.

In chapter 8 we show that there is a second Virasoro algebra on the Kerr background and we use it to compute the entropy of the near extremal Kerr. Finally in chapter 9 we propose a way how to derive the entropy of non-extremal black holes. Chapter 10 contains the summary of our results and ideas for future research.

Finally we shall introduce our unit conventions. We set $c = G = k_B = 1$ to simplify the equations but we keep the value of $\hbar$ to allow the dimensional analysis and to stress the quantum effects.

Chapter 2

Basic Facts about the Black Hole Entropy and Thermodynamics

The entropy of the black holes and the concept of information itself in their presence have always been a mystery. The black holes are the simplest macroscopical objects we know, they are uniquely determined by their mass, angular momentum and charge[1]. The information about anything that falls into the black hole seems to be lost. We cannot say of what kind of particles the black hole consists, if it is made of matter or antimatter etc. Actually we have no idea what happens to the matter inside the event horizon of the black hole. The equations of motion tell us that it falls in the centre, but there is a singularity in the centre of the black hole, where the curvature of the spacetime usually diverges. We cannot say if the singularity is physical, because when the density of matter becomes comparable to the Planck density the quantum gravity has to be applied. In string theory there are certain kinds of black holes that can be describes by D-branes and strings wrapped over the compact dimensions, so it is possible that the in-falling particles join this system. But this can be only effective description and the string theory is not proved.

At the first look it seem that the holes break the second law of thermodynamics, that the entropy of any thermodynamical system never decreases, because their entropy seems to be very small. But when we take a closer look, we can see that it is exactly the other way. By the statistical definition the entropy describes the number of microstates (more precisely the logarithm of) of an object, that look macroscopically the same. As mentioned above the black holes have only several macroscopic characteristics, so the matter inside the event horizon can do virtually

[1]This is not true for higher dimensional black holes. In five dimensions there are black rings, black saturns, black dirings and other solutions with more parameters. These black holes are characterized by their charges and rod structure (see for example [13]). In [14] there are derived new types of black holes in arbitrary dimensions, so the number of characteristics rises with the dimension of the spacetime. However it seems that the number of parameters of black holes is much lower that of any other object.

anything and the black hole still looks the same from outside, so the number of microstates is extremely high. The entropy of black hole is actually the largest of all objects of comparable volume and mass.

The entropy in the classical thermodynamics (for example of an ideal gas) is usually proportional to the volume. But there is no well defined volume of the black holes, because on a slice of constant time the radial direction becomes timelike inside the horizon. The closest analogy of volume is the area of horizon, which is a well-defined quantity. The idea that the entropy of black hole is proportional to its area was first proposed by Bekenstein in [1]. During the research of black holes it was found that the area of black holes never decreases. This is true for all classical processes like absorption of particles, extraction of energy and collision of black holes. This reminds the second law of thermodynamics, so the entropy should be a function of area. By a simple experiment involving absorbtion of particle Bekenstein showed that the minimal increase of black hole area is $8\pi\hbar$, which he compared to 1 bit of information. His result was that the entropy of black hole is $S_{Bec} = \frac{A}{2\pi\hbar \ln 2}$. Now we know that the entropy is slightly different [15]

$$S = \frac{A}{4\pi\hbar}. \tag{2.1}$$

More information can be read from another useful relation. The quantities of black hole satisfy

$$dM = \frac{\kappa dA}{8\pi} + \Omega_H dJ, \tag{2.2}$$

where $M,\ J$ are the mass and the angular momentum, κ is a surface gravity and Ω_H is an angular velocity of horizon (see section 7.1 for precise definitions). This formula can be proven generally and it does not depend on the details of the black hole.[2] It is remarkably reminding the first law of thermodynamics

$$dU = TdS + pdV. \tag{2.3}$$

We can identify the mass of the black hole with its energy and if the area is proportional to entropy, the surface gravity should be proportional to temperature, namely

$$T_H = \frac{\hbar\kappa}{2\pi}. \tag{2.4}$$

Any object at a finite temperature has to radiate black body radiation to be in thermal equilibrium with its surroundings. So at first this concept seemed to be ridiculous. The black holes are famous for the fact that nothing that falls in can go out. But this is true only on classical level, in [15] Hawking found that the black holes

[2] This is actually a form for vacuum solutions in four dimensions, because there can be other terms in presence of charges or more angular momenta in higher dimensions. Nevertheless the terms above are always present.

can radiate by quantum effects. The black hole emits radiation with temperature precisely equal to (2.4), so the temperature and the first law of thermodynamics fixes the ratio of the entropy and the area to $\frac{1}{4\pi\hbar}$.

The Hawking radiation can be interpreted in several ways. There is one of the classical explanations. Due to quantum fluctuations pairs of virtual particles appear near the horizon, one of them has positive energy and the other one negative energy. Normally the negative energy particle if forbidden, so the particles have to annihilate again. However near the horizon the negative energy particle can tunnel inside the horizon, where it can exist because of the switch of roles of the time and radial direction. So this particle falls into the black hole and decreases its mass, while the positive energy particle radiates away.

In the quantum field theory the Hawking radiation is seen in a different way. There are two different vacua: a vacuum of an observer at the asymptotic infinity and a vacuum of an observer freely falling through the horizon, which is associated with the black hole. Due to the inaccessibility of the interior of the black hole the vacuum state on the horizon is interpreted at infinity as a thermal bath of particles that form the Hawking radiation.

The radiation does not break the second law of thermodynamics although the entropy of the black hole decreases, because the sum of the entropy of the black hole and of the Hawking radiation increases. This way the black hole can in principle radiate away all its mass and disappear. In reality it is highly improbable that we will ever see such an event. The Hawking temperature and the surface gravity, which represents the "force" that tears the virtual particles apart, decrease with the mass of the black hole. In Newtonian gravity the gravitational force is proportional to $\frac{M}{r^2}$ and the horizon radius of black hole is of order of M, so we can expect that the temperature is proportional to $\frac{1}{M}$. In four dimensions this relation turns out to be correct and as a consequence the evaporation time is proportional to M^3. A black hole with a mass comparable to the Sun will evaporate approximately in 10^{67} years. The temperature of stelar black holes is so small that it is smaller then the temperature of the relict radiation, so the black holes will not start to evaporate until this temperature becomes sufficiently small.

Chapter 3

The Asymptotic Charges and Construction of Holographic Field Theory

In this chapter we will review the formalism of asymptotic charges and use it to predict a holographic field theory.

There are many approaches to the problem of the conserved charges in general relativity. There is the classical Hamiltonian formalism [16], [6], the covariant formalism of Barnich, Brandt and Compère [17], [18], [19], the Noether charge approach [20], [21], the quasilocal formalism of Brown and York [22] and others.

Is the next section we review the formalism based on Noether charges [20], [21], which is summarized in [11] and [23], because we consider it the easiest to understand, however we mostly use results of [18] and [19].

3.1 Formalism of Conserved Charges

We shall consider a system with action **S**

$$S = \int_{\mathcal{M}} \mathbf{L}, \quad \mathbf{L} = \mathcal{L}\sqrt{-g}d^D x = \mathcal{L} * \mathbf{1}, \tag{3.1}$$

where $\mathbf{L}$ is a Lagrangian, $\mathcal{L}$ is a Lagrangian density and D is the dimension of the spacetime. We denote differential forms by bold letters and $*$ is a Hodge dual. It is defined as

$$(*w)_{\mu_1\ldots\mu_k} = \frac{1}{(D-k)!}\epsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{D-k}} w^{\nu_1\ldots\nu_{D-k}} \tag{3.2}$$

with $|\epsilon| = \sqrt{-g}$. So $*\mathbf{1}$ is a volume form. The Lagrangian density is considered to be depending on fields Φ^a and their derivatives $\mathcal{L} = \mathcal{L}(\Phi^a, \partial_\mu\Phi^a, \partial_\mu\partial_\nu\Phi^a, \ldots)$.

The system has a symmetry if a variation of the action with respect to a generator of this symmetry is zero. We shall denote the generator ξ. This means that the Lagrangian $\mathbf{L}$ has to be invariant up to a total derivative.

$$\delta_\xi \mathbf{L} = d\mathbf{M}_\xi, \quad \Rightarrow \quad \delta_\xi S = \int_{\mathcal{M}} d\mathbf{M}_\xi = \oint_{\partial\mathcal{M}} \mathbf{M}_\xi = 0 \tag{3.3}$$

The variation of the Lagrangian can be decomposed as

$$\delta_\xi \mathbf{L} = E_a \delta_\xi \Phi^a * \mathbf{1} + d\mathbf{\Theta}(\Phi^a, \delta_\xi), \tag{3.4}$$

where we recognize the Euler-Lagrange equations of motion $E_a = 0$. All derivatives of $\delta_\xi \Phi^a$ have been absorbed in the outer derivative of the $D-1$ form $\mathbf{\Theta}$. For every symmetry there is a related Noether current $\mathbf{J}_\xi$

$$\mathbf{J}_\xi = \mathbf{\Theta}(\Phi^a, \delta_\xi) - \mathbf{M}_\xi. \tag{3.5}$$

We can see that the Noether current is conserved when the equations of motion are satisfied, because

$$d\mathbf{J}_\xi = d\mathbf{\Theta}(\Phi^a, \delta_\xi) - d\mathbf{M}_\xi = -E_a \delta_\xi \Phi^a * \mathbf{1} = 0. \tag{3.6}$$

So the Noether current can be written (at least locally) as $\mathbf{J}_\xi = d\mathbf{Q}_\xi$, where $\mathbf{Q}_\xi$ is a $D-2$ form. Using the Noether current we can define a conserved charge on a spacelike hypersurface Σ as

$$Q_\xi = \int_\Sigma \mathbf{J}_\xi = \int_\Sigma d\mathbf{Q}_\xi = \oint_{\partial\Sigma} \mathbf{Q}_\xi. \tag{3.7}$$

We can see that the charge can be also defined even on the boundary of Σ using $\mathbf{Q}_\xi$. The consistency requires appropriate boundary conditions to ensure that the charge is finite.

We will be mainly interested in symmetries generated by vectors fields ξ. The action of ξ on the fields is the Lie derivative $\delta_\xi \Phi^a = \pounds_\xi \Phi^a$. The variation of the Lagrangian is

$$\begin{aligned}\delta_\xi \mathbf{L} &= E_a \pounds_\xi \Phi^a * \mathbf{1} + d\mathbf{\Theta}(\Phi^a, \pounds_\xi) \\ &= \pounds_\xi \mathbf{L} = d(i_\xi \mathbf{L}) - i_\xi d\mathbf{L} = d(i_\xi \mathbf{L}).\end{aligned} \tag{3.8}$$

The corresponding Noether current (3.5) is

$$\mathbf{J}_\xi = \mathbf{\Theta}(\Phi^a, \pounds_\xi) - i_\xi \mathbf{L}. \tag{3.9}$$

We can define an analog of the symplectic form Ω^{ab} in the classical Hamiltonian mechanics by

$$\Omega(\Phi^a, \delta, \delta_\xi) = \int_\Sigma \mathbf{w}(\Phi^a, \delta, \delta_\xi), \tag{3.10}$$

where

$$\mathbf{w}(\Phi^a;\delta,\delta_\xi)=\delta\mathbf{\Theta}(\Phi^a,\delta_\xi)-\delta_\xi\mathbf{\Theta}(\Phi^a,\delta). \tag{3.11}$$

If $d\mathbf{w}(\Phi^a,\delta,\delta_\xi)=0$ the perturbations of the fields must satisfy the linearized equations of motion, because

$$0=(\delta\delta_\xi-\delta_\xi\delta)\mathbf{L}=(\delta E_a\delta_\xi\Phi^a-\delta_\xi E_a\delta\Phi^a)*\mathbf{1}+d\mathbf{w}(\Phi^a,\delta\delta_\xi), \tag{3.12}$$

so

$$d\mathbf{w}(\Phi^a,\delta,\delta_\xi)=0\Rightarrow\delta E_a\delta_\xi\Phi^a=\delta_\xi E_a\delta\Phi^a=0. \tag{3.13}$$

If these conditions are satisfied then the quantity $\Omega(\Phi^a,\delta,\delta_\xi)$ is conserved, so the form $\mathbf{w}(\Phi^a,\delta,\delta_\xi)$ is analogous to the Noether currents (3.5) and $\Omega(\Phi^a,\delta,\delta_\xi)$ to the Noether charge (3.7). We shall denote this charge δQ_ξ. Its meaning is an infinitesimal charge associated with the first symmetry computed on a variation of the fields Φ^a caused by the other symmetry.

When $d\mathbf{w}(\Phi^a,\delta,\delta_\xi)=0$ we can write $\mathbf{w}$ as

$$\mathbf{w}(\Phi^a,\delta,\delta_\xi)=d\mathbf{k}_\xi(\Phi^a,\delta), \tag{3.14}$$

where $\mathbf{k}_\xi(\Phi^a,\delta)$ is a $D-2$ form. So it is possible to express δQ_ξ as a surface integral

$$\delta Q_\xi=\oint_{\partial\Sigma}\mathbf{k}_\xi(\Phi^a,\delta). \tag{3.15}$$

The on-shell variation of the Noether current (3.9) is

$$\delta\mathbf{J}_\xi=\delta\mathbf{\Theta}(\Phi^a,\pounds_\xi)-i_\xi\delta\mathbf{L}=\delta\mathbf{\Theta}(\Phi^a,\pounds_\xi)-\pounds_\xi\mathbf{\Theta}(\Phi^a,\delta)+d(i_\xi\mathbf{\Theta}(\Phi^a,\delta)). \tag{3.16}$$

So we are able to express (3.11) as a total derivative

$$\begin{aligned}\mathbf{w}(\Phi^a;\delta,\delta_\xi)&=\delta\mathbf{\Theta}(\Phi^a,\delta_\xi)-\delta_\xi\mathbf{\Theta}(\Phi^a,\delta)=\delta\mathbf{\Theta}(\Phi^a,\delta_\xi)-\pounds_\xi\mathbf{\Theta}(\Phi^a,\delta)\\&=\delta\mathbf{J}_\xi-d(i_\xi\mathbf{\Theta}(\Phi^a,\delta))=d\delta\mathbf{Q}_\xi-d(i_\xi\mathbf{\Theta}(\Phi^a,\delta)).\end{aligned} \tag{3.17}$$

We read off that the $D-2$ form $\mathbf{k}$ is given by

$$\mathbf{k}_\xi(\Phi^a,\delta)=\delta\mathbf{Q}_\xi-i_\xi\mathbf{\Theta}(\Phi^a,\delta). \tag{3.18}$$

If we want the charge (3.15) in a finite form, we have to fix its value for some background $\bar{\Phi}^a$. The usual choice is $Q_\xi(\bar{\Phi}^a)=0$. For a different value of fields the charge is

$$Q_\xi(\Phi)=\int_{\bar{\Phi}}^{\Phi}Q_\xi(\Phi)=\int_{\bar{\Phi}}^{\Phi}\oint_{\partial\Sigma}\mathbf{k}_\xi(\Phi,\delta). \tag{3.19}$$

The integral over Φ means that we choose a path in phase space from $\bar{\Phi}$ to the final value Φ and we integrate over this path. The value of charges is independent on the choice of path if certain technical assumption are satisfied, see[19].

The symmetries usually form an algebra of the form

$$[\delta_a, \delta_b] = f_{abc}\delta_c, \tag{3.20}$$

where f_{abc} are structure constants. We want to generalize this algebra to the charges, so we define Poisson bracket in analogy with classical Hamiltonian mechanics by

$$\{Q_\xi, Q_\zeta\}_{PB} = \Omega(\Phi^a, \mathcal{L}_\xi, \mathcal{L}_\zeta) = \int_{\partial\Sigma} \mathbf{k}_\xi(\Phi^a, \mathcal{L}_\zeta). \tag{3.21}$$

It can be shown that the Poisson bracket is equal to

$$\{Q_\xi, Q_\zeta\}_{PB} = Q_{[\xi,\zeta]} + K[\xi,\zeta], \tag{3.22}$$

where $K[\xi,\zeta]$ is a central term. If the charges of the background are zero, this term is equal to

$$K[\xi,\zeta] = \{Q_\xi, Q_\zeta\}_{PB} = \int_{\partial\Sigma} \mathbf{k}_\xi(\Phi^a, \mathcal{L}_\zeta). \tag{3.23}$$

The nontrivial part of $K[\xi,\zeta]$ is not affected by a constant shift of the charges, so we can use (3.23) generally.

3.2 Construction of Holographic Field Theory and Central Charge

This formalism can be used even for analysis of perturbations of the fields and asymptotic symmetries. Suppose that we have general fields Φ^a that asymptotically approach background fields $\bar{\Phi}^a$ in the sense that the deviation $\phi^a = \Phi^a - \bar{\Phi}^a$ is small compared to the background. We require that the perturbations ϕ^a satisfy boundary conditions that say precisely how big the perturbations can be

$$\phi^a \to O(\chi^a), \tag{3.24}$$

where χ^a are some functions of the coordinates. The perturbations ϕ^a do not have to go to zero, but the usual requirement is $\frac{\phi^a}{\bar{\Phi}^a} \to 0$. By asymptotic symmetries we mean transformations that are not exact symmetries but that preserve the boundary conditions

$$\delta_\xi \Phi^a \to O(\chi^a). \tag{3.25}$$

We can define asymptotic charges associated with the asymptotic symmetries using (3.15) and (3.19). These charges are asymptotically conserved ($\partial_t Q \to 0$ near the boundary) and they satisfy Poisson bracket algebra (3.22). There is a lot of technical requirements on the generators and the boundary conditions in order to the charges are well-defined, see for example [19].

The algebra (3.22) has a representation in a form of a $D-1$ dimensional holographic quantum field theory on the boundary and this theory is dual to the perturbation theory. The canonical quantization is done by replacements of the charges by operators $\hbar Q_\xi \to L_\xi$ and the Poisson brackets by commutators $\{.,.\}_{PB} \to -\frac{i}{\hbar}[.,.]$.

In the rest of our article we will be mostly interested in vectors satisfying algebra

$$i[\xi_m, \xi_n] = (m-n)\xi_{m+n}. \tag{3.26}$$

This algebra is called Witt algebra in mathematics, but we will use a different name because it is a centerless Virasoro algebra. The charge algebra is a representation of the Virasoro algebra with central charge

$$\{Q_m, Q_n\}_{PB} = (m-n)Q_{m+n} + \frac{1}{12}\bar{c}(m^3 + \bar{B}m)\delta_{m+n,0}, \tag{3.27}$$

and after quantization it becomes

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}c(m^3 + Bm)\delta_{m+n,0}, \tag{3.28}$$

where $\bar{c} = \hbar c$ and $\bar{B} = \hbar B$. The constant B is not important because it can be absorbed in a shift of L_0. The nontrivial information in carried by the constant c, which is called central charge. The central term (3.23) is equal to

$$\int \mathbf{k}_{\xi_m}(\Phi, \xi_n) = \frac{-i}{12}\hbar c(m^3 + \bar{B}m)\delta_{m+n,0}, \tag{3.29}$$

so the exact prescription for the central charge is

$$c = \frac{1}{6}\partial_m^3 \left(\frac{12i}{\hbar} \int \mathbf{k}_{\xi_m}(\Phi, \xi_{-m}) \right). \tag{3.30}$$

3.3 Application to Gravitational Action

To get the prescription for charges in general relativity, we shall apply this formalism to the Hilbert action with cosmological constant

$$S = \frac{1}{16\pi} \int (R - 2\Lambda)\sqrt{-g}d^D x. \tag{3.31}$$

We do no consider any matter fields. The variation of the Lagrangian is

$$\delta \mathbf{L} = \frac{1}{16\pi} \left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} + \Lambda g^{\mu\nu} - \nabla^\mu \nabla^\nu + g^{\mu\nu}\nabla_\rho \nabla^\rho \right) \delta g_{\mu\nu} * \mathbf{1}. \tag{3.32}$$

We shall denote $\delta g_{\mu\nu}$ as $h_{\mu\nu}$ and $h = g^{\mu\nu}h_{\mu\nu}$. Using (3.4) we recognize Einstein's equations in the first part

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} + \Lambda g^{\mu\nu} = 0. \tag{3.33}$$

The second part is a total derivative and it gives us the form $\mathbf{\Theta}(g_{\mu\nu}, \delta)$

$$\mathbf{\Theta}(g_{\mu\nu}, \delta) = \frac{1}{16\pi}(d^{D-1}x)_{\mu}(\nabla_{\nu}h^{\mu\nu} - \nabla^{\mu}h) \tag{3.34}$$

where

$$(d^{D-k}x)_{\mu_1\ldots\mu_k} = \frac{1}{k!(D-k)!}\epsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{D-k}}dx^{\nu_1}\wedge\ldots\wedge dx^{\nu_{D-k}}. \tag{3.35}$$

After inserting ξ it becomes

$$i_{\xi}\mathbf{\Theta}(g_{\mu\nu}, \delta) = \frac{1}{16\pi}(d^{D-2}x)_{\mu\nu}2\xi^{\nu}(\nabla_{\nu}h^{\mu\nu} - \nabla^{\mu}h). \tag{3.36}$$

The Noether current (3.9) is

$$\begin{aligned}\mathbf{J}_{\xi} &= \tfrac{1}{16\pi}(d^{D-1}x)_{\mu}\left(\nabla_{\nu}h^{\mu\nu} - \nabla^{\mu}h - (R-2\Lambda)\xi^{\mu}\right)\\ &= \tfrac{1}{16\pi}(d^{D-1}x)_{\mu}\nabla_{\nu}(\nabla^{\mu}\xi^{\nu} - \nabla^{\nu}\xi^{\mu}),\end{aligned} \tag{3.37}$$

where we have used $h_{\mu\nu} = \pounds_{\xi}g_{\mu\nu} = \nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu}$ and the Einstein's equations. The form $\mathbf{Q}_{\xi}$ and its variations are

$$\mathbf{Q}_{\xi} = \frac{1}{16\pi}(d^{D-2}x)_{\mu\nu}(\nabla^{\mu}\xi^{\nu} - \nabla^{\nu}\xi^{\mu}) \tag{3.38}$$

$$\begin{aligned}\delta\mathbf{Q}_{\xi} &= \tfrac{1}{16\pi}(d^{D-2}x)_{\mu\nu}\left(\tfrac{h}{2}(\nabla^{\mu}\xi^{\nu} - \nabla^{\nu}\xi^{\mu})\right.\\ &\left.+h^{\mu\rho}\nabla_{\rho}\xi^{\mu} - h^{\nu\rho}\nabla_{\rho}\xi^{\mu} - (\nabla^{\mu}h^{\nu\rho} - \nabla^{\nu}h^{\mu\rho})\xi_{\rho}\right).\end{aligned} \tag{3.39}$$

And finally from (3.18) we get form $\mathbf{k}_{\xi}(g_{\mu\nu}, h_{\mu\nu})$

$$\begin{aligned}\mathbf{k}_{\xi}(g_{\mu\nu}, h_{\mu\nu}) &= \tfrac{1}{16\pi}(d^{D-2}x)_{\mu\nu}k_{\xi}^{\mu\nu}\\ k^{\mu\nu} &= \xi^{\nu}\nabla^{\mu}h + \xi^{\mu}\nabla_{\rho}h^{\rho\nu} + \xi_{\rho}\nabla^{\nu}h^{\rho\mu} + \tfrac{1}{2}h\nabla^{\nu}\xi^{\mu} + h^{\mu\rho}\nabla_{\rho}\xi^{\nu} - (\mu\leftrightarrow\nu).\end{aligned} \tag{3.40}$$

We will use a slightly different expression obtained in the formalism of [18] where

$$\begin{aligned}k_{\xi}^{\mu\nu} &= \xi^{\nu}\nabla^{\mu}h + \xi^{\mu}\nabla_{\rho}h^{\rho\nu} + \xi_{\rho}\nabla^{\nu}h^{\rho\mu} + \frac{1}{2}h\nabla^{\nu}\xi^{\mu} +\\ &+h^{\mu\rho}\nabla_{\rho}\xi^{\nu} + \frac{1}{2}h^{\nu\rho}(\nabla^{\mu}\xi_{\rho} + \nabla_{\rho}\xi^{\mu}) - (\mu\leftrightarrow\nu).\end{aligned} \tag{3.41}$$

These two formulas differ by a term $\frac{1}{2}h^{\nu\rho}(\nabla^{\mu}\xi_{\rho} + \nabla_{\rho}\xi^{\mu})$. This term is zero for exact Killing vectors, but it plays a role for the asymptotic ones.

Chapter 4

Review of Kerr Spacetime

4.1 General Kerr Black Hole

The Kerr metric describes rotating black hole in four dimensions. It is the most general stationary solution of vacuum Einstein's equations. The metric was discovered by Roy P. Kerr in 1963 [24].

The Kerr metric in Boyer-Lindquist coordinates reads

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)d\varphi - adt\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \quad (4.1)$$

where

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta. \quad (4.2)$$

The metric is described by four coordinates. The coordinate t corresponds to a proper time of a static observer at spatial infinity. The distance from the black hole is described by the radial coordinate r. The coordinates θ and φ are angles on a sphere of constant time and radius. θ is a polar angle and φ is angle around the rotational axis. The standard order of the coordinates will be (t, r, θ, φ).

The metric has two free parameters M and a. They have a close connection to the charges of the black hole. The spacetime is stationary and the metric does not depend on two coordinates: t and φ. So it has a time-translation and rotation symmetries generated by two Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$[1]. The Komar integrals or the ADM computation give us that the mass of the black hole is precisely equal to M in our unit conventions. The angular momentum of Kerr black hole is $J = aM$, so the parameter a is connected to its rotation.

The Kerr black hole has usually two horizons that are located at the radii where $\Delta(r) = r^2 - 2Mr + a^2 = 0$. This condition gives us

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \quad (4.3)$$

[1]There are actually no other Killing vectors, but the metric has one Killing-Yano tensor, however this tensor is not relevant for our work.

We can see that the horizon radii become complex if $|a| > M$. In this case there would be no horizons and the spacetime would contain a naked singularity. Such spacetimes are considered unphysical, so the value of a is bounded by

$$-M \leq a \leq M. \tag{4.4}$$

The Kerr spacetime contains a frame-dragging effect. It can be imagined as if the black hole drags the nearby space along, so everything has to move with it. Any physical observer has to move on a timelike worldline, so consider a stationary observer with four-velocity $u^\mu = N(1, 0, 0, \Omega)$. The four-velocity is normalized as

$$u_\mu u^\mu = -1 = N^2(g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2). \tag{4.5}$$

The normalization constant N is real, so the expression in brackets has to negative. This condition gives us

$$\omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \leq \Omega \leq \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}}, \tag{4.6}$$

where we defined $\omega = \frac{g_{t\varphi}}{g_{\varphi\varphi}}$. The expression inside the square root is proportional to Δ, so as the observer approaches the horizon the interval of possible angular velocities gets narrower and the observer begins to be dragged by the rotating space. On the horizon he is forced to move with angular velocity

$$\Omega_H = \frac{a}{2Mr_+}. \tag{4.7}$$

This quantity is called horizon angular velocity. It is a constant, so the entire horizon moves as a one solid object. The motionless observer with respect to the black hole is not the static one with $\Omega = 0$, but the one that has angular velocity $\Omega = \omega$. This observer is called zero angular momentum observer (ZAMO).

Another property of the horizon is a surface gravity. It express the acceleration of an observer freely falling through the horizon, which is redshifted to spatial infinity. The redshift in necessary to define this quantity, because the acceleration itself is infinite. The surface gravity κ is defined by equation

$$\xi^\mu {\xi^\nu}_{;\mu}\Big|_{r=r+} = \kappa \xi^\nu, \tag{4.8}$$

where ξ is Killing vector

$$\xi = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \varphi}. \tag{4.9}$$

The surface gravity of Kerr black hole is

$$\kappa = \frac{r_+ - M}{2Mr_+} \tag{4.10}$$

and the associated Hawking temperature (2.4) is

$$T_H = \frac{\hbar\kappa}{2\pi} = \frac{\hbar(r_+ - M)}{4\pi M r_+}. \tag{4.11}$$

The final important properties of the horizon we mention are its area

$$A = \int \sqrt{g_{\theta\theta} g_{\varphi\varphi}}\Big|_{r=r_+} d\theta d\varphi = 8\pi M r_+ \tag{4.12}$$

and the corresponding entropy

$$S = \frac{A}{4\hbar} = \frac{2\pi M r_+}{\hbar}. \tag{4.13}$$

4.2 Extremal Kerr

The Kerr black hole becomes extremal if both of the horizons are at the same radius, which is equivalent to a condition that the surface gravity is zero. Equation (4.3) implies that Kerr black hole is extremal if

$$M = a \;\Leftrightarrow\; J = M^2 = a^2. \tag{4.14}$$

Both horizon radii become $r_+ = r_- = a$ (generally we shall use notation r_0 for the radius of the doubled horizon) and the metric simplifies to

$$\begin{aligned} ds^2_{ext} = &-\tfrac{(r-a)^2}{r^2+a^2\cos^2\theta}\left(dt - a\sin^2\theta d\varphi\right)^2 + \tfrac{\sin^2\theta}{r^2+a^2\cos^2\theta}\left((r^2+a^2)d\varphi - a dt\right)^2 \\ &+\tfrac{r^2+a^2\cos^2\theta}{(r-a)^2}dr^2 + (r^2+a^2\cos^2\theta)d\theta^2. \end{aligned} \tag{4.15}$$

As mentioned before the Hawking temperature has to be zero

$$T_H^{ext} = 0. \tag{4.16}$$

Finally the angular velocity of the horizon and the entropy become

$$\Omega_H^{ext} = \frac{1}{2a}, \tag{4.17}$$

$$S^{ext} = \frac{2\pi a^2}{\hbar}. \tag{4.18}$$

The key differences between extremal and non-extremal Kerr is the behavior of $\Delta = r^2 - 2Mr + a^2$ near the horizon. This function becomes $\Delta = (r-a)^2$ in the extremal case, so it goes to zero much faster close to the horizon. This behavior actually causes that the Hawking temperature vanishes. Another important consequence is that the proper distance from the horizon to some finite radius R is infinite, because the expression

$$\Delta s = \int_{r_0}^{R} \sqrt{g_{rr}} dr \sim \int_{r_0}^{R} \frac{1}{r - r_0} dr = \ln(R - r_0) - \ln 0 \tag{4.19}$$

is logarithmically divergent. But a test particle still falls into the black hole in finite proper time even though in has to travel infinite distance because of the infinite time dilation given by g_{tt}.

The extremal black holes have many useful properties but they probably cannot exist in the universe, because they cannot be obtained by a finite number of steps (like an absorbtion of particle) in a finite time. However a black hole can approach arbitrarily close to the extremality.

Chapter 5

Extremal Kerr/CFT Review

This chapter contains a review of the article [8]. Our developments are summarized in the chapter 6.

The process of the computation of the entropy of extreme Kerr begins by zooming to a close neighborhood of the outer horizon. To obtain this near horizon geometry we take the metric (4.15), we apply a coordinate transformation

$$\bar{t} = \frac{\lambda}{2a}t, \qquad \bar{r} = \frac{r-a}{a\lambda}, \qquad \phi = \varphi - \frac{t}{2a} \tag{5.1}$$

and then we take the limit $\lambda \to 0$. The shift in φ "cancels" the frame dragging effect near the horizon and transforms the metric into a corotating coordinate system (meaning that a particle near the horizon can move with zero angular velocity in the new coordinates, the frame-dragging effect with respect to the spatial infinity of course remains). The rescaling of the time and radius zooms "infinitely close" to the horizon (in sense of the value of the original coordinates because the proper length to the horizon is infinite). The successive limit $\lambda \to 0$ cuts off the outer parts of the spacetime. The result of this procedure is so called near horizon extreme Kerr (NHEK) geometry

$$ds^2_{NHEK} = 2a^2\Omega^2(\theta)\left(-\bar{r}^2 d\bar{t}^2 + \frac{d\bar{r}^2}{\bar{r}^2} + d\bar{\theta}^2 + \Lambda^2(\theta)(d\phi + \bar{r}d\bar{t})^2\right), \tag{5.2}$$

where

$$\Omega^2(\theta) = \frac{1+\cos^2\theta}{2}, \qquad \Lambda(\theta) = \frac{2\sin\theta}{1+\cos^2\theta}. \tag{5.3}$$

This metric was found in [25], where it is studied in detail.

For simplicity we will omit the bars over the coordinates $\bar{r}$ and $\bar{t}$ when there is no risk of confusion. The slices of constant θ remind AdS_3 metric. The space is not geodetically complete, so it is possible to do one more coordinate transformation to global coordinates

$$r = \tilde{r} + \cos\tau\sqrt{1+\tilde{r}^2}, \tag{5.4}$$

$$t = \frac{\sin\tau\sqrt{1+\tilde{r}^2}}{r}, \tag{5.5}$$

$$\phi = \tilde{\phi} + \ln\left(\frac{\cos\tau + \tilde{r}\sin\tau}{1+\sin\tau\sqrt{1+\tilde{r}^2}}\right). \tag{5.6}$$

This transformation is analogous to AdS coordinate transformation between Poincaré and global coordinates. In the global coordinates the metric reads

$$ds^2_{NH} = 2a^2\Omega(\theta)\left(-(1+\tilde{r}^2)d\tau^2 + \frac{d\tilde{r}^2}{1+\tilde{r}^2} + d\theta^2 + \Lambda^2(\theta)(\tilde{\phi} + \tilde{r}d\tau)^2\right). \tag{5.7}$$

We want to compute the entropy using conformal field theory, so we have to define a holographic theory in the NHEK spacetime. First we need to find a proper holographic screen. The choice is not obvious. We can use boundary at $r = \infty$ in (5.2) or $\tilde{r} = \infty$ in the extended space (5.7), in the chapter 6 we consider other possibilities. In [8] they decide to work in the global extension, but for example [9] uses the non-extended spacetime. We choose the NHEK space in form (5.2), but both of the metrics (5.2) and (5.7) have the same asymptotic behavior (in their own r), so the following calculations are valid in both cases. See chapter 6 for further discussion.

Once we have chosen the holographic screen, we begin to study the perturbations and asymptotic symmetries of the NHEK metric. The choice of consistent boundary conditions is also quite problematic, because the asymptotic behavior of the metric is different from conventional spacetimes like the Minkowski or AdS. The boundary condition must be chosen in such a way that the charges (3.15) related to the generators of the symmetry transformations are finite. On the other hand we are interested only in such transformations that at least some of the corresponding charges are nonzero. This usually leaves only a narrow class of boundary conditions, however it is sometimes possible to choose which components of the boundary conditions will allows symmetries with nontrivial charges and which will not. The freedom is mainly in the off-diagonal components (in the standard coordinates). It seems that the boundary conditions can be chosen differently for different purposes.

For the NHEK spacetime we choose boundary conditions following [8]

$$h = \begin{pmatrix} O(r^2) & O(r^{-2}) & O(r^{-1}) & O(1) \\ O(r^{-2}) & O(r^{-3}) & O(r^{-2}) & O(r^{-1}) \\ O(r^{-1}) & O(r^{-2}) & O(r^{-1}) & O(r^{-1}) \\ O(1) & O(r^{-1}) & O(r^{-1}) & O(1) \end{pmatrix}. \tag{5.8}$$

The boundary conditions should be usually subleading to the metric, which is not true in this case, however they still guarantee finiteness of the charges.

The generators of the symmetry transformations of the metric are vectors fields ξ. Their action on the metric is given by the Lie derivative, which is defined as

$$h_{\mu\nu} = \pounds_\xi g_{\mu\nu} = g_{\mu\nu,\rho}\xi^\rho + g_{\mu\rho}\xi^\rho{}_{,\nu} + g_{\nu\rho}\xi^\rho{}_{,\mu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}. \tag{5.9}$$

We shall call these vector fields asymptotic Killing vector fields, although sometimes they will be defined near a boundary, which is not an asymptotic infinity.

The most general vector that preserves these boundary conditions is in the form

$$\begin{aligned}\xi_F = (C + O(r^{-3}))\,\tfrac{\partial}{\partial t} + (-rF'(\phi) + O(1))\,\tfrac{\partial}{\partial r} \\ +O(r^{-1})\tfrac{\partial}{\partial \theta} + (F(\phi) + O(r^{-2}))\,\tfrac{\partial}{\partial \phi},\end{aligned} \tag{5.10}$$

where $F(\phi)$ is an arbitrary 2π-periodic function. The subleading terms correspond to diffeomorphisms with trivial charges. So the nontrivial asymptotic symmetries are

$$\xi_F = -rF'(\phi)\frac{\partial}{\partial r} + F(\phi)\frac{\partial}{\partial \phi}. \tag{5.11}$$

These vectors generate conformal group on the circle. If we define $\xi_m = \xi_{F_m}$, where $F_m(\phi) = -e^{-im\phi}$, as a basis, the vectors satisfy Virasoro algebra

$$i[\xi_m, \xi_n] = (m - n)\xi_{m+n}. \tag{5.12}$$

The ξ_0 generator is equal to the Killing vector $\frac{\partial}{\partial \phi}$ generating rotation, so these symmetries are extension of the $U(1)$ symmetry.

Using the metric (5.2) and vectors (5.11) we obtain the integral (3.29)

$$\int k_{\xi_m}[\pounds_{\xi_n} g; g] = -ia^2(m^3 + 2m)\delta_{m+n,0} \tag{5.13}$$

and we can easily read off the central charge

$$c_L = \frac{12a^2}{\hbar} = \frac{12J}{\hbar}. \tag{5.14}$$

We denoted this central charge c_L, as left-moving, because the Kerr black hole has also a right central charge, which will be derived in chapter 8. The notation is just a matter of convention, the algebras have different origins.

Now we have to find a temperature of the field theory. We cannot use the Hawking temperature (4.16), because it corresponds to a different quantum field, so we have to find an effective temperature that describes the distribution of the gravitational perturbations created by (5.11).

To compute this temperature we first expand some quantum field Φ into eigenmodes of asymptotic energy ω and angular momentum m

$$\Phi = \sum_{\omega,m,l} \phi_{\omega,m,l} e^{-i\omega t + im\phi} f_l(r, \theta). \tag{5.15}$$

Then we rewrite the argument of the exponential into the near horizon coordinates (5.1)

$$-i\omega t + im\varphi = -\frac{i}{\lambda}\left(2a\omega - \Omega_H^{ext}\right)\bar{t} + im\phi = -in_R\bar{t} + in_L\phi. \tag{5.16}$$

This equation defines new quantum numbers n_R, n_{Lj}

$$n_L = m, \qquad n_R = \frac{1}{\lambda}(2a\omega - m). \tag{5.17}$$

In the Frolov-Thorne vacuum the particles of Hawking radiation with quantum numbers ω and m are distributed with a Boltzmann factor [26]

$$e^{-\hbar\frac{\omega-\Omega_H m}{T_H}}. \tag{5.18}$$

If we use n_L and n_R, we can write this expression as

$$e^{-\hbar\frac{\omega-\Omega_H m}{T_H}} = e^{-\frac{n_R}{T_R}-\frac{n_L}{T_L}}, \tag{5.19}$$

where T_L and T_R are new effective temperatures associated with quantum numbers n_L and n_R. These temperatures are equal to

$$T_R = \frac{r_+ - M}{2\pi\lambda r_+}, \qquad T_L = \frac{r_+ - M}{2\pi(r_+ - a)}. \tag{5.20}$$

For the extremal Kerr the right temperature is

$$T_R = 0. \tag{5.21}$$

If we try to compute the left temperature of the extremal Kerr, we end up with expression $\frac{0}{0}$. But we can use (5.20) and take a limit $M \to a$. The limit is equal to

$$T_L = \frac{1}{2\pi}. \tag{5.22}$$

We can expect that our field theory is in thermal equilibrium with the Hawking radiation, so we will use (5.22) as an effective temperature of the theory. A generalized formula for effective temperatures can be found in appendix A.

The entropy of an unitary conformal field theory with central charge c_L and temperature T_L is (for large T_L) given by Cardy formula (see appendix B)

$$S = \frac{\pi^2}{3} c_L T_L. \tag{5.23}$$

Now we insert the temperature (5.22) and the central charge (5.14) in the Cardy formula (5.23) and we get entropy

$$S = \frac{2\pi a^2}{\hbar}. \tag{5.24}$$

We see that this value is equal to (4.18) so we have reproduced the Hawking-Bekenstein entropy.

This result is not entirely rigorous, because the Cardy formula is proved only for $T_L \gg c_L$, which is not obeyed for black holes larger then Planck scale. Nevertheless the value (5.24) is correct.

Chapter 6

Extremal Kerr/CFT without the Near Horizon Limit

The calculation of the entropy of extremal Kerr black hole summarized in the previous chapter is based on performing the near horizon limit, which results into a new spacetime, which is somehow similar to AdS_3. This procedure is repeated in most of the other articles about the entropy of extremal black holes (for example [27], [10], [11]). We have found that it is possible to repeat the calculations in the original spacetime without taking the near horizon limit by postulating boundary conditions on the horizon.

We begin with a look on the metric of extreme Kerr, which is given by (4.15)

$$\begin{aligned} ds_E^2 = &-\tfrac{(r-a)^2}{r^2+a^2\cos^2\theta}\left(dt - a\sin^2\theta d\varphi\right)^2 + \tfrac{\sin^2\theta}{r^2+a^2\cos^2\theta}\left((r^2+a^2)d\varphi - adt\right)^2 \\ &+\tfrac{r^2+a^2\cos^2\theta}{(r-a)^2}dr^2 + (r^2+a^2\cos^2\theta)d\theta^2. \end{aligned} \tag{6.1}$$

As mentioned in section 4.2, the near horizon part of the spacetime has an infinite volume because of the behavior of g_{rr}. This makes possible to do the scaling limit described in chapter 5, which restricts the spacetime to a region "close" to the horizon. If the volume was finite the zooming would cover a space of smaller and smaller volume and we would end up with infinitely thin strip of space.

This procedure is not necessary because we can restrict to the near horizon region just by considering r close to r_0. Instead of (5.1) we make simply the following transformation into the coordinate system corotating with the horizon

$$\phi = \varphi - \Omega_H^{ext} t. \tag{6.2}$$

The metric changes to

$$\begin{aligned} ds^2 = &-\tfrac{(r-a)^2}{r^2+a^2\cos^2\theta}\left(\tfrac{1+\cos^2\theta}{2}dt - a\sin^2\theta d\phi\right)^2 \\ &+\tfrac{(r^2+a^2)\sin^2\theta}{r^2+a^2\cos^2\theta}\left(d\phi - \tfrac{r^2-a^2}{2a(r^2+a^2)}dt\right)^2 + \tfrac{r^2+a^2\cos^2\theta}{(r-a)^2}dr^2 + (r^2+a^2\cos^2\theta)d\theta^2. \end{aligned} \tag{6.3}$$

The first nontrivial order of the expansion of (6.4) in powers of $(r-a)$ is exactly the same as the near horizon metric (5.2) (up to a shift in r and some rescaling), so we can see that the near horizon limit just cuts off the subleading terms.

We choose the following horizon boundary conditions on the horizon

$$h=\begin{pmatrix} O((r-a)^2) & O(1) & O(r-a) & O((r-a)^2) \\ O(1) & O((r-a)^{-1}) & O(1) & O((r-a)^{-1}) \\ O(r-a) & O(1) & O(r-a) & O(r-a) \\ O((r-a)^2) & O((r-a)^{-1}) & O(r-a) & O(1) \end{pmatrix}. \tag{6.4}$$

Additionally we require that $h_{\phi\phi}$ and $h_{r\phi}$ are time independent in the first order. These boundary conditions seem to by physically equivalent to (5.8) although they look different. The terms h_{tt} and $h_{\phi\phi}$ are of the same order as g_{tt} and $g_{\phi\phi}$ in the metric (6.1), while h_{rr}, $h_{\theta\theta}$ and $h_{t\phi}$ are subleading. The relative behavior of the boundary conditions (6.4) with respect to the metric (6.1) is the same as (5.8) to (5.2).

These boundary conditions are preserved by vectors

$$\begin{aligned}\xi_F = (C+O(r-a))\,\tfrac{\partial}{\partial t} + (-(r-a)F'(\phi)+O((r-a)^3))\,\tfrac{\partial}{\partial r} \\ +O(r-a)\tfrac{\partial}{\partial\theta} + (F(\phi)+O((r-a)^2))\,\tfrac{\partial}{\partial\phi},\end{aligned} \tag{6.5}$$

which also look like (5.10). The nontrivial part of this vector field is

$$\xi = -(r-a)F'(\phi)\frac{\partial}{\partial r} + F(\phi)\frac{\partial}{\partial\phi}, \tag{6.6}$$

where $F(\phi)$ is again an arbitrary periodic function, which we expand in $F_m(\phi) = -e^{-im\phi}$, and $\xi_m = \xi_{F_m}$ represent the basis of the generators.

The computation of the central charge and effective temperature can formally proceed in the same way as in chapter 5. The integral defining the central charge (3.29) with inserted metric (6.4) and vectors (6.6) is near the horizon equal to

$$12i\int k_{\xi_m}[\pounds_{\xi_n}g;g] = (12a^2m^3+24a^2m)\delta_{m+n,0}+O(r-a). \tag{6.7}$$

As we evaluate it on the horizon, the subleading term vanishes and we see that the central charge is equal to

$$c_L = \frac{12a^2}{\hbar} = \frac{12J}{\hbar}. \tag{6.8}$$

This value is the same as (5.14).

The effective temperatures also come out identical as in the previous chapter, because the transformation of φ in (5.1) is the same as (6.2). Only this coordinate transformation is important in this calculation, because the change of r does not even appear and the rescaling of time matters only in T_R, which is zero anyway. The equation (A.7) and coordinate transformation (6.2) give us values of the effective temperatures

$$T_R = \frac{T_H}{\hbar} \to 0, \quad T_L = \frac{1}{\hbar}\frac{T_H}{\Omega_H - \Omega_H^{ext}} \to \frac{1}{2\pi}. \tag{6.9}$$

The temperature and the central charge are the same as before, so it is not surprising that we obtain the correct entropy once again

$$S = \frac{\pi^2}{3} c_L T_L = \frac{2\pi a^2}{\hbar}. \tag{6.10}$$

So what is the difference between the approaches in this and the previous chapter? We will show that they describe the same physical situation and we just look from a different point of view. The main difference between our approach and the original Kerr/CFT is of course the fact that we define the boundary conditions on the horizon of the full Kerr spacetime instead of at asymptotic infinity of the NHEK space. But the whole NHEK geometry is actually contained in the neighborhood of $r = a$ in the full spacetime. To specify this claim, only the metric (5.2) is part of Kerr solution but not the extension (5.7). The coordinate transformation between the two NHEK metrics is analogous to the same one in AdS. It was first used in [25] and the motivation for it was the geodetical completeness of the spacetime. This is relevant for computation of charges in AdS, but we see no reason why the near horizon spacetime should be geodetically complete. The near horizon geometry is a segment of the Kerr spacetime close to the horizon. Any particle in the NHEK region can escape it if it falls through the horizon in the interior of the black hole or if it flies away to some finite radius r. The particle can do this in a finite proper time so this spacetime should not be geodetically complete. The global extension is certainly correct from the mathematical point of view, but we think that it is unphysical in this context.

In chapter 5 we defined boundary conditions at infinity of a space that is "infinitely close" to a horizon. So what sense can we make of this? The clash between the two infinities suggests that the boundary conditions are defined somewhere between the end of the near horizon region and the "finite r" region, but the distances there are infinite, so it is difficult to localize the boundary conditions from the global point of view.

So we shall take a different approach. When we use the NHEK metric (5.2) and the vectors (5.11) the integral (5.13) is equal to

$$\int k_{\xi_m}[\pounds_{\xi_n} g; g] = -ia^2(m^3 + 2m)\delta_{m+n,0}. \tag{6.11}$$

This is true even away from the boundary and there are no corrections, so the integral gives us the correct central charge at any value of $\bar{r}$. We can actually demand certain behavior of the perturbations in the whole NHEK spacetime and we have still guaranteed that there are symmetries with finite and nonzero charges. So we can construct the CFT on a screen of any radius with the same characteristics and entropy. This agrees with the fact that (5.11) is the near horizon part of (6.6).

From the global point of view the metric (5.2) is just the leading order of the expansion of (6.1) as noted before. The two metrics can match only when $r-a \ll a$, so the whole NHEK geometry is contained in the infinitesimal neighborhood of $r=a$. The vector field (6.6) becomes (5.11) after the near horizon limit. So the integrals (6.11) and (6.7) differ only in a subleading term and this term vanishes when we evaluate the integral on the horizon or when we perform the near horizon limit. The left central charge computed in the whole space reduces to the one in the NHEK geometry. However this is not true for the right central charge, which is lost in the near horizon limit as we shall see in chapter 8.

There is a question if the boundary conditions on the horizon describe the right degrees of freedom. So let's take some boundary conditions and evaluate charges (3.19) on the horizon. If the boundary conditions are too weak, the charges are divergent, which means that we have deformed the Kerr metric beyond the linearized theory. If the boundary conditions are too strong the charges are zero and do not create any states in the field theory. The remaining boundary conditions, that has finite charges, correspond to the degrees of freedom of the CFT. But are the degrees of freedom really located at horizon? Any perturbation can be expanded in $(r-r_+)$ and we can see that what matters is only the part that is of the same order as the allowed boundary conditions. When the perturbation is located far away from the black hole its near horizon part much smaller that our boundary conditions and it does not correspond to any state in the field theory. When the perturbation is of the same order as the boundary conditions, it can do anything far from the horizon (go to zero, constant or infinity) and it's contribution will be still the same. So the CFT reflects only the near horizon degrees of freedom of the perturbations and ignores anything far away.

Finally we shall have a look at the advantages and disadvantages of both procedures. They give us the same results in these calculations. The original procedure is fully dependent on the existence of the scaling limit, so it cannot be generalized to non-extremal black holes (at least we do not know how). It covers the near horizon region in much more detail, so it may depict the near horizon behavior in more detail, although we do not need it. Our procedure in the whole spacetime has the advantage that it can be simply generalized to non-extremal black holes. It simplifies the idea of computation of the black hole entropy, although the numerical calculations are mostly quicker in the NHEK geometry.

Chapter 7

CFT Duals for General Extremal Black Holes

In this chapter we generalize the results of the two previous chapters to general extremal black holes in arbitrary dimension. Our results are quite similar to [11], but we show that they can be computed in different way.

7.1 Some General Properties of Higher Dimensional Black Holes

We shall consider black holes in D dimensions with $R \times U(1)^k$ isometry group. The number of rotational symmetries k is usually equal to $\left[\frac{D-1}{2}\right]$, but it can be larger if we consider black holes of toroidal horizon topology (the explicit form of the near horizon geometries of such black holes were found in [28]). The symmetries are generated by Killing vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi^i}$. We shall use the following coordinates: the time coordinate t , radial coordinate r with required property that r is constant on the horizons, k angles φ^i and $l = D - k - 2$ remaining polar angles θ^a.

We assume that the metric of our black hole takes form

$$ds^2 = -\Delta(r)A(r,\theta^a)dt^2 + \frac{C(r,\theta^a)}{\Delta(r)}dr^2 + D_{ab}(r,\theta^a)d\theta^a d\theta^b \tag{7.1}$$
$$+B_{ij}(r,\theta^a)(d\varphi^i - (\Omega^i_{H+} + (r-r_+)\omega^i(r,\theta^a))dt)(d\varphi^j - (\Omega^j_{H+} + (r-r_+)\omega^j(r,\theta^a)dt).$$

The zeros of the function Δ determine the location of horizons. We will consider Δ in the form $\Delta(r) = (r-r_+)(r-r_-)$, where $r_+,\ r_-$ are radii of the horizons, any further r-dependence can be absorbed in A and C. The functions $A(r,\theta^a)$, $B_{ij}(r,\theta^a)$, $C(r,\theta^a)$ and $D_{ab}(r,\theta^a)$ are regular functions on the horizons. The combination $\Omega^i_{H+} + (r-r_+)\omega^i(r,\theta^a)$ (where Ω^i_{H+} are constants and $\omega^i(r,\theta^a)$ are regular functions at horizon) determines the angular velocity of the ZAMO observers. We have divided this function in such a way that Ω^i_{H+} is angular velocity of the outer horizon, but it can be done in the same way for the inner horizon. Finally we require that the ratio $\frac{A(r_\pm,\theta^a)}{C(r_\pm,\theta^a)}$ is θ^a independent on the horizons in order to have a well-defined surface gravity.

In [11] there is considered a slight generalization of this metric that includes $\Delta(r)A(r,\theta^a)dt^2 \rightarrow \Delta(r)A(r,\theta^a)(dt + f_i(r,\theta^a)d\varphi^i)^2$ but the metric can be always rewritten to the form (7.1) by redefinition of ω^i, B_{ij} and A. Or we can leave the metric in this form and we will just get $A + f_i\Omega^i_{H\ ext}$ instead of A in some equations. We do not know if this is the most general form of black hole metric, but it covers all known black hole solutions.

The horizon velocities Ω^i_H of a general higher dimensional black hole are given by

$$\Omega^i_{H\pm} = g_{t\varphi_j}(g^{-1}_{\varphi\varphi})^{\varphi^i\varphi^j}\Big|_{r=r_\pm}, \tag{7.2}$$

where $(g^{-1}_{\varphi\varphi})^{\varphi^i\varphi^j}$ means ij element of the inverse matrix to the $g_{\varphi\varphi}$ part of the metric. This is a matrix generalization of $\Omega_H = \frac{g_{t\varphi}}{g_{\varphi\varphi}}$. Using this equation we can check that Ω^i_{H+} in (7.2) are really the angular velocities of the outer horizon.

The surface gravity of a black hole κ is defined by equation

$$\xi^\mu \xi^\nu{}_{;\mu}\big|_{r=r+} = \kappa\xi^\nu, \tag{7.3}$$

which is equivalent to

$$(\xi^\mu\xi_\mu)_{,\nu}|_{r=r+} = -2\kappa\xi_\nu, \tag{7.4}$$

where ξ is Killing vector

$$\xi = \frac{\partial}{\partial t} + \Omega^i_H \frac{\partial}{\partial\varphi^i}. \tag{7.5}$$

To compute the surface gravity we usually use the "square" of the equation (7.4):

$$\kappa^2 = \lim_{r\to r_+} \frac{(\xi^\mu\xi_\mu)_{,\nu}(\xi^\rho\xi_\rho)^{,\nu}}{4(\xi^\sigma\xi_\sigma)}. \tag{7.6}$$

The knowledge of surface gravity allows us to compute easily the Hawking temperature, because they are related by

$$T_H = \frac{\kappa\hbar}{2\pi}. \tag{7.7}$$

For the metric (7.1) we get that the surface gravity and the Hawking temperature are equal to

$$\kappa^2 = \frac{(\Delta')^2\frac{\Delta A^2}{C} + O((r-r_+)^2)}{4\Delta A + O((r-r_+)^2)} \overset{r\to r_+}{=} \frac{(\Delta')^2 A(r_+)}{4C(r_+)} \sim \frac{(r_+ - r_0)^2 A(r_+)}{2C(r_+)}, \tag{7.8}$$

$$T_H = \frac{\hbar(r_+ - r_0)\sqrt{A(r_+)}}{2\pi\sqrt{C(r_+)}}. \tag{7.9}$$

The Hawking temperature must be a constant, so this is why we require that the ratio of A and C is θ^a-independent.

Finally the horizon area is equal to

$$A = \int \sqrt{\det B \det D}\; d^k\varphi d^l\theta = (2\pi)^k \int \sqrt{\det B \det D}\; d^l\theta, \tag{7.10}$$

so the entropy is

$$S = \frac{(2\pi)^k}{4\hbar} \int \sqrt{\det B \det D}\; d^l\theta. \tag{7.11}$$

When the black hole is extremal both of the horizon radii become one, which we will denote r_0. The function $\Delta(r) = (r - r_0)^2$ then has a double zero, so the properties known for Kerr black hole appear again: the Hawking temperature is zero (as we can see from (7.9)) and the distance from horizon to fixed value of r is infinite. There is one property we have not mentioned before, although it is also true for Kerr spacetime. The functions ω^i_{ext} have to be θ^a independent because their meaning is the derivative of angular velocity with respect to r. In the near extremal regime we can see that

$$\omega^i_{ext} \approx \left.\frac{\partial(\Omega_H + (r - r_+)\omega^i(r, \theta^a))}{\partial r}\right|_{r=r_+} \approx \frac{\Omega^i_{H+} - \Omega^i_{H-}}{r_+ - r_-} + O(r_+ - r_-). \tag{7.12}$$

The horizon velocities $\Omega^i_{H\pm}$ are constants, so the θ dependent corrections disappear as $r_+ \to r_-$. We will use this property when we compute the effective temperatures.

7.2 Entropy in the Near Horizon Geometry

In this section we follow the chapter 5, so we have to find the near horizon geometry first. To do so we make coordinate transformations that generalize (5.1)

$$\phi^i = \varphi^i - \Omega^i_{H\ ext} t, \tag{7.13}$$

$$\bar{t} = \frac{r_0 \lambda t}{\alpha}, \tag{7.14}$$

$$\bar{r} = \frac{r - r_0}{\lambda r_0}, \tag{7.15}$$

where $\alpha = \sqrt{\frac{C}{A}}$ is a constant. Then we take the limit $\lambda \to 0$, that cuts off the terms of higher order in $r - r_0$. After these transformations the metric (7.1) becomes:

$$\begin{aligned} ds^2_{NH} = &-\bar{r}^2 C(r_0, \theta^a) d\bar{t}^2 + \tfrac{C(r_0,\theta^a)}{\bar{r}^2} d\bar{r}^2 + D_{ab}(r_0, \theta^a) d\theta^a d\theta^b \\ &+ B_{ij}(r_0, \theta^a)(d\phi^i - \omega^i_{ext}\bar{r}\sqrt{\tfrac{C}{A}} d\bar{t})(d\phi^j - \omega^j_{ext}\bar{r}\sqrt{\tfrac{C}{A}} d\bar{t}). \end{aligned} \tag{7.16}$$

We will omit the bars over the coordinates from now on.

Now we can compute central charge in the same way as in chapter 5. First we choose one of the angular coordinates, say ϕ^1, and we define the following boundary conditions. The choice of the coordinate is arbitrary but it is necessary to specify the circles on which lives the CFT.

$$h = \begin{pmatrix} O(r^2) & O(r^{-2}) & O(r^{-1}) & O(1) & O(r) \\ O(r^{-2}) & O(r^{-3}) & O(r^{-2}) & O(r^{-1}) & O(r^{-1}) \\ O(r^{-1}) & O(r^{-2}) & O(r^{-1}) & O(r^{-1}) & O(r^{-1}) \\ O(1) & O(r^{-1}) & O(r^{-1}) & O(1) & O(1) \\ O(r) & O(r^{-1}) & O(r^{-1}) & O(1) & O(r^{-1}) \end{pmatrix} \tag{7.17}$$

Here the third line corresponds to θ^a coordinates, the fourth to ϕ^1 and the fifth to the remaining ϕ^j angles. The $h_{t\phi^j}$ components are of the same order as the metric for $j \neq 1$, so the symmetry between the angles is broken and it chooses the allowed transformations. The nontrivial asymptotic Killing vectors that preserve these boundary conditions are

$$\xi_F = -rF'(\phi^1)\frac{\partial}{\partial r} + F(\phi^1)\frac{\partial}{\partial \phi^1}, \tag{7.18}$$

the function F can be again expanded into basis $F(\phi^1) = e^{im\phi^1}$ and the vectors ξ_m satisfy Virasoro algebra (5.12).

Using these vectors and metric (7.16) we find that the central charge is

$$c_L = \frac{3}{2\pi\hbar}\int r g^{\varphi^1 t}\sqrt{-\det g}\; d^k\varphi d^l\theta. \tag{7.19}$$

The element of inverse metric is $g^{\varphi^1 t} = \frac{\omega^1_{ext}\sqrt{\frac{C}{A}}}{rC}$ and the metric determinant is $-\det g = C^2 \det B \det D$ so we get

$$c_L = \frac{3(2\pi)^k}{2\pi\hbar}\int \omega^1_{ext}\sqrt{\frac{C}{A}}\sqrt{\det B \det D}\; d^l\theta. \tag{7.20}$$

To obtain effective temperature we follow appendix A. The new quantum numbers n_R and n_L^j are

$$n_L^j = m_j, \qquad n_R = \alpha\frac{\omega - \Omega^j_{H\ ext} m_j}{\lambda} \tag{7.21}$$

and the Boltzman distribution factor becomes

$$e^{-\hbar\frac{\omega - \Omega^j_H m_j}{T_H}} = e^{-\frac{n_R}{T_R} - \frac{n_L^j}{T_L^j}}, \tag{7.22}$$

where the effective temperatures are given by (A.7)

$$T_R = \frac{\alpha T_H}{\hbar\lambda}, \qquad T_L^j = \frac{1}{\hbar}\frac{T_H}{\Omega_H^j - \Omega_{H\ ext}^j}. \tag{7.23}$$

In the extremal limit the Hawking temperature vanishes, so $T_R = 0$. The temperatures T_L^j have nonzero limit but we have to be careful when we compute them. We will follow the arguments from [11]. To be concrete we will compute the required temperature T_L^1.

Suppose that a is a parameter of the black hole that corresponds to the rotation in the ϕ_1 direction. The change of the horizon radius with a goes to infinity because

$$\frac{d\Delta(r_+)}{da} = 0 = \partial_r\Delta(r_+)\partial_a\ r_+ + \partial_a\Delta(r_+) \tag{7.24}$$

$$\partial_a r_+ = -\frac{\partial_a\Delta(r_+)}{\partial_r\Delta(r_+)} = -\frac{\partial_a\Delta(r_+)}{r_+ - r_-} \to \infty \tag{7.25}$$

as $r_+ \to r_-$. This means that the main change of the Hawking temperature and the horizon velocity is caused by the change of horizon radius. The left effective temperature T_L^1 is then

$$\begin{aligned} T_L^1 &= -\left.\frac{\frac{dT_H}{da}}{\frac{d\Omega_H^1}{da}}\right|_{a_i\to a_{ext}} = -\left.\frac{\partial_{r_+}T_H\partial_a r_+ + \partial_a T_H}{\partial_{r_+}\Omega_H^1\partial_a r_+ + \partial_a\Omega_H^1}\right|_{a\to a_{ext}} \\ &= -\left.\frac{\partial_{r_+}T_H}{\partial_{r_+}\Omega_H^1}\right|_{a\to a_{ext}} = \frac{1}{2\pi\omega_{ext}^1}\sqrt{\frac{A}{C}}. \end{aligned} \tag{7.26}$$

Now we can see that the inverse of the effective temperature can be read from the off-diagonal part of (7.16). So when we are given any near horizon geometry we can compute the effective temperatures in the same way as horizon velocities in (7.2)

$$\frac{1}{T_L^i} = 2\pi g_{NHt\varphi_j}(g^{-1}_{NH\varphi\varphi})^{\varphi^i\varphi^j}. \tag{7.27}$$

We have to note that this equation is correct only when the time is rescaled in such a way that the functions A and C are the same, as in (7.16). That is why we have chosen α to be equal to $\sqrt{\frac{C}{A}}$.

Finally we can use the Cardy formula (B.6) to compute the entropy. The $\omega_{ext}^1\sqrt{\frac{C}{A}}$ term in (7.20) precisely cancels with the one in (7.26), so the entropy is

$$S = \frac{\pi^2}{3}c_L T_L = \frac{(2\pi)^k}{4\hbar}\int\sqrt{\det B\det D}\ d^l\theta. \tag{7.28}$$

This expression is equal to (7.11) as we desired.

7.3 Entropy in the Full Spacetime

Now we will compute the entropy without the near horizon limit as in chapter 6. It is quite easy, because the hardest work has been already done in the previous section with the computation of effective temperature.

Again we begin with the metric (7.1) and we make a shift in the φ^i coordinates

$$\phi^i = \varphi^i - \Omega^i_{H\ ext} t, \tag{7.29}$$

so the metric changes to

$$\begin{aligned} ds^2 = -(r-r_0)^2 A(r,\theta^a)dt^2 + \tfrac{C(r,\theta^a)}{(r-r_0^2)}dr^2 + D_{ab}(r,\theta^a)d\theta^a d\theta^b. \\ +B_{ij}(r,\theta^a)(d\phi^i - (r-r_0)\omega^i_{ext}(r,\theta^a)dt)(d\phi^j - (r-r_0)\omega^j_{ext}(r,\theta^a)dt). \end{aligned} \tag{7.30}$$

We choose the preferred angle ϕ_1 and we generalize the boundary conditions (6.4)

$$h = \begin{pmatrix} O((r-r_0)^2) & O(1) & O(r-r_0) & O((r-r_0)^2) & O(r-r_0) \\ O(1) & O((r-r_0)^{-1}) & O(1) & O((r-r_0)^{-1}) & O((r-r_0)^{-1}) \\ O(r-r_0) & O(1) & O(r-r_0) & O(r-r_0) & O(r-r_0) \\ O((r-r_0)^2) & O((r-r_0)^{-1}) & O(r-r_0) & O(1) & O(1) \\ O(r-r_0) & O((r-r_0)^{-1}) & O(r-r_0) & O(1) & O(r-r_0) \end{pmatrix}. \tag{7.31}$$

As in (7.17) the component $h_{t\phi_1}$ is special, while for the other angles the components $h_{t\phi_j}$ are of the same order as the metric. The nontrivial asymptotic Killing vectors are

$$\xi_F = -(r-r_0)F'(\phi^1)\frac{\partial}{\partial r} + F(\phi^1)\frac{\partial}{\partial \phi^1}. \tag{7.32}$$

We can easily compute the central charge associated with these vectors and we find

$$c_L = \frac{3}{2\pi\hbar}\int (r-r_0)g^{\phi^1 t}(r)\sqrt{-g}\ d^k\varphi d^l\theta = \frac{3(2\pi)^k}{2\pi\hbar}\int \omega^1_{ext}\sqrt{\frac{C}{A}}\sqrt{\det B \det D}d^l\theta, \tag{7.33}$$

so it is the same as (7.20).

The computation of effective temperatures is also the same as in the previous section up to a rescaling of time, which is not important. So we shall only repeat the results

$$T_R = 0, \qquad T^i_L = \frac{1}{2\pi\omega^i_{ext}}\sqrt{\frac{A}{C}}\Bigg|_{r=r_0}. \tag{7.34}$$

So with the same central charge and effective temperature the entropy is again equal to (7.28) and these two approaches are equivalent.

There is an interesting fact that we do not have to assume the validity of Einstein's equations. The form (3.41) includes only gravitational contribution to the central charge. However it is probable that any matter field contributes only to the term linear in m in (3.27), so that the central charge remains unchanged. This have been proven in four and five dimensions for general action including scalar fields, Maxwell fields and topological term [23]. Nevertheless the duality to conformal field theory is proved for all extremal vacuum black holes.

There is a question why we can choose arbitrary angle and compute the entropy in the CFT connected to this angle. We can define k such CFTs that have mutually commuting generators, so one could naively think that the entropy would be given by

$$S = \frac{\pi^2}{3}(c_L^1 T_L^1 + c_L^2 T_L^2 + \ldots + c_L^k T_L^k). \tag{7.35}$$

But when we think more about it we find that this cannot be correct. First, the boundary conditions (7.17) and (7.31) are stronger for the $h_{t\phi}$ component in the preferred angle, so they are incompatible. Second, the Cardy formula gives entropy of two-dimensional CFT, which lives on $\mathrm{S}^1 \times \mathrm{R}$ at given values of θ^a and the remaining angles ϕ^j. Then we have to intergrate over these angles to get the entropy of the whole horizon. So (7.35) would count the degrees of freedom k-times. We can speculate that there is a nontrivial k-dimensional conformal field theory with weaker boundary conditions and much more complicated generator algebra that has all the CFTs we considered as "sub-theories", but such theory might be too complicated to study. It would be interesting to find a symmetry (probably connected to symmetries of T^k), that would transform the two-dimensional theories to each other.

Chapter 8

Right Virasoro Algebra and Near Extremal Kerr

In this chapter we show that the extremal Kerr has a second, right moving Virasoro algebra with the central charge c_R equal to $\frac{12a^2}{\hbar}$. This result can be derived only in the full spacetime and not in the NHEK geometry. We repeat the same computation for general black holes and we find that the right central charge has a different origin from the left one and that it is connected with perturbations of the horizon radius. Then we perturbatively compute the entropy of the near-extremal black holes.

The right central charge and near extremal black holes have been considered in several other articles. In [29] and [30] the right-moving Virasoro algebra is constructed in the near-NHEK space using similar vectors. However the computation of central charge requires the formalism of Brown and York [22] and the value of the central charge depends on a cut-off in the integration. The article [31] derives the same value of central charge for extremal Kerr as we do. It is computed by using an effective two-dimensional action and the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence. They agree with our result, that the right CFT describes excitations of the black hole. The last article we mention is [32]. The right Virasoro algebra is also considered there, but is found to be centerless.

8.1 Right Moving Virasoro Algebra

We begin with the metric (6.1), but this time we choose boundary conditions different from (6.4)

$$h = \begin{pmatrix} O((r-a)^3) & O((r-a)^{-1}) & O((r-a)^2) & O((r-a)^2) \\ O((r-a)^{-1}) & O((r-a)^{-1}) & O(r-a) & O(r-a) \\ O((r-a)^2) & O(r-a) & O((r-a)) & O((r-a)^2) \\ O((r-a)^2) & O(r-a) & O((r-a)^2) & O(r-a) \end{pmatrix}. \tag{8.1}$$

The relevant vector field satisfying the boundary conditions is

$$\xi_G = \frac{1}{\Omega} G(\Omega t) \frac{\partial}{\partial t} - (r - a) G'(\Omega t) \frac{\partial}{\partial r}. \tag{8.2}$$

This vector field includes an arbitrary time dependent function $G(\Omega t)$ and it commutes with (6.6). The function $G(\Omega t)$ can be expanded into basis $(\Omega t)^{1+m}$ and then the vectors ξ_m then satisfy Virasoro algebra. The Ω is a constant of arbitrary value, because the results are Ω-independent. The Ω can be fixed by requiring that Ωt is periodic in imaginary time with some period[1]. The Ω should be proportional to $\frac{1}{a}$ in order to Ωt being dimensionless. The best choice is probably $\Omega = \frac{1}{2a}$, because it corresponds to the $\varphi - \frac{t}{2a}$ dependence of (6.6) in the original Boyer-Lindquist coordinates and after this choice both of the central charges have the same value.

Before we can proceed to the computation of the central charge we need to ensure that the charges (3.19) are time independent because the integral over the surface of constant time does not choose the zero mode. It is possible to define the new generators as the residuum in the time coordinate as in [29]

$$L_m = \frac{\Omega}{2\pi i} \oint Q_{\xi_m} dt. \tag{8.3}$$

There is also another possibility: we can go to the Euclidean regime where the expansion is taken in $e^{im\Omega t}$ and then we can define the generators as average of the charges over the time period.

Using the formula (3.30) we find that the central charge is equal to

$$c_R = \frac{24 a^3 \Omega}{\hbar}. \tag{8.4}$$

This value the same as (5.14) for $\Omega = \frac{1}{2a}$

$$c_R = \frac{12 a^2}{\hbar}. \tag{8.5}$$

The generalization to the metric (7.30) is quite straightforward. We can use the boundary conditions (8.1) with the second line used for all θ^a angles and the fourth for all ϕ^i angles. The asymptotic vector field is the same as (8.2). Then we find the central charge equal to

$$c_R = \frac{3(2\pi)^k \Omega}{2\pi\hbar} \int \sqrt{\frac{C}{A}} \partial_r \sqrt{\det B \det D}\ d^l\theta. \tag{8.6}$$

Now we can clearly see the difference between the left and the right central charge. The left one (7.20) is proportion to the integral of the square root of the metric

[1] This condition quantizes Ω and with it the central charge.

determinant, while the right one is proportional to the integral of the derivative of this quantity. So the right charge cannot be found in the near horizon geometry because all the functions in the metric become r-independent after the scaling limit.

This result stands against the hypothesis of [33] that the left and the right central charge are the same. The charges are the same for the extremal Kerr, but if we suppose that (8.6) is valid even for non-extremal Kerr we find that the charges are equal only for extremal Kerr. In chapter 9 we compute the charges of non-extremal black holes, but the they still remain different.

8.2 Entropy of Near Extremal Black Holes

Once we know the right central charge it is possible to compute the entropy of near extremal Kerr.

The mass of the near extremal Kerr can be parameterized as

$$M = a\left(1+\frac{\epsilon^2}{2}\right). \tag{8.7}$$

The change of the entropy and the Hawking temperature up to the first order in ϵ is

$$S = \frac{2\pi M r_+}{\hbar} = \frac{2\pi a^2}{\hbar}(1+\epsilon+O(\epsilon^2)), \tag{8.8}$$

$$T_H = \frac{\hbar(r_+ - M)}{4\pi M r_+} = \frac{\epsilon\hbar}{4\pi a}(1+O(\epsilon)). \tag{8.9}$$

This time the right effective temperature associated with (8.2) is nonzero and using (A.7) we get

$$T_R = \frac{T_H}{\hbar\Omega} = \frac{\epsilon}{4\pi a\Omega}(1+O(\epsilon)). \tag{8.10}$$

We have to assume that the central charges and the left effective temperature remain the same as for the extremal Kerr. This does not have to be true, because the metric changes nontrivially. But we can consider the following picture. We take the extremal Kerr as a background and we induce the non-extremality by exciting the right sector. So (6.8) and (6.9) give the entropy of the extremal Kerr, while the first perturbative correction is given by (8.4) and (8.10). The entropy consisting of the left and the right contribution is

$$S = \frac{\pi^2}{3}(c_L T_L + c_R T_R) = \frac{2\pi a^2}{\hbar}(1+\epsilon+O(\epsilon^2)), \tag{8.11}$$

so it is equal to (8.8) in the first order. The theory on the near-extremal background has to constructed in a different way, see chapter 9, and the entropy can be explained by the left CFT in this case.

For the general metric (7.30) the calculations proceed in the same way. Let's again denote the parameter measuring the deviation from extremality ϵ. Using the arguments from chapter 7 we can assume that the main changes of all quantities are caused by the change of the horizon radius. So we can identify $2\epsilon = r_+ - r_-$. The entropy of near extremal black hole is equal to

$$\begin{aligned} S = S_{ext} + \partial_r S_{ent} \Delta r_+ = \tfrac{1}{4\hbar} \int \sqrt{\det B \det D}\ d^k\varphi d^l\theta \\ + \tfrac{\epsilon}{4\hbar} \int \partial_r \sqrt{\det B \det D}\ d^k\varphi d^l\theta. \end{aligned} \tag{8.12}$$

The right effective temperature is again proportional to ϵ

$$T_R = \frac{T_H}{\hbar\Omega} = \frac{\epsilon\sqrt{A}}{2\pi\Omega\sqrt{C}}. \tag{8.13}$$

So from the Cardy formula (B.6) we can read off that the entropy is

$$S = \frac{\pi^2}{3}(c_L T_L + c_R T_R) = S_{ext} + \frac{\epsilon}{4\hbar} \int \partial_r \sqrt{\mathrm{Det}B\ \mathrm{Det}D}\ d\varphi^k d\theta^l, \tag{8.14}$$

which agrees with (8.12). From this equation we can see why the right CFT describes non-extremal excitations. The right central charge (8.6) is proportional to the change of the entropy with the horizon radius, while the temperature (8.10) measures the deviation from extremality.

Chapter 9

Entropy of Non-extremal Black Holes

In this chapter we shall finally concentrate on general non-extremal black holes. The process of computing the entropy has to be different from the one used in chapter 5, because the scaling limit (5.1) would give us $g_{tt} = 0$ and $g_{rr} = \infty$, so the metric would not be regular. However our procedure using the boundary conditions on the horizon from the chapter 6 still makes sense.

9.1 Non-extremal Kerr

Again the first thing we do is that we transform the Kerr metric (4.1) into the coordinate system corotating with the horizon. The transformation is

$$\varphi = \phi + \Omega_H t \tag{9.1}$$

and the metric becomes

$$\begin{aligned} ds^2 = &-\tfrac{\Delta}{\rho^2}\left(dt - a\sin^2\theta(d\phi + \Omega_H dt)\right)^2 + \tfrac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \\ &+\tfrac{\sin^2\theta}{\rho^2}\left((r^2+a^2)(d\phi + \Omega_H dt) - a dt\right)^2 . \end{aligned} \tag{9.2}$$

We are interested in the generalization of the procedure from chapter 6. We do not begin with the boundary conditions, but with the asymptotic vectors. The possible generalizations (6.6) and (8.2) are

$$\xi^L_F = -R(r)F'(\phi)\frac{\partial}{\partial r} + F(\phi)\frac{\partial}{\partial \phi}, \tag{9.3}$$

$$\xi^R_G = -R(r)G'(\Omega t)\frac{\partial}{\partial r} + \frac{1}{\Omega}G(\Omega t)\frac{\partial}{\partial t} \tag{9.4}$$

where $F(\phi)$ and $G(\Omega t)$ are arbitrary functions as before. They can be expanded in the fourier bases and we get two commuting Virasoro algebras. $R(r)$ is a radius-dependent function and there are two natural choices of R that generalize 6.6): $R(r) = b(r - r_+)$ and $R(r) = b\sqrt{\Delta(r)}$, where b is a constant. When we look at the perturbations of the metric given by the Lie derivative, we find that the h_{tt} component is given by $\pounds_\xi g_{tt} = h_{tt} = const\ R(r)$ while $g_{tt} = O(r - r_+)$. It follows that the second choice is unacceptable because the perturbation would be greater than the metric close to the horizon. So have to choose the first option

$$R(r) = b(r - r+). \tag{9.5}$$

Using the standard procedure of computing the central charges we find that

$$c_L = c_R = 0. \tag{9.6}$$

This is not a result we expected. It means that the procedure from chapter 6 has to be modified to get the entropy correctly. So our first step will be to find out why the central charges are zero. We define a more general vector field

$$\xi_F = b_1 F(\Omega t + \gamma\phi)\frac{\partial}{\partial t} - b_2(r - r_+)F'(\Omega t + \gamma\phi)\frac{\partial}{\partial r} + b_3 F(\Omega t + \gamma\phi)\frac{\partial}{\partial \phi}, \tag{9.7}$$

where $b_1,\ b_2,\ b_3,\ \Omega$ and γ are constants. The constants have to satisfy $b_1\Omega + b_3\gamma = 1$ to ensure that the vectors form Virasoro algebra (5.12). We want the vectors to be 2π periodic in ϕ, so γ has to be integer. If we choose $b_1 = 0,\ \Omega = 0$ or $b_3 = 0,\ \gamma = 0$ we can see that the vectors (9.3) and (9.4) are just special cases of (9.7).

The integral defining central charge associated with (9.7) is

$$\int \frac{9\ b_2\ b_3 r_+^2 (r_+ + r_-)\Omega\gamma \sin\theta}{2(r_+ - r_-)} d\theta\ m^3 \delta_{m+n,0} + O(m) + O(r - r_+). \tag{9.8}$$

In (9.3) and (9.4) either Ω or γ is zero, so the central charges vanish on the horizon. There is $r_+ - r_-$ term in the denominator that goes to infinity for $r_+ \to r_-$. In the extremal case this term becomes $r - r_0$ and changes the expansion by one order. For vectors (6.6) $\Omega = 0$, so that the divergent term disappears and the linear term becomes finite on the horizon and it gives us the central charge.

The effective temperatures are given by (A.7) and we can compute them using the transformation (9.1) and the effective coordinate in (9.7)

$$T^{eff} = \frac{T_H}{\hbar\Omega}. \tag{9.9}$$

For (9.3) and (9.4) the temperatures become

$$T_L^{eff} = \frac{T_H}{0} \to \infty, \tag{9.10}$$

$$T_R^{eff} = \frac{T_H}{\hbar\Omega}. \tag{9.11}$$

We see that the left effective temperature goes to infinity and the right one is finite. This suggests that the right moving part does not contribute to the entropy as for the extremal Kerr, but there can be a reasonable limit of $T_L c_L$. We just need to find a correct regularization. This is also similar to extremal Kerr, where we have to regularize the computation of the effective temperature, the limit just moved to the product of $c_L T_L$.

First, we define our boundary conditions and evaluate the charges on a screen at radius $r = r_+ + \epsilon$ to regularize the central charge. The constant ϵ is an infinitesimal parameter with a dimension of length. At the end of the calculations we shall take limit $\epsilon \to 0$. And as a second change we use a vector field similar to (9.3), but we add some corrections proportional to ϵ

$$\xi_F^L = -(b_1(r-r_+) + b_2\epsilon)F'(\phi + \omega(r_+ + \epsilon, \theta)t)\frac{\partial}{\partial r} + F(\phi + \omega(r_+ + \epsilon, \theta)t)\frac{\partial}{\partial \phi}, \tag{9.12}$$

where $\omega(r,\theta)$ is angular velocity function $\frac{g_{t\phi}}{g_{\phi\phi}}$ (it is different from ωs in 7.30 by a factor $r - r_+$). This function works as a regularization of the effective temperature.

The meaning of the term with ω in F is that the perturbations move with respect to ZAMO observer at $r = r_+ + \epsilon$ and not just with respect to horizon as in all previous computations. In this coordinate system ω is proportional to $r - r_+$ and its value near the screen is

$$\omega = \frac{a(3r_+^2 + 2r_+r_- - r_-^2 - r_-(r_+ - r_-)\sin^2\theta)}{r_+^2 M^3}\epsilon + O((r - r_+, \epsilon)^2). \tag{9.13}$$

The vector (9.12) becomes (9.3) when we send $\epsilon \to 0$. From now on we shall do the expansions both in $r - r_+$ and ϵ. This has to be done carefully and the notation would be messy, so we will continue to write for example $O(r - r_+)$, but the reader should also imagine ϵ there. We define the following boundary conditions to specify the vectors in more details.

$$h = \begin{pmatrix} O(r-r_+) & O(r-r_+) & O(r-r_+) & O((r-r_+)^2) \\ O(r-r_+) & O((r-r_+)^{-1}) & O(r-r_+) & O(1) \\ O(r-r_+) & O(r-r_+) & O(r-r_+) & O(r-r_+) \\ O((r-r_+)^2) & O(1) & O(r-r_+) & O(1) \end{pmatrix} \tag{9.14}$$

As for the extremal black holes some of the boundary conditions are of the same order as the metric. The requirement that $h_{t\phi} = O((r - r_+)^2)$ is once again the key factor. It fixes the values of b_1 and b_2 to $b_1 = 1$ and $b_2 = 1$.

The effective temperature associated with the coordinate $\phi + \omega(r_+ + \epsilon, \theta)t$ is

$$T_L^{eff}(\theta) = \frac{T_H}{\hbar\omega(r_+ + \epsilon, \theta)} = \frac{r_+ - M}{4\pi M r_+}\frac{1}{\omega(r_+ + \epsilon, \theta)}. \tag{9.15}$$

This temperature diverges as the value of ϵ approaches zero. It is, unlike the Hawking temperature, θ dependent, so it is only a local property. This causes no problems because the two-dimensional CFT lives on a circle of a constant θ and the integration over this angle just counts the whole set of CFTs on different angles, which have different temperatures. So we just need to move the integration after applying the Cardy formula.

The central charge[1] at constant θ is

$$c_L(\theta) = \frac{3a(r_-^2 - 3r_-r_+ + 6r_+^2 + r_-(r_- - r_+)\cos^2\theta)\sin\theta}{\hbar(r_-^2 - r_+^2)}\epsilon + O(\epsilon^2). \tag{9.16}$$

When we apply the Cardy formula the ϵ-dependence cancels in the first order and in the limit $\epsilon \to 0$ we get

$$S = \int_0^\pi \frac{\pi^2}{3} c_L(\theta) T_L(\theta) d\theta = \frac{\pi r_+(r_+ + r_-)}{\hbar} = \frac{2\pi r_+ M}{\hbar}. \tag{9.17}$$

This value is in precise agreement with (4.13). This time the Cardy formula is well defined, because the temperature is much larger than the central charge.

9.2 General Black Holes

As always we generalize the results of section 9.1 to higher dimensional black holes. The form of the metric and the coordinates is similar to (7.1).

$$\begin{aligned} ds^2 = -(r-r_+)(r-r_-)\ A(r,\theta^a)dt^2 &+ \tfrac{C(r,\theta^a)}{(r-r_+)(r-r_-)}dr^2 + D_{ab}(r,\theta^a)d\theta^a d\theta^b \\ &+B_{ij}(r,\theta^a)(d\phi^i - (\Omega_H^i + (r-r_+)\ \omega^i(r,\theta^a))dt) \\ &\times(d\phi^j - (\Omega_H^j + (r-r_+)\ \omega^j(r,\theta^a))dt) \end{aligned} \tag{9.18}$$

In corotating coordinate system the metric simplifies to

$$\begin{aligned} ds^2 = -(r-r_+)(r-r_-)\ A(r,\theta^a)dt^2 &+ \tfrac{C(r,\theta^a)}{(r-r_+)(r-r_-)}dr^2 + D_{ab}(r,\theta^a)d\theta^a d\theta^b \\ &+B_{ij}(r,\theta^a)(d\varphi^i - (r-r_+)\ \omega^i(r,\theta^a)dt) \\ &\times(d\varphi^j - (r-r_+)\ \omega^j(r,\theta^a)dt) \end{aligned} \tag{9.19}$$

where we have performed transformation $\varphi^i = \phi^i + \Omega_H^i t$.

As a holographic screen we take a hyperplane at $r = r_+ + \epsilon$. The expansions are again taken in both $r - r_+$ and ϵ. We choose the preferred angle ϕ^1 and we define

[1]We did not find a natural way how to quantize the central charge. We would have to quantize ϵ and that would lead to quantization of the surface. However this might correspond to the ideas of the loop quantum gravity.

boundary conditions

$$h = \begin{pmatrix} O(r-r_+) & O(r-r_+) & O(r-r_+) & O(r-r_+) & O(r-r_+) \\ O(r-r_+) & O((r-r_+)^{-1}) & O(r-r_+) & O(1) & O(1) \\ O(r-r_+) & O(r-r_+) & O(r-r_+) & O(r-r_+) & O(r-r_+) \\ O(r-r_+) & O(1) & O(r-r_+) & O(1) & O(1) \\ O(r-r_+) & O(1) & O(r-r_+) & O(1) & O(r-r_+) \end{pmatrix}, \quad (9.20)$$

where the third line corresponds to θ^a coordinates, the fourth to ϕ^1 and the fifth to the remaining ϕ^j angles. Note that we have relaxed the condition for $h_{t\phi^1}$, we require that only the $g_{\phi^1\phi^1}$-proportional part of this term is $O((r-r_+)^2)$. These boundary conditions are satisfied by vector field

$$\xi^L_F = -(r-r_+ + \epsilon)F'(\phi^1 - \epsilon\omega^1(r_+,\theta^a)t)\frac{\partial}{\partial r} + F(\phi^1 - \epsilon\omega^1(r_+,\theta^a)t)\frac{\partial}{\partial \phi^1}. \quad (9.21)$$

The central charge computed at constant θ^a is

$$c_L(\theta^a) = (2\pi)^k \frac{3\sqrt{C(r_+,\theta^a)}\sqrt{\det B(r_+,\theta^a)}\sqrt{\det D(r_+,\theta^a)}\omega^1(r_+,\theta^a)}{\pi\sqrt{A(r_+,\theta^a)}(r_+ - r_-)\hbar}\epsilon + O(\epsilon^2). \quad (9.22)$$

It goes to zero as ϵ approach zero as in section 9.2. The effective temperature behaves also in the same way. Using (A.7) we find that the left temperature is equal to

$$T^1_L(\theta^a) = \frac{T_H}{\epsilon\omega^1(r_+,\theta^a)\hbar}, \quad (9.23)$$

where the Hawking temperature (computed by (7.6) and (7.7)) is

$$T_H = \frac{\hbar(r_+ - r_-)}{4\pi}\sqrt{\frac{A(r_+,\theta^a)}{C(r_+,\theta^a)}}. \quad (9.24)$$

When we use the Cardy formula we can see that the ϵ dependance cancels, so we can take limit $\epsilon \to 0$, and the entropy of the black hole is

$$S = \int \frac{\pi^2}{3} c_L(\theta^a) T_L(\theta^a) d\theta^a = \frac{(2\pi)^k}{4\hbar}\int \sqrt{\det B(r_+,\theta^a)}\sqrt{\det D(r_+,\theta^a)} d^l\theta. \quad (9.25)$$

The horizon area of the black hole (9.18) is

$$A = (2\pi)^k \int \sqrt{\det B(r_+,\theta^a)}\sqrt{\det D(r_+,\theta^a)} d^l\theta, \quad (9.26)$$

so (9.25) is equal to $\frac{A}{4\hbar}$ as expected.

Finally we shall shortly mention the right central charge. We can use vector field (9.4) to compute the right central charge on the same screen as the left one. Its value is

$$c_R = \frac{3(2\pi)^k \Omega}{2\pi\hbar}\epsilon \int \sqrt{\frac{C}{A}} \partial_r \sqrt{\det B \det D} d^l\theta + O(\epsilon^2). \tag{9.27}$$

It looks similar to (8.6), so it can be used to describe perturbations changing the horizon radius. However we have to note that the two Virasoro algebras do not commute exactly, but only up to order $O(\epsilon)$. We did not have time to investigate if this can be corrected.

9.3 Limit to Extremal Kerr

At this moment we can look what happens to these computation as we approach to the extremal Kerr. We show that the conformal field theory has a smooth limit to the one obtained for the extremal black hole in chapter 6.

We have used the regularization with $\omega(r_+ + \epsilon, \theta)$, but what happens as approach the extremal regime? We can set $\epsilon = \frac{r_+ - r_-}{2} = \sqrt{M^2 - a^2}$, which cancels the diverging term in (9.16), and the central charge becomes

$$c_L = \int_0^\pi \frac{3a^2 \sin\theta}{\pi\hbar} d\theta + O(\epsilon), \tag{9.28}$$

so we can take limit $r_- \to r_+$ and the central charge becomes equal to (5.14) after the integration.

The value of ϵ also cancels the $r_+ - M$ term in (9.15) and the effective temperature is

$$T_L^{eff} = \frac{1}{2\pi} + O(\epsilon). \tag{9.29}$$

So it is equal to (6.9) when $\epsilon \to 0$. The Cardy formula gives us

$$S = \frac{\pi^2}{3} c_L T_L = \frac{2\pi a^2}{\hbar}, \tag{9.30}$$

so we get the entropy correctly.

On the other hand in the extremal case we can take the limit $\epsilon \to 0$ first. The holographic screen moves to the horizon and (9.12) becomes (6.6). So we exactly recover the theory from chapter 6.

This limit can be repeated for the general metric (9.19) in the same way. When we replace ϵ by $\frac{r_+ - r_-}{2}$ all expressions become finite and we recover the same central charge, effective temperature and entropy as in chapter 7.

9.4 Comparison to Other Approaches

The boundary conditions on the horizon have been studied by S. Carlip [34]. We shall briefly explain his approach and compare it to ours.

The first thing we should notice is that Carlip uses a Hamiltonian formalism for the asymptotic charges, but we can suppose that the results should not be formalism-dependent. The main difference between the two approaches are Carlip's conditions on the generators and symmetries. He defines different boundary conditions for the metric and he requires that his generators depend on t and r only in a retarded time $t-r_*$, where r_* is defined by $\sqrt{C}dr = \Delta\sqrt{A}dr_*$ in our notation. So his perturbations describe ingoing waves, while our perturbations describe "stationary" waves near the horizon. With these generators he derives the central charge

$$c = \frac{3A}{2\pi T_H T}, \tag{9.31}$$

where T is arbitrary constant. This value does not generally match either (7.20) or (9.22).

So we have to conclude that the only common point of these two approaches is the definition of the boundary conditions on the horizon, the rest of the calculations is different.

Next we would like to compare our results to the recent article [33] and the successive articles (for example [35], [36], [37], [37]).

In the article [33] the main object of interest is the massless wave equation instead of the holographic field theory. The radial part of the D'Alambert operator $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ commutes with six vector fields in a so called matching region[2]. These vector fields form $SL(2,R)_L \times SL(2,R)_R$ algebra, so they are supposed to be part of two hidden Virasoro algebras. The vectors of the left and right algebra depend on t and φ as

$$e^{2\pi T_L \varphi - \frac{t}{4M}} \quad \text{and} \quad e^{2\pi T_R \varphi}, \tag{9.32}$$

where

$$T_L = \frac{r_+ + r_-}{4\pi a} \quad \text{and} \quad T_R = \frac{r_+ - r_-}{4\pi a}. \tag{9.33}$$

So the vectors are not periodic in φ. This is interpreted as a symmetry breaking by the identification of φ and the quantities (9.33) are identified as temperatures of the CFTs.

The authors assume that the central charges of the extremal Kerr $c_L = c_R = \frac{12J}{\hbar}$ maintain their values even for non-extremal Kerr. These central charges and

[2]The region is restricted by $M \ll r \ll \frac{1}{\omega}$, where M is a mass of the black hole and ω is a frequency of the waves. It depends on the frequency of the waves, so it is not just a geometrical condition.

temperatures give the correct entropy for Kerr

$$S = \frac{\pi^2}{3}(c_L T_L + c_R T_R) = \frac{2\pi M r_+}{\hbar}. \tag{9.34}$$

It can be clearly seen that this approach is not consistent with ours, because the central charges of the non-extremal Kerr are different from ours. We can in principle make the central charge (9.16) to be equal to $\frac{12J}{\hbar}$ by a correct choice of ϵ, but this can be done only in the near extreme regime where ϵ can be comparable to $r_+ - r_-$.

The inconsistency of the approaches can be expected, because the symmetry vectors of the wave equation are defined in the matching region, which is far from the horizon, they do not have the periodicity of the spacetime and they are divergent near the horizon. So they do not match any asymptotic symmetries we considered in this article and they clearly describe different CFTs, which can have different central charges. We tried to compute the central charges of the Virasoro extension of the $SL(2,R)_L \times SL(2,R)_R$ algebra, but we failed to obtain well-defined charges.

Chapter 10

Summary and Discussion

Our first result is the rederivation of the entropy of the extremal Kerr (and extremal black holes generally) in the full spacetime. We think that the Kerr/CFT correspondence in the near horizon geometry can be obtained by the scaling limit in any phase of the calculations, although we have not proven for example that the boundary conditions (5.8) follow from (6.4). The main advantage of our procedure is that it does not require the near horizon limit, so it can be easily generalized.

We used this method to construct the right Virasoro algebra and to compute its central charge in chapter 8. Both of the central charges of extremal Kerr are the same (when we choose Ω properly) but we do not think that this is true generally, because the charges come from a different expressions. The right CFT does not contribute to the entropy of the extremal Kerr, but it can be used to describe non-extremal perturbations. We can compute the entropy of near-extremal black holes that are described as perturbations of the extremal ones. The whole construction seems to be not entirely rigorous, because some of the expansions in $r - r_0$ stop to make sense once the horizon radius changes, but a more detailed analysis of the consistency of the boundary conditions and charges is needed.

Finally we have found a way how to compute an entropy of non-extremal black holes. It is more complicated that in the extremal case, because the limit in the computation of the effective temperature moves in the product of the central charge and the temperature. We have found a possible regularization using a screen little bit away from horizon that gives us the the correct entropy, but we have to use central charge and the temperature depending on the θ angle and location of the screen. This suggests that these quantities are highly effective and only their product, which describes the entropy, makes sense from the global point of view.

There are also other possible regularizations that give us the correct entropy, but they have different central charges and they usually result into a different CFT in the extremal limit. We did not have enough time to check the consistency of our boundary conditions in detail and to classify the possible regularizations and their limits, so this is needed to be done in detail to understand the theory properly.

The central charges we obtained are different from the ones assumed in [33]. This is not surprising because we describe a CFT in a different region. However we can see that in the near-extremal case the central charge (9.28) remains equal to $\frac{12J}{\hbar}$ when evaluated on a proper screen. This suggests that there can be a different CFT description on a screen of a finite radius, but we do not know how to construct such theory.

Finally we shall mention that all our results concern only rotating black holes. The computations do not work for static black holes. We can describe them as a limit of rotating black holes but the construction of CFT on their background fails because the central charge is proportional to ω, so it is zero. It is possible to regularize the computations by hand but the final expression for entropy includes unfixed constants that would have to take non-intuitive values to get the entropy correctly. So the static black holes remain to be unsolved problem in this approach. However it is possible to compute the entropy of Reissner-Nordström black hole by uplifting the solution to five dimensions [38], [39].

Appendix A

Computation of Effective Temperatures

Suppose we have an arbitrary quantum field Φ in a black hole spacetime with $R \times U(1)^k$ isometry group generated by $\frac{\partial}{\partial t}$ and by k $\frac{\partial}{\partial \varphi^i}$ vectors. For simplicity we will use scalar field, but the computations can be repeated for higher spin fields.

The coordinate dependent part of the field can be expanded into eigenmodes of the Killing vectors

$$\Phi = \sum_{\omega, m_1, \ldots, m_k, l} \phi_{\omega, m_1, \ldots, m_k, l} e^{-i\omega t + i m_j \varphi^j} f_l(r, \theta^a), \tag{A.1}$$

where ω is asymptotic energy, m_j is asymptotic angular momenta and l represents all other quantum numbers . The set of coordinates (t, φ^j) and set of quantum numbers $(-\omega, m_j)$ will be denoted x^j and μ_j respectively, $j = 0, \ldots, k$. Then we may write the argument of the exponential simply as

$$-i\omega t + i m_j \varphi^j = i\mu_j x^j. \tag{A.2}$$

Suppose we perform a linear transformation of coordinates $y^j = T^j{}_i x^i$. The argument of the exponential changes to

$$i\mu_j x^j = i\mu_j (T^{-1})^j{}_l y^l = i\nu_k y^l, \tag{A.3}$$

where $\nu_k = (T^{-1})^i{}_j y^j$ are new quantum numbers associated with the coordinates y^j.

The Frolov-Thorne vacuum for general Kerr black hole is seen as a diagonal density matrix by observer at infinity [26], where the distribution of quantum modes is given by Boltzman weighting factor

$$e^{-\hbar \frac{\omega - \Omega_H m}{T_H}}, \tag{A.4}$$

where T_H is the Hawking temperature and Ω_H the horizon velocity. The generalization to higher dimensions with more angular momenta is

$$e^{-\hbar\frac{\omega-\Omega_H^1 m_1-\ldots-\Omega_H^k m_k}{T_H}} = e^{-\mu_j\beta^j}, \tag{A.5}$$

where β^j represents the set of inverse temperatures $\left(\frac{\hbar}{T_H}, -\frac{\hbar\Omega_H^j}{T_H}\right)$. We can rewrite the Boltzman factor into the new modes

$$e^{-\mu_j\beta^j} = e^{-\nu_i T^i{}_j\beta^j} = e^{-\nu_i\gamma^i} \tag{A.6}$$

and we see that the new effective temperatures are

$$T_j^{eff} = \frac{1}{\gamma^j} = \frac{1}{T^i{}_j\beta^j}. \tag{A.7}$$

Appendix B

Cardy Formula

The Cardy formula [40] tells us that the entropy of a unitary conformal field theory is given by

$$S = 2\pi\sqrt{\frac{cL_0}{6}}, \tag{B.1}$$

where c is the central charge of the CFT and L_0 is the mean value $\hat{L}_0$ operator. It is valid when $L_0 \gg c$. In our work we are not sure how to fix L_0, but we have finite temperature CFT, so we want to rewrite this formula in terms of c and T. This means a transformation from the microcanonical to canonical ensemble. We shall present here a simple derivation using the first law of thermodynamics.

The L_m operators are Fourier coefficients of a stress-energy tensor. So L_0 corresponds to the Hamiltonian and its eigenvalue to energy. We consider the first law of thermodynamics

$$dL_0 = TdS. \tag{B.2}$$

The differential of (B.1) is

$$dS = \pi\sqrt{\frac{c}{6L_0}}dL_0. \tag{B.3}$$

When we compare this equation to (B.2) we find that

$$T = \frac{1}{\pi}\sqrt{\frac{6L_0}{c}}, \tag{B.4}$$

so

$$L_0 = \frac{\pi^2 cT^2}{6}. \tag{B.5}$$

Now we can substitute this expression in (B.1) and we find that

$$S = 2\pi\sqrt{\frac{\pi^2 c^2 T^2}{36}} = \frac{\pi^2}{3}cT, \tag{B.6}$$

which is the desired form of the Cardy formula.

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