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DOCTORAL THESIS

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Symmetries of the CR sub-Laplacian

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Abstract: The aim of this work is to characterize the vector space of symmetry operators of the CR sub-Laplacian. To do this, we define a CR structure on some distinguished submanifold of $\mathbb{C}^{n+1}$ (it is in fact the big cell in the CR sphere) and write down the CR sub-Laplacian on it. We also define the symmetries of the CR sub-Laplacian in general and using the ambient construction, which we introduce in the sequel, we construct all of them.
Keywords: CR geometry, CR sub-Laplacian, symmetries of differential operator.

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Abstrakt: Cílem této práce je charakterizovat vektorový prostor symetrií CR sub-Laplaca. Za tím účelem zadefinujeme CR strukturu na jedné konkrétní podvarietě $\mathbb{C}^{n+1}$ (ve skutečnosti velké buňce v CR sféře) a napíšeme konkrétně, jak na ní vypadá CR sub-Laplace. Zadefinujeme obecně symetrie sub-Laplaca a použitím ambientní konstrukce je všechny zkonstruujeme.
Klíčová slova: CR geometrie, CR sub-Laplace, symetrie diferenciálního operátoru.

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Contents

Chapter 1

Introduction

Invariant differential operators have a long story of importance and this is particularly the case for operators of Laplace type. The conformally invariant Laplacian is the basic example in conformal geometry. A family of higher order generalizations of the conformal Laplacian with principal part a power of the Lalacian was constructed in [9]. In CR geometry, the CR invariant sub-Laplacian of Jerison-Lee ([10]) palys a role analogous to that of the conformal Laplacian. In [8] generalizations of the Jerison-Lee sub-Laplacian are defined, which are the CR analogues of the 'conformally invariant powers of the Laplacian'.

This work was inspired by the article [7] by M. Eastwood, and the diploma thesis [4] by Vít Tuček. The aim was to characterize the vector space of all symmetries of the CR sub-Laplacian. In the paper [7] author identifies the symmetry algebra of the Laplacian on the Euclidean space as an explicit quotient of the universal enveloping algebra of the Lie algebra of conformal motions and constructs analogues of these symmetries on a general conformal manifold.

The space of smooth first order linear differential operators on $\mathbb{R}^n$ that preserve harmonic functions is closed under Lie bracket. For $n \geq 3$, it is finite-dimensional (of dimension $(n^2+3n+4)/2$). Its commutator algebra is isomorphic to $\mathfrak{so}(n+1,1)$, the Lie algebra of conformal motions of $\mathbb{R}^n$. Second order symmetries of the Laplacian on $\mathbb{R}^3$ were classified by Boyer, Kalnis, and Miller in [11]. Commuting pairs of second order symmetries, as observed by Winternitz and Friš in [12], correspond to separation of variables for the Laplacian. This leads to classical coordinate systems and special functions, see [11] and [13].

General symmetries of the Laplacian on $\mathbb{R}^n$ give rise to an algebra, filtered by degree. For $n \geq 3$, the filtering subspaces are finite-dimensional and closely related to the space of conformal Killing tensors. The main result of [7] is an explicit algebraic description of this symmetry algebra. The motivation for [7] has come from physics, especially the theory of higher spin fields and their symmetries.

Next four chapters are introductory. Chapter 2 defines some basic notions we will work with, in particular the CR structures, and indicates some notation. Chapter 3 is devoted to some structure and representation theory of semisimple Lie algebras and their parabolic subalgebras. In Chapter 4 we define parabolic geometries as Cartan geometries, some underlying structures, and establish an equivalence between the Cartan geometry and the underlying structure in most cases. We also define the BGG sequences of invariant operators for parabolic geometries. Chapter 5 describes the CR structures as a special kind of parabolic geometry, and ilustrates the abstract phenomena in the concrete setting of CR structures.

The core of the work are the last two chapters. In Chapter 6 we define the ambient construction for the big cell of the CR sphere and use it to define the CR sub-Laplacian. The advantage of the ambient construction is that the ambient operators are much more simple then the induced operators on the big cell. For the ambient construction for more general CR manifolds, see [6]. The idea of the construction in our simple case is to embed the trivial $\mathbb{C}^\times$-principal nudle on the big cell into some complex vector space (called the ambient space) as a null-cone of some suitable Hermitean metric. The CR sub-Laplacian will be the restriction of the Laplace operator corresponding to the Hermitean metric to the big cell.

Chapter 7 is then devoted to the ambient construction of symmetries of the sub-Laplacian and characterization of vector space of the symmetries. To do this, we start with first order symmetries. The ambient Laplace operator is surely $U(\mathbb{V})$-invariant, where we denote by $\mathbb{V}$ the ambient space. So the generator of the infinitesimal action of $\mathfrak{u}(\mathbb{V})$, which are simply vector fields (and hence first order differential operators) on $\mathbb{V}$, commute with the Laplacian and also with the Hermitean metric on $\mathbb{V}$. So they induce first order differential operators on the big cell and we prove that these are all first order symmetries of the sub-Laplacian.

For higher order symmetries, we first compute the case of second order symmetries to get some feeling about the general case. The work has three steps. In the first step, we compute conditions on the symbol of symmetries of given order. They have the following structure: Let P be a symmetry of order d. Then we can write $P = \sum_{i+j\leq d} V^{a_1\dots a_i\bar{b}_1\dots\bar{b}_j\sigma\dots\sigma}\partial_{a_1}\dots\partial_{a_i}\partial_{\bar{b}_1}\partial_{\bar{b}_j}\partial_\sigma\partial_\sigma +$ $LOTS$, where each term has exactly d indices, in which it is trace-free and symmetric. Then $V^{\sigma\dots\sigma}$ satisfies the first BGG equation for $\overset{d}{\circ}\!-\!\overset{0}{\circ}-\cdots-\overset{0}{\circ}\!-\!\overset{d}{\circ}$. If $V^{\sigma\dots\sigma} = 0$, then all terms with $\min(i,j) = 0$ vanish and $V^{a_1\bar{b}_1\sigma\dots\sigma}$ satisfies the first BGG equation for $\overset{d-2}{\circ}\!-\!\overset{1}{\circ}-\cdots-\overset{1}{\circ}\!-\!\overset{d-2}{\circ}$ ... If $V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_k\sigma\dots\sigma} = 0$ for all $k < s$, then all terms with $\min(i,j) < s$ vanish and $V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$ satisfies the first BGG equation for $\overset{d-2s}{\circ}\!-\!\overset{s}{\circ}-\cdots-\overset{s}{\circ}\!-\!\overset{d-2s}{\circ}$. The second step is the ambient construction of the symmetries in terms of composition of generators of the $\mathfrak{u}(\mathbb{V})$-action, and the third step is the proof that we have constructed all of them.

In all of this work we use the Penrose summation convention without further comments.

Chapter 2

Basic notions and notation

2.1 Complex vector spaces and vector bundles

Definition 2.1.1. Let V be a complex vector space. Then we can form another complex vector space $\bar{V}$, such that:

- $\bar{V} = V$ as abelian group
- for $v \in \bar{V}$ $z \cdot v := \bar{z}v$, where the action on the right-hand side is in V

Lemma 2.1.1. *The identity $Id : V \to \bar{V}$ is a conjugate-linear isomorphism.*

Proof. This is clear from definition. □

We will write this isomorphism in the form

$$Id : v = v^i e_i \mapsto \bar{v} = v^{\bar{i}} e_{\bar{i}}$$

where e_i and $e_{\bar{i}}$ denote the same vector, first considered in V, and then in $\bar{V}$. It is clear that $v^{\bar{i}} = \overline{v^i}$. We will use this notation throughout this work.

Now let us denote the complex structure on V as J and form the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$. We also extend J to this complexification by complex bilinearity. The eigenvalues of J on $V \otimes_{\mathbb{R}} \mathbb{C}$ are $\pm i$, and we have a direct sum decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

where $V^{1,0}$ is $+i$-eigenspace and $V^{0,1}$ is $-i$-eigenspace. It is easy to see that

$$V^{1,0} = \{v - iJv, v \in V\} \quad V^{0,1} = \{v + iJv, v \in V\}$$

Lemma 2.1.2. *As complex vector spaces, we have $V \cong V^{1,0}$ and $\bar{V} \cong V^{0,1}$.*

Proof. For the first isomorphism consider the map $h : v \mapsto v - iJv$. It is clear that this is an isomorphism of real vector spaces. The only thing that remains to check is its behaviour with respect to complex structures on these spaces. The complex structure on V is J, and that of $V^{1,0}$ is given by multiplication by i. So we need to check that $h(Jv) = ih(v)$. But

$$h(Jv) = Jv - iJ^2v = Jv + iv = i(v - iJv)$$

For the second isomorphism it suffices to find a conjugate-linear isomorphism $conj : V^{1,0} \to V^{0,1}$. The natural choice would be mapping $conj : v - iJv \mapsto v + iJv$. This is really conjugate-linear, since

$$conj[i(v - iJv)] = conj(Jv - iJ^2v) = Jv + iJ^2v = -i(v + iJv)$$

That *conj* is an isomorphism of real vector spaces, is clear. □

In the sequel, we will often identify V with $V^{1,0}$. Via this identification, the isomorphism *conj* corresponds to the isomorphism Id above. So for $v \in V^{1,0}$, we will write $\bar{v}$ for $conj(v)$.

The same constructions can be done for complex vector bundles on smooth manifolds, and we will use the same notation as for complex vector spaces.

Definition 2.1.2. Let M be a smooth manifold, V be a distribution on M (i.e. a subbundle of the tangent bundle), which is in addition a complex vector bundle on M. We say that the complex structure J on V is integrable, if for any two sections $X, Y \in \Gamma(V)$ the following expression vanishes:

$$\mathcal{N}(X,Y) := [X,Y] - J[JX,JY] + J([JX,Y] + [X,JY])$$

2.2 Contact geometry

Definition 2.2.1. Let M be a $(2n+1)$-dimensional manifold. We say that M is *a contact manifold*, if it has a $2n$-dimensional distribution $HM \subset TM$, s.t. the *Levi bracket*

$$\mathcal{L} : HM \times HM \to TM/HM \quad \text{given by} \quad \mathcal{L}(X,Y) := p([X,Y])$$

where $p : TM \to TM/HM$ is the canonical projection, is nondegenerate.

Lemma 2.2.1. *Let M be a contact manifold. Then there locally exists a 1-form α, s.t. $\alpha(HM) = 0$ and $d\alpha(HM \times HM)$ is nondegenerate.*

Proof. Let's take as α any 1-form with kernel HM (such forms always exist locally). Then

$$d\alpha(X, Y) = -\alpha([X, Y]), \quad X, Y \in HM$$

□

Such a 1-form as above will be called *a contact form*. For every contact form there exists a unique vector field r, called the *Reeb vector field*, s.t. $\alpha(r) = 1$ and $\imath_r d\alpha = 0$. This induces a splitting of the tangent bundle of M.

2.3 CR structures

Definition 2.3.1. A *CR manifold of CR dimension n and codimension 1* is a $(2n+1)$-dimensional manifold, s.t. there is a $2n$-dimensional distribution $HM \subset TM$ carrying an integrable complex structure. The CR structure on M is called *nondegenerate*, if the Levi bracket is nondegenerate, i.e. if M is a contact manifold.

Remark 2.3.1. The integrability of the complex structure implies that the Levi bracket is of type $(1,1)$ with respect to this complex structure, i.e. $\mathcal{L}(JX, JY) = \mathcal{L}(X, Y)$.

Example 2.3.1. Let M be a real hypersurface in $\mathbb{C}^{n+1}$. Then TM is a subbundle of the complex vector bundle $T\mathbb{C}^{n+1}|_M$. We put $HM := TM \cap J(TM)$. It is an easy exercise in linear algebra that $\dim_{\mathbb{C}} HM = n$ and that this induces a CR structure on M. The integrability of the complex structure on HM is simply integrability of the complex structure of $\mathbb{C}^{n+1}$. This can be easily generalized to a real hypersurface of any complex manifold.

Since we are only interested in nondegenerate CR manifolds, we will assume from now on that M is nondegenerate. We will work locally.

Let α be some contact form on M. We define a Hermitean scalar product h on HM.

Definition 2.3.2. The *CR metric* on HM is defined by

$$h(X, Y) := id\alpha(X - iJX, Y + iJY)$$

where $d\alpha$ is extended by complex bilinearity to $HM \otimes \mathbb{C}$. This is a Hermitean metric on HM and its signature is called a *signature of the CR manifold* M.

Chapter 3

Parabolic subgroups and subalgebras

3.1 Definition

Definition 3.1.1. Let $\mathfrak{g}$ be a simple Lie algebra and $k > 0$ be an integer. A $|k|$-grading on $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

into a direct sum of subspaces such that

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where $\mathfrak{g}_i = 0$ for $|i| > k$,
- the subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated as a Lie algebra by $\mathfrak{g}_{-1}$,
- $\mathfrak{g}_{-k} \neq \{0\}$ and $\mathfrak{g}_k \neq \{0\}$.

By definition, if $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ is a $|k|$-grading, then $\mathfrak{p} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ is a subalgebra of $\mathfrak{g}$, and $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is a nilpotent ideal in $\mathfrak{p}$. Similarly, the subalgebra $\mathfrak{g}_-$ is nilpotent by the grading property. It is also easy to see that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. The central object of interest is, however, the pair $(\mathfrak{g}, \mathfrak{p})$, while $\mathfrak{g}_0$ is a rather auxiliary object, which is usually easier to deal with. One source of complications is the fact that $\mathfrak{g}_0 = \mathfrak{p}/\mathfrak{p}_+$, so $\mathfrak{g}_0$ is naturally a quotient of $\mathfrak{p}$.

Since the main object is the subalgebra $\mathfrak{p}$, the grading of $\mathfrak{g}$ will be of minor importance. The important object will be the induced filtration $\mathfrak{g} =$

$\mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \cdots \supset \mathfrak{g}^{k}$ given by $\mathfrak{g}^i := \bigoplus_{j \geq i} \mathfrak{g}_j$. This filtration is $\mathfrak{p}$-invariant, $\mathfrak{p} = \mathfrak{g}^0$ and $\mathfrak{p}_+ = \mathfrak{g}^1$. The quotients $gr_i \mathfrak{g} := \mathfrak{g}^i/\mathfrak{g}^{i+1}$ are naturally isomorphic as $\mathfrak{g}_0$-modules to $\mathfrak{g}_i$-s, extended to $\mathfrak{p}$-modules by the trivial action of $\mathfrak{p}_+$.

Lemma 3.1.1. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded simple Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ be a nondegenerate invariant bilinear form. Then we have*

1. *There is a unique element $E \in \mathfrak{g}$, called the grading element, such that $[E, X] = jX$ for all $X \in \mathfrak{g}_j$, $j = -k, \ldots, k$. The element E lies in the centre of the subalgebra $\mathfrak{g}_0 \leq \mathfrak{g}$.*

2. *the isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ provided by B is compatible with the filtration and the grading of $\mathfrak{g}$. In particular, B induces dualities of $\mathfrak{g}_0$-modules between $\mathfrak{g}_i$ and $\mathfrak{g}_{-i}$, and the filtration component $\mathfrak{g}^i$ is exactly the annihilator (with respect to B) of $\mathfrak{g}^{-i+1}$. Hence B induces a duality of $\mathfrak{p}$-modules between $\mathfrak{g}/\mathfrak{g}^{-i+1}$ and $\mathfrak{g}^i$, and in particular between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}_+$.*

3. *For $i < 0$ we have $[\mathfrak{g}_{i+1}, \mathfrak{g}_{-1}] = \mathfrak{g}_i$.*

4. *Let $A \in \mathfrak{g}_i$ with $i \geq 0$ be an element such that $[A, X] = 0$ for all $X \in \mathfrak{g}_{-1}$. Then $A = 0$.*

Proof. See [1]. □

We want to define subgroups corresponding to Lie algebras $\mathfrak{g}_0$ and $\mathfrak{p}$:

Definition 3.1.2. Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded simple Lie algebra and let G be a Lie group with Lie algebra $\mathfrak{g}$.

1. A *parabolic subgroup* of G corresponding to the given $|k|$-grading is a subgroup $P \subset G$, which lies between $\bigcap_{i=-k}^{k} N_G(\mathfrak{g}^i)$ and its connected component of the identity.

2. Given a parabolic subgroup $P \subset G$, we define the *Levi subgroup* $G_0 \subset P$ by
$$G_0 := \{g \in P : Ad(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for all } i = -k, \ldots, k\}.$$

It is easy to see, that in the definition of G_0 we can replace $g \in P$ by $g \in G$. Note also that P is a closed subgroup of G and G_0 is a closed subgroup of P.

Theorem 3.1.1. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra and let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $P \subset G$ be a parabolic subgroup for the given grading and let $G_0 \subset P$ be the Levi subgroup.*

Then $(g_0, Z) \mapsto g_0 \exp(Z)$ defines a diffeomorphism $G_0 \times \mathfrak{p}_+ \to P$, and

$$(g_0, Z_1, \dots, Z_k) \mapsto g_0 \exp(Z_1) \cdots \exp(Z_k)$$

is a diffeomorphism $G_0 \times \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k \to P$.

Proof. See [1]. □

3.2 Structure of $\mathfrak{p}$

3.2.1 Complex case

We start with complex semisimple Lie algebras. Any complex semisimple Lie algebra is determined up to isomorphism by the associated root system. For this we have to choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, i.e. a maximal abelian subalgebra such that the adjoint action $ad(H) : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable for all $H \in \mathfrak{h}$. Any two of them are conjugate by an inner automorphism of $\mathfrak{g}$. Having chosen $\mathfrak{h}$, one gets a set of roots, i.e. the finite set Δ of linear functionals $\alpha \in \mathfrak{h}^*$, such that the root space $\mathfrak{g}_\alpha = \{A \in \mathfrak{g} : [H, A] = \alpha(H)A \quad \forall H \in H\}$ is nonzero. The Lie algebra decomposes as $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. The root spaces $\mathfrak{g}_\alpha$ are one-dimensional and for $\alpha, \beta \in \Delta$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in \Delta$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ if $\alpha + \beta \notin \Delta$. From the fact that the projections on the eigenspaces of an operator are polynomials in the operator, one concludes that any subalgebra of $\mathfrak{g}$ which contains the Cartan subalgebra $\mathfrak{h}$ is (as a vector space) automatically a direct sum of $\mathfrak{h}$ and some root spaces.

The subspace $\mathfrak{h}_0 \subset \mathfrak{h}$ on which all roots are real, is a real form of $\mathfrak{h}$. Choosing an ordered basis $\{H_1, \dots, H_r\}$ of $\mathfrak{h}_0$ we define a real linear functional $\phi : \mathfrak{h}_0 \to \mathbb{R}$ to be positive, if for some $i = 1, \dots, r$ one has $\phi(H_j) = 0$ for all $j < i$ and $\phi(H_i) > 0$. Putting $\phi < \psi$ if and only if $\psi - \phi > 0$, we have a total ordering on the space of all such linear functionals. Using this, Δ decomposes as disjoint union of positive roots Δ^+ and negative roots Δ^-.

These two choices are equivalently described by a choice of a Borel subalgebra, i.e. a maximal solvable subalgebra $\mathfrak{b} \leq \mathfrak{g}$. Having chosen $\mathfrak{h}$ and Δ^+, the associated Borel subalgebra is $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_+ := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{alpha}$.

The Lie algebra $\mathfrak{n}_+$ is obviously nilpotent and we have $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{n}_+$. The Borel subalgebra $\mathfrak{b}$ is called the *standard Borel subalgebra* associated to $\mathfrak{h}$ and $\delta^+ \subset \Delta$. Any two Borel subalgebras of a complex semisimple Lie algebra are conjugate by an inner automorphism of $\mathfrak{g}$.

Definition 3.2.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A *parabolic subalgebra* $\mathfrak{p}$ of $\mathfrak{g}$ is a Lie subalgebra that contains a Borel subalgebra.

Fixing a choice of $\mathfrak{h}$ and Δ^+ for $\mathfrak{g}$, we obtain the corresponding standard Borel subalgebra. Subalgebras of $\mathfrak{g}$ containing this Borel subalgebra are called *standard parabolic subalgebras.* Since all Borel subalgebras are conjugate to each other, it suffices to deal with the standard one.

A positive root $\alpha \in \Delta^+$ is called simple, if it cannot be written as a sum of two positive roots. The set of simple roots is usually denoted by Δ^0. One may write any root $\alpha \in \Delta$ uniquely as a linear combination of simple roots with integral coefficients, which are all nonnegative for positive roots and nonpositive for negative roots.

Proposition 3.2.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \leq \mathfrak{g}$ a Cartan subalgebra, Δ the corresponding set of roots and Δ^0 the set of simple roots for some choice of a positive subsystem. Then standard parabolic subalgebras $\mathfrak{p} \leq \mathfrak{g}$ are in one-to-one correspondence with subsets $\Sigma \subset \Delta^0$.*

Explicitly, we associate to $\mathfrak{p}$ the subset $\Sigma_{\mathfrak{p}} := \{\alpha \in \Delta^0 : \mathfrak{g}_{-\alpha} \not\subseteq \mathfrak{p}\}$. Conversely, the standard parabolic subalgebra $\mathfrak{p}_\Sigma$ corresponding to a subset Σ is the sum of the standard Borel subalgebra $\mathfrak{b}$ and all negative root spaces corresponding to roots, which can be written as a linear combination of elements of $\Delta^0 \setminus \Sigma$.

Proof. See [1] $\square$

The obvious choices $\Sigma = \emptyset$ and $\Sigma = \Delta^0$ lead to the subalgebra $\mathfrak{g}$ and the standard Borel subalgebra, respectively. Note also that if $\Sigma \subset \Sigma' \subset \Delta^0$, then $\mathfrak{p}_{\Sigma'} \leq \mathfrak{p}_\Sigma$.

Suppose we have given the subset $\Sigma \subset \Delta^0 = \{\alpha_1, \ldots, \alpha_r\}$ and a root $\alpha \in \Delta$, we define the Σ-*height* $\mathrm{ht}_\Sigma(\alpha)$ of α as

$$\mathrm{ht}_\Sigma\left(\sum_i a_i\alpha_i\right) := \sum_{i:\alpha_i \in \Sigma} a_i$$

For $0 \neq i \in \mathbb{Z}$ define $\mathfrak{g}_i := \bigoplus_{\alpha:\mathrm{ht}_\Sigma(\alpha)=i} \mathfrak{g}_\alpha$ and put $\mathfrak{g}_0 := \mathfrak{h} \oplus \bigoplus_{\alpha:\mathrm{ht}_\Sigma(\alpha)=0} \mathfrak{g}_\alpha$. Recall that we have a total ordering on the set of roots. In particular, there is a maximal root in this ordering, and we define k to be a Σ-height of this root. We then have $\mathfrak{g}_i = 0$ for $|i| > k$ and $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$.

Theorem 3.2.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \leq \mathfrak{g}$ a Cartan subalgebra with corresponding roots Δ, Δ^+ a set of positive roots and $\Delta^0 \subset \Delta^+$ the set of simple roots.*

1. *For any standard parabolic subalgebra $\mathfrak{p} \leq \mathfrak{g}$ corresponding to the subset $\Sigma \subset \Delta^0$, the decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ according to Σ-height makes $\mathfrak{g}$ into a $|k|$-graded Lie algebra such that $\mathfrak{p} = \mathfrak{g}^0 = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$. Moreover, the subalgebra $\mathfrak{g}_0 \leq \mathfrak{g}$ is reductive and the dimension of its centre $\mathfrak{z}(\mathfrak{g}_0)$ coincides with the number of elements of Σ.*

2. *Conversely, for any $|k|$-grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$, the subalgebra $\mathfrak{g}^0$ is parabolic, and choosing a Cartan subalgebra and positive roots in such a way that $\mathfrak{g}^0$ is a standard parabolic subalgebra $\mathfrak{p}_\Sigma$, the grading is given by the Σ-height.*

Proof. See [1]. □

From this result we see that the situation between the subalgebras $\mathfrak{g}_-$ and $\mathfrak{p}_+$ is completely symmetric, so $\mathfrak{g}_-$ and $\mathfrak{p}_+$ are isomorphic as Lie algebras. Moreover, the filtration of $\mathfrak{g}$ is completely determined by the parabolic subalgebra $\mathfrak{p} = \mathfrak{g}^0$.

Corollary 3.2.1. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, such that no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$. Then we have:*

1. *For $i > 0$ we have $[\mathfrak{g}_{i-1}, \mathfrak{g}_1] = \mathfrak{g}_i$. In particular, the filtration component $\mathfrak{g}^i$ is the i-th power of $\mathfrak{p}_+ = \mathfrak{g}^1$ and $\mathfrak{p}_+ \supset \mathfrak{g}^2 \supset \cdots \supset \mathfrak{g}^k$ is the lower central series of $\mathfrak{p}_+$.*

2. *If for some $i \leq 0$ an element $X \in \mathfrak{g}_i$ satisfies $[X, Z] = 0$ for all $Z \in \mathfrak{g}_1$, then $X = 0$.*

3. *The filtration component $\mathfrak{g}^1 = \mathfrak{p}_+$ is the nilradical of $\mathfrak{p} = \mathfrak{g}^0$.*

4. *For any Lie group G with Lie algebra $\mathfrak{g}$, the parabolic subgroups defined in the previous section are exactly the subgroups, which lie between the normalizer $N_G(\mathfrak{p})$ and its connected component of the identity.*

Proof. See [1]. $\square$

Definition 3.2.2. Let $\mathfrak{g}$ be a complex semisimple Lie algebra endowed with a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$ and a set Δ^+ of positive roots. Then we denote the standard parabolic subalgebra $\mathfrak{p}_\Sigma \subset \mathfrak{g}$ corresponding to $\Sigma \subset \Delta^0$ as well as the corresponding $|k|$-grading by Σ-height by representing in the Dynkin diagram of $\mathfrak{g}$ the nodes corresponding to elements of Σ by a cross instead of a dot.

We know that for any given $|k|$-grading the subalgebra $\mathfrak{g}_0 \leq \mathfrak{g}$ is reductive, so it is a direct sum of a semisimple Lie algebra $\mathfrak{g}_0^{ss}$ and the centre $\mathfrak{z}(\mathfrak{g}_0)$.

Proposition 3.2.2. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a complex semisimple $|k|$-graded Lie algebra. Then the dimension of the centre of $\mathfrak{g}_0$ coincides with the number of crosses in the diagram describing the $|k|$-grading, and the Dynkin diagram of the semisimple part $\mathfrak{g}_0^{ss}$ is obtained by removing all crossed nodes and all edges connected to crossed nodes.*

Proof. See [1]. $\square$

3.2.2 Real case

The description of real $|k|$-gradings proceeds via complexification. Given a real semisimple Lie algebra $\mathfrak{g}$, one first chooses a Cartan involution θ on $\mathfrak{g}$, i.e. an involutive automorphism such that the bilinear form $B_\theta(X,Y) := -B(X,\theta Y)$ is positive definite, where B is the Killing form of $\mathfrak{g}$. To avoid confusion with parabolics, we denote the -1-eigenspace of θ by $\mathfrak{q}$, so the Cartan decomposition reads as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$. Now one looks at θ-stable Cartan subalgebras $\mathfrak{h} \leq \mathfrak{g}$, i.e. abelian subalgebras such that $\theta(\mathfrak{h}) = \mathfrak{h}$ and the complexification $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. Then $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{q})$, and $\mathfrak{h}$ is called maximally noncompact, if the dimension of $\mathfrak{a} := \mathfrak{h} \cap \mathfrak{q}$ is maximal among all θ-stable Cartan subalgebras. Having chosen θ and $\mathfrak{h}$, we look at the root system Δ associated to the Cartan subalgebra $\mathfrak{h}_\mathbb{C} \leq \mathfrak{g}_\mathbb{C}$. Let σ be the conjugation of $\mathfrak{g}_\mathbb{C}$ with respect to the real form $\mathfrak{g}$. Then σ induces an involutive automorphism $\sigma* : \Delta \to \Delta$. A positive subsystem $\Delta^+ \subset \Delta$ is called admissible, if for $\alpha \in \Delta^+$ we either have $\sigma^*\alpha = -\alpha$ or $\sigma^*\alpha \in \Delta^+$.

Definition 3.2.3. Let $\mathfrak{g}$ be a real semisimple Lie algebra with a complexification $\mathfrak{g}_{\mathbb{C}}$, θ a Cartan involution, $\mathfrak{h} \leq \mathfrak{g}$ a θ-stable maximally non-compact Cartan subalgebra, Δ the set of roots for $\mathfrak{h}_{\mathbb{C}} \leq \mathfrak{g}_{\mathbb{C}}$, and $\Delta^+ \subset \Delta$ an admissible positive subsystem.

A Lie subalgebra $\mathfrak{p} \leq \mathfrak{g}$ is called a *standard parabolic subalgebra* with respect to choices of $\mathfrak{h}$ and Δ^+, if and only if the complexification $\mathfrak{p}_{\mathbb{C}}$ is a standard parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ and Δ^+.

Consider the abelian subspace $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{q}$. For $A \in \mathfrak{a}$ we have $\theta(A) = -A$, so $\mathrm{ad}(A) : \mathfrak{g} \to \mathfrak{g}$ is symmetric for the inner product B_θ. Thus the family $\{\mathrm{ad}(A) : A \in \mathfrak{a}\}$ is simultaneously diagonalizable over $\mathbb{R}$. The corresponding eigenvalues are given by linear functionals $\lambda : \mathfrak{a} \to \mathbb{R}$ and the nonzero eigenvalues are called the *restricted roots*. The set of all restricted roots is denoted by Δ_r. The eigenspaces are called *restricted root spaces* and they define the *restricted root decomposition* of $\mathfrak{g}$. The set $\Delta_r \subset \mathfrak{a}^*$ is an abstract root system, but it is not reduced in general. Still the notions of positive and simple subsystem pose no problems for Δ_r.

Having given θ, $\mathfrak{h} \leq \mathfrak{g}$, Δ and the conjugation σ as above, we define $\Delta_c := \{\alpha : \sigma^*\alpha = -\alpha\} \subset \Delta$. Putting $\mathfrak{t} := \mathfrak{h} \cap \mathfrak{k}$, all roots are real on $i\mathfrak{t} \oplus \mathfrak{a} \subset \mathfrak{h}_{\mathbb{C}}$. Restricting the roots to $\mathfrak{h} \subset \mathfrak{h}_{\mathbb{C}}$, the map σ^* becomes complex conjugation, so $\Delta_c = \{\alpha \in \Delta : \alpha|_{\mathfrak{a}} = 0\}$. The admissibility of a positive subsystem $\Delta^+ \subset \Delta$ reads as $\sigma^*\alpha \in \Delta^+$ for all $\alpha \in \Delta^+ \setminus \Delta_c$. Passing to the associated system Δ^0, it turns out that $\Delta_c^0 := \Delta^0 \cap \Delta_c$ is a simple system for Δ_c and for any $\alpha \in \Delta^0 \setminus \Delta_c^0$ there is a unique $\alpha' \in \Delta^0 \setminus \Delta_c^0$ such that $\sigma^*\alpha - \alpha'$ is a linear combination of compact roots. Mapping α to α' defines an involutive automorphism of $\Delta^0 \setminus \Delta_c^0$. The Satake diagram of $\mathfrak{g}$ is then obtained by taking the Dynkin diagram of Δ^0 with elements of Δ_c^0 indicated by black dots $\bullet$ and elements of $\Delta^0 \setminus \Delta_c^0$ by white dots $\circ$. Moreover, for every element $\alpha \in \Delta^0 \setminus \Delta_c^0$ such that $\alpha' \neq \alpha$, one connects α and α' by an arrow.

Since all roots are real on $i\mathfrak{t}\oplus\mathfrak{a}$, the restricted roots are exactly the nonzero restrictions of roots to $\mathfrak{a} \subset \mathfrak{h}$. Thus we obtain a surjective restriction map $\Delta \setminus \Delta_c \to \Delta_r$. Since for $\alpha \in \Delta$, the restrictions to $\mathfrak{a}$ of α and $\sigma^{*\alpha}$ are conjugate, we see that $\sigma^*\alpha|_{\mathfrak{a}} = \alpha|_{\mathfrak{a}}$. For an admissible choice of $\Delta^+ \subset \Delta$, the image of Δ^+ in Δ_r is a positive subsystem. This easily implies that the corresponding simple system Δ_r^0 for Δ_r is the quotient of $\Delta^0 \setminus \Delta_c^0$ obtained by identifying each simple root α with α'.

Theorem 3.2.2. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, θ a Cartan involution with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a} \subset \mathfrak{g}$ a* *maximally noncompact θ-stable Cartan subalgebra. Put $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, let σ^* be the involutive automorphism of Δ induced by the conjugation with respect to $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ and let $\Delta^+ \subset \Delta$ be an admissible positive subsystem. Then we have:*

1. *Put $\mathfrak{m} := \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and let $\mathfrak{n} \subset \mathfrak{g}$ be the direct sum of all positive restricted root spaces. Then $\mathfrak{p}_0 := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a subalgebra of $\mathfrak{g}$, and the standard parabolic subalgebras of $\mathfrak{g}$ are exactly the subalgebras containing $\mathfrak{p}_0$.*

2. *Let Δ^0 be the set of simple roots and let $\Delta^0_r \subset \Delta_r$ be the corresponding set of simple restricted roots. Then subsets of Δ^0_r are in bijective correspondence with subsets of Δ^0 that are disjoint to Δ^0_c and stable under the involution induced by σ^*. On the other hand, the set of all subsets of Δ^0 with these two properties is in bijective correspondence with the set of all standard parabolic subalgebras of $\mathfrak{g}$.*

   Explicitly, the parabolic subalgebra corresponding to $\Sigma \subset \Delta^0$ is the sum of $\mathfrak{p}_0$ and the restricted root spaces for those negative restricted roots, which can be written as linear combination of the simple restricted roots, which are outside of the image of Σ in Δ^0_r.

Proof. See [1]. □

Proposition 3.2.3. *Let $\mathfrak{g}$ be a real semisimple Lie algebra endowed with a Cartan involution θ, a θ-stable maximally noncompact Cartan subalgebra $\mathfrak{h}$ and an admissible positive subsystem Δ^+. Then we have:*

1. *For a standard parabolic subalgebra $\mathfrak{p} \leq \mathfrak{g}$ corresponding to a subset $\Sigma \subset \Delta^0_r$, the Σ-height determines a $|k|$-grading of $\mathfrak{g}$, such that $\mathfrak{p} = \mathfrak{g}^0$. Here, k is the Σ-height of the maximal restricted root.*

2. *Given a $|k|$-grading of $\mathfrak{g}$, there is an automorphism $\phi \in (Int)(\mathfrak{g})$ such that $\phi(\mathfrak{g}^0)$ is a standard parabolic subalgebra of $\mathfrak{g}$. Denoting by $\Sigma \subset \Delta^0_r$ the corresponding subset, the given grading on $\mathfrak{g}$ corresponds to the grading given by Σ-height under ϕ.*

Proof. See [1]. □

Remark 3.2.1. As in the complex case, one may directly obtain information about $\mathfrak{g}_0$ from the Satake diagram describing the parabolic. The (real) dimension of the centre of $\mathfrak{g}_0$ again equals the number of crossed nodes in the

Satake diagram. Parallel to the complex case, one may also show that in the real case the Satake diagram of $\mathfrak{g}_0^{ss}$ is obtained by erasing all crossed nodes as well as all edges and arrows connecting to these nodes from the Satake diagram describing the parabolic subalgebra. Details can be found in [2].

To describe a standard parabolic subalgebra $\mathfrak{p} \leq \mathfrak{g}$, we consider the Satake diagram of $\mathfrak{g}$ and denote all the simple roots corresponding to elements of the subset $\Sigma \subset \Delta^0$ by crosses. Since Σ is disjoint to Δ_c, we can recover the original Satake diagram by replacing all crosses by white dots. So the classification of real parabolics can be rephrased in term of replacing white dots in a Satake diagram by crosses. The only rule one has to take into account is that two roots joined by an arrow either have to be both crossed or both uncrossed.

3.3 Representations of $\mathfrak{p}$

We will consider only those representations of $\mathfrak{p}$, which are completely reducible as representations of $\mathfrak{g}_0$.

Proposition 3.3.1. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded simple Lie algebra, $\mathfrak{p} = \mathfrak{g}^0$ the corresponding parabolic subalgebra and $E \in \mathfrak{z}(\mathfrak{g}_0)$ the grading element.*

1. *Any finite-dimensional completely reducible representation W of $\mathfrak{p}$ is obtained by trivially extending a completely reducible representation of $\mathfrak{g}_0$ to $\mathfrak{p}$. Moreover, E acts by a scalar on each irreducible component of W.*

2. *Let V be a finite-dimensional representation of $\mathfrak{p}$ such that $\mathfrak{z}(\mathfrak{g}_0)$ acts diagonalizably. Then V admits a $\mathfrak{p}$-invariant filtration $V = V^0 \supset V^1 \supset \cdots \supset V^N \supset V^{N+1} = \{0\}$ such that each of the quotients V^i/V^{i+1} is completely reducible.*

Proof. See [1]. $\square$

Assume we have chosen $\mathfrak{h}$ and Δ^+ in such a way that $\mathfrak{p}$ is a standard parabolic subalgebra. On any finite-dimensional complex representation V of $\mathfrak{g}$, the Cartan subalgebra $\mathfrak{h}$ acts diagonalizably. The corresponding eigenvalues $\lambda \in \mathfrak{h}^*$ are called the *weights* of V and the eigenspaces are called *weight spaces*. In finite-dimensional irreducible representation they are unique up to scale. A *highest weight vector* in V is an element of a weight space, which is annihilated by the action of all elements of positive root spaces. The *highest weight* of a finite-dimensional irreducible representation V is the weight of its highest weight vectors. These weights are always *dominant* and *algebraically integral*, i.e. for the inner product induced by the Killing form the expression $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is a nonnegative integer for each $\alpha \in \Delta^0$. There is a bijective correspondence between dominant integral weights and isomorphism classes of finite-dimensional complex irreducible representations of $\mathfrak{g}$. The condition of being dominant and algebraically integral can be rephrased in terms of *fundamental weights*, which are defined by $\frac{2\langle\lambda_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} = \delta_{ij}$ for $\alpha_j \in \Delta^0$. Any weight can be written as a linear combination of these weights, and the weight is dominant and algebraically integral, if all coefficients in this expansion are nonnegative integers. The Dynkin diagram notation for weights and representations is then obtained by writing the coefficient of λ_i in this expansion over the node of the Dynkin diagram of $\mathfrak{g}$ that corresponds to the simple root α_i.

Complex irreducible representations can be dealt with in a similar way. These coincide with complex irreducible representations of $\mathfrak{g}_0$, which in turn are given by irreducible representations of $\mathfrak{g}_0^{ss}$ and linear functionals on the centre $\mathfrak{z}(\mathfrak{g}_0)$. Assuming that $\mathfrak{p}$ is a standard parabolic, we obtain the corresponding set $\Sigma = \{\alpha \in \Delta^0 : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\} \subset \Delta^0$. We can naturally split the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ as $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ with $\mathfrak{h}' := \{H \in \mathfrak{h} : \alpha(H) = 0 \quad \forall \alpha \in \Delta^0 \setminus \Sigma\}$ and $\mathfrak{h}''$ the span of the elements H_α for $\alpha \in \Delta^0 \setminus \Sigma$. Then $\mathfrak{h}' = \mathfrak{z}(\mathfrak{g}_0)$, while $\mathfrak{h}''$ is a Cartan subalgebra for $\mathfrak{g}_0^{ss}$. Hence, complex irreducible representations of $\mathfrak{g}_0$ are in bijective correspondence with a set of functionals on $\mathfrak{h}$, but the dominance and integrality conditions refer only to the restriction to $\mathfrak{h}''$. In analogy to the usual notation we define a functional $\lambda : \mathfrak{h} \to \mathbb{C}$ to be $\mathfrak{p}$-dominant, respectively, $\mathfrak{p}$-algebraically integral, if $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is real and nonnegative, respectively, an integer for all $\alpha \in \Delta^0 \setminus \Sigma$.

Corollary 3.3.1. *Let $\mathfrak{p} \leq \mathfrak{g}$ be a standard parabolic subalgebra in a complex semisimple Lie algebra. Then isomorphism classes of finite-dimensional complex irreducible representations of $\mathfrak{p}$ are in bijective correspondence with weights $\lambda : \mathfrak{h} \to \mathbb{C}$, which are $\mathfrak{p}$-dominant and $\mathfrak{p}$-algebraically integral.*

The condition of $\mathfrak{p}$-dominance and integrality can again be rephrased in terms of fundamental weights as the requirement that the coefficients of all fundamental weights corresponding to simple roots not contained in Σ must be nonnegative integers. In the Dynkin diagram notation this means that the coefficients over all uncrossed nodes are nonnegative integers.

In the realm of Lie algebra representations, there is no restriction on the coefficients over the crossed nodes. However, if one wants the representation to integrate to at least one parabolic subgroup then the coefficients over the crossed nodes have to be integers in the complex case. In the real case, there are usually less (or even no) integrability conditions, since in this case it may happen that $\mathfrak{z}(\mathfrak{g}_0)$ integrates to a subgroup isomorphic to $\mathbb{R}^l$.

Proposition 3.3.2. *Let V be a finite-dimensional complex representation of $\mathfrak{g}$ and let $V^{\mathfrak{p}_+} \subset V$ be the subspace of $\mathfrak{p}_+$-invariant elements. Then there is a bijective correspondence between $\mathfrak{p}$-invariant subspaces of $V^{\mathfrak{p}_+}$ and $\mathfrak{g}$-invariant subspaces of V. In particular, if V is the irreducible representation of $\mathfrak{g}$ with highest weight λ, then $V^{\mathfrak{p}_+}$ is the irreducible $\mathfrak{p}$-representation with the same highest weight.*

Proof. See [1]. $\square$

3.4 The Hasse diagram of $\mathfrak{p}$

Consider a complex simple Lie algebra $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$ and an ordering on $\mathfrak{h}^*$, and denote by Δ, Δ^+, and Δ^0 the corresponding sets of roots, positive roots and simple roots, respectively. The real subspace $\mathfrak{h}_0 \subset \mathfrak{h}$, on which all roots are real, defines a real form of $\mathfrak{h}$, and the Killing form restricts to a positive definite inner product on $\mathfrak{h}_0$. The set Δ of roots is then a finite subset of the real dual of $\mathfrak{h}_0$, and via the duality, the Killing form induces a positive definite inner product on $\mathfrak{h}_0^*$. For any $\alpha \in \Delta$ we have the root reflection $s_\alpha : \mathfrak{h}_0^* \to \mathfrak{h}_0^*$ defined by $s_\alpha(\phi) = \phi - \frac{2\langle\phi,\alpha\rangle}{\langle\alpha,\alpha\rangle}$, which maps Δ to itself.

The Weyl group $W = W_\mathfrak{g}$ is then the subgroup of the orthogonal group $O(\mathfrak{h}_0^*)$ generated by these root reflections. We may view W as a subgroup of the permutation group of Δ, so it is a finite group. It is actually generated by the reflections s_{α_i} corresponding to the simple roots α_i. An expression of $w \in W$ as a composition of simple root reflections is called *reduced*, if it has the least possible number of factors. This number is called the *length* $\ell(w)$ of w. The *sign* $\mathrm{sgn}(w)$ of $w \in W$ is defined as the determinant of the linear map $w : \mathfrak{h}_0^* \to \mathfrak{h}_0^*$. We have $\mathrm{sgn}(w) = (-1)^\ell$.

We have to view the Weyl group W not only as a group, but also as a directed graph. The vertices of this graph are elements $w \in W$, and we have a directed edge $w \xrightarrow{\alpha} w'$ labeled with $\alpha \in \Delta^+$, if and only if $\ell(w') = \ell(w) + 1$ and $w' = s_\alpha w$. For $w \in W$ define $\Phi_w := \{\alpha \in \Delta^+ : w^{-1}(\alpha) \in -\Delta^+\}$,or equivalently $\Phi_w = w(-\Delta^+) \cap \Delta^+$. The set Φ_w is saturated and its complement $\Delta^+ \setminus \Phi_w$ is saturated, too. We define $\langle \Phi_w \rangle := \sum_{\alpha \in \Phi_w} \alpha$. The *lowest form* $\delta = \delta_{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ is defined as $\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ and can also be expressed as the sum of all fundamental weights. Hence, δ lies in the interior of the dominant Weyl chamber, and since the Weyl group W acts simply transitively on the set of all Weyl chambers, we conclude that the mapping $w \mapsto w(\delta)$ is an injection from W to $\mathfrak{h}_0^*$.

Proposition 3.4.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a chosen simple system Δ^0, Weyl group W and lowest form $\delta \in \mathfrak{h}_0^*$. Then we have:*

1. *$w(\delta) = \delta - \left\langle \Phi_(w) \right\rangle$ for all $w \in W$.*

2. *The map $w \mapsto \Phi_w$ defines a bijection between W and the set of all subsets $\Phi \subset \Delta^+$ such that both Φ and $\Delta^+ \setminus \Phi$ are saturated.*

3. *$|\Phi_w| = \ell(w)$ for all $w \in W$.*

4. *For $w \in W$ and $\alpha \in \Delta^+$, the element α is contained in exactly one of the sets $\Phi(w)$ and $\Phi_{s_\alpha w}$, and $\ell(s_\alpha w) > \ell(w)$ if and only if $\alpha \notin \Phi_w$. In particular, if $\alpha \in \Delta^0$, then $\ell(s_\alpha w) = \ell(w) + 1$ if $\alpha \notin \Phi_w$ and $\ell(s_\alpha w) = \ell(w) - 1$ if $\alpha \in \Phi_w$.*

5. *Let $w, w' \in W$ be two elements such that $|\Phi_{w'}| = |\Phi_w| + 1$. Then for $\alpha \in \Delta^+$ we have $w \xrightarrow{\alpha} w'$ if and only if $\langle \Phi_{w'} \rangle = \langle \Phi_w \rangle + k\alpha$ for some $k \in \mathbb{Z}$.*

Proof. See [1]. □

There is a unique element $w_0 \in W$ of maximal length $\ell(w_0) = |\Delta^+|$, which corresponds to $\Phi_{w_0} = \Delta^+$. For any representation of w_0 as composition of simple root reflections, the composition in the opposite order equals w_0^{-1}, so $w_0 = w_0^{-1}$. Since $w_0(\Delta^+) = -\Delta^+$, it must map the simple system Δ^0 to a simple system for $-\Delta^+$, whence $w_0(\Delta^0) = -\Delta^0$. Since w_0 exchanges positive and negative roots, we see that for $w \in W$ and $\alpha \in \Delta^+$ we have $w_0(w^{-1}(\alpha)) \in \Delta^+$ if and only if $\alpha \in \Phi_w$. Consequently, $\Phi_{ww_0} = \Delta^+ \setminus \Phi_w$.

The directed graph structure on W gives rise to a partial order on the set W, which is called the *Bruhat order*. For $w, w' \in W$ we put $w \leq w'$ if either $w = w'$ or there is a chain

$$w \overset{\alpha_1}{\to} w_1 \overset{\alpha_2}{\to} \cdots \overset{\alpha_{n-1}}{\to} w_{n-1} \overset{\alpha_n}{\to} w'$$

for some (not necessarily different) roots $\alpha_i \in \Delta^+$. The identity is the smallest element and w_0 is the largest element. $w \leq w'$ implies $\ell w \leq \ell w'$, but not conversely. Moreover, $w \leq w'$ implies $w(\lambda) \leq w'(\lambda)$ for any *dominant* weight λ.

Corollary 3.4.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Weyl group W, $w_0 \in W$ the longest element, λ a dominant integral weight and V the irreducible complex finite-dimensional representation of highest weight λ. Then the highest weight of the dual representation V^* is $-w_0(\lambda)$.*

Proof. See [1]. □

Now let $\mathfrak{p} \leq \mathfrak{g}$ be a standard parabolic subalgebra and $\Sigma \subset \Delta^0$ the corresponding subset. Then we obtain the $|k|$-grading of $\mathfrak{g}$ given by Σ-height, and in particular, the subalgebras $\mathfrak{g}_0$ and $\mathfrak{p}_+ = \mathfrak{g}^1$ of $\mathfrak{p}$. The reductive subalgebra $\mathfrak{g}_0$ splits as $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$. The Cartan subalgebra splits as $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$, where $\mathfrak{h}' = \mathfrak{z}(\mathfrak{g}_0)$ is the common kernel of all elements of $\Delta^0 \setminus \Sigma$, while $\mathfrak{h}''$ is spanned by the elements H_α for $\alpha \in \Delta^0 \setminus \Sigma$, so $\mathfrak{h}''$ is a Cartan subalgebra for $\mathfrak{g}_0^{ss}$. Let $\mathfrak{h}_0 \subset \mathfrak{h}$ be the subspace on which all roots are real. By construction, the element H_α for $\alpha \in \Delta^0 \setminus \Sigma$ lies in $\mathfrak{h}'' \cap \mathfrak{h}_0$, so we may identify this space with $\mathfrak{h}_0''$. On the other hand, $\mathfrak{h}_0' := \mathfrak{h}' \cap \mathfrak{h}_0$ is a real form of $\mathfrak{h}'$, so we get $\mathfrak{h}_0 = \mathfrak{h}_0' \oplus \mathfrak{h}_0''$. The inner product on $\mathfrak{h}_0$ induced by the Killing form satisfies $\langle H_\alpha, H \rangle = \alpha(H)$, so $\mathfrak{h}_0'$ and $\mathfrak{h}_0''$ are orthogonal. Passing to the duals, we get an orthogonal decomposition of $\mathfrak{h}_0^*$, and in particular, any simple reflection s_{α_j} with $\alpha_j \in \Delta^0 \setminus \Sigma$ acts as the identity on $(\mathfrak{h}_0')^*$. Defining $W_\mathfrak{p}$ to be the Weyl group of $\mathfrak{g}_0^{ss}$, we see that we may naturally view tis as the subgroup of $W_\mathfrak{g}$ generated by the simple reflections s_{α_j} for $\alpha \in \Delta^0 \setminus \Sigma$.

The Hasse diagram is the set of distinguished representatives for the set $W_\mathfrak{p} \backslash W_\mathfrak{g}$. Let us decompose $\Delta^+ = \Delta^+(\mathfrak{g}_0) \sqcup \Delta^+(\mathfrak{p}_+)$ according to the subalgebra containing the corresponding root space. For $\alpha \in \Delta^+$ we have $\alpha \in \Delta^+(\mathfrak{g}_0)$ if and only if $\mathrm{ht}_\Sigma(\alpha) = 0$. Since the Σ-height is additive, both $\delta^+(\mathfrak{g}_0)$ and $\Delta^+(\mathfrak{p}_+)$ are saturated. Assume that $\alpha \in \Delta^+(\mathfrak{g}_0)$ and $\beta \in \Delta^+(\mathfrak{p}_+)$. Then $s_\alpha(\beta)$ differs from β by a multiple of α and thus $\mathrm{ht}_\Sigma(s_\alpha\beta) = \mathrm{ht}_\Sigma\beta$. Thus

s_α maps $\Delta^+(\mathfrak{p}_+)$ to itself, so the same holds fro any element $w \in W_\mathfrak{p}$. Conversely, if $w \in W_\mathfrak{g}$ is such that $\Phi_w \subset \Delta^+(\mathfrak{g}_0)$, then Φ_w is saturated. Applying part 2 of the above Proposition to $\mathfrak{g}_0^{ss}$, we find an element in $W_\mathfrak{p}$ corresponding to this subset, and again by part 2 of the same Proposition we conclude that this element coincides with w, whence $w \in W_\mathfrak{p}$. Having characterized $W_\mathfrak{p}$ as those elements $w \in W$ for which $\Phi_w \subset \Delta^+(\mathfrak{g}_0)$, the following definition is natural.

Definition 3.4.1. The Hasse diagram $W^\mathfrak{p}$ of the standard parabolic subalgebra $\mathfrak{p} \leq \mathfrak{g}$ is the subset of $W_\mathfrak{g}$ consisting of all elements w such that $\Phi_w \subset \Delta^+(\mathfrak{p}_+)$. We endow $W^\mathfrak{p}$ with the structure of a directed graph induced from the structure on $W_\mathfrak{g}$.

There is a nice alternative characterization of $W^\mathfrak{p}$: Recall that a weight $\lambda \in (\mathfrak{h}_0)^*$ is $\mathfrak{g}$-dominant, if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta^0$ and $\mathfrak{p}$-dominant if the same holds for all $\alpha \in \Delta^0 \setminus \Sigma = \Delta^0 \cap \Delta^+(\mathfrak{g}_0)$. Equivalently, one can require these conditions for all elements of Δ^+, respectively, $\Delta^+(\mathfrak{g}_0)$. Since any element $w \in W$ acts as an orthogonal transformation on $(\mathfrak{h}_0)^*$, we get $\langle w(\lambda), \alpha \rangle = \langle \lambda, w^{-1}(\alpha) \rangle$. But this shows that $w(\lambda)$ is $\mathfrak{p}$-dominant for any $\mathfrak{g}$-dominant weight λ if and only if $w^{-1}(\lambda) \in \Delta^+$ for any $\alpha \in \Delta^+(\mathfrak{g}_0)$, i.e. if and only if $w \in W^\mathfrak{p}$. Hence, $w \in W^\mathfrak{p}$ if and only if $w(\lambda)$ is $\mathfrak{p}$-dominant for any $\mathfrak{g}$-dominant weight λ.

Proposition 3.4.2. *Let $w \in W$ be any element. Then there are unique elements $w_\mathfrak{p} \in W_\mathfrak{p}$ and $w^\mathfrak{p} \in W^\mathfrak{p}$ such that $w = w_\mathfrak{p} w^\mathfrak{p}$. Moreover, $\ell(w) = \ell(w_\mathfrak{p}) + \ell(w^\mathfrak{p})$.*

Proof. See [1]. □

The existence and the uniqueness of the decomposition $w = w_\mathfrak{p} w^\mathfrak{p}$ tells us that $w^\mathfrak{p}$ is the unique element in the right coset $W_\mathfrak{p} w$ that lies in $W^\mathfrak{p}$. Thus, $W^\mathfrak{p}$ is the set of distinguished representatives for the right coset space $W_\mathfrak{p} \backslash W_\mathfrak{g}$. The statement about the length then tells us that these representatives are the unique elements of minimal length in each coset.

Let us note two simple facts about the Hasse diagram. If $w, w' \in W^\mathfrak{p}$ and $w \xrightarrow{\alpha} w'$, then $\alpha \in \phi_{w'}$ whence $\alpha \in \Delta^+(\mathfrak{p}_+)$. On the other hand, since both the sets $\Delta^+(\mathfrak{p}_+)$ and $\Delta^+(\mathfrak{g}_0)$ are saturated, there is a unique longest element $w_0^\mathfrak{p} \in W^\mathfrak{p}$ with $\Phi_{w_0^\mathfrak{p}} = \Delta^+(\mathfrak{p}_+)$. Since $W_\mathfrak{p}$ is the Weyl group of a semisimple Lie algebra, it contains a unique longest element $w_\mathfrak{p}^0$, and $w_\mathfrak{p}^0 w_0^\mathfrak{p} = w_0$, the longest element in the Weyl group $W_\mathfrak{g}$.

Proposition 3.4.3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{p} \leq \mathfrak{g}$ be the standard parabolic subalgebra corresponding to a set Σ of simple roots. Let $\delta^{\mathfrak{p}}$ be the sum of all fundamental weights corresponding to elements of Σ. Then we have:*

1. *The map $w \mapsto w^{-1}(\delta^{\mathfrak{p}})$ restricts to a bijection between $W^{\mathfrak{p}}$ and the orbit of $\delta^{\mathfrak{p}}$ under $W_{\mathfrak{g}}$.*

2. *Suppose that $w \in W^{\mathfrak{p}}$ and $\alpha \in \Delta^0$ is a simple root such that $\alpha \notin \Phi_{w^{-1}}$ and $s_\alpha(w^{-1}(\delta^{\mathfrak{p}})) \neq w^{-1}(\delta^{\mathfrak{p}})$. Then $ws_\alpha \in W^{\mathfrak{p}}$, $w \overset{w(\alpha)}{\to} ws_\alpha$, and $\Phi_{ws_\alpha} = \Phi_w \cup \{w(\alpha)\}$.*

Proof. See [1]. $\square$

Recipe for determining the Hasse diagram

(A) Determine the Dynkin diagram of the parabolic, i.e. the Dynkin diagram of $\mathfrak{g}$ with those simple roots crossed whose root spaces are contained in $\mathfrak{g}_1$.

(B) Determine the elements of $W^{\mathfrak{p}}$.
Take the weight $\delta^{\mathfrak{p}}$, i.e. the weight, which has coefficient 1 over the crossed nodes and 0 over the uncrossed nodes. Apply simple reflections to this weight according to the following rules, which describe the action of the reflection corresponding to the simple root with coefficient b (all coefficients not shown in the picture remain unchanged under this reflection):

$$\cdots\!-\!\overset{a}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{c}{\circ}\!-\!\cdots \quad \to \quad \cdots\!-\!\overset{a+b}{\circ}\!-\!\!-\!\overset{-b}{\circ}\!-\!\!-\!\overset{b+c}{\circ}\!-\!\cdots$$

We don't write here the other cases, since we will not need them.
The resulting pattern gives all elements of the Hasse diagram and some of the arrows. The element in the Hasse diagram corresponding to the weight obtained by applying first $s_{\alpha_{i_1}}$, then $s_{\alpha_{i_2}}$ and so on up to $s_{\alpha_{i_\ell}}$ to $\delta^{\mathfrak{p}}$ is given by $s_{\alpha_1} \ldots s_{\alpha_\ell}$, so one has to revert the order of the compositions. Moreover, the length of this element is ℓ.

(C) For each element w in the pattern, determine the corresponding set Φ_w of roots, as well as the labels of the arrows determined so far.

Start with the empty set for the point corresponding to $\delta^{\mathfrak{p}}$. Having determined the sets for elements of length $< \ell$ and the labels of the arrows leading to these sets, consider a point corresponding to an element of length ℓ in the original diagram determined by step (B). Choose a sequence of arrows leading from δ^p to the given point, take the simple root indicated on the last arrow in the path, and apply the simple reflections corresponding to the other arrows in the path going back to $\delta^{\mathfrak{p}}$. The resulting root has to be contained in $\Delta^+(\mathfrak{p}_+)$ and the set corresponding to the chosen point is given by adding this root to the set corresponding to the source of the last arrow in the chosen sequence. Now for any of the arrows determined so far, which ends in the given element, the set corresponding to the source of the arrow has to be obtained by deleting one element from the set corresponding to the target of the arrow, and this element is the right label for the arrow.

(D) Determine the remaining arrows.
For each of the sets determined in step (C), compute the sum of all roots contained in the set. For two sets in adjacent columns, which are not yet joined by an arrow, check whether the difference between the two corresponding expressions is a multiple of a root. If yes, then add an arrow between the two sets labeled by that root.

Chapter 4

Parabolic geometries

4.1 Definition

Definition 4.1.1. Let G be a Lie group and $H \leq G$ its closed subgroup. The Cartan geometry of type (G, H) on a $(\dim G - \dim H)$-dimensional manifold M is a principal H-bundle $p : \mathcal{G} \to M$ together with a 1-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, such that:

1. $R_g^* \omega = Ad(g^{-1}) \circ \omega$ for all $g \in H$,

2. $\omega(A^*) = A$ for all $A \in \mathfrak{h}$,

3. $\forall b \in \mathcal{G}$ ω_b induces a linear isomorphism of $T_b\mathcal{G}$ onto $\mathfrak{g}$.

One obvious example of such geometry is the homogeneous space $G \to G/H$ with the Maurer-Cartan form of G as the Cartan connection. This is called the homogeneous model of the geometry.

Definition 4.1.2. Let G be a Lie group, H a closed subgroup of G and $\mathcal{G}$ a principal H-bundle over M, $\dim M = \dim G/H$. Let ω be a Cartan connection on $\mathcal{G}$. The *curvature* K of the Cartan connection ω is a 2-form on $\mathcal{G}$ with values in $\mathfrak{g}$ given by

$$K(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]$$

Definition 4.1.3. 1. The Cartan geometry $(\mathcal{G} \to M, \omega)$ is called flat, if its curvature vanishes identically.

2. The Cartan geometry $(\mathcal{G} \to M, \omega)$ is called torsion-free, if its curvature has values in $\mathfrak{h}$.

It is easy to see that the curvature measures the failure of $(\mathcal{G}, \omega)$ to be locally isomorphic to the homogeneous model.

Proposition 4.1.1. *Let G, H, $\mathcal{G}$ and M be as above. Let ω be a Cartan connection on $\mathcal{G}$ and Ω its curvature. Then K is horizontal in the sense that it vanishes upon inserting one vertical vector field.*

Proof. It suffices to prove it for ω-constant vector fields. So assume that X is the fundamental vector field corresponding to $A \in \mathfrak{h}$ and $Y = \omega^{-1}(B)$, $B \in \mathfrak{g} \setminus \mathfrak{h}$. We have

$$K(X,Y) = -\omega([X,Y]) + [\omega(X), \omega(Y)]$$

Let $h_t = \exp(tX)$ be a curve in H. Then $dh_t/dt = X$ and

$$[X,Y] = \lim_{t\to 0} \frac{1}{t}(Y - (R_{h_t})_* Y) = \lim_{t\to 0} \frac{1}{t}(\omega^{-1}A - \omega^{-1}(Ad(h_t^{-1})B)) =$$
$$\lim_{t\to 0} \frac{1}{t}\omega^{-1}(A - Ad(h_t^{-1})B) = \omega^{-1}([A,B])$$

So we see that $K(X,Y) = 0$.
Since the Cartan connection trivializes $T\mathcal{G}$, any differential form on $\mathcal{G}$ is determined by its values on the constant vector fields $\omega^{-1}(X)$. □

Definition 4.1.4. Let $\mathcal{G} \to M, \omega$ be a Cartan geometry and K its curvature form. Then the *curvature function* $\kappa : \mathcal{G} \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ is defined by

$$\kappa(u)(X,Y) = K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u))$$

or, equivalently

$$\kappa(u)(X,Y) = [X,Y] - \omega([\omega^{-1}(X), \omega^{-1}(Y)](u)).$$

Since we know that K is horizontal, we can view κ as a function on P with values in $\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$.

Definition 4.1.5. A *parabolic geometry* is a Cartan geometry of type (G,P), where G is a semisimple Lie group and $P \subset G$ is a parabolic subgroup. We will use the terminology 'parabolic geometry of type (G,P)' in this situation.

We will assume that no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$. We stay without proof that in this case the geometry is infinitesimally effective.

4.2 Regularity

4.2.1 Infinitesimal flag structures

Definition 4.2.1. Let G be a semisimple Lie group and P its parabolic subgroup and G_0 be the Levi subgroup of P. An *infinitesimal flag structure* of type (G, P) on a smooth manifold M is given by:

(i) A filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ of the tangent bundle of M, such that the rank of T^iM equals the dimension of $\mathfrak{g}^i/\mathfrak{p}$ for all $i = -k, \ldots, -1$.

(ii) A principal G_0-bundle $p : E \to M$.

(iii) A collection $\theta = (\theta_{-k}, \ldots, \theta_{-1})$ of smooth sections $\theta_i \in \Gamma(L(T^iE, \mathfrak{g}_i))$, which are G_0-equivariant in the sense that $R_g^*\theta_i = Ad(g^{-1}) \circ \theta_i$ for all $g \in G_0$, and such that for each $u \in E$ and $i = -k, \ldots, -1$ the kernel of $\theta_i(u) : T_u^iE \to \mathfrak{g}_i$ is $T_u^{i+1}E \subset T_u^iE$.

Proposition 4.2.1. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra, let G be a Lie group with Lie algebra $\mathfrak{g}$, $P \subset G$ a parabolic subgroup corresponding to the grading, and $G_0 \subset P$ the Levi subgroup.*
An infinitesimal flag structure of type (G, P) on a smooth manifold M is equivalent to a filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ of the tangent bundle of Msuch that for each i the rank of T^iM equals the dimension of $\mathfrak{g}^i/\mathfrak{p}$ and a reduction of the structure group of the associated graded bundle $gr(TM)$ to the structure group G_0 with respect to the homomorphism $Ad : G_0 \to GL_{gr}(\mathfrak{g}_-)$.

Proof. See [1]. □

Note, in particular, that this implies that for an infinitesimal flag structure $(\{T^iM\}, p : E \to M, \theta)$ we have a natural isomorphism $\mathrm{gr}_iTM \cong E \times_{G_0} \mathfrak{g}_i$. Explicitly, this identification is induced by the map $E \times \mathfrak{g}_i \to \mathrm{gr}_iTM$ defined by $(u, X) \mapsto [T_up{\cdot}\xi]$, where $\xi \in T_u^iE$ is any tangent vector such that $\theta_i(\xi) = X$ and [] denotes the class in $\mathrm{gr}_i(TM) = T^iM/T^{i+1}M$.

Consider a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P). We have the reductive subgroup G_0 and the nilpotent normal subgroup P_+ of P, which decompose P as a semidirect product. Since acts freely on $\mathcal{G}$, the same is true for P_+, so we can form the orbit space $\mathcal{G}_0 := \mathcal{G}/P_+$. The projections p factors to a smooth map $p_0 : \mathcal{G}_0 \to M$. If $U \subset M$ is open such that there is a principal bundle chart $\psi : p^{-1}(U) \to U \times P$, then ψ is equivariant for the principal right action, so it factors to a diffeomorphism $p_0^{-1}(U) = p^{-1}(U)/P_+ \to U \times (P/P_+)$. This is equivariant for the right action of G_0, so we conclude that $p_0 : \mathcal{G}_0 \to M$ is a smooth principal bundle with structure group $P/P_+ \cong G_0$. On the other hand, the inclusion of G_0 into P leads to local smooth sections of the projection $\mathcal{G} \to \mathcal{G}_0$, so this is a principal bundle with structure group P_+.

Consider the filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \cdots \supset \mathfrak{g}^k \supset \{0\}$. Since the Cartan connection ω induces an isomorphism $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$, we see that for each $i = -k, \dots, k$ we get a smooth subbundle $T^i\mathcal{G} := \omega^{-1}(\mathfrak{g}^i)$ of $T\mathcal{G}$. Since the filtration $\{\mathfrak{g}^i\}$ is P-invariant, equivariancy of ω implies that each of the subbundles $T^i\mathcal{G}$ is stable under the principal right action. Since ω reproduces the generators of fundamental vector fields, we conclude that for $i \geq 0$ the subbundle $T^i\mathcal{G}$ is spanned by the fundamental vector fields with generators in $\mathfrak{g}^i$. In particular, $T^0\mathcal{G}$ is the vertical bundle of $\mathcal{G} \to M$, while $T^1\mathcal{G}$ is the vertical bundle of $\mathcal{G} \to \mathcal{G}_0$.

Since the filtration of $T\mathcal{G}$ is stable under the principal right action, it can be pushed down to $\mathcal{G}_0$ and M, so we have filtrations $T\mathcal{G}_0 = T^{-k}\mathcal{G}_0 \supset \cdots \supset T^0\mathcal{G}_0$ and $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ by smooth subbundles. The tangent maps to all the bundle projections are filtration preserving and $T^0\mathcal{G}_0$ is exactly the vertical bundle of $p_0 : \mathcal{G}_0 \to M$. Once we have a filtration of the tangent bundle of $\mathcal{G}_0$, it makes sense to consider partially defined differential forms, i.e. sections of a bundle of the form $L(T^i\mathcal{G}_0, V)$, where V is some finite-dimensional vector space or a vector bundle over $\mathcal{G}_0$.

Proposition 4.2.2. *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) corresponding to the $|k|$-grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ of the Lie algebra $\mathfrak{g}$ of G. Let $p_0 : \mathcal{G}_0 \to M$ be the underlying G_0-principal bundle.*
Then for each $i = -k, \dots, -1$, the Cartan connection ω descends to a smooth section ω_i^0 of the bundle $L(T^i\mathcal{G}_0, \mathfrak{g}_i)$. For each $u \in \mathcal{G}_0$ and $i = -k, \dots, -1$ the kernel of $\omega_i^0 : T_u^i\mathcal{G}_0 \to \mathfrak{g}_i$ is $T_u^{i+1}\mathcal{G}_0$, and each ω_i^0 is equivariant in the sense that for $g \in G_0$ we have $R_g^*\omega_i^0 = Ad(g^{-1}) \circ \omega_i^0$.

Proof. See [1]. $\square$

This says that any parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) gives rise to an underlying infinitesimal flag structure $(\{T^iM\}, p_0 : \mathcal{G}_0 \to M, \omega^0)$ of type (G, P).

4.2.2 Regularity

Let $(\{T^iM\}, p : E \to M, \theta)$ be an infinitesimal flag structure of some fixed type (G, P). We have seen that $\mathrm{gr}_i(TM) \cong E \times_{G_0} \mathfrak{g}_i$, and thus $\mathrm{gr}(TM) \cong E \times_{G_0} \mathfrak{g}_-$. Via this identification, the Lie bracket on $\mathfrak{g}_-$ (which is preserved by the adjoint action) induces a bilinear bundle map

$$\{\ ,\ \} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM),$$

which is compatible with the grading. This makes $\mathrm{gr}(TM)$ into a bundle of nilpotent graded Lie algebras modelled on $\mathfrak{g}_-$.

Assume that the filtration $\{T^iM\}$ is compatible with the Lie bracket in the sense that for each $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$ the Lie bracket $[\xi, \eta]$ is a section of $T^{i+j}M$. Then for each $i = -k, \dots, -1$ let us denote by $q_i : T^iM \to \mathrm{gr}_i(TM)$ the natural quotient map, and consider the operator $\Gamma(T^iM) \times \Gamma(T^jM) \to \Gamma(\mathrm{gr}_{i+j}(TM))$ defined by $(\xi, \eta) \mapsto q_{i+j}([\xi, \eta])$. This mapping is bilinear over smooth functions, so it is induced by a bilinear bundle map $T^iM \times T^jM \to \mathrm{gr}_{i+j}(TM)$. Moreover, if $\xi \in T^{i+1}M$ or $\eta \in T^{j+1}M$, then $[\xi, \eta] \in T^{i+j+1}M$, so it lies in the kernel of q_{i+j}. Thus, our map further descends to a bundle map $\mathcal{L} : \mathrm{gr}_i(TM) \times \mathrm{gr}_j(TM) \to \mathrm{gr}_{i+j}(TM)$, which is compatible with the gradings. Since $\mathcal{L}$ is induced by the Lie bracket of vector fields, it follows immediately that it makes each fibre $\mathrm{gr}(T_xM)$ into a nilpotent graded Lie algebra.

Definition 4.2.2. 1. A *filtered manifold* is a smooth manifold M together with a filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ of its tangent bundle by smooth subbundles, which is compatible with the Lie bracket in the sense that $[\xi, \eta] \in \Gamma(T^{i+j}M)$ for any $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$.

2. For a filtered manifold $(M, \{T^iM\})$ the tensorial map

$$\mathcal{L} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM)$$

induced by the Lie bracket of vector fields as described above is called the *(generalized) Levi bracket*. For $x \in M$, the nilpotent graded Lie algebra $(\mathrm{gr}(T_xM), \mathcal{L}_x)$ is called the *symbol algebra* of the filtered manifold at x. The bundle $(\mathrm{gr}(TM), \mathcal{L})$ of nilpotent graded Lie algebras obtained in this way is called the *bundle of symbol algebras*.

3. An infinitesimal flag structure $(\{T^iM\}, E \to M, \theta)$ is called *regular*, if $(M, \{T^iM\})$ is a filtered manifold and the algebraic bracket $\{\ ,\ \} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM)$ coincides with the Levi bracket $\mathcal{L}$.

4. A parabolic geometry is called *regular*, if the underlying infinitesimal flag structure is regular.

Observation 4.2.1. A regular infinitesimal flag structure of type (G, P) on a smooth manifold is equivalent to:

- A filtration $\{T^iM\}$ of the tangent bundle, which makes M into a filtered manifold such that the bundle of symbol algebras is locally trivial and modelled on the nilpotent graded Lie algebra $\mathfrak{g}_-$.
- A reduction of structure group of the natural frame bundle of $\mathrm{gr}(TM)$ with respect to $Ad : G_0 \to \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$.

Proposition 4.2.3. *Let $(\{T^iM\}, p : E \to M, \theta)$ be an infinitesimal flag structure such that $(M, \{T^iM\})$ is a filtered manifold. Then the structure is regular if and only if for all $i, j < 0$ such that $i + j \geq -k$ and all sections $\xi \in \Gamma(T^iE)$ and $\eta \in \Gamma(T^jM)$ we have*

$$\theta_{i+j}([\xi, \eta]) = [\theta_i(\xi), \theta_j(\eta)].$$

Proof. See [1]. □

Proposition 4.2.4. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra, G a Lie group with Lie algebra $\mathfrak{g}$, $P \subset G$ a parabolic subgroup corresponding to the grading, and $G_0 \subset P$ the Levi subgroup. Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) with curvature function $\kappa : \mathcal{G} \to L(\Lambda^2\mathfrak{g}_-, \mathfrak{g})$, $u \in \mathcal{G}$ any point, and put $x = p(u) \in M$.*
Then there is an open neighbourhood U of $x \in M$ and a linear extension operator $T_xM \to \mathfrak{X}(U)$, written as $\xi \mapsto \tilde{\xi}$, which is compatible with all structures on TM obtained from the Cartan connection ω and has the following property: For $\xi, \eta \in T_xM$ let $X, Y \in \mathfrak{g}_-$ be the unique elements such that $T_up \cdot \omega_u^{-1}(X) = \xi$ and $T_up \cdot \omega_u^{-1}(Y) = \eta$. Then

$$[\tilde{\xi}, \tilde{\eta}] = T_up \cdot \omega_u^{-1}([X, Y] - \kappa(u)(X, Y)).$$

Proof. See [1]. □

Corollary 4.2.1. *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) with curvature form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ and curvature function $\kappa : \mathcal{G} \to L(\Lambda^2\mathfrak{g}_-, \mathfrak{g})$, and let $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ be the induced filtration of the tangent bundle TM. Then we have:*

1. *$(M, \{T^iM\})$ is a filtered manifold if and only if $K(T^i\mathcal{G}, T^j\mathcal{G}) \subset \mathfrak{g}^{i+j}$, or equivalently $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j}$ for all $i, j < 0$.*

2. *The geometry $(p : \mathcal{G} \to M, \omega)$ is regular if and only if $K(T^i\mathcal{G}, T^j\mathcal{G}) \subset \mathfrak{g}^{i+j+1}$, or equivalently $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ for all $i, j < 0$.*

Proof. See [1]. □

4.3 Normality

Let $\mathfrak{g} = -\mathfrak{k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra, G a Lie group with Lie alegbra $\mathfrak{g}$, and let B be the Killing form of $\mathfrak{g}$. B induces an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ of G-modules (and thus of P-modules), and also an isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ of P-modules. Consequently, we can identify $\Lambda^j(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ with the dual P-module of $\Lambda^j\mathfrak{p}_+^* \otimes \mathfrak{g} = C^j(\mathfrak{p}_+, \mathfrak{g})$ - the Lie algebra cohomology cochain space. The differential $\partial_\mathfrak{p}$ given by

$$\begin{aligned}\partial\phi(X_0, \ldots, X_k) \;&:=\; \sum_{i=0}^{k}(-1)^i X_i \cdot \phi(X_0, \ldots, \hat{X}_i, \ldots, X_k) \\ &+\; \sum_{i<j}(-1)^{i+j}\phi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)\end{aligned}$$

where the hat indicates omission, is a P-homomorphism and $\partial_\mathfrak{p} \circ \partial_\mathfrak{p} = 0$. Dualizing this map, we obtain a P-homomorphism $\partial^* : \Lambda^j(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to \Lambda^{j-1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, which satisfies $\partial^* \circ \partial^* = 0$ and is called the *Kostant codifferential.*

We can easily obtain a formula for ∂^* on decomposable elements. Taking into account that $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$, we can write a decomposable element of $\Lambda^{n+1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ as $Z_0 \wedge \cdots \wedge Z_n \otimes A$ with $Z_i \in \mathfrak{p}_+$ and $A \in \mathfrak{g}$. The pairing of $\psi \in C^{n+1}(\mathfrak{p}_+, \mathfrak{g})$ with that element is given by $B(\psi(Z_0, \ldots, Z_n), A)$. Hence for $\phi \in C^n(\mathfrak{p}_+, \mathfrak{g})$, the pairing of $\partial(\phi)$ with our element is given by

$$\begin{aligned}&\sum_{i=0}^{n}(-1)^i B([Z_i, \phi(Z_0, \ldots, \hat{Z}_i, \ldots, Z_n)], A) \\ +\; &\sum_{i<j}(-1)^{i+j} B(\phi([Z_i, Z_j], Z_0, \ldots, \hat{Z}_i, \ldots, \hat{Z}_j), \ldots, Z_n), A)\end{aligned}$$

Using invariance of B,we may rewrite each of the summands in the first sum as $(-1)^{i+1}B(\phi(Z_0,\dots,\hat{Z}_i,\dots,Z_n),[Z_i,A])$, whence we immediately see that

$$\partial^*(Z_0\wedge\cdots\wedge Z_n\otimes A)=\sum_{i=0}^{n}(-1)^{i+1}Z_0\wedge\cdots\wedge\hat{Z}_i\wedge\cdots\wedge Z_n\otimes[Z_i,A]$$
$$+\sum_{i<j}(-1)^{i+j}[Z_i,Z_j]\wedge Z_0\wedge\cdots\wedge\hat{Z}_i\wedge\cdots\wedge\hat{Z}_j\wedge\cdots\wedge Z_n\otimes A$$

Lemma 4.3.1. *Consider an element $\phi\in L(\Lambda^2(\mathfrak{g}/\mathfrak{p}),\mathfrak{g})$. Interpreting ϕ as a bilinear map on $\mathfrak{g}$, which vanishes if one of its entries lies in $\mathfrak{p}$, we can compute $\partial^*\phi:\mathfrak{g}/\mathfrak{p}\to\mathfrak{g}$ as follows. Choose elements $X_1,\dots,X_\ell\in\mathfrak{g}$ such that $\{X_1+\mathfrak{p},\dots,X_\ell+\mathfrak{p}\}$ is a basis of $\mathfrak{g}/\mathfrak{p}$ and let $Z_1,\dots,Z_\ell$ be the dual basis of $\mathfrak{p}_+$. Then for each $X\in\mathfrak{g}$ we get*

$$\partial^*\phi(X+\mathfrak{p})=2\sum_i[Z_i,\phi(X,X_i)]-\sum_i\phi([Z_i,X],X_i)$$

Proof. See [1]. □

Definition 4.3.1. A parabolic geometry $(p:\mathcal{G}\to M,\omega_)$ is called *normal*, if its curvature function satisfies $\partial^*\kappa=0$.

Theorem 4.3.1. *Let $\mathfrak{g}=\mathfrak{g}_{-k}\oplus\cdots\oplus\mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra such that none of the simple ideals is contained in $\mathfrak{g}_0$, and such that $H^1(\mathfrak{g}_-,\mathfrak{g})^1=0$. Suppose that G is a Lie group with Lie algebra $\mathfrak{g}$, and $P\subset G$ is a parabolic subgroup corresponding to the grading with Levi subgroup $G_0\subset P$.*
Then associating to a parabolic geometry the underlying infinitesimal flag structure and to any morphism of parabolic geometries the induced morphism of the underlying infinitesimal flag structures defines an equivalence between the category of normal regular parabolic geometries of type (G,P) and the category of regular infinitesimal flag structures of type (G,P).

Proof. See [1]. □

If $\mathfrak{g}$ does not contain a simple summand isomorphic to $\mathfrak{sl}(2)$, then we always have $H^1(\mathfrak{g}_-,\mathfrak{g})^2=0$. Moreover, $H^1(\mathfrak{g}_-,\mathfrak{g})^1\neq 0$ happens only if $\mathfrak{g}$ contains a simple factor that belongs to one of two specific series of simple graded Lie algebras. Geometrically, these correspond to classical projective structures and contact projective structures, respectively.

4.4 Weyl structures

Definition 4.4.1. A (local) *Weyl structure* for the parabolic geometry $(p : \mathcal{G} \to M, \omega)$ is a (local) smooth G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the projection $\pi : \mathcal{G} \to \mathcal{G}_0$.

Proposition 4.4.1. *For any parabolic geometry $(p : \mathcal{G} \to M, \omega)$, there exists a global Weyl structure $\sigma; \mathcal{G}_0 \to \mathcal{G}$.*
Fixing one Weyl structure σ, there is a bijective correspondence between the set of all Weyl structures and the space $\Gamma(gr(T^*M))$ of smooth sections of the associated graded of the cotangent bundle. Explicitly, this correspondence is given by mapping $\Upsilon \in \Gamma(gr(T^*M))$ with corresponding functions $\Upsilon_i : \mathcal{G}_0 \to \mathfrak{g}_i$ for $i = 1, \ldots, k$ to the Weyl structure $\hat{\sigma}(u) := \sigma(u)\exp(\Upsilon_1(u)) \cdots \exp(\mathcal{Y}_k(u))$.

Proof. See [1]. □

Given a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ for a parabolic geometry $(p : \mathcal{G} \to M, \omega)$, we can consider the pullback $\sigma^*\omega \in \Omega^1(\mathcal{G}_0, \mathfrak{g})$ of the Cartan connection. Equivariancy of σ implies the equivariancy of $\sigma^*\omega$. As a G_0-module, the Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = -\mathfrak{k} \oplus \cdots \oplus \mathfrak{g}_k$, so decomposing $\sigma^*\omega = \sigma^*\omega_{-k} \oplus \cdots \oplus \sigma^*\omega_k$ accordingly, each of the components is G_0-equivariant.

Proposition 4.4.2. *Let $\sigma : \mathcal{G}_0 \to \mathcal{G}$ be a Weyl structure on a parabolic geometry $(p : \mathcal{G} \to M, \omega)$. Then we have:*

1. *The component $\sigma^*\omega_0 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ defines a principal connection on the bundle $p_0 : \mathcal{G}_0 \to M$.*

2. *The components $\sigma^*\omega_{-k}, \ldots, \sigma^*\omega_{-1}$ can be interpreted as defining an element of $\Omega^1(M, gr(TM))$. This form determines an isomorphism*

$$TM \to gr(TM) = T^{-k}M/T^{-k+1}M \oplus \cdots \oplus T^{-1}M$$

   which is a splitting of the filtration of TM. This means that for each $i = -k, \ldots, -1$ the subbundle T^iM is mapped to $\bigoplus_{j \geq i} gr_j(TM)$ and the component in $gr_i(TM)$ is given by the canonical surjection $T^iM \to T^iM/T^{i+1}M$.

3. *The components $\sigma^*\omega_1, \ldots, \sigma^*\omega_k$ can be interpreted as a one-form $\mathrm{P} \in \Omega^1(M, gr(T^*M))$.*

Proof. See [1]. □

Definition 4.4.2. Let $\sigma : \mathcal{G}_0 \to \mathcal{G}$ be a Weyl structure for a parabolic geometry $(p : \mathcal{G} \to M, \omega)$.

1. The principal connection $\sigma^*\omega_0$ on the bundle $\mathcal{G}_0 \to M$ is called the *Weyl connection* associated to the Weyl structure σ.

2. The $\mathrm{gr}(TM)$-valued one-form on M determined by the negative components of $\sigma^*\omega$ is called the *soldering form* associated to the Weyl structure σ.

3. The one-form $\mathrm{P} \in \Omega^1(M, \mathrm{gr}(T^*M))$ induced by the positive components of $\sigma^*\omega$ is called the *Rho tensor* associated to the Weyl structure σ.

Proposition 4.4.3. *Let $(p; \mathcal{G} \to M, \omega)$ be a parabolic geometry of some fixed type (G, P), and let S be a smooth manifold endowed with a smooth left P-action. Then choosing a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ induces an isomorphism $\mathcal{G} \times_P S \cong \mathcal{G}_0 \times_{G_0} S$ and thus gives rise to a connection on the natural bundle $\mathcal{G} \times_P S$. In the case of a natural vector bundle, this connection is automatically linear.*

Proof. See [1]. □

A homomorphism $\lambda : G_0 \to \mathbb{R}$ gives rise to a Lie algebra homomorphism $\lambda' : \mathfrak{g}_0 \to \mathbb{R}$. This homomorphism vanishes on $\mathfrak{g}_0^{ss}$, so it is just a linear functional on the centre $\mathfrak{z}(\mathfrak{g}_0)$. The Killing form B of $\mathfrak{g}$ restricts to a nondegenerate bilinear form on $\mathfrak{g}_0$. The splitting of $\mathfrak{g}_0$ into its centre and semisimple part is orthogonal with respect to B, so he restriction of B to $\mathfrak{z}(\mathfrak{g}_0)$ still is nondegenerate. Given a homomorphism $\lambda : G_0 \to \mathbb{R}$, we therefore get a unique element $E_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$ such that $\lambda'(A) = B(E_\lambda, A)$.

Definition 4.4.3.
1. An element $F \in \mathfrak{z}(\mathfrak{g}_0)$ is called a *scaling element* if and only if the restriction to $\mathfrak{p}_+$ of the adjoint action $\mathrm{ad}_F : \mathfrak{g} \to \mathfrak{g}$ is injective.

2. A *bundle of scales* for parabolic geometries of type (G, P) is a natural principal $\mathbb{R}$-bundle $\mathcal{L}^\lambda$ associated to a homomorphism $\lambda : G_0 \to \mathbb{R}$, such that the corresponding element $E_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$ is a scaling element. If λ has values in $\mathbb{R}^+$, then we obtain an *oriented bundle of scales*.

3. Having chosen a bundle $\mathcal{L}^\lambda$ of scales, a *(local) scale* for a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) is a (local) smooth section of the principal $\mathbb{R}^+$-bundle $\mathcal{L}^\lambda \to M$.

Proposition 4.4.4. *1. For any type of parabolic geometry there exist natural oriented bundles of scales.*

2. A bundle of scales admits global smooth sections if and only if it is oriented.

Proof. See [1]. □

Definition 4.4.4. Let L^λ be a fixed bundle of scales.

1. A Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ is called *closed*, if and only if the induced connection ∇ on L^λ is flat.

2. A Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ is called *exact*, if and only if the induced connection ∇ on L^λ comes from a global trivialization of L^λ

Proposition 4.4.5. *Let L^λ be a fixed bundle of scales. Then the space of all Weyl structures for $(p : \mathcal{G} \to M, \omega)$ is an affine space modelled on the vector space $\Omega^1(M)$ of one-forms on M. If they exist, then the space of closed (respectively exact) Weyl structures are affine subspaces modelled on closed (respectively exact) one-forms.*

Proof. See [1]. □

The very flat Weyl structure

We will work on the homogeneous space $(G \to G/P, \omega)$. Given a parabolic pair (G, P), there is also the opposite parabolic subgroup P^{op} and the Carnot group G_- with Lie algebra $\mathfrak{g}_-$, which is a simply connected Lie group. These group are defined in a similar way to P and P_+, but with respect to the opposite filtration to $\mathfrak{g}^i$. For $X \in \mathfrak{g}$ the vector field $\omega^{-1}(X)$ is the left invariant vector field L_X, so its flow through e is simply $\exp(X)$. Now $\exp : \mathfrak{g}_- \to G_-$ is a diffeomorphism from $\mathfrak{g}_-$ onto an (actually dense) open subset of G/P called the *big cell*. The restriction of both the bundles $G \to G/P$ and $G/P_+ \to G/P$ to the big cell is canonically trivialized by the restrictions of the group multiplication to maps $G_- \times P \to G$, respectively, $G_- \times G_0 \to G$. Our Weyl structure is characterized by $\sigma(\exp(X)P_+) = \exp(X)$ for all $X \in \mathfrak{g}_-$. Under our trivializations $p^{-1}(G_-) \cong G_- \times P$ and $p_0^{-1}(G_- \times G_0)$, this section is simply given by the inclusion $G_0 \to P$. In the trivialization of $p_0^{-1}(G_-)$, σ is given by the group multiplication viewed as a map $G_- \times G_0 \to G$.

This map can be viewed as the composition of the isomorphism $G_- \times G_0 \cong P^{op}$ and the inclusion of this subgroup into G. The pullback of the Maurer-Cartan form of G along the inclusion of P^{op} clearly is the Maurer-Cartan form of P^{op}. The induced principal connection is defined by the $\mathfrak{g}_0$-component of this Maurer-Cartan form. In particular, on $\{e\} \times G_0$, this is the Maurer-Cartan form of G_0. Via the diffeomorphism $G_- \times G_0 \to P^{op}$ defined by the group multiplication, this is extended to the whole bundle, so we exactly obtain the flat connection determined by this trivialization. hence, the induced linear connection on any associated bundle is flat. On the other hand, the soldering form is induced by the Maurer-Cartan form of G_-, i.e. by trivialization of TG_- by left translations. Finally, for vectors tangent to $P^{op} \subset G$, the Maurer-Cartan form of G has values in $\mathfrak{p}^{op} = \mathfrak{g}_- \oplus \mathfrak{g}_0$, so the Rho tensor of our Weyl structure vanishes identically. Finally note, that since every bundle of scales is associated to $G/P_+ \to G/P$, our trivialization of $p_0^{-1}(G_-) \to G_-$ gives rise to a bundle of scales, which is trivial for the induced Weyl connection. Hence, the very flat Weyl structure is always an exact Weyl structure.

4.5 BGG sequences

4.5.1 Invariant operators

Given a representation of P on a vector space $\mathbb{V}$ and a parabolic geometry $(\mathcal{G} \to M, \omega)$, we can form the associated bundle $VM = \mathcal{G} \times_P \mathbb{V} \to M$. If $\Phi : \mathcal{G} \to \mathcal{G}'$ is a homomorphism of principal bundles covering a local diffeomorphism $\underline{Phi} : M \to M'$, then we get the induced homomorphism of vector bundles $VM \to VM'$ lying over the same map $\underline{\Phi}$ and restricting to a linear isomorphism on each fibre. In other words, we get a functor from the category of parabolic geometries to the category of vector bundles over manifolds of the same dimension as G/P such that the composition of the base functor with the given functor is the base functor.

Consider next a fixed category of real parabolic geometries, and two representations $\mathbb{V}$ and $\mathbb{W}$ of P. Let V and W be the corresponding natural vector bundles. A *natural linear operator* mapping sections of V to sections of W is defined to be a system of linear operators $D_{(\mathcal{G},\omega)} : \Gamma(VM) \to \Gamma(WM)$, where M is the base of $\mathcal{G}$ such that for any morphism $\Phi : (\mathcal{G},\omega) \to (\mathcal{G}',\omega')$ we have

$$\Phi^* \circ D_{(\mathcal{G}',\omega')} = D_{(\mathcal{G},\omega)} \circ \Phi^*$$

It is a classical result on Cartan connections that any flat parabolic geometry is locally isomorphic to the homogeneous model G/P. So any natural operator on the category of flat parabolic geometries is uniquely determined by its value on the homogeneous model G/P, i.e. the parabolic geometry $(G \to G/P, \omega)$. An operator on the homogeneous model extends to a natural operator on the category of flat parabolic geometries if and only if it is natural with respect to all automorphisms of (G,ω). The left multiplication by an element of G induces an automorphism of the principal bundle $G \to G/P$ and by left invariance of the Maurer-Cartan form this actually is an automorphism of the parabolic geometry (G,ω). On the other hand, the only smooth maps $G \to G$, which pullback the Maurer-Cartan form to itself, are the constant left translations. Thus G is exactly the group of all automorphisms of (G,ω). This group has a natural action on sections of all homogeneous bundles $G \times_P \mathbb{V}$ for some P-representation $\mathbb{V}$. This action is defined by $g \cdot [g', v] = [gg', v]$ and lifts the action of G on G/P. We see that an operator on the homogeneous model extends to a natural operator on the category of flat parabolic geometries if and only if it is equivariant for the G-action just defined.

Usually, the question on more general natural operators is then posed as the question of the existence of *curved analogs* of invariant operators. An invariant operator of order r is then induced by a P-module homomorphism $J^r E_o \to F_o$, which does not factor over $J^{r-1}F_o$. Now the kernel of the projection $J^r E_o \to J^{r-1}E_o$ is the bundle $S^r T^*M \otimes E$, so it corresponds to the representation $S^r\mathfrak{p}_+ \otimes \mathbb{E}$. Thus the invariant operator gives rise to a homomorphism $S^r\mathfrak{p}_+ \otimes \mathbb{E} \to \mathbb{F}$ of P-modules, which in turn gives a G-equivariant homomorphism between the corresponding homogeneous vector bundles, which is precisely the symbol of the operator we started with. But this P-module homomorphism induces a homomorphism of associated bundles on any parabolic geometry, so for any parabolic geometry $(\mathcal{G},\omega)$ over a manifold M, we get the corresponding homomorphism $S^rT^*M \otimes EM \to FM$. Now a curved analog of an invariant operator is a natural operator such that for each $(\mathcal{G},\omega)$ the symbol of $D_{(\mathcal{G},\omega)}$ is the above homomorphism.

4.5.2 Semiholonomic jets and strongly invariant operators

It is possible to define for each P-representation $\mathbb{V}$ its 1st jet prolongation $\mathcal{J}^1(\mathbb{V})$ in such a way that $J^1(\mathcal{G} \times_P \mathbb{V}) = \mathcal{G} \times_P \mathcal{J}^1(\mathbb{V})$. As a G_0-module, this decomposes as $\mathcal{J}^1(\mathbb{E}) \cong \mathbb{E} \oplus \mathfrak{g}_-^* \otimes \mathbb{V}$, see [3]. $\mathcal{J}^1$ can be made into a functor on the category of P-modules.

Since we have posed no conditions on the representation $\mathbb{V}$, we can iterate the functors J^1 on the associated vector bundles as well as the functors $\mathcal{J}^1$ on the P-modules. Then the r-th iteration $J^1 \dots J^1 M$ is an associated bundle to $\mathcal{G}$ corresponding to the P-module $\mathcal{J}^1 \dots \mathcal{J}^1\mathbb{V}$. Let us look more carefully at $\mathcal{J}^1\mathcal{J}^1\mathbb{V}$ and J^1J^1VM. There are two obvious P-module homomorphisms $\mathcal{J}^1\mathcal{J}^1\mathbb{V} \to \mathcal{J}^1\mathbb{V}$, the first one given by the projection $p_{\mathcal{J}^1\mathbb{V}}$ defined on each fist jet prolongation by projection to the first component, and the other one obtained by the action of $\mathcal{J}^1$ on $p_\mathbb{V}$. Thus there is the subbundle $\bar{\mathcal{J}}^2\mathbb{V}$ in $\mathcal{J}^1\mathcal{J}^1\mathbb{V}$, on which these two projections coincide. As a vector space and G_0-module, ew have

$$\bar{\mathcal{J}}^2\mathbb{V} = \mathbb{V} \oplus (\mathfrak{G}_-^* \otimes \mathbb{V}) \oplus (\mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathbb{V}).$$

The two P-module homomorphisms $\mathcal{J}^1(p_\mathbb{V})$ and $p_{\mathcal{J}^1\mathbb{V}}$ give rise to vector bundle homomorphisms $J^1J^1VM \to J^1VM$, which are just the two standard projections on the second nonholonomic jet prolongation. So we conclude that the second semiholonomic jet prolongation $\bar{J}^2VM$ is naturally isomorphic to $\mathcal{G} \times_P \bar{\mathcal{J}}^2\mathbb{V}$.

Iterating this procedure, we obtain the r-th semiholonomic jet prolongation and $\mathcal{J}^1(\bar{\mathcal{J}}^r\mathbb{V})$ equipped with two natural projections onto $\mathcal{J}^1(\bar{\mathcal{J}}^{r-1}\mathbb{V})$, which correspond to the usual projections on the first jet prolongation of semiholonomic jets. Their equalizer is then the subbundle $\bar{\mathcal{J}}^{r+1}\mathbb{V}$. As a G_0-module

$$\bar{\mathcal{J}}^r\mathbb{V} = \bigoplus_{i=0}^{r} (\otimes^i \mathfrak{g}_-^* \otimes \mathbb{V})$$

Suppose that $\mathbb{V}$ and $\mathbb{W}$ are representations of P and suppose that $\Phi : \bar{\mathcal{J}}^r(\mathbb{V}) \to \mathbb{W}$ is a homomorphism of P-modules. Then for any parabolic geometry $(\mathcal{G}, \omega)$ we can define a differential operator $D_{(\mathcal{G},\omega)} : \Gamma(VM) \to \Gamma(WM)$ in a canonical way, see [3]. This gives us a natural operator on the category of all parabolic geometries of order $\leq r$. The operators arising in this way are called *strongly invariant operators*.

4.5.3 Twisted covariant derivative and exterior covariant derivative

Any representation $\mathbb{V}$ of G is a representation of P by restriction. These representations have one interesting feature in the case of the homogeneous model since they give rise to trivial homogeneous bundles. To see this, we associate to any element v in a representation $\mathbb{V}$ of G a global nonzero section of the associated bundle $G \times_P \mathbb{V}$. To do this, we just have to specify a P-equivariant map $G \to \mathbb{V}$, and we define this map simply by $g \mapsto g^{-1} \cdot v$. This map is even G-equivariant and not only P-equivariant.

There is a simple generalization of this result. Suppose that $\mathbb{W}$ is any representation of P. Then sections of $W(G/P)$ are in bijective correspondence with P-equivariant maps $G \to \mathbb{W}$. Now we define a map on sections of homogeneous bundles

$$\begin{gathered}\Gamma(W(G/P)) \otimes \mathbb{V} \to \Gamma(W(G/P) \otimes V(G/P)) \\ s \otimes v \mapsto (g \in G \mapsto s(g) \otimes g^{-1} \cdot v)\end{gathered}$$

This is an isomorphism of G-modules. In particular, this implies that if $\mathbb{W}'$ is another P-representation and $D : \Gamma(W(G/P)) \to \Gamma(W'(G/P))$ is an invariant differential operator, then we can pull back

$$D \otimes \mathrm{id}_{\mathbb{V}} : \Gamma(W(G/P)) \otimes \mathbb{V} \to \Gamma(W'(G/P)) \otimes \mathbb{V}$$

along these isomorphisms to get an invariant operator

$$D_{\mathbb{V}} : \Gamma(W(G/P) \otimes V(G/P)) \to \Gamma(W'(G/P) \otimes V(G/P)).$$

This operator is called the twisted invariant operator corresponding to D and $\mathbb{V}$.

Notice that the above isomorphism between the spaces of sections of the associated bundles induces an isomorphism $\bar{\mathcal{J}}^r(\mathbb{W}) \otimes \mathbb{V} \cong \bar{\mathcal{J}}^r(\mathbb{W} \otimes \mathbb{V})$ of P-modules, for all P-modules $\mathbb{W}$ and G-modules $\mathbb{V}$ and all orders r. Thus, for strongly invariant operators D, we may extend the construction of the twisted invariant operators to natural operators $D_{\mathbb{V}}$ acting on all geometries $(\mathcal{G}, \omega)$ of type (G, P) and the resulting operators are again strongly invariant.

The standard exterior derivatives d on the differential forms on G/P are first order invariant operators, so we can apply the construction above to get the twisted exterior derivatives

$$d_{\mathbb{V}} : \Gamma(\Lambda^n T^*(G/P) \otimes V(G/P)) \to \Gamma(\Lambda^{n+1} T^*(G/P) \otimes V(G/P))$$

for $n =, \dots, \dim(G/P)$. The operators $d_{\mathbb{V}}$ are strongly invariant, so we obtain a canonical curved analog. We may obtain another curved analog as follows. For any parabolic geometry $(\mathcal{G}, \omega)$ on M, we consider the extended bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_P G$, which is a principal G-bundle over M. It is a classical observation that the Cartan connection ω induces a principal connection $\tilde{\omega}$ on $\tilde{\mathcal{G}}$. If $\mathbb{V}$ is a representation of G, we can view the corresponding natural bundle $VM = \mathcal{G} \times_P \mathbb{V}$ also as $VM = \tilde{\mathcal{G}} \times_G \mathbb{V}$, and thus we have the induced linear connection on this bundle. The covariant exterior derivative with respect to this connection gives a natural operator on VM-valued forms on M. If $s : \tilde{\mathcal{G}} \to \Lambda^k \mathfrak{p}_+ \otimes \mathbb{V}$ is the equivariant function corresponding to a k-form ϕ on M, then the value of the latter operator is a $(k+1)$-form on M, given by the formula

$$d^{\tilde{\omega}} s(u)(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i \nabla^{\tilde{\omega}}_{X_i} s(u)(X_0, \dots, \hat{X}_i, \dots, X_k) + \\ + \sum_{i<j} (-1)^{i+j} s(u)([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k)$$

where $X_0, \dots, X_k \in \mathfrak{g}_-$, $u \in \tilde{\mathcal{G}}$, $\nabla^{\tilde{\omega}}_{X_i} s(u)$ means the derivative of s in the direction of the horizontal vector at u determined by X_i, and there are standard omissions of arguments in the expressions on the right-hand side.

These operators coincide with the twisted exterior derivatives on the homogeneous space but they differ in general. The explicit general comparison is as follows:

Lemma 4.5.1. *Let $\mathbb{V}$ be a G-module, VM the corresponding natural vector bundle over a manifold M equipped with a parabolic geometry $(\mathcal{G}, \omega)$. The covariant exterior derivative $d^{\tilde{\omega}}$ on $\Lambda^k T^*M \otimes VM$, $k > 0$, and the twisted exterior derivative $d_{\mathbb{V}}$ on the same space satisfy*

$$d^{\tilde{\omega}} \phi = d_{\mathbb{V}} \phi + i_{\kappa_-} \phi$$

where κ_- is the torsion component of the curvature of ω and $i_{\kappa_-}\phi$ is the usual insertion operator, i.e. the alternation of $\phi(\kappa_-(X_0, X_1), X_2, \dots, X_k)$ over the arguments.

Proof. See [3]. □

4.5.4 BGG sequences

We have already defined the Lie algebra cohomology. Let $\mathbb{V}$ be a representation of G. Then we have the complex $(C^n(\mathfrak{g}_-, \mathbb{V}), \partial)$ with corresponding cohomology $H^n(\mathfrak{g}_-, \mathbb{V})$ and Kostant codifferential ∂^*. We define the Laplacian

$$\Box = \partial \circ \partial^* + \partial^* \circ \partial$$

Then for each n this is a G_0-endomorphism of $C^n(\mathfrak{g}_-, \mathbb{V})$. Moreover, the adjointness implies that $\ker(\Box) = \ker(\partial) \cap \ker(\partial^*)$ and we have a G_0-invariant splitting

$$C^n(\mathfrak{g}_-, \mathbb{V}) = \operatorname{im}(\partial) \oplus \ker(\Box) \oplus \operatorname{im}(\partial^*)$$

This implies that the cohomology group $H^n(\mathfrak{g}_-, \mathbb{V})$ is isomorphic as a G_0-module to the subspace $\ker(\Box) \subset C^n(\mathfrak{g}_-, \mathbb{V})$. Moreover, the situation between ∂ and ∂^* is completely symmetric, so we can as well compute the cohomology groups $H^*(\mathfrak{g}_-, \mathbb{V})$ as $\ker(\partial^*)/\operatorname{im}(\partial^*)$. This is more suitable, since ∂^* is even a P-homomorphism. This also implies that (even as G_0-module) the cohomology group $H^n(\mathfrak{g}_-, \mathbb{V})$ is dual to $H^n(\mathfrak{p}_+, \mathbb{V}^*)$. Thus we have a canonical action of P on the cohomology groups $H^n(\mathfrak{g}_-, \mathbb{V})$. This module is completely reducible, see [3].

Let us put $\mathbb{H}^n_{\mathbb{V}} := H^n(\mathfrak{g}_-, \mathbb{V})$ as a representation of P. Then it is possible to construct a sequence of strongly invariant operators

$$0 \to \Gamma(H^0_{\mathbb{V}}M) \xrightarrow{D^{\mathbb{V}}} \Gamma(H^1_{\mathbb{V}}M) \xrightarrow{D^{\mathbb{V}}} \cdots \xrightarrow{D^{\mathbb{V}}} \Gamma(H^{\dim(G/P)}_{\mathbb{V}}M) \to 0$$

called the *Bernstein-Gelfand-Gelfand sequence* or the *BGG sequence* determined by the G-module $\mathbb{V}$.

All bundles in this sequence correspond to completely reducible representations of P, so they all split into direct sum of bundles corresponding to irreducible representations. The construction applies to both real and complex setting.

Theorem 4.5.1. *Let $(\mathcal{G}, \omega)$ be a real parabolic geometry of type (G, P) on a manifold M, $\mathbb{V}$ be a G-module. If the twisted de Rham sequence*

$$0 \to \Omega^0(M; VM) \xrightarrow{d_{\mathbb{V}}} \Omega^1(M; VM) \xrightarrow{d_{\mathbb{V}}} \cdots \xrightarrow{d_{\mathbb{V}}} \Omega^{\dim(G/P)}(M; VM) \to 0$$

is a complex, then also the Bernstein-Gelfand-Gelfand sequence

$$0 \to \Gamma(H^0_{\mathbb{V}}M) \xrightarrow{D^{\mathbb{V}}} \Gamma(H^1_{\mathbb{V}}M) \xrightarrow{D^{\mathbb{V}}} \cdots \xrightarrow{D^{\mathbb{V}}} \Gamma(H^{\dim(G/P)}_{\mathbb{V}}M) \to 0$$

is a complex, and they both compute the same cohomology.

Proof. See [3]. □

Corollary 4.5.1. *Let $(\mathcal{G}, \omega)$ be a flat real parabolic geometry. Then for any representation $\mathbb{V}$ of G the BGG sequence*

$$0 \to \Gamma(H^0_{\mathbb{V}}M) \overset{D^{\mathbb{V}}}{\to} \Gamma(H^1_{\mathbb{V}}M) \overset{D^{\mathbb{V}}}{\to} \cdots \overset{D^{\mathbb{V}}}{\to} \Gamma(H^{\dim(G/P)}_{\mathbb{V}}M) \to 0$$

is a complex, which computes the twisted de Rham cohomology of M with coefficients in the bundle VM, which is defined as the cohomology of the complex given by the covariant exterior derivative with respect to the linear connection on VMinduced by the Cartan connection ω.

Proof. See [3]. □

We may always consider the obvious flat parabolic geometry on the trivial P-bundle over $\mathbb{R}^{\dim(G/P)} \cong \mathfrak{g}_-$ (i.e. the big cell). In this case, the twisted de Rham cohomologies are obviously zero, so the above Corollary provides global resolutions of the constant sheaf $\mathbb{V}$ in this case. Important for us is that the kernel of the operator $D^{\mathbb{V}}\Gamma(H^0_{\mathbb{V}}M) \to \Gamma(H^1_{\mathbb{V}}M)$ is as a G-module isomorphic to $\mathbb{V}$.

It is obvious that we can complexify the bundles and the operators above to obtain the 'complexified' BGG resolution.

4.5.5 Relation between BGG and the Hasse graph

We know that we may consider the BGG operators acting between some irreducible bundles, which correspond to some irreducible P-modules. We will denote these representations by minus the lowest weight - i.e. the highest weight of the dual representation. Similar notation will be used for representations of G.

Definition 4.5.1. Let $w \in W_{\mathfrak{g}}$ and λ be a weight for $\mathfrak{g}$. The affine action of w on λ is given by

$$w \cdot \lambda := w(\lambda + \delta) - \delta$$

Let us fix a G-module $\mathbb{V}$ with minus lowest weight λ. Then the bundles $H^i(\mathfrak{g}_-, \mathbb{V})$ decompose as the direct sum of bundles corresponding to irreducible representations of P with minus lowest weight $w \cdot \lambda$, $w \in W^{\mathfrak{p}}$, $\ell(w) = i$. These weights can be computed as follows. Choose a path in the Hasse graph from the identity to the given element w, take the weight λ, start at w and in each step apply the reflection corresponding to root labelling the arrow ending at the point you are, going back from w to the identity. This recipe can be found in [5].

Chapter 5

CR structures as parabolic geometries

5.1 Parabolic contact structures

Let M be a $(2n+1)$-dimensional contact manifold. We may say that contact manifold is a filtered manifold with $TM = T^{-2}M \supset T^{-1}M$ with $T^{-1}M$ being the contact subbundle $H \subset TM$. Its symbol algebra will be called the Heisenberg algebra $\mathfrak{h}_{2n+1}$. Let us denote $Q := TM/H$. We know that the Levi bracket $\mathcal{L} : H \times H \to Q$ is nondegenerate.

Proposition 5.1.1. *Let $H \subset TM$ be a contact structure on a smooth $(2n+1)$-dimensional manifold M with the quotient bundle $Q = TM/H$. Let $p : E \to M$ be the natural frame bundle for $H \oplus Q$ with structure group $Aut_{gr}(\mathfrak{h}_{2n+1})$. Let $\mathcal{L} : \Lambda^2 H \to Q$ be the Levi bracket and let $\Lambda^2_0 H \subset \Lambda^2 H$ be the kernel of $\mathcal{L}$.*

1. *Locally, there exists a contact form α, which has H as its contact subbundle, and thisd form is unique up to multiplication by a nowhere vanishing function. In particular, contact structures have no local invariants. There exists a global contact form α for H if and only if the quotient bundle Q is orientable and hence trivial.*

2. *Any principal connection on E is completely determined by the induced connection on H. A linear connection on H arises in this way if and only if the induced connection on $\Lambda^2 H$ preserves the subbundle $\Lambda^2_0 H$. These connections are called contact connections.*

3. *If $\alpha \in \Omega^1(M)$ is a contact form with contact subbundle H, then there is a unique vector field r on M such that $\alpha(r) = 1$ and $i_r d\alpha = 0$. In particular, α induces an isomorphism $TM \cong H \oplus Q$.*

4. *Given α as above, there is a linear connection ∇ on TM such that ∇ preserves the subbundle H, $\nabla\alpha = 0$, $\nabla d\alpha = 0$ and $\nabla r = 0$ and such that the restriction to H is induced by a principal connection on E.*

Proof. See [1]. □

Let (E, θ) be a regular infinitesimal flag structure of some type (G, P) corresponding to a contact grading. Then choosing a Weyl structure, we get an identification $TM \cong H \oplus Q$ and a contact connection ∇^H on H, which is compatible with the additional structure induced by E. This connection determines a connection on Q via $\nabla\mathcal{L} = 0$, and so a linear connection on TM. Since Q may be used as a bundle of scales for any parabolic contact geometry, we see that there is a one-to-one correspondence between exact Weyl structures and contact forms on M.

5.2 Partially integrable almost CR structures

For $p + q = n \geq 1$ we consider the real form $\mathfrak{su}(p+1, q+1)$ of $\mathfrak{sl}(n+2, \mathbb{C})$. We choose the Hermitean form on $\mathbb{C}^{n+2}$, which is given by

$$\langle (z_o, \ldots, z_{n+1}), (w_0, \ldots, w_{n+1}) \rangle = \overline{z_0} w_{n+1} + \overline{z_{n+1}} w_0 + \sum_{j=1}^{p} \overline{z_j} w_j - \sum_{j=p+1}^{n} \overline{z_j} w_j$$

Denoting by $\mathbb{I} = \mathbb{I}_{pq}$ be the $n \times n$-diagonal matrix with the first p entries equal to 1 and the remaining entries equal to -1, we can represent the Lie algebra in block form with blocks of sizes 1, n and 1 as

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z & iz \\ X & A & -\mathbb{I}Z^* \\ ix & -X^*\mathbb{I} & -\bar{a} \end{pmatrix} : \begin{matrix} A \in \mathfrak{u}_n, a \in \mathbb{C}, X \in \mathbb{C}^n, Z \in \mathbb{C}^{n*}, \\ x, z \in \mathbb{R}; a + \mathrm{tr}A - \bar{a} = 0 \end{matrix} \right\}$$

The grading components are indicated by

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1^A & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^A & \mathfrak{g}_0 & \mathfrak{g}_1^B \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^B & \mathfrak{g}_0 \end{pmatrix}$$

From this description it is easy to see that the Dynkin diagram of $\mathfrak{p}_{\mathbb{C}}$ is

$$\times\!\!-\cdots-\!\!\times$$

Rather then the splitting of $\mathfrak{g}_{\pm 1}$ into two irreducible pieces, we have a complex structure on these subspaces. After complexification, the splitting into two components is recovered as the splitting of $\mathfrak{g}_{\pm 1} \otimes \mathbb{C}$ into holomorphic and antiholomorphic part. The bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is given by $[X, Y] = Y^* \mathbb{I} X - X^* \mathbb{I} Y$, so this is minus twice the imaginary part of the standard Hermitean inner product of signature (p, q). Note that this is compatible with the complex structure in the sense that $[iX, iY] = [X, Y]$.

As a group with Lie algebra $\mathfrak{g}$, we take $G = PSU(p+1, q+1)$. The parabolic subgroup P is then the stabilizer of the isotropic complex line generated by the first basis vector. (This automatically stabilizes its orthocomplement, which is a hyperplane containing the given line.) The subgroup G_0 is given by block diagonal matrices, i.e. we have matrices $\begin{pmatrix} c & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 1/\bar{c} \end{pmatrix}$ with $c \in \mathbb{C} \setminus 0$ and $C \in U(n)$ such that $c\det(C)/\bar{c} = 1$. We have to identify matrices, which are multiples of each other, which leaves the freedom by a multiplying by a $(n+2)$nd root of unity. The adjoint action is given by $(c, C) \cdot (ix, X) = (|c|^{-2} ix, c^{-1} C X)$, which is complex linear on $\mathfrak{g}_{-1}$ and orientation preserving on $\mathfrak{g}_{-2}$. There is an p-dimensional subspace in $\mathfrak{g}_{-1}$ on which $X \mapsto [X, iX]$ is nonzero with all values of the same sign and a q-dimensional subspace, for which the same is true for the opposite sign. Hence, if $p \neq q$, then preserving the bracket and the complex structure on $\mathfrak{g}_{-1}$ implies that the orientation on $\mathfrak{g}_{-2}$ is preserved. For $p = q$, this is an additional condition.

Conversely, assume that $A : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ is a complex linear isomorphism such that $[AX, AY] = \lambda [X, Y]$ for some $\lambda > 0$. Since the standard Hermitean form of signature (p, q) is obtained as $1/2(i[X, iY] + [X, Y])$, we conclude that A has the same compatibility with that Hermitean form. In particular, $|\det(A)|^2 = \lambda^n$. Now choose $c \in \mathbb{C}$ such that $|c|^2 c^{-n-2} = \det(A)$. Then we get $|\det(A)|^2 = |c|^{-2n} = \lambda^n$, and since $\lambda > 0$, this implies $\lambda = |c|^{-2}$. Hence cA has the property that $[cAX, cAY] = [X, Y]$ and hence $cA \in U(n)$. But then A is realized as the adjoint action of (c, cA) and $c\det(cA)/\bar{c} = c^{n+2}|c|^{-2}\det(A) = 1$ as required. In this procedure c is only unique up to multiplication with an $(n+2)$nd root of unity.

A regular infinitesimal flag structure (and hence a regular normal parabolic geometry) of type (G, P) on a smooth manifold M of dimension

$2n+1$ is equivalent to a contact structure $H \subset TM$ together with a complex structure J on H such that $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all $\xi, \eta \in \Gamma(H)$. If this last condition is satisfied, then identifying the fibre Q_x over $x \in M$ with $\mathbb{R}$, the map $\mathcal{L}$ is minus the imaginary part of a Hermitean form, and one requires that this form has signature (p, q). If $p = q$, one in addition has to choose an orientation on Q (which requires Q to be trivial). Since as a complex vector bundle H is canonically oriented, this is equivalent to choosing an orientation on M. For $p \neq q$, this orientation is automatically chosen by deciding between the signature (p, q) and signature (q, p).

We rephrase this in the language of CR geometry. Given a real smooth manifold M of dimension $2n + 1$, a rank n complex subbundle (H, J) in TM is called an *almost CR structure of hypersurface type*. The almost CR structure is called *nondegenerate*, if H defines a contact structure on M. Next, we have the condition that $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all $\xi, \eta \in H$. This condition is not very often used in CR geometry, since it is implied by the integrability condition. One usual terminology for this condition is *partial integrability*. Then $\mathcal{L}$ becomes minus the imaginary part of a Hermitean form and choosing an orientation on Q, the signature of this form is called the signature of (M, H, J). Hence, regular normal parabolic geometries of type $(PSU(p+1, q+1), P)$ are equivalent to oriented nondegenerate partially integrable hypersurface type almost CR structures of signature (p, q).

Theorem 5.2.1. *The category of oriented hypersurface type CR structures of signature (p, q) is equivalent to a category of torsion-free regular normal parabolic geometries of type $(PSU(p + 1, q + 1), P)$.*

Proof. See [1]. □

5.3 The CR sub-Laplacian

We have an n-dimensional complex vector bundle $H^{1,0} \subset TM \otimes \mathbb{C}$ and its conjugate $H^{0,1} \subset TM \otimes \mathbb{C}$. Define $\Lambda^{1,0} \subset T^*M \otimes \mathbb{C}$ by $\Lambda^{1,0} = (H^{0,1})^\perp$. The *canonical bundle* $\mathcal{K} := \Lambda^{n+1}(\Lambda^{1,0})$ is a complex line bundle on M. We will assume that $\mathcal{K}$ admits an $(n + 2)$nd root (for this we have to pass to the group $SU(p + 1, q + 1)$) and we fix a bundle denoted $\mathcal{E}(1, 0)$, which is a $-1/(n+2)$-th power of $\mathcal{K}$. The bundle $\mathcal{E}(w_1, w_2) := (\mathcal{E}(1, 0))^{w_1} \otimes (\overline{\mathcal{E}(1, 0)})^{w_2}$ of (w_1, w_2)-densities is defined for $w_1, w_2 \in \mathbb{C}$ satisfying $w_1 - w_2 \in \mathbb{Z}$. If $\mathcal{E}(1, 0) \setminus \{0\}$ is viewed as a $\mathbb{C}^\times$-principal bundle, then $\mathcal{E}(w_1, w_2)$ is the bundle induced by the representation $\lambda \to \lambda^{-w_1}\bar{\lambda}^{-w_2}$. As a representation of P, this is the one-dimensional irreducible representation (and thus irreducible representation of G_0) given by $\rho((c, C)) = c^{-w_1}\bar{c}^{-w_2}$. The bundle $Q \otimes \mathbb{C}$ may be identified with $\mathcal{E}(1, 1)$.

Assume we have given a contact form α on M. Then we can view Q as a trivial $\mathbb{R}$-bundle over M. Using this identification, we may view the Levi bracket as minus twice the imaginary part of some Hermitean form h of signature (p, q), i.e. a pseudoscalar product on H, which we can use to upper and lower indices. We also have the corresponding exact Weyl structure and, in particular, the corresponding Weyl connection satisfying $\nabla\alpha = 0$, $\nabla d\alpha = 0$, $\nabla r = 0$ and $\nabla\mathcal{L} = 0$. The last equation implies that $\nabla h = 0$. This connection preserves the splitting $TM \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1} \oplus \langle r\rangle$. Let $\{X_\alpha\}$ be a basis of $H^{1,0}$ and $\{X_{\bar{\alpha}}\}$ the corresponding basis of $H^{0,1}$. We will use indices $\alpha, \bar{\alpha}, 0$ for components with respect to basis $\{X_\alpha, X_{\bar{\alpha}}, r\}$. If f is a (possibly density-valued) tensor field, we will denote the components of the (tensorial) iterated covariant derivatives of f by preceding ∇-s, e.g. $\nabla_\alpha\nabla_0 \cdots \nabla_{\bar{\beta}} f$. Such indices may alternately be interpreted abstractly.

Definition 5.3.1. Let M be a $(2n+1)$-dimensional CR manifold with contact subbundle H, a contact form α and the corresponding Weyl connection. Let w_1 and w_2 be complex numbers such that $w_1 - w_2 \in \mathbb{Z}$ and $n + w_1 + w_2 = 0$. Then the CR sub-Laplacian is the operator $\Delta : \mathcal{E}(w_1, w_2) \to \mathcal{E}(w_1-1, w_2-1)$ given by

$$\Delta(f) = \frac{1}{2}[\nabla^\alpha\nabla_\alpha + \nabla_\alpha\nabla^\alpha + i(w_1 - w_2)\nabla_0 - \frac{n}{2(n+1)}R - \frac{n(w_1 - w_2)^2}{2(n+1)(n+2)}R](f)$$

where R is the scalar curvature with respect to h.

Using the very flat Weyl structure on the big cell, the last two terms vanish, since the connection is flat. This definition coincides with that in [8].

5.4 Symmetries of the sub-Laplacian

Definition 5.4.1. The *symmetry operator* of Δ is a differential operator P such that

$$\Delta P = \delta\Delta$$

for some differential operator δ.

It is easy to see that any operator of type $\mathcal{D}\Delta$ for some differential operator $\mathcal{D}$ is a symmetry operator of Δ. But these operators act trivially on the solution space of Δ. Ergo it is useful to introduce certain relation of equivalence on the vector space of all symmetry operators of Δ.

Definition 5.4.2. Two symmetry operators P and $\tilde{P}$ of Δ are called equivalent, if there exists a differential operator $\mathcal{D}$ such that $P - \tilde{P} = \mathcal{D}\Delta$

Lemma 5.4.1. *The vector space of equivalence classes of symmetry operators forms an associative unital algebra with multiplication being the composition of differential operators.*

Proof. We need to show that the operation of composition of differential operators preserves the vector space of symmetry operators and respects their equivalence. If P_1 and P_2 are two symmetry operators, we must gind a differential operator δ such that

$$\Delta P_1 P_2 = \delta\Delta \tag{5.1}$$

From the definition of the symmetry operator we see that there are differential operators δ_1 and δ_2 satisfying the relations

$$\Delta P_1 = \delta_1\Delta \quad \Delta P_2 = \delta_2\Delta$$

Substituting these relations into 5.1

$$\Delta P_1 P_2 = \delta_1\Delta P_2 = \delta_1\delta_2\Delta$$

we see that we can put $\delta = \delta_1\delta_2$.

The second part states that the composition $\hat{P}_1\hat{P}_2$ of two symmetry operators $\hat{P}_1$, $\hat{P}_2$ (equivalent to P_1 and P_2, respectively) is equivalent to P_1P_2. In other words, we are looking for some differential operator $\mathcal{D}$ such that $P_1P_2 - \hat{P}_1\hat{P}_2 = \mathcal{D}\Delta$. If we establish operators $\mathcal{D}_1$, $\mathcal{D}_2$ and δ_2 in an obvious manner, we get

$$\begin{aligned} P_1P_2 - \hat{P}_1\hat{P}_2 &= (\mathcal{D}_1\Delta + \hat{P}_1)P_2 - \hat{P}_1(P_2 - \mathcal{D}_2\Delta) = \\ &= \mathcal{D}_1\Delta P_2 + \hat{P}_1\mathcal{D}_2\Delta = \\ &= (\mathcal{D}_1\delta_2 + \hat{P}_1\mathcal{D}_2)\Delta \end{aligned}$$

and hence we can put $\mathcal{D} = \mathcal{D}_1\delta_2 + \hat{P}_1\mathcal{D}_2$. $\square$

5.5 BGG sequences for CR structures

We will need BGG sequences for representations of type $\overset{a}{\circ}\!\!-\!\!\!-\!\!\overset{b}{\circ}\!- \cdots -\!\overset{b}{\circ}\!\!-\!\!\!-\!\!\overset{a}{\circ}$. This representation is dual to itself, so minus the lowest weight is simply the highest weight. To get a feeling how the Hasse graph looks like in general, we write it down for $n = 1, 2$. In the sets we only note the indices of the roots. The order of positive roots is given form top to bottom in the matrix representation of $\mathfrak{g}$.

For $n = 1$, we have two simple roots α_1 and α_2, and we number the additional positive root by $\alpha_3 = \alpha_1 + \alpha_2$:

$$\begin{tikzcd}
 & \{1\} \arrow[r, "\alpha_3"] \arrow[ddr, "\alpha_1"'] & \{1,3\} \arrow[dr, "\alpha_2"] & \\
\emptyset \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"'] & & & \{1,2,3\} \\
 & \{2\} \arrow[r, "\alpha_3"] \arrow[uur, "\alpha_2"] & \{2,3\} \arrow[ur, "\alpha_1"] &
\end{tikzcd}$$

For $n = 2$, we have three simple roots α_1, α_2 and α_3, and we number the additional positive roots by $\alpha_4 = \alpha_1+\alpha_2$, $\alpha_5 = \alpha_2+\alpha_3$ and $\alpha_6 = \alpha_1+\alpha_2+\alpha_3$:

$$\begin{tikzcd}
 & & \{1,4\} \arrow[r, "\alpha_6"] \arrow[ddr, "\alpha_4"'] & \{1,4,6\} \arrow[dr, "\alpha_3"] & & \\
 & \{1\} \arrow[ur, "\alpha_4"] \arrow[dr, "\alpha_3"] & & & \{1,3,4,6\} \arrow[dr, "\alpha_5"] & \\
\emptyset \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_3"'] & & \{1,3\} \arrow[r, "\alpha_6"] \arrow[uur, "\alpha_3"] \arrow[ddr, "\alpha_1"'] & \{1,3,6\} \arrow[ur, "\alpha_4"] \arrow[dr, "\alpha_5"] & & \{1,3,4,5,6\} \\
 & \{3\} \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_5"] & & & \{1,3,5,6\} \arrow[ur, "\alpha_4"] & \\
 & & \{3,5\} \arrow[r, "\alpha_6"] \arrow[uur, "\alpha_5"] & \{3,5,6\} \arrow[ur, "\alpha_1"] & &
\end{tikzcd}$$

From these two examples we see that the first operators in the complexified BGG sequence correspond to reflections according to the two simple roots represented by the extremal nodes of the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$. Knowing how to compute the simple root reflections of any weight, we see that the beginning of the BGG sequence corresponding to the representation $\overset{a}{\circ}\!-\!\overset{b}{\circ}\!-\cdots-\!\overset{b}{\circ}\!-\!\overset{a}{\circ}$ looks like

$$\overset{a}{\circ}\!-\!\overset{b}{\circ}\!-\cdots-\!\overset{b}{\circ}\!-\!\overset{a}{\circ} \hookrightarrow \overset{a}{\times}\!-\!\overset{b}{\circ}\!-\cdots-\!\overset{b}{\circ}\!-\!\overset{a}{\times} \begin{matrix} \nearrow \; \overset{-a-2}{\times}\!-\!\overset{a+b+1}{\circ}\!-\cdots-\!\overset{b}{\circ}\!-\!\overset{a}{\times} \\ \searrow \; \overset{a}{\times}\!-\!\overset{b}{\circ}\!-\cdots-\!\overset{a+b+1}{\circ}\!-\!\overset{-a-2}{\times} \end{matrix}$$

The section of the bundle $\overset{-a-2}{\times}\!-\!\overset{a+b+1}{\circ}\!-\cdots-\!\overset{b}{\circ}\!-\!\overset{a}{\times}$ is (modulo weights) a tensor field $V^{c_1\ldots c_b\bar{d}_1\ldots\bar{d}_b}$ from $S^bHM \otimes S^b(HM^*)$ (HM^* is up to weight isomorphic to $\overline{HM}$), which is totally trace-free. The upper operator looks like

$$V^{c_1\ldots c_b)\bar{d}_1\ldots\bar{d}_b} \mapsto \text{the trace-free part of} \quad \partial^{(c_1}\ldots\partial^{c_{a+1}}V^{c_{a+2}\ldots c_{a+b+1})\bar{d}_1\ldots\bar{d}_b}$$

modulo curvature terms and the lower operator is the conjugate one:

$$V^{c_1\ldots c_b\bar{d}_1\ldots\bar{d}_b} \mapsto \text{the trace-free part of} \quad \partial^{(\bar{d}_1}\ldots\partial^{\bar{d}_{a+1}}V^{\bar{d}_{a+2}\ldots\bar{d}_{a+b+1})c_1\ldots c_b}$$

modulo curvature terms.
It can be easily seen from the fact, that if $\overset{-a-2}{\times}\!-\!\overset{a+b+1}{\circ}\!-\cdots-\!\overset{b}{\circ}\!-\!\overset{a}{\times}$ is an irreducible P-representation, then it is an irreducible G_0-representation, and the corresponding G_0^{ss}-representation is given by deleting all crossed nodes and edges connecting to them in the Dynkin diagram notation. We shall note that the order of these operators is always $a+1$.

Chapter 6

The ambient construction

We want to describe symmetries of the CR sub-Laplacian on the CR quadric. We will work on the big cell with the very flat Weyl structure. Since none of what will follow depends on the signature of the CR structure, we will consider general signature.

Let's consider $\mathbb{C}^{n+1}(z^1, \dots, z^n, z^\infty)$ with Hermitean metric of the form

$$\begin{pmatrix} g_{\bar{a}b} & 0 \\ 0 & 0 \end{pmatrix}$$

and consider a submanifold M of $\mathbb{C}^{n+1}$ given by $\sum_{i=1}^n z^a z_a + z^\infty + \bar{z}^\infty = 0$. On the manifold M we shall define a CR structure. In coordinates the submanifold M looks like $(z^1, \dots, z^n, -\sum_{i=1}^n z^a z_a/2 + i\sigma)$. In terms of coordinates on $\mathbb{C}^{n+1}$, the coordinate vector fields on M look like $\{\partial_{z^a} - \frac{z_a}{2}\partial_{z^\infty} - \frac{z_a}{2}\partial_{z^{\bar{\infty}}}\}$, their conjugates, and $\partial_\sigma = i(\partial_{z^\infty} - \partial_{z^{\bar{\infty}}})$. The contact subbundle $HM^{1,0} \subset (T\mathbb{C}^{n+1})^{1,0}$ has basis $\{\partial_a = \partial_{z^a} - z_a\partial_{z^\infty}, a = 1, \dots, n\}$, since it has to formed by complex linear combinations of coordinate vector fields on M, which form holomorphic vector fields on $\mathbb{C}^{n+1}$. The commutator $[\partial_{\bar{a}}, \partial_b] = [\partial_{z^{\bar{a}}} - z_{\bar{a}}\partial_{z^{\bar{\infty}}}, \partial_{z^b} - z_b\partial_{z^\infty}] = ig_{\bar{a}b}\partial_\sigma$. This is the only nontrivial commutator. We will work mostly with vector fields $\partial_{\bar{a}}$, ∂_b, the coordinate vector fields $\partial_{z^{\bar{a}}}$, ∂_{z^b} will be without use.

Definition 6.0.1. Let M be as above. The *ambient space* for M is $\mathbb{C}^{n+2}$ $(z^0, z^1, \dots, z^n, z^\infty)$ with non-degenerate Hermitean metric $g_{\bar{A}B}$ of the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & g_{\bar{a}b} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We will denote
$$x^A = \begin{pmatrix} x^0 \\ x^a \\ x^\infty \end{pmatrix}$$

The term *ambient* will be used when referring to the objects defined on some open subset of $\mathbb{C}^{n+2}$. The ambient Laplace operator will be distinguished by tilde $\tilde{\Delta} f = g^{A\bar{B}} \partial_A \partial_{\bar{B}}$.

Definition 6.0.2. Let
$$r = g_{A\bar{B}} x^A x^{\bar{B}} \tag{6.1}$$
be the quadratic form associated to the ambient metric $g_{A\bar{B}}$. The *null cone* $\mathcal{N}$ is the zero set of r.
$$\mathcal{N} = \left\{ x \in \mathbb{C}^{n+2} | r(x) = 0 \right\}$$

Now consider the mapping $\phi : M \to \mathbb{C}^{n+2}$ given by
$$(z^a, i\sigma) \mapsto \begin{pmatrix} 1 \\ z^a \\ -\frac{z^a z_a}{2} + i\sigma \end{pmatrix} =: \phi^A$$

The mapping ϕ is actually a restriction to M of the embedding $\imath : \mathbb{C}^{n+1} \to \mathbb{C}^{n+2}$ given by $(z^1, \ldots, z^n, z^\infty) \mapsto (1, z^1, \ldots, z^n, z^\infty)$. It is easily seen that $\phi(M)$ lies on the null cone and that this characterizes M in $\mathbb{C}^{n+1}$.

Definition 6.0.3. Let $z^0 \in \mathbb{C}$, $\rho, \sigma \in \mathbb{R}$ and $z^a \in \mathbb{C}^n$.

$$X^A = \begin{pmatrix} z^0 \\ z^0 z^a \\ z^0(\rho - \frac{z^a z_a}{2} + i\sigma) \end{pmatrix} \qquad Y_b^A = \partial_b X^A = \begin{pmatrix} 0_b \\ z^0 \delta_b^a \\ -z^0 z_b \end{pmatrix}$$

$$Y_{\bar{b}}^A = \partial_{\bar{b}} X^A = 0 \qquad Z^A = -\frac{1}{n} \partial^b Y_b^A = \begin{pmatrix} 0 \\ 0^a \\ z^0 \end{pmatrix}$$

$$X^{\bar{A}} = \begin{pmatrix} z^{\bar{0}} \\ z^{\bar{0}} z^{\bar{a}} \\ z^{\bar{0}}(\rho - \frac{z^a z_a}{2} - i\sigma) \end{pmatrix} \qquad Y_{\bar{b}}^{\bar{A}} = \partial_{\bar{b}} X^{\bar{A}} = \begin{pmatrix} 0_{\bar{b}} \\ z^{\bar{0}} \delta_{\bar{b}}^{\bar{a}} \\ -z^{\bar{0}} z_{\bar{b}} \end{pmatrix}$$

$$Y_b^{\bar{A}} = \partial_b X^{\bar{A}} = 0 \qquad Z^{\bar{A}} = -\frac{1}{n}\partial^{\bar{b}} Y_{\bar{b}}^{\bar{A}} = \begin{pmatrix} 0 \\ 0^{\bar{a}} \\ z^{\bar{0}} \end{pmatrix}$$

$$X_A = \left(z^{\bar{0}}(\rho - \frac{z^a z_a}{2} - i\sigma), z^{\bar{0}} z_a, z^{\bar{0}} \right) \qquad Z_A = \left(z^{\bar{0}}, 0_a, 0 \right)$$

$$Y_{Ab} = 0 \qquad Y_{A\bar{b}} = \left(-z^{\bar{0}} z_{\bar{b}}, z^{\bar{0}} g_{a\bar{b}}, 0_{\bar{b}} \right)$$

$$X_{\bar{A}} = \left(z^0(\rho - \frac{z^a z_a}{2} + i\sigma), z^0 z_{\bar{a}}, z^0 \right) \qquad Z_{\bar{A}} = \left(z^0, 0_{\bar{a}}, 0 \right)$$

$$Y_{\bar{A}\bar{b}} = 0 \qquad Y_{\bar{A}b} = \left(-z^0 z_b, z^0 g_{\bar{a}b}, 0_b \right)$$

Lemma 6.0.1.

$$|z^0|^2 \delta_B^A = (X^A + \rho Z^A) Z_B + Z^A (X_B + \rho Z_B) + Y_c^A Y_B^c \tag{6.2}$$

The mapping

$$\Phi(z^0, z^a, z^\infty) = \begin{pmatrix} z^0 \\ z^0 z^a \\ z^0 (z^\infty - \frac{z^a z_a}{2}) \end{pmatrix} = \begin{pmatrix} y^0 \\ y^a \\ y^\infty \end{pmatrix}$$

where $z^\infty = \rho + i\sigma$, defines a change of coordinates, which, however, is smooth, but not holomorphic. We see that $\phi(z^a, i\sigma) = \Phi(1, z^a, i\sigma)$ and the identity (6.2) simplifies on the image of ϕ to

$$\delta_B^A = X^A Z_B + Z^A X_B + Y_c^A Y_B^c$$

Similarly for $\delta_{\bar{B}}^{\bar{A}}$:

$$\delta_{\bar{B}}^{\bar{A}} = X^{\bar{A}} Z_{\bar{B}} + Z^{\bar{A}} X_{\bar{B}} + Y_{\bar{c}}^{\bar{A}} Y_{\bar{B}}^{\bar{c}}$$

Lemma 6.0.2. *The operator $\mathbb{E} = x^C \partial_C$ in the new coordinates is equal to*

$$\mathbb{E} = z^0 \partial_{z^0} \tag{6.3}$$

Proof. For $f(y^A, y^{\bar{A}}) \in \mathcal{C}^\infty(\mathbb{C}^{n+2})$ we have

$$\begin{aligned} z^0\partial_{z^0} f &= z^0(\frac{\partial f}{\partial y^0} + z^a\frac{\partial f}{\partial y^a} + (z^\infty - \frac{z^a z_a}{2})\frac{\partial f}{\partial y^\infty}) \\ &= y^0(\frac{\partial f}{\partial y^0} + \frac{y^a}{y^0}\frac{\partial f}{\partial y^a} + \frac{y^\infty}{y^0}\frac{\partial f}{\partial y^\infty}) \\ &= (y^A\partial_A f)\circ\Phi \end{aligned}$$

□

Here are some identities we will need later:

$$\begin{aligned} Y_q^A\partial_A Y_{\bar{r}}^{\bar{B}} &= \begin{pmatrix} 0_q \\ z^0\delta_q^a \\ -z^0 z_q \end{pmatrix} \begin{pmatrix} \partial_{z^0} - \frac{z^a}{z^0}\partial_{z^a} - \frac{z^\infty}{z^0}\partial_{z^\infty} - \frac{z^a z_a}{2z^0}\partial_{z^{\bar\infty}} \\ \frac{1}{z^0}\partial_{z^a} + \frac{z_a}{2z^0}\partial_{z^\infty} + \frac{z_a}{2z^0}\partial_{z^{\bar\infty}} \\ \frac{1}{z^0}\partial_{z^\infty} \end{pmatrix} \begin{pmatrix} 0_{\bar{r}} \\ z^{\bar{0}}\delta_{\bar{r}}^{\bar{b}} \\ -z^{\bar{0}} z_{\bar{r}} \end{pmatrix} \\ &= \left(\partial_{z^q} + \frac{z_q}{2}\partial_{z^{\bar\infty}} - \frac{z_q}{2}\partial_{z^\infty}\right) \begin{pmatrix} 0_{\bar{r}} \\ z^{\bar{0}}\delta_{\bar{r}}^{\bar{b}} \\ -z^{\bar{0}} z_{\bar{r}} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ -z^{\bar{0}} g_{q\bar{r}} \end{pmatrix} = -g_{q\bar{r}} Z^{\bar{B}} \end{aligned} \tag{6.4}$$

Similarly,

$$Y_{\bar{r}}^{\bar{B}}\partial_{\bar{B}} Y_q^A = -g_{q\bar{r}} Z^A \tag{6.5}$$

$$Z^B\partial_B = \partial_{z^\infty} \tag{6.6}$$

$$Z^{\bar{B}}\partial_{\bar{B}} = \partial_{z^{\bar\infty}} \tag{6.7}$$

Definition 6.0.4. Suppose that F is a smooth complex-valued function defined on the neighbourhood of the origin in M. Then for any pair $(w_1, w_2) \in \mathbb{C}^2$, s.t. $w_1 - w_2 \in \mathbb{Z}$

$$f(\Phi(z^0, z^a, i\sigma)) = (z^0)^{w_1}(z^{\bar{0}})^{w_2} F(z^a, i\sigma) \tag{6.8}$$

defines a smooth function on a conical neighbourhood of $(1, 0, 0)$ in the null-cone $\mathcal{N}$. Conversely, F may be recovered from f by setting $z^0 = 1$. Hence, for fixed (w_1, w_2), the functions F and f are equivalent.

Remark 6.0.1. If we view $\mathcal{N} \setminus \{x = (z^0, \dots, z^\infty) \in \mathcal{N} : z^0 = 0\}$ as a principal $\mathbb{C}^\times$-bundle over M, then we can represent the sections of $\mathcal{E}(w_1, w_2)$ as equivariant functions on the null-cone. But these are exactly the (w_1, w_2)-homogeneous functions as defined above. Having in mind that we are working with the very flat Weyl structure on M, we can identify densities of arbitrary weights with functions when working on M.

We want to use the ambient construction to represent differential operators on M by much simpler ambient differential operators. In order to be able to apply ambient differential operators to f, we need to extend it from the null-cone to the whole space or at least to some open (in $\mathbb{C}^{n+2}$) neighbourhood of $(1, 0, 0)$. There are infinitely many choices for such an extension even if we restrict ourselves to the homogeneous ones. Nevertheless, any two such extensions will differ by a very convenient factor.

Lemma 6.0.3. *Let f and $\hat{f}$ be two smooth (w_1, w_2)-homogeneous extensions of F on some open neighbourhood of $(1, 0, 0)$. Then there exists a smooth $(w_1 - 1, w_2 - 1)$-homogeneous function hsuch that $(f - \hat{f})(y^A) = r(y^A)h(y^A)$, where r is defined by (6.1).*

Proof. If we perform coordinate transformation

$$(y^0, y^a, y^\infty) \mapsto (y^0, y^a, r + ip) = (y^0, y^a, y^0 y^{\bar{\infty}} + y^a y_a)$$

we will be dealing with 2 functions equal on the real hyperplane $r = 0$. For any smooth complex-valued function k on $\mathbb{C}^{n+2}$ holds

$$k(y^0, y^a, r + ip) = k(y^0, y^a, ip) + \int_0^1 \frac{d}{dt} k(y^0, y^a, tr + ip)dt =$$

$$= k(y^0, y^a, ip) + r \int_0^1 \frac{\partial k}{\partial(r + ip)}(y^0, y^a, tr + ip) + \frac{\partial k}{\partial(r - ip)}(y^0, y^a, tr + ip)dt$$

So if we take k as the difference of two (w_1, w_2)-homogeneous extensions of F, we will have $k(y^0, y^a, ip) = 0$ and thus it follows that $f - \hat{f} = rh$. This h has homogeneity $(w_1 - 1, w_2 - 1)$, because r has homogeneity $(1, 1)$. $\square$

Remark 6.0.2. The classical chain rule formula gives

$$\partial_a F = \partial_a(f \circ \phi) = (\partial_a \phi^B \partial_B f + \partial_a \phi^{\bar{B}} \partial_{\bar{B}} f) \circ \phi =$$
$$(Y_a^B \partial_B f) \circ \phi = (\partial_a f) \circ \phi$$

$$\partial_{\bar a}F = \partial_{\bar a}(f\circ\phi) = (\partial_{\bar a}\phi^B\partial_B f + \partial_{\bar a}\phi^{\bar B}\partial_{\bar B} f)\circ\phi =$$
$$(Y_{\bar a}^{\bar B}\partial_{\bar B} f)\circ\phi = (\partial_{\bar a} f)\circ\phi$$
$$\partial_\sigma F = \partial_\sigma(f\circ\phi) = (\partial_\sigma\phi^A\partial_A f + \partial_\sigma\phi^{\bar A}\partial_{\bar A} f)\circ\phi =$$
$$(i\partial_{z^\infty} f - i\partial_{\bar z^\infty} f)\circ\phi = (\partial_\sigma f)\circ\phi$$

Lemma 6.0.4. *For homogeneous function h on $\mathbb{C}^{n+2}$ of bidegree (w_1-1, w_2-1) holds*

$$\tilde\Delta(rh) = r\tilde\Delta h + (n + w_1 + w_2)h$$

Proof.

$$\begin{aligned}
\tilde\Delta(rh) &= g^{A\bar B}\partial_A\partial_{\bar B}(rh) = g^{A\bar B}\partial_A(x_{\bar B}h + r\partial_{\bar B}h) = \\
&= g^{A\bar B}(g_{A\bar B}h + x_{\bar B}\partial_A h + x_A\partial_{\bar B}h + r\partial_A\partial_{\bar B}h) = \\
&= (n+2)h + x^A\partial_A h + x^{\bar B}\partial_{\bar B}h + r\tilde\Delta h = \\
&= (n+2)h + (w_1-1)h + (w_2-1)h + r\tilde\Delta h = \\
&= r\tilde\Delta h + (n + w_1 + w_2)h
\end{aligned}$$

□

It immediately follows that for $n + w_1 + w_2 = 0$, then $\tilde\Delta f|_{\mathcal N}$ depends only on the restriction of f to the null-cone and hence it depends only on F. This defines a differential operator on M.

Theorem 6.0.1. *Let F be a smooth complex-valued function on some open neighbourhood of $0 \in M$ and let f be the smooth homogeneous function of bidegree (w_1, w_2) that corresponds to F via (6.8) and is defined on some open neighbourhood of $(1, 0, 0) \in \mathbb{C}^{n+2}$. Then the following equality holds*

$$(\tilde\Delta f)\circ\phi = \Delta F$$

where Δ is the CR sub-Laplacian.

Proof. Using the equation (6.2) we obtain

$$(g^{A\bar B}\partial_A\partial_{\bar B} f)\circ\phi =$$
$$= [(X^A Z^{\bar B} + Z^A X^{\bar B})\partial_A\partial_{\bar B} f + g^{q\bar r}Y_q^A Y_{\bar r}^{\bar B})\partial_A\partial_{\bar B} f]\circ\phi$$

Now the first term gives

$$(X^A Z^{\bar B} + Z^A X^{\bar B})\partial_A\partial_{\bar B} =$$

$$-Z^A(\partial_A X^{\bar B})\partial_{\bar B} + Z^A\partial_A X^{\bar B}\partial_{\bar B} - Z^{\bar B}(\partial_{\bar B}X^A)\partial_A + Z^{\bar B}\partial_{\bar B}X^A\partial_A =$$
$$Z^A\partial_A X^{\bar B}\partial_{\bar B} + Z^{\bar B}\partial_{\bar B}X^A\partial_A =$$
$$Z^A\partial_A\bar{\mathbb{E}} + Z^{\bar B}\partial_{\bar B}\mathbb{E}$$

applied to f and evaluated on the image of ϕ. Let us recall that $Z^A\partial_A = \partial_{z^\infty}$ and $Z^{\bar B}\partial_{\bar B} = \partial_{z^{\bar\infty}}$. The second term is

$$\left(g^{q\bar r}Y_q^A Y_{\bar r}^{\bar B}\partial_A\partial_{\bar B}f\right)\circ\phi =$$
$$= \left(g^{q\bar r}[Y_q^A\partial_A Y_{\bar r}^{\bar B}\partial_{\bar B} - Y_q^A(\partial_A Y_{\bar r}^{\bar B})\partial_{\bar B}]f\right)\circ\phi =$$
$$= \left(g^{q\bar r}Y_q^A\partial_A Y_{\bar r}^{\bar B}\partial_{\bar B}f + \frac{n}{2}Z^{\bar B}\partial_{\bar B}f\right)\circ\phi =$$
$$= \left(g^{q\bar r}\partial_q\partial_{\bar r}f + \frac{n}{2}\partial_{z^{\bar\infty}}f\right)\circ\phi$$

Another way to compute the second term is

$$\left(g^{q\bar r}Y_{\bar r}^{\bar B}Y_q^A\partial_{\bar B}\partial_A f\right)\circ\phi =$$
$$= \left(g^{q\bar r}[Y_{\bar r}^{\bar B}\partial_{\bar B}Y_q^A\partial_A - Y_{\bar r}^{\bar B}(\partial_{\bar B}Y_q^A)\partial_A]f\right)\circ\phi =$$
$$= \left(g^{q\bar r}Y_{\bar r}^{\bar B}\partial_{\bar B}Y_q^A\partial_A f + \frac{n}{2}Z^A\partial_A f\right)\circ\phi =$$
$$= \left(g^{q\bar r}\partial_{\bar r}\partial_q f + \frac{n}{2}\partial_{z^\infty}f\right)\circ\phi$$

We will take as the second term one half of their sum

$$\left([\frac{1}{2}g^{q\bar r}(\partial_{\bar r}\partial_q + \partial_q\partial_{\bar r}) + \frac{n}{2}(\partial_{z^{\bar\infty}} + \partial_{z^\infty})]f\right)\circ\phi$$

Altogether we get

$$(\tilde\Delta f)\circ\phi = (w_1\partial_{z^{\bar\infty}}f + w_2\partial_{z^\infty}f)\circ\phi +$$
$$+\left([\frac{g^{q\bar r}}{2}(\partial_{\bar r}\partial_q + \partial_q\partial_{\bar r}) + \frac{n}{2}(\partial_{z^{\bar\infty}} + \partial_{z^\infty})]f\right)\circ\phi =$$
$$= \left([\frac{g^{q\bar r}}{2}(\partial_q\partial_{\bar r} + \partial_{\bar r}\partial_q) + \frac{n+w_1+w_2}{2}\partial_\rho + i\frac{w_1-w_2}{2}\partial_\sigma]f\right)\circ\phi =$$
$$= \frac{1}{2}(\partial^a\partial_a + \partial_a\partial^a)F + \frac{i(w_1-w_2)}{2}\partial_\sigma F$$

which completes the proof. We have only used the fact that $z^\infty = \rho + i\sigma$. $\square$

Chapter 7

Ambient construction of symmetries of the sub-Laplacian

In this chapter we first characterize the symbol of a symmetry of the sub-Laplacian (modulo equivalence) and then we will construct ambient differential operators commuting with $\tilde{\Delta}$, which also commute with multiplication by r, and so inducing operators (symmetries of the sub-Laplacian) on M. We shall prove that we can construct all of them this way. Sometimes we will for brevity write 'symmetry' instead of 'symmetry of the sub-Laplacian'.

7.1 First order symmetries

Lemma 7.1.1. *The first order operators $x_{\bar{A}}\partial_B - x_B\partial_{\bar{A}}$ commute with $\tilde{\Delta}$ and with r.*

Proof.

$$g^{C\bar{D}}\partial_C\partial_{\bar{D}}(x_{\bar{A}}\partial_B - x_B\partial_{\bar{A}}) =$$
$$= g^{C\bar{D}}\partial_C(x_{\bar{A}}\partial_B\partial_{\bar{D}} - g_{\bar{D}B}\partial_{\bar{A}} - x_B\partial_{\bar{A}}\partial_{\bar{D}}) =$$
$$= g^{C\bar{D}}(g_{\bar{A}C}\partial_B\partial_{\bar{D}} + x_{\bar{A}}\partial_B\partial_C\partial_{\bar{D}} - g_{\bar{D}B}\partial_{\bar{A}}\partial_C - x_B\partial_{\bar{A}}\partial_C\partial_{\bar{D}}) =$$
$$= g^{C\bar{D}}(x_{\bar{A}}\partial_B - x_B\partial_{\bar{A}})\partial_C\partial_{\bar{D}} + \partial_B\partial_{\bar{A}} - \partial_{\bar{A}}\partial_B$$

Similarly, using that $r = x^A x_A = x^{\bar{A}}x_{\bar{A}}$,

$$(x_{\bar{A}}\partial_B - x_B\partial_{\bar{A}})x^C x_C =$$

$$x_{\bar{A}}x_B + x_{\bar{A}}x^C x_C\partial_B - x_B x_{\bar{A}} - x_B x^C x_C \partial_{\bar{A}}$$

□

Now we know that any complex linear combination of such operators, i.e. any operator of the form $V^{B\bar{A}}(x_{\bar{A}}\partial_B - x_B\partial_{\bar{A}})$ commutes with $\tilde{\Delta}$ and r, and hence induces a symmetry of the sub-Laplacian on M. For convenience, we give a description of the real operators between them:

Lemma 7.1.2. *The operator of the form $D = V^{B\bar{A}}(x_{\bar{A}}\partial_B - x_B\partial_{\bar{A}})$ is real ($D = \overline{D}$), if and only if the matrix $V^{B\bar{A}}$ is skew-Hermitean (represents an element of $\Lambda^{1,1}\mathbb{C}^{n+2}$), i.e. $\overline{V^{B\bar{A}}} = -V^{A\bar{B}}$.*

Proof. If we substitute for D into the equation $D = \overline{D}$, we get

$$\overline{V^{B\bar{A}}}(x_A\partial_{\bar{B}} - x_{\bar{B}}\partial_A) = V^{B\bar{A}}(x_{\bar{A}}\partial_B - x_B\partial_{\bar{A}})$$

After relabeling the indices, we have

$$\overline{V^{B\bar{A}}}(x_A\partial_{\bar{B}} - x_{\bar{B}}\partial_A) = V^{A\bar{B}}(x_{\bar{B}}\partial_A - x_A\partial_{\bar{B}})$$

which completes the proof. □

The vector space of first order operators we have found so far, is clearly isomorphic to $\mathfrak{u}(\mathbb{C}^{n+2}, g)$. The operator corresponding to the central element is

$$ig^{B\bar{A}}(x_{\bar{A}}\partial_B - x_B\partial_{\bar{A}}) = i(x^B\partial_B - x^{\bar{A}}\partial_{\bar{A}})$$

which induces on M scalar multiplication by $i(w_1 - w_2)$ on functions with weight (w_1, w_2). Since scalar multiplication is not very interesting operator, we will restrict ourselves to operators corresponding to $\mathfrak{su}(\mathbb{C}^{n+2}, g)$.

Every real first order symmetry of the sub-Laplacian can be written in the form

$$V^c\partial_c + V^{\bar{c}}\partial_{\bar{c}} + V^\sigma\partial_\sigma + V$$

where the reality condition reads as $\overline{V^c} = V^{\bar{c}}$, $\overline{V^\sigma} = V^\sigma$, $\overline{V} = V$. Composing it with the sub-Laplacian, we get:

$$[\frac{1}{2}g^{q\bar{r}}(\partial_g\partial_{\bar{r}} + \partial_{\bar{r}}\partial_q) + \frac{i(w_1 - w_2)}{2}\partial_\sigma, V^c\partial_c + V^{\bar{c}}\partial_{\bar{c}} + V^\sigma\partial_\sigma + V] =$$
$$= \frac{1}{2}g^{q\bar{r}}[2iV^c\partial_q g_{\bar{r}c}\partial_\sigma - 2iV^{\bar{c}}\partial_{\bar{r}}g_{\bar{c}q}\partial_\sigma] +$$

$$
\begin{aligned}
&\frac{1}{2}g^{q\bar{r}}[(\partial_q V^c)\partial_{\bar{r}}\partial_c + (\partial_{\bar{r}} V^c)\partial_q\partial_c + (\partial_q\partial_{\bar{r}} V^c)\partial_c]+\\
&+\frac{1}{2}g^{q\bar{r}}[(\partial_{\bar{r}} V^c)\partial_q\partial_c + (\partial_q V^c)\partial_{\bar{r}}\partial_c + (\partial_{\bar{r}}\partial_q V^c)\partial_c]+\\
&+\frac{1}{2}g^{q\bar{r}}[(\partial_q V^{\bar{c}})\partial_{\bar{r}}\partial_{\bar{c}} + (\partial_{\bar{r}} V^{\bar{c}})\partial_q\partial_{\bar{c}} + (\partial_q\partial_{\bar{r}} V^{\bar{c}})\partial_{\bar{c}}]+\\
&+\frac{1}{2}g^{q\bar{r}}[(\partial_{\bar{r}} V^{\bar{c}})\partial_q\partial_{\bar{c}} + (\partial_q V^{\bar{c}})\partial_{\bar{r}}\partial_{\bar{c}} + (\partial_{\bar{r}}\partial_q V^{\bar{c}})\partial_{\bar{c}}]+\\
&+\frac{1}{2}g^{q\bar{r}}[(\partial_q V^\sigma)\partial_{\bar{r}}\partial_\sigma + (\partial_{\bar{r}} V^\sigma)\partial_q\partial_\sigma + (\partial_q\partial_{\bar{r}} V^\sigma)\partial_\sigma]+\\
&+\frac{1}{2}g^{q\bar{r}}[(\partial_{\bar{r}} V^\sigma)\partial_q\partial_\sigma + (\partial_q V^\sigma)\partial_{\bar{r}}\partial_\sigma + (\partial_{\bar{r}}\partial_q V^\sigma)\partial_\sigma]+\\
&+\frac{1}{2}g^{q\bar{r}}[(\partial_q V)\partial_{\bar{r}} + (\partial_{\bar{r}} V)\partial_q + (\partial_q\partial_{\bar{r}} V)]+\\
&+\frac{1}{2}g^{q\bar{r}}[(\partial_{\bar{r}} V)\partial_q + (\partial_q V)\partial_{\bar{r}} + (\partial_{\bar{r}}\partial_q V)]+\\
&+\frac{i(w_1-w_2)}{2}[(\partial_\sigma V^c)\partial_c + (\partial_\sigma V^{\bar{c}})\partial_{\bar{c}} + (\partial_\sigma V^\sigma)\partial_\sigma + (\partial_\sigma V)]
\end{aligned}
$$

The leading term of the commutator is

$$
(\partial^q V^c)\partial_q\partial_c + (\partial^{\bar{r}} V^{\bar{c}})\partial_{\bar{r}}\partial_{\bar{c}} + \frac{1}{2}(\partial^q V^{\bar{r}} + \partial^{\bar{r}} V^q)(\partial_q\partial_{\bar{r}} + \partial_{\bar{r}}\partial_q)+ \tag{7.1}
$$

$$
+(\partial^q V^\sigma + iV^q)\partial_q\partial_\sigma + (\partial^{\bar{r}} V^\sigma - iV^{\bar{r}})\partial_{\bar{r}}\partial_\sigma \tag{7.2}
$$

Lemma 7.1.3. *Let $V^c\partial_c + V^{\bar{c}}\partial_{\bar{c}} + V^\sigma\partial_\sigma$ be a symbol of some symmetry. Then we have*

$$
\begin{aligned}
V^a &= i\partial^a V^\sigma \qquad (7.3)\\
V^{\bar{a}} &= -i\partial^{\bar{a}} V^\sigma\\
0 = \partial^{(a}V^{b)} &= i\partial^{(a}\partial^{b)} V^\sigma\\
0 = \partial^{(\bar{a}}V^{\bar{b})} &= -i\partial^{(\bar{a}}\partial^{\bar{b})} V^\sigma
\end{aligned}
$$

Proof. This is an easy consequence of 7.1. We want the leading term be of the form $\frac{1}{2}g^{b\bar{a}}\mu(\partial_{\bar{a}}\partial_b + \partial_b\partial_{\bar{a}})$ for some function μ. This means that the coefficients at $\partial_q\partial_c$, $\partial_{\bar{r}}\partial_{\bar{c}}$, $\partial_q\partial_\sigma$, and $\partial_{\bar{r}}\partial_\sigma$ to vanish. But this is equivalent to the statement. □

Remark 7.1.1. We have

$$
\partial^a V^{\bar{b}} + \partial^{\bar{b}} V^a = i[\partial^{\bar{b}}, \partial^a]V^\sigma = g^{a\bar{b}}\partial_\sigma V^\sigma.
$$

so $\mu = \partial_\sigma V^\sigma$.

We see that the symmetry (its symbol) is completely determined by V^σ. Now we write down the operator induced by $V^{B\bar{A}}(X_{\bar{A}}\partial_B - X_B\partial_{\bar{A}})$ on M:

$$\begin{gathered} V^{B\bar{A}}(X_{\bar{A}}\partial_B - X_B\partial_{\bar{A}}) = V^{B\bar{A}}(X_{\bar{A}}\delta_B^C\partial_C - X_B\delta_{\bar{A}}^{\bar{D}}\partial_{\bar{D}}) = \\ = V^{B\bar{A}}X_{\bar{A}}(X^C Z_B + Z^C X_B + Y_e^C Y_B^e)\partial_C - \\ -V^{B\bar{A}}X_B(X^{\bar{D}}Z_{\bar{A}} + Z^{\bar{D}}X_{\bar{A}} + Y_{\bar{e}}^{\bar{D}}Y_{\bar{A}}^{\bar{e}})\partial_{\bar{D}} = \\ = V^{B\bar{A}}X_{\bar{A}}Y_B^e Y_e^C\partial_C - V^{B\bar{A}}X_B Y_{\bar{A}}^{\bar{e}} Y_{\bar{e}}^{\bar{D}}\partial_{\bar{D}} + \\ +V^{B\bar{A}}X_{\bar{A}}X_B(Z^C\partial_C - Z^{\bar{D}}\partial_{\bar{D}}) + \\ +V^{B\bar{A}}X_{\bar{A}}Z_B\mathbb{E} - V^{B\bar{A}}X_B Z_{\bar{A}}\bar{\mathbb{E}} \end{gathered}$$

If we want it to be in the form $V^e\partial_e + V^{\bar{e}}\partial_{\bar{e}} + V^\sigma\partial_\sigma + V$, we shall put

$$\begin{aligned} V^e &= V^{B\bar{A}}X_{\bar{A}}Y_B^e \\ V^{\bar{e}} &= -V^{B\bar{A}}X_B Y_{\bar{A}}^{\bar{e}} \\ iV^\sigma &= V^{B\bar{A}}X_{\bar{A}}X_B \end{aligned} \tag{7.4}$$

These functions fulfill the conditions (7.3) on the symbol of a symmetry. Here we have used that $\partial_e = \partial_{z^e} - \frac{z_e}{2}\partial_{z^\infty} + \frac{z_e}{2}\partial_{z^{\bar{\infty}}} = Y_e^C\partial_C$.

It may be interesting to write down the basis of first order symmetries using the ambient construction. We write the real basis of real symmetries, and it can of course be used as complex basis of complax symmetries. All the symmetries are of the form $V^{B\bar{A}}(X_{\bar{A}}\partial_B - X_B\partial_{\bar{A}})$, where $V^{B\bar{A}}$ is a skew-Hermitean matrix and

$$\begin{aligned} X_{\bar{A}} &= (z^0(z^\infty - \frac{z^a z_a}{2}), z^0 z_{\bar{a}}, z^0) = (X_{\bar{0}}, X_{\bar{a}}, X_{\bar{\infty}}) \\ X_B &= (z^{\bar{0}}(z^{\bar{\infty}} - \frac{z^a z_a}{2}), z^{\bar{0}} z_a, z^{\bar{0}}) = (X_0, X_a, X_\infty) \end{aligned}$$

Here we shall emphasis that the coordinate vector fields ∂_{z^i} are *not* holomorphic, whereas the generators of the contact subbundle $\partial_j = \partial_{z^j} + i\frac{z_j}{2}\partial_\sigma$ *are* holomorphic. As the matrix $V^{B\bar{A}}$ we will take in turn all the basis vectors of the trace-free skew-Hermitean matrices:
Let $V^{\infty\bar{\infty}} = i$ be the only nonzero entry. Then

$$V^{B\bar{A}}(X_{\bar{A}}\partial_B - X_B\partial_{\bar{A}}) =$$

$$= i(\partial_{z^\infty} - \partial_{z^{\bar\infty}}) =$$
$$= \partial_\sigma$$

Let $V^{a\bar\infty} = -V^{\bar a\infty} = 1$ be the only nonzero entries. Then

$$V^{B\bar A}(X_{\bar A}\partial_B - X_B\partial_{\bar A}) =$$
$$\partial_{z^a} + \frac{z_a}{2}\partial_{z^\infty} + \frac{z_a}{2}\partial_{z^{\bar\infty}} - z_a\partial_{z^{\bar\infty}} - z_{\bar a}\partial_{z^\infty} + \partial_{z^{\bar a}} + \frac{z_{\bar a}}{2}\partial_{z^\infty} + \frac{z_{\bar a}}{2}\partial_{z^{\bar\infty}} =$$
$$= \partial_{z^a} + \partial_{z^{\bar a}} + \frac{i(z_{\bar a} - z_a)}{2}\partial_\sigma =$$
$$= \partial_a + \partial_{\bar a} + \frac{z_a - z_{\bar a}}{i}\partial_\sigma$$

Let $V^{a\bar\infty} = V^{\bar a\infty} = i$ be the only nonzero entries. Then

$$V^{B\bar A}(X_{\bar A}\partial_B - X_B\partial_{\bar A}) =$$
$$i\partial_{z^a} + i\frac{z_a}{2}\partial_{z^\infty} + i\frac{z_a}{2}\partial_{z^{\bar\infty}} - iz_a\partial_{z^{\bar\infty}} + iz_{\bar a}\partial_{z^\infty} - i\partial_{z^{\bar a}} - i\frac{z_{\bar a}}{2}\partial_{z^\infty} - i\frac{z_{\bar a}}{2}\partial_{z^{\bar\infty}} =$$
$$= i(\partial_{z^a} - \partial_{z^{\bar a}}) + \frac{z_a + z_{\bar a}}{2}\partial_\sigma =$$
$$= i(\partial_a - \partial_{\bar a}) + (z_a + z_{\bar a})\partial_\sigma$$

Let $V^{0\bar\infty} = -V^{\bar 0\infty} = 1$ be the only nonzero entries. Then

$$V^{B\bar A}(X_{\bar A}\partial_B - X_B\partial_{\bar A}) =$$
$$= z^0\partial_{z^0} - z^a\partial_{z^a} - z^\infty\partial_{z^\infty} - \frac{z^a z_a}{2}\partial_{z^{\bar\infty}} - (z^{\bar\infty} - \frac{z^a z_a}{2})\partial_{z^{\bar\infty}} -$$
$$-(z^\infty - \frac{z^a z_a}{2})\partial_{z^\infty} + z^{\bar 0}\partial_{z^{\bar 0}} - z^{\bar a}\partial_{z^{\bar a}} - z^{\bar\infty}\partial_{z^{\bar\infty}} - \frac{z^a z_a}{2}\partial_{z^\infty} =$$
$$= w_1 + w_2 - z^a\partial_{z^a} - z^{\bar a}\partial_{z^{\bar a}} - 2\sigma\partial_\sigma =$$
$$= -z^a\partial_a - z^{\bar a}\partial_{\bar a} - 2\sigma\partial_\sigma + w_1 + w_2$$

Let $V^{0\bar\infty} = V^{\bar 0\infty} = -i$ and $V^{a\bar a} = 2i$ for fixed a be the only nonzero entries. Then

$$V^{B\bar A}(X_{\bar A}\partial_B - X_B\partial_{\bar A}) =$$
$$= -iz^0\partial_{z^0} + iz^a\partial_{z^a} + iz^\infty\partial_{z^\infty} + i\frac{z^a z_a}{2}\partial_{z^{\bar\infty}} + i(z^{\bar\infty} - \frac{z^a z_a}{2})\partial_{z^{\bar\infty}} -$$
$$-i(z^\infty - \frac{z^a z_a}{2})\partial_{z^\infty} + iz^{\bar 0}\partial_{z^{\bar 0}} - iz^{\bar a}\partial_{z^{\bar a}} - iz^{\bar\infty}\partial_{z^{\bar\infty}} - i\frac{z^a z_a}{2}\partial_{z^\infty} +$$

$$+2iz_{\bar a}\partial_{z^a}+2iz_{\bar a}\frac{z_a}{2}\partial_{z^\infty}+2iz_{\bar a}\frac{z_a}{2}\partial_{z^{\bar\infty}}-2iz_a\partial_{z^{\bar a}}-2iz_a\frac{z_{\bar a}}{2}\partial_{z^\infty}-2iz_a\frac{z_{\bar a}}{2}\partial_{z^{\bar\infty}}=$$
$$=i(w_2-w_1)+i(z^a+2z_{\bar a})\partial_{z^a}-i(z^{\bar a}+2z_a)\partial_{z^{\bar a}}=$$
$$=i(z^a+2z_{\bar a})\partial_a-i(z^{\bar a}+2z_a)\partial_{\bar a}+(z^az_a+2z_az_{\bar a})\partial_\sigma+i(w_2-w_1)$$

Let $V^{a\bar b}=-V^{\bar a b}=1$ for fixed $a<b$ be the only nonzero entries. Then

$$V^{B\bar A}(X_{\bar A}\partial_B-X_B\partial_{\bar A})=$$
$$=z_{\bar b}\partial_{z^a}+z_{\bar b}\frac{z_a}{2}\partial_{z^\infty}+z_{\bar b}\frac{z_a}{2}\partial_{z^{\bar\infty}}-z_a\partial_{z^{\bar b}}-z_a\frac{z_{\bar b}}{2}\partial_{z^\infty}-z_a\frac{z_{\bar b}}{2}\partial_{z^{\bar\infty}}-$$
$$-z_{\bar a}\partial_{z^b}-z_{\bar a}\frac{z_b}{2}\partial_{z^\infty}-z_{\bar a}\frac{z_b}{2}\partial_{z^{\bar\infty}}+z_b\partial_{z^{\bar a}}+z_b\frac{z_{\bar a}}{2}\partial_{z^\infty}+z_b\frac{z_{\bar a}}{2}\partial_{z^{\bar\infty}}=$$
$$=z_{\bar b}\partial_{z^a}-z_a\partial_{z^{\bar b}}-z_{\bar a}\partial_{z^b}+z_b\partial_{z^{\bar a}}=$$
$$=z_{\bar b}\partial_a-z_a\partial_{\bar b}-z_{\bar a}\partial_b+z_b\partial_{\bar a}+i(z_{\bar a}z_b-z_az_{\bar b})\partial_\sigma$$

Let $V^{a\bar b}=V^{\bar a b}=i$ for fixed $a<b$ be the only nonzero entries. Then

$$V^{B\bar A}(X_{\bar A}\partial_B-X_B\partial_{\bar A})=$$
$$=iz_{\bar b}\partial_{z^a}+iz_{\bar b}\frac{z_a}{2}\partial_{z^\infty}+iz_{\bar b}\frac{z_a}{2}\partial_{z^{\bar\infty}}-iz_a\partial_{z^{\bar b}}-iz_a\frac{z_{\bar b}}{2}\partial_{z^\infty}-iz_a\frac{z_{\bar b}}{2}\partial_{z^{\bar\infty}}+$$
$$+iz_{\bar a}\partial_{z^b}+iz_{\bar a}\frac{z_b}{2}\partial_{z^\infty}+iz_{\bar a}\frac{z_b}{2}\partial_{z^{\bar\infty}}-iz_b\partial_{z^{\bar a}}-iz_b\frac{z_{\bar a}}{2}\partial_{z^\infty}-iz_b\frac{z_{\bar a}}{2}\partial_{z^{\bar\infty}}=$$
$$=iz_{\bar b}\partial_{z^a}-iz_a\partial_{z^{\bar b}}+iz_{\bar a}\partial_{z^b}-iz_b\partial_{z^{\bar a}}=$$
$$=iz_{\bar b}\partial_a-iz_a\partial_{\bar b}+iz_{\bar a}\partial_b-iz_b\partial_{\bar a}+(z_az_{\bar b}+z_{\bar a}z_b)\partial_\sigma$$

Let $V^{a\bar 0}=-V^{\bar a 0}=1$ for fixed a be the only nonzero entries. Then

$$V^{B\bar A}(X_{\bar A}\partial_B-X_B\partial_{\bar A})=$$
$$=(z^\infty-\frac{z^cz_c}{2})\partial_{z^a}+(z^\infty-\frac{z^cz_c}{2})\frac{z_a}{2}\partial_{z^\infty}+(z^\infty-\frac{z^cz_c}{2})\frac{z_a}{2}\partial_{z^{\bar\infty}}-$$
$$-z_az^{\bar 0}\partial_{z^{\bar 0}}+z^{\bar a}z_a\partial_{z^{\bar a}}+z_az^{\bar\infty}\partial_{z^{\bar\infty}}+z_a\frac{z^cz_c}{2}\partial_{z^\infty}-$$
$$-z_{\bar a}z^0\partial_{z^0}+z^az_{\bar a}\partial_{z^a}+z_{\bar a}z^\infty\partial_{z^\infty}+z_{\bar a}\frac{z^cz_c}{2}\partial_{z^{\bar\infty}}+$$
$$+(z^{\bar\infty}-\frac{z^cz_c}{2})\partial_{z^{\bar a}}+(z^{\bar\infty}-\frac{z^cz_c}{2})\frac{z_{\bar a}}{2}\partial_{z^\infty}+(z^{\bar\infty}-\frac{z^cz_c}{2})\frac{z_{\bar a}}{2}\partial_{z^{\bar\infty}}=$$

$$=-z_aw_2-z_{\bar a}w_1+(-\frac{z^cz_c}{2}+i\sigma)\partial_{z^a}+(-\frac{z^az_a}{2}-i\sigma)\partial_{z^{\bar a}}+$$
$$+z_{\bar a}z^c\partial_{z^c}+z_az^{\bar c}\partial_{z^{\bar c}}+(\sigma\frac{z_a+z_{\bar a}}{2}-i\frac{z^cz_c}{2}\frac{z_a-z_{\bar a}}{2})\partial_\sigma=$$

$$= (-\frac{z^c z_c}{2} + i\sigma)\partial_a + (-\frac{z^a z_a}{2} - i\sigma)\partial_{\bar a} + z_{\bar a} z^c \partial_c + z_a z^{\bar c}\partial_{\bar c} +$$
$$+(\sigma(z_a + z_{\bar a}) - i\frac{z^c z_c}{2}(z_{\bar a} - z_a))\partial_\sigma - z_a w_2 - z_{\bar a} w_1$$

Let $V^{a\bar 0} = V^{\bar a 0} = i$ for fixed a be the only nonzero entries. Then

$$V^{B\bar A}(X_{\bar A}\partial_B - X_B \partial_{\bar A}) =$$
$$= i(z^\infty - \frac{z^c z_c}{2})\partial_{z^a} + i(z^\infty - \frac{z^c z_c}{2})\frac{z_a}{2}\partial_{z^\infty} + i(z^\infty - \frac{z^c z_c}{2})\frac{z_a}{2}\partial_{z^{\bar\infty}} -$$
$$-iz_a z^{\bar 0}\partial_{z^{\bar 0}} + iz^{\bar a} z_a \partial_{z^{\bar a}} + iz_a z^{\bar\infty}\partial_{z^{\bar\infty}} + iz_a \frac{z^c z_c}{2}\partial_{z^\infty} +$$
$$+iz_{\bar a} z^0 \partial_{z^0} - iz^a z_{\bar a}\partial_{z^a} - iz_{\bar a} z^\infty \partial_{z^\infty} - iz_{\bar a}\frac{z^c z_c}{2}\partial_{z^{\bar\infty}} -$$
$$-i(z^{\bar\infty} + i\frac{z^c z_c}{2})\partial_{z^{\bar a}} - i(z^{\bar\infty} + i\frac{z^c z_c}{2})\frac{z_{\bar a}}{2}\partial_{z^\infty} - i(z^{\bar\infty} + i\frac{z^c z_c}{2})\frac{z_{\bar a}}{2}\partial_{z^{\bar\infty}} =$$
$$= i(z_{\bar a} w_1 - z_a w_2) - (\sigma + i\frac{z^c z_c}{2})\partial_{z^a} - iz_a z^c \partial_{z^c} -$$
$$-(\sigma - i\frac{z^c z_c}{2})\partial_{z^{\bar a}} + iz_a z^{\bar c}\partial_{z^{\bar c}} + (\frac{z^c z_c}{2}\frac{z_a + z_{\bar a}}{2} + i\sigma\frac{z_a - z_{\bar a}}{2})\partial_\sigma =$$
$$= -(\sigma + i\frac{z^c z_c}{2})\partial_a - iz_a z^c \partial_c - (\sigma - i\frac{z^c z_c}{2})\partial_{\bar a} + iz_a z^{\bar c}\partial_{\bar c} +$$
$$+(-\frac{z^c z_c}{2}(z_a + z_{\bar a}) + i\sigma(z_a - z_{\bar a}))\partial_\sigma + i(z_{\bar a} w_1 - z_a w_2)$$

Let $V^{0\bar 0} = i$ be the only nonzero entry. Then

$$V^{B\bar A}(X_{\bar A}\partial_B - X_B \partial_{\bar A}) =$$
$$iz^0(z^\infty - \frac{z^a z_a}{2})(\partial z^0 - \frac{z^a}{z^0}\partial_{z^a} - \frac{z^\infty}{z^0}\partial_{z^\infty} - \frac{z^a z_a}{2z^0}\partial_{z^{\bar\infty}}) -$$
$$-iz^{\bar 0}(z^{\bar\infty} - \frac{z^a z_a}{2})(\partial z^{\bar 0} - \frac{z^{\bar a}}{z^{\bar 0}}\partial_{z^{\bar a}} - \frac{z^{\bar\infty}}{z^{\bar 0}}\partial_{z^{\bar\infty}} - \frac{z^a z_a}{2z^{\bar 0}}\partial_{z^\infty}) =$$
$$= -(\sigma + i\frac{z^a z_a}{2})w_1 - (\sigma - i\frac{z^a z_a}{2})w_2 - iz^a(-\frac{z^a z_a}{2} + i\sigma)\partial_{z^a} +$$
$$+iz^{\bar a}(-\frac{z^a z_a}{2} - i\sigma)\partial_{z^{\bar a}} + (\sigma^2 - \frac{z^a z_a}{2}\frac{z^a z_a}{2})\partial_\sigma =$$
$$= -iz^a(-\frac{z^a z_a}{2} + i\sigma)\partial_a + iz^{\bar a}(-\frac{z^a z_a}{2} - i\sigma)\partial_{\bar a} +$$
$$+(\sigma^2 + \frac{z^a z_a}{2}\frac{z^a z_a}{2})\partial_\sigma - (\sigma + i\frac{z^a z_a}{2})w_1 - (\sigma - i\frac{z^a z_a}{2})w_2$$

7.2 Higher order symmetries

Composing such first order operators, we get higher order operators with the same properties. Concretely, we may write them like this:

$$V_1^{B_1\bar{A}_1}\dots V_s^{B_s\bar{A}_s}(x_{\bar{A}_1}\partial_{B_1}-x_{B_1}\partial_{\bar{A}_1})\dots(x_{\bar{A}_s}\partial_{B_s}-x_{B_s}\partial_{\bar{A}_s}) \tag{7.5}$$

From now on, we will denote the product of V_i-s by one tensor field $V^{B_1\bar{A}_1\dots B_s\bar{A}_s}$. The rest of the expression (7.5) will be simplified and from the simplification we get symmetries of the tensor $V^{B_1\bar{A}_1\dots B_s\bar{A}_s}$.

First, we compute the commutator of two first order operators. For this purpose, we rewrite them as $V_A^B(x^A\partial_B - x_B\partial^A)$, where $V_A^B \in \mathfrak{su}(\mathbb{C}^{n+2}, g)$:

$$\begin{gathered}
V_{A_1}^{B_1}W_{A_2}^{B_2}(x^{A_1}\partial_{B_1}-x_{B_1}\partial^{A_1})(x^{A_2}\partial_{B_2}-x_{B_2}\partial^{A_2})- \\
-W_{A_2}^{B_2}V_{A_1}^{B_1}(x^{A_2}\partial_{B_2}-x_{B_2}\partial^{A_2})(x^{A_1}\partial_{B_1}-x_{B_1}\partial^{A_1})= \\
=V_{A_1}^{B_1}W_{A_2}^{B_2}(x^{A_1}\delta_{B_1}^{A_2}\partial_{B_2}+x^{A_1}x^{A_2}\partial_{B_1}\partial_{B_2}-x^{A_1}x_{B_2}\partial_{B_1}\partial^{A_2})- \\
-V_{A_1}^{B_1}W_{A_2}^{B_2}(x_{B_1}x^{A_2}\partial^{A_1}\partial_{B_2}-x_{B_1}\delta_{B_2}^{A_1}\partial^{A_2}-x_{B_1}x_{B_2}\partial^{A_1}\partial^{A_2})- \\
-W_{A_2}^{B_2}V_{A_1}^{B_1}(x^{A_2}\delta_{B_2}^{A_1}\partial_{B_1}+x^{A_2}x^{A_1}\partial_{B_2}\partial_{B_1}-x^{A_2}x_{B_1}\partial_{B_2}\partial^{A_1})+ \\
+W_{A_2}^{B_2}V_{A_1}^{B_1}(x_{B_2}x^{A_1}\partial^{A_2}\partial_{B_1}-x_{B_2}\delta_{B_1}^{A_2}\partial^{A_1}-x_{B_2}x_{B_1}\partial^{A_2}\partial^{A_1})= \\
=V_{A_1}^{C}W_{C}^{B_2}x^{A_1}\partial_{B_2}+V_{C}^{B_1}W_{A_2}^{C}x_{B_1}\partial^{A_2}-W_{A_2}^{C}V_{C}^{B_1}x^{A_2}\partial_{B_1}-W_{C}^{B_2}V_{A_1}^{C}x_{B_2}\partial^{A_1}= \\
=(V_A^CW_C^B-V_C^BW_A^C)(x^A\partial_B-x_B\partial^A)
\end{gathered} \tag{7.6}$$

So we see that taking commutator doesn't enlarge the vector space of symmetries. Therefore we can restrict ourselves to such tensors $V^{B_1\bar{A}_1\dots B_s\bar{A}_s}$, which are symmetric in pairs $B_i\bar{A}_i$. We will restrict to tensors, which are totally trace-free.

7.2.1 Second order symmetries

Any second order operator on M can be written in the form

$$\begin{gathered}
P=V^{c_1c_2}\partial_{c_1}\partial_{c_2}+V^{\bar{c}_1c_2}(\partial_{\bar{c}_1}\partial_{c_2}+\partial_{c_2}\partial_{\bar{c}_1})+V^{\bar{c}_1\bar{c}_2}\partial_{\bar{c}_1}\partial_{\bar{c}_2}+ \\
+V^{c\sigma}\partial_c\partial_\sigma+V^{\bar{c}\sigma}\partial_{\bar{c}}\partial_\sigma+V^{\sigma\sigma}\partial_\sigma\partial_\sigma+V^c\partial_c+V^{\bar{c}}\partial_{\bar{c}}+V^\sigma\partial_\sigma+V
\end{gathered}$$

Commuting it with the sub-Laplacian, we get

$$[g^{\bar{a}b}(\partial_{\bar{a}}\partial_b+\partial_b\partial_{\bar{a}})+i(w_1-w_2)\partial_\sigma,$$

$$
\begin{aligned}
&V^{c_1c_2}\partial_{c_1}\partial_{c_2}+V^{\bar{c}_1c_2}(\partial_{\bar{c}_1}\partial_{c_2}+\partial_{c_2}\partial_{\bar{c}_1})+V^{\bar{c}_1\bar{c}_2}\partial_{\bar{c}_1}\partial_{\bar{c}_2}+\\
&+V^{c\sigma}\partial_c\partial_\sigma+V^{\bar{c}\sigma}\partial_{\bar{c}}\partial_\sigma+V^{\sigma\sigma}\partial_\sigma\partial_\sigma+LOTS]=\\
&=2(\partial^bV^{c_1c_2})\partial_b\partial_{c_1}\partial_{c_2}+2(\partial^{\bar{a}}V^{\bar{c}_1\bar{c}_2})\partial_{\bar{a}}\partial_{\bar{c}_1}\partial_{\bar{c}_2}+\\
&+(\partial^{\bar{a}}V^{c_1c_2}+\partial^{c_2}V^{\bar{a}c_1})\partial_{\bar{a}}\partial_{c_1}\partial_{c_2}+(\partial^{c_2}V^{\bar{a}c_1}+\partial^{c_1}V^{\bar{a}c_2})\partial_{c_1}\partial_{\bar{a}}\partial_{c_2}+\\
&+(\partial^{\bar{a}}V^{c_1c_2}+\partial^{c_1}V^{\bar{a}c_2})\partial_{c_1}\partial_{c_2}\partial_{\bar{a}}+(\partial^{\bar{c}_2}V^{\bar{c}_1b}+\partial^bV^{\bar{c}_1\bar{c}_2})\partial_b\partial_{\bar{c}_1}\partial_{\bar{c}_2}+\\
&+(\partial^{\bar{c}_1}V^{\bar{c}_2b}+\partial^{\bar{c}_2}V^{\bar{c}_1b})\partial_{\bar{c}_1}\partial_b\partial_{\bar{c}_2}+(\partial^{\bar{c}_1}V^{\bar{c}_2b}+\partial^bV^{\bar{c}_1\bar{c}_2})\partial_{\bar{c}_1}\partial_{\bar{c}_2}\partial_b+\\
&+(4iV^{c_1c_2}+2\partial^{c_1}V^{c_2\sigma})\partial_{c_1}\partial_{c_2}\partial_\sigma+(-4iV^{\bar{c}_1\bar{c}_2}+2\partial^{\bar{c}_1}V^{\bar{c}_2\sigma})\partial_{\bar{c}_1}\partial_{\bar{c}_2}\partial_\sigma+\\
&+(\partial^{\bar{c}_1}V^{c_2\sigma}+\partial^{c_2}V^{\bar{c}_1\sigma})\partial_{\bar{c}_1}\partial_{c_2}\partial_\sigma+(\partial^{c_1}V^{\bar{c}_2\sigma}+\partial^{\bar{c}_2}V^{c_1\sigma})\partial_{c_1}\partial_{\bar{c}_2}\partial_\sigma+\\
&+(2iV^{c\sigma}+2\partial^cV^{\sigma\sigma})\partial_c\partial_\sigma\partial_\sigma+(-2iV^{\bar{c}\sigma}+2\partial^{\bar{c}}V^{\sigma\sigma})\partial_{\bar{c}}\partial_\sigma\partial_\sigma+\\
&+LOTS=
\end{aligned}
$$

$$
\begin{aligned}
&=2(\partial^bV^{c_1c_2})\partial_b\partial_{c_1}\partial_{c_2}+2(\partial^{\bar{a}}V^{\bar{c}_1\bar{c}_2})\partial_{\bar{a}}\partial_{\bar{c}_1}\partial_{\bar{c}_2}+ \qquad (7.7)\\
&+(\partial^{\bar{c}_1}V^{ac_2}+\partial^aV^{\bar{c}_1c_2}+\partial^{c_2}V^{\bar{c}_1a})\partial_a(\partial_{\bar{c}_1}\partial_{c_2}+\partial_{c_2}\partial_{\bar{c}_1})+\\
&+(\partial^{\bar{c}_1}V^{\bar{a}c_2}+\partial^{c_2}V^{\bar{a}\bar{c}_1}+\partial^{\bar{a}}V^{\bar{c}_1c_2})\partial_{\bar{a}}(\partial_{\bar{c}_1}\partial_{c_2}+\partial_{c_2}\partial_{\bar{c}_1})+\\
&+(4iV^{c_1c_2}+2\partial^{c_1}V^{c_2\sigma})\partial_{c_1}\partial_{c_2}\partial_\sigma+(-4iV^{\bar{c}_1\bar{c}_2}+2\partial^{\bar{c}_1}V^{\bar{c}_2\sigma})\partial_{\bar{c}_1}\partial_{\bar{c}_2}\partial_\sigma+\\
&+(\partial^{\bar{c}_1}V^{c_2\sigma}+\partial^{c_2}V^{\bar{c}_1\sigma})\partial_\sigma(\partial_{\bar{c}_1}\partial_{c_2}+\partial_{c_2}\partial_{\bar{c}_1})+\\
&+(2iV^{c\sigma}+2\partial^cV^{\sigma\sigma})\partial_c\partial_\sigma\partial_\sigma+(-2iV^{\bar{c}\sigma}+2\partial^{\bar{c}}V^{\sigma\sigma})\partial_{\bar{c}}\partial_\sigma\partial_\sigma+LOTS \qquad (7.8)
\end{aligned}
$$

We may assume that $V^{c_1c_2}$ and $V^{\bar{c}_1\bar{c}_2}$ are symmetric and $V^{\bar{c}_1c_2}=V^{c_2\bar{c}_1}$ is trace-free because of equivalence.

Lemma 7.2.1. *Let P be a symmetry of second order. Then we have:*

1. *$V^{\sigma\sigma}$ is a solution of the first BGG operator corresponding to representation $\overset{2}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{2}{\circ}=S^2\mathbb{C}^{n+2}\boxtimes S^2(\mathbb{C}^{n+2})^*$ and*

$$
\begin{gathered}
\partial^{(b}V^{c_1c_2)}=0\quad \partial^{(\bar{a}}V^{\bar{c}_1\bar{c}_2)}=0\\
(\exists F^a)\partial^{\bar{c}_1}V^{ac_2}+\partial^aV^{\bar{c}_1c_2}+\partial^{c_2}V^{\bar{c}_1a}=g^{\bar{c}_1c_2}F^a+g^{\bar{c}_1a}F^{c_2} \qquad (7.9)\\
(\exists F^{\bar{a}})\partial^{\bar{c}_1}V^{\bar{a}c_2}+\partial^{c_2}V^{\bar{a}\bar{c}_1}+\partial^{\bar{a}}V^{\bar{c}_1c_2}=g^{\bar{c}_1c_2}F^{\bar{a}}+g^{\bar{a}c_2}F^{\bar{c}_1}\\
V^{c_1c_2}=\frac{i}{2}\partial^{(c_1}V^{c_2)\sigma}\quad V^{\bar{c}_1\bar{c}_2}=-\frac{i}{2}\partial^{(\bar{c}_1}V^{\bar{c}_2)\sigma}\\
V^{c\sigma}=i\partial^cV^{\sigma\sigma}\quad V^{\bar{c}\sigma}=-i\partial^{\bar{c}}V^{\sigma\sigma}
\end{gathered}
$$

2. *If $V^{\sigma\sigma}=0$, then $V^{\bar{c}_1c_2}$ is a solution of the first BGG operator corresponding to $\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{1}{\circ}\!-\cdots-\!\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}=\Lambda^2\mathbb{C}^{n+2}\boxtimes\Lambda^2(\mathbb{C}^{n+2})^*$.*

Proof. We know that the commutator must be of the form $\mu\Delta$ for some first order differential operator μ. Looking at the leading term, the coefficients at $\partial_b\partial_{c_1}\partial_{c_2}$, $\partial_{\bar a}\partial_{\bar c_1}\partial_{\bar c_2}$, $\partial_{c_1}\partial_{c_2}\partial_\sigma$, $\partial_{\bar c_1}\partial_{\bar c_2}\partial_\sigma$, $\partial_c\partial_\sigma\partial_\sigma$, and $\partial_{\bar c}\partial_\sigma\partial_\sigma$ have to vanish. This is in turn equivalent to

$$\begin{aligned} \partial^{(b}V^{c_1c_2)} &= 0 \quad & \partial^{(\bar a}V^{\bar c_1\bar c_2)} &= 0 \\ V^{c_1c_2} &= \frac{i}{2}\partial^{(c_1}V^{c_2)\sigma} \quad & V^{\bar c_1\bar c_2} &= -\frac{i}{2}\partial^{(\bar c_1}V^{\bar c_2)\sigma} \\ V^{c\sigma} &= i\partial^c V^{\sigma\sigma} \quad & V^{\bar c\sigma} &= -i\partial^{\bar c}V^{\sigma\sigma} \end{aligned}$$

Moreover, the coefficient at $\partial_a(\partial_{\bar c_1}\partial_{c_2}+\partial_{c_2}\partial_{\bar c_1})$ must be of the form $g^{a\bar c_1}F^{c_2} + g^{c_2\bar c_1}F^a$ for some tensor F^c, since the part skew-symmetric in a and c_2 is of lower order. Similarly, the coefficient at $\partial_{\bar a}(\partial_{\bar c_1}\partial_{c_2}+\partial_{c_2}\partial_{\bar c_1})$ must be of the form $g^{c_2\bar c_1}F^{\bar a} + g^{c_2\bar a}F^{\bar c_1}$. But these are exactly the two remaining equations.

Using the equations 1, we may express $V^{c_1c_2}$ as $V^{c_1c_2} = -\frac{1}{2}\partial^{(c_1}\partial^{c_2)}V^{\sigma\sigma}$, and similarly $V^{\bar c_1\bar c_2} = -\frac{1}{2}\partial^{(\bar c_1}\partial^{\bar c_2)}V^{\sigma\sigma}$. But then we have

$$\begin{aligned} \partial^{(b}V^{c_1c_2)} &= -\frac{1}{2}\partial^{(b}\partial^{c_1}\partial^{c_2)}V^{\sigma\sigma} = 0 \\ \partial^{(\bar a}V^{\bar c_1\bar c_2)} &= \partial^{(\bar a}\partial^{\bar c_1}\partial^{\bar c_2)}V^{\sigma\sigma} = 0 \end{aligned}$$

what is exactly the first BGG operator corresponding to $\overset{2}{\circ}\!\!-\!\!-\!\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ}\!\!-\!\!-\!\!\overset{2}{\circ}$, as described in section 5.5.

If $V^{\sigma\sigma}=0$, then the second and third equation in 1 are equivalent to

$$\begin{aligned} &\text{the trace-free part of} \quad \partial^a V^{\bar c_1c_2} + \partial^{c_2}V^{\bar c_1 a} = 0 \\ &\text{the trace-free part of} \quad \partial^{\bar c_1} V^{\bar a c_2} + \partial^{\bar a}V^{\bar c_1 c_2} = 0 \end{aligned}$$

what is exactly the first BGG operator corresponding to $\overset{0}{\circ}\!\!-\!\!-\!\!\overset{1}{\circ}\!-\cdots-\!\overset{1}{\circ}\!\!-\!\!-\!\!\overset{0}{\circ}$, as described in section 5.5. $\square$

Remark 7.2.1.

$$\partial^{\bar c_1}V^{c_2\sigma} + \partial^{c_2}V^{\bar c_1\sigma} = i(\partial^{\bar c_1}\partial^{c_2} - \partial^{c_2}\partial^{\bar c_1})V^{\sigma\sigma} = -g^{\bar c_1 c_2}\partial_\sigma V^{\sigma\sigma}$$

Now we look on the ambient construction of second-order symmetries. We start with investigating symmetries of the form

$$V^{B_1\bar A_1B_2\bar A_2}(x_{\bar A_1}\partial_{B_1} - x_{B_1}\partial_{\bar A_1})(x_{\bar A_2}\partial_{B_2} - x_{B_2}\partial_{\bar A_2})$$

where $V^{B_1\bar{A}_1B_2\bar{A}_2}$ is symmetric in pairs $B_i\bar{A}_i$. First, we compute

$$(x_{\bar{A}_1}\partial_{B_1} - x_{B_1}\partial_{\bar{A}_1})(x_{\bar{A}_2}\partial_{B_2} - x_{B_2}\partial_{\bar{A}_2}) =$$
$$= x_{\bar{A}_1}x_{\bar{A}_2}\partial_{B_1}\partial_{B_2} - x_{\bar{A}_1}x_{B_2}\partial_{B_1}\partial_{\bar{A}_2} - x_{B_1}x_{\bar{A}_2}\partial_{\bar{A}_1}\partial_{B_2} + x_{B_1}x_{B_2}\partial_{\bar{A}_1}\partial_{\bar{A}_2} +$$
$$+ x_{\bar{A}_1}g_{\bar{A}_2B_1}\partial_{B_2} + x_{B_1}g_{\bar{A}_1B_2}\partial_{\bar{A}_2}$$

We are only interested in that part of this product, which is symmetric in pairs $B_i\bar{A}_i$ and trace-free, i.e.

$$x_{\bar{A}_1}x_{\bar{A}_2}\partial_{B_1}\partial_{B_2} - x_{\bar{A}_1}x_{B_2}\partial_{B_1}\partial_{\bar{A}_2} - x_{B_1}x_{\bar{A}_2}\partial_{\bar{A}_1}\partial_{B_2} + x_{B_1}x_{B_2}\partial_{\bar{A}_1}\partial_{\bar{A}_2}$$

So using the ambient construction, we have second order symmetries of the form

$$V^{B_1\bar{A}_1B_2\bar{A}_2}[x_{\bar{A}_1}x_{\bar{A}_2}\partial_{B_1}\partial_{B_2} - x_{\bar{A}_1}x_{B_2}\partial_{B_1}\partial_{\bar{A}_2} -$$
$$- x_{B_1}x_{\bar{A}_2}\partial_{\bar{A}_1}\partial_{B_2} + x_{B_1}x_{B_2}\partial_{\bar{A}_1}\partial_{\bar{A}_2}]$$

Looking at the induced operator, we have

$$\begin{aligned}
&V^{B_1\bar{A}_1B_2\bar{A}_2}[X_{\bar{A}_1}X_{\bar{A}_2}\partial_{B_1}\partial_{B_2} - X_{\bar{A}_1}X_{B_2}\partial_{B_1}\partial_{\bar{A}_2} -\\
&- X_{B_1}X_{\bar{A}_2}\partial_{\bar{A}_1}\partial_{B_2} + X_{B_1}X_{B_2}\partial_{\bar{A}_1}\partial_{\bar{A}_2}] =\\
&= V^{B_1\bar{A}_1B_2\bar{A}_2}[X_{\bar{A}_1}X_{\bar{A}_2}\delta^{D_1}_{B_1}\partial_{D_1}\delta^{D_2}_{B_2}\partial_{D_2} - X_{\bar{A}_1}X_{B_2}\delta^{D_1}_{B_1}\partial_{D_1}\delta^{\bar{C}_2}_{\bar{A}_2}\partial_{\bar{C}_2} -\\
&- X_{B_1}X_{\bar{A}_2}\delta^{\bar{C}_1}_{\bar{A}_1}\partial_{\bar{C}_1}\delta^{D_2}_{B_2}\partial_{D_2} + X_{B_1}X_{B_2}\delta^{\bar{C}_1}_{\bar{A}_1}\partial_{\bar{C}_1}\delta^{\bar{C}_2}_{\bar{A}_2}\partial_{\bar{C}_2}] =\\
&= V^{B_1\bar{A}_1B_2\bar{A}_2}[X_{\bar{A}_1}X_{\bar{A}_2}(X^{D_1}Z_{B_1} + Z^{D_1}X_{B_1} + Y^{D_1}_cY^c_{B_1})\partial_{D_1}\\
&(X^{D_2}Z_{B_2} + Z^{D_2}X_{B_2} + Y^{D_2}_cY^c_{B_2})\partial_{D_2} -\\
&- X_{\bar{A}_1}X_{B_2}(X^{D_1}Z_{B_1} + Z^{D_1}X_{B_1} + Y^{D_1}_cY^c_{B_1})\partial_{D_1}\\
&(X^{\bar{C}_2}Z_{\bar{A}_2} + Z^{\bar{C}_2}X_{\bar{A}_2} + Y^{\bar{C}_2}_{\bar{c}}Y^{\bar{c}}_{\bar{A}_2})\partial_{\bar{C}_2} -\\
&- X_{B_1}X_{\bar{A}_2}(X^{\bar{C}_1}Z_{\bar{A}_1} + Z^{\bar{C}_1}X_{\bar{A}_1} + Y^{\bar{C}_1}_{\bar{c}}Y^{\bar{c}}_{\bar{A}_1})\partial_{\bar{C}_1}\\
&(X^{D_2}Z_{B_2} + Z^{D_2}X_{B_2} + Y^{D_2}_cY^c_{B_2})\delta^{D_2}_{B_2}\partial_{D_2} +\\
&+ X_{B_1}X_{B_2}(X^{\bar{C}_1}Z_{\bar{A}_1} + Z^{\bar{C}_1}X_{\bar{A}_1} + Y^{\bar{C}_1}_{\bar{c}}Y^{\bar{c}}_{\bar{A}_1})\partial_{\bar{C}_1}\\
&(X^{\bar{C}_2}Z_{\bar{A}_2} + Z^{\bar{C}_2}X_{\bar{A}_2} + Y^{\bar{C}_2}_{\bar{c}}Y^{\bar{c}}_{\bar{A}_2})\partial_{\bar{C}_2}]
\end{aligned} \tag{7.10}$$

Using the equations

$$\partial_a = Y^C_a\partial_C \quad \partial_{\bar{a}} = Y^{\bar{C}}_{\bar{a}}\partial_{\bar{C}} \quad \partial_\sigma = i(Z^C\partial_C - Z^{\bar{C}}\partial_{\bar{C}})$$

we see that we have to put

$$
\begin{aligned}
V^{\sigma\sigma} &:= -V^{B_1\bar{A}_1B_2\bar{A}_2}X_{\bar{A}_1}X_{\bar{A}_2}X_{B_1}X_{B_2}\circ\phi \\
V^{a\sigma} &:= -iV^{B_1\bar{A}_1B_2\bar{A}_2}X_{\bar{A}_1}X_{\bar{A}_2}(X_{B_1}Y^a_{B_2}+Y^a_{B_1}X_{B_2})\circ\phi \\
V^{\bar{a}\sigma} &:= iV^{B_1\bar{A}_1B_2\bar{A}_2}X_{B_1}X_{B_2}(X_{\bar{A}_1}Y^{\bar{a}}_{\bar{A}_2}+Y^{\bar{a}}_{\bar{A}_1}X_{\bar{A}_2})\circ\phi \\
V^{ab} &:= \frac{1}{2}V^{B_1\bar{A}_1B_2\bar{A}_2}X_{\bar{A}_1}X_{\bar{A}_2}(Y^a_{B_1}Y^b_{B_2}+Y^b_{B_1}Y^a_{B_2})\circ\phi \\
V^{\bar{a}\bar{b}} &:= \frac{1}{2}V^{B_1\bar{A}_1B_2\bar{A}_2}(Y^{\bar{a}}_{\bar{A}_1}Y^{\bar{b}}_{\bar{A}_2}+Y^{\bar{b}}_{\bar{A}_1}Y^{\bar{a}}_{\bar{A}_2})X_{B_1}X_{B_2}\circ\phi \\
V^{\bar{a}b} &:= -\frac{1}{2}V^{B_1\bar{A}_1B_2\bar{A}_2}(X_{\bar{A}_1}X_{B_2}Y^{\bar{a}}_{\bar{A}_2}Y^b_{B_1}+X_{\bar{A}_2}X_{B_1}Y^{\bar{a}}_{\bar{A}_1}Y^b_{B_2})\circ\phi
\end{aligned}
\tag{7.11}
$$

The functions 7.11 satisfy the equations 1.

7.2.2 General case

We first note, that the zeroth order symmetries are simply the constants, since they have to satisfy $\partial^{\bar{a}}V = 0$ and $\partial^b V = 0$ for $a, b = 1, \ldots, n$. Now we look at the symmetries of general order d.

Theorem 7.2.1. *Let*

$$
P = \sum_{k+l\leq d} V^{a_1\ldots a_k\bar{b}_1\ldots\bar{b}_l\sigma\ldots\sigma}\partial_{a_1}\ldots\partial_{a_k}\partial_{\bar{b}_1}\ldots\partial_{\bar{b}_l}\partial_\sigma\ldots\partial_\sigma + LOTS
$$

where each V has exactly d indices, be a symmetry of order d. Then we have:

1. *$V^{\sigma\ldots\sigma}$ is a solution of the first BGG operator corresponding to* $\overset{d}{\circ}\!-\!\overset{0}{\circ}- \cdots -\overset{0}{\circ}\!-\!\overset{d}{\circ}$ *and*

$$
\begin{aligned}
V^{a_1\ldots a_k\sigma\ldots\sigma} &= \frac{i^k}{k!}\partial^{(a_1}\ldots\partial^{a_k)}V^{\sigma\ldots\sigma} \\
V^{\bar{b}_1\ldots\bar{b}_k\sigma\ldots\sigma} &= \frac{(-i)^k}{k!}\partial^{(\bar{b}_1}\ldots\partial^{\bar{b}_k)}V^{\sigma\ldots\sigma}
\end{aligned}
$$

2. *If $V^{\sigma\ldots\sigma} = 0$, then $V^{a_1\bar{b}_1\sigma\ldots\sigma}$ is a solution of the first BGG operator corresponding to* $\overset{d-2}{\circ}\!-\!\overset{1}{\circ}- \cdots -\overset{1}{\circ}\!-\!\overset{d-2}{\circ}$ *and*

$$
\begin{aligned}
V^{a_1\ldots a_{k+1}\bar{b}_1\sigma\ldots\sigma} &= \frac{i^k}{k!}\partial^{(a_1}\ldots\partial^{a_k}V^{a_{k+1})\bar{b}_1\sigma\ldots\sigma} \\
V^{a_1\bar{b}_1\ldots\bar{b}_{k+1}\sigma\ldots\sigma} &= \frac{(-i)^k}{k!}\partial^{(\bar{b}_1}\ldots\partial^{\bar{b}_k}V^{\bar{b}_{k+1})a_1\sigma\ldots\sigma}
\end{aligned}
$$

3. *Let* $2s \leq d$*. If* $V^{\sigma\dots\sigma} = V^{a_1\bar{b}_1\sigma\dots\sigma} = \dots = V^{a_1\dots a_{s-1}\bar{b}_1\dots\bar{b}_{s-1}\sigma\dots\sigma} = 0$*, then* $V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma} = 0$ *and*

$$V^{a_1\dots a_{k+s}\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma} = \frac{i^k}{k!}\partial^{(a_1}\dots\partial^{a_k}V^{a_{k+1}\dots a_{k+s})\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$$
$$V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_{k+s}\sigma\dots\sigma} = \frac{(-i)^k}{k!}\partial^{(\bar{b}_1}\dots\partial^{\bar{b}_k}V^{\bar{b}_{k+1}\dots\bar{b}_{k+s})a_1\dots a_s\sigma\dots\sigma}$$

Proof. Every symmetry of order d can be written as

$$P = \sum_{k+l\leq d} V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_l\sigma\dots\sigma}\partial_{a_1}\dots\partial_{a_k}\partial_{\bar{b}_1}\dots\partial_{\bar{b}_l}\partial_\sigma\dots\partial_\sigma + LOTS$$

where each V is totally symmetric (because the commutator of two derivatives gives a term of lower order), trace-free (because the trace gives some trivial symmetry) and has exactly d indices. If we commute it with the sub-Laplacian, the leading term of the commutator consists of terms of two types. Every $V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_l\sigma\dots\sigma}$ gives rise to a term

$$\partial^{(a}V^{a_1\dots a_k)\bar{b}_1\dots\bar{b}_l\sigma\dots\sigma}\partial_a\partial_{a_1}\dots\partial_{a_k}\partial_{\bar{b}_1}\dots\partial_{\bar{b}_l}\partial_\sigma\dots\partial_\sigma+$$
$$+\partial^{(\bar{b}}V^{\bar{b}_1\dots\bar{b}_l)a_1\dots a_k\sigma\dots\sigma}\partial_{a_1}\dots\partial_{a_k}\partial_{\bar{b}}\partial_{\bar{b}_1}\dots\partial_{\bar{b}_l}\partial_\sigma\dots\partial_\sigma$$

coming from commuting $V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_l\sigma\dots\sigma}$ with derivatives of the sub-Laplacian. Commuting the derivatives of the sub-Laplacian with the derivatives of P, we get

$$i(k-l)V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_l\sigma\dots\sigma}\partial_{a_1}\dots\partial_{a_k}\partial_{\bar{b}_1}\dots\partial_{\bar{b}_l}\partial_\sigma\dots\partial_\sigma\partial_\sigma$$

So together the leading term of the commutator is

$$\sum_{k+l=d+1}(\partial^{(a_1}V^{a_2\dots a_k)\bar{b}_1\dots\bar{b}_l} + \partial^{(\bar{b}_1}V^{\bar{b}_2\dots\bar{b}_l)a_1\dots a_k})\partial_{a_1}\dots\partial_{a_k}\partial_{\bar{b}_1}\dots\partial_{\bar{b}_l}+$$
$$+\sum_{1\leq k\leq d}(ikV^{a_1\dots a_k\sigma\dots\sigma} + \partial^{(a_1}V^{a_2\dots a_k)\sigma\dots\sigma})\partial_{a_1}\dots\partial_{a_k}\partial_\sigma\dots\partial_\sigma+$$
$$+\sum_{1\leq l\leq d}(-ilV^{\bar{b}_1\dots\bar{b}_k\sigma\dots\sigma} + \partial^{(\bar{b}_1}V^{\bar{b}_2\dots\bar{b}_l)\sigma\dots\sigma})\partial_{\bar{b}_1}\dots\partial_{\bar{b}_l}\partial_\sigma\dots\partial_\sigma+$$
$$+\sum_{\substack{k\geq1,l\geq1\\k+l\leq d}} i(k-l)V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_l\sigma\dots\sigma}\partial_{a_1}\dots\partial_{a_k}\partial_{\bar{b}_1}\dots\partial_{\bar{b}_l}\partial_\sigma\dots\partial_\sigma+$$

$$+\sum_{\substack{k\geq 1, l\geq 1\\ k+l\leq d}} (\partial^{(a_1}V^{a_2\dots a_k)\bar{b}_1\dots\bar{b}_l\sigma\dots\sigma} + \partial^{(\bar{b}_1}V^{\bar{b}_2\dots\bar{b}_l)a_1\dots a_k\sigma\dots\sigma})\partial_{a_1}\dots\partial_{a_k}\partial_{\bar{b}_1}\dots\partial_{\bar{b}_l}\partial_\sigma\dots\partial_\sigma$$

where the second and third row are special cases of the last two rows for $l = 0$ and $k = 0$, respectively. This should be the leading term of some operator of the form $\delta\Delta$. This is only possible, if

$$\partial^{(a_1}V^{a_2\dots a_k)\bar{b}_1\dots\bar{b}_l} + \partial^{(\bar{b}_1}V^{\bar{b}_2\dots\bar{b}_l)a_1\dots a_k} = g^{(a_1\bar{b}_1}\lambda^{a_2\dots a_k\bar{b}_1\dots\bar{b}_l)} \tag{7.12}$$

for some tensor λ and $k, l \geq 1$, $k + l = d + 1$,

$$\partial^{(a_1}V^{a_2\dots a_{d+1})} = 0 \qquad \partial^{(\bar{b}_1}V^{\bar{b}_2\dots\bar{b}_{d+1})} = 0 \tag{7.13}$$

$$ikV^{a_1\dots a_k\sigma\dots\sigma} + \partial^{(a_1}V^{a_2\dots a_k)\sigma\dots\sigma} = 0 \tag{7.14}$$

for $1 \leq k \leq d$,

$$-ilV^{\bar{b}_1\dots\bar{b}_k\sigma\dots\sigma} + \partial^{(\bar{b}_1}V^{\bar{b}_2\dots\bar{b}_l)\sigma\dots\sigma} = 0 \tag{7.15}$$

for $1 \leq l \leq d$,

$$\begin{aligned} i(k-l)V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_l\sigma\dots\sigma} + \partial^{(a_1}V^{a_2\dots a_k)\bar{b}_1\dots\bar{b}_l\sigma\dots\sigma} + \partial^{(\bar{b}_1}V^{\bar{b}_2\dots\bar{b}_l)a_1\dots a_k\sigma\dots\sigma} &= \\ = g^{(a_1|(\bar{b}_1}\lambda^{|a_2\dots a_k)|\bar{b}_1\dots\bar{b}_l)\sigma\dots\sigma} \end{aligned} \tag{7.16}$$

for $1 \leq k$, $1 \leq l$, $k + l \leq d$ and some tensor λ.

We will proceed by induction. From equations 7.14, 7.15 and 7.13 we see that

$$\begin{aligned} \partial^{(a_1}\dots\partial^{a_{d+1})}V^{\sigma\dots\sigma} &= 0 \\ \partial^{(\bar{b}_1}\dots\partial^{\bar{b}_{d+1})}V^{\sigma\dots\sigma} &= 0 \end{aligned} \tag{7.17}$$

what is exactly that $V^{\sigma\dots\sigma}$ lies in the kernel of first BGG operator corresponding to $\overset{d}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ}\!-\!\!-\!\overset{d}{\circ}$, as described in section 5.5. We also see that the terms $V^{a_1\dots a_k\sigma\dots\sigma}$ and $V^{\bar{b}_1\dots\bar{b}_k\sigma\dots\sigma}$ only depend on $V^{\sigma\dots\sigma}$. To compute the dependence explicitly, we use k times the equation 7.14 and 7.15, respectively. We get

$$V^{a_1\dots a_k\sigma\dots\sigma} = \frac{i^k}{k!}\partial^{(a_1}\dots\partial^{a_k)}V^{\sigma\dots\sigma}$$

$$V^{\bar{b}_1\ldots\bar{b}_k\sigma\ldots\sigma} = \frac{(-i)^k}{k!}\partial^{(\bar{b}_1}\ldots\partial^{\bar{b}_k)}V^{\sigma\ldots\sigma}$$

Putting $V^{\sigma\ldots\sigma}=0$ (this implies $V^{a_1\ldots a_k\bar{b}_1\ldots\bar{b}_l\sigma\ldots\sigma}=0$ for $\min(k,l)=0$), from equations 7.16 we see that

$$\begin{aligned} &\text{the trace-free part of} \quad \partial^{(a_1}\ldots\partial^{a_{d-1}}V^{a_d)\bar{b}_1\sigma\ldots\sigma}=0 \\ &\text{the trace-free part of} \quad \partial^{(\bar{b}_1}\ldots\partial^{\bar{b}_{d-1}}V^{\bar{b}_d)a_1\sigma\ldots\sigma}=0 \end{aligned} \tag{7.18}$$

what is exactly that $V^{a_1\bar{b}_1\sigma\ldots\sigma}$ lies in the kernel of the first BGG operator corresponding to $\overset{d-2}{\circ}\!\!-\!\!\!-\!\!\overset{1}{\circ}\!-\cdots-\!\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{d-2}{\circ}$, as described in section 5.5. We also see that the terms $V^{a_1\ldots a_k\bar{b}_1\ldots\bar{b}_l\sigma\ldots\sigma}$ with $\min(k,l)=1$ only depend on $V^{a_1\bar{b}_1\sigma\ldots\sigma}$. To compute the dependence explicitly, we use k times equation 7.16. We get

$$\begin{aligned} V^{a_1\ldots a_{k+1}\bar{b}_1\sigma\ldots\sigma} &= \frac{i^k}{k!}\partial^{(a_1}\ldots\partial^{a_k}V^{a_{k+1})\bar{b}_1\sigma\ldots\sigma} \\ V^{a_1\bar{b}_1\ldots\bar{b}_{k+1}\sigma\ldots\sigma} &= \frac{(-i)^k}{k!}\partial^{(\bar{b}_1}\ldots\partial^{\bar{b}_k}V^{\bar{b}_{k+1})a_1\sigma\ldots\sigma} \end{aligned}$$

Continuing this way, we see for each s such that $0<2s\leq d$, that putting $V^{\sigma\ldots\sigma}=\cdots=V^{a_1\ldots a_{s-1}\bar{b}_1\ldots\bar{b}_{s-1}\sigma\ldots\sigma}=0$, we have $V^{a_1\ldots a_k\bar{b}_1\ldots\bar{b}_l\sigma\ldots\sigma}=0$ for $\min(k,l)<s$. From equations 7.16 we see that

$$\begin{aligned} &\text{the trace-free part of} \quad \partial^{(a_1}\ldots\partial^{a_{d+1-2s}}V^{a_{d+2-2s}\ldots a_{d+1-s})\bar{b}_1\ldots\bar{b}_s\sigma\ldots\sigma}=0 \\ &\text{the trace-free part of} \quad \partial^{(\bar{b}_1}\ldots\partial^{\bar{b}_{d+1-2s}}V^{\bar{b}_{d+2-2s}\ldots\bar{b}_{d+1-s}a_1\ldots a_s\sigma\ldots\sigma}=0 \end{aligned} \tag{7.19}$$

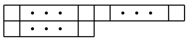
what is exactly the first BGG operator corresponding to $\overset{d-2s}{\circ}\!\!-\!\!\!-\!\!\overset{s}{\circ}\!-\cdots-\!\overset{s}{\circ}\!\!-\!\!\!-\!\!\overset{d-2s}{\circ}$, as described in section 5.5. We also see that the terms $V^{a_1\ldots a_k\bar{b}_1\ldots\bar{b}_l\sigma\ldots\sigma}$ with $\min(k,l)=s$ only depend on $V^{a_1\ldots a_s\bar{b}_1\ldots\bar{b}_s\sigma\ldots\sigma}$. To compute the dependence explicitly, we use k times equation 7.16. We get

$$\begin{aligned} V^{a_1\ldots a_{k+s}\bar{b}_1\ldots\bar{b}_s\sigma\ldots\sigma} &= \frac{i^k}{k!}\partial^{(a_1}\ldots\partial^{a_k}V^{a_{k+1}\ldots a_{k+s})\bar{b}_1\ldots\bar{b}_s\sigma\ldots\sigma} \\ V^{a_1\ldots a_s\bar{b}_1\ldots\bar{b}_{k+s}\sigma\ldots\sigma} &= \frac{(-i)^k}{k!}\partial^{(\bar{b}_1}\ldots\partial^{\bar{b}_k}V^{\bar{b}_{k+1}\ldots\bar{b}_{k+s})a_1\ldots a_s\sigma\ldots\sigma} \end{aligned}$$

□

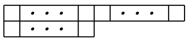
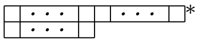
Remark 7.2.2. The representation $\overset{a}{\circ}\!\!-\!\!\!-\!\!\overset{b}{\circ}\!-\cdots-\!\overset{b}{\circ}\!\!-\!\!\!-\!\!\overset{a}{\circ}$ of $SU(p+1,g+1)$ is simply the Cartan product of $\overset{a}{\circ}\!\!-\!\!\!-\!\!\overset{b}{\circ}\!-\cdots-\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}$ with its dual/conjugate. The representation $\overset{a}{\circ}\!\!-\!\!\!-\!\!\overset{b}{\circ}\!-\cdots-\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}$ is an irreducible subrepresentation of $\otimes^d\mathbb{C}^{n+2}$

with symmetries given by Young tableau



with b columns with two boxes and a columns with one box (the total number of boxes is d).

Theorem 7.2.2. *The vector space of symmetries of the sub-Laplacian Δ is as a $SU(p+1, q+1)$ representation isomorphic to the direct sum of*

 $\boxtimes$  (7.20)

for all possible tableaux of this shape including the empty one for symmetries of order zero. For symmetries of order $\leq d$, we only consider tableaux with $\leq d$ boxes.

Proof. It suffices to prove the second statement. We proceed by induction. Assume that the statement holds for orders $< d$. We have proved that for fixed order d, the vector space of possible symbols is isomorphic to 7.20 for all tableaux of this shape with d boxes. Considering two symmetries of order d with the same symbol, we substract one from the other and we get a symmetry of lower order, so we can use the induction hypothesis. $\square$

Now we look at the ambient construction of the symmetries. Let I, J be ordered subsets of $\{1, \dots, d\}$ such that $I \cup J = \{1, \dots, d\}$ and $I \cap J = \emptyset$. Composing d first order symmetries, we get

$$V^{B_1\bar{A}_1 \dots B_d\bar{A}_d} \prod_{i=1}^{d} (X_{\bar{A}_i}\partial_{B_i} - X_{B_i}\partial_{\bar{A}_i}) =$$

$$= \sum_{\substack{|I|=k, |J|=l \\ k+l=d}} (-1)^l V^{B_1\bar{A}_1 \dots B_d\bar{A}_d} X_{\bar{A}_{i_1}} X_{\bar{A}_{i_k}} X_{B_{j_1}} \dots X_{B_{j_l}} \partial_{B_{i_1}} \dots \partial_{B_{i_k}} \partial_{\bar{A}_{j_1}} \dots \partial_{\bar{A}_{j_l}}$$

where $V^{B_1\bar{A}_1 \dots B_d\bar{A}_d}$ is totally trace-free and symmetric in pairs $B_i\bar{A}_i$. Looking at the induced operator, we get

$$\sum_{\substack{|I|=k, |J|=l \\ k+l=d}} (-1)^l V^{B_1\bar{A}_1 \dots B_d\bar{A}_d} X_{\bar{A}_{i_1}} X_{\bar{A}_{i_k}} X_{B_{j_1}} \dots X_{B_{j_l}} \partial_{B_{i_1}} \dots \partial_{B_{i_k}} \partial_{\bar{A}_{j_1}} \dots \partial_{\bar{A}_{j_l}} =$$

$$= \sum_{\substack{|I|=k, |J|=l \\ k+l=d}} (-1)^l V^{B_1\bar{A}_1 \dots B_d\bar{A}_d} X_{\bar{A}_{i_1}} X_{\bar{A}_{i_k}} X_{B_{j_1}} \dots X_{B_{j_l}} \delta^{D_{i_1}}_{B_{i_1}} \partial_{D_{i_1}} \dots \delta^{D_{i_k}}_{B_{i_k}} \partial_{D_{i_k}}.$$

$$\cdot\delta^{\bar{C}_{j_1}}_{\bar{A}_{j_1}}\partial_{\bar{C}_{j_1}}\dots\delta^{\bar{C}_{j_l}}_{\bar{A}_{j_l}}\partial_{\bar{C}_{j_l}} =$$

$$\sum_{\substack{|I|=k,|J|=l\\k+l=d}}(-1)^l V^{B_1\bar{A}_1\dots B_d\bar{A}_d}X_{\bar{A}_{i_1}}X_{\bar{A}_{i_k}}X_{B_{j_1}}\dots X_{B_{j_l}}\cdot$$

$$\cdot(X^{D_{i_1}}Z_{B_{i_1}}+Z^{D_{i_1}}X_{B_{i_1}}+Y^{D_{i_1}}_{a_1}Y^{a_1}_{B_{i_1}})\partial_{D_{i_1}}\dots(X^{D_{i_k}}Z_{B_{i_k}}+Z^{D_{i_k}}X_{B_{i_k}}+Y^{D_{i_k}}_{a_k}Y^{a_k}_{B_{i_k}})\partial_{D_{i_k}}\cdot$$

$$\cdot(X^{\bar{C}_{j_1}}Z_{\bar{A}_{j_1}}+Z^{\bar{C}_{j_1}}X_{\bar{A}_{j_1}}+Y^{\bar{C}_{j_1}}_{\bar{b}_1}Y^{\bar{b}_1}_{\bar{A}_{j_1}})\partial_{\bar{C}_{j_1}}\dots(X^{\bar{C}_{j_l}}Z_{\bar{A}_{j_l}}+Z^{\bar{C}_{j_l}}X_{\bar{A}_{j_l}}+Y^{\bar{C}_{j_l}}_{\bar{b}_l}Y^{\bar{b}_l}_{\bar{A}_{j_l}})\partial_{\bar{C}_{j_l}}$$

Knowing that $Y^D_a\partial_D=\partial_a$, $Y^{\bar{C}}_{\bar{b}}\partial_{\bar{C}}=\partial_{\bar{b}}$, $X^D\partial_D=\mathbb{E}$, $X^{\bar{C}}\partial_{\bar{C}}=\bar{\mathbb{E}}$ and $Z^D\partial_D-Z^{\bar{C}}\partial_{\bar{C}}=-i\partial_\sigma$, we see we have to put

$$V^{a_1\dots a_d} := \sum_{\sigma\in\mathfrak{S}_d}\frac{1}{d!}V^{B_1\bar{A}_1\dots B_d\bar{A}_d}X_{\bar{A}_1}\dots X_{\bar{A}_d}Y^{a_{\sigma(1)}}_{B_1}\dots Y^{a_{\sigma(d)}}_{B_d}\circ\phi \tag{7.21}$$

$$V^{\bar{b}_1\dots\bar{b}_d} := \sum_{\tau\in\mathfrak{S}_d}\frac{1}{d!}V^{B_1\bar{A}_1\dots B_d\bar{A}_d}X_{B_1}\dots X_{B_d}Y^{\bar{b}_{\tau(1)}}_{\bar{A}_1}\dots Y^{\bar{b}_{\tau(d)}}_{\bar{A}_d}\circ\phi$$

$$V^{a_1\dots a_k\sigma\dots\sigma} :=$$

$$= (-i)^{d-k}\sum_{|I|=k}\sum_{\sigma\in\mathfrak{S}_k}\frac{1}{k!}V^{B_1\bar{A}_1\dots B_d\bar{A}_d}X_{\bar{A}_1}\dots X_{\bar{A}_d}Y^{a_{\sigma(1)}}_{B_{i_1}}\dots Y^{a_{\sigma(k)}}_{B_{i_k}}X_B\dots X_B\circ\phi$$

$$V^{\bar{b}_1\dots\bar{b}_l\sigma\dots\sigma} :=$$

$$= (-1)^l(-i)^{d-l}\sum_{|J|=l}\sum_{\tau\in\mathfrak{S}_l}\frac{1}{l!}V^{B_1\bar{A}_1\dots B_d\bar{A}_d}X_{B_1}\dots X_{B_d}Y^{\bar{b}_{\tau(1)}}_{\bar{A}_{j_1}}\dots Y^{\bar{b}_{\tau(l)}}_{\bar{A}_{j_l}}X_{\bar{A}}\dots X_{\bar{A}}\circ\phi$$

$$V^{\sigma\dots\sigma} := (-i)^d V^{B_1\bar{A}_1\dots B_d\bar{A}_d}X_{\bar{A}_1}\dots X_{\bar{A}_d}X_{B_1}\dots X_{B_d}\circ\phi$$

$$V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_l\sigma\dots\sigma} :=$$

$$= (-1)^l(-i)^{d-k-l}\sum_{\substack{|I|=k,|J|=l\\I\cap J=\emptyset}}\sum_{\substack{\sigma\in\mathfrak{S}_k\\\tau\in\mathfrak{S}_l}}\frac{1}{k!l!}V^{B_1\bar{A}_1\dots B_d\bar{A}_d}X_{\bar{A}_{i_1}}\dots X_{\bar{A}_{i_k}}X_{B_{j_1}}\dots X_{B_{j_l}}X_{\bar{A}}\dots X_{\bar{A}}\cdot$$

$$\cdot Y^{a_{\sigma(1)}}_{B_{i_1}}\dots Y^{a_{\sigma(k)}}_{B_{i_k}}Y^{\bar{b}_{\tau(1)}}_{\bar{A}_{j_1}}\dots Y^{\bar{b}_{\tau(l)}}_{\bar{A}_{j_l}}X_B\dots X_B\circ\phi$$

where I and J are ordered subsets of $\{1,\dots,d\}$ and all indices not specified are simply the remaining indices. We will always assume this in the sequel. These functions satisfy the conditions 7.12, 7.13, 7.14, 7.15 and 7.16.

Theorem 7.2.3. *For every given symmetry P of the sub-Laplacian Δ, we can construct a symmetry of Δ with the same symbol using the ambient construction.*

Proof. We know from above that every symmetry of order d can be written as a sum of symmetries with $V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}\neq 0$ and $V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_k\sigma\dots\sigma}=0$ for all $k<s$ and $0\leq 2s\leq d$. So it suffices to construct for each $0\leq 2s\leq d$ and each function $V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$ satisfying the first BGG equation corresponding to $\overset{d-2s}{\circ}\!\!-\!\!\!-\!\!\overset{s}{\circ}\!-\cdots-\!\overset{s}{\circ}\!\!-\!\!\!-\!\!\overset{d-2s}{\circ}$ a tensor $V^{B_1\bar{A}_1\dots B_d\bar{A}_d}$ inducing a symmetry with given (up to possible nonzero constant multiple) $V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$ and such that for all $k<s$ $V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_k\sigma\dots\sigma}=0$. Since contracting with X-s and Y-s is equivariant map and for each s the set of possible $V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$-s (with $V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_k\sigma\dots\sigma}=0$ for all $k<s$) forms an irreducible representation of $SU(p+1,q+1)$, it suffices to construct the tensor $V^{B_1\bar{A}_1\dots B_d\bar{A}_d}$ for one such function for each s.

For $s=0$ we put $V^{\sigma\dots\sigma}=1$ and the only nontrivial component of $V^{B_1\bar{A}_1\dots B_d\bar{A}_d}$ will be $V^{\infty\bar{\infty}\dots\infty\bar{\infty}}=1$. This doesn't depend on d (only the constant factor does).

For $s>0$, we have the mapping

$$V^{B_1\bar{A}_1\dots B_d\bar{A}_d}\mapsto V^{B_1\bar{A}_1\dots B_d\bar{A}_d}(-1)^s(-i)^{d-2s}\sum_{\substack{|I|=s,|J|=s\\ I\cap J=\emptyset}}\sum_{\substack{\sigma\in\mathfrak{S}_s\\ \tau\in\mathfrak{S}_s}}\frac{1}{(s!)^2}X_{\bar{A}_{i_1}}\dots X_{\bar{A}_{i_s}}\cdot$$
$$\cdot X_{B_{j_1}}\dots X_{B_{j_s}}X_{\bar{A}}\dots X_{\bar{A}}Y^{a_{\sigma(1)}}_{B_{i_1}}\dots Y^{a_{\sigma(s)}}_{B_{i_s}}Y^{\bar{b}_{\tau(1)}}_{\bar{A}_{j_1}}\dots Y^{\bar{b}_{\tau(s)}}_{\bar{A}_{j_s}}X_B\dots X_B\circ\phi=$$
$$=V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$$

We fix some *constant* tensor field $V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$ and we put

$$V^{a_1\bar{\infty}\infty\bar{b}_1\dots a_s\bar{\infty}\infty\bar{b}_s\infty\bar{\infty}\dots\infty\bar{\infty}}=V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$$

This will be for now the only nonzero component up to symmetry in $B_i\bar{A}_i$-s. This tensor surely satisfies the corresponding first BGG equation. It is easy to see that the symmetry induced by using this tensor has nonzero $V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$ (it is in fact, up to nonzero constant, our chosen one). But the symbol parts $V^{a_1\dots a_k\bar{b}_1\dots\bar{b}_k\sigma\dots\sigma}=0$ for all $k<s$ are also nonzero. In order to make them vanish, we define some other components to be possibly nonzero. These components will not influence the symbol part $V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$.

Let's fix some $s>0$. We define

$$V^{a_1\bar{\infty}\infty\bar{b}_1\dots a_s\bar{\infty}\infty\bar{b}_s\infty\bar{\infty}\dots\infty\bar{\infty}}=V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$$
$$V^{a_1\bar{\infty}\infty\bar{b}_1\dots a_{s-1}\bar{\infty}\infty\bar{b}_{s-1}a_s\bar{b}_s\infty\bar{\infty}\dots\infty\bar{\infty}}=x_1V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$$
$$V^{a_1\bar{\infty}\infty\bar{b}_1\dots a_{s-2}\bar{\infty}\infty\bar{b}_{s-2}a_{s-1}\bar{b}_{s-1}a_s\bar{b}_s\infty\bar{\infty}\dots\infty\bar{\infty}}=x_2V^{a_1\dots a_s\bar{b}_1\dots\bar{b}_s\sigma\dots\sigma}$$

. . . .

$$V^{a_1\bar{b}_1\ldots a_s\bar{b}_s\infty\bar{\infty}\ldots\infty\bar{\infty}} = x_s V^{a_1\ldots a_s\bar{b}_1\ldots\bar{b}_s\sigma\ldots\sigma}$$

These will be the only nonzero components of $V^{B_1\bar{A}_1\ldots B_d\bar{A}_d}$ up to symmetry in $B_i\bar{A}_i$-s. We will call them 'types'-in each row is one particular representant of one type. For brevity, we will write '$V^{a_1\ldots a_s\bar{b}_1\ldots\bar{b}_s\sigma\ldots\sigma} = 1$'. To make the symbol parts $V^{a_1\ldots a_k\bar{b}_1\ldots\bar{b}_k\sigma\ldots\sigma} = 0$ for all $k < s$ vanish, they have to satisfy the following system of linear equations:

To make $V^{a_1\ldots a_k\bar{b}_1\ldots\bar{b}_k\sigma\ldots\sigma}$ vanish, we must have

$$\binom{d}{2s}\binom{2s}{s}a^s_{k+1,0} + \binom{d}{2s-2}\binom{2s-2}{s-1}\binom{d-2s+2}{1}a^s_{k+1,1}x_1 + \tag{7.22}$$
$$+\binom{d}{2s-4}\binom{2s-4}{s-2}\binom{d-2s+4}{2}a^s_{k+1,2}x_2 + \cdots + \binom{d}{0}\binom{0}{0}\binom{d}{s}a^s_{k+1,s} = 0$$

where the terms $\binom{d}{2s-2i}\binom{2s-2i}{s-i}\binom{d-2s+2i}{i}$ express the number of components of one type (it is always nonzero) and the numbers $a^s_{k+1,i}$ express the contribution of one component of corresponding type to the symbol part $V^{a_1\ldots a_k\bar{b}_1\ldots\bar{b}_k\sigma\ldots\sigma}$ modulo the greatest common divisor, which is some polynomial. It is easy to see that

$$a^s_{k+1,i} = \binom{s-i}{k}\binom{s}{k} + \binom{s-i}{k-1}\binom{i}{1}\binom{s-1}{k} +$$
$$+\binom{s-i}{k-2}\binom{i}{2}\binom{s-2}{k} + \cdots + \binom{s-i}{k-j}\binom{i}{j}\binom{s-j}{k} + \cdots$$

This expression is always finite. So we have a system of linear equations in x_i-s, and we only need to prove the existence of some solution. We don't need the explicit expression. To prove the existence, we prove that the matrix of this system has nonzero determinant. First, the determinant is linear in columns, so it is a polynomial in d times the determinant of matrix with entries $a^s_{k+1,i}$. We will prove that this last determinant is nonzero by induction on s.

We claim that $a^{s+1}_{k+2,i} - a^{s+1}_{k+2,i+1} = a^s_{k+1,i}$, where we put $a^s_{0,i} = 0$. This means that

$$\sum_{j\geq 0}\binom{s+1-i}{k+1-j}\binom{i}{j}\binom{s+1-j}{k+1} =$$
$$= \sum_{j\geq 0}\binom{s-i}{k-j}\binom{i}{j}\binom{s-j}{k} + \sum_{j=0}^{k+1}\binom{s-i}{k+1-j}\binom{i+1}{j}\binom{s+1-j}{k+1}$$

But

$$\binom{s+1-i}{k+1-j}\binom{i}{j}\binom{s+1-j}{k+1} =$$
$$= \left(\binom{s-i}{k-j} + \binom{s-i}{k+1-j}\right)\binom{i}{j}\binom{s+1-j}{k+1} =$$
$$= \binom{s-i}{k+1-j}\binom{i}{j}\binom{s+1-j}{k+1} + \binom{s-i}{k-j}\binom{i}{j}\binom{s-j}{k} +$$
$$+ \binom{s-i}{k-j}\binom{i}{j}\binom{s-j}{k+1}$$

Summing over j we get

$$\sum_{j\geq 0}\binom{s+1-i}{k+1-j}\binom{i}{j}\binom{s+1-j}{k+1} =$$
$$= \sum_{j\geq 0}\binom{s-i}{k-j}\binom{i}{j}\binom{s-j}{k} + \sum_{j=0}^{k+1}\binom{s-i}{k+1-j}\binom{i}{j}\binom{s+1-j}{k+1} +$$
$$+ \sum_{j\geq 1}\binom{s-i}{k-j}\binom{i}{j-1}\binom{s+1-j}{k+1} =$$
$$\sum_{j\geq 0}\binom{s-i}{k-j}\binom{i}{j}\binom{s-j}{k} + \sum_{j=0}^{k+1}\binom{s-i}{k+1-j}\binom{i+1}{j}\binom{s+1-j}{k+1}$$

what is exactly what we claimed. Now we note that $a^s_{1,i} = 1$ for all s and i. Using this, we get

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a^s_{2,1} & a^s_{2,2} & \cdots & a^s_{2,s-1} & a^s_{2,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^s_{s,1} & a^{s,2} & \cdots & a^s_{s,s-1} & a^s_{s,s} \end{vmatrix} = \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ a^s_{2,1}-a^s_{2,2} & a^s_{2,2}-a^s_{2,3} & \cdots & a^s_{2,s-1}-a^s_{2,s} & a^s_{2,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^s_{s,1}-a^s_{s,2} & a^s_{s,2}-a^s_{s,3} & \cdots & a^s_{s,s-1}-a^s_{s,s} & a^s_{s,s} \end{vmatrix} =$$
$$= (-1)^s \begin{vmatrix} a^{s-1}_{1,1} & a^{s-1}_{1,2} & \cdots & a^{s-1}_{1,s-1} \\ \vdots & \vdots & \ddots & \vdots \\ a^{s-1}_{s-1,1} & a^{s-1}_{s-1,2} & \cdots & a^{s-1}_{s-1,s-1} \end{vmatrix}$$

So this determinant is nonzero for any s, if and only if it is nonzero for $s = 1$. But in this case the determinant is 1, since it is the determinant of matrix with one entry, concretely $a^1_{1,1} = 1$. $\square$

Knowing this, we can construct any given symmetry as a sum of those constructed by the ambient construction. Assume that we can construct every symmetry of order $\leq k$ as a sum of symmetries constructed by the ambient construction. This is trivial for symmetries of order zero, which are simply the constants. First, we construct some symmetry with the same symbol. The difference is again a symmetry, but of lower order, so we can use the induction hypothesis to write the given symmetry as a sum of those constructed by the ambient construction.

Chapter 8

Conclusion

In the previous chapter we have constructed the symmetries of the sub-Laplacian and characterized the vector space of them. Using the ambient construction as described in [6], it should be possible to do the same in a more general setting. The next natural aim should be, as in [7], to compute the multiplicative structure of the algebra of symmetries. In the conformal case, the symmetry algebra is isomorphic to the tensor algebra of $\mathfrak{so}(n+1,1)$ modulo the two-sided ideal generated by irreducible pieces in the second tensor power, which don't occure in the description of symmetries.

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