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MASTER THESIS

Volatility Modeling: Evidence from CEE Stock Markets

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Declaration of Authorship

The author hereby declares that she compiled this thesis independently, using only the listed resources and literature.

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Abstract

The thesis applies newly developed heterogenous autoregressive model of realized volatility on high frequency data of three stock market indices: Prague, Budapest and Warsaw with the aim to capture behavior of three different market participants and to quantify their role in forecasting daily realized volatility. Also, the presence of jumps in volatility is investigated and the predictive power assessed. In addition, wavelet analysis is used to detect periods and frequencies of comovements between the three indices.

The main contribution of the thesis lies especially in its primary empirical analysis conducted in CEE region.

The estimation results indicate that future realized volatility is determined very similarly in all markets with an insignificant impact of participants trading on monthly basis. Moreover, occurrence of a jump proves to be of a high relevance when predicting future volatility. Moreover, wavelet analysis indicates a strong degree of comovement at a frequency of few months across the whole period examined.

JEL Classification	C22, C50, F30, G15
Keywords	realized volatility, heterogenous autoregressive model, wavelet analysis, stock markets

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Abstrakt

Práce aplikuje heterogenní autoregresní modely realizované volatility na vysokofrekvenční data akciových indexů v Praze, Budapešti a Varšavě. Jejím cílem je zachytit chování tří skupin účastníků trhu obchodujících na denní, týdenní a měsíční bázi a odhadnout jejich roli v predikci denní realizované volatility. Předmětem výzkumu jsou také přístupy zaměřující se na detekci skoků ve volatilitě a jejich vlivu na predikční schopnost modelů. V další části práce specifikujeme pomocí wavelet analýzy období a frekvenci, během které indexy vykazují společný vývoj.

Přínos této práce spočívá především v primární aplikaci heterogenních autoregresních modelů realizované volatility na vybrané akciové indexy v oblasti střední a východní Evropy.

Výstupy modelů ukazují, že budoucí realizovaná volatilita je na trzích určena velmi podobně a ve všech případech vylučuje vliv těch účastníků trhu, kteří obchodují v horizontu jednoho měsíce. Navíc se potvrzuje přítomnost skoků ve volatilitě jako vysoce relevantního ukazatele pro její budoucí vývoj. Výsledky wavelet analýzy potvrzují tendenci trhů vyvíjet se stejným směrem v období přibližně čtyř měsíců.

Klasifikace JEL	C22, C50, F30, G15
Klíčová slova	realizovaná volatilita, heterogenní autoregresní model, wavelet analýza, akciové trhy

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Master Thesis Proposal

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Topic characteristics With the onset of electronic trading and computer sciences development, it has become possible to collect and store a large volume of financial data. They are in the form of "tick-by-tick" data, where a tick is a unit of information, such as price or volume in a transaction. In liquid markets, thousands of ticks are generated every trading day, hence offering a vast database of information to researchers. Compared to past financial time series modelling studies, nowadays it is not necessary to restrict the number of observations to one per day only. Using all the intra-day trading information, new scopes for analyses and modelling emerge. Based on these facts, the aim of this thesis is to apply neural networks on high frequency financial data.

Research Hypotheses This thesis revolves around the research question whether the application of neural networks on high frequency financial data can better capture the patterns present in the data and thus help us understand the world financial markets.

Data Data preprocessing is crucial for financial time series modelling using high frequency data. These data are randomly and irregularly distributed in time with occasional trading gaps. Moreover, a lot of asymmetries and nonlinearities are expected to be present. As most statistical methods are designed to data equally spaced in time, appropriate statistical processing methods should be considered.

Methodology As in other methods, neural networks relate a set of input variables to a set of output variables. However, this process is not performed directly, but through one or more hidden layers, in which the input information is transformed. Put another way, some input variables generate output variables, but these output variables, in turn, stand for input variables into another output, which allows for modelling the nonlinearities in data.

This structure and behaviour of interconnected variables is based on the universal approximation theorem. It claims, that neural networks are able to approximate any function. Therefore, they represent a robust and efficient approach to dealing with nonlinearities and chaotic systems, which can be present in the approximated function.

Thanks to these properties, neural networks are able to learn, thus they can simulate the learning process of economic agents. This characteristic is especially important for financial markets, where agents create expectations as a result of their learning process. In addition, as neural networks can deal with nonlinearities, they are also able to deal with another pattern of agent's behaviour, which is unproportional reaction to changes in external variables.

Based on the characteristics of neural networks that conform with typical aspects of financial markets, we believe, that this method will help us better understand the functioning of world financial markets.

Outline

1. Introduction
2. Neural Network
3. High Frequency Data Analysis
4. Application
   - In-Sample Performance
   - Out-of-Sample Performance
   - Interpretation of Results
5. Conclusion

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Chapter 1

Introduction

Volatility, defined as a risk of a financial instrument over a specified time period, is a central topic in risk management and asset allocation. It is followed by every investor and also shapes trading decisions, thus can be regarded as a key concept in financial market theory.

Most of the research focusing on volatility modeling treated volatility as unobservable variable and approached it through GRACH and stochastic-volatility models. The disadvantage of these models lies especially in their inability to reproduce main features of financial data and in their short memory.

However, with the onset of electronic trading and computer sciences development, it has become possible to collect and store a large volume of financial data, which can be recorded at various time intervals. The smallest unit is a tick. It is a piece of information, such as price or volume in a transaction. In liquid markets, thousands of ticks are generated every trading day, hence offering a vast database of information to researchers. The use of high frequency data thus gives new scope for modeling as we are not restricted to one observation per day.

Andersen *et al.* (2001) therefore suggest to construct an observable volatility with the use of high-frequency data. They call it Realized Volatility.

In the light of Heterogenous Market Hypothesis suggested by Müller *et al.* (1997) and newly defined realized volatility, Corsi (2003) presents a new model of three volatility components, each created by different type of market agents. He calls this model Heterogenous Autoregressive model of Realized Volatility (HAR-RV). This simple AR-type model, though not formally belonging to a class of long memory models, can capture volatility persistence and therefore gives rise to many succeeding extensions.

First, Corsi *et al.* (2005) investigates behavior of residuals and concludes, that residuals exhibit non-Gaussianity and volatility clustering. They therefore propose to consider normal inverse Gaussian distribution instead and to specify a GARCH process to account for volatility clustering in the squared residuals. Afterwards, the discussion turns solely to the presence of jumps and their impact on predicting future realized volatility. This shift is motivated by research work of Barndorff-Nielsen & Shephard (2004) and Barndorff-Nielsen & Shephard (2006), who specify so-called bi-power variation measures, which become the key concept in separating jump part from continuous time series. Consequently, a lot of studies on the role of jumps emerge, comprising Corsi *et al.* (2008) and Andersen *et al.* (2007). An overall list of references can be found in a literature survey Barndorff-Nielsen & Shephard (2007).

Given the background of the thesis, we now turn to its objectives and contributions. The thesis applies newly developed HAR-RV models on high frequency data of three stock market indices: Prague, Budapest and Warsaw in years 2008 and 2009, thus focusing on period of financial turmoil. It tries to capture behavior of three different market participants in time of high uncertainty. In addition, financial crisis is typical of many unexpected jumps in volatility; this fact stays at forefront of our interest. To our best knowledge, HAR-RV models have not been applied to stock market indices in CEE region yet, we therefore present completely primary results in this area, which is our first contribution.

Except for estimation of HAR-RV models, we also investigate relationships between indices from realized volatility perspective. We employ wavelet analysis which is able to expand time series into a time-frequency space and therefore identify periods of strong and significant comovements of two time series in a specified frequency band, i.e. within a given time interval. This analysis represents our second contribution as we extend the range of existing applications of wavelet analysis.

Overall, the thesis complements the existing applications of HAR-RV models and wavelet analysis, and derives new implications for risk management.

The thesis is organized as follows. Chapter 2 provides theoretical background for HAR-RV models. In this section, six models are presented that are applied on data. Chapter 3 describes high frequency data employed in this work, construction of variables, jump detection procedure and descriptive statistics. Chapter 4 presents empirical results of HAR-RV models, evaluates the in-sample and out-of-sample performance and identifies common features of all indices. Chapter 5 introduces wavelet analysis, covers basic theory behind it and comments on empirical results. Chapter 6 concludes.

Chapter 2

Theoretical Background

The purpose of this chapter is to introduce both quadratic variation theory and heterogenous autoregressive models.

Quadratic variation is one type of a variation of a process which is broadly used in analysis of stochastic processes. The mathematics behind the quadratic variation stems from the research work of Back (1991).

Quadratic variation is first introduced by Andersen *et al.* (2003), who build the concept of realized volatility on this theory. In this form, quadratic variation theory defines an arbitrage-free setting in which price process and return process are viewed as special semimartingales. Using the properties of special semimartingales, the quadratic variation of each process is constructed and relationship between conditional return covariance matrix and conditional quadratic return variation matrix is described under several conditions. Finally, employing martingale representation theorem, distributional characteristics of return are derived.

Currently, the quadratic variation theory is in a process of extension and generalization such that it is able to split up the individual components of quadratic variation into that due to the continuous part and that due to jumps, as proposed by Barndorff-Nielsen & Shephard (2004), who define the concept of realized bipower variation. The theory is also extended by tests designed to detect the presence of jump, as elaborated in Barndorff-Nielsen & Shephard (2006).

The second part of the chapter deals with heterogenous autoregressive models of realized volatility called HAR-RV. The first model was proposed by Corsi (2003) and was inspired by theory of Müller *et al.* (1997) about Heterogenous Market Hypothesis. However, the technical derivation of the model rests on quadratic variation theory, which suggests a way of approximation of quadratic variation called realized volatility.

We first start with the quadratic variation theory.

2.1 Quadratic Variation Theory

The quadratic variation theory presented here comes from Andersen *et al.* (2003). In their study they summarize the theory and explain the links to realized volatility modeling, thus creating a standard setting in this area. Therefore, the following propositions and definitions are taken from them. We start with definition of the setting.

We consider an n-dimensional price process defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, evolving in continuous time over the interval $[0, T]$, where T denotes a positive integer. We further consider an information filtration, i.e. an increasing family of σ-fields, $(\mathfrak{F}_t)_{t\in[0,T]} \subseteq \mathfrak{F}$, which satisfies the usual conditions of P-completeness and right continuity. Finally, we assume that the asset prices through time t, including the relevant state variables, are included in the information set $\mathfrak{F}_t$.

However, before we jump directly to the quadratic variation theory, we devote some lines to semimartingales.

A real valued process X defined on a filtered probability space is called a semimartingale if it can be decomposed into a sum of local martingale and a predictable finite variation process starting at zero. Put another way, if X has a finite mean, then $X = A + M$ where A has finite variation, i.e. for every finite time interval the variation is bounded, and also has a predictable mean. M denotes local martingale which is a type of stochastic process. However, this decomposition does not have to be unique.

To ensure uniqueness of decomposition, we need to consider special semimartingales. Every special semimartingale is a semimartingale, but every semimartingale is a special semimartingale if and only if X is locally integrable. This holds for example when M and A are continuous processes.

Also, semimartingales have several useful properties, such as: linear combination of semimartingales is a semimartingale, product of semimartingales is a semimartingale, smooth transformation of a semimartingale is a semimartingale and quadratic variation exists for every semimartingale.

Now let us turn to the quadratic variation theory starting with Proposition 2.1.

Proposition 2.1. *For any n-dimensional arbitrage free vector price process with finite mean, the logarithmic vector price process p may be written uniquely as the sum of a finite variation and predictable mean component $A = (A_1, A_2, ..., A_n)$ and a local martingale $M = (M_1, M_2, ..., M_n)$. These may each be decomposed into a continuous sample-path and jump part*

$$p(t) = p(0) + A(t) + M(t) = p(0) + A^c(t) + \Delta A(t) + M^c(t) + \Delta M(t) \quad (2.1)$$

where the finite variation predictable components $A^c(t) + \Delta A(t)$ are respectively continuous and pure jump processes, while the local martingales $M^c(t) + \Delta M(t)$ are respectively continuous sample-path and compensated jump processes, and by definition $M(0) \equiv A(0) \equiv 0$. Moreover, the predictable jumps are associated with genuine jump risk, in the sense that if $\Delta A(t) \neq 0$, then

$$P[sgn(\Delta A(t)) = -sgn(\Delta A(t) + \Delta M(t))] > 0. \quad (2.2)$$

where $sgn(x) \equiv 1$ for $x \geq 0$ and $sgn(x) \equiv 0$ for $x < 0$.

Proposition 2.1 claims that the price process belongs to a class of special semimartingales and, as mentioned earlier, can be uniquely decomposed into a component having finite variation and predictable mean and into a local martingale. Both components are assumed to have continuous and jump parts, as suggested by (2.1). In addition, the price process is arbitrage free, i.e. there is no way of predicting future development that would result in additional profit. Concretely, the arbitrage-free condition is ensured by the presence of jumps, as indicated by (2.2). If there is a predictable jump in the component A, i.e. if $\Delta A(t) \neq 0$, then there is scope for arbitrage opportunities, and therefore the predictable jump has to be compensated by another jump, which is unexpected, i.e. $\Delta M(t) \neq 0$.

The definition of return follows.

Definition 2.1. Continuously compounded return over $[t-h, t]$ is denoted by $r(t,h) = p(t) - p(t-h)$, while the cumulative return process from $t = 0$ onward is denoted $r(t) \equiv r(t,t) = p(t) - p(0) = A(t) + M(t)$

The definition of return states, that return overtakes all the main properties from price process, thanks to the characteristics of special semimartingales. It can also be decomposed into predictable and integrable mean component A and into local martingale M. Both components have their continuous and dis-continuous paths, where A^c must have smooth sample path, while the non-constant continuous martingale M^c may be driven by Brownian motion, for instance. In addition, there are two types of jumps. If there is a predictable jump $\Delta A(t) \neq 0$, i.e. timing and magnitude of the jump are known in advance, the equation (2.2) holds to ensure no-arbitrage environment. This kind of jumps occurs when new information are released and the market has already created expectations about the information. The second option is that the jump is purely unanticipated, i.e. $\Delta M(t) \neq 0$. Such jumps are usually associated with unexpected events that influence the market.

To introduce the quadratic variation theory, it is important to say that quadratic variation is one type of many variation processes and is used to analyze stochastic processes such as martingales or Brownian motion. An already mentioned property of special semimartingale process is that is has a corresponding quadratic variation process. Proposition 2.2 defines quadratic variation of a return process.

Proposition 2.2. *For any n-dimensional arbitrage-free price process with finite mean, the quadratic variation $n \times n$ matrix process of the associated return process $[r,r] = [r,r]_{t,t\in[0,T]}$ is well defined. The i'th diagonal element is called the quadratic variation process of the i'th asset return while the ij'th off-diagonal element $[r_i, r_j]$ is called the quadratic covariation process between asset returns i and j. The quadratic variation and covariation processes have the following properties:*

(i) *For an increasing sequence of random partitions of [0,T], $0 = \tau_{m,0} \leq \tau_{m,1} \leq ...$, such that $sup_{j>1}(\tau_{m,j} - \tau_{m,j-1}) \to 0$ and $sup_{j>1}\tau_{m,j} \to T$ for $m \to \infty$ with probability one, we have that*

$$\lim_{m\to\infty} \sum_{j>1} [r(t \wedge \tau_{m,j}) - r(t \wedge \tau_{m,j-1})][r(t \wedge \tau_{m,j}) - r(t \wedge \tau_{m,j-1})]' \to [r,r]_t \tag{2.3}$$

where $t \wedge \tau \equiv min(t,\tau), t \in [0,T]$ and the convergence is uniform on $[0,T]$ in probability.

(ii) *If the finite variation component A in the canonical return decomposition in definition 2.1 is continuous, then*

$$[r_i, r_j]_t = [M_i, M_j]_t = [M_i^c, M_j^c]_t + \sum_{0\leq s\leq t} \Delta M_i(s) \Delta M_j(s). \tag{2.4}$$

Proposition 2.2 shows two important implications. First, it defines a practical way of approximating quadratic variation. We can understand it as follows: Let us have a real valued stochastic process X and let us assume we know all its values on a time interval from $t-1$ to t, then the corresponding quadratic variation $[X]_t$ is given by

$$[X]_t = \lim_{||P|| \to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2 \tag{2.5}$$

where n denotes number of intervals between $t-1$ and t and P defines the length of an interval corresponding to n. The equation shows that as P goes the zero, the number of intervals increases to infinity and the sum of squared difference of the variable X whose value is recorded at every infinitely small moment approaches quadratic variation denoted as $[X]_t$ in probability. In Proposition 2.2, we deal with returns. In practice, the quadratic variation obtained through this method is called realized volatility.

The second implication of Proposition 2.2 is that quadratic variation is determined by the squared jump size only, as the quadratic variation of continuous finite-variation process is zero. In addition, if the finite variation component A is continuous, then the quadratic variation process is exclusively determined by the innovations to the return process. It also suggests that jump components contribute to the quadratic covariation if there are simultaneous jumps in the price path of the i'th and j'th asset.

We now turn to the relation between quadratic variation and return volatility process. We start with Proposition 2.3.

Proposition 2.3. *Consider an n-dimensional square-integrable arbitrage-free logarithmic price process with a continuous mean return, as in property (ii) of Proposition 2.2. The conditional return covariance matrix at time t over* $[t, t+h]$, *where* $0 \leq t \leq t+h \leq T$, *is then given by*

$$\begin{aligned} cov(r(t+h,h)|\mathfrak{F}_t) &= E([r,r]_{t+h} - [r,r]_t|\mathfrak{F}_t) + \Gamma_A(t+h,h) \\ &\quad + \Gamma_{AM}(t+h,h) + \Gamma'_{AM}(t+h,h) \end{aligned} \tag{2.6}$$

where $\Gamma_A(t+h,h) = cov(A(t+h,h) - A(t)|\mathfrak{F}_t)$ *and*
$\Gamma_{AM}(t+h,h) = E(A(t+h)[M(t+h) - M(t)]'|\mathfrak{F}_t)$.

The Proposition 2.3 explains in general the covariance matrix of returns conditional on an information set $\mathfrak{F}_t$ available at time t. The condition about continuous mean requires regards return to be decomposed into component A and M with no jump part in A so that the whole component is continuous. Under this condition, the component A does not enter the quadratic variation. The conditional covariance matrix has the following form

$$\begin{pmatrix} V(t) & C \\ C & V(t+h) \end{pmatrix} \tag{2.7}$$

where $V(t)$ denotes variance of $r(t)$, $V(t+h)$ denotes variance of $r(t+h)$ and C denotes covariance between $r(t)$ and $r(t+h)$. In every time moment and conditional on the information available in time t, each element of the matrix consists of an expectation of the difference between quadratic variation at time t and $t+h$, on variance of the difference in the continuous component A and finally on the expectation of the product of A and difference in M between t and $t+h$. The following Proposition 2.4 shows how the construction evolves as more conditions are added.

Proposition 2.4. *Consider an n-dimensional square-integrable arbitrage-free logarithmic price process, as described in Proposition 2.3. If the mean process $A(s) - A(t)_{s\in[t,t+h]}$, conditional on information at time t is independent of the return innovation process $M(u)_{u\in[t,t+h]}$, then the conditional return covariance matrix reduces to the conditional expectation of the quadratic return variation plus the conditional variance of the mean component, i.e. for $0 \leq t \leq t+h \leq T$,*

$$cov(r(t+h,h)|\mathfrak{F}_t) = E([r,r]_{t+h} - [r,r]_t|\mathfrak{F}_t) + \Gamma_A(t+h,h). \tag{2.8}$$

If the mean process $A(s) - A(t)_{s\in[t,t+h]}$ conditional on information at time t is a predetermined function over $[t, t+h]$, then the conditional return covariance matrix equals the conditional expectation of the quadratic return variation process, i.e. $0 \leq t \leq t+h \leq T$

$$cov(r(t+h,h)|\mathfrak{F}_t) = E([r,r]_{t+h} - [r,r]_t|\mathfrak{F}_t). \tag{2.9}$$

The Proposition 2.4 defines two cases, under which the specification of conditional return covariance matrix reduces. In the first case, we assume that the mean process is independent of the return innovation process. As the vectors are in orthogonal position, their product is zero. Since innovations are generally assumed to be independent of each other and of other variables, too, we do not regard this condition as binding.

In the second case, an assumption that the mean process is a predetermined function is added. This assumption implies that the difference $A(s) - A(t)$ is a constant and therefore its covariance is equal to zero. As the continuous process A is assumed to have a predictable mean, we again do not consider this condition to be very strong.

To sum up, Proposition 2.3 and Proposition 2.4 define quadratic variation as the main driving force in determining return volatility. Under certain assumptions, which in practice hold for a large set o data, they directly relate return volatility to the expectation of quadratic variation difference in time, both conditional on information set available in time t.

We now state the Martingale representation theorem in order to create a benchmark, that would allow us to make a step towards derivation of specific distributional results of the return generating process. However, we start with a bit restrictive conditions requiring continuous price process without any jumps, i.e. $(\Delta M = 0)$.

Proposition 2.5. *For any n-dimensional square-integrable arbitrage-free logarithmic price process p with continuous sample path and a full rank of the associated $n \times n$ quadratic variation process $[r, r]_t$, we have a.s. (P) for all $0 \leq t \leq T$,*

$$r(t, t+h) = p(t+h) - p(t) = \int_0^h u_{t+s} ds + \int_0^h \sigma_{t+s} dW(s) \tag{2.10}$$

where u_s denotes an integrable predictable nx1 dimensional vector, $\sigma_s = (\sigma_{(i,j),s})_{i,j=1,\ldots,n}$ is a $n \times n$ matrix, $W(s)$ is a $n \times 1$ dimensional standard Brownian motion, integration of a matrix or vector w.r.t. a scalar denotes component-wise integration, so that

$$\int_0^h \sigma_{t+s} ds = \left(\int_0^h \sigma_{1,t+s} ds, ..., \int_0^h \sigma_{n,t+s} ds \right)' \tag{2.11}$$

and integration of a matrix w.r.t. a vector denotes component-wise integration of the associated vector, so that

$$\int_0^h \sigma_{t+s} dW(s) = \left(\int_0^h \sum_{j=1,\ldots,n} \sigma_{(1,j),t+s} dW_j(s), ..., \int_0^h \sum_{j=1,\ldots,n} \sigma_{(n,j),t+s} dW_j(s) \right) \tag{2.12}$$

Moreover, we have

$$P\left[\int_0^h (\sigma_{(i,j),t+s})^2 ds < \infty\right] = 1, 1 \leq i, j \leq n. \tag{2.13}$$

Finally, letting $\Omega_s = \sigma_s \sigma_s'$*, the increments to the quadratic return variation process take the form*

$$[r,r]_{t+h} - [r,r]_t = \int_0^h \Omega_{t+s} ds. \tag{2.14}$$

The Martingale representation theorem states that under the conditions specified in Proposition 2.5, there is a a predictable process σ_{t+s}. Therefore, we may now infer some results about return distribution.

Proposition 2.6. *For any n-dimensional square-integrable arbitrage-free price process with continuous sample paths satisfying Proposition 5 and thus representation (2.10) with conditional mean and volatility process* u_s *and* σ_s *independent of the innovation process* $W(s)$ *over* $[t, t+h]$*, we have*

$$\frac{r(t,t+h)}{\sigma(u_{t+s}, \sigma_{t+s})_{s\in[0,h]}} \sim N\left(\int_0^h u_{t+s} ds, \int_0^h \Omega_{t+s} ds\right) \tag{2.15}$$

where $\sigma(u_{t+s}, \sigma_{t+s})_{s\in[0,h]}$ *denotes* σ*-field generated by* $(u_{t+s}, \sigma_{t+s})_{s\in[0,h]}$.

Proposition 2.6 shows distributional characteristics of the return process conditional on the ex-post sample-path realization for u_{t+s} and σ_{t+s}. At first sight, the result might seem to be of little practical relevance as the realizations are usually not observable. However, thanks to Proposition 2.2 which suggests how to approximate quadratic variation and thanks to equation (2.14) which specifies difference in quadratic variation relating to two time points, we can approximate Ω_{t+s} from equation (2.15) to obtain the values. Therefore, we can conclude that daily returns are normally distributed with the daily realized quadratic return variation being the major determinant of the distribution.

To sum up, the properties of price process as a special semimartingale are summarized in Proposition 2.1 and the associated quadratic variation in Proposition 2.2. In addition, property (i) and equation (2.4) in Proposition 2.2 suggests a formula on approximation of the quadratic variation. Proposition 2.3 and Proposition 2.4 uncover the relationship between quadratic variation process and return volatility process, concluding that under some reasonable assumptions the return volatility process is driven by the expectation of difference in quadratic variation processes related to two moments of time. Finally, for the continuous sample path case, we derive the distributional characteristics of return, as described in Proposition 2.6. The quadratic return variation reduces to $\int_0^h \Omega_{t+s} ds$, which is often referred to as the integrated volatility.

2.2 HAR-RV models

Corsi (2003) proposes in his study a new approach to volatility modeling and forecasting. He is inspired by the Heterogenous Market Hypothesis developed by Müller *et al.* (1997) in which each group of market participants follows their own investment strategy. This approach also corresponds with another view of Lux & Marchesi (1999) in Interacting Agent View. The idea of various volatility components was also studied by Andersen & Bollerslev (1997).

The aim of the heterogenous market hypothesis is to explain the strong positive causality between volatility and market participation, as proved by empirical investigations. However, if homogenous setting of a market is assumed, i.e. all market agents are identical, then the more agents, the faster the price convergence to its real value and volatility diminishes. Therefore, there should be a negative correlation between volatility and the amount of market participants which contradicts the empirical results. On the other hand, under heterogenous market setting where each trader follows his own trading strategy, volatility increases because market participants react differently to market news.

The heterogeneity usually stems from various characteristics such as volume of financial sources, available information, prior beliefs, institutional constraints, geographical location, risk profile etc. In the following analysis we focus on the differences in time horizons, so that market is created by participants with various trading frequency. Speculative traders react to new information on a daily basis or even faster, while pension fund managers adjust the portfolio weekly and central banks can remain in a position for a month. All these agents have different reaction paths and thus create different types of volatility. In the HAR-RV model, 3 types of agents are present: short-term representing daily trading, medium-term representing weekly trading and long-term standing for monthly trading.

It is also important to explain the link between quadratic variation theory and heterogenous autoregressive models. First, the theory suggests that realized volatility can be estimated in probability for time intervals approaching zero. Second, it states that realized volatility is the major factor in determining conditional return covariance and finally, under the condition of purely continuous processes, suggests that returns are approximately normally distributed with integrated volatility having the highest impact on the shape of the distribution.

The following subchapters describe the HAR-RV model and their extenstions, as suggested by Corsi (2003), Corsi *et al.* (2005), Corsi *et al.* (2008) and Andersen *et al.* (2007), and pick up the models which we estimate.

2.2.1 HARV-RV model

The standard HAR-RV model proposed by Corsi (2004) consideres the following stochastic volatility process:

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) \tag{2.16}$$

where $p(t)$ is a logarithmic price process, $\mu(t)$ is a continuous finite variation process, $dW(t)$ is a Brownian motion and $\sigma(t)$ is a stochastic process independent of $dW(t)$. We also define integrated volatility related to day t as an integral of the stochastic volatility process over a whole day $1d$.

$$\sigma_t^{(d)} = \left(\int_{t-1d}^{t} \sigma^2(\omega)d\omega \right)^{1/2} \tag{2.17}$$

The theory behind the relationship was first developed in Back (1991) and later generalized for a class of finite semi-martingales in Andersen *et al.* (2003) described in the previous subsection. It states, that the integrated volatility of a Brownian motion can be approximated by a sum of intra-day squared returns, thus allows us to construct an error free estimate of the actual volatility, i.e. realized volatility, defined below.

$$RV_t^{(d)} = \left(\sum_{j=0}^{M-1} r_{t-j\delta}^2 \right)^{1/2} \tag{2.18}$$

where $\delta = 1d/M$ is the number of observations during one day and $r_{t-j\delta} = p(t-j\delta) - p(t-(j+1)\delta)$ defines the intra-day return of the price process for the sampling frequency δ. Under these conditions, the realized volatility becomes an unbiased volatility estimator. When constructing the estimator, only two parameters have to be specified. First, the sampling frequency δ, which depends on the data, second, the aggregation period $1d$. In the equation, a daily volatility was defined and is used as an input variable into weekly and monthly volatility estimators defined as:

$$RV_t^{(w)} = \frac{1}{5}\left(RV_{t-1d}^{(d)} + RV_{t-2d}^{(d)} + RV_{t-3d}^{(d)} + RV_{t-4d}^{(d)} + RV_{t-5d}^{(d)}\right) \tag{2.19}$$

for the weekly volatility which takes the last 5 working days into consideration. Monthly volatility estimator is defined in the same way using the last 22 observations.

In the next step, the term partial volatility is introduced. Every market participant is assigned a volatility component which only he or she creates. The unobserved partial volatility related to one day measures is denoted $\tilde{\sigma}_t^{(d)}$ etc. Moreover, it is assumed, that the market volatility is given by the component representing the highest trading frequency, i.e. one day. Thus $\tilde{\sigma}_t^{(d)} = \sigma_t^{(d)}$.

In the model, it is assumed that the unobservable partial volatility depends on its observable past realization RV_t and on the expectation of the future partial volatility of longer measure. However, the monthly partial volatility is determined by its historical realizations only. This assumption stems from the empirical observation that market agents trading on high frequency are influenced by their expectation of volatility created by agents trading at lower frequency. The three equations are set as follows:

$$\tilde{\sigma}_{t+1m}^{(m)} = c^{(m)} + \phi^{(m)} RV_t^{(m)} + \tilde{\omega}_{t+1m}^{(m)} \tag{2.20}$$

$$\tilde{\sigma}_{t+1w}^{(w)} = c^{(w)} + \phi^{(w)} RV_t^{(w)} + \gamma^{(w)} E_t[\tilde{\sigma}_{t+1m}^{(m)}] + \tilde{\omega}_{t+1w}^{(w)} \tag{2.21}$$

$$\tilde{\sigma}_{t+1d}^{(d)} = c^{(d)} + \phi^{(d)} RV_t^{(d)} + \gamma^{(d)} E_t[\tilde{\sigma}_{t+1w}^{(w)}] + \tilde{\omega}_{t+1d}^{(d)} \tag{2.22}$$

Substituting equation (2.20) into equation (2.21) and further into (2.22) and using the fact that daily partial volatility is equal to the daily volatility, we obtain:

$$\sigma_{t+1wd}^{(d)} = c + \beta^{(d)} RV_t^{(d)} + \beta^{(w)} RV_t^{(w)} + \beta^{(m)} RV_t^{(m)} + \tilde{\omega}_{t+1d}^{(d)} \tag{2.23}$$

Moreover, an assumption that the daily volatility is equal to the realized daily volatility plus an innovation term is added.

$$\sigma^{(d)}_{t+1wd} = RV^{(d)}_{t+1d} + \omega^{(d)}_{t+1d} \tag{2.24}$$

The innovation term $\omega^{(d)}_{t+1d}$ enters the equation because in practice not all assumptions are satisfied and a measurement error can occur mainly due to data properties, especially microstructure effects. Combining equation (2.23) and (2.24), we arrive at

$$RV^{(d)}_{t+1d} = c + \beta^{(d)} RV^{(d)}_t + \beta^{(w)} RV^{(w)}_t + \beta^{(m)} RV^{(m)}_t + \omega_{t+1d} \tag{2.25}$$

where $\omega_{t+1d} = \tilde{\omega}^{(d)}_{t+1d} - \omega^{(d)}_{t+1d}$ is considered as a disturbance term in the regression.

To sum up, equation (2.25) introduces an AR-type model of realized volatility whose inputs represent different market participants and which can be estimated using historical observations. We call this model HAR-RV and estimate it under the same name.

2.2.2 HARV-RV-GARCH model

Corsi (2003) consideres the residuals to be Gaussian and independently identically distributed. However, analyses of residuals of the HAR-RV models applied on various financial data have shown that they do not behave as was initially assumed. The empirical results point to the iid and gaussianity violation and also confirm volatility clustering. As a result, Corsi *et al.* (2005) try to account for these properties that not only distort the HAR-RV model estimation, but also diminish its forecasting accuracy. Their work presents a new version of HAR-RV model in which they give residuals more flexibility through more flexible normal inverse Gaussian distribution. They also specify a garch process to account for volatility clustering in the squared residuals.

Corsi *et al.* (2005) suggest to perform a GARCH-LM test to justify the following extension of the HAR-RV model by the GARCH(q,p) component.

$$RV^{(d)}_{t+1d} = \beta_0 + \beta^{(d)} RV^{(d)}_t + \beta^{(w)} RV^{(w)}_t + \beta^{(m)} RV^{(m)}_t + h_t u_t \tag{2.26}$$

$$h^2_t = \omega + \sum_{j=1}^{q} \alpha_j u^2_{t-j} + \sum_{j=1}^{p} \beta_j h^2_{t-j} \tag{2.27}$$

$$u_t|\Omega_{t-1}(0,1) \tag{2.28}$$

In this version, the error term $h_t u_t$ follows a conditional density with time-varying variance.

Corsi *et al.* (2005) test the performance of four models: the initial HAR-RV model, its extension for GARCH component, another extension for residuals distribution and finally both extensions together. They find out that the prediction ability improves in every case. This model is estimated under the name HAR-RV-GARCH.

2.2.3 HARV-RV-J models

Thus far, the approaches have been built on the assumption that a price process exhibits only continuous sample-path. However, as suggested by many parametric studies, the practical properties of high frequency data often indicate jumps in volatility, thus breaking the assumption. Results of the studies reveal that many price processes are best described by combination of a smooth and very slowly mean-reverting continuous sample path process and a much less persistent jump component. Introducing jumps into the parametric models have shown to improve the predictive performance and therefore created incentive for non-parametric realized volatility model to implement jumps, too.

The main idea of the HAR-RV-J type of models lies in the decomposition of the sample path into its continuous and discontinuous parts and in using the jump components as inputs into the model. This investigation is encouraged by new theoretical results of Barndorff-Nielsen & Shephard (2004) and Barndorff-Nielsen & Shephard (2006) who define so-called bi-power variation. The new technique was first employed by Andersen *et al.* (2007) and later in Corsi *et al.* (2008).

This section sets forth to describe the separation procedure and its implementation in HAR-RV models.

Following Andersen *et al.* (2007), equation (2.16) transforms to

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) + \kappa(t)dq(t) \tag{2.29}$$

where $0 \leq t \leq T$, $\mu(t)$ is a continuous and locally bounded variation process, $\sigma(t)$ is a strictly possitive stochastic volatility process with a sample path that is right continuous and has well defined left limits, $W(t)$ is a standard Brownian motion and $q(t)$ is a counting process with a time-varying intensity $\lambda(t)$, and $\kappa(t)$ refers to the size of corresponding discrete jumps in the price process.

Therefore, the quadratic variation associated with this return process is

$$[r,r]_t = \int_0^t \sigma^2(s)ds + \sum_{0<s\leq t} \kappa^2(s) \tag{2.30}$$

Having Δ sampling frequency and defining the Δ-period return process as $r_{t,\Delta} = p(t) - p(t-\Delta)$, the realized volatility can be computed in the following way:

$$RV_{t+1}(\Delta) = \sum_{j=1}^{1/\Delta} r^2_{t+j\Delta,\Delta} \tag{2.31}$$

From the quadratic variation theory it follows that realized variance converges in probability to the increment of the quadratic variation process as the sampling frequency increases, i.e.

$$RV_{t+1}(\Delta) \rightarrow \int_t^{t+1} \sigma^2(s)ds + \sum_{t<s\leq t+1} \kappa^2(s) \tag{2.32}$$

for $\Delta \rightarrow 0$. The equation suggests that in the absence of jumps, realized volatility converges in probability to the quadratic variation. Barndorff-Nielsen & Shephard (2004) define the standardized realized bi-power variation as follows:

$$BV_{t+1}(\Delta) = \mu_1^{-2} \sum_{j=2}^{1/\Delta} |r_{t+j\Delta,\Delta}||r_{t+(j-1)\Delta,\Delta}| \tag{2.33}$$

where μ_1 denotes the mean of the absolute value of standard normally distributed random variable Z[1]. Then, it can be shown that as the sampling frequency increases to infinity, i.e. as the intervals go to zero,

$$BV_{t+1}(\Delta) \rightarrow \int_t^{t+1} \sigma^2(s)ds \tag{2.34}$$

Hence, subtracting equation (2.34) from equation (2.32), we arrive at an expression that allows us to consistently estimate the jumps in the price process, given that $\Delta \rightarrow 0$.

$$\begin{aligned} RV_{t+1}(\Delta) - BV_{t+1}(\Delta) &= \textstyle\int_t^{t+1} \sigma^2(s)ds + \sum_{t<s\leq t+1} \kappa^2(s) - \int_t^{t+1} \sigma^2(s)ds \\ &= \textstyle\sum_{t<s\leq t+1} \kappa^2(s) \end{aligned} \tag{2.35}$$

[1]The exact value of the mean is 0.7979, which we use in the calculations.

However, this theoretical approach cannot prevent the empirical figures of jumps from being negative. The following definition ensures positiveness of jumps.

$$J_{t+1}(\Delta) = max[RV_{t+1}(\Delta) - BV_{t+1}(\Delta), 0] \tag{2.36}$$

In order to split the process into the continuous and discontinuous part, Andersen *et al.* (2007) use Z statistics and compare it with a critical value. The Z statistics is defined as follows:

$$Z_{t+1}(\Delta) = \Delta^{-1/2} \frac{[RV_{t+1}(\Delta) - BV_{t+1}(\Delta)]RV_{t+1}(\Delta)^{-1}}{(\mu_1^{-4} + 2\mu_1^{-2} - 5)max[1, TQ_{t+1}(\Delta)BV_{t+1}(\Delta)^{-2}]^{1/2}} \tag{2.37}$$

where TQ denotes the standardized tri-power quarticity measure which can be consistently estimated even in the presence of jumps.

$$TQ_{t+1}(\Delta) = \Delta^{-1}\mu_{4/3}^{-3} \sum_{j=3}^{1/\Delta} |r_{t+j\Delta,\Delta}|^{4/3}|r_{t+(j-1)\Delta,\Delta}|^{4/3}|r_{t+(j-2)\Delta,\Delta}|^{4/3} \tag{2.38}$$

To decide whether a jump is statistically significant, indicator function I is employed. If the Z statistics exceeds a critical value of Φ_α, which depends on a selected significance level α, the function I yields 1, otherwise zero. Hence, jumps are identified in the following way:

$$J_{t+1,\alpha}(\Delta) = I[Z_{t+1}(\Delta) > \Phi_\alpha][RV_{t+1}(\Delta) - BV_{t+1}(\Delta)]^+ \tag{2.39}$$

After specifying the jump component, Andersen *et al.* (2007) estimate the HAR-RV-J model of these forms:

$$RV_{t+1d}^{(d)} = \beta_0 + \beta^{(d)}RV_t^{(d)} + \beta^{(w)}RV_t^{(w)} + \beta^{(m)}RV_t^{(m)} + \beta^{(j)}J_t + \epsilon_t \tag{2.40}$$

and

$$RV_{t+1d}^{(d)} = \beta_0 + \beta^{(cd)}C_t^{(d)} + \beta^{(cw)}C_t^{(w)} + \beta^{(cm)}C_t^{(m)} + \beta^{(jd)}J_t^d + \beta^{(jw)}J_t^w + \beta^{(jm)}J_t^m + \epsilon_t \tag{2.41}$$

where

$$C_{t+1,\alpha}(\Delta) = I[Z_{t+1}(\Delta) \leq \Phi_\alpha]RV_{t+1}(\Delta) + I[Z_{t+1}(\Delta) > \Phi_\alpha]BV_{t+1}(\Delta) \tag{2.42}$$

and

$$J_t^w = \frac{1}{5}[J_{t-1}^d + J_{t-2}^d + J_{t-3}^d + J_{t-4}^d + J_{t-5}^d] \tag{2.43}$$

The term J_t^m is determined analogously. We call the model depicted in equation (2.40) with J specified in (2.36) as HAR-RV-J and model represented by equation (2.41) with only one jump regressor (daily) as HAR-RV-CJ1 and model fully represented by (2.41) we denote as HAR-RV-CJ3.

Following Corsi *et al.* (2008), the HAR-RV models incorporating jumps develop further. First, bi-power variation can be considered as a consistent estimator of integrated volatility as time interval between observations decreases to infinity, this does not hold for finite samples and therefore causes large underestimation of discontinuous component. Second, a concept of threshold multipower variation is introduced to overcome the difficulties in presence of jumps. Third, Corsi *et al.* (2008) also introduce an updated test for jump detection called $C-Tz$ statistics which is based on threshold multipower variation. The statistics has much more power than previously used z statistics, especially when applied to data consisting of jumps and when the jumps are consecutive. Finally, Corsi *et al.* (2008) apply the newly developed model on equity data and concludes, that jumps have positive and highly significant impact on future volatility.

Corsi *et al.* (2008) start with equations presented in (2.29)-(2.31). The multi-power variation suggested by Barndorff-Nielsen & Shephard (2006) is defined as

$$MPV_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]} = \delta^{1-\frac{1}{2}(\gamma_1+\ldots\gamma_M)} \sum_{j=M}^{[T/\delta]} \prod_{k=1}^{M} |\Delta_{j-k+1}X|^{\gamma_k} \tag{2.44}$$

Its purpose is to disentangle the continuous and jump part. Its behavior was studied in presence of jumps and in absence of jumps and the following can be inferred.

$$p - \lim_{\delta\to 0} MPV_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]} = \left(\prod_{k=1}^{M} \mu_{\gamma_k}\right) \int_t^{t+T} \sigma_s^{\gamma_1+\ldots\gamma_M} ds \tag{2.45}$$

where μ_γ is expectation of absolute value of standard normal variable to the power γ. In practice, the multi-power variation is used for estimation of integrated volatility and integrated quarticity. Given this framework, bi-power variation can be defined as follows:

$$BPV_\delta(X)_t = \mu_1^{-2} MPV_\delta(X)_t^{[1,1]} = \mu_1^{-2} \sum_{j=2}^{[T/\delta]} |\Delta_{j-1}X||\Delta_j X| \rightarrow_{\delta\to 0} \int_t^{t+T} \sigma_s^2 ds \tag{2.46}$$

However, to account for threshold, Corsi *et al.* (2008) adjust the equations as follows:

$$TBPV_\delta(X)_t = \mu_1^{-2} TMPV_\delta(X)_t^{[1,1]} = \mu_1^{-2} \sum_{j=2}^{[T/\delta]} |\Delta_{j-1}X||\Delta_j X| I_{[|\Delta_{j-1}X| \le \vartheta_{j-1}]} I_{[|\Delta_j X| \le \vartheta_j]} \tag{2.47}$$

where ϑ is the key part of threshold. It is based on an estimate of continuous local variance by means of an iterating smoothing filter which excludes jumps using the threshold function itself (Corsi *et al.* (2008), p. 7). It is constructed as follows:

$$\hat{V}_t^Z = \frac{\sum_{i=-L, i\neq -1,0,1}^{L} K\left(\frac{i}{L}\right) (\Delta_{t+i}X)^2 \, I_{[(\Delta_{t+i}X)^2 \le c_\vartheta^2 \hat{V}_t^{Z-1}]}}{\sum_{i=-L, i\neq -1,0,1}^{L} K\left(\frac{i}{L}\right) I_{[(\Delta_{t+i}X)^2 \le c_\vartheta^2 \hat{V}_t^{Z-1}]}} \tag{2.48}$$

The $\hat{V}_t^z$ statistics is a result of several iterations. Parameter c_ϑ is a dimensionless parameter used to scale the threshold, usually it is equal to 3. L denotes width of a time window for which the iterations are carried out, usually set equal to 25. The kernel function $K(.)$ is defined as

$$K(y) = \left(\frac{1}{\sqrt{2\pi}}\right) exp\left(\frac{-y^2}{2}\right) \tag{2.49}$$

The final value of the threshold is given by

$$\vartheta_t = c_\vartheta^2 \hat{V}_t^Z \tag{2.50}$$

To account for finite δ, a correction of indicator function is necessary, which yields

$$C - TMPV_\delta(X)_t^{[\gamma_1,\dots,\gamma_M]} = \delta^{1-\frac{1}{2}(\gamma_1+\dots\gamma_M)} \sum_{j=M}^{[T/\delta]} \prod_{k=1}^{M} Z_{\gamma_k}\left(\Delta_{j-k+1}X, \vartheta_{j-k+1}\right) \tag{2.51}$$

where function Z is defined as

$$Z = |x|^\gamma \tag{2.52}$$

if

$$x^2 \leq y \tag{2.53}$$

and

$$Z = \frac{1}{2N(-c_\vartheta)\sqrt{\pi}} \left(\frac{2}{c_\vartheta^2} y\right)^{\frac{\gamma}{2}} \Gamma\left(\frac{\gamma+1}{2}, \frac{c_\vartheta^2}{2}\right) \tag{2.54}$$

if

$$x^2 > y \tag{2.55}$$

Therefore, the corrected version of bi-power variation with incorporated threshold is given by

$$\begin{aligned} C - TBPV_\delta(X)_t \quad = \quad & \mu_1^{-2} TMPV_\delta(X)_t^{[1,1]} \\ & = \mu_1^{-2} \sum_{j=2}^{[T/\delta]} Z_1\left(\Delta X_j, \vartheta_j\right) Z_1\left(\Delta X_{j-1}, \vartheta_{j-1}\right) \end{aligned} \tag{2.56}$$

Finally, the test for jump detection which employs threshold value and corrections is defined as

$$C - Tz = \delta^{-\frac{1}{2}} \frac{\left(RV_\delta(X)_T - C - TBPV_\delta(X)_T\right) RV_\delta(X)_T^{-1}}{\sqrt{\vartheta max\left[1, \frac{C-CTTriPV_\delta(X)_T}{(C-TBPV_\delta(X)_T)^2}\right]}} \tag{2.57}$$

Given this new framework, Corsi *et al.* (2008) suggests to estimate HAR-RV model with threshold corrected jumps, therefore named HAR-RV-TCJ, as follows

$$RV_{t+1} = \beta_0 + \beta_d \hat{TC}_d + \beta_w \hat{TC}_w + \beta_m \hat{TC}_m + \beta_j \hat{TJ}_t + \varepsilon_t \tag{2.58}$$

where the jump component is defined as

$$\hat{TJ}_t = I_{t,J}\left(RV_t - TBPV_t\right)^+ \tag{2.59}$$

and the corresponding continuous part of the time series as

$$\hat{TC}_t = RV_t - \hat{TJ}_t \tag{2.60}$$

The continuous part is equal to realized volatility if there is no significant jump or to the threshold bi-power variation for days with significant jumps. We estimate the model under the same name.

Chapter 3

High Frequency Data

According to the theory, infinitely small time intervals should ensure a consistent and unbiased estimator of daily integrated volatility. However, empirical data differ largely from the theoretical arbitrage free continuous time process due to the presence of market microstructure effects. They cause the data to deviate from their optimal values. Corsi *et al.* (2001) suggest that under small time intervals, the unbiasedness of volatility estimator is induced by a systematic error. For example, for a foreign exchange market this error is positive and ranges from 30% at 1-minute intervals to 80% for tick-by-tick data.

On the other hand, the shorter time intervals, the higher number of observations per day and therefore the more effective the stochastic error of measurement. To span the gap between these characteristics leading to the opposite interval selection, two steps can be taken. First, to select such an interval that is not significantly influenced by microstructure effects while simultaneously ensuring a reasonable stochastic error. The second option is to eliminate the microstructure effects at the tick-by-tick level.

The first approach can be represented by Andersen *et al.* (2001) who explore this way of dealing with the bias. They work with foreign exchange market data and use 30-minute intervals for the most liquid currencies. Despite the relatively low frequency they obtain 48 observations per day as the liquid world currencies are traded 24 hours per day. The successive studies on which we build our analysis also employ this method: Corsi (2003), who uses 5-minute USD/CHF data; Corsi *et al.* (2005) employ 5-minute data of S&P500 index futures; Andersen *et al.* (2007) make use of 5-minute data of DM/$ exchange rate, S&P500 market index and 30-year US Treasury bond yield and finally Corsi *et al.* (2008) who apply 5-minute returns of S&P500 futures index, single stocks and US bond yields.

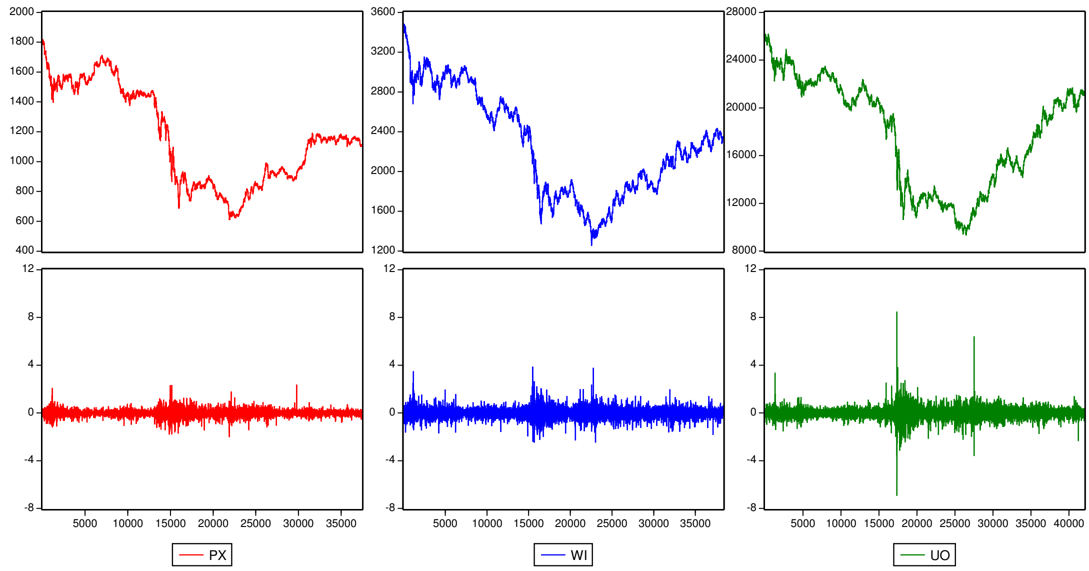
The second approach of data transformation to eliminate microstructure effects was suggested by Corsi *et al.* (2001). They identify the presence of the effects by scaling and autocorrelation analysis. The scaling analysis is based on the comparison of volatility expectations obtained from daily returns and returns taken from small time intervals. Results suggests that 1-minute return volatility can be 60% above the daily return volatility for foreign exchange markets, resp. 60% below the daily return volatility for stock market indices. Autocorrelation analysis reveals a very strong negative first-lag autocorrelation for FX markets, while stock market indices exhibit gradually decaying positive autocorrelation. The difference between both markets is explained by lagged adjustment model. It states that in the market there are leading stocks that are able to absorb new information very quickly and thus react immediately, while some other stocks react with a delay or react according to the behavior of the leading stocks. As a stock market index is comprised of both kinds of stocks, the autocorrelation is much more persistent. Given the different behavior patterns of FX and stock markets, the direct elimination of the bias is not generally applicable. The filter is based on exponential moving averages and for FX data yields nearly unbiased estimators of high precision. For stock markets data the application is much more complicated due to the presence of gradually decaying positive autocorrelation and is therefore not used in any study succeeding Corsi *et al.* (2001).

3.1 Data description

We estimate the HAR-RV models on three stock market indices: Prague, Budapest and Warsaw. All series cover a period from January 1, 2008 till November 30, 2009, thus focusing on the period of financial crisis. In total, there are 474, 475 and 485 trading days, respectively. In addition, for every trading day we have the information on its open and close price as well as the highest and the lowest, recorded at tick-by-tick, 1-minute, 5-minute, 10-minute, 30-minute, 60-minute and daily frequency.

The graphs below show index price and corresponding returns of the three markets covering the whole period. It can be seen that it was a period of high price and return volatility. Also, volatility clustering is apparent.

Figure 3.1: Index values and corresponding returns



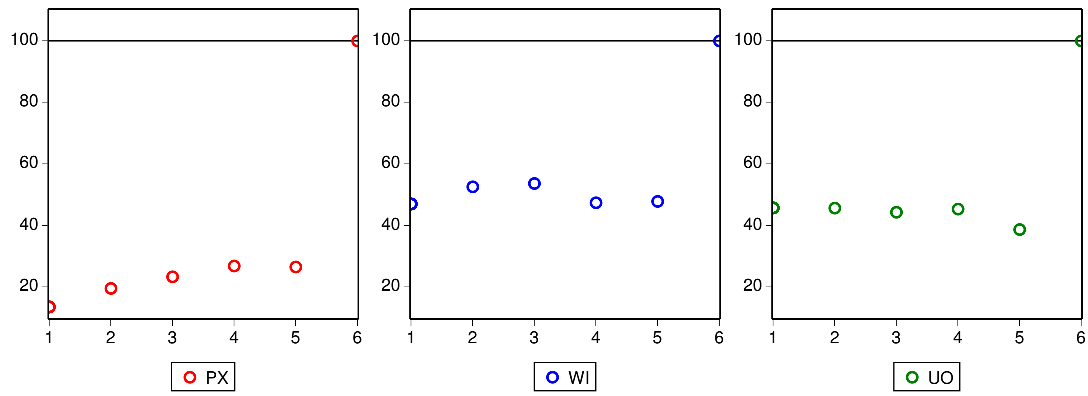
Source: Own calculations

According to the scaling analysis proposed by Corsi *et al.* (2001), which is performed in Figure 3.2, the biasedness of volatility obtained with intraday returns is almost the same for all time intervals and represents approximately 80% of daily volatility for PX index and 50% of daily volatility for UO and WI index. We refer to the deviation of daily volatility computed from intraday observations to observed daily volatility as a bias because it may alter the view of market participants. The agents consider daily volatility as a measure of one day risk and its different estimates may influence their investment strategies. However, as seen from the figure, the bias is the same for all time intervals, thus allows us to select the 5-minute interval for further computations as suggested in previous literature.

3.2 Construction of variables

The data were not subject to cleaning, as index price represents the whole market and it is comprised of the most liquid stocks. Therefore, it should not exhibit any sharp jumps or periods of low trading. However, overnight returns were removed from the construction of realized volatility, realized bi-power

Figure 3.2: Scaling analysis of annualized volatilities (in%)



Source: Own calculations

variation and realized tri-power variation and quarticity because they do not reflect any realistic trading.

The variable realized volatility was constructed using equation (2.48). In literature, three different computations of realized volatility exist. The first one appears in Corsi *et al.* (2001).

$$RV_t^{(d)} = \left(\frac{1}{M} \sum_{j=0}^{M-1} r_{t-j\delta}^2 \right)^{1/2} \tag{3.1}$$

The second type is used in Corsi (2003)

$$RV_t^{(d)} = \left(\sum_{j=0}^{M-1} r_{t-j\delta}^2 \right)^{1/2} \tag{3.2}$$

We use the last version defined in Corsi *et al.* (2008):

$$RV_t^{(d)} = \left(\sum_{j=0}^{M-1} r_{t-j\delta}^2 \right) \tag{3.3}$$

where r denotes logarithmic return of the variable close after overnight cleaning. It was annualized using the number of observations in 2008. For every time point recorded, logarithmic return was squared and then summed over one day. The variable bi-power variation was created in a similar way by summing two consecutive logarithmic returns over one day and multiplying it with μ_1, an expectation of absolute value of standard normally distributed variable[1]. In a similar way, we construct realized tri-power quarticity as a sum of three consecutive logarithmic returns in absolute value to the power of 4/3 adjusted by the presence of $\mu_{4/3}$ and sampling frequency[2]. Having defined the realized volatility and bi-power and tri-power variation, Z-statistics was constructed according to equation (2.37). If the Z statistics isgreater than a selected threshold corresponding to α, the difference between realized volatility and bi-power variation is called jump. This jump becomes an input into three models: HAR-RV-J, HAR-RV-CJ1 and HAR-RV-CJ3. The difference between HAR-RV-J and HAR-RV-CJ1 lies in its approach to the continuous part. The HAR-RV-CJ3 defines the continuous part as a difference between realized volatility and jump and employs weekly and monthly jump component, too.

To estimate the last model HAR-RV-TCJ, threshold bi-power variation was needed. It was constructed using equation (2.47), where the threshold was a multiple of $\hat{V}_t^z$ statistics and a dimensionless parameter c_ϑ to scale the threshold. A typical value of c_ϑ is 3 which we employed. The $\hat{V}_t^z$ statistics is a result of several iterations. First, infinity enters the equation as the initial value $\hat{V}_0^z$, thus no jumps are identified (the indicator function I is always 1) in a time window of 47 time points. The value of $\hat{V}_1^z$ is computed, again entering the equation. In this point, the value is much smaller thus making it possible that some jumps are identified. As soon as a jump is identified, the computation is repeated in one more step. If the jumps identified in the last two rounds are the same, the final value of $\hat{V}_t^z$ enters the threshold.

For high frequency data, 2 or 3 iterations are usually enough to constitute a threshold. In the case of Prague and Warsaw stock index, the same jumps were identified after 3 iterations. With Budapest stock market index the situation was much more difficult as first identical jumps were found after 51 interations, which is not typical of high frequency data. The reason might be a relatively low number of jumps present in the index in general.

The final value of $\hat{V}_t^z$ is multiplied by c_ϑ to create the threshold, which is used in computation of $TBPV$ in equation (2.47). A significant positive difference between realized volatility and threshold bi-power variation defines jump, a difference between realized volatility and jump identifies the continuous part.

The descriptive statistics of realized volatility and jumps estimates for the

[1] $\mu_1 \simeq 0.7979$

[2] $\mu_{4/3} \simeq 0.8309$

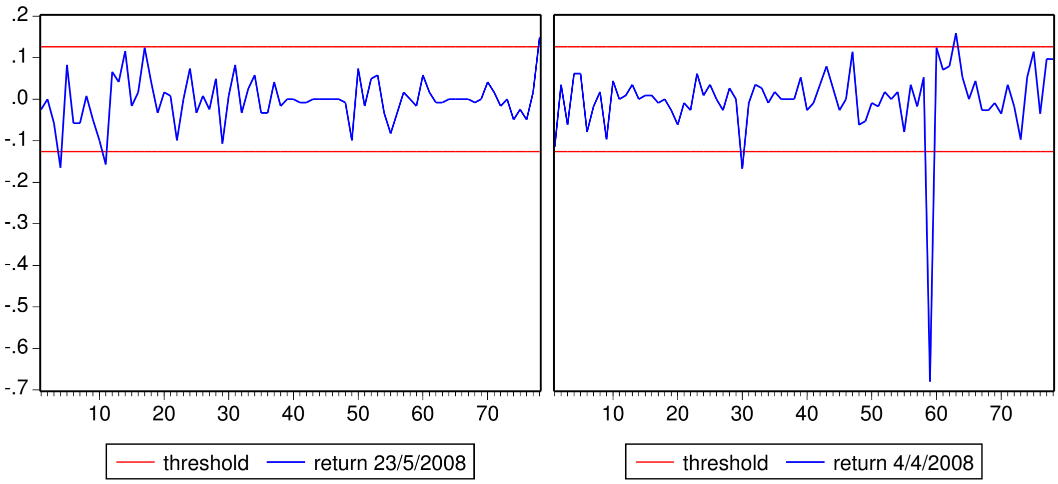
three indices can be found in table A.1 and A.2 in Appendix A. Realized volatility as well as its square root and logarithmic form of all three indices are not normally distributed, with the logarithmic version being the closest to it. All series exhibit strong serial correlation. This pattern is typical of stock market indices as suggested by lagged adjustment model. The Budapest index is the most volatile, followed by Warsaw and Prague. The threshold jump statistics TJ behaves similarly across indices and the null hypothesis of normal distribution is very strongly rejected. Median indicates that less than a half of the data exhibits jumps. Again, Budapest has the broadest distribution, followed by Warsaw and Prague.

3.3 Jump detection

There are two types of jumps. The first one was constructed according to Andersen *et al.* (2007) and comprises J from equation (2.36), $CJ1$ from equation (2.39) and $CJ3$ defined in (2.43). All of them are based on the use of Z statistics. Equation (2.37) describes its computation from realized volatility, bi-power and tri-power variation. The value of Z statistics is then compared with a normal quantile representing a significance level α. If, for the observation, the value is greater, there is a significant jump. Otherwise no jump is identified and the observation is classified as purely continuous.

The second option comes from Corsi *et al.* (2008), who suggest to use a corrected threshold z statistics $C-Tz$ defined in (2.57). The statistics employs corrected multi-power variation, threshold resulting from iterations in $\hat{V}_t^z$ and the frequency of time intervals. Therefore, it is able to reflect all main characteristics of the data. In practice, the time series is split into its continuous and jump part using the threshold $\vartheta_t = c_\vartheta^2 \hat{V}_t^z$. However, to assess the significance of the jump, the $C-Tz$ statistics is contrasted with a significance level represented by quantile of standard normal distribution. Figure 3.3 illustrates the jump detection process. The red line represents the threshold estimated for PX index, which has a value of 0.126. On May 23, 2008, the 5-minute logarithmic returns behave within the band specified by the threshold thus give no scope for the presence of a jump. This is confirmed by $C-Tz$ statistics whose value for this day is 0.001503, therefore rejecting presence of jump at any reasonable level. On the other hand, on April 4, 2008, there is an evident jump outside the threshold region. The $C-Tz$ statistics reaches a value of 3.727077 and confirms a significant jump at 99.9 % level.

Figure 3.3: Jump detection



Source: Own calculations

However, the crucial assumption behind the comparison of $C - Tz$ or Z statistics with standard normal quantile is that these statistics are normally distributed, too. If the condition is not satisfied, the significance of jumps and potential regression results should be taken with caution. Table 3.1 summarizes descriptive statistics of $C - Tz$ and Z, revealing that Budapest index might have above described difficulties. We should keep this fact in mind when assessing the results and performance of estimated HAR models.

Table 3.1: Descriptive statistics of C-Tz and Z statistics

	PX		UO		WI	
	$C - Tz$	Z	$C - Tz$	Z	$C - Tz$	Z
mean	2.145	0.066	0.801	0.031	1.049	0.056
median	2.180	0.068	0.746	0.037	1.037	0.055
max	5.467	0.383	3.223	0.294	3.131	0.308
min	-0.867	-0.216	-0.909	-0.457	-0.773	-0.227
st.dev.	1.212	0.093	0.671	0.089	0.693	0.088
skew	0.097	-0.038	0.508	-0.641	0.232	0.004
kurt	2.588	3.203	3.276	5.113	2.802	3.233
J-B	4.101	0.931	21.963	120.902	5.099	1.091
prob	0.129	0.628	0.000	0.000	0.078	0.580

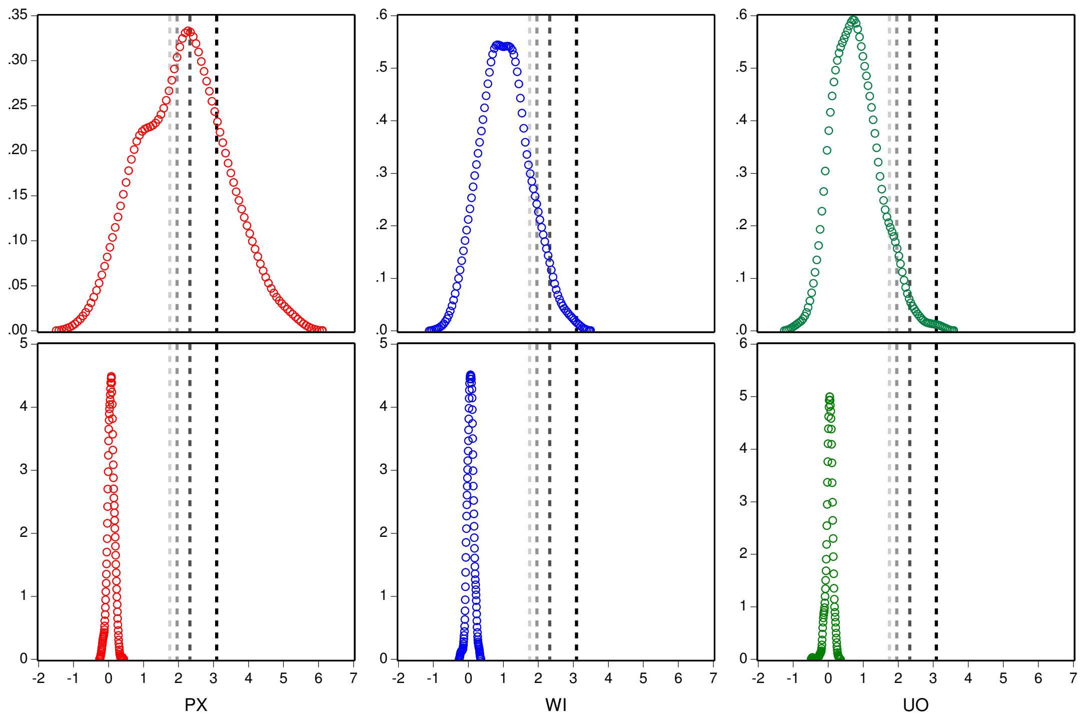
Source: Own calculations

In addition, Table 3.1 suggests that Z statistics in all indices, even normally distributed, exhibits very narrow range corresponding to approximately 60% or

65% significance of jumps detection at the best. From this reason, we exclude the Z statistics from further use and employ the $C-Tz$ statistics only even in the regressions where Z statistics was initially suggested.

Figure 3.4 illustrates the distribution of the statistics, where the four grey lines indicate α at 96%, 97.5%, 99% and 99.9% significance level. It is apparent that the significance level of 99.9% is almost unreachable for Budapest and Warsaw indices. Therefore, α used for Prague is expected to be higher than α used for Budapest and Warsaw.

Figure 3.4: C-Tz and Z statistics

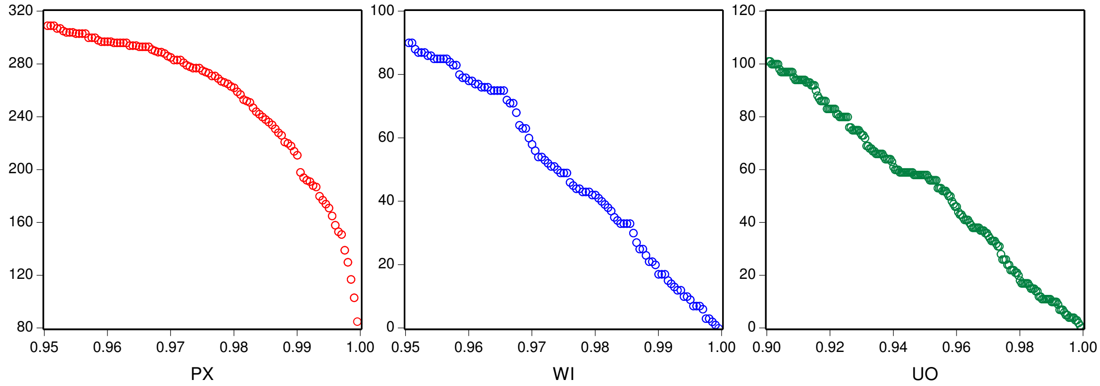


Source: Own calculations

We further look into sensitivity of jumps to the significance level α. Figure 3.5 shows the dependence between the significance level α and the number of jumps identified.

Prague stock market index reveals the highest number of jumps for all significance levels. As α increases, the number of jumps diminishes exponentially with large difference in jump number among the highest significance levels. For Warsaw stock market index, the relationship between α and the presence of jumps is linear and seems to be uniquely distributed. In Budapest, the

Figure 3.5: Importance of α in jump detection



Source: Own calculations

amount of jumps found is the lowest of all three indices so we decided to consider range from 90%. As in Warsaw, the relationship is also linear and almost uniquely distributed.

Chapter 4

Applications

The whole sample of 23 months was divided into in-sample period covering the first 20 months, i.e. from January 1, 2008 to August 31, 2009 and into out-of-sample period of the remaining three months, i.e. from September 1, 2009 to November 20, 2009.

In total, we estimate the following six models: HAR-RV from equation (2.25), HAR-RV-GARCH represented by equation (2.26) and (2.27), HAR-RV-J model as in (2.40) with J estimated by (2.36), then HAR-RV-CJ1 model estimated from (2.41) with J^d being the only jump regressor, HAR-RV-CJ3 model represented by (2.41) and finally HAR-RV-TCJ model depicted in (2.58). All HAR models were estimated on the in-sample data and the results were applied on the out-of-sample period to compare the forecasting accuracy of the models.

All models except for HAR-RV-GARCH were estimated by ordinary least squares. However, as HAR models can be classified as models with generated regressors, it leads to the covariance matrix of the disturbance term being non-spherical, with both non-zero off-diagonal and non-constant diagonal elements. Put another way, this kind of models generally produces heteroscedasticity and serial correlation in disturbance terms. To overcome this difficulty, we employ Newey-West correction which should ensure that the standard errors and therefore t-statistics and corresponding p-values are correctly estimated. The HAR-RV-GARCH model was estimated with maximum likelihood estimation with its garch(1,1) component which seemed to be the most appropriate, as indicated by LM test. Also, the use of garch(1,1) is suggested by Corsi *et al.* (2005).

The model performance is evaluated on the basis of Regression R^2, Root

Mean Square Errors (RMSE), Mean Absolute Error (MAE), Mean Absolute Percentage Error (MAPE), Theil's Inequality coefficient, Information Criterions and R^2 of Mincer-Zarnowitz regression.

$$RMSE = \sqrt{E((a - \hat{a})^2)} \tag{4.1}$$

$$MAE = E|a - \hat{a}| \tag{4.2}$$

$$MAPE = E|\frac{a - \hat{a}}{a}| \tag{4.3}$$

$$U = \frac{\sqrt{E((a - \hat{a})^2)}}{\sqrt{E(a^2)} + \sqrt{E(\hat{a}^2)}} \tag{4.4}$$

$$RV_t^d = b_0 + b_1 E_{t-1}[RV_t^d] + error \tag{4.5}$$

RMSE penalizes larger errors more than MAE, therefore RMSE tends to be higher than MAE. The greater difference between them, the greater the variance in the individual errors in the sample. MAPE is useful for comparison of accurancy among models as it is measure free. Theil Inequality Coefficient also measures a degree to which one time series differs from another. It is particularly useful for comparison of various models. The value ranges from 0 to 1 and the lowest, the better. Values close to 1 indicate that naive forecast is of similar accuracy. Mincer-Zarnowitz regression investigates the relationship between actual value of realized volatility and its forecasted value in time $t-1$. The higher the R^2, the higher the accuracy. We also look at regression information criterion, specifically at Akaike Information Criterion, Schwarz Information Criterion and Hannan-Quinn Information Criterion.

In order to better understand the model's accuracy after occurrence of a jump, we divide the sample into a part consisting of days immediately following a jump and into part consisting of days which follow days with no jumps. We calculate $RMSE$ for both parts and denote them as $RMSE_J$ for after-jump period and $RMSE_C$ for continuous period.

According to Corsi *et al.* (2008), we focus mostly on $RMSE$, Theil's Inequality Coefficient and R^2 of Mincer-Zarnowitz regression. The regression information criterions are regarded as additional information that should not shape out performance decisions.

Before progressing further, we find it important to summarize results obtained in previous studies.

First, Corsi (2003) estimates the classical HAR-RV model in its standard form with all parameters significant and the daily coefficient having the highest value and the daily coefficient the lowest.

Second, Corsi *et al.* (2005) estimate HAR-RV-GARCH model with a garch(1,1) component. They find out that all parameters are highly significant with the highest coefficient at weekly component, followed by daily and monthly component.

Third, Andersen *et al.* (2007) estimate HAR-RV-J model in its standard, logarithmic and square root form. They find β_d, β_w and β_m highly significant with daily coefficient being having the largest impact in all three versions. In all cases, the jump component is overwhelmingly significant but attains negative values, again in all three forms. Therefore, they conclude that the presence of jumps diminishes the predictive power of continuous variables so that the following day's prediction approaches zero.

Fourth, Corsi *et al.* (2008) comments on results obtained by Andersen *et al.* (2007) and explain the reason of negative jump coefficient. Realized variance is a sum of continuous and jump time series. If tomorrow's volatility is regressed on today's volatility and a jump component, the jump is already present in the today's volatility regressor, which makes the interpretation of the impact of jump nontrivial. Therefore, Corsi *et al.* (2008) suggest to separate the two processes and regress tomorrow's volatility on continuous and jump part independently. They also expects jump component to have positive influence on future volatility. Corsi *et al.* (2008) estimate HAR-RV-CJ1 model in three versions: standard, logarithmic and square root. Results reveal that all parameters are significant including the jump component, but its value is still negative.

Fifth, Andersen *et al.* (2007) estimate HAR-RV-CJ3 model which incorporates weekly and monthly jump components, compared to HAR-RV-CJ1. The model is built on the assumption that both continuous and dis-continuous regressors should have the same time structure. However, most of the jump coefficients are insignificant while the daily, monthly and weekly volatility components prove to be highly significant. This results confirms previous presumptions that only continuous processes can help predict future volatility.

Last, Corsi *et al.* (2008), driven by previous results, identify the difficulties in the role of jumps in incorrect separation of jump part from continuous part.

They suggest new model HAR-RV-TCJ. The estimation indicates significant role of all regressors with a positive impact of jumps on future volatility. They also find improved predictive power of the model compared to previous versions.

In light of the above described results, we expect the HAR-RV-TCJ model to perform most accurately in in-sample and out-of-sample period.

4.1 PX

4.1.1 Estimation

The HAR-RV models were estimated in standard, logarithmic and square root version with α equal to 99%. All results are reported in Appendix A in Table A.3 to Table A.5. The decision about value of α stems from the distribution of $C-Tz$ statistics indicating a sufficient amount of jumps. We have conducted regression analyses for α ranging from 96% to 99.9% and can conclude that the outcome are reasonably robust to the selected significance level. Figure A.1 in Appendix A gives more details.

The standard model suggests a significant impact of last day volatility on today's volatility which can be quantified in a range from 35% to 84%. Significance of weekly component is not confirmed in CJ-type of models, which, in turn, regard monthly component as important. The presence of jump plays role on daily basis and in two models only. J-model indicates a negative impact of a jump, while TCJ-model suggests a smaller and positive impact. As claimed by Corsi *et al.* (2008) we expect that jumps have highly significant and positive impact on a following day volatility. Almost all in-sample performance criterions favor garch-type of model, even though is has lower R^2 than the maximum obtained by J model.

The logarithmic version yields more stable results: last day volatility is significant for all models and ranges from 35% to 45%, volatility measured weekly is also significant for all models, its contribution to future daily volatility is equal to one fifth of the coefficient value which can be quantified between 50% and 68%. Volatility measured monthly does not play any role, as confirmed across the models. Last day jumps are significant for the same models as in the standard version, but the coefficient in J-model is much smaller and still negative. In-sample performance reveals preference towards TCJ-type of model, which scores the best in the three key measures. However, it does not provide any superior performance after a jump occurs.

In the square root form, J and CJ1-models do not have any significant regressors and therefore are excluded from further evaluation. For the remaining models, daily volatility component is highly significant and reaches a value around 40%, resp. 60% for TCJ model. The week part is also significant, with the coefficient value in TCJ-model being twice as large as in other models. Again, monthly volatility component does not contribute to volatility forecasting at all and jump is significant and positive for TCJ-model only. Inspecting the in-sample performance, TCJ model performs best in all aspects. It is also able to improve its accuracy in days immediately following a jump occurrence. Mincer-Zarnowitz regression indicates a very high level of accuracy.

By in-sample comparison of all three models on the key measures basis, the square root model of HAR-RV-TCJ yields the best performance. First, significance of parameters is unchanged across models and jumps are always positive with only small deviations in its coefficient value. Second, it reaches the highest scoring in $RMSE$ with a proven ability to improve after-jump accuracy, and lowest values in mean absolute percentage error and Theil's inequality coefficient.

4.1.2 Performance

The out-of-sample comparison reveals that in the standard version, HAR-GARCH is the most accurate predictive model. However, in the other two versions, HAR-TCJ clearly outperforms the remaining models and is the only model whose forecasting accuracy is even improved after a jump is present. However, despite the more than satisfactory results, we should keep in mind that there still exists a non-linear relationship among residuals that can not be rejected. The out-of-sample performance of all versions is summarized in Table A.12 in Appendix A.

4.2 UO

4.2.1 Estimation

HAR models applied on Budapest stock market index are estimated in standard, logarithmic and square root form with α equal to 96%. All regression results are summarized in Appendix A in Table A.6 for standard version to Table A.8 for square root version. The value of α was selected as a result of

$C - Tz$ statistics distribution, sensitivity of jumps to α and as a result of regression results sensitivity to α reported in Figure A.3 in Appendix A. As can be seen from the figure, for 99% significance level results are much more volatile than for 96% significance level around which the regression results seem to be stable.

Results of the standard HAR-RV estimation suggest that the major determinant of volatility one day ahead is the present daily volatility. Its impact is significantly higher for model incorporating jumps. Moreover, weekly volatility is not important for CJ-type of models and the relevance of volatility computed on monthly basis is completely rejected in all regressions. However, previous day jumps are found significant, but except for TCJ-model indicate negative relationship, which is not compatible with our expectations. Investigation of in-sample criteria does not provide any clear implications; RMSE is the lowest for TCJ model but reveals that it performs poorly after a jump, compared to other jump-type models. In total, TCJ behaves slightly better, but the overall assessment is not satisfactory.

The logarithmic version is much more explicit. β_d plays key role as it is significant across all models and the coefficient value is relatively stable, ranging from 50% to 64%. A volatility component assigned to one week is also important with its impact around 30% on weekly basis, while volatility measured on monthly basis does not contribute to model quality in general. Jumps present in the previous day are also significant, but again attain negative levels, except for TCJ model. The R^2 of regression is generally a lot higher than in the standard form, with only modest deviations across models. In-sample performance uniquely favors HAR-RV-TCJ model as it scores best in the three key criterions: RMSE, Theil's inequality coefficient and R^2 of Mincer-Zarnowitz regression. In addition, it also yields higher accuracy in days immediately following a jump.

From qualitative point of view, square root form does not differ from the logarithmic version. Daily volatility sustains its key role and its impact is approximately twice larger in models incorporating jump components, ranging from 65% to 75%. Weekly volatility also stays significant and keeps its stable level of coefficient values around 35% or 40%. Monthly volatility is not considered as determinant of daily volatility, while daily jumps indicate high significance. However, as in the logarithmic version, only TCJ model exhibits positive relationship. The characteristics are similar also in the remaining criterions: $RMSE$ in whole sample as well as in days following a jump, MAE,

$MAPE$, Theil's U and highest R^2 of Mincer-Zarnowitz regression.

Comparing the three versions, the logarithmic and the square root form perform very similarly and exhibit a lot better patterns than the standard version. By closer inspection of criterion values, one may favor the logarithmic version more as it has a half of RMSE compared to the square root form and a bit higher R^2 of Mincer-Zarnowitz regression. It is also important to note that in the square root model, residuals still include some linear and non-linear relationships, while in the logarithmic form only non-linear patterns are present. However, despite the pros and cons of both approaches, the final model is selected in the out-of-sample performance investigation.

4.2.2 Performance

The quality of standard model out-of-sample performance is uncomparable to the other two versions, as indicated by Table A.13 in Appendix A, and therefore we do not consider it in the following analysis. Results of the log version are very similar to the square root version and posses an important common feature: they cannot raise the forecasting accuracy in day immediately following a jump. This characteristics is very striking given the fact, that we have selected the HAR-RV-TCJ model as the most appropriate. The reason behind can lie in the non-normality of $C-Tz$ statistics which indicates the significance of jumps. As we mentioned by Table 3.1, rejecting the null hypothesis of normal distribution of $C-Tz$ statistics can have further consequences for regression results and model accuracy. From this reason, we select the model with lower error measurements - the square root model, which outperforms the logarithmic one in all out-of-sample criterions.

4.3 WI

4.3.1 Estimation

As by previous indices, also Warsaw stock market index is estimated in its standard, logarithmic a square root version with regression results reported in Appendix A in tables Table A.9 to Table A.11 and α being equal to 0.96. This value stems from the comparison of $C-Tz$ statistics values and quantiles of standard normal distribution representing significance level as depicted in

Figure 3.4. Also, regression results do not fluctuate a lot around this level, as indicated by Figure A.2 in Appendix A, which is not true for α around 99%.

The standard model of Warsaw stock market index provides different outcomes than other indices. First, daily volatility component is not significant for J- and CJ-type of models. In addition, the coefficient value fluctuates. CJ-models also reject the presence of weekly component as important regressor of the model. Coefficient values in the remaining regressions are also unstable. Surprisingly, CJ-models identify the monthly component as being highly significant. This statement is rejected by other models. Daily jump is important only in the TCJ model. Due to these confusing results, the in-sample performance is relatively poor with the best model being HAR-RV-GARCH.

In the logarithmic version, CJ-models identify the weekly volatility measure as the only significant explanatory variable. We do not regard this regression as satisfactory and therefore exclude it from further assessment. The remaining models find last day volatility highly important for the prediction of today's volatility with an impact around 35% or 40%. They also consider weekly volatility measure as being significantly important for determining present day volatility. The impact can be quantified at level closely below 50% on a weekly basis. Monthly measure of volatility can be ignored as it does not prove to be of a large relevance. Finally, jump is found significant for TCJ model only with a positive influence. It is this model which scores best in all in-sample criteria, giving scope for better accuracy in after-jump period.

The square root version gives similar results as the logarithmic version. CJ models significantly stand on weekly and monthly volatility component and we consider them as unsatisfactory. The remaining models pick daily and monthly volatility as key determinants of daily volatility one day ahead, with daily volatility ranging from 30% to 48%, resp. from 37% to 64% for weekly volatility measure. As in previous models, volatility measured on monthly basis does not contribute to the quality of the model. The only significant jump identified is incorporated in TCJ model. As far as in-sample performance is concerned, HAR-TCJ model reveals its strengths in $RMSE$ and its ability to raise performance accuracy in days following a jump. However, HAR-GARCH model slightly outperforms HAR-TCJ in R^2 of Mincer-Zarnowitz regression.

4.3.2 Performance

Turning to the out-of-sample model comparison, which is reported in Table A.14 in Appendix A, HAR-GARCH is the leading model in the standard version as it scores best in all key criteria. In the logarithmic version, HAR-CJ3 would be the most appropriate model if we did not exclude it due to unrealistic significance of regressors. Finally, HAR-TCJ performs best in square root version despite the fact that it exhibits a bit larger inaccuracies in period following a jump than in the whole sample. However, this characteristics is apparent in all models of Warsaw stock market index. There are two possible reasons. First, the non-normality of $C-Tz$ statistics as in Budapest stock market index. This idea stems from the descriptive statistics in Table 3.1 which indicates that non-normality is rejected only weakly. Second, no model is able to ensure this behavior in the out-of-sample period, thus it might be a natural characteristics of the time series.

4.4 Conclusion

The empirical analysis uncovers a lot of features that are common for all three stock market indices. Regressions show that the most appropriate and accurate model for prediction of future realized volatility is HAR-RV-TCJ model estimated in square root form.

In all cases, monthly volatility component is not a significant determinant of future volatility. This fact is implied by all other models, too, and contrasts with previous literature which finds monthly volatility component important. We suggest that this result corresponds clearly with the time frame in which it is analyzed, as it covers the period of financial crisis that hit also the CEE markets. Within this framework, the insignificant monthly component reveals an outflow of trading strategies in one month horizon, or put another way, outflow of market participants trading on a monthly basis. However, this fact might be a long term characteristics of CEE stock market indices, but it would be necessary to investigate it over a longer period.

Moreover, all indices find daily jumps in realized volatility highly influential for predicting realized volatility one day ahead. This variable has positive impact and can increase tomorrow's volatility by 20% or 30% in days immediately following a jump. In addition, the HAR-RV-TCJ model is able to improve the predictive power in these days in in-sample period. As times of financial turmoil are usually escorted by unexpected jumps in stock market indices, we regard this pattern as very interesting and of great relevance.

Finally, the fact that all three markets can be best described by the same model with very similar parameters indicates that these markets evolve very narrowly. To test this assumption is the task of the following chapter.

Chapter 5

Wavelet Analysis

Wavelet analysis represents a powerful tool of analyzing time series from two perspectives simultaneously - from frequency and time point of view. Put another way, wavelet transform expands time series into time-frequency space and is therefore able to capture localized intermittent periodicities. In general, there are two classes of wavelet transforms - a continuous denoted CWT and a discrete DWT. We deal with continuous version only as the aim is to extract time-frequency features. In addition, there are two extensions of CWT that enable to conduct analysis of two time series together. The first one, Cross Wavelet Transform (XWT), yields areas of common power of the two time series, and second, Wavelet Coherence (WTC) finds areas of significant coherence of the two time series using the information from XWT. It is Wavelet Coherence that stands at forefront of our interest.

The wavelet theory is primarily built on research work of Mallat (1989) and Daubechies (1990). An introductory overview can be found in Lau & Weng (1995) and foundations of statistical significance testing in wavelet analysis were first presented in Wang & Wang (1996). Since then, wavelet analysis has been applied in many areas. Initially, it was geophysics, climate and meteorology represented by works of Torrence & Compo (1998) and Grinsted *et al.* (2004) and many others. In the last decade, wavelets have been applied in economic issues, mainly to business cycles and levels of integration to Euro Area, as elaborated in Crowley & Lee (2005) and in Crowley *et al.* (2006). Lastly, an application on stock markets emerged with Rua & Nunes (2009), who discuss comovements of returns of stock market indices in Germany, US, UK and Japan. They conclude that there is a significant long term comovement of US, UK and German indices on a frequency around 8 years which spreads into higher frequencies of about 6 months in the last decade or two.

In light of the previous research in this topic, we would like to contribute to the existing literature by applying wavelet analysis on realized volatility. To our knowledge, this application has not been conducted yet and we therefore expect to obtain prime results in this field.

We first present the theory behind wavelet transform, explain the extensions to XWT and WTC, and then apply it on realized volatility data of stock market indices in Prague, Budapest and Warsaw.

5.1 Theoretical Background

The continuous wavelet transform splits an original time series into wavelets $\Psi_{\tau,s}(t)$, also called daughter wavelets. Wavelets can be imagined as waves whose magnitude depends on time period for a given frequency. These wavelets are products of mother wavelet $\Psi(t)$ which is dependent on time position τ (translation parameter) and frequency (scale parameter s). Wavelets are defined as follows:

$$\Psi_{\tau,s}(t) = \frac{1}{\sqrt{s}}\Psi\left(\frac{t-\tau}{s}\right) \tag{5.1}$$

where $\frac{1}{\sqrt{s}}$ denotes a normalization factor which ensures comparability of wavelets across time frequencies.

Also, mother wavelet $\Psi(t)$ has to satisfy three conditions. Firstly, its mean has to be zero.

$$\int_{-\infty}^{+\infty} \Psi(t)dt = 0 \tag{5.2}$$

Secondly, its square integrates to unity

$$\int_{-\infty}^{+\infty} \Psi^2(t)dt = 1 \tag{5.3}$$

Lastly, the so-called admissibility condition has to hold in order to ensure that the original time series $x(t)$ can be reconstructed from its continuous wavelet transform.

$$0 < C_\Psi = \int_{0}^{+\infty} \frac{|\hat{\Psi}(\omega)|^2}{\omega} d\omega < +\infty \tag{5.4}$$

where $\hat{\Psi}(\omega)$ is the Fourier transform of $\Psi(t)$.

The continuous wavelet transform of a time series $x(t)$ with respect to $\Psi(t)$

is given by

$$W_x(\tau, s) = \int_{-\infty}^{+\infty} x(t)\Psi^*_{\tau,s}(t)dt = \frac{1}{\sqrt{s}}\int_{-\infty}^{+\infty} x(t)\Psi^*\left(\frac{t-\tau}{s}\right)dt \tag{5.5}$$

where * denotes the complex conjugate. The wavelet power is defined as $|W_x(\tau, s)|^2$ and its values appear in the time-frequency space of CWT.

However, from practical reasons it is more convenient to conduct the wavelet transform in Fourier space, we use Morlet wavelet as mother wavelet. It is defined as follows:

$$\Psi(t) = \pi^{\frac{-1}{4}} e^{i\omega_0 t} e^{\frac{-t^2}{2}} \tag{5.6}$$

where the corresponding Fourier transform is given by

$$\hat{\Psi}(\omega) = \pi^{\frac{1}{4}}\sqrt{2}e^{\frac{-1}{2}(\omega-\omega_0)^2} \tag{5.7}$$

In practice, ω_0 is set to 6 as it provides a good balance between time and frequency localization.

The wavelet transform can be viewed as a set of band-pass filters applied to a time series consecutively with the wavelet scale linearly related to the characteristic period of the filter. However, CWT is not completely localized in time due to the increasing length of band for which the wavelet power is computed. Therefore, a cone of influence (COI) is introduced in which the wavelet power has dropped to e^{-2} of the value at the edge. COI also delimitates the space within which results are interpreted.

Cross wavelet transform (XWT) of two time series $x(t)$ and $y(t)$ is defined as $W_{xy}(\tau, s) = W_x(\tau, s)W_y^*(\tau, s)$ where * denotes complex conjugate. The cross wavelet power is defined as $|W_{xy}(\tau, s)|$. Cross wavelet transform can provide us with information about how two time series together behave in time and in frequency, with cross wavelet power being the resulting measure.

Finally, **Wavelet Coherence** (WTC) indicates a degree of coherence between two time series using the information from cross wavelet transform. Wavelet coherence is defined as

$$R_n^2(s) = \frac{|S(s^{-1}W_{xy}(s))|^2}{S(s^{-1}|W_x(s)|^2)S(s^{-1}|W_y(s)|^2)} \tag{5.8}$$

where S is a smoothing operator. It is apparent that the definition of $R_n^2(s)$ is close to the definition of correlation coefficient. Therefore, it is intuitive to consider wavelet coherence as a localized correlation coefficient within a time-frequency space. Likewise correlation coefficient, it yield values in range from 0 to 1, with 1 indicating a very strong comovement. To assess the significance of wavelet coherence, Monte Carlo simulation is used. It generates a large set of data set pairs with the same length as the input datasets and for each pair, wavelet coherence is calculated. The significance level for each scale is estimated using only out-of-COI values. However, empirical results show that the characteristics of smoothing operators play significant role.

5.2 Empirical Results

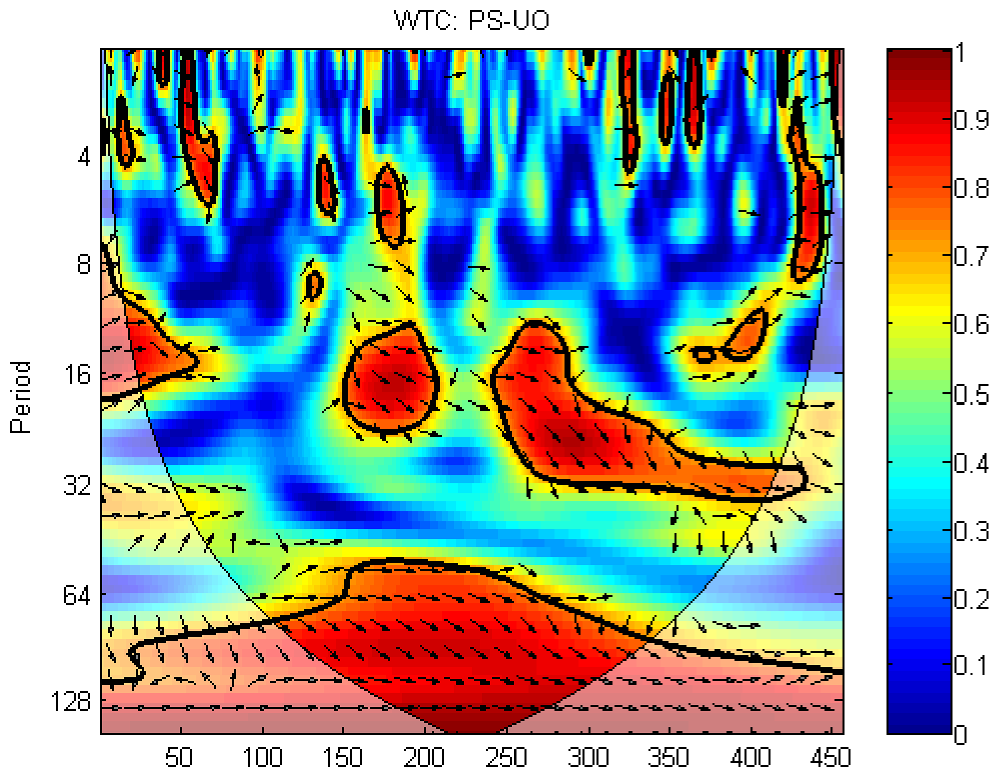
The aim of empirical analysis is to identify key periods and frequencies at which two stock market indices show high level of coherence. Therefore, we adjust the data set so that the observations match, i.e. for each date there is a value of all three indices. If this condition is not satisfied, the date is deleted. In total, 35 observations are removed leaving us with 455 for empirical part. The results are obtained with use of Matlab code written as a part of Grinsted *et al.* (2004) and made available at http://www.pol.ac.uk/home/research/waveletcoherence/.

The wavelet coherence is depicted in a contour plot and consists of three dimensions. The horizontal axis refers to time dimension covering period from January 1, 2008 to November 30, 2008, a total of 455 observations. The vertical axis stands for frequency in days and colors represent degree of comovement with red being the highest and blue the lowest. Areas of statistical significance at 5% level are bounded with thick black contour while thin black contour indicates the cone of influence COI.

When interpreting the results, it is important to distinguish between red areas at the bottom (top) showing comovements across all time in a low (high) frequency and red areas on the left (right) indicating a comovement across all frequencies but only at the beginning (end) of the time period examined.

5.2.1 PX - UO

Figure 5.1: Wavelet Coherence between PX and UO indices



Source: Own calculations

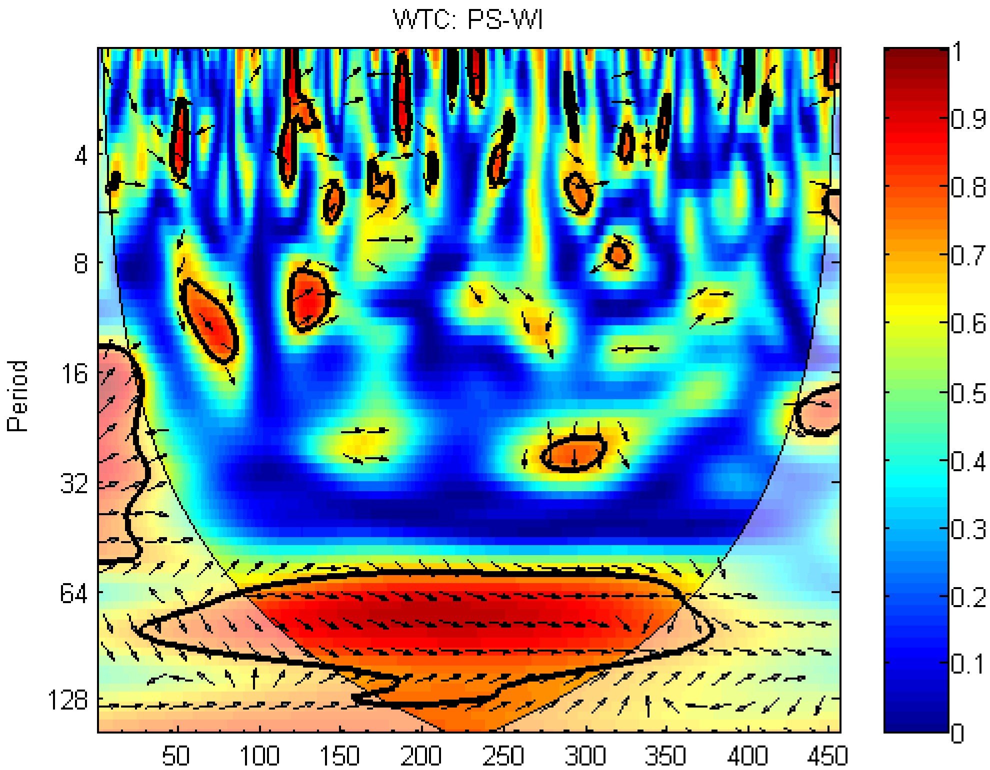
Starting with the wavelet coherence between Prague and Budapest stock index, we can see three significant areas of comovement. The first one occurs in frequency around 10 to 20 days and is relevant to a period from the middle of August 2008 till the end of October 2008. The second field starts at 10-day frequency which diminished in time to 20-day frequency with the endpoint in 30 days. It has a longer duration of almost 200 days covering the period of end January 2009 till the beginning of November 2009. Finally, the largest area is apparent at frequency of 50 days and lower across all influential period defined by COI. The degree of comovement reaches 90% and increases as frequency decreases.

The arrows indicate relationship between both indices. In most of the significant periods, arrows point to the right with an occasional slight downward direction. This indicates that both indices evolve in the same direction, i.e.

volatility increases (or decreases) in Prague and Budapest, too. However, the way of transmission is not explicitly clear as arrows pointing to the right do not constitute any leading and following index, while arrows pointing down suggest Prague to be the leading index. We could infer that in a 4-month horizon realized volatility of Prague index encourages realized volatility of Budapest index while in a longer horizon there is no particular way of influence.

5.2.2 PX - WI

Figure 5.2: Wavelet Coherence between PX and WI indices



Source: Own calculations

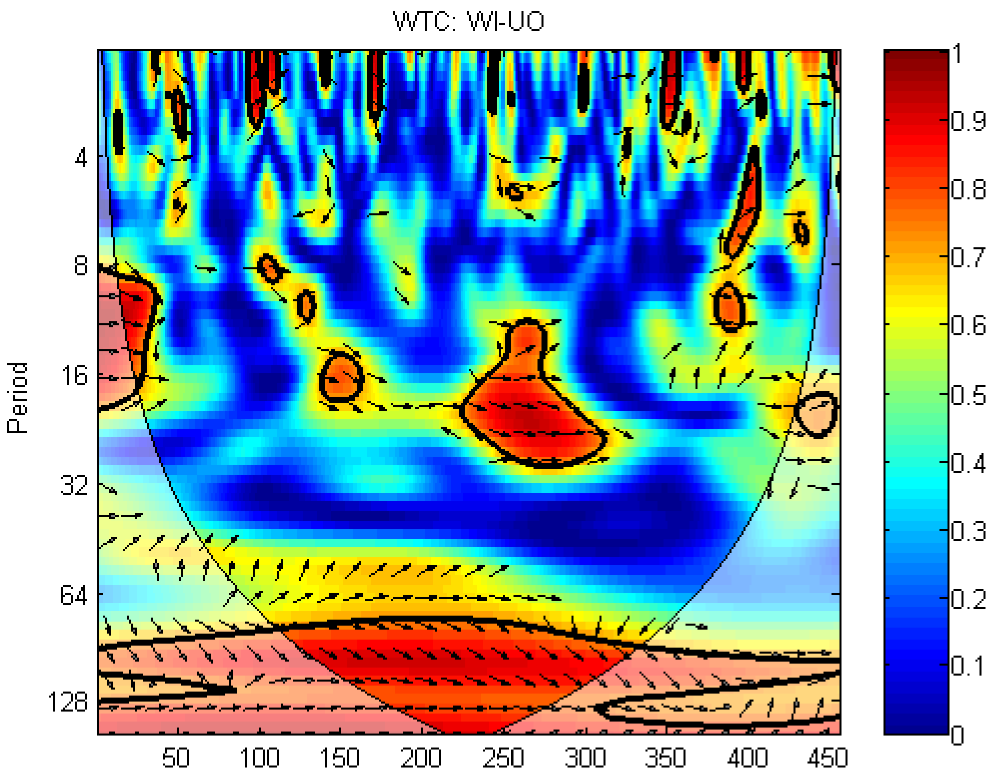
Examining the relationship between Prague and Warsaw stock market realized volatility, we can see that in a frequency band up to 60 days, there are several small islands of significant comovement. However, all of them cover a period of few dozens of days with a strength of about 70% or 80%. Also, as there are large areas of dark blue, we do not regard these islands of significance as important. In addition, each of them suggests a bit different relationship as seen from the arrows directions.

However, in a frequency band from 60 to 120 days there is a large field of significant comovements located across the whole time period defined by COI. Therefore, there is a very strong relationship between daily realized volatility of Prague and Warsaw stock market indices with peak located in 80 to 90-day frequency and reaching values above 90%.

As arrows indicate, the transmission channel is very unclear and therefore we are not able to say whether realized volatility of one stock market index is influenced by the other index. What can be inferred is that realized volatilities of both indices move in the same way.

5.2.3 UO - WI

Figure 5.3: Wavelet Coherence between UO and WI indices



Source: Own calculations

Turning to Budapest and Warsaw stock market indices, two areas of significant comovement can be identified. First, in a 10 to 30-day frequency region there is an area of almost 90% significance corresponding to period between December 2008 and May 2009. Second, as found in previous markets analyses, a band from 80 to 130-day frequency shows to be very powerful in the whole influential period given by COI. In these two areas, both indices indicate a high level of comovement.

Arrows directions uncover that realized volatilities of both stock market indices evolve in the same direction over time. In addition, a frequency of 100 days reveals that realized volatility of Budapest stock market index is a leading indicator for realized volatility of Warsaw stock market index. However, as frequency decreases, arrows tend to point more to the right than down and thus suggest an unclear way of determination. The latter relationship is also apparent in the first significant region identified. Therefore, we conclude that the relationship between realized volatilities of both indices is unclear in general with probable impacts of Warsaw stock market on Budapest stock market.

5.2.4 Conclusion

Wavelet transform has shown to be a useful tool for analyzing significant comovements of two time series. Its application on daily realized volatility of Prague, Budapest and Warsaw stock market indices shows that realized volatilities evolve highly similarly across time in a low frequency band starting at 60 days and lower. In general, no stock market index proves to be the leading indicator.

This result has broad implications for risk management. It suggests that 4-months investments into central european stock market indices are subject to very similar realized volatility.

Chapter 6

Conclusion

The thesis revolves around the topic of daily realized volatility of three stock market indices: Prague, Budapest and Warsaw. It first describes the quadratic variation theory which is a cornerstone of HAR-RV models and then turns to the application on high frequency data. It tries to explain behavior of three groups of market agents and to quantify the role of unexpectedly occuring jumps in predicting future realized volatility. The investigation is conducted on 2008 and 2009 data set, thus covering the period of financial crisis. The second part of the thesis is devoted to wavelet analysis, through which relationships between indices can be uncovered in a time-frequency space.

Results indicate that all three markets can be best described by the same type of model which in general reveals large similarity across the markets.

In addition, all models classify monthly volatility component as having zero impact on future daily volatility, a result that contrasts with previous studies. This can reflect unwillingness of market participants to trade in longer horizons, probably due to high degree of uncertainty at the markets. Prediction of future daily realized volatility is therefore driven by daily and weekly volatility component.

Moreover, all indices find daily jumps in realized volatility highly influential for predicting realized volatility one day ahead. Results also show that correct separation of jump part from continuous part in time series can increase model's forecasting accuracy in days immediately following a jump. As the impact is positive with relatively high coefficient, we regard this pattern as very interesting and of great relevance, especially in period of financial instability.

Results of wavelet transform also reveal very common features. Analyses of three pairs of stock market indices uniquely suggest that markets evolve in the same direction in a longer term of 60 days and more with no market being the leading determinant.

To sum up, HAR-RV models identify shorter term variables as driving future daily volatility, while wavelet transform suggests that realized volatility comovements of the three markets are the strongest in a horizon of few months.

We consider the empirical results to have large implications for risk management. The gain might be even doubled by the fact that this is the first empirical analysis of realized volatility applied on CEE stock market indices.

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Appendix A

Results of Regressions

Table A.1: Descriptive statistics of RV

	PX			UO			WI		
	RV	$RV^{1/2}$	$logRV$	RV	$RV^{1/2}$	$logRV$	RV	$RV^{1/2}$	$logRV$
mean	2.476	1.385	0.423	6.913	2.321	1.491	5.546	2.182	1.429
median	1.416	1.190	0.348	4.133	2.033	1.419	3.991	1.998	1.384
max	30.820	5.551	3.428	207.144	14.392	5.333	53.090	7.286	3.972
min	0.175	0.418	-1.744	0.586	0.766	-0.534	0.712	0.844	-0.340
st.dev.	3.421	0.748	0.922	12.377	1.237	0.830	5.848	0.888	0.700
skew	4.424	2.084	0.468	10.373	3.523	0.717	4.367	2.185	0.543
kurt	28.860	9.274	3.097	152.088	26.003	4.115	27.900	10.493	3.838
J-B	14754.3	1120.5	17.5	448433.2	11455.3	65.3	13928.6	1504.8	37.6
prob	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
obs	474	474	474	475	475	475	480	480	480
LB_{10}	1054.6	1667.7	1845.7	511.8	1667.7	1808.5	821.7	1221.1	1325.9

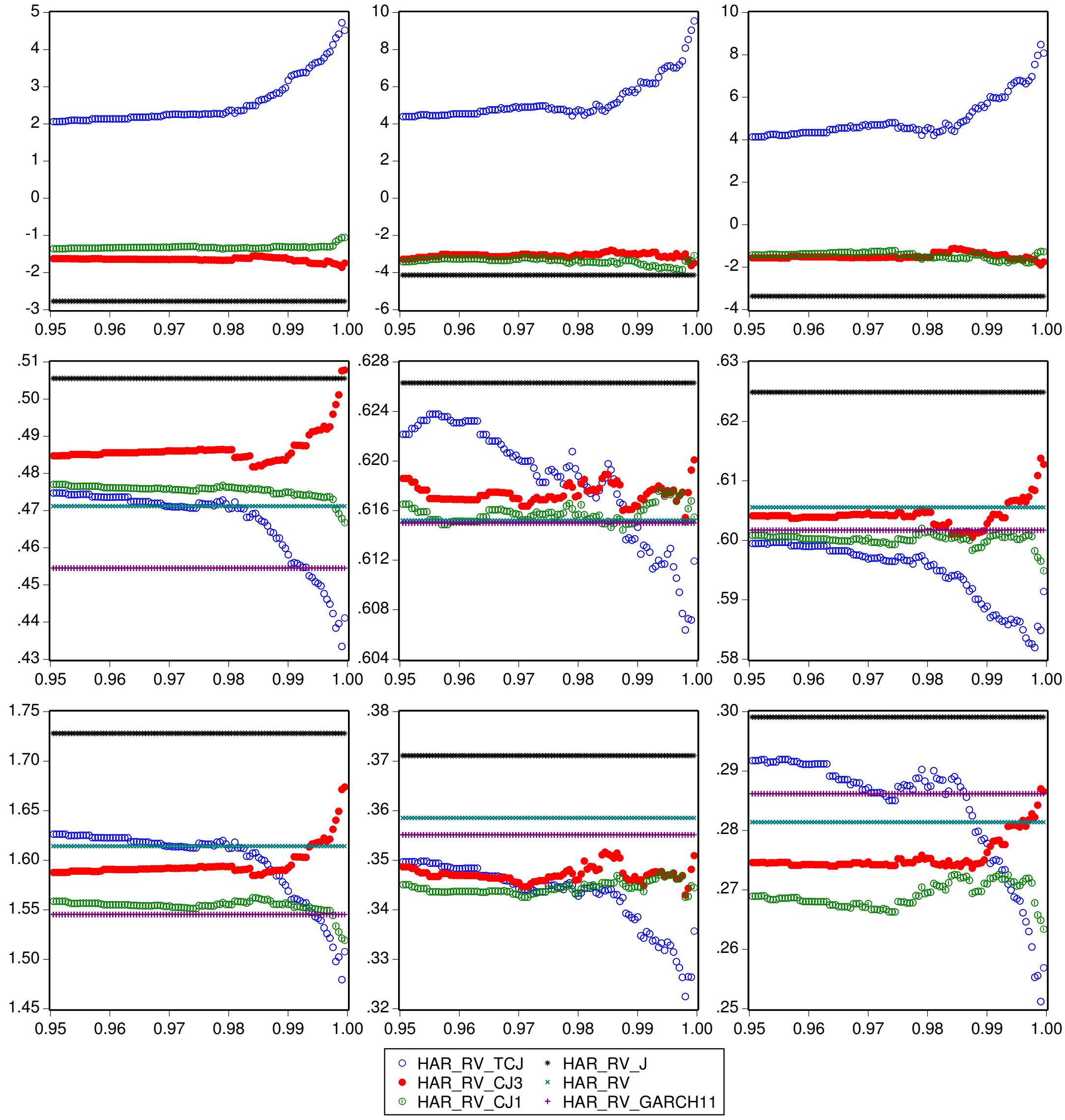
Source: Own calculations

Table A.2: Descriptive statistics of TJ

	PX			UO			WI		
	TJ	$TJ^{1/2}$	$logTJ$	TJ	$TJ^{1/2}$	$logTJ$	TJ	$TJ^{1/2}$	$logTJ$
mean	1.150	0.617	0.450	1.924	0.383	0.258	1.307	0.419	0.312
median	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
max	26.032	5.102	3.297	191.200	13.828	5.259	44.714	6.687	3.822
min	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
st.dev.	2.800	0.878	0.660	10.843	1.335	0.821	4.545	1.065	0.762
skew	5.146	1.806	1.661	12.625	4.681	3.193	5.608	2.890	2.420
kurt	36.617	7.146	5.588	203.633	32.440	12.569	40.393	11.921	8.054
J-B	24410.8	597.2	350.1	809303.2	18888.6	2619.2	30481.5	2260.0	979.5
prob	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
obs	474	474	474	475	475	475	480	480	480
LB_{10}	753.5	1103.3	1251.2	267.0	828.0	906.2	232.3	215.6	194.5

Source: Own calculations

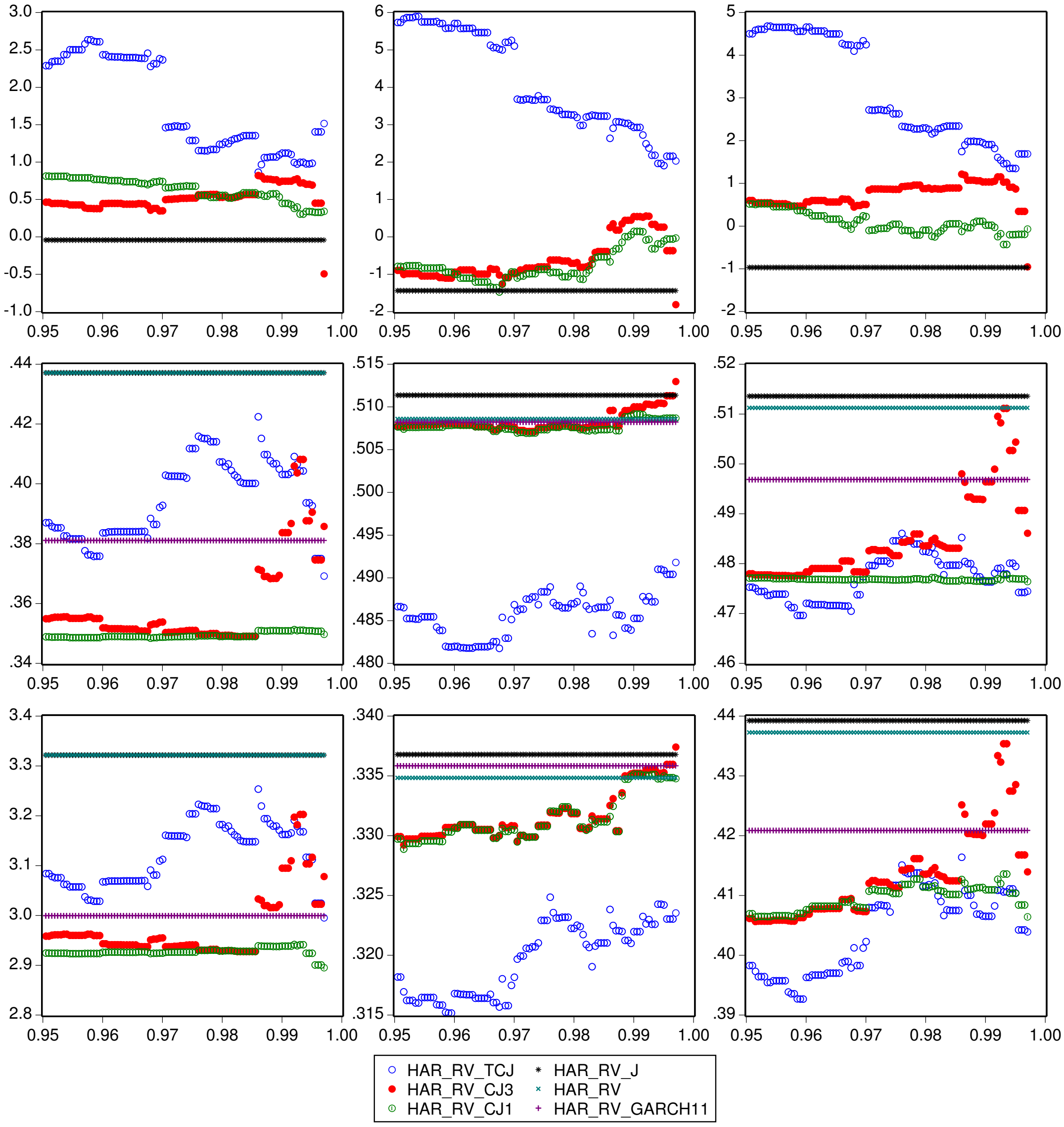
Figure A.1: PX sensitivity to α



Source: Own calculations

The first row shows sensitivity to alpha of β_j t-statistics, the second row indicates sensitivity to R^2 and the third row reveals sensitivity of α to RMSE. The first column reports results for standard HAR-RV model, while the second column show its logarithmic form and the last column the square root form.

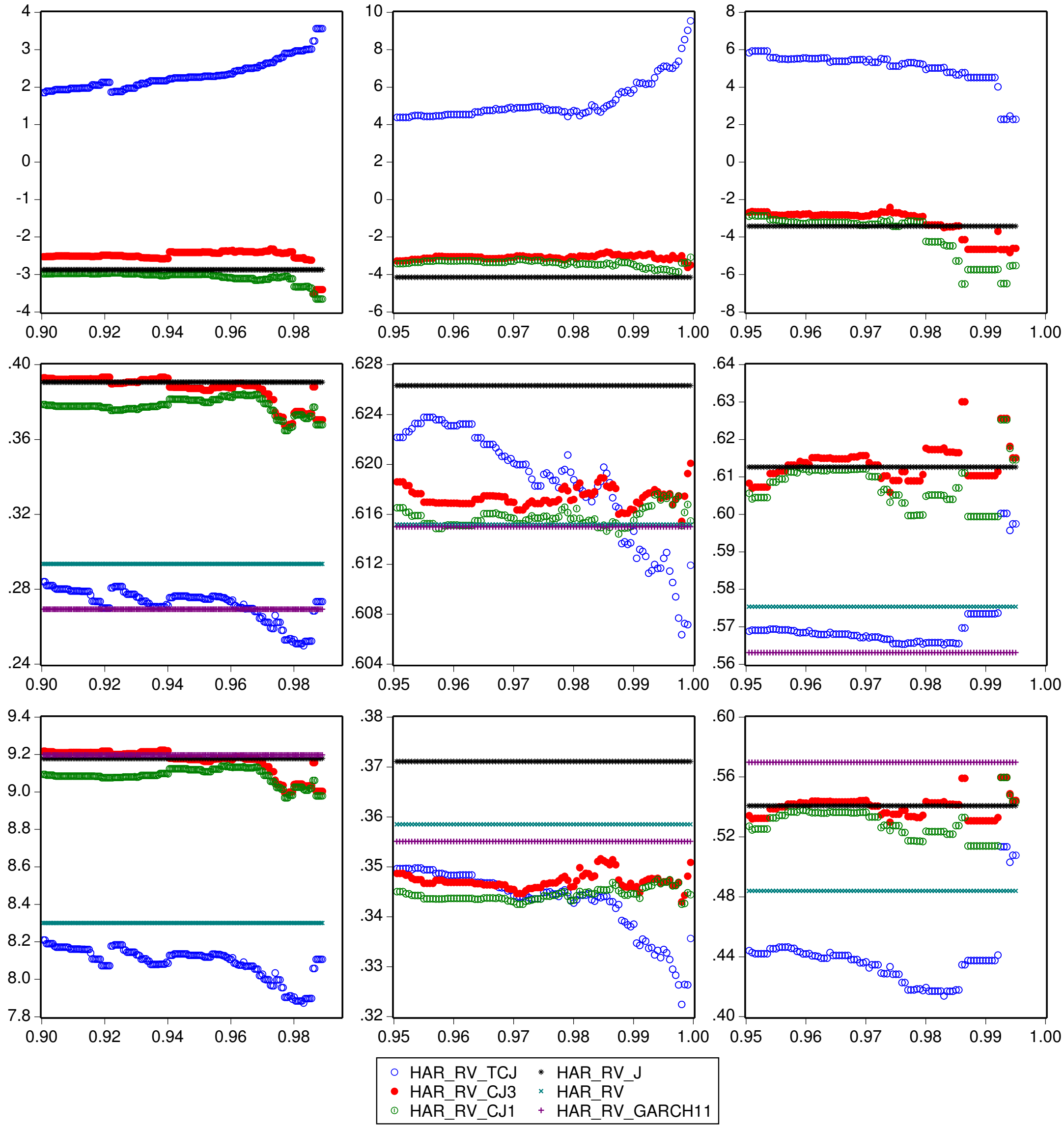
Figure A.2: WI sensitivity to α



Source: Own calculations

The first row shows sensitivity to alpha of β_j t-statistics, the second row indicates sensitivity to R^2 and the third row reveals sensitivity of α to RMSE. The first column reports results for standard HAR-RV model, while the second column show its logarithmic form and the last column the square root form.

Figure A.3: UO sensitivity to α



Source: Own calculations

The first row shows sensitivity to alpha of β_j t-statistics, the second row indicates sensitivity to R^2 and the third row reveals sensitivity of α to RMSE. The first column reports results for standard HAR-RV model, while the second column show its logarithmic form and the last column the square root form.

Table A.3: Regression results for PX and $\alpha = 0.99$

Daily PX Regression (C-Tz statistics) RV_{t+1} for $\alpha = 0.99$						
	HAR	HAR-garch	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
β_0	0.317 **	0.201 **	0.489 ***	0.264	0.365	-1.076 **
	(0.143)	(0.087)	(0.142)	(0.258)	(0.237)	(0.484)
β_d	0.346 ***	0.407 ***	0.425 ***	0.597 ***	0.538 ***	0.837 ***
	(0.117)	(0.053)	(0.116)	(0.169)	(0.170)	(0.264)
β_w	0.428 ***	0.291 ***	0.471 ***	-0.006	-0.002	1.373 **
	(0.137)	(0.048)	(0.140)	(0.014)	(0.013)	(0.545)
β_m	0.092	0.087 **	0.093	0.452 ***	0.318 ***	0.145
	(0.092)	(0.039)	(0.089)	(0.156)	(0.113)	(0.355)
β_{jd}			-1.575 ***	-0.624	-0.965 *	0.332 ***
			(0.566)	(0.463)	(0.580)	(0.108)
β_{jw}					0.769	
					(1.233)	
β_{jm}					1.836	
					(2.016)	
R^2	0.454	0.438	0.491	0.458	0.468	0.442
LB_{10}	0.009	0.946	0.151	0.000	0.000	0.000
LB_{10}^2	0.000	0.957	0.000	0.000	0.000	0.000
$RMSE$	1.736	1.652	1.860	1.672	1.712	1.690
$RMSE_J$			1.690	1.117	1.008	0.906
$RMSE_C$			2.412	1.566	1.630	0.778
MAE	0.832	0.702	0.994	0.892	0.853	0.828
$MAPE$	0.446	0.294	0.573	0.535	0.480	0.432
$Theil$	21.624	21.511	22.999	20.817	21.266	21.117
AIC	4.765	3.481	4.701	4.763	4.754	4.793
SC	4.806	3.553	4.752	4.814	4.825	4.844
HQC	4.781	3.510	4.721	4.783	4.782	4.813
$M - ZR^2$	0.806	0.885	0.746	0.827	0.808	0.829

Source: Own calculations

Table A.4: Regression results for logarithmic PX and $\alpha = 0.99$

Daily PX Regression (C-Tz statistics) RV_{t+1} for $\alpha = 0.99$						
	HAR	HAR-garch	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
β_0	-0.005	-0.019	0.065	0.063	0.054	0.165
	(0.043)	(0.043)	(0.044)	(0.043)	(0.045)	(0.042)
β_d	0.359 ***	0.365 ***	0.433 ***	0.400 ***	0.382 ***	0.394 ***
	(0.057)	(0.052)	(0.063)	(0.058)	(0.059)	(0.069)
β_w	0.521 ***	0.514 ***	0.522 ***	0.506 ***	0.514 ***	0.676 ***
	(0.091)	(0.076)	(0.094)	(0.089)	(0.088)	(0.132)
β_m	0.006	0.010	0.003	0.014	-0.021	-0.018
	(0.086)	(0.077)	(0.089)	(0.085)	(0.085)	(0.126)
β_{jd}			-0.466 ***	-0.081	-0.167	0.313 ***
			(0.111)	(0.101)	(0.113)	(0.054)
β_{jw}					-0.105	
					(0.264)	
β_{jm}					0.493	
					(0.365)	
R^2	0.605	0.605	0.619	0.618	0.621	0.603
LB_{10}	0.843	0.790	0.755	0.880	0.879	0.547
LB_{10}^2	0.236	0.449	0.165	0.187	0.188	0.130
$RMSE$	0.367	0.364	0.381	0.389	0.388	0.342
$RMSE_J$			0.413	0.366	0.355	0.300
$RMSE_C$			0.390	0.399	0.397	0.354
MAE	0.281	0.279	0.291	0.302	0.301	0.258
$MAPE$	1.259	1.192	1.473	1.616	1.567	1.356
$Theil$	19.132	19.038	19.832	20.225	20.194	17.866
AIC	1.711	1.711	1.681	1.683	1.687	1.721
SC	1.751	1.782	1.731	1.734	1.758	1.772
HQC	1.727	1.740	1.701	1.703	1.715	1.742
$M-ZR^2$	0.854	0.858	0.836	0.828	0.827	0.881

Source: Own calculations

Table A.5: Regression results for square root of PX and $\alpha = 0.99$

Daily PX Regression (C-Tz statistics) RV_{t+1} for $\alpha = 0.99$						
	HAR	HAR-garch	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
β_0	0.151 **	0.173 **	0.188	0.126	0.171 ***	-0.333 **
	(0.060)	(0.068)	(0.053)	(0.058)	(0.053)	(0.133)
β_d	0.396 ***	0.394 ***	0.468	0.463	0.439 ***	0.611 ***
	(0.082)	(0.063)	(0.084)	(0.079)	(0.082)	(0.112)
β_w	0.439 ***	0.395 ***	0.446	0.433	0.443 ***	0.799 ***
	(0.109)	(0.084)	(0.108)	(0.100)	(0.106)	(0.209)
β_m	0.040	0.046	0.040	0.055	0.011	0.009
	(0.082)	(0.068)	(0.085)	(0.081)	(0.086)	(0.183)
β_{jd}			-0.348	-0.105	-0.229	0.231 ***
			(0.103)	(0.081)	(0.161)	(0.041)
β_{jw}					0.070	
					(0.220)	
β_{jm}					0.230	
					(0.288)	
R^2	0.586	0.582	0.608	0.606	0.611	0.569
LB_{10}	0.833	0.975	0.799	0.831	0.814	0.125
LB^2_{10}	0.000	0.726	0.000	0.000	0.000	0.000
$RMSE$	0.298	0.305	0.318	0.311	0.318	0.293
$RMSE_J$			0.339	0.294	0.287	0.179
$RMSE_C$			0.358	0.292	0.300	0.193
MAE	0.198	0.196	0.223	0.221	0.217	0.176
$MAPE$	0.145	0.139	0.169	0.166	0.160	0.122
$Theil$	9.351	9.700	9.970	9.753	9.968	9.208
AIC	1.396	1.078	1.346	1.350	1.348	1.440
SC	1.436	1.149	1.396	1.401	1.419	1.491
HQC	1.412	1.106	1.366	1.370	1.376	1.461
$M-ZR^2$	0.869	0.879	0.837	0.846	0.836	0.880

Source: Own calculations

Table A.6: Regression results for UO and $\alpha = 0.96$

Daily PX Regression (C-Tz statistics) RV_{t+1} for $\alpha = 0.96$						
	HAR	HAR-garch	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
β_0	1.778 ***	0.718 ***	1.142 ***	-0.355	-0.453	-1.625 **
	(0.638)	(0.123)	(0.373)	(0.933)	(0.769)	(0.750)
β_d	0.211 ***	0.132 ***	1.033 ***	1.241 ***	1.278 ***	1.077 ***
	(0.051)	(0.043)	(0.334)	(0.415)	(0.400)	(0.236)
β_w	0.539 ***	0.562 ***	0.271 ***	0.000	-0.005	1.294 *
	(0.096)	(0.094)	(0.081)	(0.003)	(0.005)	(0.760)
β_m	0.018	-0.012	-0.097	0.044	0.043	-0.729
	(0.083)	(0.053)	(0.127)	(0.148)	(0.167)	(0.692)
β_{jd}			-3.427 ***	-2.780 ***	-2.528 **	0.212 **
			(1.189)	(0.911)	(1.079)	(0.092)
β_{jw}					1.549	
					(1.761)	
β_{jm}					-0.261	
					(1.041)	
R^2	0.284	0.262	0.384	0.375	0.379	0.265
LB_{10}	0.022	0.120	0.011	0.002	0.015	0.000
LB_{10}^2	0.996	0.982	1.000	1.000	1.000	0.928
$RMSE$	8.962	10.038	9.915	9.859	9.903	8.766
$RMSE_J$			2.914	3.881	3.648	4.966
$RMSE_C$			5.019	4.572	4.567	3.527
MAE	2.938	2.953	2.842	2.437	2.349	2.829
$MAPE$	0.525	0.335	0.430	0.220	0.202	0.373
$Theil$	34.992	42.049	37.501	37.390	37.500	34.452
AIC	7.722	5.270	7.578	7.592	7.595	7.754
SC	7.763	5.342	7.629	7.643	7.666	7.805
HQC	7.738	5.299	7.598	7.612	7.623	7.774
M-Z R^2	0.621	0.518	0.460	0.468	0.462	0.666

Source: Own calculations

Table A.7: Regression results for logarithmic UO and $\alpha = 0.96$

Daily PX Regression (C-Tz statistics) RV_{t+1} for $\alpha = 0.96$						
	HAR	HAR-garch	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
β_0	0.108 ** (0.044)	0.092 (0.067)	0.097 ** (0.044)	0.049 (0.047)	0.077 (0.049)	0.018 (0.055)
β_d	0.519 *** (0.057)	0.483 *** (0.055)	0.636 *** (0.074)	0.601 *** (0.071)	0.603 *** (0.071)	0.573 *** (0.071)
β_w	0.282 *** (0.081)	0.317 *** (0.079)	0.283 *** (0.082)	0.293 *** (0.085)	0.270 *** (0.094)	0.309 *** (0.107)
β_m	0.109 (0.069)	0.108 (0.076)	0.065 (0.078)	0.073 (0.082)	0.074 (0.093)	0.118 (0.095)
β_{jd}			-0.274 *** (0.062)	-0.143 ** (0.060)	-0.178 *** (0.061)	0.235 *** (0.037)
β_{jw}					0.071 (0.179)	
β_{jm}					0.042 (0.154)	
R^2	0.682	0.681	0.699	0.694	0.695	0.673
LB_{10}	0.832	0.872	0.883	0.824	0.828	0.499
LB_{10}^2	0.070	0.931	0.800	0.489	0.329	0.040
$RMSE$	0.249	0.267	0.272	0.263	0.266	0.233
$RMSE_J$			0.214	0.210	0.202	0.162
$RMSE_C$			0.282	0.244	0.246	0.209
MAE	0.186	0.199	0.191	0.179	0.177	0.168
$MAPE$	0.869	0.880	0.746	0.693	0.723	0.708
$Theil$	7.100	7.666	7.748	7.506	7.589	6.644
AIC	1.383	1.371	1.335	1.351	1.358	1.415
SC	1.424	1.442	1.385	1.402	1.429	1.466
HQC	1.399	1.399	1.355	1.371	1.386	1.435
M-Z R^2	0.934	0.920	0.911	0.920	0.917	0.948

Source: Own calculations

Table A.8: Regression results for square root of UO and $\alpha = 0.96$

Daily PX Regression (C-Tz statistics) RV_{t+1} for $\alpha = 0.96$						
	HAR	HAR-garch	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
β_0	0.313 ***	0.257 ***	0.287 ***	-0.050	0.065	-0.146
	(0.079)	(0.072)	(0.073)	(0.120)	(0.104)	(0.129)
β_d	0.423 ***	0.323 ***	0.653 ***	0.732 ***	0.730 ***	0.749 ***
	(0.070)	(0.051)	(0.098)	(0.116)	(0.111)	(0.110)
β_w	0.374 ***	0.497 ***	0.344 ***	0.352 ***	0.344 **	0.393 **
	(0.092)	(0.098)	(0.084)	(0.088)	(0.141)	(0.193)
β_m	0.064	0.012	-0.001	-0.025	-0.070	-0.031
	(0.071)	(0.069)	(0.088)	(0.111)	(0.174)	(0.178)
β_{jd}			-0.520 ***	-0.463 ***	-0.527 ***	0.287 ***
			(0.154)	(0.169)	(0.182)	(0.051)
β_{jw}					-0.089	
					(0.481)	
β_{jm}					0.320	
					(0.486)	
R^2	0.567	0.553	0.608	0.614	0.617	0.561
LB_{10}	0.082	0.246	0.076	0.082	0.093	0.000
LB_{10}^2	0.131	0.936	0.772	0.833	0.836	0.070
$RMSE$	0.515	0.617	0.578	0.583	0.591	0.470
$RMSE_J$			0.402	0.568	0.557	0.204
$RMSE_C$			0.493	0.421	0.420	0.256
MAE	0.297	0.343	0.336	0.257	0.253	0.198
$MAPE$	0.124	0.129	0.138	0.090	0.090	0.072
$Theil$	9.612	11.792	10.759	10.849	11.001	8.766
AIC	2.554	1.841	2.462	2.445	2.447	2.573
SC	2.595	1.912	2.513	2.496	2.518	2.624
HQC	2.571	1.869	2.482	2.465	2.475	2.593
M-Z R^2	0.878	0.815	0.820	0.815	0.808	0.913

Source: Own calculations

Table A.9: Regression results for WI and $\alpha = 0.96$

Daily PX Regression (C-Tz statistics) RV_{t+1} for $\alpha = 0.96$						
	HAR	HAR-garch	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
β_0	1.099 **	1.000 ***	1.105 **	1.974 ***	1.927 ***	-0.556
	(0.436)	(0.227)	(0.492)	(0.424)	(0.455)	(0.671)
β_d	0.190 **	0.358 ***	0.193	0.174	0.180	0.548 ***
	(0.089)	(0.050)	(0.119)	(0.198)	(0.202)	(0.124)
β_w	0.710 ***	0.271 ***	0.709 ***	0.013 *	0.010	1.053 ***
	(0.149)	(0.080)	(0.151)	(0.007)	(0.009)	(0.400)
β_m	-0.088	0.082 *	-0.089	0.357 ***	0.361 ***	-0.237
	(0.115)	(0.050)	(0.117)	(0.104)	(0.104)	(0.336)
β_{jd}			-0.020	0.367	0.228	0.254 **
			(0.494)	(0.500)	(0.524)	(0.108)
β_{jw}					0.375	
					(1.351)	
β_{jm}					0.984	
					(2.147)	
R^2	0.428	0.373	0.428	0.336	0.340	0.375
LB_{10}	0.093	0.383	0.093	0.000	0.000	0.000
LB_{10}^2	0.000	0.967	0.000	0.000	0.000	0.000
$RMSE$	3.554	3.067	3.554	3.126	3.145	3.291
$RMSE_J$			2.491	2.461	2.449	1.846
$RMSE_C$			3.854	2.525	2.489	2.462
MAE	1.962	1.490	1.963	1.835	1.799	1.715
$MAPE$	0.411	0.274	0.411	0.444	0.435	0.327
$Theil$	23.001	20.994	23.001	20.532	20.646	21.483
AIC	5.826	4.875	5.831	5.979	5.984	5.920
SC	5.866	4.946	5.882	6.029	6.055	5.970
HQC	5.842	4.903	5.851	5.999	6.012	5.940
M-Z R^2	0.652	0.878	0.652	0.817	0.809	0.744

Source: Own calculations

Table A.10: Regression results for logarithmic WI and $\alpha = 0.96$

Daily PX Regression (C-Tz statistics) RV_{t+1} for $\alpha = 0.96$						
	HAR	HAR-garch	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
β_0	0.165 **	0.179 **	0.172 **	0.144 **	0.144 *	0.102
	(0.068)	(0.074)	(0.067)	(0.069)	(0.074)	(0.090)
β_d	0.342 ***	0.345 ***	0.390 ***	-0.123	-0.128	0.403 ***
	(0.053)	(0.055)	(0.068)	(0.113)	(0.120)	(0.060)
β_w	0.499 ***	0.480 ***	0.493 ***	0.488 ***	0.493 ***	0.498 ***
	(0.087)	(0.084)	(0.085)	(0.084)	(0.094)	(0.107)
β_m	0.020	0.024	0.006	0.024	0.019	0.062
	(0.073)	(0.074)	(0.075)	(0.079)	(0.090)	(0.111)
β_{jd}			-0.112	-0.091	-0.088	0.187 ***
			(0.079)	(0.080)	(0.099)	(0.034)
β_{jw}					-0.046	
					(0.212)	
β_{jm}					0.032	
					(0.245)	
R^2	0.531	0.531	0.534	0.531	0.531	0.503
LB_{10}	0.708	0.543	0.609	0.670	0.672	0.214
LB_{10}^2	0.243	0.819	0.340	0.308	0.297	0.109
$RMSE$	0.323	0.322	0.325	0.320	0.320	0.306
$RMSE_J$			0.309	0.281	0.281	0.251
$RMSE_C$			0.358	0.308	0.307	0.294
MAE	0.254	0.253	0.256	0.250	0.250	0.242
$MAPE$	0.371	0.371	0.367	0.360	0.359	0.351
$Theil$	10.067	10.065	10.146	9.988	9.986	9.549
AIC	1.403	1.400	1.401	1.408	1.418	1.466
SC	1.443	1.471	1.451	1.458	1.488	1.516
HQC	1.419	1.428	1.421	1.428	1.446	1.486
M-Z R^2	0.823	0.829	0.817	0.827	0.827	0.860

Source: Own calculations

Table A.11: Regression results for square root of WI and $\alpha = 0.96$

Daily PX Regression (C-Tz statistics) RV_{t+1} for $\alpha = 0.96$						
	HAR	HAR-garch	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
β_0	0.344 ***	0.397 ***	0.356 ***	1.007 ***	0.992 ***	-0.082
	(0.116)	(0.106)	(0.119)	(0.168)	(0.202)	(0.189)
β_d	0.282 ***	0.359 ***	0.318 ***	0.091	0.093	0.480 ***
	(0.062)	(0.055)	(0.082)	(0.122)	(0.131)	(0.074)
β_w	0.587 ***	0.367 ***	0.579 ***	0.097 ***	0.103 ***	0.636 ***
	(0.103)	(0.086)	(0.102)	(0.024)	(0.030)	(0.169)
β_m	-0.034	0.061	-0.041	0.203 ***	0.200 **	-0.019
	(0.085)	(0.067)	(0.086)	(0.076)	(0.081)	(0.166)
β_{jd}			-0.088	0.034	0.122	0.189 ***
			(0.103)	(0.102)	(0.217)	(0.042)
β_{jw}					0.023	
					(0.359)	
β_{jm}					-0.235	
					(0.454)	
R^2	0.511	0.497	0.513	0.472	0.474	0.473
LB_{10}	0.731	0.518	0.715	0.006	0.013	0.032
LB_{10}^2	0.000	0.833	0.000	0.000	0.000	0.000
$RMSE$	0.452	0.419	0.454	0.417	0.416	0.410
$RMSE_J$			0.372	0.320	0.331	0.275
$RMSE_C$			0.503	0.384	0.386	0.347
MAE	0.320	0.283	0.323	0.304	0.305	0.270
$MAPE$	0.147	0.127	0.149	0.148	0.148	0.121
$Theil$	9.529	8.941	9.569	8.801	8.783	8.639
AIC	1.929	1.659	1.930	2.011	2.018	2.009
SC	1.969	1.730	1.980	2.061	2.088	2.060
HQC	1.945	1.687	1.950	2.031	2.046	2.029
M-Z R^2	0.774	0.857	0.770	0.836	0.836	0.846

Source: Own calculations

Table A.12: Out of sample performance for PX and $\alpha = 0.99$

	HAR	HAR-GARCH	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
PX 0.99						
$RMSE$	0.402	0.302	0.513	0.400	0.367	0.395
$RMSE_J$			0.654	0.433	0.378	0.292
$RMSE_C$			0.387	0.381	0.352	0.406
MAE	0.346	0.231	0.454	0.360	0.340	0.325
$MAPE$	0.633	0.371	0.838	0.653	0.623	0.466
$Theil$	20.622	16.970	25.135	19.869	18.300	24.613
$M-ZR^2$	0.681	0.863	0.247	0.686	0.820	0.593
log PX 0.99						
$RMSE$	0.384	0.377	0.370	0.400	0.404	0.365
$RMSE_J$			0.355	0.369	0.375	0.340
$RMSE_C$			0.336	0.401	0.404	0.367
MAE	0.327	0.322	0.316	0.329	0.331	0.296
$MAPE$	0.901	0.916	0.925	0.809	0.807	0.760
$Theil$	38.081	36.961	36.090	39.642	40.402	34.967
$M-ZR^2$	0.573	0.588	0.602	0.487	0.490	0.577
sqrt PX 0.99						
$RMSE$	0.168	0.163	0.190	0.186	0.181	0.121
$RMSE_J$			0.243	0.173	0.170	0.106
$RMSE_C$			0.149	0.185	0.178	0.121
MAE	0.139	0.134	0.162	0.151	0.147	0.098
$MAPE$	0.180	0.171	0.207	0.195	0.192	0.117
$Theil$	9.307	9.054	10.473	10.224	9.927	6.921
$M-ZR^2$	0.702	0.742	0.459	0.522	0.584	0.766

Source: Own calculations

Table A.13: Out of sample performance for UO and $\alpha = 0.96$

	HAR	HAR-GARCH	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
UO 0.99						
$RMSE$	1.737	1.433	1.903	1.711	1.497	1.470
$RMSE_J$			1.686	0.299	0.215	2.132
$RMSE_C$			1.210	1.724	1.509	1.457
MAE	1.508	1.057	1.356	0.715	0.651	1.183
$MAPE$	0.778	0.414	0.546	0.174	0.155	0.467
$Theil$	22.025	21.802	25.174	22.783	20.041	19.404
$M-ZR^2$	0.435	0.321	0.197	0.369	0.479	0.713
log UO 0.99						
$RMSE$	0.271	0.286	0.246	0.250	0.250	0.242
$RMSE_J$			0.290	0.373	0.362	0.335
$RMSE_C$			0.233	0.247	0.247	0.240
MAE	0.223	0.234	0.195	0.196	0.196	0.197
$MAPE$	0.596	0.609	0.514	0.491	0.499	0.520
$Theil$	12.101	12.854	11.030	11.183	11.148	10.758
$M-ZR^2$	0.885	0.851	0.897	0.879	0.889	0.902
sqrt UO 0.99						
$RMSE$	0.285	0.308	0.304	0.254	0.258	0.146
$RMSE_J$			0.344	0.365	0.372	0.290
$RMSE_C$			0.273	0.252	0.256	0.143
MAE	0.240	0.247	0.243	0.154	0.158	0.122
$MAPE$	0.165	0.157	0.161	0.093	0.097	0.077
$Theil$	7.948	8.852	8.525	7.170	7.255	4.116
$M-ZR^2$	0.768	0.609	0.620	0.719	0.716	0.921

Source: Own calculations

Table A.14: Out of sample performance for WI and $\alpha = 0.96$

	HAR	HAR-GARCH	HAR-J	HAR-CJ1	HAR-CJ3	HAR-TCJ
WI 0.99						
$RMSE$	1.295	0.912	1.296	1.338	1.311	1.064
$RMSE_J$			1.480	1.329	1.299	0.978
$RMSE_C$			1.406	1.338	1.312	1.069
MAE	1.080	0.744	1.081	1.158	1.132	0.854
$MAPE$	0.496	0.325	0.497	0.568	0.557	0.342
$Theil$	17.999	13.370	18.009	18.001	17.705	14.878
$M-ZR^2$	0.343	0.832	0.343	0.677	0.700	0.521
log WI 0.99						
$RMSE$	0.322	0.320	0.318	0.308	0.308	0.313
$RMSE_J$			0.339	0.381	0.381	0.369
$RMSE_C$			0.369	0.303	0.302	0.309
MAE	0.258	0.256	0.256	0.246	0.245	0.252
$MAPE$	0.889	0.887	0.873	0.852	0.851	0.825
$Theil$	14.118	14.034	13.932	13.508	13.489	13.673
$M-ZR^2$	0.714	0.729	0.711	0.760	0.760	0.681
sqrt WI 0.99						
$RMSE$	0.302	0.259	0.304	0.311	0.308	0.247
$RMSE_J$			0.338	0.326	0.321	0.270
$RMSE_C$			0.347	0.310	0.307	0.246
MAE	0.247	0.212	0.250	0.260	0.257	0.205
$MAPE$	0.162	0.138	0.165	0.176	0.174	0.131
$Theil$	8.382	7.259	8.445	8.577	8.487	6.886
$M-ZR^2$	0.566	0.781	0.549	0.763	0.758	0.688

Source: Own calculations