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BAKALÁŘSKÁ PRÁCE

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Aplikace Baireovy věty

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Prohlašuji, že jsem svou bakalářskou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

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Abstrakt: Cieľom tejto práce je v troch rôznych prípadoch dokázať, že množina prvkov s danou vlastnosťou je množina typických prvkov. Najskôr dokážeme, že typická spojitá funkcia, definovaná na intervale $[0, 1]$, nemá deriváciu v žiadnom bode. Potom dokážeme, že typická kompaktná podmnožina $\mathbb{R}^2$ je diskontinuum. A napokon ukážeme, že typiclé rovinné kontinuum je nerozložiteľné. Dôležitým nástrojom bude Baireova veta, ktorej použitie nám okrem hustosti zaistí zároveň aj to, že daná množina je spočetným prienikom otvorených množín.
Kl'účové slová: typický prvok, spojitá funkcia, derivácia, kompaktná množina, diskontinuum, kontinuum, nerozložiteľné kontinuum

Title: Applications of Baire theorem
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Abstract: The aim of this work is to show, having three different spaces and a set of elements with some common property in each one of them that the given set is the set of typical elements in that space. First we will show that a typical continuous function defined on the interval $[0, 1]$ is a nowhere differentiable one. Then we will show that a typical compact set in $\mathbb{R}^2$ is a discontinuum. And lastly, we will show that a typical planar continuum is an indecomposable one. A valuable tool will be the Baire theorem, the use of which will ensure, besides the density, also the fact that the given set is a countable intersection of open sets.
Keywords: typical element, continuous function, derivative, compact set, discontinuum, continuum, indecomposable continuum

Introduction

Most people would say that typical are things which are easy to be found and are nice in some way. even in mathematics such belief was common in the past centuries. However, in some cases this appeared to be wrong. The notion of what is typical needed to be defined in a more precise way in order to be used in mathematics and here the Baire cathegory came handy. A set of typical elements was defined so that its complement has to be meagre, in other words, in order for a set to be the set of typical elements in some space, it has to be a countable intersection of open dense sets in that space. In a complete metric space such a set is dense itself, due to the Baire theorem. Therefore a set of typical elements in a complete metric space, defined as a dense set, which is as well a countable intersection of open sets, is defined in a very natural way, close to what we would instinctively hold for typical. An easy example of such a set is the set of irrational numbers in $\mathbb{R}$, since they form a dense set, which can be writen as the intersection of all sets $\mathbb{R} \setminus \{r\}$, where $r \in \mathbb{Q}$ and this intersection is obviously countable. This example was easy and still, the irrational numbers are a more complicated construction than the rationals and were discovered only by a student of Pythagoras and until then people didn't believe that such numbers might exist. But in some spaces, there are elements so bizarre that mathematicians did not believe that such things could exist, let alone be typical, and they were discovered only lately, but they still are typical. In this bachelor thesis I will show three examples of such objects. First I will show that typical continuous functions on $[0, 1]$ are nowhere differentiable. Then I will show that in the space of all compact sets in $\mathbb{R}^2$, the discontinua are typical objects. And lastly, in the last chapter I will show that typical planar continua are the indecomposable ones. Examples of each of these objects were given only in the last two centuries and the proofs that they are typical are even more recent. And they are so bizarre that before these examples were introduced, most mathematicians believed that such objects simply can't exist.

Chapter 1

Definitions and Baire Theorem

1.1 Definitions

First I will list the general definitions which I will use in this work and afterwards the definitions needed in the single chapters. The general definitions come from [1] and the definitions related to the third and fourth chapter are from [2]. The historical notes at the beginning of each chapter are mostly borrowed from [3].

A *metric space* is an ordered pair (X, d), where X is a nonempty set and d is a function from $X \times X$ to $\mathbb{R}$, satisfying following conditions:

$$\begin{aligned}
&d(x,y) \geq 0 \qquad \& \qquad d(x,y) = 0 \Leftrightarrow x = y \\
&d(x,y) = d(y,x) \\
&d(x,z) \leq d(x,y) + d(y,z)
\end{aligned}$$

The last condition is called the *triangle inequality.*

A *topological sapce* is an ordered pair (X, τ), where X is a nonempty set and $\tau \subseteq \mathcal{P}(X)$ is a set of subsets of X, satysfying following conditions:

$$\begin{aligned}
&\emptyset \in \tau \qquad \& \qquad X \in \tau \\
&\text{for } \forall A, B \in \tau \text{ , their intersection } A \cap B \in \tau \\
&\text{for } \forall \mathcal{A} \subseteq \tau \text{ , the union } \bigcup \mathcal{A} \in \tau
\end{aligned}$$

X is called an *underlying set* in both cases. d is called a *metric on* X and τ is called a *topology* on X. The sets in τ are called open in (X, τ).

From now on I will write only X instead of (X,d) or (X,τ) and d will denote the metric on X and τ will denote the topology on X unless specified otherwise.

A set $\mathcal{B}$ is called a *base* of a topology τ, if $\left\{\bigcup\mathcal{A};\mathcal{A}\subseteq\mathcal{B}\right\}\cup\{\emptyset,X\}=\tau$. We can easily see that having a metric space (X,d) we can define a topology on X with the base $\mathcal{B}=\{B_d(x,\varepsilon);\, x\in X,\, \varepsilon>0\}$. This topology is said to be *induced* by the metric d.

Having two topological spaces (X,τ) and (Y,σ), a mapping $f\colon X\to Y$ is called a *homeomorphism* if it satisfies following conditions:

$$
\begin{aligned}
&f \text{ is both surjective and injective}\\
&\text{for } \forall A\in\tau,\quad f(A)\in\sigma\\
&\text{for } \forall B\in\sigma,\quad f^{-1}(B)\in\tau
\end{aligned}
$$

Let now X be a nonempty set and τ and σ two topologies on X. We say that τ and σ *coincide* on X, if the identity mapping $id_X : X\to X$ is a homeomorphism of (X,τ) and (X,σ).

Remark. *Let's have a set $X\neq\emptyset$ and τ and σ topologies on X, generated by bases $\mathcal{B}_\tau$ and $\mathcal{B}_\sigma$, respectively. If for $\forall x\in X$ and $\forall\, A\in\mathcal{B}_\tau$, such that $x\in A$, there $\exists\, B\in\mathcal{B}_\sigma$, such that $x\in B\subseteq A$ and if for $\forall x\in X$ and $\forall\, B\in\mathcal{B}_\sigma$, such that $x\in B$, there $\exists\, A\in\mathcal{B}_\tau$, such that $x\in A\subseteq B$, then τ and σ coincide.*

Proof. Let $A\in\tau$ be any set, open in τ. Then $A=\bigcup\limits_{A_\alpha\in\mathcal{A}}A_\alpha$ for some system $\mathcal{A}\subseteq\mathcal{B}_\tau$. Now for $\forall A_\alpha\in\mathcal{A}$ and for $\forall x\in A_\alpha$, there $\exists B_{x,\alpha}\in\mathcal{B}_\sigma$, such that $x\in B_{x,\alpha}\subseteq A_\alpha$, hence $A_\alpha\subseteq\bigcup\limits_{x\in A_\alpha}B_{x,\alpha}\subseteq A_\alpha$ and therefore $A=\bigcup\limits_{\substack{A_\alpha\in\mathcal{A}\\ x\in A_\alpha}}B_{x,\alpha}$, which is open in σ. Similarly, any set, open in σ, will be open in τ as well and therefore id_X is a homeomorphism of (X,τ) and (X,σ). □

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be *Cauchy*, if for every real $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that for $\forall\, m,n\in\mathbb{N}$, where $m,n\geq n_0$, the distance $d(x_m,x_n)<\varepsilon$.

A *complete metric space* is a metric space X, in which every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to a limit x. In other words, for every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X there is a point $x\in X$, such that for every real $\varepsilon>0$ there is an $n_0\in\mathbb{N}$, such that for every integer $n\geq n_0$, the distance $d(x_n,x)<\varepsilon$.

A set $A \subseteq X$ is said to be *dense* in X if for every nonempty and open set $Y \subseteq X$, $A \cap Y \neq \emptyset$.

A set $A \subseteq X$ is said to be G_δ if there exists a collection of open sets $\{A_n \subseteq X\}_{n\in\mathbb{N}}$ such that $A = \bigcap\limits_{n\in\mathbb{N}} A_n$.

Now we can define a typical element of a complete metric space as follows:

Having a complete metric space X and $A \subseteq X$ which is G_δ and dense in X, we call all the elements of A *typical elements* of X. A is called the *set of typical elements* of X.

Let (X, d) be a complete metric space, τ_d topology on X induced by d, (Y, σ) a topological space and $f : (X, \tau_d) \to (Y, \sigma)$ a homeomorphism. Then for any set $A \subseteq Y$, which is dense in (Y, σ), $f^{-1}(A)$ will be dense in (X, τ_d), since for $\forall B \in \tau_d$, there $\exists f(B) \in \sigma$, hence $f(B) \cap A \neq \emptyset$ and then $\emptyset \neq f^{-1}\big(f(B) \cap A\big) = B \cap f^{-1}(A)$. Also for any G_δ set $A \subseteq Y$, the set $f^{-1}(A)$ will be G_δ in (X, τ_d), because $f^{-1}(A) = f^{-1}\big(\bigcap\limits_{n\in\mathbb{N}} A_n\big) = \bigcap\limits_{n\in\mathbb{N}} f^{-1}(A_n)$, where $f^{-1}(A_n)$ are open in (X, τ_d).
Therefore if we find a set which is dense and G_δ in a space that is homeomorphic with a compact space, then the homeomorphic inverse image of A will be the set of typical elements in the compact space X.

In the following chapters I will search for the typical elements in some particular spaces. Following definitions will apply to these.

In the second chapter I will show that nowhere differentiable functions are the typical elements of the space of continuous functions, defined on the closed unit interval $[0, 1]$.

A function $f : [0, 1] \to \mathbb{R}$ is said to be *continuous* if for $\forall\, x \in [0, 1]$ and $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that for $\forall\, y \in \mathbb{R}$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

The space consisting of all continuous functions, defined on $[0, 1]$ is denoted $\mathcal{C}([0, 1])$ and endowed with a metric d, defined as follows:

$$f, g \in \mathcal{C}([0,1]) \quad \text{then} \quad d(f, g) = \sup_{x\in[0,1]} \left(|f(x) - g(x)|\right)$$

For a function $f \in \mathcal{C}([0, 1])$ and a point $t \in [0, 1]$ if following right-hand limit exists and is finite

$$\frac{d^+ f(t)}{dt} = \lim_{h\to 0_+} \left| \frac{f(t+h) - f(t)}{h} \right|$$

then $\frac{d^+ f(t)}{dt}$ is called the *right derivative* of f in the point t and f is said to be *right differentiable* in t. If following left-hand limit exists and is finite

$$\frac{d^- f(t)}{dt} = \lim_{h\to 0_-} \left| \frac{f(t+h) - f(t)}{h} \right|$$

then $\frac{d^- f(t)}{dt}$ is called the *left derivative* of f in t and f is said to be *left differentiable* in t. If $\frac{d^+ f(t)}{dt} = \frac{d^- f(t)}{dt} = \frac{d\,f(t)}{dt}$, $\frac{d\,f(t)}{dt}$ is called the *derivative* of f in t and f is said to be *differentiable* in t.

A function $f \in \mathcal{C}[0,1]$ is said to be nowhere differentiable, if it is not differentiable in any point $x \in [0,1]$.

In the third chapter I will show that discontinua are the typical elements of the space of all nonempty compact sets in the plane. Since in this chapter we will work in $\mathbb{R}^2$, the definitions can be restricted to the Euclidean case.

A set $A \subseteq \mathbb{R}^2$ is said to be *closed* if $\mathbb{R}^2 \setminus A$ is open.
A is said to be *bounded* if A is contained in some ball of finite radius in $\mathbb{R}^2$.
A is said to be *compact* if A is both closed and bounded.

Denote $\mathfrak{X}^2$ the space consisting of all nonempty compact sets in $\mathbb{R}^2$ and denote $E_\varepsilon(A) = \bigcup_{x\in A} B(x,\varepsilon)$ the union of all open ε-balls around points in A.
Now we can define distance in $\mathfrak{X}^2$ as follows:

$A, B \in \mathfrak{X}^2$ then $d_H(A,B) = \inf\{\varepsilon; E_\varepsilon(A) \supset B \text{ and } E_\varepsilon(B) \supset A\}$

d_H is obviously a metric and it is called the *Hausdorff metric.* Distance in the space of all nonempty bounded sets in $\mathbb{R}^2$ can also be defined this way, but it won't be a metric, because it does not satisfy the condition $d_H(x,y) = 0 \;\Rightarrow\; x = y$.

For $k \in \mathbb{N}$ and $\{A_1, \dots, A_k\}$ a collection of open subsets of $\mathbb{R}^2$, put

$$V(A_1, \dots, A_k) = \Big\{ B \in \mathfrak{X}^2; \text{ such that } B \subseteq \bigcup_{n=1}^{k} A_n \text{ and } B \cap A_n \neq \emptyset \text{ for } \forall n = 1, \dots, k \Big\}$$

Now put $\mathcal{B}_V = \big\{ V(A_1, \dots, A_k);\ k \in \mathbb{N}, U_n \subseteq \mathbb{R}^2 \text{ open for } \forall n = 1, \dots, k \big\}$. $\mathcal{B}_V$ is obviously base of a topology on $\mathfrak{X}^2$. This topology is called the *Vietoris topology.*

The Vietoris topology and the topology induced by the Hausdorff metric coincide on any space of compact subsets of $\mathbb{R}^2$. The proof of this is in the third chapter, in statement 5.

A metric space D is said to be perfect, if D contains no isolated points. A metric space D is totally disconnected, if any connected subset of D contains only one point.
A totally disconnected, perfect compact metric space is called a *discontinuum.*
Therefore in $\mathbb{R}^2$ a discontinuum is a compact set $D \in \mathfrak{X}^2$, such that D contains no isolated points and any connected subset of D is a singleton.

In the fourth chapter I will show that indecomposable continua are typical in the space of all continua in plane. In this chapter we will work in $\mathbb{R}^2$, so the definitions restricted to the Euclidean case will be sufficient again.

A metric space P is said to be *connected* if for $\forall\ B, C \subseteq P$, where B, C are open, $B \cap C = \emptyset$ and $B \cup C = P$, either $B = P$ and $C = \emptyset$, or $C = P$ and $B = \emptyset$. We can easily see that if the sets B, C are closed, we get an equvalent definition.
A set $A \subseteq \mathbb{R}^2$ is said to be *connected* if it is a connected metric space as a subspace of $\mathbb{R}^2$.
A set $\emptyset \neq A \subset \mathbb{R}^2$ is called a *continuum* if A is both compact and connected. Since A is compact, it is also closed in $\mathbb{R}^2$ and hence any set $B \subseteq A$, closed in A is closed in $\mathbb{R}^2$, too. Therefore we can define a continuum in $\mathbb{R}^2$ equivalently, as a set $A \subset \mathbb{R}^2$, which is compact and for $\forall B, C \subset \mathbb{R}^2$, where B, C are closed, $B \cap C = \emptyset$ and $B \cup C = A$, either $B = A$ and $C = \emptyset$, or $C = A$ and $B = \emptyset$.

Furthermore, it is obvious that a compact set $A \subset \mathbb{R}^2$ is a continuum if and only if for $\forall U, V \subseteq \mathbb{R}^2$, where U, V are open, such that $U \cap V = \emptyset$ and $U \cup V \supset A$, either $U \cap A = \emptyset$ and $V \cap A = A$, or $U \cap A = A$ and $V \cap A = \emptyset$.

Denote $\mathcal{C}$ the set of all continua in $\mathbb{R}^2$. Endowed with the Hausdorff metric, restricted to $\mathcal{C}$, $\mathcal{C}$ is a metric space.

A continuum $K \subset \mathbb{R}^2$ is said to be *indecomposable* if for $\forall\, C_1, C_2 \subseteq K$, where C_1, C_2 are continua in $\mathbb{R}^2$, such that $K = C_1 \cup C_2$, at least one of C_1, C_2 is equal to K.

For two topological spaces X, Y, a mapping $f : X \to Y$ is said to be *continuous*, if the inverse image of any open set in Y is open in X.
Having a continuous mapping $f : [0,1] \to X$, the set $U = f([0,1])$ is called a *path* in X. If $f(0) = x \in X$ and $f(1) = y \in X$, then U is the path, joining x and y.

1.2 Baire Theorem

Theorem (Baire)**.** *Let X be a complete metric space and $\{A_n \subseteq X\}_{n\in\mathbb{N}}$ a collection of sets that are open and dense in X. Then $\bigcap_{n\in\mathbb{N}} A_n$ is dense in X.*

Proof. Let G_1 be any nonempty and open subset of X. $G_1 \cap A_1 \neq \emptyset$, because A_1 is dense. Therefore there exists $x_1 \in G_1 \cap A_1$. Moreover, $G_1 \cap A_1$ is open, because it is the intersection of two open sets. Therefore there exists a real number $\varepsilon_1 \in (0, \frac{1}{2^2})$ such that both the open ball $G_2 = \{x \in X; d(x_1, x) < \varepsilon_1\}$ and the closed ball $\overline{G}_2 = \{x \in X;\ d(x_1, x) \leq \varepsilon_1\}$ lie in $G_1 \cap A_1$.

G_2 is a nonempty and open subset of X and hence we can repeat the construction using the sets G_2 and A_2. This way we obtain the point x_2, the real number $\varepsilon_2 \in (0, \frac{1}{2^3})$ and the set G_3. Recursively we will obtain the following sequences:

the sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X

the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of real numbers, such that for $\forall n \in \mathbb{N}$, $\varepsilon_n \in (0, \frac{1}{2^{n+1}})$

and the sequence $\{G_n\}_{n\in\mathbb{N}}$ of open subsets of X, such that $\overline{G}_{n+1} \subseteq G_n$

We can easily see that $\bigcap_{n\in\mathbb{N}} G_n \subseteq \bigcap_{n\in\mathbb{N}} \overline{G}_n \subseteq \bigcap_{n\in\mathbb{N}} A_n \cap G_1$.

Now for $\forall \varepsilon > 0$ we can find $n_0 \in \mathbb{N}$, such that $\dfrac{1}{2^{n_0}} < \varepsilon$. Then for $\forall n \geq n_0$, $n \in \mathbb{N}$, the point x_n lies in $G_n \subseteq G_{n_0}$. Therefore for $\forall m, n \geq n_0$, $m, n \in \mathbb{N}$,

$$d(x_n, x_m) < diam(G_{n_0}) = 2\,\varepsilon_{n_0} < 2\frac{1}{2^{n_0+1}} = \frac{1}{2^{n_0}} < \varepsilon$$

Which means that the sequence $\{x_n\}$ is Cauchy and because X is complete, it converges to a limit $x \in X$.
From the construction above we have that for $\forall n \in \mathbb{N}$, $\{x_m\}_{m>n} \subseteq G_n \subseteq \overline{G}_n$, which is closed. Hence $\bigcap\limits_{n\in\mathbb{N}} \overline{G}_n$ is also closed and $x \in \bigcap\limits_{n\in\mathbb{N}} \overline{G}_n \subseteq \bigcap\limits_{n\in\mathbb{N}} A_n \cap G_1$, which means that $\bigcap\limits_{n\in\mathbb{N}} A_n \cap G_1 \neq \emptyset$ and hence $\bigcap\limits_{n\in\mathbb{N}} A_n$ is dense in X.

□

Now let (X, d) be a metric space and (Y, τ) a topological space, which is homeomorphic to the space (X, τ_d) with topology induced by d and let $\{A_n \subseteq Y\}_{n\in\mathbb{N}}$ be a collection of sets that are open and dense in (Y, τ). We now have $\{f^{-1}(A_n) \subseteq X\}_{n\in\mathbb{N}}$, a collection of sets that are open and dense in (X, τ_d), which is complete and hence open and complete in (X, d). Therefore $\bigcap\limits_{n\in\mathbb{N}} f^{-1}(A_n) = f^{-1}(\bigcap\limits_{n\in\mathbb{N}} A_n)$ is dense in (X, d) and thus in (X, τ_d) and therefore $\bigcap\limits_{n\in\mathbb{N}} A_n$ is dense in (Y, τ). Obviously, the completness of the space X in the Baire theorem is a needlessly strong condition and we can use a stronger version of the theorem, which is borrowed from [4].

Theorem (Baire)**.** *Let X be a topological space, which is homeomorphic to some complete metric space and $\{A_n \subseteq X\}_{n\in\mathbb{N}}$ a collection of sets that are open and dense in X. Then $\bigcap\limits_{n\in\mathbb{N}} A_n$ is dense in X.*

Chapter 2

Nowhere differentiable functions in $\mathcal{C}([0,1])$

Until the early 19th century the general belief among mathematicians was that any continuous function is differentiable except maybe at a few isolated points. In 1806 André-Marie Ampère even tried to prove this hypothesis. However, on July, 1872 an example of a continuous, nowhere differentiable function on an interval was presented by the German mathematician Karl Theodor Wilhelm Weierstrass (1815-1897) in a lecture at the Royal Academy of Science in Berlin. This function was defined by the formula

$$W(x) = \sum_{k=0}^{\infty} a^k cos(b^k \pi x)$$

where $a \in (0,1)$, b is an odd integer and $ab > 1 + \frac{3\pi}{2}$. This function was first published in 1875, which made it the first example of a continuous, nowhere differentiable function to be published. Still, it was not the first such construction. The earliest such function is due to the Czech mathematician Bernard Bolzano (1781-1848), who constructed it around 1830, yet was not published until 1922. Unlike many other constructions, Bolzano function is constructed as the limit of a sequence of continuous functions defined as follows.

1) Let $[a_0, b_0]$ be the desired domain and A_0, B_0 two real numbers. Then $f_0(x) = A_0 + \dfrac{B_0 - A_0}{b_0 - a_0}(x - a_0)$
2) Let $\{[a_i, b_i]\}_{i=1}^{2^n}$ be the maximum intervals, where the function f_{n-1} is linear and $\{A_i, B_i\}_{i=1}^{2^n}$ real numbers, such that $A_i = f_{n-1}(a_i)$ and

$B_i = f_{n-1}(b_i)$.

Now $B_n(x)$ is defined as a piecewise linear function, which is linear on the intervals

$$\left[a_i, a_i + \frac{3}{8}(b_i - a_i)\right] \qquad \left[a_i + \frac{3}{8}(b_i - a_i), \frac{1}{2}(b_i + a_i)\right]$$

$$\left[\frac{1}{2}(b_i + a_i), a_i + \frac{7}{8}(b_i - a_i)\right] \qquad \left[\frac{7}{8}(b_i - a_i), b_i\right]$$

with following values at the endpoints

$$f_n(a_i) = A_i\,, \quad f_n(a_i + \frac{3}{8}(b_i - a_i)) = A_i + \frac{5}{8}(B_i - A_i),$$
$$f_n(\frac{1}{2}(b_i + a_i)) = A_i + \frac{1}{2}(B_i - A_i),$$
$$f_n(\frac{7}{8}(b_i - a_i)) = B_i + \frac{1}{8}(B_i - A_i), \qquad f_n(b_i) = B_i$$

Then Bolzano function is defined $B = \lim_{n\in\mathbb{N}} f_n$.

Another example of a continuous nowhere differentiable function was

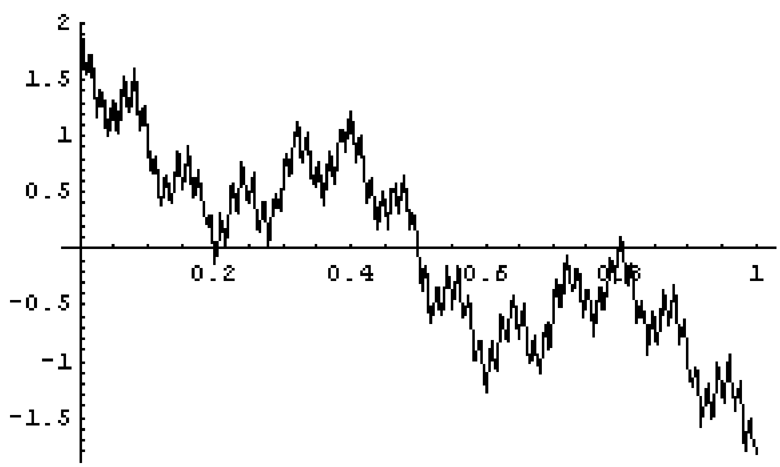


Figure 2.1: Weierstrass function with the parameter values $a = 1/2,\ b = 5$

given by the Swiss mathematician Charles Cellérier (1818-1889) around 1860, but was first published in 1890. This function is very similar to the example given by Weierstrass and it is defined by the formula

$$C(x) = \sum_{k=0}^{\infty} \frac{1}{a^k} sin(a^k x)$$

where a is a sufficiently large ($a > 1000$) even number.

Many other examples were given later. In this chapter I will show that nowhere differentiable functions are typical elements of $\mathcal{C}([0,1])$.

In order to be able to use the Baire theorem, we need the following statement.

Statement 1. $\mathcal{C}([0,1])$ *is a complete metric space.*

Proof. Let $\{f_n \in \mathcal{C}([0,1])\}_{n\in\mathbb{N}}$ be a Cauchy sequence of functions in $\mathcal{C}([0,1])$, which means that for $\forall\varepsilon > 0$, there $\exists n_0 \in \mathbb{N}$ such that for $\forall m, n \in \mathbb{N}$ and $m, n \geq n_0$, $\varepsilon > d(f_m, f_n) = \sup_{x\in[0,1]} |f_m(x) - f_n(x)| \geq |f_m(x) - f_n(x)|$ for every $x \in [0,1]$.
So now we have sequences $\{f_n(x)\}_{n\in\mathbb{N}}$ of real numbers that are Cauchy and because $\mathbb{R}$ is a complete metric space, each of them converges to a limit $\lim_{n\to\infty} f_n(x) \in \mathbb{R}$. We can define a function $f : [0,1] \to \mathbb{R}$ pointwise as $f(x) = \lim_{n\to\infty} f_n(x)$.
Since $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, for $\forall\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that for $\forall m, n \geq n_0$ and for $\forall x \in [0,1]$, $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$. Since $f(x) = \lim_{n\to\infty} f_n(x)$, for $\forall n \geq n_0$, $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$. Because this was true for $\forall x \in [0,1]$, we get that $\sup_{x\in[0,1]} |f_n(x) - f(x)| = d(f_n, f) < \varepsilon$ for every $n \geq n_0$. And thus the function f is the limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$.
Now we just need to show that $f \in \mathcal{C}([0,1])$. Let ε be any positive real number. Now because the function f is the limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$, there exists an $n \in \mathbb{N}$, such that $d(f_n, f) < \frac{\varepsilon}{3}$, which means that for every $x \in [0,1]$ $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$. Furthermore, since f_n is continuous, there is a real number $\delta > 0$, such that for $\forall y \in [0,1]$ $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$.
Now using triangle inequality we get

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(y) - (f_n(x) - f_n(y)) + f_n(x) - f_n(y)| \leq \\ &\leq |f(x) - f_n(x)| + |f(y) - f_n(y)| + |f_n(x) - f_n(y)| < \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

For any real number $\varepsilon > 0$ we have found $\delta > 0$ such that for $\forall y \in [0,1]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$, which means that the function f is

continuous and thus $f \in \mathcal{C}([0,1])$.
We found a limit for any Cauchy sequence in $\mathcal{C}([0,1])$ and thus proved that $\mathcal{C}([0,1])$ is a complete metric space.

□

Now denote A the set of all nowhere differentiable functions in $\mathcal{C}([0,1])$. In order to show that it is the set of typical elements of $\mathcal{C}([0,1])$, I will need following statements.

Statement 2. *The set $B \subset \mathcal{C}([0,1])$ of all functions from $\mathcal{C}([0,1])$ for which $\lim\limits_{h \to 0_+} \left| \frac{f(t+h)-f(t)}{h} \right| = \infty$ at each point $t \in [0,1)$ is dense in $\mathcal{C}([0,1])$.*

Proof. For each $n \in \mathbb{N}$ denote
$B_n = \left\{ f \in \mathcal{C}([0,1]); \forall t \in [0, 1-\frac{1}{n}], \forall \alpha \in (0, 1-t], \exists h \in (0,\alpha], \left| \frac{f(t+h)-f(t)}{h} \right| > n \right\}$.
We can easily see that $B = \bigcap\limits_{n \in \mathbb{N}} B_n$.
Now for $\forall f \in B_n$, $\forall t \in [0, 1-\frac{1}{n}]$ and for $\forall \alpha \in (0, 1-t]$, there $\exists h \in (0, \alpha]$, such that we have $\left| \frac{f(t+h)-f(t)}{h} \right| > n$. Because $h > 0$, $|f(t+h) - f(t)| > h.n$, which means that there exists a real number $\varepsilon > 0$, such that $|f(t+h) - f(t)| > h.n + \varepsilon > h.n$.
Now let $G_{\varepsilon,f}$ be the open ball $\left\{ g \in \mathcal{C}([0,1]);\ d(f,g) < \frac{\varepsilon}{2} \right\}$. For $\forall g \in G_{\varepsilon,f}$ we have

$$\frac{\varepsilon}{2} > d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)| \geq |f(x) - g(x)| \qquad \text{for } \forall x \in [0,1].$$

Since for $\forall n \in \mathbb{N}$ is $t \in [0,1]$ and $t+h \in [0,1]$, $|f(t+h) - g(t+h)| < \dfrac{\varepsilon}{2}$ and $|f(t) - g(t)| < \dfrac{\varepsilon}{2}$.
Now, using the triangle inequality we get the following

$$\begin{aligned} |g(t+h) - g(t)| &= |g(t+h) - g(t) - (f(t+h) - f(t)) + f(t+h) - f(t)| \geq \\ &\geq |f(t+h) - f(t)| - |f(t+h) - g(t+h)| - |f(t) - g(t)| > \\ &> hn + \varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = hn \end{aligned}$$

and hence for every $t \in [0, 1-\frac{1}{n}]$ and for every $\alpha \in (0, 1-t]$ we have found an $h \in (0, \alpha]$, such that $\left| \dfrac{g(t+h) - g(t)}{h} \right| > n$. Since this is true for $\forall g \in G_{\varepsilon,f}$, $G_{\varepsilon,f} \subseteq B_n$ for $\forall f \in \mathcal{C}([0,1])$ and thus B_n is open for $\forall n \in \mathbb{N}$.

Now we just need to prove that each B_n is dense. Suppose that B_n is not dense. Then there will exist an open set $F \subseteq \mathcal{C}([0,1]) \setminus B_n$. Because the set of all polynomials from $[0,1]$ to $\mathbb{R}$ is dense in $\mathcal{C}([0,1])$, there has to exist a polynomial $p \in F$ and since F is open, there also has to exists a real number $\varepsilon > 0$, such that the open ball $F_{\varepsilon,p} = \{f \in \mathcal{C}([0,1]);\ d(f,p) < \varepsilon\} \subseteq F$. Now we take a function $g \in \mathcal{C}([0,1])$, satisfying following conditions

$$\sup_{x\in[0,1]} |\, g(x)| < \varepsilon$$

$$\text{for } \forall t \in [0,1), \qquad \exists \lim_{h\to 0_+} \left|\frac{g(t+h) - g(t)}{h}\right| \neq \infty$$

for $\forall t \in [0,1), \exists \alpha \in (0, 1-t], \forall h \in (0,\alpha]$,

$$\left|\frac{g(t+h) - g(t)}{h}\right| > \lim_{h\to 0_+} \left|\frac{p(t+h) - p(t)}{h}\right| + n$$

An example of such function g is a piecewise linear function constructed using the folowing construction.

Since p is a polynomial, there is a real nubber m, such that

$$\lim_{h\to 0_+} \left|\frac{p(t+h) - p(t)}{h}\right| < m$$

.

Then define $g : [0,1] \to \mathbb{R}$

$$\text{for } t \in \left[\frac{2k\varepsilon}{m+n}, \frac{\varepsilon + 2k\varepsilon}{m+n}\right], \qquad g(t) = (m+n)\left(t - \frac{2k\varepsilon}{m+n}\right) - \frac{\varepsilon}{2}$$

$$\text{for } t \in \left[\frac{\varepsilon + 2k\varepsilon}{m+n}, \frac{2\varepsilon + 2k\varepsilon}{m+n}\right], \qquad g(t) = -(m+n)\left(t - \frac{2\varepsilon + 2k\varepsilon}{m+n}\right) - \frac{\varepsilon}{2}$$

for $k \in \mathbb{N}$. g is linear in each interval $\left[\frac{2k\varepsilon}{m+n}, \frac{\varepsilon+2k\varepsilon}{m+n}\right]$ and $\left[\frac{\varepsilon+2k\varepsilon}{m+n}, \frac{2\varepsilon+2k\varepsilon}{m+n}\right]$.

$$\begin{aligned} g\left(\frac{2k\varepsilon}{m+n}\right) &= (m+n)\left(\frac{2k\varepsilon}{m+n} - \frac{2k\varepsilon}{m+n}\right) - \frac{\varepsilon}{2} = -\frac{\varepsilon}{2} = \\ &= -(m+n)\left(\frac{2k\varepsilon}{m+n} - \frac{2\varepsilon + 2(k-1)\varepsilon}{m+n}\right) - \frac{\varepsilon}{2} = g\left(\frac{2k\varepsilon}{m+n}\right) \\ g\left(\frac{\varepsilon + 2k\varepsilon}{m+n}\right) &= (m+n)\left(\frac{\varepsilon + 2k\varepsilon}{m+n} - \frac{2k\varepsilon}{m+n}\right) - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} = \\ &= -(m+n)\left(\frac{\varepsilon + 2k\varepsilon}{m+n} - \frac{2\varepsilon + 2k\varepsilon}{m+n}\right) - \frac{\varepsilon}{2} = g\left(\frac{\varepsilon + 2k\varepsilon}{m+n}\right) \end{aligned}$$

Hence $g \in \mathcal{C}([0,1])$. Moreover, $g(t) \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ and so $\sup_{x\in[0,1]} |\, g(x)| = \frac{\varepsilon}{2} < \varepsilon$.
$\lim_{h\to 0_+} \left|\frac{g(t+h) - g(t)}{h}\right| = m + n > \lim_{h\to 0_+} \left|\frac{p(t+h) - p(t)}{h}\right| + n$ for all $t \in [0,1)$, hence g constructed like this satisfies the conditions.
Now we put $f = g + p$.

$$d(f,p) = \sup_{x\in[0,1]} |f(x) - p(x)| = \sup_{x\in[0,1]} |\, g(x) + p(x) - p(x)| = \sup_{x\in[0,1]} |\, g(x)| < \varepsilon$$

Hence $f \in F_{\varepsilon,p} \subseteq F$.

But we also have

$$\begin{aligned}
\lim_{h\to 0_+} \left|\frac{f(t+h) - f(t)}{h}\right| &= \lim_{h\to 0_+} \left|\frac{(g+p)(t+h) - (g+p)(t)}{h}\right| = \\
&= \lim_{h\to 0_+} \left|\frac{g(t+h) + p(t+h) - g(t) - p(t)}{h}\right| \geq \\
&\geq \lim_{h\to 0_+} \left(\left|\frac{g(t+h) - g(t)}{h}\right| - \left|\frac{p(t+h) - p(t)}{h}\right|\right) = \\
&= \lim_{h\to 0_+} \left|\frac{g(t+h) - g(t)}{h}\right| - \lim_{h\to 0_+} \left|\frac{p(t+h) - p(t)}{h}\right| \geq \\
&\geq \lim_{h\to 0_+} \left|\frac{g(t+h) - g(t)}{h}\right| - \lim_{h\to 0_+} \left|\frac{p(t+h) - p(t)}{h}\right| > \\
&> \lim_{h\to 0_+} \left|\frac{p(t+h) - p(t)}{h}\right| + n - \lim_{h\to 0_+} \left|\frac{p(t+h) - p(t)}{h}\right| = n
\end{aligned}$$

We have $\lim_{h\to 0_+} \left|\frac{f(t+h) - f(t)}{h}\right| > n$, which means that there exists a real number $\varepsilon > 0$, such that $\lim_{h\to 0_+} \left|\frac{f(t+h) - f(t)}{h}\right| > n+\varepsilon > n$. There also exists $\delta > 0$, such that for $\forall h \in (0,\delta)$, $\left|\lim_{h\to 0_+} \left|\frac{f(t+h) - f(t)}{h}\right| - \left|\frac{f(t+h) - f(t)}{h}\right|\right| < \varepsilon$, thus $\left|\frac{f(t+h) - f(t)}{h}\right| > n$ for $\forall h \in (0,\delta)$, which means that $f \in B_n$.
We have found a function $f \in F \cap B_n$ which was supposed to be empty, which is a contradiction and each B_n has to be dense.
Now we can use Baire theorem and thus $B = \bigcap_{n\in\mathbb{N}} B_n$ is dense in $\mathcal{C}([0,1])$.

□

Statement 2., together with some parts of the proof was borrowed from [5].

All the functions in B are obviously nowhere differentiable, so $B \subseteq A$. Since B is dense, A is also dense. Now we just need to prove that A is G_δ.

Statement 3. *A is G_δ .*

Proof. For $\forall n \in \mathbb{N}$ denote

$$A_n = \Big\{ f \in \mathcal{C}([0,1]); \forall t \in \left[\tfrac{1}{n}, 1 - \tfrac{1}{n}\right], \forall \alpha \in (0, 1-t], \forall \beta \in (0, t]$$

$$\exists h \in (0, \alpha], \quad \left| \frac{f(t+h) - f(t)}{h} \right| > n$$

$$\text{or } \exists h \in [-\beta, 0), \quad \left| \frac{f(t+h) - f(t)}{h} \right| > n$$

$$\text{or } \exists h_1 (0, \alpha] \text{ and } h_2 \in [-\beta, 0),$$

$$\left| \frac{f(t+h_1) - f(t)}{h_1} - \frac{f(t+h_2) - f(t)}{h_2} \right| > \frac{1}{n} \Big\}$$

$$A_n^1 = \Big\{ f \in \mathcal{C}([0,1]); \forall t \in \left[\tfrac{1}{n}, 1 - \tfrac{1}{n}\right], \forall \alpha \in (0, 1-t]$$

$$\exists h \in (0, \alpha], \quad \left| \frac{f(t+h) - f(t)}{h} \right| > n \Big\}$$

$$A_n^2 = \Big\{ f \in \mathcal{C}([0,1]); \forall t \in \left[\tfrac{1}{n}, 1 - \tfrac{1}{n}\right], \forall \beta \in (0, t]$$

$$\exists h \in [-\beta, 0), \quad \left| \frac{f(t+h) - f(t)}{h} \right| > n \Big\}$$

$$A_n^3 = \Big\{ f \in \mathcal{C}([0,1]); \forall t \in \left[\tfrac{1}{n}, 1 - \tfrac{1}{n}\right], \forall \alpha \in (0, 1-t], \forall \beta \in (0, t]$$

$$\exists h_1 (0, \alpha] \text{ and } h_2 \in [-\beta, 0),$$

$$\left| \frac{f(t+h_1) - f(t)}{h_1} - \frac{f(t+h_2) - f(t)}{h_2} \right| > \frac{1}{n} \Big\}$$

We can easily see that $A = \bigcap\limits_{n \in \mathbb{N}} A_n$ and $A_n = A_n^1 \cup A_n^2 \cup A_n^3$ for $\forall n \in \mathbb{N}$.

Let f be any function in A_n^1. Then for $\forall t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ and $\forall \alpha \in (0, 1-t]$ there is $h \in (0, \alpha]$, such that $|f(t+h) - f(t)| > hn$. Hence there exists a real number $\varepsilon > 0$, such that $|f(t+h) - f(t)| > hn + \varepsilon > hn$. Now let $G_{\varepsilon,f}$

be the open ball $\{g \in \mathcal{C}([0,1]); d(f,g) < \frac{\varepsilon}{2}\}$. For $\forall g \in G_{\varepsilon,f}$ we have

$$\frac{\varepsilon}{2} > d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| \geq |f(x) - g(x)| \qquad \text{for } \forall x \in [0,1].$$

Since for $\forall n \in \mathbb{N}$ is $t \in [0,1]$ and $t{+}h \in [0,1]$, $|f(t{+}h) - g(t{+}h)| < \dfrac{\varepsilon}{2}$ and $|f(t) - g(t)| < \dfrac{\varepsilon}{2}$.
Now, using the triangle inequality we get the following

$$\begin{aligned}|g(t{+}h) - g(t)| &= |g(t{+}h) - g(t) - (f(t{+}h) - f(t)) + f(t{+}h) - f(t)| \geq \\ &\geq |f(t{+}h) - f(t)| - |f(t{+}h) - g(t{+}h)| - |f(t) - g(t)| > \\ &> \quad hn + \varepsilon \quad - \quad \frac{\varepsilon}{2} \quad - \quad \frac{\varepsilon}{2} \quad = \quad hn\end{aligned}$$

and hence for every $t \in [\frac{1}{n}], 1 - \frac{1}{n}]$ and for every $\alpha \in (0, 1-t]$ we have found an $h \in (0,\alpha]$, such that $\left|\dfrac{g(t{+}h) - g(t)}{h}\right| > n$. Since this is true for $\forall g \in G_{\varepsilon,f}$, $G_{\varepsilon,f} \subseteq A^1_n$ for $\forall f \in \mathcal{C}([0,1])$ and thus A^1_n is open for $\forall n \in \mathbb{N}$.

Now let f ba any function in A^2_n. For $\forall t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ and for $\forall \beta \in (0,t]$ there is $h \in [-\beta, 0)$, such that $|f(t{+}h) - f(t)| > |h|\, n$. And again, there is a real number $\varepsilon > 0$, such that $|f(t{+}h) - f(t)| > |h|\, n + \varepsilon > |h|\, n$. Now for $\forall g \in G_{\varepsilon,f}$ we have

$$\frac{\varepsilon}{2} > d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| \geq |f(x) - g(x)| \qquad \text{for } \forall x \in [0,1].$$

Since for $\forall n \in \mathbb{N}$ is $t \in [0,1]$ and $t{+}h \in [0,1]$, $|f(t{+}h) - g(t{+}h)| < \dfrac{\varepsilon}{2}$ and $|f(t) - g(t)| < \dfrac{\varepsilon}{2}$.
Now again, using the triangle inequality we get

$$\begin{aligned}|g(t{+}h) - g(t)| &= |g(t{+}h) - g(t) - (f(t{+}h) - f(t)) + f(t{+}h) - f(t)| \geq \\ &\geq |f(t{+}h) - f(t)| - |f(t{+}h) - g(t{+}h)| - |f(t) - g(t)| > \\ &> \quad |h|\, n + \varepsilon \quad - \quad \frac{\varepsilon}{2} \quad - \quad \frac{\varepsilon}{2} \quad = \quad |h|\, n\end{aligned}$$

For every $t \in [\frac{1}{n}], 1 - \frac{1}{n}]$ and for every $\beta \in (0,t]$ we have found an $h \in [-\beta, 0)$, such that $\left|\dfrac{g(t{+}h) - g(t)}{h}\right| > n$. Since this is true for $\forall g \in G_{\varepsilon,f}$, $G_{\varepsilon,f} \subseteq A^2_n$

for $\forall f \in \mathcal{C}([0,1])$ and thus A_n^2 is open for $\forall n \in \mathbb{N}$.

And similarly for A_n^3. Let f ba any function in A_n^3. For $\forall t \in \left[\frac{1}{n}, 1-\frac{1}{n}\right]$ and for $\forall \alpha \in (0, 1-t]$ and $\forall \beta \in (0,t]$ there are $h_1 \in (0,\alpha]$ and $h_2 \in [-\beta, 0)$, such that $|h_2 f(t+h_1) - h_2 f(t) - h_1 f(t+h_2) + h_1 f(t)| > |h_1 h_2|\, n$. And again, there is a real number $\varepsilon > 0$, such that
$|h_2 f(t+h_1) - h_2 f(t) - h_1 f(t+h_2) + h_1 f(t)| > |h_1 h_2|\, n + \varepsilon > |h_1 h_2|\, n$. Now let $F_{\varepsilon,f}$ be the open ball $\{g \in \mathcal{C}([0,1]); d(f,g) < \frac{\varepsilon}{4}\}$. For $\forall g \in F_{\varepsilon,f}$ we have

$$\frac{\varepsilon}{2} > d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| \geq |f(x) - g(x)| \qquad \text{for } \forall x \in [0,1].$$

For $\forall n \in \mathbb{N}$ is $t \in [0,1]$, $t+h_1 \in [0,1]$ and $t+h_2 \in [0,1]$, thus
$|f(t+h_1) - g(t+h_1)| < \dfrac{\varepsilon}{2}$, $|f(t+h_2) - g(t+h_2)| < \dfrac{\varepsilon}{2}$ and $|f(t) - g(t)| < \dfrac{\varepsilon}{2}$.
And again, using the triangle inequality we get

$$\begin{aligned}
&|h_2 g(t+h_1) - h_2 g(t) - h_1 g(t+h_2) + h_1 g(t)| = \\
&\quad = |h_2 g(t+h_1) - h_2 g(t) - h_1 g(t+h_2) + h_1 g(t) - \\
&\qquad - \big(h_2 f(t+h_1) - h_2 f(t) - h_1 f(t+h_2) + h_1 f(t)\big) + \\
&\qquad + h_2 f(t+h_1) - h_2 f(t) - h_1 f(t+h_2) + h_1 f(t)| \geq \\
&\quad \geq |h_2 f(t+h_1) - h_2 f(t) - h_1 f(t+h_2) + h_1 f(t)| - |h_2|\,|f(t+h_1) - g(t+h_1)| - \\
&\qquad - |h_2|\,|f(t) - g(t)| - |h_1|\,|f(t+h_2) - g(t+h_2)| - |h_1|\,|f(t) - g(t)| > \\
&\quad > \quad |h_1 h_2|\, n + \varepsilon \quad - \quad \frac{\varepsilon}{4} \quad - \quad \frac{\varepsilon}{4} \quad - \quad \frac{\varepsilon}{4} \quad - \quad \frac{\varepsilon}{4} \quad = \quad |h_1 h_2|\, n
\end{aligned}$$

For every $t \in [\frac{1}{n}], 1-\frac{1}{n}]$ and for every $\alpha \in (0, 1-t]$ and $\beta \in (0,t]$ we have found $h_1 \in (0,\alpha]$ and $h_2 \in [-\beta, 0)$, such that
$|h_2 g(t+h_1) - h_2 g(t) - h_1 g(t+h_2) + h_1 g(t)| > |h_1 h_2|\, n$. Since this is true for $\forall g \in F_{\varepsilon,f}$, $\quad F_{\varepsilon,f} \subseteq A_n^3$ for $\forall f \in \mathcal{C}([0,1])$ and thus A_n^3 is open for $\forall n \in \mathbb{N}$.

$A_n = A_n^1 \cup A_n^2 \cup A_n^3$ is the union of three open sets and hence open. $A = \bigcap_{n\in\mathbb{N}} A_n$ is G_δ.

□

The set A of all nowhere differentiable functions in $\mathcal{C}([0,1])$ is the set of typical elements of $\mathcal{C}([0,1])$.

Chapter 3

Discontinua in the space of nonempty compact sets in $\mathbb{R}^2$

The first known example of a discontinuum was discovered in 1875 by a British mathematician Henry John Stephen Smith (1826-1883) and introduced in 1883 by a German mathematician Georg Ferdinand Ludwig Phillip Cantor (1845-1918). It is known as the Cantor ternary, the Cantor set or the Cantor discontinuum. There are more ways how to describe this set. We can get it from the closed unit interval $[0, 1]$, using the following constuction: in the first step we delete the open interval $(\frac{1}{3}, \frac{2}{3})$, in the second step we delete the open intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ and so on, in each step deleting the open middle thirds of all line segments that are left. The points left after this process form the Cantor set.

Another way how to describe this set is through considering the points

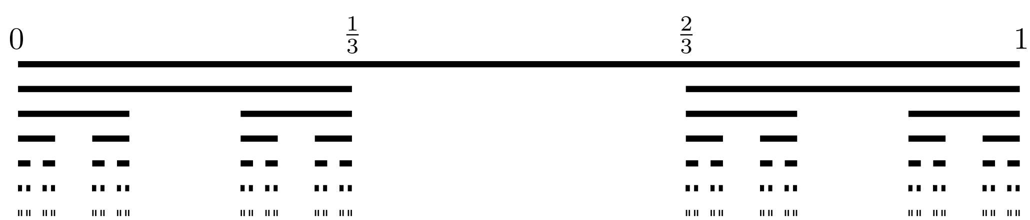


Figure 3.1: First few steps of the construction of the Cantor ternary set

in the interval $[0, 1]$ in the ternary notation. The number $\frac{1}{3}$ can be writen as $0{,}1_3 = 0{,}222\ldots_3$ and $\frac{2}{3}$ can be writen as $0{,}1222\ldots_3 = 1{,}2_3$. Since in the first step of the construcion all the points between these two numbers are removed, all the remaining numbers can be written in the ternary notation with 0 or 2 on the first decimal place. Number $\frac{1}{9}$ can be writen as $0{,}01_3 = 0{,}00222\ldots_3$ and $\frac{2}{9}$ as $0{,}01222\ldots_3 = 1{,}02_3$. Therefore after the first step, where the intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{2}{3}+\frac{1}{9}, \frac{2}{3}+\frac{2}{9})$ are removed, all the remaining numbers can be writen with 0's and 2's on the first two decimal places. The Cantor ternary set is hence the set of all numbers in $[0, 1]$ that can be writen in ternary notation with only 0's and 2's on their decimal places. And yet another way how to describe the Cantor set is as a space that is homeomorphic to $\{0, 1\}^\omega$, which is obvious from the ternary notation.

Another example of a discontinuum is called the Smith-Volterra-Cantor set or the fat Cantor set. It is constructed similarly to the Cantor ternary set, but instead of deleting the middle thirds of intervals, we delete the open middle quarters. This gives a space that is homeomorphic to the Cantor set,

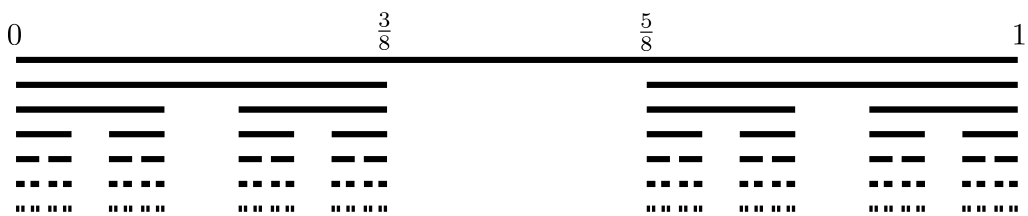


Figure 3.2: First few steps of the construction of the Smith-Volterra-Cantor set

but, unlike the Cantor ternary set, which has the Lebesque measure zero, the fat Cantor set has the Lebesque measure $\frac{1}{2}$.

In 1970 Stephen Willard proved that any two totally disconnected, perfect compact metric spaces are homeomorphic [2, p. 216-217] and thus, since the Cantor set is a totally disconnected, perfect compact metric space, that it is the only such space up to a homeomorphism. Which gives an alternative definition of a discontinuum as a topological space, which is homeomorphic to the Cantor set. One might think that there can't be many sets homeomorphic to such a bizarre space, but opposite is the truth. In fact, discontinua form a set of typical elements of $\mathfrak{X}^2$, as I will show in this chapter.

In order to be able to use the Baire theorem and to define the set of typical elements, we need to prove the following statements.

Statement 4. *The space $\mathfrak{X}^2$, with the Hausdorff metric, is complete.*

Proof. Let $\{A_n; A_n \in \mathfrak{X}^2\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathfrak{X}^2$ and let B be the set of all Cauchy sequences $\{a_n; a_n \in A_n\}_{n\in\mathbb{N}}$. Since $\mathbb{R}^2$ is complete, all the sequences in B have limits in $\mathbb{R}^2$. Therefore we can define set $A = \{a = \lim_{n\in\mathbb{N}} a_n;\ \{a_n\}_{n\in\mathbb{N}} \in B\}$.

Now we will show that the closure $\overline{A}$ is the limit of the sequence $\{A_n\}_{n\in\mathbb{N}}$. Since $\{A_n\}_{n\in\mathbb{N}}$ is Cauchy, for $\forall\varepsilon > 0$ we have $n_0 \in \mathbb{N}$ such that for $\forall m, n \geq n_0$ the distance $d(A_m, A_n) < \varepsilon$. Since d is symmetric, we can put $m \leq n$. Hence $E_\varepsilon(A_m) \supseteq A_n$ for $\forall n \geq m \geq n_0$, thus for every Cauchy sequence $\{a_n;\ a_n \in A_n\}_{n\in\mathbb{N}}$ its elements a_n are in $E_\varepsilon(A_m)$ for $\forall n \geq m$ and so the limit $\lim_{n\in\mathbb{N}} a_n \in \overline{E_\varepsilon(A_m)}$. Since this is true for every $\{a_n\}_{n\in\mathbb{N}} \in B$, we get $A \subseteq \overline{E_\varepsilon(A_m)} \subset E_{2\varepsilon}(A_m)$ for $\forall m \geq n_0$.

In order to prove that $E_{2\varepsilon}(A) \supseteq A_m$ for $\forall m \geq n_0$ for some $n_0 \in \mathbb{N}$, we need to find a sequence $\{a_n; a_n \in A_n\}_{n\in\mathbb{N}}$ for each $x \in A_m$, such that $\left|x - \lim_{n\in\mathbb{N}} a_n\right| < 2\varepsilon$. First we will pick a strictly increasing sequence of integers $\{m_k\}_{k\in\mathbb{N}}$, such that for $\forall m, n \geq m_k$ the distance $d(A_m, A_n) < \frac{\varepsilon}{2^k}$. Such sequence can be found, since $\{A_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Now we put $n_0 = m_1$ and for an arbitrary $m \geq m_1$ and $x \in A_m$ we put $a_m = x$ and pick $a_{m_2} \in A_{m_2}$, such that $|x - a_{m_2}| < \frac{\varepsilon}{2}$. Such a_{m_2} can be found, because $m_2, m \geq m_1$ and thus $d(A_m, A_{m_2}) < \frac{\varepsilon}{2}$ follows from the definition of the sequence $\{m_k\}_{k\in\mathbb{N}}$, hence $E_{\frac{\varepsilon}{2}}(A_{m_2}) \supseteq A_m$ and so there is a point $a_{m_2} \in A_{m_2}$, such that $|x - a_{m_2}| < \frac{\varepsilon}{2}$. All the other elements a_n for $n < m_2$ can be picked arbitrary.

We define the rest of the sequence using the induction. Let's assume that we already have all a_m for $m \leq m_k$. Having an integer $m_k \leq n \leq m_{k+1}$, we know that $d(A_n, A_{m_k}) < \frac{\varepsilon}{2^k}$ from the definition of the sequence $\{m_k\}_{k\in\mathbb{N}}$ and so $E_{\frac{\varepsilon}{2^k}}(A_n) \supseteq A_{m_k}$ and so we can find a point $a_n \in A_n$, such that $|a_{m_k} - a_n| < \frac{\varepsilon}{2^k}$.

We can easily see that for $\forall m, n \in \mathbb{N}$, such that $m_k \leq m, n \leq m_{k+1}$ the distance $|a_m - a_n| = |a_m - a_{m_k} + a_{m_k} - a_n| \leq |a_m - a_{m_k}| + |a_{m_k} - a_n| < \frac{\varepsilon}{2^k} + \frac{\varepsilon}{2^k} = \frac{\varepsilon}{2^{k-1}}$ for $k \geq 2$, which implies that $|a_{m_k} - a_{m_{k+1}}| < \frac{\varepsilon}{2^{k-1}}$. Furthermore, for $\forall n \geq m \geq m_k$ we can find a $m_j \in \mathbb{N}$, such that $m_j \leq n \leq m_{j+1}$ and thus we get the following

$$
\begin{aligned}
|a_m - a_n| &= \left|a_m - a_{m_k} + a_{m_k} - a_{m_{k+1}} + \cdots + a_{m_{j-1}} - a_{m_j} + a_{m_j} - a_n\right| \leq \\
&\leq |a_{m_k} - a_m| + \left|a_{m_k} - a_{m_{k+1}}\right| + \cdots + \left|a_{m_{j-1}} - a_{m_j}\right| + \left|a_{m_j} - a_n\right| < \\
&< \quad \frac{\varepsilon}{2^k} \quad + \quad \sum_{i=k}^{j} \frac{\varepsilon}{2^i} \quad < \quad \frac{\varepsilon}{2^{k-2}}
\end{aligned}
$$

Therefore $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence of points in $\mathbb{R}^2$ and hence converges to a limit $a = \lim\limits_{n\in\mathbb{N}} a_n$ and so we can find $m \in \mathbb{N}$, such that for $\forall n \geq m$, $|a_n - a| < \frac{\varepsilon}{2}$. We take $i \geq 2$, such that $m_i \geq m$. Now we get the following inequality

$$
\begin{aligned}
|x-a| &= \left|x - a_{m_2} + a_{m_2} - a_{m_3} + \cdots + a_{m_{i-1}} - a_{m_i} + a_{m_i} - a\right| \leq \\
&\leq |x - a_{m_2}| + |a_{m_2} - a_{m_3}| + \cdots + \left|a_{m_{i-1}} - a_{m_i}\right| + |a_{m_i} - a| < \\
&< \quad \frac{\varepsilon}{2} \quad + \quad \sum_{k=2}^{i-1} \frac{\varepsilon}{2^{k-1}} \quad + \quad \frac{\varepsilon}{2} \quad < \quad 2\,\varepsilon
\end{aligned}
$$

For $n_0 = m_1$ and for each $x \in A_m$ we have found a sequence $\{a_n; a_n \in A_n\}_{n\in\mathbb{N}}$, such that $\left|x - \lim\limits_{n\in\mathbb{N}} a_n\right| < 2\varepsilon$ and thus for $\forall m \geq n_0$, $E_{2\varepsilon}(A) \supseteq A_m$.

Hence for a sufficiently large $n_0 \in \mathbb{N}$ and for $\forall m \geq n_0$, $d(A, A_m) < 2\varepsilon$. We can easiy see that $d(A, \overline{A}) < \delta$ for $\forall \delta > 0$ and so for $\forall m \geq n_0$ we get $d(A_m, \overline{A}) \leq d(A_m, A) + d(A, \overline{A}) < 3\varepsilon$. Since ε was an arbitrary positive real number, we can put $\overline{A} = \lim\limits_{n\in\mathbb{N}} A_n$. It is obviously closed and because $E_{3\varepsilon}(A_m) \supseteq \overline{A}$ for some $m \in \mathbb{N}$, it is also bounded and hence $\overline{A} \in \mathfrak{X}^2$. For an arbitrary Cauchy sequence in $\mathfrak{X}^2$ we have found its limit in $\mathfrak{X}^2$ and hence proven that the space $\mathfrak{X}^2$ of all nonempty compact sets in $\mathbb{R}^2$, endowed with the Hausdorff metric, is complete.

$\square$

Statement 5. *Let X be any space of compact subsets of $\mathbb{R}^2$, endowed with the Hausdorff metric. Then the Vietoris topology τ_V and the topology τ_H induced by the Hausdorff metric d_H coincide on X.*

Proof. $\mathcal{B}_H = \Big\{B(A,\varepsilon) = \{B \in X; E_\varepsilon(A) \supseteq B \;\&\; E_\varepsilon(B) \supseteq A\}; A \in X, \varepsilon > 0\Big\}$ and $\mathcal{B}_V = \Big\{V(A_1,\ldots,A_k) = \{B \in X; \bigcup\limits_{n=1}^{k} A_n \supseteq B \;\&\; B \cap A_n \neq \emptyset \text{ for } \forall n = 1,\ldots,k\}; k \in \mathbb{N}, A_n \subseteq \mathbb{R}^2 \text{ open in } \mathbb{R}^2\Big\}$ are the bases of topologies τ_H and τ_V, respectively.

If for $\forall B \in \mathfrak{X}^2$ and for $\forall B(A,\varepsilon) \in \mathcal{B}_H$, such that $B \in B(A,\varepsilon)$, we could find $V(A_1,\ldots,A_k) \in \mathcal{B}_V$, such that $B \in V(A_1,\ldots,A_k) \subseteq B(A,\varepsilon)$ and for $\forall B \in \mathfrak{X}^2$ and for $\forall V(A_1,\ldots,A_k) \in \mathcal{B}_V$, such that $B \in V(A_1,\ldots,A_k)$, we could find $B(A,\varepsilon) \in \mathcal{B}_H$, such that $B \in B(A,\varepsilon) \subseteq V(A_1,\ldots,A_k)$, the two topologies would coincide.

Let $B \in X$ be an arbitrary set and $B(A, \varepsilon) \in \mathcal{B}_H$ any open ball, such that $B \in B(A, \varepsilon)$. Now put $A_1 = E_\varepsilon(A)$ and pick $A_n \subseteq \mathbb{R}^2$ for $n = 2, \ldots, k$ such that

$$\text{for } \forall\, n{=}2, \ldots, k,\ \exists\, a_n \in \mathbb{R}^2, \text{ such that } A_n = B(a_n, \tfrac{\varepsilon}{2})$$
$$\text{for } \forall\, n{=}2, \ldots, k,\ A_n \cap A \neq \emptyset$$
$$\text{for } \forall\, n{=}2, \ldots, k,\ A_n \cap B \neq \emptyset$$
$$\bigcup_{n=2}^{k} \supseteq A$$

Such sets can be found, because $B \in B(A, \varepsilon)$, hence $A \subseteq E_\varepsilon(B)$ and therefore for $\forall\, a \in A$, there $\exists\, b \in B$, such that $|\, a - b| < \varepsilon$ and so there $\exists\, x \in \mathbb{R}^2$, such that $|\, a - x| < \frac{\varepsilon}{2} > |\, x - b|$. Therefore we can find a cover of A with $\frac{\varepsilon}{2}$-balls, which are intersecting B. Since A is bounded, A_n can be picked in such a way that k is finite. Furthermore, $A_n \subseteq E_\varepsilon(A)$ for $n{=}2, \ldots, k$, thus $\bigcup\limits_{n=1}^{k} A_n = E_\varepsilon(A)$.

Since $B \in B(A, \varepsilon)$, it must be $B \subseteq E_\varepsilon(A) = \bigcup\limits_{n=1}^{k} A_n = A_1$. The sets A_n for $n = 2, \ldots, k$ were picked in such a way that $A_n \cap B \neq \emptyset$, so we get $B \in V(A_1, \ldots, A_k)$.

Having $C \in V(A_1, \ldots, A_k)$, we know that $C \subseteq A_1 = E_\varepsilon(A)$. For $\forall a \in A$, there $\exists\, A_n$, such that $a \in A_n$ and since $C \cap A_n \neq \emptyset$, there $\exists\, c \in C \cap A_n$. Therefore $|\, a - c| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, hence $A \subseteq E_\varepsilon(C)$, which means that the set $C \in B(A, \varepsilon)$ and $V(A_1, \ldots, A_k) \subseteq B(A, \varepsilon)$.

Let $B \in X$ be an arbitrary set and $V(A_1, \ldots, A_k) \in \mathcal{B}_V$ any set, such that $B \in V(A_1, \ldots, A_k)$. Pick $\varepsilon > 0$, such that

$$\text{for } \forall n = 1, \ldots, k, \text{ there } \exists a_n \in A_n \cap B, \text{ such that } B(a_n, 2\varepsilon) \subseteq A_n$$
$$E_\varepsilon(B) \subseteq \bigcup_{n=1}^{k} A_n$$

Such ε can be found, since A_n are all open and $B \subseteq \bigcup\limits_{n=1}^{k} A_n$ is closed and $B \subset \mathbb{R}^2$ is nonempty and bounded, so it cannot be both closed and open. Now put $A = B \cup \bigcup\limits_{n=1}^{k} \overline{B(a_n, \frac{\varepsilon}{2})}$, where a_n are the points from the first condition for ε.

Obviously, A is closed, bounded and nonempty. Furthermore $A \subseteq E_\varepsilon(B)$ and $E_\varepsilon(A) \subseteq A \subseteq B$, thus $B \in B(A, \varepsilon)$.

Having $C \in B(A, \varepsilon)$, we know $C \subseteq E_\varepsilon(A) = E_\varepsilon(B) \cup \bigcup_{n=1}^{k} \overline{B(a_n, \frac{3\varepsilon}{2})} \subseteq \bigcup_{n=1}^{k} A_n$. Moreover, for $\forall\, n = 1, \dots, k$, there $\exists\, a_n \in A \cap A_n$, such that $B(a_n, 2\varepsilon) \subseteq A_n$ and since $A \subseteq E_\varepsilon(C)$, there $\exists\, c \in C$, such that $|\, a_n - c| < \varepsilon$ and hence $c \in A_n$. Therefore $C \cap A_n \neq \emptyset$ for $\forall\, n = 1, \dots, k$, which means that $C \in V(A_1, \dots, A_k)$ and $B(A, \varepsilon) \subseteq V(A_1, \dots, A_k)$.

□

Since this statement was true for any space of compact subsets of $\mathbb{R}^2$, it is true for $\mathfrak{X}^2$ as well. Now we have everything ready for the following statement.

Statement 6. *Denote A the set of all discontinua in the plane. Then A is dense and G_δ in the space $\mathfrak{X}^2$ of all compact sets in the plane with the Vietoris topology.*

Proof. For $n \in \mathbb{N}$ put $\mathcal{B}_n^1 = \left\{B(x, \frac{1}{4n}); x \in \mathbb{R}^2\right\}$ and $\mathcal{A}_n^1 = \Big\{\{A_1, \dots, A_k\} \subset \mathcal{B}_n^1;$ such that for $\forall A_i$, there $\exists\, A_j$, such that $A_i \cap A_j = \emptyset$ and $A_j \subseteq E_{\frac{1}{2n}}(A_i)\Big\}$ and then put $A_n^1 = \bigcup_{\{A_1,\dots,A_k\} \in \mathcal{A}_n^1} V(A_1, \dots, A_k)$. It is obvious that A_n^1 is the set of all $X \in \mathfrak{X}^2$, such that for $\forall x \in X$ the point $x \in E_{\frac{1}{n}}(X \setminus \{x\})$, because any set with this property can be covered with a finite number of $\frac{1}{4n}$-balls, satysfying the condition in $\mathcal{A}_n^1$, but if $\exists\, x \in X$, such that $x \notin E_{\frac{1}{n}}(X \setminus \{x\})$, then there has to be a $\frac{1}{4n}$-ball B, such that $x \in B$, hence $B \subseteq B(x, \frac{1}{2n})$ and $E_{\frac{1}{2n}}(B) \subseteq B(x, \frac{1}{n})$, which is disjoint with $X \setminus \{x\}$ and so any $\frac{1}{4n}$-ball in $E_{\frac{1}{2n}}(B)$, disjoint with B, is disjoint with X and therefore $X \notin A_n^1$.
A_n^1 is obviously open for $\forall n \in \mathbb{N}$. Moreover, having any $V(B_1, \dots, B_k) \in \mathcal{B}_V$, we can pick points $x_i, y_i \in B_i$, such that $|x_i - y_i| < \frac{1}{n}$ for $\forall i = 1, \dots, k$, since B_i are all open. Then we can easily find open $\frac{1}{4n}$-balls B_i^1, B_i^2, such that $x_i \in B_i^1$, $y_i \in B_i^2$, $B_i^1 \subseteq E_{\frac{1}{2n}}(B_i^2)$ and $B_i^2 \subseteq E_{\frac{1}{2n}}(B_i^1)$. Therefore we get that $\{B_1^1, B_1^2, \dots, B_k^1, B_k^2\} \in \mathcal{A}_n^1$ and hence $\{x_1, y_1, \dots, x_k, y_k\}$ lies in $V(B_1, \dots, B_k) \cap A_n^1$, which is therefore nonempty. Since any set $X \in \tau_V$ is a union of sets from $\mathcal{B}_V$, the intersection $X \cap A_n^1$ will be nonempty as well, hence the set A_n^1 is dense for $\forall n \in \mathbb{N}$.
It is obvious that the set $A^1 = \bigcap_{n \in \mathbb{N}} A_n^1$ is the set of all $X \in \mathfrak{X}^2$, such that X

contains no isolated point. Since all the A^1_n are open and dense in τ_V, A^1 is G_δ and due to the Baire theorem also dense in τ_V.
Now for $n \in \mathbb{N}$ put $\mathcal{B}^2_n = \left\{B(x, \frac{1}{n}); x \in \mathbb{R}^2\right\}$ and $\mathcal{A}^2_n = \left\{\{A_1, \ldots, A_k\} \subset \mathcal{B}^2_n;\right.$ such that $A_i \cap A_j = \emptyset$ for any $\left. i \neq j\right\}$ and then put $A^2_n = \bigcup_{\{A_1,\ldots,A_k\} \in \mathcal{A}^2_n} V(A_1, \ldots, A_k)$.
A^2_n is obviously open for $\forall n \in \mathbb{N}$. Moreover, having any $V(B_1, \ldots, B_k) \in \mathcal{B}_V$, we can pick a point $x_i \in B_i$ for $\forall i = 1, \ldots, k$. We get a finite number of isolated points, therefore we can easily find disjoint open $\frac{1}{n}$-balls $B_1, \ldots, B_m$, such that for $\forall i = 1, \ldots, k$, there $\exists j$, such that $x_i \in B_j$. $\{B_1, \ldots, B_m\} \in \mathcal{A}^2_n$ and thus $\{x_1, \ldots, x_k\}$ lies in $V(B_1, \ldots, B_k) \cap A^2_n$, which is hence nonempty. Since any set $X \in \tau_V$ is a union of sets from $\mathcal{B}_V$, the intersection $X \cap A^2_n$ will be nonempty as well, therefore the set A^2_n is dense for $\forall n \in \mathbb{N}$.
It is obvious that the set $A^2 = \bigcap_{n \in \mathbb{N}} A^2_n$ is the set of all $X \in \mathfrak{X}^2$, such that any connected subset of X consists of just one point. Since all the A^2_n are open and dense in τ_V, A^2 is G_δ and due to the Baire theorem also dense in τ_V.
Since $A = A^1 \cap A^2$ and both A^1 and A^2 are dense and G_δ in τ_V, the set A, consisting of all discontinua in $\mathbb{R}^2$ is also dense and G_δ in τ_V.

□

Some ideas in this proof are borrowed from [6].

Since the set A of all discontinua in $\mathbb{R}^2$ is G_δ and dense in τ_V, it is also G_δ and dense in τ_H and hence in the metric space of all compact subsets of $\mathbb{R}^2$ with the Hausdorff metric, which means that it is the set of typical elements of $\mathfrak{X}^2$.

Chapter 4

Indecomposable continua in the space of all continua in $\mathbb{R}^2$

The first known example of an indecomposable continuum comes from 1910 and is due to the Dutch mathematician Luitzen Egbertus Jan Brouwer (1881-1966). He constructed it as a counterexample to a claim made in 1904 by A. Schoenflies, who stated that there do not exist three distinct regions in the plane with a common boundary. Brouwer constructed planar continua which were the common boundary of three regions and showed that they are indecomposable.
One of the simplest examples of a planar indecomposable continuum is called the Knaster's bucket handle, named after the Polish mathematician Bronisław Knaster (1893-1990), who discovered it. It is constructed from a square by an infinite process, repeatedly removing parts of the original square. Another examples of planar indecomposable continua are the pseudoarc, which was discovered by B. Knaster as well and the Lakes of Wada continuum.

In this chapter I will show that indecomposable continua are the typical elements of the space $\mathcal{C}$ of all planar continua.

In order to be able to use the Baire theorem and to define the set of typical elements of $\mathcal{C}$, endowed with the Hausdorff metric d, we need to prove that $(\mathcal{C}, d_H)$ is a complete metric space. Furthermore, the Vietoris topology τ_V coincides with the topology τ_H, induced by the Hausdorff metric on the space $\mathcal{C}$, since these two topologies coincide on any space of compact subsets of $\mathbb{R}^2$, as was proved in the previous chapter in the statement 5. and $\mathcal{C}$ is a space of all planar continua, which are nonempty connected compact subsets of $\mathbb{R}^2$. Therefore we will be able to use the Vietoris topology to prove that indecomposable continua are typical elements of $\mathcal{C}$. So all we now need is the following statement.

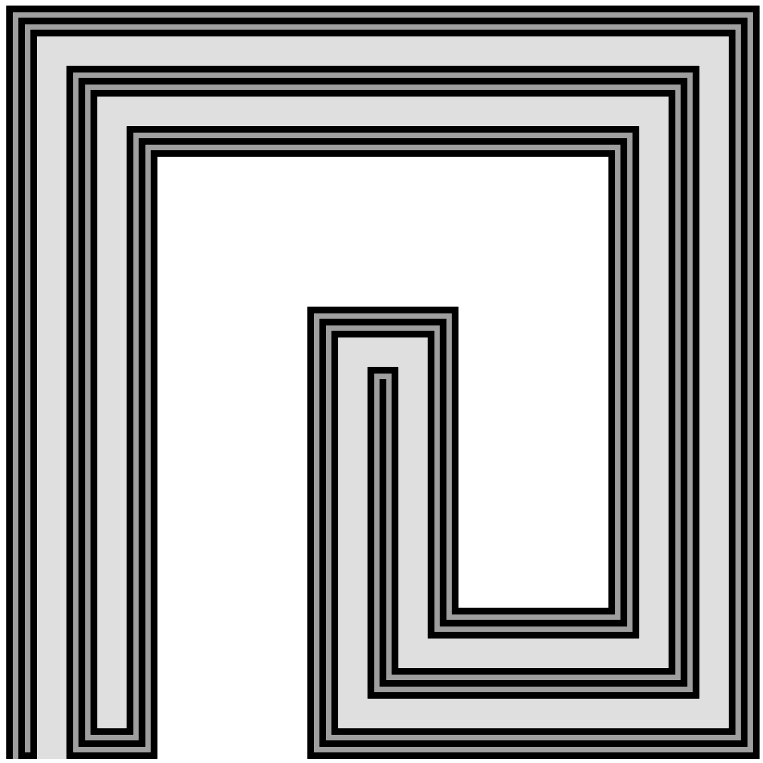


Figure 4.1: The Knaster's bucket handle

Statement 7. *The space $\mathcal{C}$, consisting of all planar continua, together with the Hausdorff metric, is complete.*

Proof. Let $\{A_n; A_n \in \mathcal{C}\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}$. In the previous chapter, in statement 4., we have already proven that a set $\overline{A}$, defined as the closure of a set A, consisting of the limit points of all Cauchy sequences $\{a_n; a_n \in A_n\}_{n\in\mathbb{N}}$, is a limit of the sequence $\{A_n\}_{n\in\mathbb{N}}$ and $\overline{A}$ is compact. All we need, to prove the statement, is that if all A_n are connected, $\overline{A}$ will be connected as well.
Let's assume that all A_n are connected, while $\overline{A}$ is not. Then $\exists\, B, C \subset \mathbb{R}^2$ closed sets, such that $B \cap C = \emptyset$, $B \cup C = \overline{A}$ and $B \neq \emptyset \neq C$ and $\exists\, \varepsilon > 0$,

such that $E_\varepsilon(B) \cap E_\varepsilon(C) = \emptyset$, $E_\varepsilon(B) \cup E_\varepsilon(C) \supset \overline{A}$, $E_\varepsilon(B) \cap \overline{A} \neq \emptyset$ and $E_\varepsilon(C) \cap \overline{A} \neq \emptyset$. Since $\overline{A}$ is the limit of $\{A_n\}_{n\in\mathbb{N}}$, there $\exists n_0 \in \mathbb{N}$, such that for $\forall m \geq n_0$ the set $A_m \subset E_\varepsilon(\overline{A}) = E_\varepsilon(B) \cup E_\varepsilon(C)$. Since A_m is connected, it must be a subset of either $E_\varepsilon(B)$ or $E_\varepsilon(C)$. Let's assume that $A_m \subset E_\varepsilon(B)$ and $A_m \cap E_\varepsilon(C) = \emptyset$. Then $E_\varepsilon(A_m) \subseteq E_\varepsilon\big(E_\varepsilon(B)\big) \subseteq \mathbb{R}^2 \smallsetminus C$. Hence $E_\varepsilon(A_m) \cap C = \emptyset$ and since $C \subset \overline{A}$, we get that $\overline{A} \nsubseteq E_\varepsilon(A_m)$ and therefore $d(A_m, \overline{A}) > \varepsilon$, which is contradiction. Thus $\overline{A} = \lim\limits_{n\in\mathbb{N}} A_n$ has to be connected as well and $\overline{A} \in \mathcal{C}$, hence the space $\mathcal{C}$, consisting of all planar continua, together with the Hausdorff metric is complete.

□

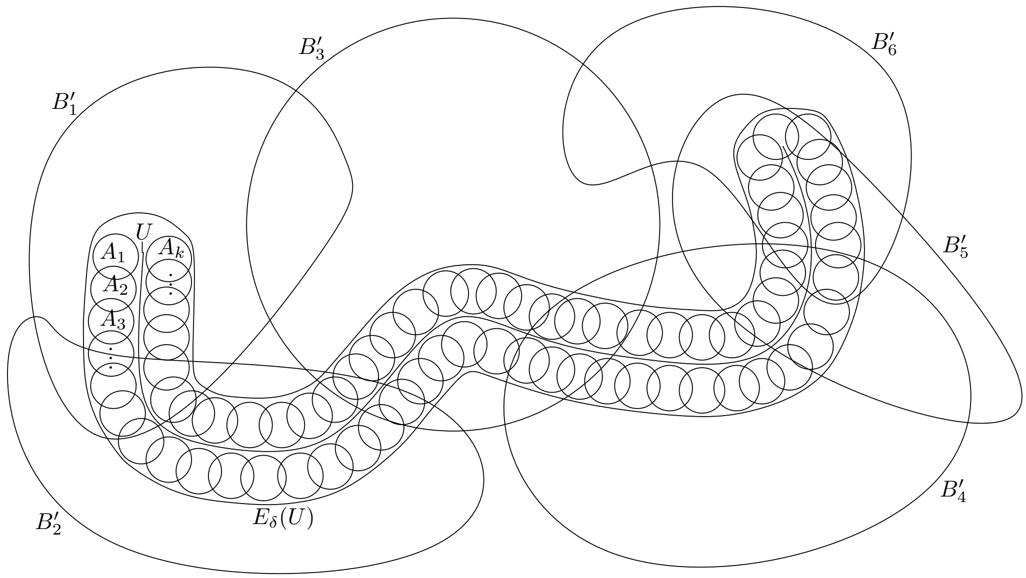
Now we have everything we need to formulate the following statement.

Statement 8. *Denote A the set of all indecomposable continua in the plane. Then A is dense and G_δ in the space $\mathcal{C}$ of all compact sets in plane with the Vietoris topology.*

Proof. For $n \in \mathbb{N}$ put $\mathcal{B}_n^1 = \big\{B(x,\varepsilon); x \in \mathbb{R}^2, \varepsilon \in (0, \frac{1}{4n}]\big\}$. Let's say that a finite collection $\{A_1, \dots, A_k\} \subset \mathcal{B}_n^1$, such that $\bigcup\limits_{i=1}^{k} A_i$ is connected, has a property $\mathfrak{P}_n$ if, having any $\mathcal{A}_1, \mathcal{A}_2 \subseteq \{A_1, \dots, A_k\}$, such that the sets $\bigcup \mathcal{A}_1$ and $\bigcup \mathcal{A}_2$ are both connected and $\mathcal{A}_1 \cup \mathcal{A}_2 = \{A_1, \dots, A_k\}$, for $\forall A_i \in \mathcal{A}_1$, there $\exists A_j \in \mathcal{A}_2$, such that $A_j \subseteq E_{\frac{1}{2n}}(A_i)$ or for $\forall A_j \in \mathcal{A}_2$, there $\exists A_i \in \mathcal{A}_1$, such that $A_i \subseteq E_{\frac{1}{2n}}(A_j)$.

Now put $\mathcal{A}_n^1 = \Big\{\{A_1, \dots, A_k\} \subset \mathcal{B}_n^1;$ such that $\{A_1, \dots, A_k\}$ has the property $\mathfrak{P}_n\Big\}$ and then put $A_n^1 = \bigcup\limits_{\{A_1,\dots,A_k\}\in\mathcal{A}_n^1} V(A_1, \dots, A_k)$.

A_n^1 is obviously open for $\forall n \in \mathbb{N}$. Moreover, having any $V(B_1, \dots, B_m) \in \mathcal{B}_V$, if $V(B_1, \dots, B_m)$ is nonempty, then there are open connected sets $B_i' \subseteq B_i$ for $\forall i = 1, \dots, m$, such that $\bigcup\limits_{i=1}^{m} B_i'$ is connected. Pick $a_i \in B_i'$ for $\forall i = 1, \dots, m$, such that $a_i \neq a_j$ for $i \neq j$. Since $\bigcup\limits_{i=1}^{m} B_i'$ is open and connected, we can find a path $U_1 \subseteq \bigcup\limits_{i=1}^{m} B_i'$, joining a_1 and a_2 and $\bigcup\limits_{i=1}^{m} B_i' \smallsetminus U_1$ will still be open and connected and $a_2 \in \overline{\bigcup\limits_{i=1}^{m} B_i' \smallsetminus U_1}$. Therefore we can find a path U_2, joining a_2 and a_3. This way we get paths $U_1, \dots, U_{m-1}$ and the path $U = \bigcup\limits_{i=1}^{m-1} U_i$, which is joining a_1 and a_m and going through a_i for $i=2,\ldots,m-1$. This is closed and hence there $\exists\,\delta \in (0, \frac{1}{2n}]$, such that $E_\delta(U) \subseteq \bigcup_{i=1}^{m} B'_i$. If we



now put $A_1, \ldots, A_k$ in a way shown in the picture above, we can easily see that $V(A_1, \ldots, A_k) \subseteq V(B_1, \ldots, B_m) \cap A^1_n$, which is therefore nonempty and hence A^1_n is also dense in the space $\mathcal{C}$.

Now if we have a continuum $C \in A^1_n$, then there are open $\frac{1}{4n}$-balls $A_1, \ldots, A_k$, such that $C \in V(A_1, \ldots, A_k)$. Whenever we have two continua $C_1, C_2 \subseteq C$, such that $C_1 \cup C_2 = C$, then there are $\mathcal{A}_1, \mathcal{A}_2 \subseteq \{A_1, \ldots, A_k\}$, such that $C_1 \in V(\mathcal{A}_1)$ and $C_2 \in V(\mathcal{A}_2)$. Moreover $\mathcal{A}_1 \cup \mathcal{A}_2 = \{A_1, \ldots, A_k\}$, since $C_1 \cup C_2 = C$ and both $\bigcup \mathcal{A}_1$ and $\bigcup \mathcal{A}_2$ are connected, because C_1 and C_2 are connected. Therefore, due to the way, in which $A_1, \ldots, A_k$ were picked, for $\forall A_i \in \mathcal{A}_1$, there $\exists A_j \in \mathcal{A}_2$, such that $A_j \subseteq E_{\frac{1}{2n}}(A_i)$ and hence $\bigcup \mathcal{A}_1 \subseteq E_{\frac{1}{2n}}(\bigcup \mathcal{A}_2)$, or for $\forall A_j \in \mathcal{A}_2$, there $\exists A_i \in \mathcal{A}_1$, such that $A_i \subseteq E_{\frac{1}{2n}}(A_j)$ and hence $\bigcup \mathcal{A}_2 \subseteq E_{\frac{1}{2n}}(\bigcup \mathcal{A}_1)$. Since $C_1 \cap A_i \neq \emptyset$ for $\forall A_i \in \mathcal{A}_1$ and all the $A_i \in \mathcal{A}_1$ are open $\frac{1}{4n}$-balls, we get that $\bigcup \mathcal{A}_1 \subseteq E_{\frac{1}{2n}}(C_1)$. Similarly, $\bigcup \mathcal{A}_2 \subseteq E_{\frac{1}{2n}}(C_2)$. Therefore we get $C_1 \subseteq \bigcup \mathcal{A}_1 \subseteq E_{\frac{1}{2n}}(\bigcup \mathcal{A}_2) \subseteq E_{\frac{1}{n}}(C_2)$ or $C_1 \subseteq \bigcup \mathcal{A}_1 \subseteq E_{\frac{1}{2n}}(\bigcup \mathcal{A}_2) \subseteq E_{\frac{1}{n}}(C_2)$. Hence for any continuum $C \in \bigcap_{n \in \mathbb{N}} A^1_n$, if we have two continua $C_1, C_2 \subseteq C$, such that $C_1 \cup C_2 = C$, then $C_1 \subseteq \overline{C_2}$ or $C_2 \subseteq \overline{C_1}$. But since both C_1 and C_2 are closed, we get that $C_1 \subseteq C_2$

or $C_2 \subseteq C_1$ and because $C_1 \cup C_2 = C$, one of them has to be equal to C, which means that C is indecomposable. Therefore the set $\bigcap_{n\in\mathbb{N}} A_n^1$ contains only indecomposable continua.

On the other hand, if we have an indecomposable continuum $C \in \mathcal{C}$ and two continua $C_1, C_2 \subseteq C$, such that $C_1 \cup C_2 = C$, then $C_1 = C$ or $C_2 = C$, and hence $C_2 \subseteq C_1$ or $C_1 \subseteq C_2$. Now let $n \in \mathbb{N}$ be an arbitrary integer. For any $\{A_1, \ldots, A_k\} \subseteq \mathcal{B}_n^1$, such that $C \in V(A_1, \ldots, A_k)$, the union $\bigcup_{i=1}^{k} A_i$ is obviously connected. Moreover, having any $\mathcal{A}_1, \mathcal{A}_2 \subseteq \{A_1, \ldots, A_k\}$, such that the sets $\bigcup \mathcal{A}_1$ and $\bigcup \mathcal{A}_2$ are both connected and $\mathcal{A}_1 \cup \mathcal{A}_2 = \{A_1, \ldots, A_k\}$, we can easily find continua $C_1, C_2 \subseteq C$, such that $C_1 \cup C_2 = C$, $C_1 \in V(\mathcal{A}_1)$ and $C_2 \in V(\mathcal{A}_2)$. For $\forall A_i \in \mathcal{A}_1$, there $\exists\, x_i \in A_i \cap C_1$. If $C_1 \subseteq C_2$, then $x_i \in C_2$ and therefore there $\exists A_j \in \mathcal{A}_2$, such that $x_i \in A_j \cap C_2$. Since A_i is a $\frac{1}{4n}$-ball, we get $A_i \subseteq E_{\frac{1}{2n}}(\{x_i\}) \subseteq E_{\frac{1}{2n}}(A_j)$. Similarly, if $C_2 \subseteq C_1$, then for $\forall A_j \in \mathcal{A}_2$, ther $\exists A_i \in \mathcal{A}_1$, such that $A_j \subseteq E_{\frac{1}{2n}}(A_i)$. Therefore the system $\{A_1, \ldots, A_k\}$ has the property $\mathfrak{P}_n$ and it is in $\mathcal{A}_n^1$ and thus $C \in A_n^1$. Since $n \in \mathbb{N}$ was an arbitrary number, the continuum $C \in \bigcap_{n\in\mathbb{N}} A_n^1$. This is true for any indecomposable continuum $C \in \mathcal{C}$ and therefore all indecomposable planar continua are in the set $\bigcap_{n\in\mathbb{N}} A_n^1$.

Hence the set $A = \bigcap_{n\in\mathbb{N}} A_n^1$ is the set of all indecomposable continua in $\mathbb{R}^2$ and it is G_δ and due to the Baire theorem also dense in the space $\mathcal{C}$, consisting of all planar continua, endowed with the Vietoris topology.

$\square$

Some ideas in this proof are borrowed from [7].

Since the set A of all indecomposable continua in $\mathbb{R}^2$ is G_δ and dense in the space of all planar continua with τ_V, it is also G_δ and dense in $\mathcal{C}$ with τ_H and hence with the Hausdorff metric, which means that it is the set of typical elements of $\mathcal{C}$.

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