Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

BAKALÁŘSKÁ PRÁCE

Ondřej Bílka

Konvexní otázky pro konečné množiny bodů v rovině

Katedra aplikovane matematiky

Vedoucí bakalářské práce: doc. RNDr. Pavel Valtr, Dr.,
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V Praze dne Ondřej Bílka

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Vedoucí bakalářské práce: doc. RNDr. Pavel Valtr, Dr.
e-mail vedoucího: valtr@kam.mff.cuni.cz

Abstract

Velikost konvexně nezavisle množiny je dobrě prostudovana area. V roce 1994 Emo Welzl položil následující otazku: Lze v každé n-prvkové podmnožině polynomialně velké mřížky vždy najít konvexně nezavislou množinu velikosti $\Omega(n^\epsilon)$? My tuto hypotezu vyvrátíme. I v mřížce $n^{1+\epsilon} \times n^{1+\epsilon}$ existuje n-prvková množina která neobsahuje větší slabě konvexně nezávislou množinu než $O(log^{1+1/\epsilon}n)$. Také zodpovíme otázku Eisenbranda Pacha Rothvosse and Sophera o velikosti konvexně nezávislé množiny minkovskeho součtu dvou množin v rovině. Naše konstrukce dává těsný odhad.

Klíčová slova: konvexně nezavisle množiny ,minkovskeho soucet, mrizka

Title: Convexity questions for finite planar point sets
Author: Ondřej Bílka
Department: Department of Applied Mathematics
Supervisor: doc. RNDr. Pavel Valtr, Dr.
Supervisor's e-mail address: valtr@kam.mff.cuni.cz

Abstract

Size of a convex independent subset of n points in the plane has been frequently studied area. Emo Welzl asked in 1994 if in n points of polynomialy sized grid we can always find a convex independent subset of size $\Omega(n^\epsilon)$. We answer this question by showing that for any $\epsilon > 0$ and for any integer n, even in $n^{1+\epsilon} \times n^{1+\epsilon}$ grid we can still find an n-point subset such that weakly convex independent subset size is bounded by $O(log^{1+1/\epsilon}n)$. We also answer question of Eisenbrand, Pach, Rothvoss and Sopher about size of the sum of two finite point sets in the plane. We show a construction which gives us convex independent sets with asymptotically tight size.

Keywords: convex independent set, minkowski sum, grid

Chapter 1

Introduction

1.1 Definitions

We study convex independance and related concepts.

We restrict ourself to planar case. In subsequent text by set we implicitly mean set of points in the plane.

We begin by reviewing basic notations. One of basic concepts is general position.

Definition 1.1 (general position)**.** *We say set S is set of points in general position if no three points from S are collinear.*

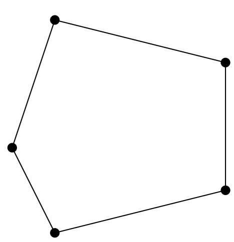
Majority of theorems in combinatorial geometry requires general position.

Another widely used notion is convexity.

Definition 1.2 (convexity)**.** *We call set S convex if following holds: For any pair of points $a, b \in S$ line segment ab is contained in S.*

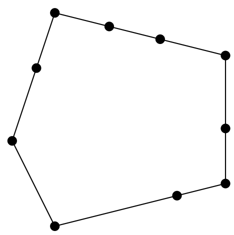
Note that area above convex function is convex set.

Definition 1.3 (convex independence)**.** *We call a finite subset of R^2* convex independent *if it is a set of vertices of a convex polygon.*



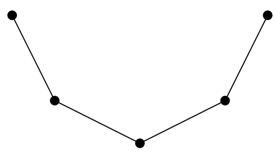
We can relax this notion to:

Definition 1.4 (weak convex independence)**.** *We call a finite subset of* R^2 weakly convex independent *if it is a set of points that lie on the boundary of a convex polygon.*

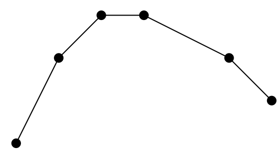


Or strenghten *convex independence*:

Definition 1.5 (cup)**.** *We call a set* $U \subset R^2$ *a* cup *resp* cap *if* U *is a subset of the graph of a convex resp. concave function.*



cup



cap

To simplify notation we define:

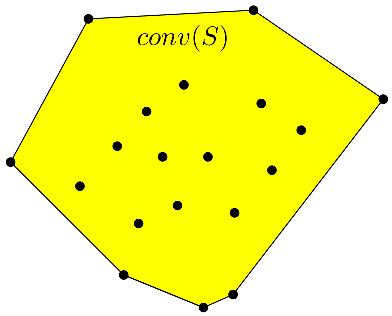
Definition 1.6 (direction)**.** *By* direction $\operatorname{dir} xy$ *of an oriented line* xy *we mean the size of counterclockwise angle between the positive x-axis* $(0,0)(0,1)$ *and the line* xy*.*

Alternate description of convex independence will come handy.

Observation 1.1. *In the plane a set* C *is convex independent if and only if it can be ordered as* $C = \{c_1, c_2, \ldots c_n, c_{n+1} = c_1\}$ *such that* $\operatorname{dir} c_i c_{i+1} < \operatorname{dir} c_{i+1} c_{i+2}$ *holds for every* i*. which is relaxed in weakly convex independent case to* $\operatorname{dir} c_i c_{i+1} \leq \operatorname{dir} c_{i+1} c_{i+2}$.

One of basic computional geometry problems is finding of convex hull.

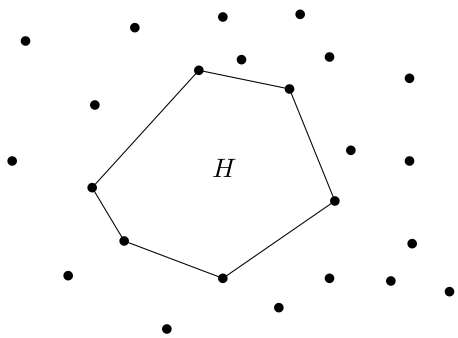
Definition 1.7 (convex hull)**.** *Given a planar point set* S *by convex hull* $conv(S)$ *we mean minimal convex set containing* S*.*



Note that if S is finite then $conv(S)$ is convex polygon.

We also study related concept of hole:

Definition 1.8 (hole)**.** *For given set $S \subset R^2$ we call its subset $H \subset S$ a* hole *if H is convex independent and $H = S \cap conv(H)$.*

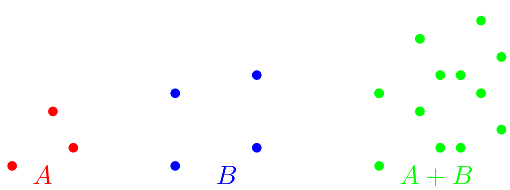


As we relaxed convex independence we can relax hole.

Definition 1.9 (weak hole)**.** *For given set $S \subset R^2$ we call its subset $H \subset S$ a* weak hole *if H is weakly convex independent and $H = S \cap conv(H)$.*

Definition 1.10 (grid)**.** *By $t \times t$* grid *we mean following set of t^2 points: $\{(x, y)|x, y \in Z, 0 \leq x, y < t\}$.*

Definition 1.11 (minkovski sum)**.** *By the* minkowski sum *$A + B$ of sets A and B we mean the set $\{a + b|a \in A, b \in B\}$.*



Note that resulting set is dependent to origin but resulting sets are translations of each other.

Chapter 2

Known results

2.1 Convex independent sets

One of combinatorial geometry results is the Erdős-Szekeres theorem [4]

Theorem 2.1 (Erdős-Szekeres)**.** *For any s there exist $es(s)$ such that any set of at least $es(s)$ points in general position in plane contain a convex independent subset with size s.*

Original Erdős and Szekeres [4] upper bound on $es(s)$ was obtained by following application of ramsey theory:

We color quadruples of points. We color quadruple red if its convex independent and blue otherwise. Observe that every set of 5 points contains convex independent quadruple. By ramsey theorem there is monochromatic clique. So this clique must be red which corresponds to convex independent set.

There is even simpler proof: Fix order of points. We color triples of points. we color triple red if its points form triangle in clockwise order, blue otherwise. By ramsey theorem we get a clique which is again convex independent set.

Later Erdős and Szekeres [5] improved upper bound:

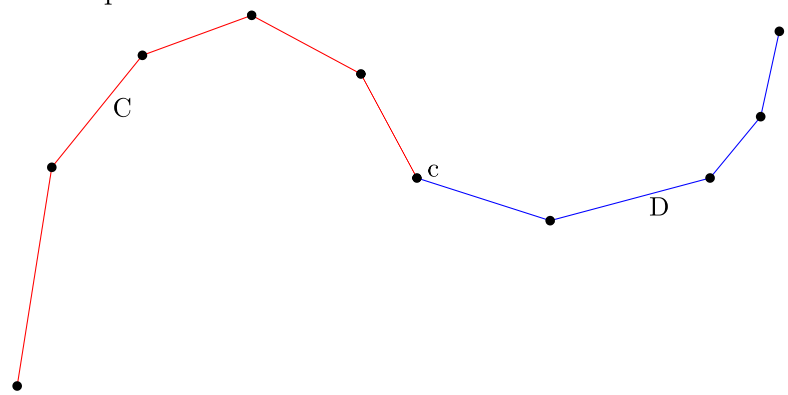
For any $a, b > 2$ let $f(a, b)$ denote minimal size such that every set of $f(a, b)$ points in general position contains either cup with size a or cap with size b. Then $f(a, b) \leq \binom{a+b-2}{a-2} - 1$.

We get bound $es(n) \leq f(n, n)$.

Proof. Proof is by induction.

Obviously $f(k,3) = f(3,k) = k$. General inequality follows from reccurence $f(a,b) \leq f(a,b-1) + f(a-1,b) - 1$. Now we prove recurrence.

Let F be set with $f(a,b-1) + f(a-1,b) - 1$ points. Let X be set of rigthmost points of cups with size $a-1$. If $|F \backslash X| \geq f(a-1,b)$ then $F \backslash X$ can't contain cup of size $a-1$ so it contains cap with size b. So $|X| \geq f(a,b-1)$. If X contains a-cup we are done. Otherwise it contains $b-1$ cap C. Considering leftmost c point of C and $(a-1)$-cup D whose rightmost point is c we observe that we can extend C or D.



□

Best known bound $es(n) \leq f(n-1, n-1)$ was given by Valtr by refining former argument.

By probablistic argument Erdős [5] showed that for every n there exist set of 2^{n-1} points without convex independent set of size n. He also conjectured that this lower bound is tight.

2.2 Holes

Erdős question if analogue of Erdős-Szekeres's theorem for holes holds was also solved. Horton [8] gave following upper bound:

Theorem 2.2. *For any n there exist an n-point set that doesn't contain a* 7*-hole.*

We describe generalization by Valtr [13]. First we need some definitions.

Definition 2.1 (deep above)**.** *For any two sets finite A and B we say set A is deep above set B if for any $b_1, b_2 \in B$ set A lies entirely in upper halfplane induced by line b_1b_2 and for any $a_1, a_2 \in A$ set B entirely in lower halfplane induced by line a_1a_2*

Definition 2.2 (r-closed)**.** *We say set S is* r-upper *closed resp.* r-lower closed *if S does not contain empty r-cup resp. r-cap. We say S is* r-closed *if S is both r-upper-closed and r-lower closed.*

Now we can say what is Horton set

Definition 2.3. *A set S is Horton if it can be constructed in following way:*
1. A point is Horton set.
2 If A and B are Horton and A lies deep above B then $A \cup B$ is Horton set.

To prove Horton set does not contain 7-hole we need to show that any Horton set is 4-closed since by Dirichet's principle any 7-hole contains empty 4-cup or 4-cap.

Recenty Tobias Gerken[6] gave tight lower bound:

Theorem 2.3. *There is an integer c such that any set of at least c points in general position contains a* 6*-hole.*

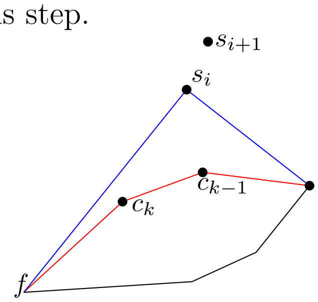
2.3 Convex hull

One of basic algorithms of computionial geometry is convex hull problem. Problem is given a finite n-point set S we want to find conv(S) represented as enumeration of conv(S) vertices in order of observation 1.1.

In general case convex hull of n-point set could be find in the time $O(nlogn)$.

It is well known that convex hull problem is equivalent to sorting. To sort numbers by using convex hull algorithm we replace number x by point (x, x^2) and convex hull gives us sorted list of points. We can find convex hull of the S by sorting. First pick leftmost $f \in S$ and translate and rotate S such that f has coordinates (0,0) and is unique point at y axis. By tangents of the point (x,y) we mean the ratio y/x. We sort points of S
f by their tangents. We label them as $s_1, s_2, \ldots, s_n$ in order of sorting.

Algorithm has n steps. In i-th step we constructed convex $f, c_1, c_2, \ldots, c_k$ hull of points $f, s_1, s_2, \ldots, s_{i-1}$. We want add point c_i to former convex hull. If oriented angle $f s_i c_k$ is greater than 180 then s_i is inside convex hull of $f, s_1, s_2, \ldots, s_{i-1}$. Otherwise we continue. If oriented angle $s_i c_k c_{k-1}$ is less than 180° then $f, c_1, c_2, \ldots, c_k, s_i$ is convex hull of $f, s_1, s_2, \ldots, s_i$. Otherwise c_k lies inside convex hull of $f, s_1, s_2, \ldots, s_{i-1}$ and we delete c_k and repeat this step.



Upto sorting algorithm is linear because we add or delete point in constant time and we can delete at most n points.

Theorem 2.4. *Given n points subset S of $n^k \times n^k$ grid we can find convex hull of S in time $O(n)$*

Proof. It suffices to show we can sort S by tangents in linear time. If $(x_1, y_1), (x_2, y_2)$ are distinct points of grid then the difference of their tangents is $\frac{y_1}{x_1} - \frac{y_2}{x_2} \geq \frac{1}{x_1 x_2} \geq \frac{1}{n^{2k}}$. So to point (x,y) we assign number $\lfloor x n^{2k}/y \rfloor$ and previous observation guarantes us that distinct points get distinct labels. We can sort this integers by radixsort in linear time. □

Chapter 3

New results

3.1 Convexity of grid subsets

3.1.1 Introduction

We studied if stronger variant of Erdős-Szekeres theorem holds in grid subsets. Emo Welzl [11] (see also [2]) asked:

Question 3.1. *Fix $d \geq 1$, is it true that for some $\epsilon > 0$ any n-point subset of $n^d \times n^d$ grid in general position contains a convex independent subset of size $\Omega(n^\epsilon)$.*

The reason of requiring the general position is the existence of degenerate configurations like point sets on the line whose maximal convex independent set has size 2. We relax convex independence to weak convex independence.

We solve Question 3.1 negatively:

Theorem 3.1. *For any $\epsilon > 0$ and any integer n, there exists a n-point set S in $n^{1+\epsilon} \times n^{1+\epsilon}$ grid such that size of any weakly convex independent subset of S is bounded by $O(\ln^{1+\epsilon} n)$.*

If general position is required we need larger grid:

Theorem 3.2. *For any $\epsilon > 0$ and any integer n, there exists a n-point set S of points in general position in $n^{3+2\epsilon} \times n^{3+2\epsilon}$ grid such that size of any convex independent subset of S is bounded by $O(\ln^{1+\epsilon} n)$.*

We can extend this construction to smaller grids:

Corollary 3.1. *For any $n, 0 \le \delta < \frac{1}{2}, \epsilon > 0$ (note case $\delta = 0$) there exists an n-point set in a $n^{1-\delta} \times n^{1-\delta}$ grid with size of any weakly convex independent subset bounded by $O(n^{\delta+\epsilon} \ln^{1+1/\epsilon} n)$.*

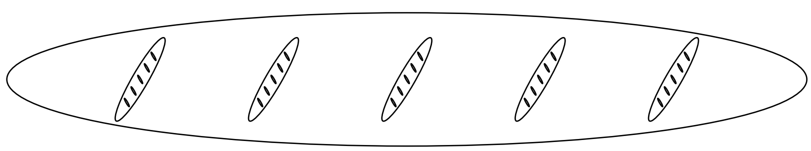
3.1.2 Construction

Our construction consists from ℓ levels where ℓ will be chosen later. For $i \in N$ by O_i we mean an interval $(\pi i/\ell, \pi(i+1)/\ell)$ and $I_i = O_i \cup O_{i+\ell}$.

Lemma 3.1. *For any three nonnegative integers i, ℓ, s there exists a set S_i of integer points such that these three conditions are satisfied:*
1. $|S_i| = s^i$,
2. $diam(S_i) \le d_i = (\ell s)^i$, and
3. For any $k, 1 \le k \le i$ and any set $D = (d_1, d_2, \dots, d_n), D \subset S_i$ with $\operatorname{dir} d_j d_{j+1} \in O_k, 1 \le j < n$ has at most s elements. For $i < k \le \ell$ no two points x,y have $\operatorname{dir} xy \in O_k$. Same holds for $O_{k+\ell}$.

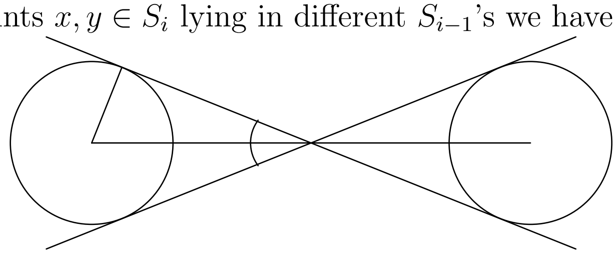
Proof. By induction on ℓ.
1. $S_0 = \{(0,0)\}$ with diameter bounded by $d_0 = 1$.
2. Assume we constructed S_{i-1} with diameter bounded by $d_{i-1} = (\ell s)^{i-1}$.
Let $v = (\lfloor \ell d_{i-1} \cos(\pi(i+1/2)/\ell) \rfloor, \lfloor \ell d_{i-1} \sin(\pi(i+1/2)/\ell) \rfloor)$ then S_i consists from s shifted copies of S_{i-1}:

$$S_i = \bigcup_{j=0}^{s-1} (S_{i-1} + jv).$$



Conditions 1,2 are obvious, now we prove 3.

Since $2 \arcsin 1/\ell < \pi/\ell$, for $\ell = 1$ there is no rounding error of v and for bigger ℓ it can change $\operatorname{dir} v$ at most by ℓ^{-2} and from picture for any two points $x, y \in S_i$ lying in different S_{i-1}'s we have $\operatorname{dir} xy \in I_i$.



From induction any two points in the same copy of S_{i-1} have $\operatorname{dir} xy \in I_k$ for $k < i$. Only the case when $d_j d_{j+1} \in I_i$ remains. But $d_j d_{j+1} \in I_i$ if and only if $d_j \in S_{i-1} + av, d_{j+1} \in S_{i-1} + bv, b > a$ and because $1 < a, b < s$ it has size $|D| \leq s$. □

Now we are ready to prove Theorem 3.1:

Proof. Set $\ell := \epsilon \ln n / \ln(\epsilon \ln n)$ and $s = \lceil n^{1/\ell} \rceil$. By lemma 1 we get $s^\ell \geq n$ point set S_ℓ and it is not hard to check that diameter $d_\ell = (\ell s)^\ell = n\ell^\ell < n^{1+\epsilon}$. Let $C = \{c_1, c_2, \ldots, c_n, cn+1 = c_1\}$ be any weakly convex independent subset of S_ℓ. By observation 1 the set C is weakly convex independent if and only if the directions $\operatorname{dir} c_i c_{i+1}$ form nondecreasing sequence. By condition 3 for each $O_k, 1 \leq k \leq 2\ell$ there are at most s consecutive c_i such that $\operatorname{dir} c_i c_{i+1} \in O_k$. Since O_k partitions directions and there are 2ℓ of them we conclude that weakly convex independent subset size is bounded by

$$2\ell s = 2\ell e^{\ln n/\ell} = 2\ell e^{1/\ln(\epsilon \ln n)/\epsilon} = 2\epsilon \ln n / \ln(\epsilon \ln n)(\epsilon \ln n)^{1/\epsilon} = O(\ln^{1+1/\epsilon} n).$$

□

Proof of Corollary 3.1 follows:

Proof. Use Theorem 3.1 with $n' = n^{1-2(\delta+\epsilon)}$ and with the grid size $n^{1-2\delta-\epsilon} = n^\epsilon n^{1-2(\delta+\epsilon)} > n'^{1+\epsilon}$. Now enlarge this set $n^{\delta+\epsilon}$ times and replace each point by an $n^{\delta+\epsilon} \times n^{\delta+\epsilon}$ subgrid of $n^{1-\delta} \times n^{1-\delta}$ grid. In former construction point had diameter bounded by 1 and subgrid has diameter 1 too (before enlarging). In any weakly convex independent subset we visit $O(\ln^{1+1/\epsilon} n)$ subgrids and in each subgrid $O(n^{\delta+\epsilon})$ points which sums up to $O(n^{\delta+\epsilon} \ln^{1+1/\epsilon} n)$ points. □

This is up to polylogarithmic factor tight with the lower bound n^δ from Dirichet's principle.

3.1.3 Set in general position

Now we will construct set satisfying theorem 3.2.

We first need this lemma:

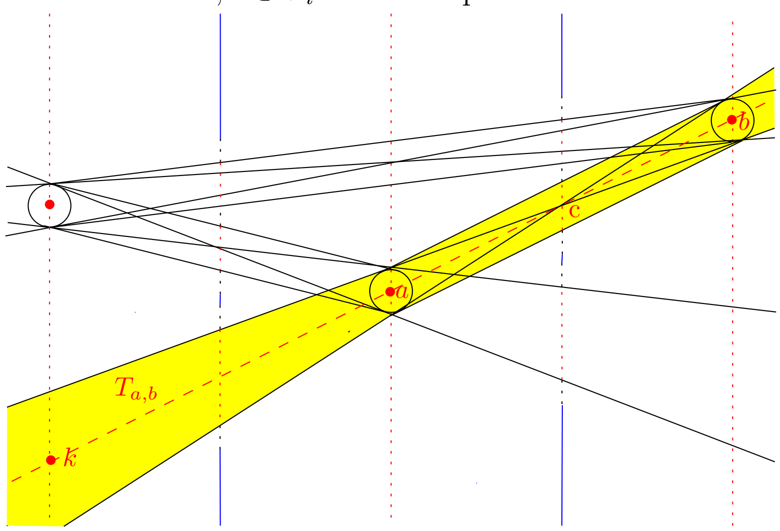
Lemma 3.2. *We again use parameters $\ell, s, \ell > 1$. First we need to define $f(s) = cs^2 \ln s$ where c is suitable constant. Let L be union of following $2s$ line segments $(\ell f(\ell, s)i, 0) - (\ell f(\ell, s)i, f(\ell, s))$.*

Then we can pick set $G \subset L$ of s points, no two at same line segment satisfying:For any triple of points $a, b, c \in S$ there does not exist line which distance to a, b and c is at most 1.

Proof. For any $S \subset L$ by measure $\mu(S)$ of S we mean the sum of Lebegue measures of S in invidual line segments.

We construct G by greedy algorithm. Let G_i be set of i points from L satisfying lemma condition. By $A_{i+1} \subset L$ we mean the set of possible choices of $i+1$-th point g_{i+1} of G if first i points are already choosen. Complement of A_i is set F_i of forbidden points.

We can describe F_i explicitly. Forbid lines containing point of G_i. Construct around each point circle with diameter 1. For any two points $a, b \in G_i$ let $T_{a,b}$ be set of points between common tangets to circles around a and b. It is easy to see that $T_{a,b}$ is union of all lines whose distance to a and b is at most 1. For each $a, b \in G_i$ we forbid points with distance at most 1 to $T_{a,b}$.



We show that $\mu(F_{i+1}) \leq 2f(s)i$ so we can always choose a point.

1. Measure $\mu(F_1) = 0$.
2. We show that $\mu(F_{i+1}) \leq \mu(F_i) + 2f(s)$.

At i-th step we forbid one line segment with measure at most $f(s)$.

We need to estimate size of $\mu(L \cap T_{a,b})$ when a and b are k lines of L apart. Let $c = \frac{a+b}{2}$. Because $\ell > 1$ directions in $T_{a,b}$ are restricted to $(-45°, 45°)$ so measure in line segment of L containing a is at most $\sqrt(2)$. By homothety measure in other line segment l of L is at most $\sqrt(2) max(|p-c|/|a-c|, 1)$ where p is intersection point of l and ab. So $\mu(L \cap T_{a,b}) \leq 2 \sum_{i=1}^{2s} \sqrt(2)(i/k + 1) < c's^2/k + 4s$ for suitable constant c'.

Because directions in $T_{a,b}$ are restricted we can bound measure of points with distance at most 1 to $T_{a,b}$ is $c's^2/k + 8s$.

Now we use that for any k at most two points from G_i are k lines apart from g_i. So newly forbidden area has measure bounded by $f(s) + 2\sum_{k=0}^{i/2}(c's^2/k+4s) = f(s)+8s^2+2c's^2\sum_{k=0}^{i/2} 1/k \leq f(s)+8s^2+2c's^2 \ln s \leq 2f(s)$

□

Note that if we needed only general position we could use construction of Erdős: For any p prime the set $\{(i^2 \mod p, i) | 0 \leq i < p\}$ is set of points at general position in $p \times p$ grid.

We generalize construction from Lemma 3.1.

Lemma 3.3. *For any three nonnegative integers i, ℓ, s there exists a set S_i of integer points such that these three conditions are satisfied:*

1. $|S_i| = s^i$,

2. $diam(S_i) \leq d_i = (f(s)\ell s)^i$, and

3. For any $k, 1 \leq k \leq i$ and any set $D = (d_1, d_2, \ldots, d_n), D \subset S_i$ with $\operatorname{dir} d_j d_{j+1} \in O_k, 1 \leq j < n$ has at most s elements. For $i < k \leq \ell$ no two points x,y have $\operatorname{dir} xy \in O_k$. Same holds for $O_{k+\ell}$.

4. S_i is set of points in general position.

Proof. We again use induction. 1. S_0 is again a point.

Assume we constructed S_{i-1} satisfying conditions. As in Lemma 3.1 let $v = (\lfloor \ell f(\ell, s) d_{i-1} \cos(\pi(i + 1/2)/\ell) \rfloor, \lfloor \ell f(\ell, s) d_{i-1} \sin(\pi(i + 1/2)/\ell) \rfloor)$. We first construct G from Lemma 3.2. We rotate and scale G to G' in such way that vector $(1, 0)$ in G corresponds to v in G'. We round points of G' to nearest integer point. At each point of G' we place copy of S_{i-1}. So $S_i = S_{i-1} + G'$. Conditions 1,2 and 3 have exact same proof as in Lemma 3.1 with diameter $f(\ell, s)d_{i-1}$.

To prove condition 4 assume we have three collinear points a', b', c' at line l. Look in which O_k is direction of line l. By condition 3 if $\operatorname{dir} l\ inO_k, k < i$ we apply induction otherwise is $\operatorname{dir} l \in O_i$. Then points a', b', c' lie at disjoint copies of S_i. We map points a', b', c' to a, b, c by rotating and scaling G' to G. Distance of points a, b, c from G is less than 1 and they are collinear which contradicts lemma 3.2.

To get desired diameter set s and t as in theorem 3.1. Diameter is then $(f(s)\ell s)^\ell = (s^3 \ln s\ell)^\ell < ((s^3\ell^2)^\ell = n^{3+2\epsilon}$ □

3.1.4 Remarks

The set S_ℓ can be seen as a suitable projection of s^ℓ-hypercube. It is interesting that Horton sets could be too described in this way. Some bound could be derived from using the fact that preimage of convex independent set is convex independent set (but this does not hold for weakly convex independent sets).

Weaker version of Corollary 1 can be also deduced from compression lemma which states.

Lemma 3.4. *Partition set P of points in the plane to k classes $P_1, P_2, \ldots, P_k$. Then for any set $f_1, f_2, \ldots, f_k$ of affine mappings following holds: $conv(P) \geq conv(\bigcup f_i(P_i))/k$.*

Proof. Dirichet's principle. □

Plausible conjecture is: Assume we have n points in $k \times k$ grid. Divide that grid to $\ell \times \ell$ subgrids and let G_{ij} be points of our set contained in (i,j) subgrid. For each define mapping to $\ell \times \ell$ grid by $f_{ij}((x, y)) = (x + a \mod l, y + b \mod l)$ where a, b are constants to maximize next expression. Suppose that we (for k, ℓ functions depending on n) found α such that $|\bigcup f_{ij}(G_{ij}))/k| \geq \alpha n$.

If we have lower bound $f(\ell)$ in $\ell \times \ell$ grid then by conjecture lemma we extend this to the lower bound for $k \times k$ grid: $\frac{f(\ell)}{4(k/\ell)^2}$ for n/α points in $k \times k$ grid.

3.2 Convex sets in minkowski sums

Halman et al. [7] studied maximal number $E(n)$ of straight-line segments between n points in the plane such that their midpoints form a convex independent set. They asked if $E(n)$ is linear or quadratic.

Motivated by this question Eisenbrand et al. [3] studied a more general question: What is the maximal size $M(m,n)$ of convex independent subset of $P+Q, |P|=m, |Q|=n$?

This directly relates to previous question because we can write set of midpoints of P as $1/2(P+P)$. If we have convex independent set in $P+Q$ then midpoints of $2(P \cup Q)$ contain this set too. Therefore, $M(n,n) \geq E(n)$ and $M(n,n) \leq E(2n)$. Eisenbrand at al. showed the following upper bound on the maximum size of convex independent set:

$$M(m,n) = O(m^{2/3}n^{2/3} + m + n).$$

They mentioned they don't know any superlinear lower bound of $M(m,n)$. Swanepoel and Valtr [10] gave a superlinear lower bound $M(n,n) \geq E(n) \geq \Omega(n\sqrt{\log n})$. We will prove that when $m=n$ the above bound $M(n,n) = O(n^{4/3})$ is tight.

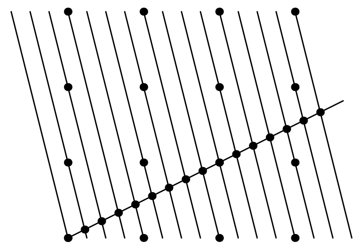
Theorem 3.3. *For every integer n there exist sets J and K of size $|J| = |K| = n$ such that the sum $J+K$ contains a convex independent subset of size $\Omega(n^{4/3})$.*

Proof. We put $K := \phi(G)$. The sum $J+K$ will consist of n shifted copies of K. Take the set S of n lines of G with largest sizes. For any direction a/b consider the set $S_{a/b}$ of lines from S with direction a/b.

Claim 3.1. *For any direction $a/b \in (0,1)$ there exists a mapping $t : S_{a/b} \to R^2$ such that the set $U_{a/b} = \bigcup_{L \in S_{a/b}} \{\ell + t(L) | \ell \in L\}$ is a cup. Moreover no two points of $U_{a/b}$ coincide.*

Proof. First we define linear mapping $f : R^2 -> R^2$ such that

$$f(x,y) = \left(a\frac{mx+y}{ma+b}, b\frac{mx+y}{ma+b} \right).$$



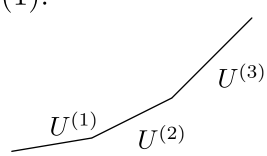
For any line $L \in S_{a/b}$, $f(L)$ is a translation of L which follows from $f(a,b) = (a,b)$.

Now observe that for any $x, y \in R$

$$\phi(x,y) - (x,y) = (0, z(mx+y)) = \phi(f(x,y)) - f(x,y).$$

Thus, $\phi(f(L))$ is the translation of $\phi(L)$ we seek. The image of ϕf coincides with the graph of the convex function $y = b/ax + \epsilon 3^{(m+b/a)x}$. Therefore $U_{a/b}$ is a cup. No two points of $U_{a/b}$ coincide because $mx + y$ restricted to G is an injective mapping. □

Now let $U^{(1)}, U^{(2)}, \ldots, U^{(k)}$ be the sets $U_{a/b}$ sorted by increasing direction. We shift these sets in such a way that the rightmost point of $U^{(i)}$ coincides with the leftmost point of $U^{(i+1)}$. Then the set $U^{(1)} \cup \ldots \cup U^{(k)}$ is a cup due to (1).



Now we need to estimate the size of this set. For the direction a/b the set $U_{a/b}$ has $\frac{bm}{2}$ lines starting at $(i,j), 0 \le i < n/2, 0 \le j < b$. Each of these lines has at least $\frac{m}{2b}$ points. Since the number of divisors of b is at most $b/2$ the number of directions with given b is at least $\frac{b}{2}$. The number of lines we use is at most $\sum_{b=1}^{k} 1/2 \sum_{a=0}^{b} |U_{a/b}| = 1/2 \sum_{b=1}^{k} \frac{b}{2}\frac{bm}{2}$. Set $k = m^{1/3}$ to ensure we use less than $n = m^2$ lines. Note that by shifting $U^{(i)}$ we identify at most m^2 points. The number of points on these lines is at least $\sum_{b=1}^{k} \frac{m}{2b} b/2 \frac{bm}{2} - m^2 = \sum_{b=1}^{m^{1/3}} bm^2/8 - m^2 = \Omega(m^{2/3}m^2) = \Omega(n^{4/3})$.

□

Chapter 4

Unsolved problems

4.1 Number of edges in geometric graphs without convex quadrangle

We studied problem of Nara, Sakai and Urrutia [9].

We recall a Turan's theorem:

Theorem 4.1. *The maximal number of edges in simple graphs of order n not containing a complete graph of order r is $t_{r-1}(n)$ and $T_{r-1}(n)$ is the unique graph of order n with $t_{r-1}(n)$ edges that does not contain a complete graph of r points.*

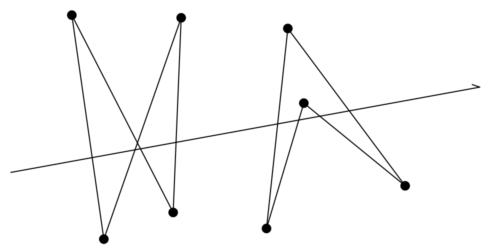
Question 4.1. *Is it true that given a point set P_n of n points in general position, there is always a geometric graph containing no convex quadrilaterals (resp. hexagons) whose vertex set is P_n with $t_3(n)$ (resp. $t_5(n)$) edges?*

Observe this bound is tight for P_n in convex position by Turan's theorem.

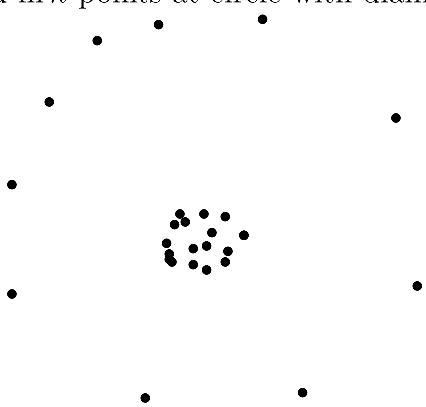
In Nara et. al. [9] also proved that if we are interested how large can geometric graph without convex k-gon be tight bound is $t_{es(k)-1}$. Indeed if graph has at least $t_{es(k)-1}$ edges then it contains complete graph with $es(k)$ vertices so it contains convex k-gon. On other hand take counterexample with $es(k)-1$ points and construct complete $es(k)-1$ partite graph where points of each partity lie near corresponding point of counterexample.

Lemma 4.1. *For any point set P_n of n points in general position, there is always a geometric graph containing no convex quadrilaterals with at least $\frac{1}{2}\binom{n}{2}$ edges.*

Proof. Pick a line which splits P_n into sets A,B of size $n/2$. Observe that complete bipartite graph with partities A and B does not contain convex quadrilateral. □



We were unsuccesfull in proving better bound than $1/2\binom{n}{2}$. We to split g into more areas and use complete n-partite graph. All our attempts failed at following couterexample: Consider union of $n - \ln n$ points at unit circle and $\ln n$ points at circle with diameter n^2.



Problem with this set and sets with inner points at unit circle is that any halving line must pass through inner points. So it divides outer points to halves. Problem is we cant divide outer points to more parts otherwise we would have convex quadrangle.

4.2 Holes in grid subsets

We also studied how large hole can subset of grid contain. We did not managed better bound than because (weak) hole is (weakly) convex independent we can can bound by our construction. Weak hole concept was defined in hope we find construction in grid. For subsets of plane even weaker conjecture is open.

Definition 4.1. *We say a see b in set S if line segment ab does not contain any points from S except a and b.*

Conjecture 4.1 (Kara, Por, Wood)**.** *For all integers k, l there is an integer n such that every set of at least n points in the plane contains at least k collinear points or l points that see each other.*

Conjecture was solved for $k < 6$ by Abel et al [1]. Proof is very similar to proof of empty hexagon theorem [6].

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