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BAKALÁŘSKÁ PRÁCE

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Zakázané minory pro apexové třídy grafů

Forbidden minors of apex classes of graphs

Katedra aplikované matematiky

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Děkuji vedoucímu za trpělivost a ochotu, se kterou mi pomáhal při řešení a psaní práce, a Tomáši Gavenčiakovi za podporu ve všem, co dělám.

Prohlašuji, že jsem svou bakalářskou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 6.8.2009 Tereza Klimošová

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Abstrakt: V předložené práci se zabýváme hledáním minimálních zakázaných minorů, neboli obstrukcí, pro třídu apexů částečných 2-stromů. Jelikož je tato třída uzavřená na minory, má podle Robertson-Seymourovy věty konečnou množinu obstrukcí. Množina obstrukcí je jedna z možných charakterizací každé třídy uzavřené na minory.

V práci analyzujeme strukturu obstrukcí pro třídu apexů částečných 2-stromů a díky její znalosti nacházíme všechny obstrukce s výjimkou speciálního typu obstrukcí, které mají path-width 3. Při hledání obstrukcí využíváme znalosti obstrukcí pro příbuzné třídy grafů.

Klíčová slova: teorie grafů, minimální zakázaný minor, částečný 2-strom, apex

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Abstract: In the present work we search for the minimal forbidden minors (also called obstructions) for the class of apexes of partial 2-trees. Because this class is minor closed, by Robertson-Seymour's theorem it has a finite set of obstructions. The set of obstructions is one of the possible characterizations of every minor closed class.

We analyze a structure of possible obstructions for the class of apexes of partial 2-trees and thanks to this knowledge, we can classify all the obstructions for the class of apexes of partial 2-trees except for some special type of them with path-width three. We use the knowledge of obstructions for related classes of graphs.

Keywords: graph theory, minimal forbidden minor, partial 2-tree, apex,

Contents

Chapter 1

Introduction

The main topic of the thesis is classifying obstructions for the class of apexes of partial 2-trees, an open problem raised by Bogdan Oporowski during the BIRS graph minor workshop held in Banff, AB, in September 2008. In the thesis, we present structural results that allow us to list all obstructions for the class of apexes of K_4-minor-free graphs except for some special types of obstructions with path-width three, but these remaining obstructions can be easily generated by a computer and thus the presented results form one part of a complete solution of Oporowski's problem. The list of obstructions we have identified is given in Figures 1.1 and 1.2.

The class of apexes of partial 2-trees is minor closed. Many natural graph classes are minor closed, for example planar and outerplanar graphs, graphs embeddable on surfaces, graphs with bounded tree-width etc. Partial 2-trees form an intermediate class between outerplanar and planar graphs. Apexes of planar graphs play an important role because of their relation to deep open problems in graph theory, Hadwiger's conjecture, in particular. Since the classification of obstructions for apexes of planar graphs is a long-standing open problem, Oporowski asked to classify obstructions for apexes of some simpler subclasses of planar graphs.

Graph minors and minor closed graph classes were studied in the series of papers by Robertson and Seymour [4]. One of the many results contained in the series is the proof of Wagner's conjecture, that every infinite set of finite graphs contains two graphs such that one of them contains the other one as a minor. For minor closed graph classes the following corollary of this theorem is very important. Every minor closed class can be characterized by a finite set of obstructions, i.e., graphs that are not contained in the class, but all their proper minors are contained in the class. Hence, the number of obstructions for the class of apexes of partial 2-trees is finite.

Let us give some examples of classification results for other minor closed classes of graphs. The set of obstructions for the class of planar graphs consists of the complete graph K_5 on five vertices and the complete bipartite graph $K_{3,3}$ with partities of size three (this statement is equivalent to Kuratowski's theorem). The set of obstructions for outerplanar graphs consists of the complete graph K_4 on four vertices and the complete bipartite graph $K_{2,3}$ with partities of size two and three.

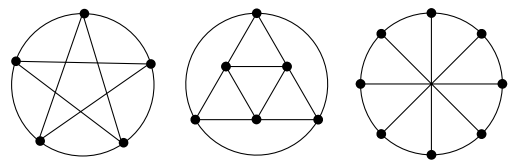


Figure 1.1: Obstructions with tree-width 4.

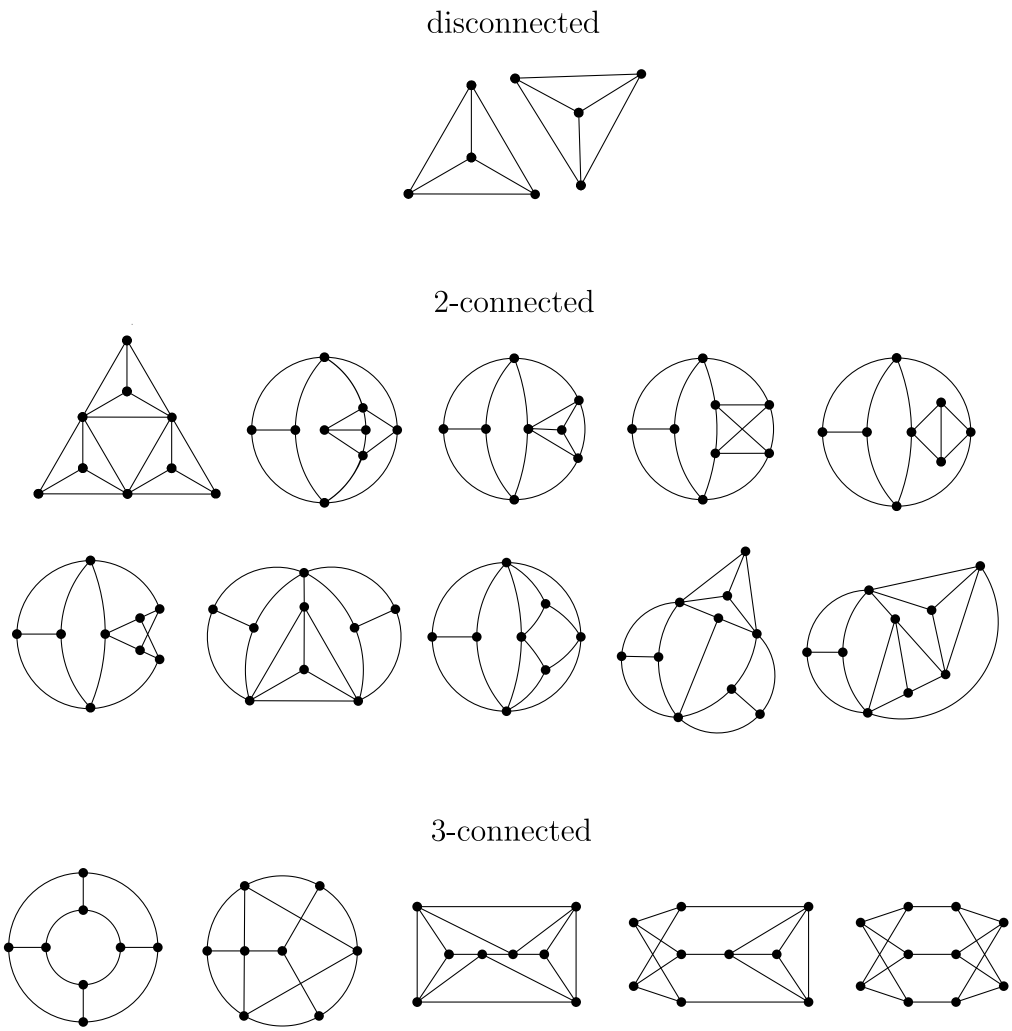


Figure 1.2: Obstructions with tree-width 3.

The set of obstructions for the class of the graphs embeddable on projective plane was classified by Archdeacon in [1]. There exists only one obstruction for the class of partial 2-trees—the complete graph K_4 on four vertices.

An explicit knowledge of a set of obstructions for a particular minor closed class of the graphs is also important from the algorithmic point of view. Another result from the graph minor series [6] asserts that it can be tested in a cubic time whether an input graph contains a fixed graph as a minor. In particular, every minor-closed class of graphs can be recognized in polynomial time and the explicit knowledge of the obstructions often leads to an algorithm with a better running time.

1.1 Basic definitions

In this section, we survey basic definitions, notation and some theorems from graph theory in particular, there related to graph minors and tree-width, which are used throughout the thesis. Most of these topics are described in more detail, e.g., in Diestel's book [3].

Definition 1. A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a set of vertices and $E(G) \subseteq \binom{V(G)}{2}$ is a set of edges.

A graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is a *subgraph* of G.

An edge $e = \{u, v\}$ is usually denoted as uv and the vertices u end v are called *end-vertices* of the edge e. Two vertices $u, v \in V(G)$ are *adjacent* if there exists an edge $e = uv$ in $E(G)$. Vertices adjacent to v are also called *neighbors* of v and the set of all vertices adjacent to v is the *neighborhood* of v.

We say that a vertex v has *degree* k if v is contained in k edges in G.

Definition 2. Graphs G and H are *isomorphic* if there exists a bijection $\varphi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$.

In the thesis, we consider isomorphic graphs to be equal and graph classes to be closed under isomorphism.

Definition 3. A graph with an edge between every two vertices, i.e. $(V, \binom{V}{2})$, is called a *complete graph* and is denoted K_n, where $n = |V|$.

Definition 4. A *path* P of length n is a graph with $n + 1$ vertices $v_0, \ldots, v_n$ and edges $e_i = v_i v_{i+1}$ for $i = 0, \ldots, n - 1$. The vertices v_0 and v_n are *ends* of the path P and the vertices $v_1, \ldots, v_{n-1}$ are *internal vertices* of the path. A path in a graph G between vertices u and v or from u to v is a subgraph of G which is a path with ends u and v.

Definition 5. A *cycle* C of length n, where $n \geq 3$, is comprised of a path of length $n - 1$ with ends u and v and an edge uv.

Definition 6. A graph obtained from G by *subdividing* an edge $uv \in V(G)$ is the graph obtained by adding a vertex w into $V(G)$ and replacing the edge uv by edges uw and vw. A graph H is a *subdivision* of a graph G, if H can be obtained from G by none, one or more subdividing of its edges.

Definition 7. Let $G = (V, E)$ be a graph, $e = uv$ an edge and w a vertex in G. Then the graph obtained from G by *deleting the edge* e is the graph $G \setminus e = (V, E \setminus e)$. The graph obtained from G by *deleting the vertex* w is the graph $G \setminus w = (V \setminus \{w\}, E \cap \binom{V \setminus \{w\}}{2})$.

Definition 8. A graph obtained from a graph G by *contracting an edge* $e = uv$ is a graph obtained by deleting the edge e and identifying the vertices u and v. The resulting vertex $\overline{uv}$ is adjacent to all the vertices which are adjacent to u or to v in $G \setminus e$.

Definition 9. A graph H is a *minor* of a graph G, if H can be obtained from G by a sequence of vertex deletions, edge deletions and edge contractions. If this sequence is nonempty, H is a *proper minor* of G.

Throughout our considerations we often use the word minor in the context when some of the vertices of graphs G and H are distinguished; we then require that the components of G containing the distinguished vertices correspond to the appropriate distinguished vertices of H

A class of graphs $\mathcal{G}$ is *minor-closed* if every minor of every graph $G \in \mathcal{G}$ belongs to $\mathcal{G}$.

An *obstruction* for a minor-closed class of graphs $\mathcal{G}$ is a graph G such that $G \notin \mathcal{G}$ but every proper minor of G is contained in $\mathcal{G}$.

Definition 10. A graph G is *connected* if there exists a path between any two vertices in G. Maximal connected subgraphs of a graph G are called *connected components*.

A *vertex cut* in a connected graph G is a set W of vertices of G, such that graph $G \setminus W$ obtained from G by deleting vertices of W is not connected. A vertex cut W is called a *k-cut* if its size is k, i.e., $|W| = k$. The only vertex in 1-cut is called an *articulation*. G is called *k-connected* if its minimum vertex cut has size at least k. *The connectivity* of a graph G is the size of its minimum vertex cut.

Definition 11. Let G be a graph, W a vertex cut in G and C any connected component of the graph $G \setminus W$. Then the graph $M = (V(W) \cup V(C), E(G) \cap \binom{V(M)}{2})$ is a *bridge* in the graph G produced by the vertex cut W. Note that this definition is different from the standard definition of a bridge, which does not include edges between vertices of the vertex cut in the bridge.

Menger's theorem is a well-known result about connectivity of graphs. There exist several versions of the theorem. In the thesis the following version is used.

Theorem 1 (Menger,1927)**.** *Let G be a finite k-connected graph. Then for any pair of vertices u and v, there exist at least k (internally) vertex-disjoint paths between u and v.*

Corollary 2. *If G is a graph of connectivity k, $W = \{w_1, \dots w_k\}$ is a minimum vertex cut and v a vertex not contained W, then there exists a path P_i between vertex v and every vertex $w_i \in W$ for $i = 1, \dots, k$ such that paths $P_1, \dots, P_k$ are internally vertex-disjoint.*

Proof. The graph $G \setminus W$ has at least 2 connected components. Let u be a vertex in $G \setminus W$ which belongs to a different connected component than v. By Menger's theorem, there exist k internally vertex-disjoint paths between u and v. Because u and v are in different components of $G \setminus W$, every path between u and v contains at least one vertex of W. Thus, if there are k internally vertex-disjoint paths, each of them contains one of the k vertices in W. □

Definition 12. A *tree* is a connected graph that does not contain a cycle as a subgraph.

Definition 13 (Robertson & Seymour [5])**.** A *tree decomposition* of a graph G is a pair $(T, \mathcal{V})$ where T is a tree (a *decomposition tree*) and $\mathcal{V} = \{V_t\}_{t \in V(T)}$ is a system of subsets $V_t \subseteq V(G)$ with the following properties:

- $\bigcup_{t \in V(T)} V_t = V(G)$
- for every edge $uv \in E(G)$ there exists $t \in V(T)$ such that $\{u, v\} \subseteq V_t$
- $V_t \cap V_{t'} \subseteq V_{t''}$, if $t, t', t'' \in V(T)$ and t'' is on the path between t and t'

To distinguish between the vertices of a graph and the vertices of its tree decomposition, we call vertices of the tree decomposition *nodes*. Though, we do not strictly distinguish between a node $t \in V(T)$ and the associated set V_t.

The *width* of a tree decomposition $(T, \mathcal{V})$ is the size of the largest set V_t in the tree decomposition decreased by 1, i.e., $\max_{t \in V(T)}(|V_t| - 1)$.

The *tree-width* of a graph G is the minimum width of a tree decomposition of G.

The *path-width* of a graph G is the minimum with of a tree decomposition of G such that its decomposition tree is a path.

Definition 14. Graphs with tree-width at most k are called *partial k-trees.* The class of partial k-trees is denoted by $\mathcal{T}_k$.

The following proposition shows a relation between tree-width of a graph and tree-width of its minors.

Proposition 3 (Robertson & Seymour [5])**.** *If a graph H is a minor of a graph G, then the graph H has tree-width smaller or equal than tree-width of G.*

Proof. We prove that an edge contraction, an edge deletion and a vertex deletion cannot increase tree-width. Let $(T, \mathcal{V})$ be a tree decomposition of G with the minimum width, uv an edge in $E(G)$ and w a vertex in $V(G)$. Then, the tree decomposition $(T, \mathcal{V})$ is also a tree decomposition of $G \setminus uv$, the tree decomposition $(T, \mathcal{V}')$ where the vertices u and v in the nodes of $(T, \mathcal{V})$ are replaced the by vertex $\overline{uv}$ is a tree decomposition of G with the edge uv contracted, and $(T, \mathcal{V}'')$ where the vertex w in the nodes of $(T, \mathcal{V})$ is deleted is a tree decomposition of $G \setminus w$. The decompositions $(T, \mathcal{V}')$ and $(T, \mathcal{V}'')$ do not have greater width than $(T, \mathcal{V})$. □

The proposition gives us the following result for the class of partial k-trees:

Corollary 4. *The class $\mathcal{T}_k$ of partial k-trees is minor closed for every k.*

Definition 15. Let $\mathcal{G}$ be a class of graphs. A graph H is contained in the class of graphs $\mathcal{G}^{apex}$ if there exists an *apex vertex* a in H such that the graph $H \setminus a$ is contained in $\mathcal{G}$. We say that the graph H is an *apex* of $H \setminus a$.

Observation 5. *If a class of graphs $\mathcal{G}$ is minor closed then $\mathcal{G} \subseteq \mathcal{G}^{apex}$.*

Proof. If G is a graph in $\mathcal{G}$, then $G \setminus v$ is in $\mathcal{G}$ for any vertex $v \in V(G)$, because $G \setminus v$ is minor closed. Therefore G is in $\mathcal{G}^{apex}$. □

Observation 6. *If a graph class $\mathcal{G}$ is minor closed, then the class $\mathcal{G}^{apex}$ is minor closed.*

Proof. Let G be a graph in $\mathcal{G}^{apex}$ and a an apex vertex in G. As $\mathcal{G}$ is minor closed, the vertex a is an apex vertex in every graph G' obtained from G by deleting a vertex in $G \setminus a$ or deleting or contracting an edge in $G \setminus a$, because $G' \setminus a$ is a minor of $G \setminus a$.

Let v be a vertex adjacent to a in G. Then the vertex a is an apex vertex in every graph obtained by deleting the edge av, because $(G \setminus av) \setminus a = G \setminus a$. In the graph G'' obtained by contracting the edge av, the vertex $\overline{av}$ is an apex vertex because the graph $G'' \setminus \overline{av}$ equals $(G \setminus a) \setminus v$ which is in $\mathcal{G}$. Because the graph $G \setminus a$ is in $\mathcal{G}$, it is in $\mathcal{G}^{apex}$, by Observation 5. □

1.2 Overview

In the thesis, we present structural results that allows us to list all obstructions for $\mathcal{T}_2^{apex}$ except for some special types of them with path-width three. By the following proposition, the class $\mathcal{T}_2^{apex}$ is equivalent to the class of apexes of K_4-minor-free graphs.

Proposition 7. *[3] A graph has tree-width smaller than 3 if and only if it does not contain K_4 as a minor.*

Lemma 8. *The class $\mathcal{T}_2^{apex}$ is minor closed.*

Proof. Follows from Corollary 4 and Observation 6. □

Since the class $\mathcal{T}_2^{apex}$ is minor closed, the following theorem guarantees that the set of obstructions for $\mathcal{T}_2^{apex}$ is finite although it does not give any estimate on the number of the obstructions. The theorem is one of many results contained in the graph minor series of Robertson and Seymour [4].

Theorem 9 (Robertson & Seymour[7])**.** *The set of obstructions for every minor closed class of graphs is finite.*

We prove that $\mathcal{T}_2^{apex}$ and their obstructions have tree-width bounded by 4 (see observations below). Because all graphs with tree-width two are, by Observation 5, in $\mathcal{T}_2^{apex}$, all obstructions have tree-width either three or four. This two cases are discussed in following two chapters. In Chapter 2 we describe all obstructions with tree-width 4 from the known set of obstructions for the class of partial 3-trees. Chapter 3 deals with obstructions with tree-width three. Since graphs with bounded tree-width have bounded connectivity, we classify obstructions by their connectivity.

The following two observations provide us upper bounds on tree-width of graphs in $\mathcal{T}_2^{apex}$ and obstructions for $\mathcal{T}_2^{apex}$.

Observation 10. *Every graph in $\mathcal{T}_2^{apex}$ has tree-width at most 3.*

Proof. If a graph G is contained in $\mathcal{T}_2^{apex}$, it has an apex vertex a such that $H \setminus a$ is in $\mathcal{T}_2$. Let $(T, (V_t)_{t \in V(T)})$ be a tree decomposition of $G \setminus a$ with width at most 2. Then $(T, (V'_t)_{t \in V(T)})$, where $V'_t = V_t \cup \{a\}$ for every $t \in V(T)$, is a tree decomposition of the graph G with width at most 3. □

Observation 11. *Obstructions for $\mathcal{T}_2^{apex}$ have tree-width at most 4.*

Proof. We first prove that a vertex deletion decreases tree-width of a graph G at most by one. Suppose that the graph G has tree-width k and there exists a vertex $v \in V(G)$ such that $G \setminus v$ has tree-width at most $k-2$. Then there exists a tree decomposition of $G \setminus v$ with width at most $k-2$. By adding the vertex v into every node of this decomposition, we obtain a tree decomposition of G with width at most $k-1$. That contradicts the assumption that G has tree-width k.

By successive deleting a vertex we can decrease tree-width of any graph to 0. Thus, every graph G with tree-width greater than four has a proper minor H with tree-width 4. The graph H is not in $\mathcal{T}_2^{apex}$ because tree-width of all graphs in $\mathcal{T}_2^{apex}$ is at most 3. Then, G cannot be an obstruction for $\mathcal{T}_2^{apex}$. □

Chapter 2

Obstructions with tree-width 4

In this chapter, we describe all obstruction for $\mathcal{T}_2^{apex}$ of tree-width 4. By Observation 11, this is the maximum tree-width for obstructions for $\mathcal{T}_2^{apex}$. Every such obstruction is also an obstruction for the class $\mathcal{T}_3$ of partial 3-trees as states the following observation.

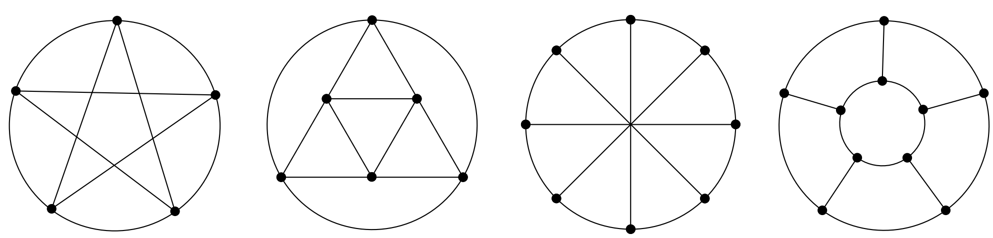
Observation 12. *Every obstruction G of $\mathcal{T}_2^{apex}$ with tree-width 4 is also an obstruction for the class $\mathcal{T}_3$.*

Proof. After any deletion or contraction, G becomes an apex of a K_4–minor–free graph and therefore a graph with tree-width at most 3. Hence, G must be an obstruction for $\mathcal{T}_3$. □

The set of all obstructions for graphs of tree-width 3 (Figure 2.1) has been classified by Arnborg, Proskurowski and Corneil [2]. All we need to do, looking for the obstructions for $\mathcal{T}_2^{apex}$ with tree-width 4, is to check whether any of the four obstructions for $\mathcal{T}_4$ is also an obstruction for $\mathcal{T}_2^{apex}$.

Lemma 13. *The graphs K_5, M_6 and M_8 are obstructions for $\mathcal{T}_2^{apex}$. The graph M_{10} is not.*

Proof. While checking all the possible deletions and contractions, we can omit deletions of a vertex because if a graph becomes an apex of a K_4-minor-free graph after deleting an edge uv, it becomes an apex after deleting either u or v, too.



K_5 M_6 M_8 M_{10}

Figure 2.1: Obstructions for the class $\mathcal{T}_4$.

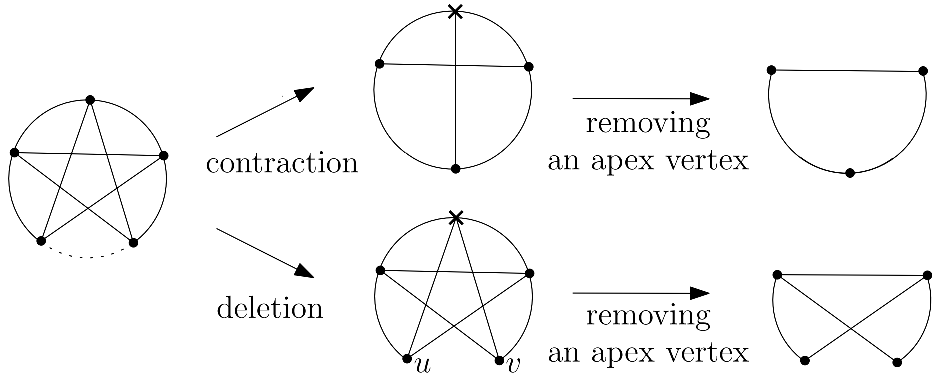


Figure 2.2: Maximal proper minors of K_5.

In what follows, we will use the observation: *If a graph G has less than four vertices of degree at least 3, then G is K_4-minor-free.*

In the following pictures we exhibit all maximal proper minors of K_5, M_6 and M_8. Deletions and contractions of dotted edges are shown and an apex vertex is marked with a cross.

- Since any graph H obtained from K_5 by contracting an edge has at most four vertices, removing any vertex from H results into a 3-vertex graph which cannot contain K_4 as a minor. Hence, we focus on minors obtained by deleting an edge. After deletion of an edge uv, we have to choose the apex vertex carefully—as a vertex not adjacent to the end-vertex of the deleted edge. The graph G obtained from K_5 by deleting the edge uv and the apex vertex has four vertices and an edge between two of them is missing (see Figure 2.2). Thus G does not contain K_4 as a minor.

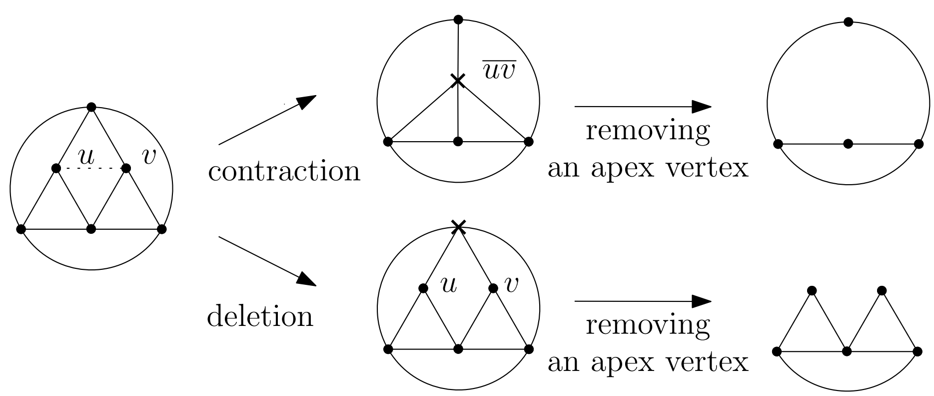


Figure 2.3: Maximal proper minors of M_6.

- Since M_6 is edge-transitive (i.e. for every two edges uv and $u'v'$ in M_6, there is an isomorphism $\varphi : M_6 \to M_6$, such that $\varphi(u)\varphi(v) = u'v'$), it is enough to check its minors obtained by contracting or deleting a fixed edge. The graph obtained by deleting an edge uv in M_6 has four vertices of degree 4 and two vertices of degree 3 (u and v). If we choose as the apex vertex one of two vertices adjacent to both the vertices of degree 3 (see Figure 2.3), the graph G obtained from M_6 by deleting the edge uv and the apex vertex contains two vertices of degree 2 and only three vertices of degree at least 3. Hence, G cannot contain K_4 as a minor.

The graph H obtained by contracting the edge uv has three vertices of degree four and two vertices of degree three (because every two adjacent vertices in M_6 have two common neighbors). Then the vertex $\overline{uv}$ has degree 4 and we can choose it is an apex vertex in H. Since $H \setminus \overline{uv}$ contains only two vertices of degree 3, it does not contain K_4 as a minor.

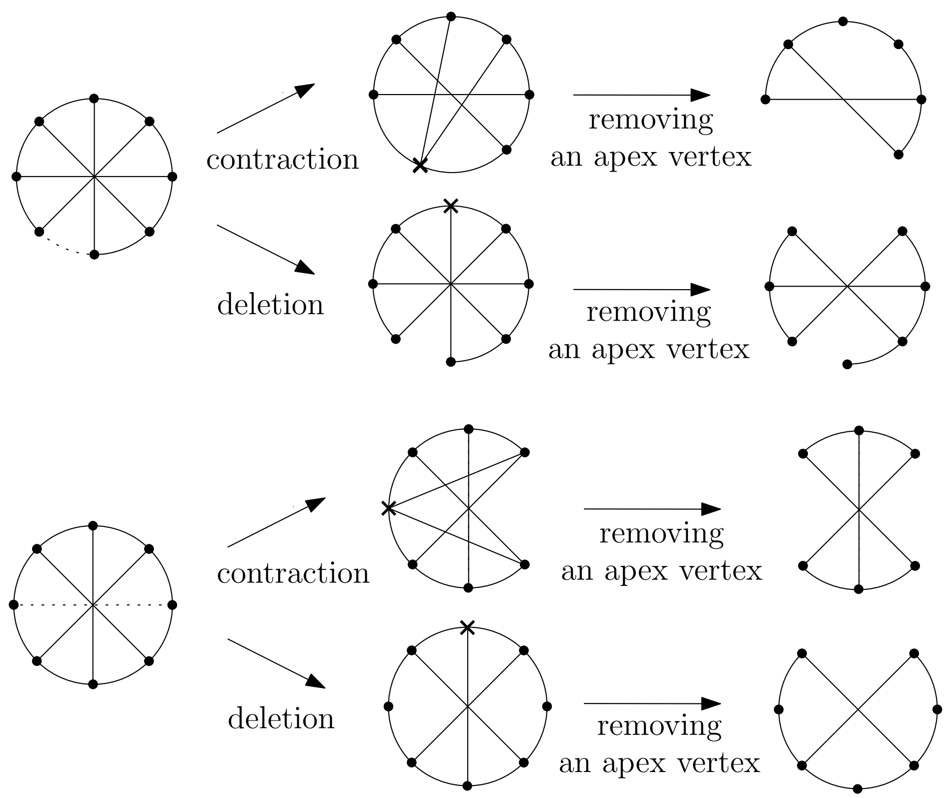


Figure 2.4: Maximal proper minors of M_8.

- All the vertices of the graph M_8 have degree 3. Therefore a graph G obtained by deleting an edge uv in M_8 contains 6 vertices of degree 3. Choose a vertex of degree 3 as the apex vertex. As shown in Figure 2.4, a graph obtained from G by deleting the apex vertex contains only 3 vertices of degree 3. Hence, M_8 becomes after deleting an edge an apex of a K_4-minor-free graph.

  In the case of a contraction of an edge there arises one vertex of degree 4 and the degrees of the other 6 vertices remain 3. Choose the new vertex obtained by contraction to be the apex vertex. After deleting the apex vertex, the degree of four of the remaining vertices decreases and there will be only 2 vertices of degree 3. In Figure 2.4 the situation for two asymmetric types of edges is shown.

- Unlike the three previous graphs, M_{10} is not an obstruction for $\mathcal{T}_2^{apex}$: after deleting an edge, the graph is not an apex of a K_4-minor-free graph. In other words after removing any vertex, the graph still contains K_4 as a minor—this is obvious for the vertices adjacent to the removed edge; other two cases, up to symmetry, are depicted in Figure 2.5.

□

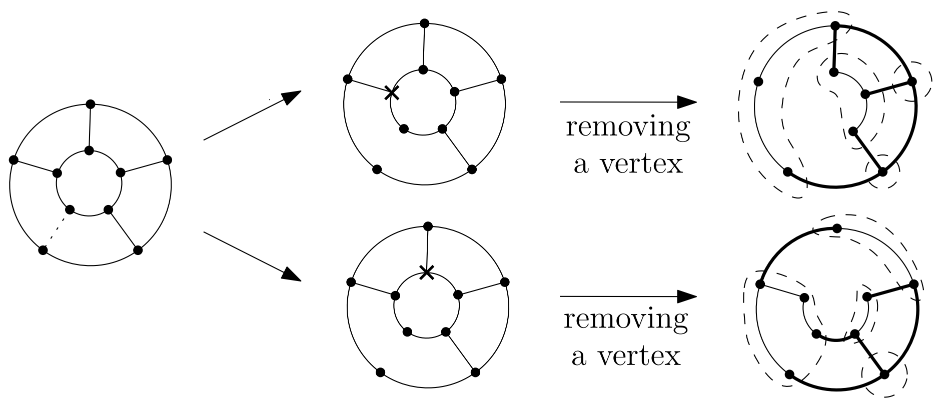


Figure 2.5: The graph M_{10} after deleting an edge.

At this point, we have found all obstructions for $\mathcal{T}_2^{apex}$ with tree-width 4. We present a proof that K_5, M_6 and M_8 are obstructions for $\mathcal{T}_2^{apex}$ by checking all possible edge deletions and contractions. Because this checking is rather mechanical and for some graphs it can be quite tedious, we do not provide this verification in the rest of the thesis (although these proofs are necessary and we did them), hoping that the reader could do it himself if he wishes.

Chapter 3

Obstructions with tree-width 3

In this chapter, we search for obstructions with tree-width at most three. We discuss several cases according to the vertex-connectivity of obstructions, using the following proposition and observations:

Proposition 14. *[3] Let H be a graph with maximum degree at most 3. Then if a graph G contains H as a minor, G contains a subdivision of H as a subgraph.*

In particular, if a graph G contains K_4 as a minor, G contains a subdivision of K_4 as a subgraph. This fact is frequently used throughout the thesis. Since K_4 is the only obstruction for the class $\mathcal{T}_2$, every obstruction for $\mathcal{T}_2^{apex}$ must contain a subdivision of K_4 after removing any vertex.

Observation 15. *If G is an obstruction for $\mathcal{T}_2^{apex}$, every vertex and every edge in G is contained in at least one subdivision of K_4.*

Proof. Suppose that a vertex v in G is not contained in any subdivision of K_4. Then $G \setminus v$ is an apex of a K_4-minor-free graph, i.e., there exists an apex vertex a, which is contained in every subdivision of K_4 in $G \setminus v$. But since every subdivision of K_4 in G is contained in $G \setminus v$, too, all subdivisions of K_4 in G contain the vertex a. Therefore, G is an apex of a K_4-minor free graph with a as the apex vertex. Hence, the obstruction G does not contain any vertex that is not contained in any subdivision of K_4. The proof that the obstruction G does not have any edge that is not contained in any subdivision of K_4 is similar. $\square$

Observation 16. *Every graph of treewidth 3 is at most 3-connected.*

Observation 17. *There exists exactly one disconnected obstruction for $\mathcal{T}_2^{apex}$, which consists of two disjoint K_4's.*

Proof. Each connected component of an obstruction G has to contain K_4 as a minor. If $G \notin \mathcal{T}_2^{apex}$ contains a component G' that does not contain K_4 as a minor, then $G \setminus G' \notin \mathcal{T}_2^{apex}$. Hence, G cannot be an obstruction for $\mathcal{T}_2^{apex}$. $\square$

This implies, that any graph which is not equal to $2K_4$ contains two disjoint subdivisions of K_4 is not an obstruction for $\mathcal{T}_2^{apex}$ (because it contains $2K_4$ as a proper minor). Besides, the intersection of all subdivisions of K_4 in an obstruction G for $\mathcal{T}_2^{apex}$ have to be empty (any vertex in this intersection would be an apex vertex in G). Using this facts we obtain the following two observations for connected obstructions:

Observation 18. *Every two subdivisions of K_4 in a connected obstructions have at least one common vertex and there are at least 3 distinct subdivisions of K_4 in every connected obstruction.*

Observation 19. *Every connected obstruction for $\mathcal{T}_2^{apex}$ is 2-connected.*

Proof. Suppose that G is an connected obstruction for $\mathcal{T}_2^{apex}$ that contains an articulation w. Because every subdivision of K_4 is 2-connected, every subdivision of K_4 in G is whole contained in one of the bridges produced by w. By the Observation 18 any two of subdivisions of K_4 in G intersect. Therefore, if there exist subdivisions of K_4 in at least two different bridges, all subdivisions of K_4 in G contain the articulation vertex. But then, G is an apex of a K_4-minor-free graph with the articulation being an apex vertex. Thus, all subdivisions of K_4 in G are in the same bridge. But as the vertex-cut w produces at least two bridges, at least one of them does not contain any subdivision of K_4 and therefore, by Observation 15, G is not an obstruction for $\mathcal{T}_2^{apex}$. Hence, connected obstructions for $\mathcal{T}_2^{apex}$ with 1-cuts do not exist. □

3.1 Obstructions of connectivity 2

For the reminder of this section, let $\mathbb{G}$ be a 2-connected obstruction for $\mathcal{T}_2^{apex}$ with a vertex cut $\{x, y\}$. Let $\tilde{K}_4$ be a subdivision of K_4. The intersection of $\tilde{K}_4$ with the vertex cut $\{x, y\}$ is denoted as an upper index. For example $\tilde{K}_4^{\emptyset}$ means that a subdivision of K_4 does not contain any of the vertices x, y, $\tilde{K}_4^{x}$ means that a subdivision of K_4 contains the vertex x and does not contain the vertex y. $\tilde{K}_4^{xy}$ contains both the vertices x and y. We say that a bridge M contains $\tilde{K}_4^{xy}$ if M contains all vertices of degree three of this subdivision of K_4.

In the following two observations, we introduce some of basic properties of obstructions $\mathbb{G}$ that are used in several lemmas further in the section.

Observation 20. *There exists at least one $\tilde{K}_4$ in $\mathbb{G}$ that does not contain x and at least one $\tilde{K}_4$ in $\mathbb{G}$ that does not contain y.*

Proof. Otherwise $\mathbb{G}$ is an apex of a K_4-minor-free graph with x or y being an apex vertex. □

Observation 21. *Every bridge of the obstruction $\mathbb{G}$ contains at least one $\tilde{K}_4$.*

Proof. Suppose that $\mathbb{G}$ has a bridge M that does not contain any $\tilde{K}_4$. As M is connected, it contains a path between x and y thus, it contains an edge xy as a proper minor (recall that $M \setminus \{x, y\}$ is a nonempty graph). Therefore, a graph H obtained from $\mathbb{G}$ by replacing the bridge M by an edge between x and y is an apex of a K_4-minor-free graph. Let a be an apex vertex in H. The graph $\mathbb{G} \setminus a$ does not contain any $\tilde{K}_4^{\emptyset}$, $\tilde{K}_4^x$ or $\tilde{K}_4^y$, because every such subdivision of K_4 is contained in $H \setminus a$, too. If there exists $\tilde{K}_4^{xy}$ in $\mathbb{G} \setminus a$, then, in $H \setminus a$, there exists $\tilde{K}_4^{xy}$ with the same vertices of degree 3, that contains the edge xy instead of some path in M. Thus, $\mathbb{G}$ does not have any bridge that does not contain any $\tilde{K}_4$. □

The following lemma describe two possible types of bridges produced by the 2-cut $\{x, y\}$ in an obstruction $\mathbb{G}$. In the reminder of the chapter we study an arrangement of $\tilde{K}_4$'s in these bridges in detail.

Lemma 22. *There exists exactly one bridge M_0 in $\mathbb{G}$ that contains $\tilde{K}_4^{\emptyset}$ or both $\tilde{K}_4^x$ and $\tilde{K}_4^y$. Consequently, there exists at least one bridge M_1 in $\mathbb{G}$ such that every $\tilde{K}_4$ in M_1 contains both x and y.*

Proof. Let us consider a bridge M_0 that contains $\tilde{K}_4^{\emptyset}$, $\tilde{K}_4^x$ or $\tilde{K}_4^y$. There exists at least one such bridge in G, otherwise x and y are apex vertices.

If M_0 contains $\tilde{K}_4^{\emptyset}$, every $\tilde{K}_4$ in any other bridge must contain both x and y: otherwise, it is disjoint with $\tilde{K}_4^{\emptyset}$ in M_0. By Observation 20, there exist $\tilde{K}_4$ that does not contain x and $\tilde{K}_4$ that does not contain y. Thus, if $\mathbb{G}$ does not contain any $\tilde{K}_4^{\emptyset}$, it contains $\tilde{K}_4^x$ and $\tilde{K}_4^y$. Suppose that M_0 contains $\tilde{K}_4^x$. Then every $\tilde{K}_4^y$ has to be contained in M_0 as well (otherwise $\tilde{K}_4^x$ and $\tilde{K}_4^y$ are disjoint). It follows that all $\tilde{K}_4^x$ are contained in M_0. Hence, any other bridge can contain only $\tilde{K}_4^{xy}$'s. By Observation 21, every such bridge contains at least one $\tilde{K}_4^{xy}$. □

The bridge containing $\tilde{K}_4^{\emptyset}$ or $\tilde{K}_4^x$ and $\tilde{K}_4^y$ is denoted by M_0. Any other bridge contains only $\tilde{K}_4^{xy}$'s and is denoted M_1. Let M_1 be one such bridge.

We first focus on a bridge M_1. Let bridges shown in Figures 3.1a and 3.1b be denoted by ☾ and ☾| respectively. We show that M_1 is isomorphic to ☾ or ☾| and there is only one bridge different from M_0 in the obstruction $\mathbb{G}$.

Observation 23. *☾ or ☾| is a minor of a bridge M_1.*

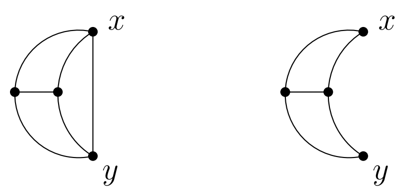


Figure 3.1: The only possible bridges isomorphic to M_1.

Lemma 24. *If is a minor of a bridge M_1, then M_1 is equal to .*

Proof. If is a proper minor of M_1, replacing M_1 by produces an apex of a K_4-minor-free graph. An apex vertex has to be in $M_0 \setminus \{x, y\}$ eliminate all $\tilde{K}_4^{\emptyset}$'s, $\tilde{K}_4^{x}$'s and $\tilde{K}_4^{y}$'s in M_0, but at the same time, an apex vertex has to eliminate K_4 in , which is not possible. Thus, is not a proper minor of M_1. □

Lemma 25. *If is not a minor of a bridge M_1, then M_1=.*

Proof. By Lemma 23, a bridge M_1 contains as a minor. Suppose that is a proper minor of M_1. Then the graph G' obtained by replacing M_1 in $\mathbb{G}$ by ,is an apex of a K_4-minor-free graph (as $\mathbb{G}$ is an obstruction). An apex vertex has to be in $M_0 \setminus \{x, y\}$, because M_0 contains $\tilde{K}_4^{\emptyset}$ or $\tilde{K}_4^{x}$ and $\tilde{K}_4^{y}$. An apex vertex has to cut off all paths between x and y in M_0, otherwise exists $\tilde{K}_4$ consisting of a path between x and y and . But if there exists such an apex vertex, its removal eliminates all $\tilde{K}_4$ in the original bridge M_1 and thus it eliminates all $\tilde{K}_4$ in $\mathbb{G}$. This contradicts that $\mathbb{G}$ is an obstruction for $\mathcal{T}_2^{apex}$. □

Lemma 26. *A graph $\mathbb{G}$ contains 2 bridges.*

Proof. We observed that the bridge M_0 is unique and all other bridges are either or (all of them are of the same type). Let us suppose there are at least two such bridges, M_1 and M_1'. Then by replacing M_1' by an edge xy, we obtain an apex of a K_4-minor-free graph. But as M_1 with the edge xy compose $\tilde{K}_4$ and M_0 contains $\tilde{K}_4^{\emptyset}$ or $\tilde{K}_4^{x}$ and $\tilde{K}_4^{y}$, there cannot be any apex vertex. Therefore the graph $\mathbb{G}$ contains only one bridge or in addition to M_0. □

Keeping the preceding notation, in the following two lemmas we describe the only obstruction that has the bridge M_1 equal to and in the rest of the section we investigate the obstructions with M_1 equal to.

Lemma 27. *If M_1 equals , then M_0 contains $\tilde{K}_4^{x}$ and $\tilde{K}_4^{y}$ and has an cut-vertex.*

Proof. As $\mathbb{G}$ does not contain two disjoint $\tilde{K}_4$, M_0 cannot contain $\tilde{K}_4^{\emptyset}$, therefore it has to contain $\tilde{K}_4^{x}$ and $\tilde{K}_4^{y}$. Suppose that M_0 is 2-connected. Then $\mathbb{G}$ is not an obstruction, because $\mathbb{G}$ without the edge xy is not an apex of a K_4-minor-free graph—since $\mathbb{G}$ contains $\tilde{K}_4^{x}$ and $\tilde{K}_4^{y}$ in M_0, no vertex in M_1 (including x and y) can be an apex vertex. Since M_0 is 2-connected, after deleting any vertex in M_0 except x,y, there remains a path between x and y in M_0. This path and M_1 composes $\tilde{K}_4$. □

Lemma 28. *The graph $\mathbb{G}$ in Figure 3.2 is the only obstruction with the bridge M_1 equal to .*

Proof. The graph $\mathbb{G}$ from Figure 3.2 is the minimal graph containing M_1 equal to and satisfying conditions on M_0 from Lemma 27. Thus, it is a minor of every obstruction for $\mathcal{T}_2^{apex}$ with M_1 equal to . Since $\mathbb{G}$ is an obstruction for $\mathcal{T}_2^{apex}$, it is not a proper minor of any other obstruction for $\mathcal{T}_2^{apex}$. Therefore, $\mathbb{G}$ is the only obstruction with M_1 equal to . □

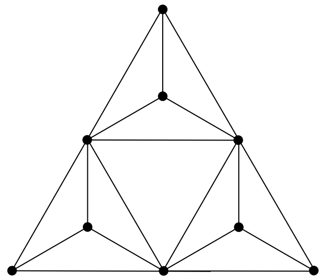


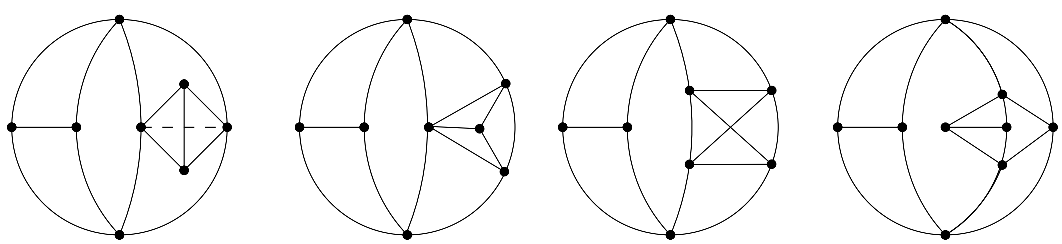
Figure 3.2: The only obstruction with the xy.

Lemma 29. *If* M_1 *equal to* and M_0 *contains* $\tilde{K}_4^{\emptyset}$*, there are exactly two disjoint paths from* x *to* y *in* M_0*.*

Proof. If there are more than two paths from x to y, after deleting an edge containing x or y in one of these paths, there remain at least two paths from x to y and we cannot cut off all of them by a single apex vertex in M_0. Thus every apex vertex has to be in M_1 to eliminate $\tilde{K}_4$ consisting of M_1 and one of the paths between x and y in M_0. At the same time every apex vertex has to be in $M_0 \setminus \{x, y\}$, to eliminate $\tilde{K}_4^{\emptyset}$.

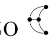
If there exists only one path between x and y in M_0, there is an articulation vertex $z \neq x, y$ in M_0. As G is 2-connected, z divides M_0 into two parts: the part M_{0x} containing x and the part M_{0y} containing y. Suppose that without loss of generality, $\tilde{K}_4^{\emptyset}$ is in the part M_{0x}. Then, if we consider the vertex cut $\{x, z\}$ instead $\{x, y\}$ and apply Lemma 22, we get that as M_{0x} contains $\tilde{K}_4^{\emptyset}$ or $\tilde{K}_4^{z}$, the other bridge consisting of M_1 equal to and M_{0y} contains only $\tilde{K}_4^{xz}$ and equals to or , which is impossible. □

Lemma 29 gives us four possible minors of obstructions in Figure 3.3, i.e., every graph satisfying Lemma 29 have at least one of the graphs in Figure 3.3 as a minor. None of these four graphs is a K_4-minor-free graph, but in the case a) the graph is not an obstruction—we can remove the dashed edge and the resulting graph is not an apex of a K_4-minor-free graph. The graphs in Figures 3.3b, c and d are obstructions.



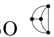
a) b) c) d)

Figure 3.3: Obstructions of connectivity 2 with $\tilde{K}_4^{\emptyset}$.

Now we are going to describe obstructions with M_1 equal to  that do not contain $\tilde{K}_4^{\emptyset}$. They have more complex structure than those containing $\tilde{K}_4^{\emptyset}$. Although there exist only five such obstructions, it takes several pages to describe all of them and prove that no other obstruction exists.

Lemma 30. *If $\mathbb{G}$ consists of M_1 equal to  and M_0 that does not contain $\tilde{K}_4^{\emptyset}$ (and therefore it contains $\tilde{K}_4^x$ and $\tilde{K}_4^y$), M_0 is 2-connected and every two $\tilde{K}_4^x$ and $\tilde{K}_4^y$ share at least two vertices.*

Proof. If there is an articulation vertex z in M_0, it is contained in both $\tilde{K}_4^x$ and $\tilde{K}_4^y$ (they must not be disjoint). It is also contained in every path from x to y in M_0 and consequently in every $\tilde{K}_4^{xy}$, thus z is contained in all $\tilde{K}_4$ of the graph $\mathbb{G}$. But then $\mathbb{G}$ is not an obstruction. Hence, M_0 is 2-connected.

$\tilde{K}_4^x$ and $\tilde{K}_4^y$ must share at least one vertex. If they share exactly one vertex z, by Menger's theorem there exists a path P from a vertex $u \neq z$ in $\tilde{K}_4^x$ to a vertex $v \neq z$ in $\tilde{K}_4^y$ avoiding $\tilde{K}_4^x$ and $\tilde{K}_4^y$ (i.e. inner vertices of P are not in any of the graphs $\tilde{K}_4^x$ and $\tilde{K}_4^y$). In fact, the path P has to be only an edge because if we replace P by an edge uv, the graph G does not become an apex of a K_4-minor-free graph. Then we have to distinguish the following cases: If the edge $uv = xy$, then M_1 is equal to  (which the lemma asserts not to be the case). If $u \neq x$ and $v \neq y$, the graph obtained from G by contracting the edge uv is not an apex of a K_4-minor-free graph.

If the edge uv equals xv, we can consider the vertex cut $\{x, z\}$ with bridges M_0' and M_1'. The vertex y has to be in M_0' thus $M_1' = \tilde{K}_4^x$, then M_1' is necessarily  and the graph is the known obstruction in Figure 3.2. For the edge uy, we obtain the same conclusion using symmetric arguments.

□

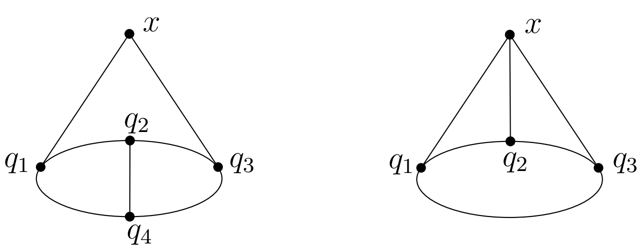


Figure 3.4: Possible types of a structure of $\tilde{K}_4^x$ in an obstruction.

The vertices x in $\tilde{K}_4^x$ and y in $\tilde{K}_4^y$ have degrees 2 or 3, i.e. the obstruction $\mathbb{G}$ has one of the graphs in Figure 3.4 as a minor.

In the following lemmas, $\mathbb{G}$ is an obstruction such that M_1 equals , M_0 does not contain $\tilde{K}_4^{\emptyset}$ and the vertex x has degree 2 in every $\tilde{K}_4^x$ (see Figure 3.4a). The case that x has degree three in $\tilde{K}_4^x$ is dealt later.

Observation 31. *For every $\tilde{K}_4^x$ (with vertices denoted as in Figure 3.4a) and $\tilde{K}_4^y$ in $\mathbb{G}$, $\tilde{K}_4^y$ contains at least one vertex of the path between q_1 and x and at least one vertex of the path between q_3 and x.*

Proof. Suppose that $\tilde{K}_4^y$ does not contain any vertex of the path between x and q_1, or q_3 respectively. Then the graph H obtained from $\mathbb{G}$ by contracting the whole path between x and q_1 or q_3 (but only one of them) into x is not a member of $\mathcal{T}_2^{apex}$: neither $\tilde{K}_4^x$ nor $\tilde{K}_4^y$ was eliminated by the contraction. Since any vertex of the $\tilde{K}_4^y$ was not contracted into x, $\tilde{K}_4^x$ and $\tilde{K}_4^y$ share at least 2 vertices. Thus, the bridge M_0 in H is 2-connected. Therefore there cannot exist any vertex that eliminates $\tilde{K}_4^x$, $\tilde{K}_4^y$ and cuts off all paths between x and y in M_0 at the same time. Hence, the graph H does not have any apex vertex. □

Lemma 32. *Let vertices of some $\tilde{K}_4^x$ in $\mathbb{G}$ be denoted as in Figure 3.4a. The vertices x, q_2 and q_4 divide $\tilde{K}_4^x$ into three connected components (see Figure 3.5). Every path between vertices of different components in M_0 contains at least one of the vertices x, y, q_2, q_4 (otherwise, $\mathbb{G}$ contains $\tilde{K}_4^{\emptyset}$). Moreover, there exist internally vertex-disjoint paths from y to vertices in at least two of the components of $\tilde{K}_4^x \setminus \{x, q_2, q_4\}$ in M_0.*

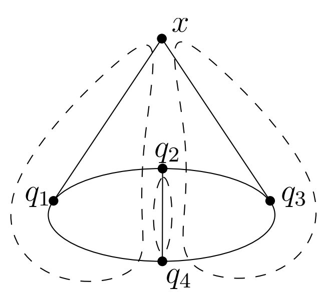


Figure 3.5: Vertices x, q_2 and q_4 divide $\tilde{K}_4^x$ into three components.

Proof. Suppose that all paths between y and vertices of $\tilde{K}_4^x$ in M_0 that avoid $\tilde{K}_4^x$, i.e., do not have any internal vertex in $\tilde{K}_4^x$, have their ends in the same component C_1 or in q_2 or q_4. By Observation 31, there are vertices of $\tilde{K}_4^y$ in the paths between x and q_1 and between x and q_3, thus vertices of $\tilde{K}_4^y$ occurs in at least two different components of $\tilde{K}_4^x \setminus \{x, q_1, q_2\}$. Every path between different components contains x, y, q_2 or q_4, because we consider an obstruction $\mathbb{G}$ that does not contain $\tilde{K}_4^{\emptyset}$. Thus, vertices q_2 and q_4 form 2-cut in $\tilde{K}_4^y$, because $\tilde{K}_4^y$ does not contain the vertex x and we suppose that there does not exist paths avoiding $\tilde{K}_4^x$ between y and vertices of different components of $\tilde{K}_4^x \setminus \{x, q_2, q_4\}$.

The 2-cut $\{q_2, q_4\}$ yields two bridges in $\tilde{K}_4^y$. Observe that neither of these bridges contains vertices of more than one component of $\tilde{K}_4^x \setminus \{x, q_2, q_4\}$. All vertices of degree 3 in $\tilde{K}_4^y$ are in the same bridge B_1. The other bridge B_2 consists only of a path between q_2 and q_4. The vertex y and vertices of $\tilde{K}_4^y$ in C_1 are contained the bridge B_1: y and vertices of $\tilde{K}_4^y$ in C_1 must be in the same bridge and if they are in B_2, there exists $\tilde{K}_4^{\emptyset}$ consisting of B_1 and a path between q_2 and q_4 in C_1.

The bridge B_1 does not contain any vertex in two of the components of $\tilde{K}_4^x \setminus \{x, q_2, q_4\}$, C_2 and C_3, different from C_1. Since the both components C_2 and C_3 contain a path between the vertices q_2 and q_4, there exist $\tilde{K}_4^y$ avoiding C_2 and $\tilde{K}_4^y$ avoiding C_3 in $\mathbb{G}$. Thus, there exists $\tilde{K}_4^y$ avoiding the component containing the vertex q_1 (and internal vertices of the path between q_1 and x) or $\tilde{K}_4^y$ avoiding the component containing q_3. That contradicts the previous observation. □

Every graph satisfying the conclusion of Lemma 32 contains one of the graphs in Figure 3.6 as a minor. Both of the graphs in Figure 3.6 are obstructions and these are all obstructions without $\tilde{K}_4^{\emptyset}$ that contains $\tilde{K}_4^x$ with x of degree 2 or $\tilde{K}_4^y$ with y of degree 2.

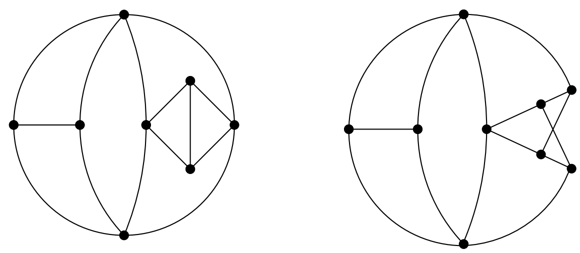
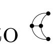


Figure 3.6: Obstructions that contain $\tilde{K}_4^x$ with x of degree two.

In what follows, we assume that $\mathbb{G}$ is an obstruction such that M_1 is equal to , M_0 does not contain $\tilde{K}_4^{\emptyset}$ and vertices x or y have degree 3 in $\tilde{K}_4^x$ or $\tilde{K}_4^y$ respectively (see Figure 3.4b). Vertices of degree 3 in $\tilde{K}_4^x$ except x are denoted by q_1, q_2 and q_3 as in Figure 3.4b. The path between the vertices x and q_i in $\tilde{K}_4^x$, that does not contain any of the remaining two vertices of degree 3, is denoted Q_i. Let $\tilde{K}_4^{y \setminus x}$ denote graph $\tilde{K}_4^y$ without edges contained in $\tilde{K}_4^x$.

Observation 33. $\tilde{K}_4^{y \setminus x}$ *in* $\mathbb{G}$ *contains a graph in Figure 3.7a, Figure 3.7d or Figure 3.7e as a minor.*

Proof. As $\tilde{K}_4^x$ and $\tilde{K}_4^y$ share at least two vertices, there exist at least two internally vertex-disjoint paths P_1 and P_2 from y to distinct vertices p_1 and p_2 in $\tilde{K}_4^x$. Because y has degree 3 in $\tilde{K}_4^y$ and $\tilde{K}_4^y$ is 2-connected, there exists a path P_3 from y to a vertex p_3 in $\tilde{K}_4^x$ (Figure 3.7a) or in one of paths P_1, P_2 (without loss of generality suppose P_2)—as shown in Figures 3.7b and c. Note that the graph in Figure 3.7c contains the graph in the Figure 3.7b as a minor.

Suppose that p_3 is in P_2. Because $\tilde{K}_4$ does not contain parallel edges, there has to exist a path P_4 avoiding P_1, P_2, P_3 and $\tilde{K}_4^x$ with ends u and v of the following properties: The vertex u is an internal vertex of P_3 or an internal vertex of the subpath of P_2 between y and p_3. Without loss of generality, suppose that u is in the

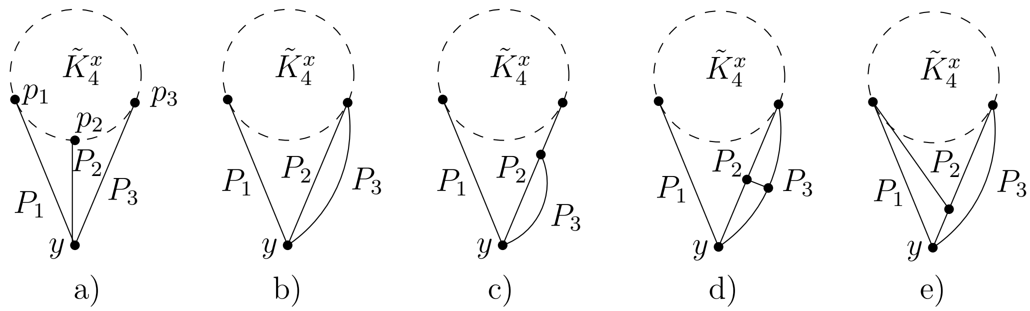


Figure 3.7: Possible minors of $\tilde{K}_4^{y\setminus x}$.

path P_2. The vertex v is either in $\tilde{K}_4^x$, in P_3 or in P_1 and then $\tilde{K}_4^{y\setminus x}$ in $\mathbb{G}$ contains Figure 3.7a, d or e respectively as a minor. □

In the following lemmas, we classify obstructions containing the graphs in Figures 3.7a, d and e as a minors of $\tilde{K}_4^{y\setminus x}$. When deciding whether a minor-minimal graph that contains a particular minor of $\tilde{K}_4^{y\setminus x}$ is an obstruction, we use the fact that if a graph contains an obstruction as a proper minor, the graph is not an obstruction.

Lemma 34. *There are two minor-minimal graphs containing the graph from Figure 3.7e as a minor of $\tilde{K}_4^{y\setminus x}$. These are the two graphs depicted in Figure 3.8. None of these graphs is contained in $\mathcal{T}_2^{apex}$. The first graph (given in Figure 3.8a), is not an obstruction (it is possible to delete the dashed edge) and the second graph is an obstruction.*

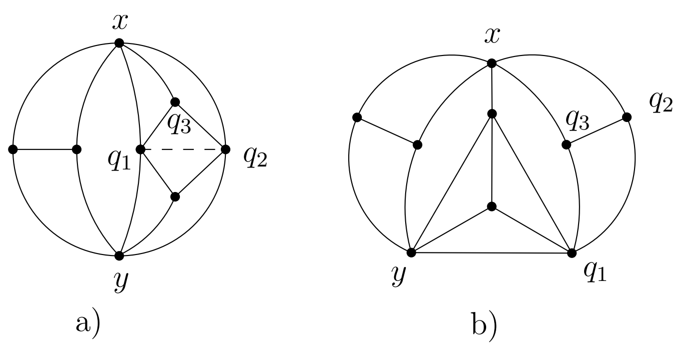
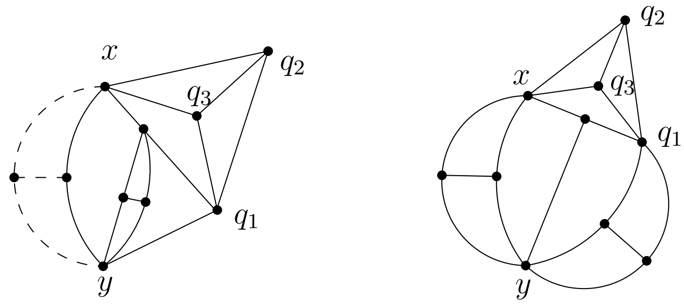


Figure 3.8: Obstructions that contain the graph in Figure 3.7e as a minor of $\tilde{K}_4^{y\setminus x}$.

Proof. If the vertices p_1 and p_2 belong to the same path Q_i, we obtain the graph in Figure 3.8b. In all other cases it is possible to contract vertices p_1 and p_2 into two distinct vertices among the vertices q_1, q_2 and q_3, which results in the graph in Figure 3.8a with the dashed edge present. □

Lemma 35. *There are three minimal graphs containing the graph from Figure 3.7d as a minor of $\tilde{K}_4^{y\backslash x}$. These are the two graphs depicted in Figure 3.9 and the graph from Figure 3.8b. None of these graphs is contained in $\mathcal{T}_2^{apex}$. The graph in Figure 3.9a is not an obstruction (by deleting of the dashed edges, we obtain the first obstruction in Figure 3.6). The graph in Figure 3.9b is an obstruction.*

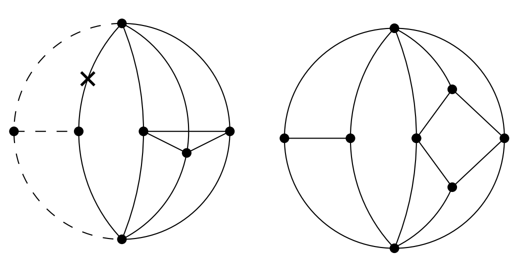


a) b)

Figure 3.9: Obstructions that contain the graph in Figure 3.7d as a minor of $\tilde{K}_4^{y\backslash x}$.

Proof. If the vertices p_1 and p_2 belong to the same path Q_i between q_i and x in $\tilde{K}_4^x$, we obtain one of the graphs in Figure 3.9, in all other cases it is possible to contract vertices p_1 and p_2 into two distinct vertices among the vertices q_1, q_2 and q_3, which results to graphs isomophic to the graph in Figure 3.8b. □

Lemma 36. *There are two minor-minimal graphs containing the graph from Figure 3.7a as a minor of $\tilde{K}_4^{y\backslash x}$, such that no two of the vertices p_1, p_2 and p_3 are contained in the same path Q_i in $\tilde{K}_4^x$. These are the two graphs depicted in Figure 3.10. The graph in Figure 3.10a is not an obstruction (by deleting dashed edges and contracting crossed edge we obtain K_5). The graph in Figure 3.10b is an obstruction.*



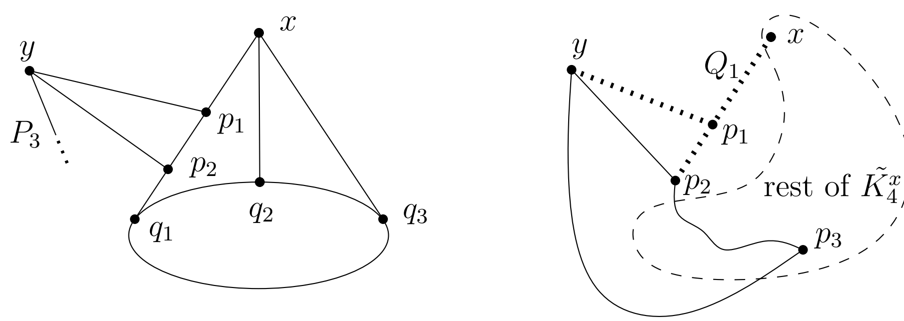
a) b)

Figure 3.10: Obstructions that contain the graph in Figure 3.7d as a minor of $\tilde{K}_4^{y\backslash x}$.

Proof. Consider an obstruction $\mathbb{G}$ the graph from Figure 3.7a as a minor of $\tilde{K}_4^{y\setminus x}$, such that no two of the vertices p_1, p_2 and p_3 are contained in the same path Q_i in $\tilde{K}_4^x$. If the vertex p_i is contained in the path Q_j, we contract p_i into q_j. If some of the vertices p_1, p_2 and p_3 is not contained in any of the paths Q_1, Q_2 and Q_3, they are contained in the cycle $q_1q_2q_3$. We contract edges of this cycle as long as it is possible to keep vertices p_1, p_2 and p_3 distinct, and vertices q_1, q_2 and q_3 distinct. By the contractions, we obtain one the graphs in Figure 3.10. □

In the following lemmas, we suppose that an obstruction $\mathbb{G}$ contains the graph from Figure 3.7a as a minor of $\tilde{K}_4^{y\setminus x}$ and at least two paths from y end on the same path Q_i. Without loss of generality, paths P_1 and P_2 end on Q_1, p_1 is nearer to x than p_2 (see Figure 3.11a). If P_3 ends on Q_1, p_3 is nearer to q_1 than both p_1 and p_2.

T denotes the part of the obstruction $\mathbb{G}$ that consists of the path P_1 without the vertex y and the subpath of the path Q_1 between x and p_2 without its ends. This part is marked by dots in Figure 3.11b. In the following observation and two lemmas, we show some properties of paths between vertices of T and vertices of $M_0 \setminus T$. Later, we use this properties for identifying obstructions that satisfy assumptions above.



a) b)

Figure 3.11:

Observation 37. *If a graph contain three internally disjoint paths between two vertices and an path between internal vertices in two of these three paths, the graph contains K_4 as a minor.*

Lemma 38. *Every path in $\tilde{K}_4^y$ from a vertex in P_2 to $M_0 \setminus T$ contains the vertex y or p_2.*

Proof. Otherwise it is possible to contract vertices p_1 and p_2 into a single vertex, eliminating neither $\tilde{K}_4^x$ nor $\tilde{K}_4^y$. Then, the resulting graph is not an apex of a K_4-minor-free graph (see Figure 3.11b and apply the observation above to three paths between x and p_2). □

Lemma 39. *There exists a path R between the vertex p_1 and a vertex in $M_0 \setminus T$ avoiding T, which does not contain any of vertices p_2, y and x.*

Proof. There must exist a path R' between a vertex in T and a vertex in $M_0 \setminus T$ which does not contain any of vertices p_2, y, x, otherwise vertices p_2 and y form a 2-cut which divides $\tilde{K}_4^y$ into three components, such that inner vertices of each of the paths P_1, P_2 and P_3 are contained in different component (note that x is not in $\tilde{K}_4^y$). Two of these components have to represent the same edge in K_4. Therefore it is possible to delete edges and inner vertices of one of the paths P_1, P_2 and P_3 without eliminating $\tilde{K}_4^y$.

The path R' must have a subpath R avoiding T. The end of the path R in T has to be the vertex p_1, otherwise it can be contracted into vertex p_1 without yielding an apex of a K_4-minor-free graph. □

In the following two lemmas deal with obstructions such that they contain the graph from Figure 3.7a as a minor of $\tilde{K}_4^{y\setminus x}$ and the vertices p_1 and p_2 are contained in the path Q_1. Note that that these obstructions must satisfy conclusions of Lemma 38 and Lemma 39. We classify the obstructions by the position of the vertex p_3 in $\tilde{K}_4^x$.

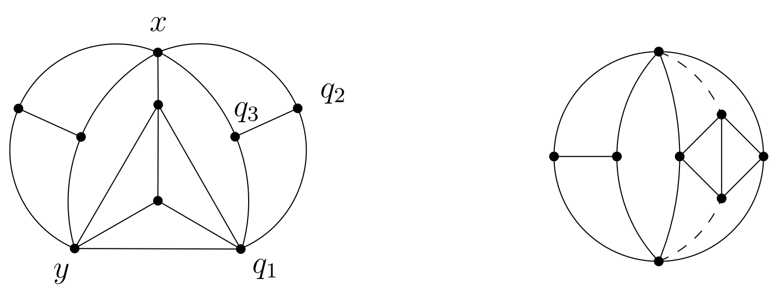


Figure 3.12: Obstructions that contain the graph in Figure 3.7a as a minor of $\tilde{K}_4^{y\setminus x}$ with the vertices p_1, p_2 and p_3 on Q_1.

Lemma 40. *There are two minor-minimal graphs containing the graph from Figure 3.7a as a minor of $\tilde{K}_4^{y\setminus x}$, such that the vertices p_1, p_2 and p_3 are contained in the path Q_1. These are the two graphs depicted in Figure 3.12. The first graph is an obstruction which we have already identified (see Figure 3.8b). The second graph is not an obstruction (after deleting the dashed edges, we obtain the obstruction depicted in Figure 3.6).*

Proof. Note that none of the graphs in Figure 3.13 is contained in $\mathcal{T}_2^{apex}$ thus, they are not proper minors of an obstruction. By Lemma 39, there exists a path R from the vertex p_1 to a vertex u in $M_0 \setminus T$ avoiding T and the vertices x, y and p_2. If u is an inner vertex of the path P_2, $\mathbb{G}$ contains the first of the graphs depicted in Figure 3.12 as a minor (obtained by contracting p_2 and p_3 into q_1). If u is not contained in P_2, it equals to one of the vertices q_1, q_2 and q_3, otherwise it is possible to contract u into one of the vertices q_1, q_2 and q_3 and the graph obtained by such contraction contains as a minor one of the graphs depicted in Figure 3.12. □

Lemma 41. *There are three minor-minimal graphs containing the graph from Figure 3.7a as a minor of* $\tilde{K}_4^{y\setminus x}$*, such that the vertices* p_1 *and* p_2 *are contained in the path* Q_1 *and* p_3 *is not contained in* Q_1*. These are the three graphs depicted in Figure 3.13. The first graph, Figure 3.13a, is an obstruction. The remaining two graphs are not obstructions (by deleting the dashed edges in Figure 3.13b, we obtain the first obstruction in Figure 3.6 and by contracting the dashed edge in Figure 3.13c, we obtain the first obstruction in Figure 3.10).*

Proof. Note that none of the graphs in Figure 3.13 is contained in $\mathcal{T}_2^{apex}$ thus, they are not proper minors of an obstruction.

Suppose first that $p_3 = q_2$. By Lemma 39, there exists a path R from the vertex p_1 to a vertex u in $M_0 \setminus T$ avoiding T and the vertices x, y and p_2. Then, if the vertex u is contained in the path P_2, $\mathbb{G}$ contains the graph depicted in Figure 3.13a as a minor. If u is not contained in P_2, it is equal to q_2 or q_3, otherwise u can be contracted into one of these vertices and the graph obtained by such contraction is contains one of the graphs in Figures 3.13b and 3.13c as a minor.

For $p_3 = q_3$ we obtain isomorphic results. Because we suppose that p_3 is not equal to q_1, p_3 is always equal to q_2 or q_3, otherwise it can be contracted into q_2 or q_3, and the resulting graph contains one of the graphs in Figure 3.13 as a minor. □

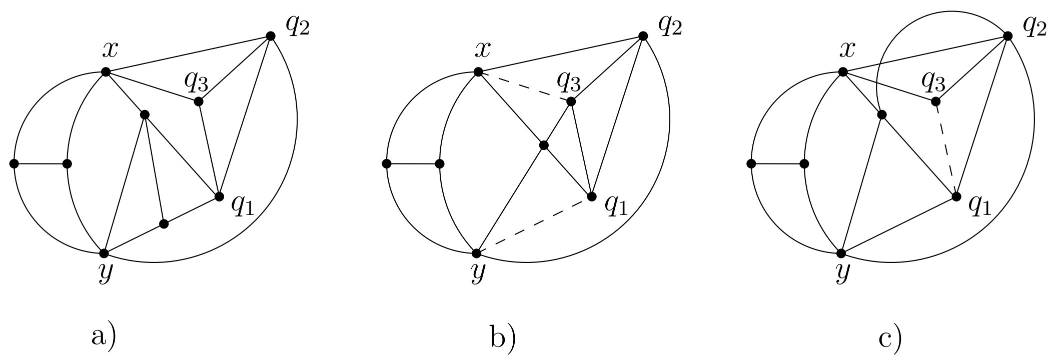


Figure 3.13: Obstructions that contain the graph in Figure 3.7a as a minor of $\tilde{K}_4^{y\setminus x}$ with not all the vertices p_1, p_2 and p_3 on Q_1.

3.2 Obstructions of connectivity 3

In this section we assume that the graph $\mathbb{G}$ is a 3-connected obstruction for $\mathcal{T}_2^{apex}$ with a vertex-cut $\{x, y, z\}$. Subdivisions of K_4 are denoted in the same way as in Section 3.1.

The following two observations show some basic properties of 3-connected obstructions. Note that the second observation is analogical to Observation 20 for 2-connected obstructions. In fact, they are both special cases of a more general observation, that there exists $\tilde{K}_4$ that does not contain v for every vertex v in an obstruction.

Observation 42. *In every obstruction $\mathbb{G}$, there exists at least one $\tilde{K}_4$ that does not contain x, at least one $\tilde{K}_4$ that does not contain y and at least one $\tilde{K}_4$ that does not contain z.*

Proof. Otherwise $\mathbb{G}$ is an apex of a K_4-minor-free graph with x,y or z being an apex vertex. □

a) a trivial bridge b) a minor of a nontrivial bridge

Figure 3.14: Minors of bridges in 3-connected obstruction.

We say that a bridge that consists of only four vertices, i.e., vertices x, y and z and one more vertex (see Figure 3.14a), is *trivial.*

Observation 43. *Every bridge in $\mathbb{G}$ contains a trivial bridge as a minor. Every nontrivial bridge in $\mathbb{G}$ contains the graph from Figure 3.14b as a minor.*

Proof. Every bridge contains at least one vertex v different from the vertices x, y and z. By Corollary 2, it contains tree internally vertex-disjoint paths between the vertex v and the vertices x, y and z. By contracting each of these paths into a single edge (and deleting all other vertices and edges) we obtain the trivial bridge.

If a bridge B is not trivial, it contains at least two vertices different from x, y and z. Let v_1 be one such vertex. By Corollary 2, B contains tree internally vertex-disjoint paths P_x, P_y and P_z between the vertex v_1 and the vertices x, y and z respectively. Suppose first that at least one of the paths P_x, P_y and P_z, without loss of generality P_x, has length at least 2. Then, there must exist a path between an internal vertex of P_x and a vertex u in $P_y \setminus v_1$ or $P_z \setminus v_1$ avoiding P_x, otherwise the vertices v_1 and

x form a 2-cut in $\mathbb{G}$. By contracting u into the vertex y or z we obtain a subdivision of the graph depicted in Figure 3.14b.

Suppose that all the paths P_x, P_y and P_z have length one. Then, there exist a vertex v_2 in B, distinct from the vertices x, y, z and v_1. By Corollary 2, there exist tree internally vertex-disjoint paths P'_x, P'_y and P'_z between the vertex v_2 and the vertices x, y and z respectively. If the vertex v_1 is contained in one of these paths, B contains the graph from Figure 3.14b as a minor. Suppose that v_1 is not contained in any of the paths P'_x, P'_y and P'_z. Since $B \setminus \{x, y, z\}$ is connected, there exists a path Q between the vertices v_1 and v_2 in $B \setminus \{x, y, z\}$. Let Q' be a subpath of Q avoiding the paths P'_x, P'_y and P'_z between the vertex v_1 and a vertex w contained in one of the paths P'_x, P'_y and P'_z, without loss of generality in P'_x. Then, B after contracting the vertex w into v_2, contains a subdivision of the graph depicted in Figure 3.14b as a subgraph. □

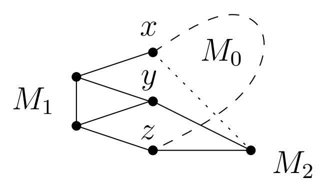


Figure 3.15: A nontrivial bridge forms $\tilde{K}_4^{xyz}$ with a trivial bridge, even after contracting the dotted edge.

Lemma 44. *If the 3-cut in the obstruction $\mathbb{G}$ yields more than 2 bridges, only one of them is nontrivial.*

Proof. Suppose that the 3-cut $\{x, y, z\}$ in $\mathbb{G}$ yields three bridges M_0, M_1 and M_2, and M_0 and M_1 are nontrivial. We first prove that every $\tilde{K}_4$ contains at least two vertices of the 3-cut.

Any two bridges form a $\tilde{K}_4$ containing $\{x, y, z\}$ if at least one of them is not trivial (see Figure 3.15 and Observation 43). Thus, every $\tilde{K}_4$ in $\mathbb{G}$ contains at least one vertex of the 3-cut (otherwise there exist two disjoint $\tilde{K}_4$).

Suppose that there is $\tilde{K}_4$ containing only one vertex of the 3-cut, without loss of generality, let $\tilde{K}_4^x$ be in M_0. Because $\mathbb{G} \setminus x$ still contains $\tilde{K}_4$, there exists $\tilde{K}_4^y$, $\tilde{K}_4^z$ or $\tilde{K}_4^{yz}$ and it has to be in M_0, too (otherwise there exist two disjoint $\tilde{K}_4$). But then $\mathbb{G}$ is not an obstruction, because M_2 contains two edges between vertices x, y and z, for example edges xy and xz, as a proper minor (because it contains a trivial bridge as a minor) and the nontrivial bridge M_1 forms with these two edges $\tilde{K}_4^{xyz}$ (see Figure 3.15).

Thus, every $\tilde{K}_4$ in $\mathbb{G}$ contain at least two of the vertices x, y, and z. Then, by Observation 42, $\mathbb{G}$ has to contain $\tilde{K}_4^{xy}, \tilde{K}_4^{xz}$ and $\tilde{K}_4^{yz}$, otherwise x, y or z is an apex vertex. These three $\tilde{K}_4$ cannot be in the same bridge, argue as in the previous paragraph. Suppose that each of $\tilde{K}_4^{xy}, \tilde{K}_4^{xz}$ and $\tilde{K}_4^{yz}$ is contained in different bridge, without loss of generality, $\tilde{K}_4^{xy}$ in M_0, $\tilde{K}_4^{xz}$ in M_1 and $\tilde{K}_4^{yz}$ in M_2. Then, by deleting edges containing x in M_2, we obtain the 2-connected graph H with 2-cut $\{y, z\}$, that is not an apex of K_4-minor-free graph: there exists a path between x and y in M_1 and a path between x and z in M_0 thus $\tilde{K}_4^{xy}$ and $\tilde{K}_4^{xz}$ are not eliminated by the deletion. Since there are two disjoint paths from y to z in M_0 and M_1, H contains $\tilde{K}_4^{yz}$ in M_2 and there does not exist any apex vertex in H (H contains the graph in Figure 3.16a as a minor).

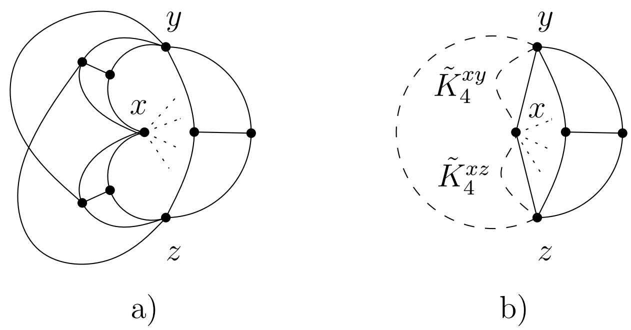


Figure 3.16: Proper minors of the graphs discussed in the proof of Lemma 44.

The only remaining possibility is that two of $\tilde{K}_4^{xy}, \tilde{K}_4^{xz}$ and $\tilde{K}_4^{yz}$, without loss of generality $\tilde{K}_4^{xy}$ and $\tilde{K}_4^{xz}$ are contained in the same bridge, suppose in M_0 and $\tilde{K}_4^{yz}$ in an other bridge, say in M_1. But then, the graph G' obtained by contracting the whole bridge M_2 to edges xy and xz is not contained in $\mathcal{T}_2^{apex}$—the M_1 with the edges xy and xz form $\tilde{K}_4$, thus if there exists an apex vertex in G', it must be contained in M_1. The only vertex in M_1 that eliminates $\tilde{K}_4^{xy}$ and $\tilde{K}_4^{yz}$ in M_0 is the vertex x. Since there exists a path between y and z in M_0 that does not contain x (otherwise $\{xy\}$ or $\{xz\}$ is a 2-cut), the graph $G' \setminus x$ contains $\tilde{K}_4^{yz}$ (see Figure 3.16b). □

Lemma 45. *Every 3-cut in $\mathbb{G}$ yields at most 3 bridges.*

Proof. Suppose that there are at least four bridges. Let M_0 be the nontrivial one. Let us contract an edge in one of trivial bridges, without loss of generality, an edge containing the vertex z. In the graph G' obtained from $\mathbb{G}$ by this contraction there exists an apex vertex. The apex vertex has to be one of the vertices x, y and z: any two bridges with an edge xz or yz form $\tilde{K}_4$ in G' (see Figure 3.17a).

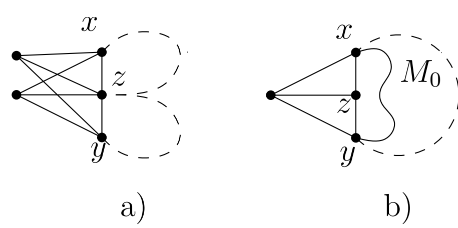


Figure 3.17: Impossible positions of an apex vertex in the graph G'.

But if an apex vertex is any of the vertices x, y, z, that vertex is an apex vertex in the original graph $\mathbb{G}$, too. Suppose that the vertex x is an apex vertex in G'. Then M_0 does not contain any $\tilde{K}_4^{\emptyset}$, $\tilde{K}_4^{y}$, $\tilde{K}_4^{z}$ and $\tilde{K}_4^{yz}$ but then, all $\tilde{K}_4$ in $\mathbb{G}$ contain x. Hence, x is an apex vertex. □

In the following part, we classify obstructions by the maximal degree of a node in their tree decompositions. For this purpose, we consider a tree decomposition $(T, \mathcal{V})$ with width 3 with the smallest maximal degree of a node, the smallest number of nodes of maximal degree and the smallest number of all nodes, where $V_t \neq V_{t'}$ for every two nodes $t, t' \in V(T)$.

Observation 46. *In every tree decomposition of a 3-connected graph of treewidth 3, every two adjacent nodes contain three common vertices.*

Proof. If two adjacent nodes in a tree decomposition share 2 or less vertices, then the vertices form a 2(or less)-cut in the graph. □

Lemma 47. *Let $\mathbb{G}$ be an obstruction with maximal degree in its tree decomposition T at least 3. Let t be a node with maximal degree in a tree decomposition of $\mathbb{G}$. Then every two nodes adjacent to t share at most 2 vertices.*

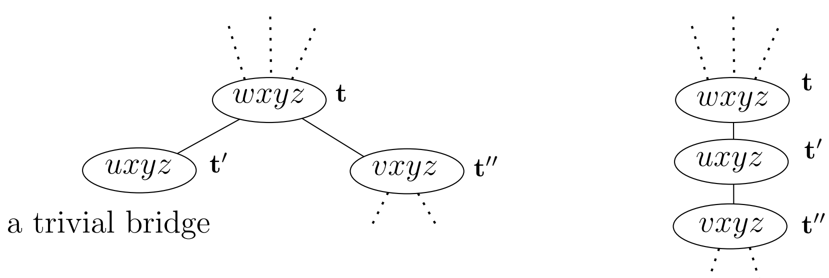


Figure 3.18: Two neighbors of a node cannot share three vertices.

Proof. By the definition of the tree-width, every vertex that is contained in at least two neighbors of t is also contained in t. Since every node contains at most 4 vertices and we suppose that every two different nodes contain different sets of vertices, any two nodes cannot share more than 3 vertices. Suppose that two neighbors t' and t'' of t share 3 vertices x, y, z. Then the vertices x, y, z form a 3-cut in $\mathbb{G}$ that produces 3 bridges (by Lemma 45 it cannot produce more bridges). The vertices of every bridge are contained in a different component of $T \setminus \{tt'', tt''\}$.

By Lemma 44, only one of these three bridges is nontrivial, therefore at least one of the nodes t' and t'', without loss of generality suppose that t', contains only vertices of the trivial bridge and has degree one in T. Then, the tree decomposition T' obtained from T by replacing the edge tt'' by $t't''$ (as shown in Figure 3.18) is a tree decomposition of $\mathbb{G}$, that has smaller number of vertices of maximal degree. That contradicts our choice of T. □

Observation 48. *The maximal degree of a node in a tree decomposition of an obstruction* $\mathbb{G}$ *is at most* 4.

Proof. Suppose that there exists a node t of degree greater than 4. Since any two adjacent nodes share three vertices and node t contains at most four vertices, there exist at least two neighbors of t that share the same triple of vertices. By Lemma 47, this is impossible. $\square$

Lemma 49. *If a tree decomposition of an obstruction* $\mathbb{G}$ *contains a node of degree* 4*,* $\mathbb{G}$ *is isomorphic to the graph in Figure 3.19b.*

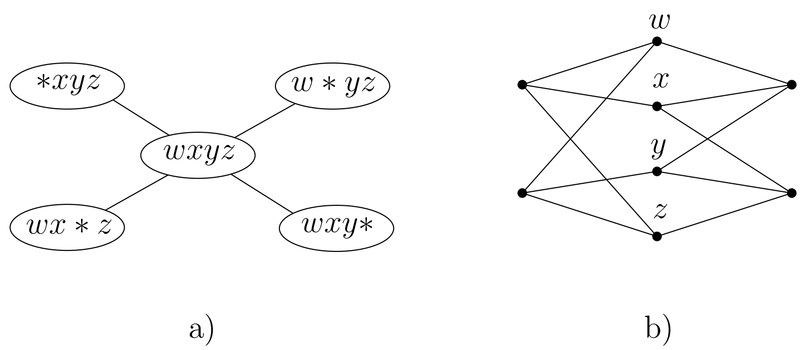


Figure 3.19: The obstruction with a node of degree 4 in its tree decomposition.

Proof. Let t be a node of degree 4 and $V_t = \{wxyz\}$. By Lemma 47, every neighbor of the node t contains a different triple of vertices w, x, y, z. Thus triples $\{w, x, y\}$, $\{w, x, z\}$, $\{w, y, z\}$ and $\{x, y, z\}$ are 3-cuts in $\mathbb{G}$. Then $\mathbb{G}$ contains the graph in Figure 3.19b as a minor. This graph is an obstruction for $\mathcal{T}_2^{apex}$. $\square$

We are now going to study obstructions such that the maximal degree of a node in their tree decompositions is 3. In the following lemmas, let t be a node of degree 3 of a tree decomposition of an obstruction $\mathbb{G}$, $V_t = \{w, x, y, z\}$ and the neighborhood of t is as shown in Figure 3.20. Observe that the vertices x, y and z are in symmetric. Therefore the next lemmas hold for every permutation of x, y and z.

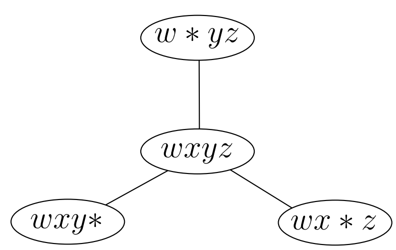
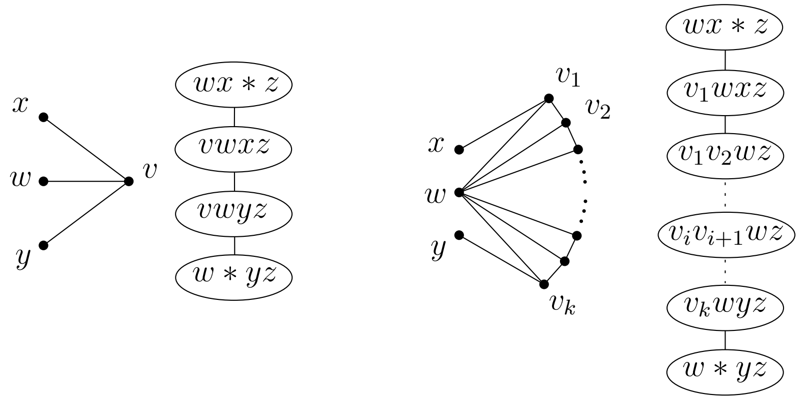


Figure 3.20: A neighborhood of a node of degree 3.

Lemma 50. *If xy is not an edge in $\mathbb{G}$ and the vertex cut $\{w, x, y\}$ yields only two bridges, then the bridge B given by the 3-cut $\{w, x, y\}$ such that B does not contain z is not trivial or equal to the bridge depicted in Figure 3.21b.*

Proof. Since there is no edge between vertices x and y, they do not have to occur in the same node. Therefore if B is trivial or equal to the graph depicted in Figure 3.21b, it is possible to rearrange vertices contained in the bridge B into several nodes of degree two as shown in Figures 3.21a and 3.21b. The tree decomposition T' of $\mathbb{G}$ obtained from T by rearranging has smaller number of vertices of degree three than T. That contradicts out choice of T. □



a) b)

Figure 3.21: Vertices of the 3-cut that yields these bridges do not have to be in the same node of a tree decomposition.

Lemma 51. *Let B be a bridge in $\mathbb{G}$ produced by the 3-cut $\{w, x, y\}$ such that B does not contain z. If the vertex cut $\{w, x, y\}$ produces only two bridges, B contains the graph*  *depicted in Figure 3.22 as a minor.*

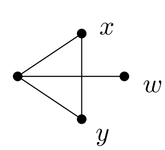
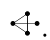
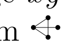


Figure 3.22: A minor of a bridge in $\mathbb{G}$ with a node of degree 3 in a tree decomposition.

Proof. Suppose that B does not contain . Then B does not contain an edge xy, because B has a trivial bridge as a minor and this minor with the edge xy form . Because B does not contain the edge xy, B must be nontrivial by the previous lemma. Thus, B contains at least one of the graphs in Figures 3.23a, c and d as a minor. The graph  is a minor of the graphs in Figures 3.23c and 3.23d (obtained by contracting the dashed edge), thus B does not contain any of the graphs in Figures 3.23c and d as a minor. Then B contains the graph in Figure 3.23a as a minor. By Lemma 50, B is not equal. Moreover, B does not contain the graph in Figure 3.23b as a minor (the graph in Figure 3.23b contains  as a minor obtained by contracting the dashed edges).

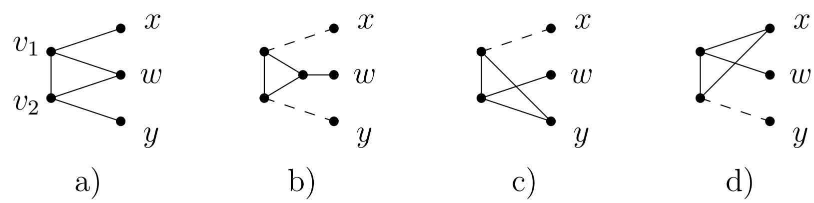
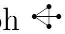
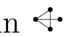


Figure 3.23: Some of possible minors of the bridge B that contains the graph  as a minor.

Thus, B contains a subgraph M that is a subdivision of the graph in Figure 3.23a. Denote P_{v_1x} the path in M between x and v_1, $P_{v_1v_2}$ the path in M between v_1 and v_2 that does not contain w and similarly P_{v_1w}, P_{v_2w} and P_{v_2y}.

Suppose that there exists u that is an inner vertex of the path P_{v_1x}. From the 3-connectivity of $\mathbb{G}$ and Corollary 2, there exist three interally vertex-disjoint paths between u and vertices w, x and y. Since u has degree 2 in M, there exists a path P between u and some vertex $u' \in M \setminus P_{v_1x}$ avoiding M, i.e., all inner vertices of P are in $B \setminus M$ (P can be also a path with no inner vertices—an edge). The vertex u' is not equal to y or v_2 and is not an inner vertex of paths $P_{v_2y}, P_{v_1v_2}$ and P_{v_2w}, otherwise B contains one of the graph in Figures 3.23b and c as a minor. The vertex u' cannot be an inner vertex of P_{v_1w}, because otherwise B would contain  as a minor (see Figure 3.24a). Thus, the only possibility is, that $u' = w$.

Similarly, if there exists u that is an inner vertex of the path P_{v_2y}, there exists a path from u to w avoiding M and there is no such path to any other vertex in $M \setminus P_{v_2y}$.

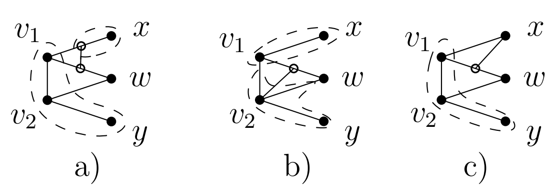


Figure 3.24: If B has an internal vertex in the path P_{v_1w}, it contains  as a minor.

Suppose that there exists u that is an inner vertex of P_{v_1w}. Then there exists a path from u to $u' \in M \setminus P_{v_1w}$ avoiding M. By the previous reasoning, we know that u' cannot be an inner vertex of P_{v_1x} or P_{v_2y}, because otherwise B contains one of the graphs in Figures 3.23b and 3.23c as a minor. By the same argument $u' \neq y$. If u' is v_2, an inner vertex of $P_{v_1v_2}, P_{v_2w}$ or y, B contains  as a minor (see Figure 3.24). Therefore such a vertex u exist. Similarly, there is no inner vertex of P_{v_2w}.

Suppose that there exists u that is an inner vertex of $P_{v_1v_2}$. By the previous reasoning, we know that there exists a path from u to $u' \in M \setminus P_{v_1v_2}$ avoiding M and u' is not an inner vertex of any of the paths P_{v_1w}, P_{v_2w}, P_{v_1x} and P_{v_2y}. If u' equals x of y, B contains one of the graphs in Figures 3.23b and 3.23c as a minor, thus the only possibility is $u' = w$.

We conclude that there are no inner vertices in P_{v_1w} and P_{v_2w} and the only possible paths from inner vertices of $P_{v_1v_2}$, P_{v_2y} and P_{v_1x} avoiding M, are paths to the vertex w and paths with both ends in the same path. Consequently, there must exist at least one vertex u in $B \setminus M$, because otherwise B is equal to the graph in Figure 3.21b (which is impossible by the previous lemma). By Corollary 2 there exist internally vertex-disjoint paths from the vertex u to w, x and y in B, thus there exist internally vertex-disjoint paths P_1, P_2, P_3 from u to distinct vertices $u_1, u_2, u_3 \in M$ respectively, such that all inner vertices of the paths P_1, P_2, P_3 are in $B \setminus M$.

The set $\{u_1, u_2, u_3\}$ cannot be equal to $\{w, x, y\}$, otherwise u is not a part of the bridge B and the the 3-cut $\{w, x, y\}$ produces 3 bridges. Note that paths P_1, P_2 and P_3 create paths between u_1 and u_2, u_1 and u_3 and u_2 and u_3, that avoids M. Therefore any two of the vertices u_1, u_2 and u_3 cannot be equal to v_1 and y or to v_2 and x (otherwise B contains the graph in Figure 3.23b or the graph in Figure 3.23c as a minor). Thus, if none of u_1, u_2 and u_3 is an inner vertex of the path $P_{v_1v_2}$, P_{v_2y} or P_{v_1x}, they are equal to x, v_1 and w, to v_2, y and w or to v_1, v_2 and w. Then B contains  as a minor, as demonstrated in Figures 3.25a and b.

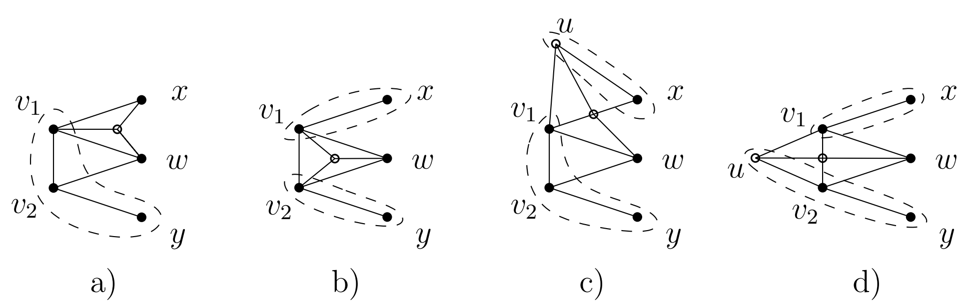
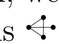
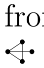


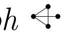
Figure 3.25: If B contains a vertex u, it has  as a minor.

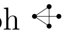
Without loss of generality, we assume that u_1 is an inner vertex of one of the paths $P_{v_1v_2}$, P_{v_2y} and P_{v_1x} and the vertices u_2 and u_3 are either contained in the same path or equal to w. Then, if $u_3 = w$, the vertices u_1 and u_2 are, in the same path. By contracting them into distinct ends of the path, we obtain one of the cases already analyzed (see Figures 3.25a,b). Thus, B contains  as a minor.

The only remaining possibility is, that all the three vertices u_1, u_2 and u_3 are contained in the same path P which is one of the paths $P_{v_1v_2}$, P_{v_2y} and P_{v_1x}. Without loss of generality, suppose that vertex u_2 is between vertices u_1 and u_3 in the path P. Then there must exist an inner vertex w' in the path between u_1 and u_3, and a path from w' to w avoiding M, otherwise ends of P form a 2-cut. But then B contains  as a minor: We can contract vertices w' and u_2 into single vertex and vertices u_1

and u_2 to the ends of P and then, by contractions shown in Figures 3.25c and d, we obtain .

So we proved that bridge B must contain the graph  as a minor. □

Observation 52. *If the vertex cut $\{w, x, y\}$ produces three bridges, then two of them that do not contain the vertex z form the a subgraph that contains the graph*  *in Figure 3.22 as a minor.*

Proof. One bridge contains the trivial bridge as a minor and the other contains the edge xy as a minor. These substructures form the graph . □

Lemma 53. *Let B be a bridge in $\mathbb{G}$ produced by the 3-cut $\{w, x, y\}$ such that B does not contain z. If B contains $\tilde{K}_4$ avoiding the vertex w, i.e., B contains $\tilde{K}_4^{\emptyset}$, $\tilde{K}_4^{x}$, $\tilde{K}_4^{y}$ or $\tilde{K}_4^{xy}$, B contains the graph H in Figure 3.26 as a minor.*

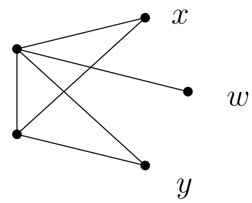


Figure 3.26: A minor of a bridge with $\tilde{K}_4$ that not containing w.

Proof. Suppose first that B contains $\tilde{K}_4^{xy}$. Consider $\tilde{K}_4^{xy}$ with vertices of degree three denoted v_1, v_2, v_3 and v_4 and the paths between x and v_1 and between y and v_3 denoted P_x and P_y respectively, such that sum of lengths of paths P_x and P_y is minimal. Let the subgraph of $\tilde{K}_4^{xy}$ consisting of the vertices v_1, v_2, v_3 and v_4 and paths between them that do not contain vertices x and y be denoted C' (see Figure 3.27).

There is no path between a vertex in $C' \setminus v_1$ and a vertex v_1' on $P_x \setminus v_1$ avoiding $\tilde{K}_4^{xy}$, otherwise there exists $\tilde{K}_4^{xy}$ with v_1' as a vertex of degree 3 instead of the vertex v_1 (see Figure 3.28). Such $\tilde{K}_4^{xy}$ cannot exist by the choice of $\tilde{K}_4^{xy}$ (the subpath of P_x

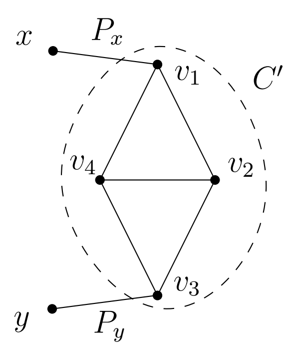


Figure 3.27: Notation used in the proof of Lemma 53 for $\tilde{K}_4^{xy}$.

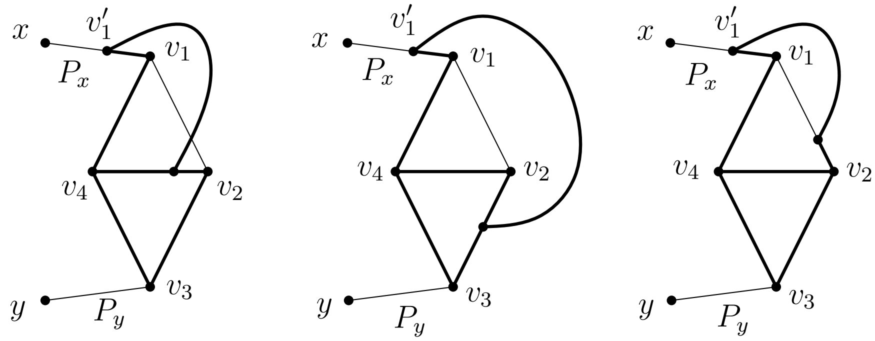


Figure 3.28: A path between the vertex v_1' and a vertex in C' yields $\tilde{K}_4^{xy}$ with shorter path from x.

between v_1' and x is shorter than P_x). A symmetric argument yields that there is no path between a vertex in $C' \setminus v_3$ and a vertex v_3' on $P_y \setminus v_3$ avoiding $\tilde{K}_4^{xy}$. Thus, there exists a path from w to a vertex $u' \in C' \setminus \{v_1, v_3\}$ avoiding $\tilde{K}_4^{xy}$, otherwise vertices v_1 and v_3 form a 2-cut. But then, by contracting the whole paths P_x and P_y into vertices x and y respectively and contracting u into one of the vertices v_2 and v_4, we obtain a subdivision of the graph depicted in Figure 3.26.

Suppose that the bridge B contains $\tilde{K}_4^{\emptyset}$. Let the vertices of degree 3 of $\tilde{K}_4^{\emptyset}$ be denoted v_1, v_2, v_3 and v_4 as shown in Figure 3.29, let $P_{v_iv_j}$ denote a path between v_i and v_j in $\tilde{K}_4^{\emptyset}$ corresponding to the edge between v_i and v_j in $\tilde{K}_4^{\emptyset}$. Since $\mathbb{G}$ is 3-connected, from Corollary 2, it follows that there exist paths P_w, P_x and P_y from vertices w, x and y respectively to three distinct vertices in $\tilde{K}_4^{\emptyset}$, such that P_x, P_y and P_w avoid $\tilde{K}_4^{\emptyset}$. If there exist P_w, P_x and P_y such that the vertices their ends are not in the same path $P_{v_iv_j}$, it is possible to contract the ends of P_w, P_x and P_y in $\tilde{K}_4^{\emptyset}$ into three distinct vertices among the vertices v_1, v_2, v_3 and v_4. Then, by contracting paths P_x and P_y we obtain a subdivision of H depicted in Figure 3.26.

Now suppose that B does not contain any $\tilde{K}_4^{\emptyset}$ such that there exist paths P_w, P_x and P_y, that do not have their ends in the same path $P_{v_iv_j}$ in $\tilde{K}_4^{\emptyset}$.

Then, all paths from vertices x, y and w to a vertex in $\tilde{K}_4^{\emptyset}$ avoiding $\tilde{K}_4^{\emptyset}$ have their ends in the same path $P_{v_iv_j}$. Without loss of generality suppose that in $P_{v_1v_2}$.

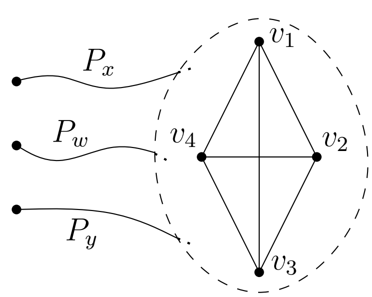


Figure 3.29: Notation used in the proof of Lemma 53 for $\tilde{K}_4^{\emptyset}$.

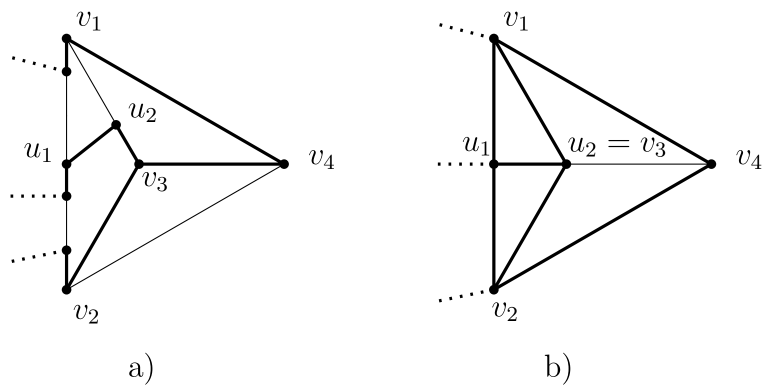


Figure 3.30: Three internally vertex disjoint paths to v_3.

Since there must exist internally vertex disjoint paths from w, x and y to the vertex v_3, by Corollary 2, there exists a path Q from a vertex u_1 in $P_{v_1v_2}$ to a vertex u_2 in $\tilde{K}_4^{\emptyset} \setminus P_{v_1v_2}$ avoiding $\tilde{K}_4^{\emptyset}$. Moreover, there exists at least one path P_w, P_x, or P_y with an end $e_1 \neq u_1$ in the subpath of $P_{v_1v_2}$ between v_1 and u_1. Symmetrically, there exists at least one path P_w,P_x, or P_y with an end $e_2 \neq u_1$ in the subpath of $P_{v_1v_2}$ between v_2 and u_1 (see Figure 3.30a). Then the ends of the paths P_w,P_x, and P_y can be contracted into distinct vertices among v_1, v_2 and u_1. Note that vertex u_2 can always be contracted into v_3 or v_4. Without loss of generality suppose, that $u_2 = v_3$. Then there exists $\tilde{K}_4^{\emptyset}$ such that the ends of the paths P_w, P_x and P_y are not in the same path $P_{v_iv_j}$, as shown in Figure 3.30b.

Observe that if you consider only one path P_x consisting of a single vertex x or P_y consisting of y in the proof for $\tilde{K}_4^{\emptyset}$, i.e., the K_4-minor is $\tilde{K}_4^{x}$ or $\tilde{K}_4^{y}$, the same conclusion can be obtained. □

Lemma 54. *If a tree decomposition of an obstruction* $\mathbb{G}$ *contains a node of degree* 3, $\mathbb{G}$ *is isomorphic to the graph* G *shown in Figure 3.31.*

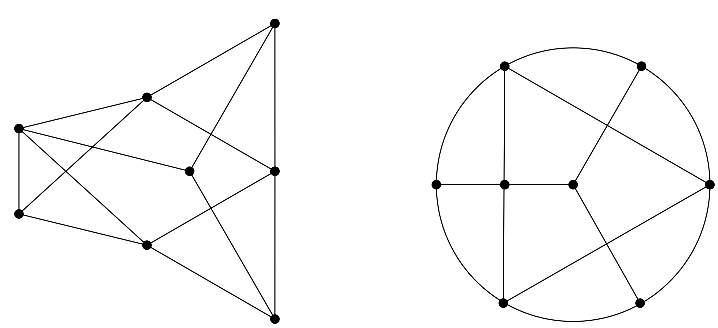
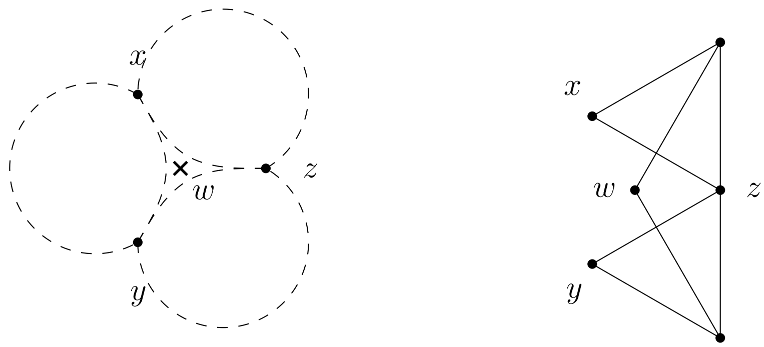


Figure 3.31: The only obstruction with a node of degree 3 in its tree decomposition and its (more symmetric) redrawing.



a) 2-cuts in the graph $\mathbb{G} \setminus w$. b) A minor of the bridge produced by $\{w, x, y\}$.

Figure 3.32:

Proof. The graph G is an obstruction for $\mathcal{T}_2^{apex}$. It is enough to prove that every obstruction $\mathbb{G}$ for $\mathcal{T}_2^{apex}$ with a node of degree 3 in its tree decomposition contains G as a minor, because an obstruction cannot contain another obstruction as a proper minor.

Since $\mathbb{G}$ is an obstruction, the graph G' obtained from $\mathbb{G}$ by deleting the vertex w contains $\tilde{K}_4$. Vertices x and y, x and z and y and z form 2-cuts in G', every $\tilde{K}_4$ in G' has all vertices of degree 3 in the same bridge produced by one of these 2-cuts that does not contain all three vertices x, y and z (see Figure 3.32a). Without loss of generality, suppose that $\tilde{K}_4$ is contained in the bridge not containing z determined by 2-cut $\{x, y\}$. Then the bridge B not containing z determined by the 3-cut $\{w, x, y\}$, contains $\tilde{K}_4$ that does not contain the vertex w. By Lemma 53, B contains the graph given in Figure 3.26 as a minor. By Lemma 51 and Observation 52 applied to the bridges given by the cuts $\{w, x, z\}$ and $\{w, y, z\}$, the bridge not containing z given the by 3-cut $\{w, x, y\}$ has the graph in Figure 3.32b as a minor. Thus, $\mathbb{G}$ contains G as a minor. □

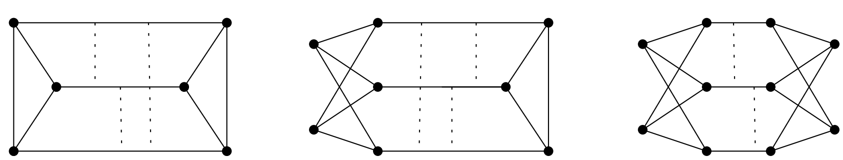


Figure 3.33: Structure of 3-connected obstructions with path-width three.

At this point, we have classified all 3-connected obstructions with nodes of degree greater than two in a tree decomposition. Thus, all remaining obstructions have path-width three. Moreover, it can be shown that such obstructions are comprised of three vertex-disjoint paths with some chords between them such that the paths interconnect triangles or two vertices of degree 3 (see Figure 3.33). All such obstructions can be generated by a computer.

We have found the three 3-connected obstructions with path-width three depicted in Figure 3.34, but we have neither an independent program to verify the correctness nor a computer-free proof.

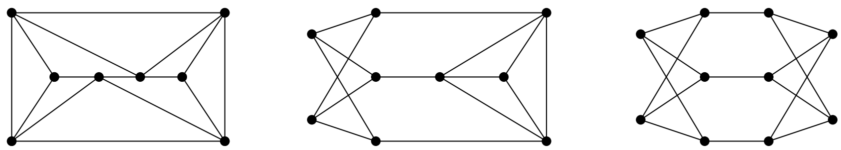


Figure 3.34: Found 3-connected obstructions with path-width three.

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