Charles University in Prague
Faculty of Mathematics and Physics

# BACHELOR THESIS

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## Finite element method for flow in time-dependent domains

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I confirm having prepared the bachelor thesis by my own, and having listed all used sources of information in the bibliography. I agree with lending the bachelor thesis.

In Prague                                                                 Lukáš Korous

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Název práce: Metoda konečných prvků pro proudění v časově závislých oblastech
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Abstrakt: Práce se zabývá numerickou simulací proudění v časově závislých oblastech. Jsou prezentovány základní rovnice popisující proudění a vyložena ALE (arbitrary Lagrangian-Eulerian) metoda, na tyto rovnice aplikovaná. Dále se práce věnuje specifikaci úlohy nestlačitelného proudění na oblasti s pohyblivou hranicí a jejímu numerickému řešení metodou konečných prvků. Ta je v práci popsána i s patřičnou stabilizací metodou SUPG a linearizací za pomoci Oseenova iteračního procesu. Numerický experiment je pak proveden na příkladu pohybujícího se upraveného leteckého profilu NACA0012 za pomoci softwaru vyvinutého na MFF UK a Fakultě Strojní ČVUT.

Klíčová slova: numerická simulace, metoda konečných prvků, Navier-Stokesovy rovnice, ALE metoda, Oseenův model

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Abstract: Work is concerned with numerical simulation of flow in time-dependent domains. The overview of principal equations of fluid dynamics is given and the ALE (arbitrary Lagrangian-Eulerian) method is presented and applied to the equations. Further, the work attends to the incompressible model considered in time-dependent domain and to its numerical solution by the finite element method. Description of this method is presented, together with the appropriate stabilization by the SUPG method and linearization with the aid of Oseen's iterative process. The numerical experiment is carried out on example of moving modified NACA0012 airfoil with the use of software developed at the Faculty of Mathematics and Physics of Charles University and the Faculty of Mechanical Engineering of Czech Technical University.

Keywords: numerical simulation, finite element method, Navier-Stokes equations, ALE method, Oseen's model

# Introduction

Computational fluid dynamics (CFD) as a branch of fluid dynamics, plays an important role in many technical disciplines - airplane industry, bladed devices (turbines, compressors, pumps), civil engineering (stability of bridges, landing stages, oil platforms), medicine (blood flow in vessels, air flow in vocal chords) etc. Analytical methods are capable of computing a solution only for general and not too complicated problems. In real applications however, there is a need for more complex models that describe real situations more closely. There exist commercial pieces of software, designed to solve such problems numerically (ANSYS, NASTRAN, FLUENT, etc.). For the numerical simulation carried out and described in this work, we use FEMFLUID [9] software developed by doc. RNDr. Petr Sváček, Ph.D from the Faculty of Mechanical Engineering of Czech Technical University.

Mathematical model of flow, represented by the system of partial differential equations, is described in Chapter 1, which closely follows [1], [2]. In order to handle time-dependency of a computational domain, the arbitrary Lagrangian-Eulerian method is described and applied to the system of equations describing the fluid flow in Chapter 2. For the viscous incompressible model, derived in Chapter 3, this system consists of the Navier-Stokes equations and the continuity equation. The discretization of the ALE formulation of this system is thoroughly described in Chapter 4.

From the range of numerical techniques that can be applied to solve this problem, we opt for the finite element method (FEM) for the space discretization and the finite difference method (FDM) for the time discretization, which turns out to be suitable approach for a problem with complicated geometry and mixed boundary conditions. Among other techniques that can be successfully applied we can mention e.g. the spectral method or the finite volume method. Since our model example is the flow around a moving airfoil, where the Reynolds number (described in Section 3.3) can be rather high, we need to employ suitable stabilization into our discrete model.

Application of the stabilized FEM to the system of equations describing the viscous incompressible flow in the ALE formulation leads to a system of nonlinear algebraic equations. However, the process of linearization known as Oseen's iterative process is applied in order to obtain a sequence of systems of linear algebraic equations. These equations are then solved by a sufficiently fast linear solver.

In our model example we limit ourselves to a lower Reynolds number in order to make the computation possible on an average personal computer. Graphic outputs

from the computation are presented at the end of the work in Chapter 5 and on the attached DVD.

# Chapter 1

# Equations describing the flow

We consider a time interval $(0, T)$ and space domain $\Omega_t \subset \mathbb{R}^3$ occupied by a fluid at time $t$. By $\mathcal{M}$ the space-time domain in consideration is denoted:

$$\mathcal{M} = \{(\boldsymbol{x}, t) \, ; \boldsymbol{x} \in \Omega_t, t \in (0, T)\} \, . \tag{1.1}$$

Moreover we assume that $\mathcal{M}$ is an open set.

## 1.1 Description of the flow

In Computational Fluid Dynamics there exist two classical approaches to the description of the flow, the *Lagrangian description* and the *Eulerian description*. Third, more modern approach, the so-called *arbitrary Lagrangian-Eulerian* description, combines the two above, making use of their respective advantages, while diminishing their drawbacks.

The idea of the Lagrangian description is to monitor each fluid particle along its pathline (i.e. the curve which the particle traverses in time). If we wanted to set a computational mesh using this description, it would mean to firmly connect nodes of the mesh with certain particles (i.e. the node and the particle would have to share their space coordinates) and move the mesh accordingly to the motion of the fluid as to preserve the node - corresponding particle connection at each time instant. The obvious drawback is the necessity of perform re-meshing operations very frequently, especially when dealing with large distortion of the fluid.

The Eulerian description focuses on fluid particles that move through fixed points within a computational domain. In other words, whereas in the Lagrangian description the particle was fixed and the point in space it was currently occupying was changing, now it is the point in space that holds still and the particle in consideration is changing and is always corresponding to the one that is currently occupying

the considered point in space. From this idea it follows that a computational mesh for this description would be fixed with respect to time.

It is not difficult to imagine that formulation of some basic mechanical principles could be easier for the moving particle, that is using the Lagrangian approach. However, the Eulerian description is used for the formulation of conservation laws as will be seen in the following sections. Lastly, using this approach, large distortions of fluid domain can be handled with relative ease.

The suitable description for our purposes is the arbitrary Lagrangian-Eulerian one, but we shall proceed now with basics of the Lagrangian and the Eulerian descriptions and their relation. We shall present the equations describing the flow derived from conservation laws in their integral forms using the Eulerian approach, which is then easy to reformulate using the arbitrary Lagrangian-Eulerian description in the next chapter.

**Lagrangian description**

We specify the particle in consideration using the mapping

$$\boldsymbol{\varphi}\left(\boldsymbol{X}, t_0; t\right) \tag{1.2}$$

which determines the current (at time $t$) position $\boldsymbol{x} \in \Omega_t$ of the particle that occupies the point $\boldsymbol{X}$ at time $t_0$, i.e.

$$\boldsymbol{x} = \boldsymbol{\varphi}\left(\boldsymbol{X}, t_0; t\right), \;\; \boldsymbol{X} \in \Omega_{t_0}, \tag{1.3}$$

where we can omit the reference time $t_0$ and write

$$\boldsymbol{x} = \boldsymbol{\varphi}\left(\boldsymbol{X}, t\right). \tag{1.4}$$

Customarily the components $X_1, X_2, X_3$ of the reference point $\boldsymbol{X}$ are called the *Lagrangian coordinates* and the components $x_1, x_2, x_3$ of the point $\boldsymbol{x}$ in the current configuration $\Omega_t$ are called the *Eulerian coordinates*. The *velocity* and *acceleration* of the particle given by the reference point $\boldsymbol{X}$ are defined as

$$\hat{v}\left(\boldsymbol{X}, t\right) = \frac{\partial \boldsymbol{\varphi}}{\partial t}\left(\boldsymbol{X}, t_0; t\right), \tag{1.5}$$

$$\hat{a}\left(\boldsymbol{X}, t\right) = \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2}\left(\boldsymbol{X}, t_0; t\right), \tag{1.6}$$

provided the above derivatives exist.

**Eulerian description**

Using once again the mapping $\boldsymbol{\varphi}$ defined in (1.2), relation (1.4) and Lagrangian definition of velocity (1.5), we can express the *velocity* of the fluid particle passing through the point $\boldsymbol{x}$ at time $t$:

$$\boldsymbol{v}\left(\boldsymbol{x}, t\right) = \hat{\boldsymbol{v}}\left(\boldsymbol{X}, t\right) = \frac{\partial \boldsymbol{\varphi}}{\partial t}\left(\boldsymbol{X}, t\right), \tag{1.7}$$

where $\boldsymbol{x} = \boldsymbol{\varphi}\left(\boldsymbol{X}, t\right)$.

We shall demand the following regularity from the velocity function:

$$\boldsymbol{v} \in \left[\mathcal{C}^1\left(\mathcal{M}\right)\right]^3. \tag{1.8}$$

We can pass to the Eulerian coordinates from the Lagrangian ones by solving the following initial value problem:

$$\frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{v}\left(\boldsymbol{x}, t\right), \quad \boldsymbol{x}\left(t_0\right) = \boldsymbol{X}. \tag{1.9}$$

Under assumption (1.8), the problem (1.9) has exactly one maximal solution $\boldsymbol{\varphi}\left(\boldsymbol{X}, t_0, t\right)$ for each $\left(\boldsymbol{X}, t_0\right) \in \mathcal{M}$ defined for $t$ from a certain subinterval of $(0, T)$. Moreover, in its domain of definition, the mapping $\boldsymbol{\varphi}$ has continuous first order derivatives with respect to $X_1, X_2, X_3, t_0, t$ and continuous second order derivatives $\partial^2 \boldsymbol{\varphi}/\partial t \partial X_i$, $\partial^2 \boldsymbol{\varphi}/\partial t_0 \partial X_i$, $i = 1, 2, 3$. These statements result from theory of classical solutions of ordinary differential equations.

Under assumption (1.8), the *acceleration* of the particle passing through the point $\boldsymbol{x}$ at time $t$ can be expressed as

$$\boldsymbol{a}\left(\boldsymbol{x}, t\right) = \frac{\partial \boldsymbol{v}}{\partial t}\left(\boldsymbol{x}, t\right) + \sum_{i=1}^{3} v_i\left(\boldsymbol{x}, t\right) \frac{\partial \boldsymbol{v}}{\partial x_i}\left(\boldsymbol{x}, t\right). \tag{1.10}$$

This, written in a short form reads

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \left(\boldsymbol{v} \cdot \mathsf{grad}\right) \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} + \left(\boldsymbol{v} \cdot \nabla\right) \boldsymbol{v}, \tag{1.11}$$

where the differentiation represented by the symbol $\nabla$ is with respect to the spatial variables $x_1, x_2, x_3$.

## 1.2 The Transport theorem

We wish to study some physical quantity which is transported by fluid particles in our space-time domain $\mathcal{M}$. Let a function $F = F\left(\boldsymbol{x}, t\right) : \mathcal{M} \longrightarrow \mathbb{R}$ represent some physical quantity in the Eulerian coordinates, and let us consider a system

of fluid particles filling a bounded domain $\mathcal{V}(t) \subset \Omega_t$ at time $t$. By $\mathcal{F}$ we denote the total amount of the quantity represented by the function $F$ contained in $\mathcal{V}(t)$:

$$\mathcal{F}(t) = \int_{\mathcal{V}(t)} F(\boldsymbol{x}, t) \, d\boldsymbol{x}. \tag{1.12}$$

For the formulation of fundamental equations describing the flow we need to calculate the rate of change of the quantity $\mathcal{F}$ bound on the system of particles considered. In other words we shall be interested in the derivative

$$\frac{d\mathcal{F}(t)}{dt} = \frac{d}{dt} \int_{\mathcal{V}(t)} F(\boldsymbol{x}, t) \, d\boldsymbol{x}. \tag{1.13}$$

In the following we shall suppose that (1.8) holds.

**Lemma 1.1**
*Let $t_0 \in (0, T)$, $\mathcal{V}(t_0)$ be a bounded domain and let $\overline{\mathcal{V}(t_0)} \subset \Omega_{t_0}$. Then there exist an interval $(t_1, t_2) \subset (0, T)$, $t_0 \in (t_1, t_2)$ such that the following conditions are satisfied:*

a) *The mapping 't $\in (t_1, t_2)$, $\boldsymbol{X} \in \mathcal{V}(t_0) \longrightarrow \boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t_0; t) \in \mathcal{V}(t)$' has continuous first order derivatives with respect to $t, X_1, X_2, X_3$ and continuous second order derivatives $\partial^2 \boldsymbol{\varphi} / \partial t \partial X_i$, $i = 1, 2, 3$.*

b) *The mapping '$\boldsymbol{X} \in \mathcal{V}(t_0) \longrightarrow \boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t_0; t) \in \mathcal{V}(t)$' is for all $t \in (t_1, t_2)$ a continuously differentiable one-to-one mapping of $\mathcal{V}(t_0)$ onto $\mathcal{V}(t)$ with continuous and bounded Jacobian $\mathcal{J}(\boldsymbol{X}, t)$ which satisfies the condition*

$$\mathcal{J}(\boldsymbol{X}, t) > 0 \quad \forall \boldsymbol{X} \in \mathcal{V}(t_0), \ \forall t \in (t_1, t_2).$$

c) *The inclusion*

$$\left\{ (\boldsymbol{x}, t) \, ; \ t \in [t_1, t_2], \boldsymbol{x} \in \overline{\mathcal{V}(t)} \right\} \subset \mathcal{M}$$

*holds and therefore the mapping $\boldsymbol{v}$ has continuous and bounded first order derivatives on $\{ (\boldsymbol{x}, t) \, ; \ t \in (t_1, t_2), \boldsymbol{x} \in \mathcal{V}(t) \}$ with respect to all variables.*

d) *$\boldsymbol{v}(\boldsymbol{\varphi}(\boldsymbol{X}, t_0; t), t) = \frac{\partial \boldsymbol{\varphi}}{\partial t}(\boldsymbol{X}, t_0; t) \ \forall \boldsymbol{X} \in \mathcal{V}(t_0), \ \forall \, t \in (t_1, t_2)$.*

For proof, see [1].

**Theorem 1.2 - The transport theorem**
*Let conditions from Lemma 1.1, a)-d) be satisfied and let the function $F = F(\boldsymbol{x}, t)$ have continuous and bounded first order derivatives on the set*

$\{(\boldsymbol{x}, t) ; t \in (t_1, t_2) , \; x \in \mathcal{V}(t)\}$. Let $\mathcal{F}(t) = \int_{\mathcal{V}(t)} F(\boldsymbol{x}, t) \, d\boldsymbol{x}$.
Then for each $t \in (t_1, t_2)$ there exists a finite derivative

$$
\begin{aligned}
\frac{d\mathcal{F}}{dt}(t) &= \frac{d}{dt} \int_{\mathcal{V}(t)} F(\boldsymbol{x}, t) \, d\boldsymbol{x} \\
&= \int_{\mathcal{V}(t)} \left[ \frac{\partial F}{\partial t}(\boldsymbol{x}, t) + \operatorname{div}(F\boldsymbol{v})(\boldsymbol{x}, t) \right] d\boldsymbol{x}.
\end{aligned} \tag{1.14}
$$

For proof, see [2].

## 1.3   The continuity equation

The *density of fluid is* a function

$$
\rho : \mathcal{M} \longrightarrow (0, +\infty)
$$

which allows us to determine the mass $m(\mathcal{V}; t)$ of the fluid contained in any sub-domain $\mathcal{V} \subset \Omega_t$ :

$$
m(\mathcal{V}; t) = \int_{\mathcal{V}} \rho(\boldsymbol{x}, t) \, d\boldsymbol{x}. \tag{1.15}
$$

**Assumptions 1.3**
In what follows, let $\rho \in \mathcal{C}^1(\mathcal{M})$ and as before let $\boldsymbol{v} \in [\mathcal{C}^1(\mathcal{M})]^3$. We shall consider an arbitrary time instant $t_0 \in (0, T)$ and a moving piece of fluid formed by the same particles at each time instant and filling at time $t_0$ a bounded domain $\mathcal{V} \subset \overline{\mathcal{V}} \subset \Omega_{t_0}$ with a Lipschitz-continuous boundary $\partial \mathcal{V}$ called the *control volume* in the domain $\Omega_{t_0}$. By $\mathcal{V}(t)$ we denote the domain occupied by this piece of fluid at time $t \in (t_1, t_2)$, where $(t_1, t_2)$ is a sufficiently small interval containing $t_0$ with properties from Lemma 1.1.

Since the domain $\mathcal{V}(t)$ is formed by the same particles at each time instant, the *conservation of mass* can be formulated in the following way: *The mass of the piece of fluid represented by the domain $\mathcal{V}(t)$ does not depend on time $t$.* This means that

$$
\frac{dm(\mathcal{V}(t); t)}{dt} = 0, \; t \in (t_1, t_2), \tag{1.16}
$$

where with respect to (1.15) we have

$$
m(\mathcal{V}(t); t) = \int_{\mathcal{V}(t)} \rho(\boldsymbol{x}, t) \, d\boldsymbol{x}. \tag{1.17}
$$

From Theorem 1.2 for a function $F := \rho$ we get the identity

$$\int_{\mathcal{V}(t)} \left[ \frac{\partial \rho}{\partial t} (\boldsymbol{x}, t) + \operatorname{div}(\rho \boldsymbol{v})(\boldsymbol{x}, t) \right] d\boldsymbol{x} = 0, \quad t \in (t_1, t_2). \tag{1.18}$$

If we substitute $t := t_0$ and take into account that $\mathcal{V}(t_0) = \mathcal{V}$, we conclude that

$$\int_{\mathcal{V}} \left[ \frac{\partial \rho}{\partial t} (\boldsymbol{x}, t_0) + \operatorname{div}(\rho \boldsymbol{v})(\boldsymbol{x}, t_0) \right] d\boldsymbol{x} = 0 \tag{1.19}$$

for an arbitrary $t_0 \in (0, T)$ and an arbitrary control volume $\mathcal{V} \subset \Omega_{t_0}$. We use the following Lemma in order to derive the differential form of the law of conservation of mass:

**Lemma 1.4**
*Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f \in \mathcal{C}(\Omega)$. Then the following holds:*

*$f \equiv 0$ in $\Omega$ if and only if $\int_{\mathcal{V}} f(\boldsymbol{x}) \, d\boldsymbol{x} = 0$ for any bounded open set $\mathcal{V} \subset \overline{\mathcal{V}} \subset \Omega$.*

Now we use Lemma 1.4 and obtain the differential form of the law of conservation of mass called the *continuity equation*:

$$\frac{\partial \rho}{\partial t} (\boldsymbol{x}, t) + \operatorname{div}(\rho(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t)) = 0, \quad \boldsymbol{x} \in \Omega_t, \ t \in (0, T). \tag{1.20}$$

## 1.4 The equations of motion

We proceed by deriving basic dynamical equations describing flow motion from the *law of conservation of momentum* which can be formulated in this way:

*The rate of change of the total momentum of a piece of fluid formed by the same particles at each time and occupying the domain $\mathcal{V}(t)$ at time instant $t$ is equal to the force acting on $\mathcal{V}(t)$.*

Let assumptions 1.3 be satisfied. The total momentum of particles contained in $\mathcal{V}(t)$ is given by

$$\boldsymbol{\mathcal{H}}(\mathcal{V}(t)) = \int_{\mathcal{V}(t)} \rho(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t) \, d\boldsymbol{x}. \tag{1.21}$$

Moreover, denoting by $\boldsymbol{\mathcal{F}}(\mathcal{V}(t))$ the force acting on the volume $\mathcal{V}$, the law of conservation of momentum reads

$$\frac{d\boldsymbol{\mathcal{H}}(\mathcal{V}(t))}{dt} = \boldsymbol{\mathcal{F}}(\mathcal{V}(t)), \quad t \in (t_1, t_2). \tag{1.22}$$

Using Theorem 1.2 for functions $F := \rho v_i$, $i = 1, 2, 3$, we get

$$\int_{\mathcal{V}(t)} \left[ \frac{\partial}{\partial t} \left( \rho \left( \boldsymbol{x}, t \right) v_i \left( \boldsymbol{x}, t \right) \right) + \mathrm{div} \left( \rho \left( \boldsymbol{x}, t \right) v_i \left( \boldsymbol{x}, t \right) \boldsymbol{v} \left( \boldsymbol{x}, t \right) \right) \right] \, d\boldsymbol{x} = \mathcal{F}_i \left( \mathcal{V} \left( t \right) \right), \quad (1.23)$$

$$i = 1, 2, 3, \ t \in (t_1, t_2).$$

Taking into account that $t_0 \in (0, T)$ is an arbitrary time instant and $\mathcal{V}_{t_0} = \mathcal{V} \subset \overline{\mathcal{V}} \subset \Omega_{t_0}$, where $\mathcal{V}$ is an arbitrary control volume, we get the law of conservation of momentum in the form where we write $t$ instead of $t_0$:

$$\int_{\mathcal{V}} \left[ \frac{\partial}{\partial t} \left( \rho \left( \boldsymbol{x}, t \right) v_i \left( \boldsymbol{x}, t \right) \right) + \mathrm{div} \left( \rho \left( \boldsymbol{x}, t \right) v_i \left( \boldsymbol{x}, t \right) \boldsymbol{v} \left( \boldsymbol{x}, t \right) \right) \right] \, d\boldsymbol{x} = \mathcal{F}_i \left( \mathcal{V}; t \right), \quad (1.24)$$

$i = 1,2,3$, for an arbitrary $t \in (0, T)$ and an arbitrary control volume $\mathcal{V} \subset \overline{\mathcal{V}} \subset \Omega_t$.

According to [1], the components $\mathcal{F}_i \left( \mathcal{V}; t \right)$, $i = 1, 2, 3$, of the vector $\boldsymbol{\mathcal{F}} \left( \mathcal{V}; t \right)$ can be expressed as

$$\mathcal{F}_i \left( \mathcal{V}; t \right) = \int_{\mathcal{V}} \rho \left( \boldsymbol{x}, t \right) f_i \left( \boldsymbol{x}, t \right) dx + \int_{\partial \mathcal{V}} \sum_{j=1}^{3} \tau_{ji} \left( \boldsymbol{x}, t \right) n_j \left( \boldsymbol{x} \right) dS, \ i = 1, 2, 3, \quad (1.25)$$

assuming that $\tau_{ij} \in \mathcal{C}^1 \left( \mathcal{M} \right)$ and $f_i \in \mathcal{C} \left( \mathcal{M} \right)$, $(i, \, j = 1, 2, 3)$. Here $\tau_{ji}$ are components of the *stress tensor* $\mathcal{T}$ and $f_i$ are components of the *density of the volume force* $\boldsymbol{f}$. Substituting this into (1.24), we get

$$\int_{\mathcal{V}} \left[ \frac{\partial}{\partial t} \left( \rho \left( \boldsymbol{x}, t \right) v_i \left( \boldsymbol{x}, t \right) \right) + \mathrm{div} \left( \rho \left( \boldsymbol{x}, t \right) v_i \left( \boldsymbol{x}, t \right) \boldsymbol{v} \left( \boldsymbol{x}, t \right) \right) \right] dx = \quad (1.26)$$

$$\int_{\mathcal{V}} \rho \left( \boldsymbol{x}, t \right) f_i \left( \boldsymbol{x}, t \right) dx + \int_{\partial \mathcal{V}} \sum_{j=1}^{3} \tau_{ji} \left( \boldsymbol{x}, t \right) n_j \left( \boldsymbol{x} \right) dS, \quad i = 1, 2, 3, \quad (1.27)$$

for each $t \in (0, T)$ and an arbitrary control volume $\mathcal{V}$ in $\Omega_t$. Moreover, applying Green's theorem and Lemma 1.4, we obtain the desired *equation of motion of a general fluid in the differential conservative form*

$$\frac{\partial}{\partial t} \left( \rho v_i \right) + \mathrm{div} \left( \rho v_i \boldsymbol{v} \right) = \rho f_i + \sum_{j=1}^{3} \frac{\partial \tau_{ji}}{\partial x_j}, \quad i = 1, 2, 3. \quad (1.28)$$

This can be written as

$$\frac{\partial}{\partial t} \left( \rho \boldsymbol{v} \right) + \mathrm{div} \left( \rho \boldsymbol{v} \otimes \boldsymbol{v} \right) = \rho \boldsymbol{f} + \mathrm{div} \, \mathcal{T}, \quad (1.29)$$

where $\otimes$ denotes the *tensor product*:

$$\boldsymbol{a} \otimes \boldsymbol{b} = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix},$$

and div $(\boldsymbol{a} \otimes \boldsymbol{b})$ is a vector quantity:

$$\mathrm{div}\,(\boldsymbol{a} \otimes \boldsymbol{b}) = \left( \sum_{i=1}^{3} \frac{\partial}{\partial x_i} a_i b_1, \ \ \sum_{i=1}^{3} \frac{\partial}{\partial x_i} a_i b_2, \ \ \sum_{i=1}^{3} \frac{\partial}{\partial x_i} a_i b_3 \right)^{T}.$$

## 1.5   The Navier-Stokes equations

The relation between the stress tensor and other quantities describing fluid flow, the velocity and its derivatives in particular, represent the so-called *rheological equations* of the fluid. For the derivation of the Navier-Stokes equations we shall use

$$\mathcal{T} = (-p + \lambda \mathrm{div}\,\boldsymbol{v})\,\mathbb{I} + 2\mu \mathbb{D}\,(\boldsymbol{v}), \tag{1.30}$$

where $\mathbb{D}$ is the deformation velocity tensor:

$$\mathbb{D} = \mathbb{D}\,(\boldsymbol{v}) = (d_{ij})_{i,j=1}^{3}, \ \ d_{ij} = \frac{1}{2}\left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \tag{1.31}$$

$\lambda, \mu$ are constants or scalar functions of thermodynamical quantities, $\lambda$ and $\mu$ are called the *first* and the *second viscosity coefficient* respectively. For the assumptions under which we can write (1.30) see [1]. Altough viscosity coefficients can be functions of thermodynamical quantities (most important of which is $\theta$, the absolute temperature) we shall treat them as if they were constants. Let assumptions 1.3 be satisfied and let us assume that

$$\frac{\partial \boldsymbol{v}}{\partial t} \in [\mathcal{C}\,(\mathcal{M})]^{3}, \ \ \frac{\partial^2 \boldsymbol{v}}{\partial x_i \partial x_j} \in [\mathcal{C}\,(\mathcal{M})]^{3} \ \ (i, j = 1, 2, 3). \tag{1.32}$$

Now let us substitute relation (1.30) into the general equations of motion (1.29) with the assumption of constant viscosity coefficients and assumptions (1.32). We come to the Navier-Stokes equations in the form

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \mathrm{div}\,(\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \rho \boldsymbol{f} - \nabla\, p + \mu\,\triangle \boldsymbol{v} + (\mu + \lambda)\,\nabla \mathrm{div}\,\boldsymbol{v}. \tag{1.33}$$

For details see [2].

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## 1.6 The energy equation

Now let us derive a differential equation equivalent to the *law of conservation of energy*. As in the preceding sections, we consider a piece of fluid represented by a control volume $\mathcal{V}(t)$ satisfying assumptions 1.3. The law of conservation of energy can be formulated as follows:

*The rate of change of the total energy of the fluid particles, occupying the domain $\mathcal{V}(t)$ at time t, is equal to the sum of powers of the volume force acting on the volume $\mathcal{V}(t)$ and the surface force acting on the surface $\partial\mathcal{V}(t)$, and of the amount of heat transmitted to $\mathcal{V}(t)$.*

Now let us denote by $\mathcal{E}(\mathcal{V}(t))$ the total energy of the fluid particles contained in the domain $\mathcal{V}(t)$ and by $\mathcal{Q}(\mathcal{V}(t))$ the amount of heat transmitted to $\mathcal{V}(t)$ at time $t$. Taking into account the character of volume and surface forces involved, we get the identity representing the law of conservation of energy:

$$\frac{d}{dt}\mathcal{E}(\mathcal{V}(t)) = \int_{\mathcal{V}(t)} \rho(\boldsymbol{x},t)\boldsymbol{f}(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x},t)\,d\boldsymbol{x} \tag{1.34}$$

$$+ \int_{\partial\mathcal{V}(t)} \sum_{i,j=1}^{3} \tau_{ji}(\boldsymbol{x},t)\, n_j(\boldsymbol{x})\, v_i(\boldsymbol{x},t)\,dS + \mathcal{Q}(\mathcal{V}(t)).$$

Following relations hold:

$$a) \quad \mathcal{E}(\mathcal{V}(t)) = \int_{\mathcal{V}(t)} E(\boldsymbol{x},t)\,d\boldsymbol{x}, \tag{1.35}$$

$$b) \quad E = \rho\left(e + \frac{|\boldsymbol{v}|^2}{2}\right),$$

$$c) \quad \mathcal{Q}(\mathcal{V}(t)) = \int_{\mathcal{V}(t)} \rho(\boldsymbol{x},t)\,q(\boldsymbol{x},t)\,d\boldsymbol{x} - \int_{\partial\mathcal{V}(t)} \boldsymbol{\phi}_q(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x})\,dS.$$

Here $E$ is the total energy, $e$ is the density of the specific internal energy (related to the unit mass) associated with molecular and atomic behavior, $|\boldsymbol{v}|^2/2$ is the density of the kinetic energy, $q$ represents the density of heat sources (again related to the unit mass) and $\boldsymbol{\phi_q}$ is the heat flux.

Let assumptions 1.3 hold and further let $\tau_{ij}, (\boldsymbol{\phi}_q)_i \in \mathcal{C}^1(\mathcal{M})$ and $f_i, q \in \mathcal{C}(\mathcal{M})$ $(i,j=1,2,3)$. Using this, relations (1.35) a)-c), Theorem 1.2, Green's theorem and Lemma 1.4, we derive from (1.34) the differential *energy equation*, where we take advantage of (1.30):

$$\frac{\partial E}{\partial t} + \operatorname{div}(E\boldsymbol{v}) = \rho\boldsymbol{f}\cdot\boldsymbol{v} - \operatorname{div}(p\boldsymbol{v}) + \operatorname{div}(\lambda\boldsymbol{v}\operatorname{div}\boldsymbol{v}) + \operatorname{div}(2\mu\mathbb{D}(\boldsymbol{v})\,\boldsymbol{v}) + \rho q - \operatorname{div}\boldsymbol{\phi}_q. \tag{1.36}$$

For details see [2].

# 1.7 Thermodynamical relations

In order to complete the equations describing the flow, some other relations shall be added. The system now contains seven unknown quantities: $v_1, v_2, v_3, \rho, e, \theta, p$, but only 5 equations (scalar continuity equation, vector Navier-Stokes equations and scalar energy equation), i.e. $(1 + 3 + 1) = 5$.
From this we see, that additional two equations should to be included.

**Basic Thermodynamical Quantities**
The absolute temperature $\theta$, the density $\rho$ and the pressure $p$ are called the *state variables*. All these quantities are positive scalar functions. We consider only the so-called *perfect gas* or *ideal gas* whose state variables satisfy the following *equation of state*

$$p = R\theta\rho, \tag{1.37}$$

where $R$ is the *gas constant*, which is defined as

$$R = c_p - c_v. \tag{1.38}$$

Here $c_p$ denotes the *specific heat at constant pressure*, i.e. the ratio of the increment of the amount of heat related to the unit mass, to the increment of temperature at constant pressure. Analogously $c_v$ denotes the *specific heat at constant volume*. Experiments show that $c_p > c_v$, so that $R > 0$, and that $c_p$ and $c_v$ can be treated like constants for a relatively large range of temperature. The internal energy related to the unit mass is defined by

$$e = c_v\theta, \tag{1.39}$$

which explains the meaning of the internal energy: it is the amount of heat it would have to be transmitted out of the fluid so that its temperature would reach (absolute) zero, volume being kept constant during the whole process.

With respect to the above relations, we can express the internal energy as

$$e = c_p\theta - R\theta. \tag{1.40}$$

Another important characteristic of the flow is so-called *Mach number*, which is defined as

$$M = \frac{|\boldsymbol{v}|}{a}, \tag{1.41}$$

where $\boldsymbol{v}$ is the flow velocity and $a$ is the speed of sound.

**The complete system of equations describing the flow**

The complete system now reads

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \boldsymbol{v}) \;=\; 0, \tag{1.42}$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \mathrm{div}\,(\rho \boldsymbol{v} \otimes \boldsymbol{v}) \;=\; \rho \boldsymbol{f} - \nabla p + \mu \triangle \boldsymbol{v} + (\mu + \lambda)\, \nabla \mathrm{div}\,\boldsymbol{v}, \tag{1.43}$$

$$\frac{\partial E}{\partial t} + \mathrm{div}\,(E\boldsymbol{v}) \;=\; \rho \boldsymbol{f} \cdot \boldsymbol{v} - \mathrm{div}\,(p\boldsymbol{v}) + \mathrm{div}\,(\lambda \boldsymbol{v}\, \mathrm{div}\,\boldsymbol{v}) + \tag{1.44}$$

$$+ \;\; \mathrm{div}\,(2\mu \mathbb{D}\,(\boldsymbol{v})\,\boldsymbol{v}) + \rho q - \mathrm{div}\,\boldsymbol{\phi}_q,$$

$$p \;=\; (c_p/c_v - 1)\left(E - \rho\,|\boldsymbol{v}|^2/2\right), \tag{1.45}$$

$$\theta \;=\; \left(E/\rho - |\boldsymbol{v}|^2/2\right)/c_v. \tag{1.46}$$

This system is simply called the *compressible Navier-Stokes equations* for a heat-conductive perfect gas. Equations (1.45) and (1.46) follow from (1.37) - (1.40) and (1.35).

# Chapter 2

# Formulation of the problem in a time-dependent domain

In this chapter we shall be concerned with the *arbitrary Lagrangian-Eulerian (ALE) method* proposed in [5] to reformulate the above equations in time-dependent domain. Henceforth, the symbol $\Omega_0$ shall represent the space domain occupied by a fluid at time 0.

## 2.1   The ALE method

Let us denote by $\Omega_{ref}$ a suitably chosen domain, which will be further referred to as the *reference domain* or the *reference configuration*. We can introduce the family of diffeomorphic mappings $\mathcal{A}_t$ for $t \in [0, T)$:

$$\mathcal{A}_t : \overline{\Omega}_{ref} \subset \mathbb{R}^3 \xrightarrow{onto} \overline{\Omega}_t \subset \mathbb{R}^3, \quad \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{y}, t) = \mathcal{A}_t(\boldsymbol{y}) \in \overline{\Omega}_t, \boldsymbol{y} \in \overline{\Omega}_{ref}. \qquad (2.1)$$

This means that for each $t \in [0, T)$ we have

$$\begin{aligned}
\forall \boldsymbol{y} \in \overline{\Omega}_{ref} \; \exists! \, \boldsymbol{x} \in \overline{\Omega}_t, \; \boldsymbol{x} &= \mathcal{A}_t(\boldsymbol{y}), & (2.2) \\
\forall \boldsymbol{x} \in \overline{\Omega}_t \; \exists! \, \boldsymbol{y} \in \overline{\Omega}_{ref}, \; \boldsymbol{y} &= \mathcal{A}_t^{-1}(\boldsymbol{x}), \\
\mathcal{A}_t &\in \left[\mathcal{C}^1\left(\overline{\Omega}_{ref}\right)\right]^3, \\
\mathcal{A}_t^{-1} &\in \left[\mathcal{C}^1\left(\overline{\Omega}_t\right)\right]^3.
\end{aligned}$$

Further, we assume that for all $t \in [0, T)$ the mapping $\mathcal{A}_t$ possesses the following two properties:

$$\partial \mathcal{A}_t / \partial t \in \mathcal{C}\left(\overline{\Omega}_{ref}\right) \qquad (2.3)$$

and

$$\mathcal{A}_t\left(\partial \Omega_{ref}\right) = \partial \Omega_t. \qquad (2.4)$$

The mapping $\mathcal{A}_t$ thus specified is called the *arbitrary Lagrangian-Eulerian (ALE) mapping*.
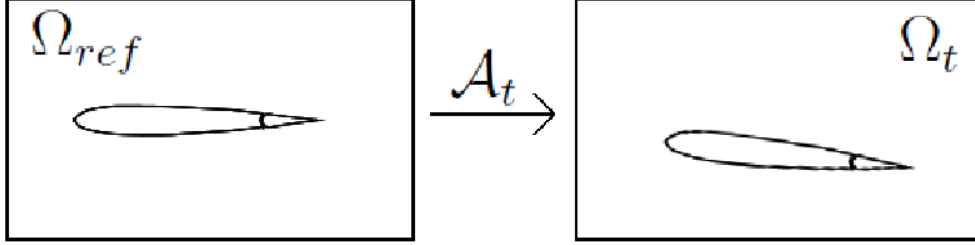
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Figure 2.1: The ALE mapping

The Jacobian matrix $\mathbf{J}_{\mathcal{A}_t}$ of the ALE mapping $\mathcal{A}_t$ exists and

$$\mathcal{J}_{\mathcal{A}_t}(\boldsymbol{y}) = \det\left(\mathbf{J}_{\mathcal{A}_t}(\boldsymbol{y})\right), \text{ where } \mathbf{J}_{\mathcal{A}_t}(\boldsymbol{y}) = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{pmatrix}(\boldsymbol{y}, t) \qquad (2.5)$$

is the Jacobian determinant of the mapping $\mathcal{A}_t$. We define the domain velocity $\hat{\boldsymbol{w}}$ as

$$\hat{\boldsymbol{w}}(\boldsymbol{y}, t) = \frac{\partial \boldsymbol{x}(\boldsymbol{y}, t)}{\partial t}, \ \ \boldsymbol{y} \in \overline{\Omega}_{ref}, \ t \in [0, T). \qquad (2.6)$$

It can be expressed in terms of current domain coordinates, using the inverse mapping $\mathcal{A}_t^{-1}(\boldsymbol{x}) : \overline{\Omega}_t \to \overline{\Omega}_{ref}$, as

$$\boldsymbol{w}(\boldsymbol{x}, t) = \hat{\boldsymbol{w}}\left(\mathcal{A}_t^{-1}(\boldsymbol{x}), t\right) = \frac{\partial \boldsymbol{x}}{\partial t}\left(\mathcal{A}_t^{-1}(\boldsymbol{x}), t\right), \ \ \boldsymbol{x} \in \overline{\Omega}_t, \ t \in [0, T). \qquad (2.7)$$

We define the *ALE trajectory* $T_{\boldsymbol{y}}$ for every $\boldsymbol{y} \in \overline{\Omega}_{ref}$ as

$$T_{\boldsymbol{y}} = \{\boldsymbol{x}(\boldsymbol{y}, t), t \in [0, T)\}.$$

Let $f : \{(\boldsymbol{x}, t); \boldsymbol{x} \in \Omega_t, t \in [0, T)\} \to \mathbb{R}$. In what follows we shall denote by $\hat{f}$ the corresponding function defined in the ALE reference domain, i.e.

$$\hat{f} : \Omega_{ref} \times [0, T) \to \mathbb{R}, \ \ \hat{f}(\boldsymbol{y}, t) = f(\mathcal{A}_t(\boldsymbol{y}), t). \qquad (2.8)$$

We define the ALE derivative of the function $f$ as the time derivative along $T_{\boldsymbol{y}}$

$$\frac{D^{\mathcal{A}}}{Dt} f(\boldsymbol{x}, t) := \frac{\partial}{\partial t}\hat{f}\left(\mathcal{A}_t^{-1}(\boldsymbol{x}), t\right), \ \ \boldsymbol{x} \in \Omega_t, \ t \in [0, T), \qquad (2.9)$$

provided the derivative on the right-hand side exists. By the chain rule we get

$$\frac{D^{\mathcal{A}}}{Dt} f(\boldsymbol{x}, t) = \frac{\partial}{\partial t} f(\boldsymbol{x}, t) + \nabla f(\boldsymbol{x}, t) \cdot \frac{\partial \boldsymbol{x}\left(\mathcal{A}_t^{-1}(\boldsymbol{x}), t\right)}{\partial t} \tag{2.10}$$

$$= \frac{\partial}{\partial t} f(\boldsymbol{x}, t) + \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(\boldsymbol{x}, t) \, w_i(\boldsymbol{x}, t)$$

$$= \frac{\partial}{\partial t} f(\boldsymbol{x}, t) + \boldsymbol{w} \cdot \nabla f(\boldsymbol{x}, t).$$

The important relation between the Jacobian determinant $\mathcal{J}_{\mathcal{A}_t}$ and the domain velocity $\boldsymbol{w}$ is given by the following theorem.

**Theorem 2.1**
*Let the domain velocity $\hat{\boldsymbol{w}}(\boldsymbol{y}, t)$ have continuous first order derivatives with respect to $y_1$, $y_2$, $y_3$ for $\boldsymbol{y} \in \Omega_{ref}$, $t \in [0, T)$. Then the function $\mathcal{J}_{\mathcal{A}_t}(\boldsymbol{y})$ has a continuous partial derivative $\partial \mathcal{J}_{\mathcal{A}_t}(\boldsymbol{y})/\partial t$ for $\boldsymbol{y} \in \Omega_{ref}, t \in [0, T)$ and*

$$\frac{1}{\mathcal{J}_{\mathcal{A}_t}(\boldsymbol{y})} \frac{\partial \mathcal{J}_{\mathcal{A}_t}}{\partial t}(\boldsymbol{y}) = \operatorname{div} \boldsymbol{w}(\boldsymbol{x}, t), \quad \text{where } \boldsymbol{x} = \mathcal{A}_t(\boldsymbol{y}), \tag{2.11}$$

*holds for all $\boldsymbol{y} \in \Omega_{ref}$, $t \in [0, T)$.*

**Proof**
The Jacobian determinant $\mathcal{J}_{\mathcal{A}_t}$ can be expanded by its $i$-th row:

$$\mathcal{J}_{\mathcal{A}_t}(\boldsymbol{y}) = \sum_{\alpha=1}^{3} \frac{\partial x_i}{\partial y_\alpha}(\boldsymbol{y}, t) \, \mathrm{D}_{i,\alpha}(\boldsymbol{y}, t), \tag{2.12}$$

where $\mathrm{D}_{i,\alpha}$ denotes the co-factor of the element $\partial x_i/\partial y_\alpha$. As the co-factors of elements on the $i$-th row do not depend on any other element on the $i$-th row, from (2.12) we get

$$\frac{\partial \mathcal{J}_{\mathcal{A}_t}}{\left(\frac{\partial x_i}{\partial y_\alpha}\right)} = \mathrm{D}_{i,\alpha}. \tag{2.13}$$

In order to calculate the derivative $\partial \mathcal{J}_{\mathcal{A}_t}/\partial t$, we use (2.12), where the derivatives $\partial x_i/\partial y_\alpha$ depend on $t$:

$$\frac{\partial \mathcal{J}_{\mathcal{A}_t}}{\partial t} = \sum_{i,\alpha=1}^{3} \frac{\partial \mathcal{J}_{\mathcal{A}_t}}{\partial \left(\frac{\partial x_i}{\partial y_\alpha}\right)} \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial y_\alpha}\right) = \sum_{i,\alpha=1}^{3} \mathrm{D}_{i,\alpha} \frac{\partial^2 x_i}{\partial y_\alpha \partial t}. \tag{2.14}$$

Now, using (2.6), (2.7) and the assumption, we can write the following:

$$\frac{\partial^2 x_i}{\partial y_\alpha \partial t}(\boldsymbol{y}, t) = \frac{\partial}{\partial y_\alpha} w_i(\mathcal{A}_t(\boldsymbol{y}), t)$$

$$= \sum_{j=1}^{3} \frac{\partial w_i}{\partial x_j}(\mathcal{A}_t(\boldsymbol{y}), t) \frac{\partial x_j}{\partial y_\alpha}(\boldsymbol{y}, t).$$

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Substituting into (2.14), we have

$$\frac{\partial \mathcal{J}_{\mathcal{A}_t}}{\partial t} = \sum_{i,\alpha=1}^{3} \mathrm{D}_{i,\alpha} \sum_{j=1}^{3} \frac{\partial x_j}{\partial y_\alpha} \frac{\partial w_i}{\partial x_j} = \sum_{i,j=1}^{3} \left( \sum_{\alpha=1}^{3} \frac{\partial x_j}{\partial y_\alpha} \mathrm{D}_{i,\alpha} \right) \frac{\partial w_i}{\partial x_j}.$$

In view of (2.12), from the theory of determinants we obtain

$$\sum_{\alpha=1}^{3} \frac{\partial x_j}{\partial y_\alpha} \mathrm{D}_{i,\alpha} = \mathcal{J}_{\mathcal{A}_t} \delta_{ij},$$

and finally

$$\frac{\partial \mathcal{J}_{\mathcal{A}_t}}{\partial t} = \mathcal{J}_{\mathcal{A}_t} \sum_{i,j=1}^{3} \delta_{ij} \frac{\partial w_i}{\partial x_j} = \mathcal{J}_{\mathcal{A}_t} \sum_{i=1}^{3} \frac{\partial w_i}{\partial x_i} = \mathcal{J}_{\mathcal{A}_t} \mathrm{div}\, \boldsymbol{w}. \qquad (2.15)$$

$\square$

In the following, we shall assume that conditions of Theorem 2.1 are fulfilled, i.e. that the domain velocity $\hat{\boldsymbol{w}}(\boldsymbol{y}, t)$ has continuous first order derivatives with respect to $y_1$, $y_2$, $y_3$ for $\boldsymbol{y} \in \Omega_{ref}$, $t \in [0, T)$.

## 2.2 Equations describing the flow in the ALE form

In what follows, we shall set $\Omega_{ref} = \Omega_0$, and the ALE mapping $\mathcal{A}_t$ for $t = 0$ shall be chosen as the identity mapping of $\Omega_0$ onto itself. This allows us to prove the following lemma:

**Lemma 2.2**
*For all $\boldsymbol{y} \in \Omega_0$ and $t \in (0, T)$*
$$\mathcal{J}_{\mathcal{A}_t}(\boldsymbol{y}) > 0.$$

**Proof**
Since we choose the ALE mapping at time $t = 0$ to be the identity mapping of $\Omega_0$ onto itself, we have
$$\mathcal{J}_{\mathcal{A}_t}(\boldsymbol{y}) = 1, \ \boldsymbol{y} \in \Omega_0, \ t = 0.$$
The domain velocity $\hat{\boldsymbol{w}}(\boldsymbol{y}, t)$ has continuous first order derivatives with respect to $y_1$, $y_2$, $y_3$ for $\boldsymbol{y} \in \Omega_{ref}$, $t \in [0, T)$. From this it follows that the Jacobian $\mathcal{J}_{\mathcal{A}_t}$ is continuous with respect to time for $t \in [0, T)$. This means that it is sufficient to show that
$$\mathcal{J}_{\mathcal{A}_t}(\boldsymbol{y}) \neq 0, \ \boldsymbol{y} \in \Omega_0, t \in (0, T).$$
But since the ALE mapping $\mathcal{A}_t$ is a diffeomorphism for each $t \in (0, T)$, the Jacobi matrix $\mathbf{J}_{\mathcal{A}_t}(\boldsymbol{y})$ has for $\boldsymbol{y} \in \Omega_0$ full rank and thus the previous holds.

$\square$

The following theorem is the ALE version of Theorem 1.2.

**Theorem 2.3 - The ALE transport theorem**
*Let $t \in (0, T)$, $\mathcal{V}_0 \subset \overline{\mathcal{V}_0} \subset \Omega_0$ be a bounded domain, and let $\mathcal{V}(t) \subset \Omega_t$ be its image under the mapping $\mathcal{A}_t$. Let a function $f : \mathcal{M} \to \mathbb{R}$ be continuously differentiable with respect to both variables in its domain of definition. Then*

$$\frac{d}{dt} \int_{\mathcal{V}(t)} f(\boldsymbol{x}, t) \, d\boldsymbol{x} = \int_{\mathcal{V}(t)} \left( \frac{D^{\mathcal{A}} f}{Dt} + f \operatorname{div} \boldsymbol{w} \right) (\boldsymbol{x}, t) \, d\boldsymbol{x}. \tag{2.16}$$

**Proof**
First we use the substitution theorem with the ALE mapping $\mathcal{A}_t$:

$$\int_{\mathcal{V}(t)} f(\boldsymbol{x}, t) \, d\boldsymbol{x} = \int_{\mathcal{V}_0} f(\mathcal{A}_t(\boldsymbol{y}), t) \, \mathcal{J}_{\mathcal{A}_t}(\boldsymbol{y}) \, d\boldsymbol{y}. \tag{2.17}$$

Then, having fixed reference domain $\mathcal{V}_0$, we can use the theorem on differentiation of an integral with respect to a parameter:

$$\frac{d}{dt} \int_{\mathcal{V}(t)} f(\boldsymbol{x}, t) \, d\boldsymbol{x} = \int_{\mathcal{V}_0} \left[ \left( \frac{\partial f}{\partial t}(\mathcal{A}_t(\boldsymbol{y}), t) + \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(\mathcal{A}_t(\boldsymbol{y}), t) \frac{\partial x_i}{\partial t}(\boldsymbol{y}, t) \right) \times \right.$$
$$\left. \times \quad \mathcal{J}_{\mathcal{A}_t}(\boldsymbol{y}) + f(\mathcal{A}_t(\boldsymbol{y}), t) \frac{\partial \mathcal{J}_{\mathcal{A}_t}}{\partial t}(\boldsymbol{y}) \right] d\boldsymbol{y}.$$

Now, using (2.7) and Theorem 2.1, we get

$$\frac{d}{dt} \int_{\mathcal{V}(t)} f(\boldsymbol{x}, t) \, d\boldsymbol{x} = \int_{\mathcal{V}_0} \left[ \left( \frac{\partial f}{\partial t}(\mathcal{A}_t(\boldsymbol{y}), t) + f \operatorname{div} \boldsymbol{w}(\mathcal{A}_t(\boldsymbol{y}), t) + \right. \right.$$
$$\left. \left. + \quad \boldsymbol{w} \cdot \nabla f(\mathcal{A}_t(\boldsymbol{y}), t) \right) \mathcal{J}_{\mathcal{A}_t}(\boldsymbol{y}) \right] d\boldsymbol{y}.$$

We adjust the expression on the right-hand side using (2.10), and to the result we shall apply the inverse substitution theorem obtaining

$$\frac{d}{dt} \int_{\mathcal{V}(t)} f(\boldsymbol{x}, t) \, d\boldsymbol{x} = \int_{\mathcal{V}(t)} \left( \frac{D^{\mathcal{A}} f}{Dt} + f \operatorname{div} \boldsymbol{w} \right) (\boldsymbol{x}, t) \, d\boldsymbol{x}.$$

$\square$

We shall proceed now with the derivation of equations describing the flow in the ALE form. Till the end of this chapter, the independent variables are omitted for readability.

**The ALE form of the continuity equation**

In Section 1.3 we derived the continuity equation in the form (1.20), which can be rewritten using (2.10) in the following way:

$$\frac{D^{\mathcal{A}}\rho}{Dt} + \text{div}\,(\rho\boldsymbol{v}) - \boldsymbol{w} \cdot \nabla\rho = 0 \tag{2.18}$$

We want, however, to get rid of the convective derivative $(\boldsymbol{w} \cdot \nabla\rho)$ in order to obtain the equation in the conservative form. By an easy manipulation we get

$$\frac{D^{\mathcal{A}}\rho}{Dt} + \text{div}\,(\rho\,(\boldsymbol{v} - \boldsymbol{w})) + \rho\,\text{div}\,\boldsymbol{w} = 0. \tag{2.19}$$

We shall inspect this relation a little closer, by presenting its physical context. First, relation (2.19) holds in $\Omega_t$, therefore we can consider a set $\mathcal{V}(t) \subset \Omega_t$, an image of a bounded domain $\mathcal{V}_0 \subset \overline{\mathcal{V}}_0 \subset \Omega_0$ under the ALE mapping $\mathcal{A}_t$. Then we can integrate both sides of the equation (2.19) over $\mathcal{V}(t)$:

$$\int_{\mathcal{V}(t)} \frac{D^{\mathcal{A}}\rho}{Dt} + \text{div}\,(\rho\,(\boldsymbol{v} - \boldsymbol{w})) + \rho\,\text{div}\,\boldsymbol{w}\,d\boldsymbol{x} = 0. \tag{2.20}$$

We continue by using Theorem 2.3. In view of assumptions 1.3, $\rho$ meets the requirements of the regularity. Thus,

$$\begin{aligned} 0 &= \int_{\mathcal{V}(t)} \frac{D^{\mathcal{A}}\rho}{Dt} + \rho\,\text{div}\,\boldsymbol{w}d\boldsymbol{x} + \int_{\mathcal{V}(t)} \text{div}\,(\rho\,(\boldsymbol{v} - \boldsymbol{w}))\,d\boldsymbol{x} \tag{2.21} \\ &= \frac{d}{dt}\int_{\mathcal{V}(t)} \rho\,d\boldsymbol{x} + \int_{\mathcal{V}(t)} \text{div}\,(\rho\,(\boldsymbol{v} - \boldsymbol{w}))\,d\boldsymbol{x}. \tag{2.22} \end{aligned}$$

Rewriting the obtained equality with the aid of Green's formula, we get

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \rho\,d\boldsymbol{x} = -\int_{\partial\mathcal{V}(t)} \rho\,(\boldsymbol{v} - \boldsymbol{w}) \cdot \boldsymbol{n}\,dS. \tag{2.23}$$

From the physical point of view, the derived relation can be formulated in the following way:

*The rate of change of the mass of the fluid piece represented by the domain $\mathcal{V}(t)$ is equal to the physical flux of the mass through the boundary $\partial\mathcal{V}(t)$ inwards, diminished by the non-physical flux induced by moving of the boundary $\partial\mathcal{V}(t)$ via the ALE mapping.*

**The ALE form of the Navier-Stokes equations**

Now we want to reformulate (1.33) in the ALE reference frame. For now, we shall use the equations in the form (1.29) in order not to have too complicated expressions on the right-hand side. The similarity with the derivation of the continuity equation in the ALE form can be seen easier when operating with scalar equations ( i.e. (1.29) written componentwise ):

$$\frac{\partial}{\partial t}\left(\rho v_i\right) + \mathrm{div}\left(\rho v_i \boldsymbol{v}\right) = \rho f_i + \left(\mathrm{div}\mathcal{T}\right)_i, \quad i = 1, 2, 3. \tag{2.24}$$

As in the previous, we shall use (2.10) for rewriting the above equations with the ALE derivative:

$$\frac{D^{\mathcal{A}}\left(\rho v_i\right)}{Dt} - \boldsymbol{w} \cdot \nabla\left(\rho v_i\right) + \mathrm{div}\left(\rho v_i \boldsymbol{v}\right) = \rho f_i + \left(\mathrm{div}\mathcal{T}\right)_i, \quad i = 1, 2, 3. \tag{2.25}$$

By a manipulation we get

$$\frac{D^{\mathcal{A}}\left(\rho v_i\right)}{Dt} + \mathrm{div}\left(\rho v_i\left(\boldsymbol{v} - \boldsymbol{w}\right)\right) + \left(\rho v_i\right)\mathrm{div}\,\boldsymbol{w} = \rho f_i + \left(\mathrm{div}\mathcal{T}\right)_i, \quad i = 1, 2, 3. \tag{2.26}$$

Going back to the vector notation, we obtain

$$\frac{D^{\mathcal{A}}\left(\rho\boldsymbol{v}\right)}{Dt} + \mathrm{div}\left(\rho\boldsymbol{v} \otimes \left(\boldsymbol{v} - \boldsymbol{w}\right)\right) + \left(\rho\boldsymbol{v}\right)\mathrm{div}\,\boldsymbol{w} = \rho\boldsymbol{f} + \mathrm{div}\mathcal{T}, \tag{2.27}$$

which is the vector equation of motion in the conservative ALE form. If we express the right-hand side as in Section 1.5, we get the Navier-Stokes equations in the conservative ALE form:

$$\frac{D^{\mathcal{A}}\left(\rho\boldsymbol{v}\right)}{Dt} + \mathrm{div}\left(\rho\boldsymbol{v} \otimes \left(\boldsymbol{v} - \boldsymbol{w}\right)\right) + \left(\rho\boldsymbol{v}\right)\mathrm{div}\,\boldsymbol{w} = \rho\boldsymbol{f} - \nabla p + \mu\,\triangle\boldsymbol{v} + \left(\mu + \lambda\right)\nabla\mathrm{div}\,\boldsymbol{v}. \tag{2.28}$$

**The ALE form of the energy equation**

Derivation of the conservative ALE form of the energy equation again follows the same principle used in the derivation of the continuity equation and the Navier-Stokes equations in the ALE form. The right-hand side does not change, only the left-hand one does, and the resulting equation reads

$$\frac{D^{\mathcal{A}}E}{Dt} + \mathrm{div}\left(E\left(\boldsymbol{v} - \boldsymbol{w}\right)\right) + E\,\mathrm{div}\,\boldsymbol{w} = \rho\boldsymbol{f} \cdot \boldsymbol{v} - \mathrm{div}\left(p\boldsymbol{v}\right) + \mathrm{div}\left(\lambda\boldsymbol{v}\,\mathrm{div}\,\boldsymbol{v}\right) + \tag{2.29}$$
$$+ \mathrm{div}\left(2\mu\mathbb{D}\left(\boldsymbol{v}\right)\boldsymbol{v}\right) + \rho q - \mathrm{div}\boldsymbol{\phi}_q.$$

**Complete system of equations of heat-conductive perfect gas in the ALE form**

Since the relations from thermodynamics presented in Section 1.7 are state equations, i.e. do not contain time derivatives, they do not change under the ALE mapping. Therefore the full system consists of the following equations:

$$\frac{D^{\mathcal{A}}\rho}{Dt} + \operatorname{div}\left(\rho\left(\boldsymbol{v} - \boldsymbol{w}\right)\right) + \rho\operatorname{div}\boldsymbol{w} \;=\; 0, \tag{2.30}$$

$$\frac{D^{\mathcal{A}}\left(\rho\boldsymbol{v}\right)}{Dt} + \operatorname{div}\left(\rho\boldsymbol{v}\otimes\left(\boldsymbol{v} - \boldsymbol{w}\right)\right) + \left(\rho\boldsymbol{v}\right)\operatorname{div}\boldsymbol{w} \;=\; \rho\boldsymbol{f} - \nabla p + \mu\triangle\boldsymbol{v} +$$
$$+ \;\left(\mu + \lambda\right)\nabla\operatorname{div}\boldsymbol{v}, \tag{2.31}$$

$$\frac{D^{\mathcal{A}}E}{Dt} + \operatorname{div}\left(E\left(\boldsymbol{v} - \boldsymbol{w}\right)\right) + E\operatorname{div}\boldsymbol{w} \;=\; \rho\boldsymbol{f}\cdot\boldsymbol{v} - \operatorname{div}\left(p\boldsymbol{v}\right) \tag{2.32}$$
$$+\operatorname{div}\left(\lambda\boldsymbol{v}\operatorname{div}\boldsymbol{v}\right) + \operatorname{div}\left(2\mu\mathbb{D}\left(\boldsymbol{v}\right)\boldsymbol{v}\right) \;+\; \rho q - \operatorname{div}\boldsymbol{\phi}_q,$$

$$p \;=\; \left(\gamma - 1\right)\left(E - \rho\left|\boldsymbol{v}\right|^2/2\right), \tag{2.33}$$

$$\theta \;=\; \left(E/\rho - \left|\boldsymbol{v}\right|^2/2\right)/c_v, \tag{2.34}$$

which all hold for $\boldsymbol{x} \in \Omega_t$ and $t \in (0, T)$.

# Chapter 3

# Simplified model specification

In what follows, we shall confine our considerations to the case of two-dimensional flow. This means that we shall assume that for $t \in [0, T)$ we have $\Omega_t \subset \mathbb{R}^2$ and $v_3(\boldsymbol{x}, t) = w_3(\boldsymbol{x}, t) = 0$, $\boldsymbol{x} \in \Omega_t$. Furthermore we have $\partial/\partial x_3 = 0$ and all functions depend on $x_1, x_2, t$ only.

## 3.1   Derivation of incompressible viscous model

In the previous chapter we have derived the system of equations of compressible heat-conductive perfect gas in the ALE formulation. As a special case we shall consider incompressible fluid flow with neglected volume force $f$ and neglected heat transfer. This model can be used for the simulation of flow of liquids, but also for gas flow under the assumption that the velocity of the gas is not too high. There is an experience that the incompressible model can be used for compressible flow, if the Mach number (see Section 1.7) does not exceed 0.4. Mathematically, our assumptions can be summarized in this way:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) &= 0, & &\text{(3.1)}\\
\frac{\partial \rho}{\partial x_i}(\boldsymbol{x}, t) &= 0, & i &= 1, 2, &\text{(3.2)}\\
f_i(\boldsymbol{x}, t) &= 0, & i &= 1, 2, &\text{(3.3)}\\
q(\boldsymbol{x}, t) &= 0, & &\text{(3.4)}\\
(\phi_q(\boldsymbol{x}, t))_i &= 0, & i &= 1, 2, &\text{(3.5)}\\
\boldsymbol{x} &\in \Omega_t, \ t \in (0, T). & &\text{(3.6)}
\end{aligned}
$$

From the first two of the above assumptions, we can see that the density $\rho$ is a constant, independent of time and space variables.

With these assumptions, the system of equations becomes simpler. We do not need the thermodynamical relations any more, because the absolute temperature is

superfluous for our considerations, and for the remaining variables (components of velocity $\boldsymbol{v}$ and the pressure $p$) we have full number of equations using the Navier-Stokes equations together with the continuity equation. Therefore, we can omit the energy equation as well.

With $\rho$ being a positive constant, we can rewrite the equation (2.30) in the form

$$\operatorname{div} \boldsymbol{v} = 0. \tag{3.7}$$

Note that the independent variables in this section are omitted for the sake of readability where possible. Since $\rho$ is a positive constant, the Navier-Stokes equations in the ALE form (2.31) can be written as

$$\rho \left[ \frac{D^{\mathcal{A}} \boldsymbol{v}}{Dt} + \operatorname{div}\left(\boldsymbol{v} \otimes (\boldsymbol{v} - \boldsymbol{w})\right) + \boldsymbol{v} \operatorname{div} \boldsymbol{w} \right] = \rho \boldsymbol{f} - \nabla p + \mu \triangle \boldsymbol{v} + (\mu + \lambda) \, \nabla \operatorname{div} \boldsymbol{v}.$$

Now using (3.3) and (3.7) we have

$$\rho \left[ \frac{D^{\mathcal{A}} \boldsymbol{v}}{Dt} + \operatorname{div}\left(\boldsymbol{v} \otimes (\boldsymbol{v} - \boldsymbol{w})\right) + \boldsymbol{v} \operatorname{div} \boldsymbol{w} \right] = -\nabla p + \mu \triangle \boldsymbol{v}. \tag{3.8}$$

Passing now to the kinematic pressure, i.e. the dynamic pressure $p$ divided by the density $\rho$ of the fluid, which will be henceforth denoted by $p$ and to the kinematic viscosity $\nu$, again the dynamic viscosity $\mu$ divided by the density of the fluid, we come to the system of equations composed by the conservative ALE form of the continuity equation and the Navier-Stokes equations:

$$\nabla \cdot \boldsymbol{v} = 0 \tag{3.9}$$

$$\frac{D^{\mathcal{A}} \boldsymbol{v}}{Dt} + \operatorname{div}\left(\boldsymbol{v} \otimes (\boldsymbol{v} - \boldsymbol{w})\right) + \boldsymbol{v} \operatorname{div} \boldsymbol{w} + \nabla p - \nu \triangle \boldsymbol{v} = 0, \tag{3.10}$$

which describes the incompressible flow.

For discretization however, the non-conservative form offers easier manipulation. We can write

$$\begin{aligned}
\operatorname{div}\left(\boldsymbol{v} \otimes (\boldsymbol{v} - \boldsymbol{w})\right) + \boldsymbol{v} \operatorname{div} \boldsymbol{w} &= -(\boldsymbol{w} \cdot \nabla) \boldsymbol{v} + \operatorname{div}\left(\boldsymbol{v} \otimes \boldsymbol{v}\right) \\
&= [(\boldsymbol{v} - \boldsymbol{w}) \cdot \nabla] \boldsymbol{v}, \tag{3.11}
\end{aligned}$$

and thus we get

$$\begin{aligned}
\nabla \cdot \boldsymbol{v}(\boldsymbol{x}, t) &= 0 \\
\frac{D^{\mathcal{A}} \boldsymbol{v}(\boldsymbol{x}, t)}{Dt} + [(\boldsymbol{v}(\boldsymbol{x}, t) - \boldsymbol{w}(\boldsymbol{x}, t)) \cdot \nabla] \boldsymbol{v}(\boldsymbol{x}, t) + \nabla p(\boldsymbol{x}, t) - \nu \triangle \boldsymbol{v}(\boldsymbol{x}, t) &= 0 \\
\forall \boldsymbol{x} \in \Omega_t, \ \forall t \in (0, T). \tag{3.12}
\end{aligned}$$

## 3.2 Initial and Boundary conditions

We need to complete our problem by the specification of appropriate boundary and initial conditions.

**Boundary conditions**
We divide the boundary of the computational domain into several mutually disjoint parts (see Figure 3.1):

    a) $\Gamma_D$ representing the inlet and the fixed part of the boundary.

    b) $\Gamma_O$ representing the outlet, i.e. the part of the boundary where the fluid generally leaves the domain $\Omega_t$.

    c) $\Gamma_{W_t}$ representing the moving part of the boundary, in our case a moving airfoil, at time $t$.
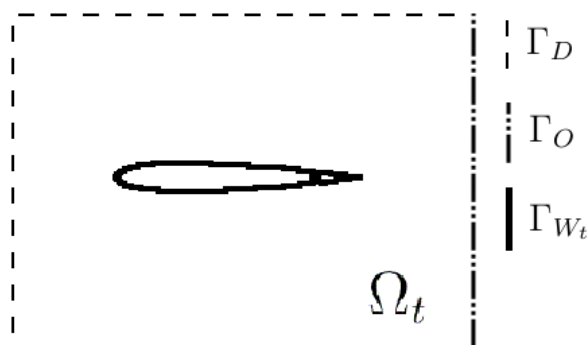


Figure 3.1: Parts of the boundary

On $\Gamma_D$ we prescribe the Dirichlet boundary condition

$$\boldsymbol{v}|_{\Gamma_D} = \boldsymbol{v}_D, \tag{3.13}$$

where $\boldsymbol{v}_D : \Gamma_D \to \mathbb{R}^2$ is a prescribed function.
On $\Gamma_{W_t}$ we assume that the fluid velocity corresponds with the domain velocity:

$$\boldsymbol{v}|_{\Gamma_{W_t}} = \boldsymbol{w}|_{\Gamma_{W_t}}. \tag{3.14}$$

Finally, on the outlet part $\Gamma_O$ we prescribe the 'do-nothing' boundary condition:

$$-(p - p_{ref})\,\boldsymbol{n} + \nu\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = 0, \tag{3.15}$$

where $p_{ref}$ is a prescribed value of the reference pressure. This boundary condition turns out to be suitable for the description of the flow on the outlet.

**Initial condition, pressure conditions**

We set up the Dirichlet initial condition for the velocity:

$$\boldsymbol{v}\left(\boldsymbol{x}, 0\right) = \boldsymbol{v}_0\left(\boldsymbol{x}\right), \quad \boldsymbol{x} \in \Omega_0. \tag{3.16}$$

Noteworthy is the fact, that we have not explicitly prescribed any boundary or initial conditions for the pressure. Since only its gradient is accounted for in the Navier-Stokes equations, the pressure is determined by the reference do-nothing condition for the velocity on $\Gamma_O$.

## 3.3   Reynolds number

Often the goal is to compare two flows with different parameters such as sizes and material characteristics. For this objective, the important constant turns out to be the *Reynolds number*, which can be expressed in the following way:

$$Re = \frac{U\,L}{\nu},$$

where $U$ is a reference speed (e.g. magnitude of the far field velocity or the average speed on the 'inlet' part of the boundary), $L$ is a reference length, e.g. in our case the length of the chord of the airfoil and $\nu$ is the kinematic viscosity. Brief explanation of the origin of this constant follows.

Let us take a look at the following simple form of the Navier-Stokes equations

$$\frac{\partial \boldsymbol{v}}{\partial t} + \left(\boldsymbol{v} \cdot \nabla\right) \boldsymbol{v} = -\nabla\,p + \nu \triangle \boldsymbol{v}. \tag{3.17}$$

Each term in this equation has its own physical unit, but we would like to render this equation dimensionless. We can set

$$\boldsymbol{v}' = \frac{\boldsymbol{v}}{U}, \quad p' = \frac{p}{U^2}, \quad \frac{\partial}{\partial t'} = \frac{L}{U}\frac{\partial}{\partial t}, \quad \nabla' = L\nabla, \tag{3.18}$$

where we multiplied each term by a factor with inverse units and so we can rewrite the equation (3.17) in the dimensionless form

$$\frac{\partial \boldsymbol{v}'}{\partial t'} + \left(\boldsymbol{v}' \cdot \nabla'\right) \boldsymbol{v}' = -\nabla'\,p' + \frac{1}{Re}\triangle' \boldsymbol{v}', \tag{3.19}$$

where $\left(\triangle' \boldsymbol{v}'\right)_i = \operatorname{div}\left(\nabla' v_i\right)$, $i = 1, 2, 3$. It can be seen now, that for two flows with the same Reynolds number, in domains geometrically similar, we have formally the same problems and having a solution of equation (3.17), the solution of (3.19) can be obtained by the scaling given by (3.18).

# Chapter 4

# Discretization of the problem

Let us summarize our problem, which we are about to solve numerically. Methods of discretization described in this chapter were used e.g. in [6], [7].
We want to find $\boldsymbol{v}$ and $p$ satisfying the equations

$$\nabla \cdot \boldsymbol{v}\left(\boldsymbol{x}, t\right) = 0,$$

$$\frac{D^{\mathcal{A}}\boldsymbol{v}\left(\boldsymbol{x}, t\right)}{Dt} + \left[\left(\boldsymbol{v}\left(\boldsymbol{x}, t\right) - \boldsymbol{w}\left(\boldsymbol{x}, t\right)\right) \cdot \nabla\right] \boldsymbol{v}\left(\boldsymbol{x}, t\right) + \nabla p\left(\boldsymbol{x}, t\right) - \nu \triangle \boldsymbol{v}\left(\boldsymbol{x}, t\right) = 0,$$

$$\boldsymbol{x} \in \Omega_t, \ t \in \left(0, T\right),$$

where $\boldsymbol{w}\left(\boldsymbol{x}, t\right)$ is defined in (2.7) and the ALE derivative $\frac{D^{\mathcal{A}}}{Dt}$ is defined in (2.10). This system is equipped with initial and boundary conditions

$$\begin{aligned}
\boldsymbol{v}\left(\boldsymbol{x}, 0\right) &= \boldsymbol{v}_0\left(\boldsymbol{x}\right), \quad \boldsymbol{x} \in \Omega_0, & (4.1) \\
\boldsymbol{v}\left(\boldsymbol{x}, t\right) &= \boldsymbol{v}_D\left(\boldsymbol{x}, t\right), \quad \boldsymbol{x} \in \Gamma_D, & (4.2) \\
\boldsymbol{v}\left(\boldsymbol{x}, t\right) &= \boldsymbol{w}\left(\boldsymbol{x}, t\right), \quad \boldsymbol{x} \in \Gamma_{W_t}, & (4.3) \\
-\left(p\left(\boldsymbol{x}, t\right) - p_{ref}\right)\boldsymbol{n}\left(\boldsymbol{x}\right) + \nu\frac{\partial \boldsymbol{v}\left(\boldsymbol{x}, t\right)}{\partial \boldsymbol{n}} &= 0, \quad \boldsymbol{x} \in \Gamma_O, & (4.4) \\
t &\in \left(0, T\right),
\end{aligned}$$

where $\boldsymbol{v}_0 : \Omega_0 \to \mathbb{R}^2$, $\boldsymbol{v}_D : \Gamma_D \to \mathbb{R}^2$ are known functions and $p_{ref}$ is a prescribed value of the reference pressure.
We shall discretize this problem separately in time and then in space.

## 4.1   Time discretization

We use the finite difference method for the time discretization. Let us consider a partition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = \tilde{T}$, $t_n = n\tau$ with a step $0 < \tau = \frac{\tilde{T}}{N}$ of the time interval $[0, \tilde{T}] \subset [0, T)$. We shall approximate the solution $\boldsymbol{v}\left(\boldsymbol{x}, t_{n+1}\right), p\left(\boldsymbol{x}, t_{n+1}\right)$ at time $t_{n+1}$ by $\boldsymbol{v}^{n+1}\left(\boldsymbol{x}\right), p^{n+1}\left(\boldsymbol{x}\right)$ defined in $\Omega_{t_{n+1}}$, $n = 0, ..., N-1$. At time $t_0 = 0$ we use the initial condition (4.1).

Now we shall approximate the ALE derivative, which is defined by

$$\frac{D^{\mathcal{A}}\boldsymbol{v}\left(\boldsymbol{x}, t_{n+1}\right)}{Dt} = \frac{\partial}{\partial t}\hat{\boldsymbol{v}}\left(\boldsymbol{X}, t_{n+1}\right), \quad \boldsymbol{X} = \mathcal{A}_{t_{n+1}}^{-1}\left(\boldsymbol{x}\right), \; \boldsymbol{x} \in \Omega_{t_{n+1}}.$$

In derivation of the finite difference scheme, we shall use the Taylor expansion: For a function $\boldsymbol{\varphi} \in \left[\mathcal{C}^3\left(\left[t_{n-1}, t_{n+1}\right]\right)\right]^2$, $n = 1, ..., N - 1$ we have

$$\boldsymbol{\varphi}\left(t_n\right) = \boldsymbol{\varphi}\left(t_{n+1}\right) - \tau\frac{\partial\boldsymbol{\varphi}}{\partial t}\left(t_{n+1}\right) + \tau^2\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2}\left(t_{n+1}\right) + O\left(\tau^3\right)$$

$$\boldsymbol{\varphi}\left(t_{n-1}\right) = \boldsymbol{\varphi}\left(t_{n+1}\right) - 2\tau\frac{\partial\boldsymbol{\varphi}}{\partial t}\left(t_{n+1}\right) + 4\tau^2\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2}\left(t_{n+1}\right) + O\left(\tau^3\right).$$

Multiplying the first equation by 4 and subtracting the second one from the result, we obtain

$$4\boldsymbol{\varphi}\left(t_n\right) - \boldsymbol{\varphi}\left(t_{n-1}\right) = 3\boldsymbol{\varphi}\left(t_{n+1}\right) - 2\tau\frac{\partial\boldsymbol{\varphi}}{\partial t}\left(t_{n+1}\right) + O\left(\tau^2\right),$$

which yields the approximation

$$\frac{\partial\boldsymbol{\varphi}}{\partial t}\left(t_{n+1}\right) = \frac{3\boldsymbol{\varphi}\left(t_{n+1}\right) - 4\boldsymbol{\varphi}\left(t_n\right) + \boldsymbol{\varphi}\left(t_{n-1}\right)}{2\tau} + O\left(\tau^2\right).$$

Omitting the last term, writing $\approx$ instead of $=$ and using this approximation for our problem, we come to the implicit second order accurate two-step backward difference formula

$$\frac{\partial}{\partial t}\hat{\boldsymbol{v}}\left(\boldsymbol{X}, t_{n+1}\right) \approx \frac{3\hat{\boldsymbol{v}}^{n+1}\left(\boldsymbol{X}\right) - 4\hat{\boldsymbol{v}}^n\left(\boldsymbol{X}\right) + \hat{\boldsymbol{v}}^{n-1}\left(\boldsymbol{X}\right)}{2\tau}.$$

Thus,

$$\begin{aligned}\frac{D^{\mathcal{A}}\boldsymbol{v}\left(\boldsymbol{x}^{n+1}, t_{n+1}\right)}{Dt} &\approx \frac{3\hat{\boldsymbol{v}}^{n+1}\left(\boldsymbol{X}\right) - 4\hat{\boldsymbol{v}}^n\left(\boldsymbol{X}\right) + \hat{\boldsymbol{v}}^{n-1}\left(\boldsymbol{X}\right)}{2\tau} \\ &= \frac{3\boldsymbol{v}^{n+1}\left(\boldsymbol{x}^{n+1}\right) - 4\boldsymbol{v}^n\left(\boldsymbol{x}^n\right) + \boldsymbol{v}^{n-1}\left(\boldsymbol{x}^{n-1}\right)}{2\tau},\end{aligned} \tag{4.5}$$

where we use the notation $\boldsymbol{x}^n = \mathcal{A}_{t_n}\left(\boldsymbol{X}\right)$. This result leads us to the implicit backward scheme for the unknown functions $\boldsymbol{v}^{n+1} : \Omega_{t_{n+1}} \to \mathbb{R}^2$ and $p^{n+1} : \Omega_{t_{n+1}} \to \mathbb{R}$:

$$\frac{3\boldsymbol{v}^{n+1}\left(\boldsymbol{x}^{n+1}\right) - 4\boldsymbol{v}^n\left(\boldsymbol{x}^n\right) + \boldsymbol{v}^{n-1}\left(\boldsymbol{x}^{n-1}\right)}{2\tau} + \tag{4.6}$$

$$\begin{aligned}+ \left(\left(\boldsymbol{v}^{n+1}\left(\boldsymbol{x}^{n+1}\right) - \boldsymbol{w}^{n+1}\left(\boldsymbol{x}^{n+1}\right)\right) \cdot \nabla\right)\boldsymbol{v}^{n+1}\left(\boldsymbol{x}^{n+1}\right) & - \\ -\nu\triangle\boldsymbol{v}^{n+1}\left(\boldsymbol{x}^{n+1}\right) + \nabla p^{n+1}\left(\boldsymbol{x}^{n+1}\right) &= 0 \\ \nabla \cdot \boldsymbol{v}^{n+1}\left(\boldsymbol{x}^{n+1}\right) &= 0, \tag{4.7}\end{aligned}$$

where $\boldsymbol{w}^{n+1}(\boldsymbol{x}) \approx \boldsymbol{w}(\boldsymbol{x}, t_{n+1})$. Function $\boldsymbol{v}^{n+1}$ has to satisfy the boundary conditions (4.2) - (4.4).

We consider the above equations for all $n = 1, ..., N-1$. In order to completely transform equations into the domain $\Omega_{t_{n+1}}$ we recall the following property of the ALE mapping:

$$\mathcal{A}_{t_{n+1}}\left(\mathcal{A}_{t_i}^{-1}\left(\boldsymbol{x}^i\right)\right) \in \Omega_{t_{n+1}}, \quad i = n-1, n. \tag{4.8}$$

Therefore we can define $\tilde{\boldsymbol{v}}^i = \boldsymbol{v}^i \circ \mathcal{A}_{t_i} \circ \mathcal{A}_{t_{n+1}}^{-1}$, $i = n-1, n$, and write the transformed equations:

$$\frac{3\boldsymbol{v}^{n+1}\left(\boldsymbol{x}^{n+1}\right) - 4\tilde{\boldsymbol{v}}^n\left(\boldsymbol{x}^{n+1}\right) + \tilde{\boldsymbol{v}}^{n-1}\left(\boldsymbol{x}^{n+1}\right)}{2\tau} + \tag{4.9}$$

$$+ \left(\left(\boldsymbol{v}^{n+1}\left(\boldsymbol{x}^{n+1}\right) - \boldsymbol{w}^{n+1}\left(\boldsymbol{x}^{n+1}\right)\right) \cdot \nabla\right)\boldsymbol{v}^{n+1}\left(\boldsymbol{x}^{n+1}\right) -$$

$$-\nu \triangle \boldsymbol{v}^{n+1}\left(\boldsymbol{x}^{n+1}\right) + \nabla p^{n+1}\left(\boldsymbol{x}^{n+1}\right) = 0,$$

$$\nabla \cdot \boldsymbol{v}^{n+1}\left(\boldsymbol{x}^{n+1}\right) = 0, \quad \forall \boldsymbol{x}^{n+1} \in \Omega_{t_{n+1}}. \tag{4.10}$$

Again this system is equipped with the boundary conditions (4.2) - (4.4). For obtaining the solution at the first time level $t_1$ we use a simple one-step backward finite difference scheme, where we use the initial condition (4.1).

## 4.2 Weak formulation

For simplicity, in what follows, we shall write $\boldsymbol{v}$, $\boldsymbol{w}$, $p$, $t$, $\Omega$ instead of $\boldsymbol{v}^{n+1}$, $\boldsymbol{w}^{n+1}$, $p^{n+1}$, $t_{n+1}$, $\Omega_{t_{n+1}}$.

As we want to employ the finite element method for the space discretization, we need to reformulate equations (4.9), (4.10), considered in the domain $\Omega$, in a weak sense. We need to set up appropriate function spaces of test functions and also the spaces in which the solution will be sought. The Dirichlet conditions on the parts $\Gamma_D$, $\Gamma_{W_t}$ of the boundary tell us that the test functions from the velocity space will be set zero in the sense of traces on these parts of the boundary. Since we have a second order problem with respect to the velocity and the first order problem with respect to the pressure, we shall use the following spaces.

The pressure space is defined by

$$\mathcal{Q} = \mathcal{L}^2\left(\Omega\right) = \left\{q : \Omega \to \mathbb{R}; q \text{ Lebesgue measurable}, \int_\Omega |q\left(\boldsymbol{x}\right)|^2 \mathrm{d}\boldsymbol{x} < \infty\right\}$$

The velocity spaces are defined as

$$\mathcal{V} = \left[\mathcal{H}^1\left(\Omega\right)\right]^2, \quad \mathcal{H}^1\left(\Omega\right) = \left\{v \in \mathcal{L}^2\left(\Omega\right); \frac{\partial v}{\partial x_i} \in \mathcal{L}^2\left(\Omega\right), i = 1, 2\right\},$$

$$\mathcal{Y} = \left\{\boldsymbol{y} \in \mathcal{V}; \boldsymbol{y}|_{\Gamma_D \cup \Gamma_{W_t}} = 0\right\}.$$

Derivatives in definition of the Sobolev space $\mathcal{H}^1$ are weak derivatives.

Now we can proceed with multiplying the equation (4.9) by a function $\boldsymbol{y} \in \mathcal{Y}$ and the equation (4.10) by a function $q \in \mathcal{Q}$. We get

$$\frac{3\boldsymbol{v} - 4\tilde{\boldsymbol{v}}^n + \tilde{\boldsymbol{v}}^{n-1}}{2\tau} \cdot \boldsymbol{y} + (((\boldsymbol{v} - \boldsymbol{w}) \cdot \nabla)\,\boldsymbol{v}) \cdot \boldsymbol{y} + \nabla p \cdot \boldsymbol{y} - \nu \triangle \boldsymbol{v} \cdot \boldsymbol{y} \;=\; 0,$$

$$(\nabla \cdot \boldsymbol{v})\, q \;=\; 0.$$

Integration over the domain $\Omega$ yields

$$\int_\Omega \frac{3\boldsymbol{v} - 4\tilde{\boldsymbol{v}}^n + \tilde{\boldsymbol{v}}^{n-1}}{2\tau} \cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} + \int_\Omega (((\boldsymbol{v} - \boldsymbol{w}) \cdot \nabla)\,\boldsymbol{v}) \cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \;+ \tag{4.11}$$

$$+ \int_\Omega \nabla p \cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} - \nu \int_\Omega \triangle \boldsymbol{v} \cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \;=\; 0,$$

$$\int_\Omega (\nabla \cdot \boldsymbol{v})\, q\,\mathrm{d}\boldsymbol{x} \;=\; 0. \tag{4.12}$$

Using Green's theorem and the boundary condition (4.4), we obtain

$$-\nu \int_\Omega \triangle \boldsymbol{v} \cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \;=\; -\nu \int_{\partial\Omega} \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \cdot \boldsymbol{y}\,\mathrm{d}S + \nu \int_\Omega \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{y}\,\mathrm{d}\boldsymbol{x}$$

$$=\; -\nu \int_{\Gamma_O} \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \cdot \boldsymbol{y}\,\mathrm{d}S + \nu \int_\Omega \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{y}\,\mathrm{d}\boldsymbol{x}$$

$$=\; -\int_{\Gamma_O} (p - p_{ref})\,\boldsymbol{n} \cdot \boldsymbol{y}\,\mathrm{d}S + \nu \int_\Omega \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{y}\,\mathrm{d}\boldsymbol{x}$$

$$\int_\Omega \nabla p \cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \;=\; \int_{\partial\Omega} p\,(\boldsymbol{n} \cdot \boldsymbol{y})\,\mathrm{d}S - \int_\Omega p\,(\nabla \cdot \boldsymbol{y})\,\mathrm{d}\boldsymbol{x}$$

$$=\; \int_{\Gamma_O} p\,(\boldsymbol{n} \cdot \boldsymbol{y})\,\mathrm{d}S - \int_\Omega p\,(\nabla \cdot \boldsymbol{y})\,\mathrm{d}\boldsymbol{x},$$

where $\int_\Omega \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = \sum_{i=1}^2 \int_\Omega \nabla v_i \cdot \nabla y_i\,\mathrm{d}\boldsymbol{x}$. These relations imply that

$$\frac{3}{2\tau} \int_\Omega \boldsymbol{v} \cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \;+\; \int_\Omega (((\boldsymbol{v} - \boldsymbol{w}) \cdot \nabla)\,\boldsymbol{v}) \cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\,(\nabla \cdot \boldsymbol{y})\,\mathrm{d}\boldsymbol{x} +$$

$$+\; \nu \int_\Omega \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} + \int_\Omega (\nabla \cdot \boldsymbol{v})\, q\,\mathrm{d}\boldsymbol{x} =$$

$$=\; \frac{1}{2\tau} \int_\Omega \left(4\tilde{\boldsymbol{v}}^n - \tilde{\boldsymbol{v}}^{n-1}\right) \cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} - \int_{\Gamma_O} p_{ref}\,(\boldsymbol{n} \cdot \boldsymbol{y})\,\mathrm{d}S,$$

where we have summed up the equations (4.11) and (4.12). Now we shall introduce the weak formulation of our problem. We want to find $V = (\boldsymbol{v}, p) \in \mathcal{V} \times \mathcal{Q}$ such that

$$a\,(V, V, Y) \;=\; f\,(Y) \quad \forall Y = (\boldsymbol{y}, q) \in \mathcal{Y} \times \mathcal{Q}, \tag{4.13}$$

where

$$a\left(V^*, V, Y\right) = \frac{3}{2\tau} \int_\Omega \boldsymbol{v} \cdot \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \left(\left(\left(\boldsymbol{v}^* - \boldsymbol{w}\right) \cdot \nabla\right) \boldsymbol{v}\right) \cdot \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} - \quad (4.14)$$

$$- \int_\Omega p \left(\nabla \cdot \boldsymbol{y}\right) \, \mathrm{d}\boldsymbol{x} + \nu \int_\Omega \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \left(\nabla \cdot \boldsymbol{v}\right) q \, \mathrm{d}\boldsymbol{x},$$

$$f\left(Y\right) = \frac{1}{2\tau} \int_\Omega \left(4\tilde{\boldsymbol{v}}^n - \tilde{\boldsymbol{v}}^{n-1}\right) \cdot \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} - \int_{\Gamma_O} p_{ref} \left(\boldsymbol{n} \cdot \boldsymbol{y}\right) \, \mathrm{d}S,$$

$V^* = \left(\boldsymbol{v}^*, p\right) \in \mathcal{V} \times \mathcal{Q}$, and $\boldsymbol{v}$ satisfies the boundary conditions (4.2) and (4.3). Then we call $V = \left(\boldsymbol{v}, p\right) \in \mathcal{V} \times \mathcal{Q}$ the weak solution of the problem (4.9), (4.10), (4.2) - (4.4).

## 4.3  Space discretization - finite element method

**Triangulation**
First step in the process of the finite element discretization is to divide the computational domain $\overline{\Omega} = \overline{\Omega}_{t_{n+1}}$ into finite number of subsets with properties described below. These subsets form the set, further denoted by $\mathcal{T}_h$, called the *triangulation of the domain* $\Omega$. The parameter $h > 0$ of the triangulation usually represents maximum of diameters of all elements $K \in \mathcal{T}_h$.
Properties of $\mathcal{T}_h$:

a) Each $K \in \mathcal{T}_h$ is closed and connected with its interior $K^\circ \neq \emptyset$.

b) Each $K \in \mathcal{T}_h$ has a Lipschitz boundary.

c) $\cup_{K \in \mathcal{T}_h} K = \overline{\Omega}$

d) If $K_1, K_2 \in \mathcal{T}_h$, $K_1 \neq K_2$, then $K_1^\circ \cap T_2^\circ = \emptyset$.

In our case of the two-dimensional problem, we assume that the domain $\Omega$ is obtained as a polygonal approximation of the original computational domain (also denoted by $\Omega$), and the triangulation is chosen accordingly to the following attributes:

A) Each $K \in \mathcal{T}_h$ is a closed nondegenerate triangle

B) For $K_1, K_2 \in \mathcal{T}_h$, $K_1 \neq K_2$ we have either $K_1 \cap K_2 = \emptyset$ or $K_1, K_2$ share one side or $K_1, K_2$ share one vertex.

C) $\cup_{K \in \mathcal{T}_h} K = \overline{\Omega}$.

**Finite element spaces**

Let us assume that for $h > 0$ we have a given triangulation $\mathcal{T}_h$. This means that objects introduced in the following sections depending on $h$ depend also on this triangulation $\mathcal{T}_h$. The next step is to construct finite dimensional function spaces $\mathcal{Q}_h, \mathcal{V}_h, \mathcal{Y}_h$ which would approximate spaces $\mathcal{Q}, \mathcal{V}, \mathcal{Y}$. We shall consider the so-called *conforming finite element method*, in which the approximate spaces are subspaces of the spaces from the continuous problem, i.e. $\mathcal{Q}_h \subset \mathcal{Q}, \mathcal{V}_h \subset \mathcal{V}, \mathcal{Y}_h \subset \mathcal{Y}$. In our case we even have $\mathcal{Q}_h \subset \mathcal{C}\left(\overline{\Omega}\right), \mathcal{V}_h \subset \left[\mathcal{C}\left(\overline{\Omega}\right)\right]^2, \mathcal{Y}_h \subset \left[\mathcal{C}\left(\overline{\Omega}\right)\right]^2$ and also we have $\mathcal{Y}_h \subset \mathcal{V}_h$. Since $\mathcal{V}_h, \mathcal{Y}_h$ are spaces of two dimensional vector-valued functions, we shall write $\mathcal{V}_h = \tilde{\mathcal{V}}_h \times \tilde{\mathcal{V}}_h, \ \mathcal{Y}_h = \tilde{\mathcal{Y}}_h \times \tilde{\mathcal{Y}}_h$, where $\tilde{\mathcal{V}}_h$ and $\tilde{\mathcal{Y}}_h$ are spaces of scalar functions.

On each element $K$ of triangulation $\mathcal{T}_h$ we define the so-called *finite element* $(K, P_K, \Sigma_K)$, where $P_K$ is a finite dimensional set of functions defined on $K$ and $\Sigma_K$ is a set of functionals over $P_K$, which is called the set of *degrees of freedom*. Every function $p \in P_K$ must be fully determined by its values $(\phi(p), \ \phi \in \Sigma_K)$. This property is called $P_K$-*unisolvence*.

**The inf-sup condition**

In order to develop a stable FEM scheme, the couple of test function spaces $(\mathcal{Y}_h, \mathcal{Q}_h)$ has to satisfy the *Babuška-Brezzi (BB, inf-sup)* condition:

$$\exists \beta > 0 \ : \ \beta \leq \sup_{0 \neq \boldsymbol{y} \in \mathcal{Y}_h} \frac{|\left(q, \nabla \cdot \boldsymbol{y}\right)|}{|\boldsymbol{y}|_{\mathcal{H}^1(\Omega)} \, ||q||_{\mathcal{L}^2(\Omega)}}, \ \forall 0 \neq q \in \mathcal{Q}_h, \tag{4.15}$$

where $(\cdot, \cdot)$ stands for the scalar product in $\mathcal{L}^2(\Omega)$ and $|\cdot|_{\mathcal{H}^1(\Omega)}$ for the seminorm in $\mathcal{H}^1(\Omega)$: $|f|_{\mathcal{H}^1(\Omega)} = \left(\int_\Omega |\nabla f|^2 \, \mathrm{d}\boldsymbol{x}\right)^{1/2}$.

**Construction of the spaces**

In our case we choose $P_K = P_k(K)$, i.e. the spaces of polynomial functions up to a degree $k \in \mathbb{N}$ defined on an element $K$.

We now derive the set $\Sigma_K$ for our needs. For $k \in \mathbb{N}$, and for an element $K$ with vertices $a_1, a_2, a_3$ we define the set

$$L_k(K) = \left\{\boldsymbol{x} : \boldsymbol{x} = \sum_{i=1}^3 \lambda_i \boldsymbol{a}_i; \ \lambda_i \in \left\{0, \frac{1}{k}, ..., \frac{k-1}{k}, 1\right\}, \ i = 1, 2, 3; \ \sum_{i=1}^3 \lambda_i = 1\right\}, \tag{4.16}$$

and the set

$$\Sigma_k(K) = \left\{\phi_{\boldsymbol{x}} : \ P_k(K) \to \mathbb{R}, \ \phi_{\boldsymbol{x}}(p) = p(\boldsymbol{x}), \ \boldsymbol{x} \in L_k(K)\right\}. \tag{4.17}$$

E.g. for $k = 1$ the set $L_k(K)$ consists of vertices $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ of $K$ as can be seen on Figure 4.1
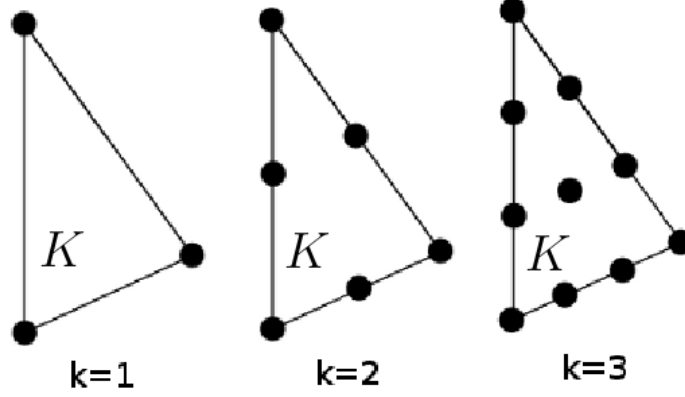


Figure 4.1: Visualization of the sets $L_k(K)$

It is possible to check that

$$\dim(P_k(K)) = \dim(L_k(K)) = \dim(\Sigma_k(K)),$$

and that every function $p \in P_k(K)$ is uniquely determined by the values $(\phi(p), \; \phi \in \Sigma_k(K))$. Then we can create the basis of $P_k(K)$ 'dual' to $\Sigma_k(K)$ :

$$\{p_i : \phi_i(p_i) = 1, \; \phi_j(p_i) = 0, \; i \neq j; \; \phi_i, \phi_j \in \Sigma_k(K)\}_{i=1}^{dim(\Sigma_k(K))}.$$

Linear independence is obvious, the fact that it is a basis follows from the equality between dimensions.

We consider $k = 1, 2$ and for each $K \in \mathcal{T}_h$ we get finite elements $(K, P_1(K), \Sigma_1(K))$ and $(K, P_2(K), \Sigma_2(K))$.
We shall denote $\mathcal{M}_h^k = \bigcup_{K \in \mathcal{T}_h} L_k(K)$, $k = 1, 2$ (for $k = 1$ this is the set of vertices of all triangles forming the triangulation $\mathcal{T}_h$, for $k = 2$ it includes midpoints of their sides as well) and we can write $\dim(\mathcal{M}_h^k) = \mathcal{N}_h^k$, $\mathcal{M}_h^k = \{\boldsymbol{z}_i\}_{i=1}^{\mathcal{N}_h^k}$, $k = 1, 2$ and for $\boldsymbol{z}_i \in \mathcal{M}_h^k$, $k = 1, 2$ it is possible to define the set of all elements of triangulation sharing the node $\boldsymbol{z}_i$:

$$\mathcal{T}_h^i = \{K \in \mathcal{T}_h; \; \boldsymbol{z}_i \in K\}.$$

The finite element spaces can be now defined as

$$\mathcal{Q}_h = \{v_h \in \mathcal{L}^2\left(\Omega\right) ;\ v_h|_K \in P_1\left(K\right) \forall K \in \mathcal{T}_h;\ v_h|_{K^1}\left(\boldsymbol{z}_i\right) = v_h|_{K^2}\left(\boldsymbol{z}_i\right)$$
$$\forall K^1, K^2 \in \mathcal{T}_h^i,\ i = 1, ..., \mathcal{N}_h^1\}.$$

$$\tilde{\mathcal{V}}_h = \{v_h \in \mathcal{L}^2\left(\Omega\right) ;\ v_h|_K \in P_2\left(K\right) \forall K \in \mathcal{T}_h;\ v_h|_{K^1}\left(\boldsymbol{z}_i\right) = v_h|_{K^2}\left(\boldsymbol{z}_i\right)$$
$$\forall K^1, K^2 \in \mathcal{T}_h^i,\ i = 1, ..., \mathcal{N}_h^2\}.$$

It can be shown that the resulting spaces can be summarized in this way:

$$\begin{aligned}
\mathcal{Q}_h &= \left\{q_h \in \mathcal{C}\left(\overline{\Omega}\right) ;\ q_h|_K \in P_1\left(K\right)\ \forall K \in \mathcal{T}_h\right\}, \\
\tilde{\mathcal{V}}_h &= \left\{v_h \in \mathcal{C}\left(\overline{\Omega}\right) ;\ v_h|_K \in P_2\left(K\right)\ \forall K \in \mathcal{T}_h\right\}, \\
\mathcal{V}_h &= \tilde{\mathcal{V}}_h \times \tilde{\mathcal{V}}_h, \\
&= \left\{\boldsymbol{v}_h \in \left[\mathcal{C}\left(\overline{\Omega}\right)\right]^2 ;\ \boldsymbol{v}_h|_K \in \left[P_2\left(K\right)\right]^2\ \forall K \in \mathcal{T}_h\right\}, \\
\mathcal{Y}_h &= \left\{\boldsymbol{y}_h \in \mathcal{V}_h :\ \boldsymbol{y}_h|_{\Gamma_D \cup \Gamma_{W_t}} = 0\right\}.
\end{aligned}$$

This defines the so-called *Taylor-Hood* $P_2/P_1$ elements, for which the couple $(\mathcal{Y}_h, \mathcal{Q}_h)$ satisfies the BB condition. See, e.g. [3], Corollary 4.1.

In order to construct bases of these function spaces we can proceed in a similar way in which we have constructed the basis of the space $P_k\left(K\right)$ dual to $\Sigma_k\left(K\right)$ on one element of triangulation. We can put

$$p_i,\ i = 1, ..., \mathcal{N}_h^k :\ p_i\left(z_j\right) = \delta_{ij},\ z_j \in \mathcal{M}_h^k,\ j = 1, ..., \mathcal{N}_h^k,\ k = 1, 2.$$

The functions $p_i,\ i = 1, ..., \mathcal{N}_h^k$ now form the basis of the space $\mathcal{Q}_h$ for $k = 1$ and the basis of the space $\tilde{\mathcal{V}}_h$ for $k = 2$ respectively. The basis of $\mathcal{V}_h$ will be constructed from the basis of $\tilde{\mathcal{V}}_h$ in this way: being

$$\left\{\tilde{v}^1, ..., \tilde{v}^{\mathcal{N}_h^2}\right\}$$

the basis of $\tilde{\mathcal{V}}_h$, the functions

$$\left\{\left(\tilde{v}^1, 0\right), ..., \left(\tilde{v}^{\mathcal{N}_h^2}, 0\right), \left(0, \tilde{v}^1\right), ..., \left(0, \tilde{v}^{\mathcal{N}_h^2}\right)\right\},$$

will form the basis of $\mathcal{V}_h$.

At the end of this section let us define the approximate solution:
The pair $V_h = \left(\boldsymbol{v}_h, p_h\right) \in \left(\mathcal{V}_h \times \mathcal{Q}_h\right)$ is an approximate solution of (4.9), (4.10) with the boundary conditions (4.2) - (4.4), if

$$a\left(V_h, V_h, Y_h\right) = f\left(Y_h\right)\ \ \forall Y_h = \left(\boldsymbol{y}_h, q_h\right) \in \mathcal{Y}_h \times \mathcal{Q}_h, \tag{4.18}$$

and $\boldsymbol{v}_h$ fulfills the following approximations of the boundary conditions (4.2) and (4.3):

$$\begin{aligned}
\boldsymbol{v}_h\left(\boldsymbol{z}\right) &= \boldsymbol{v}_D\left(\boldsymbol{z}, t\right),\ \forall \boldsymbol{z} \in \mathcal{M}_h^2 \cap \Gamma_D, \tag{4.19} \\
\boldsymbol{v}_h\left(\boldsymbol{z}\right) &= \boldsymbol{w}\left(\boldsymbol{z}\right),\ \forall \boldsymbol{z} \in \mathcal{M}_h^2 \cap \Gamma_{W_t}, \\
t &= t_{n+1}.
\end{aligned}$$

This discrete problem has two major difficulties to handle. First, for large Reynolds numbers (see Section 3.3), as $10^5 - 10^6$, which appear e.g. in aerodynamics of airplanes, the standard discretization suffers from the so-called *Gibbs phenomenon*, manifested by the presence of nonphysical spurious oscillations. Moreover, the flow characterized by such high Reynolds number is usually turbulent, but in this work we are concerned with classical Navier-Stokes equations without any turbulence model. The second problem is the nonlinearity of the convective term $(((\boldsymbol{v} - \boldsymbol{w}) \cdot \nabla) \boldsymbol{v})$, which restrains us from direct transformation of the problem into systems of linear equations. We shall deal with these problems in the next two sections.

## 4.4 Stabilization of FEM

In order to diminish the effects of the Gibbs phenomenon, we can employ the suitable stabilization to our discrete scheme. Following [4], [6], [7] we use the streamline-upwind/Petrov-Galerkin (SUPG, streamline-diffusion) method combined with the div-div stabilization that both consist of adding certain terms to our discrete equation (4.18). The stabilization terms are defined as

$$\mathcal{L}_h \left(V^*, V, Y\right) = \sum_{K \in \mathcal{T}_h} \delta_K \left( \frac{3}{2\tau} \boldsymbol{v} - \nu \triangle \boldsymbol{v} + \left(\overline{\boldsymbol{w}} \cdot \nabla\right) \boldsymbol{v} + \nabla p, \left(\overline{\boldsymbol{w}} \cdot \nabla\right) \boldsymbol{y} \right)_K, \quad (4.20)$$

$$\mathcal{F}_h \left(Y\right) = \sum_{K \in \mathcal{T}_h} \delta_K \left( \frac{1}{2\tau} \left(4\tilde{\boldsymbol{v}}^n - \tilde{\boldsymbol{v}}^{n-1}\right), \left(\overline{\boldsymbol{w}} \cdot \nabla\right) \boldsymbol{y} \right)_K, \quad (4.21)$$

$$\mathcal{P}_h \left(V, Y\right) = \sum_{K \in \mathcal{T}_h} \tau_K \left(\nabla \cdot \boldsymbol{v}, \nabla \cdot \boldsymbol{y}\right)_K, \quad (4.22)$$

where $V = \left(\boldsymbol{v}, p\right)$, $V^* = \left(\boldsymbol{v}^*, p\right)$, $Y = \left(\boldsymbol{y}, q\right)$, $(\cdot, \cdot)_K$ represents both the scalar product in $\mathcal{L}^2 \left(K\right)$ and in $\left[\mathcal{L}^2 \left(K\right)\right]^2$ and $\overline{\boldsymbol{w}} = \boldsymbol{v}^* - \boldsymbol{w}$ is the transport velocity appearing in the convective terms. Further, $\delta_K \geq 0$, $\tau_K \geq 0$ are suitable parameters.

The *stabilized discrete problem* now reads: Find $V_h = \left(\boldsymbol{v}_h, p_h\right) \in \left(\mathcal{V}_h \times \mathcal{Q}_h\right)$ such that $\boldsymbol{v}_h$ satisfies the approximations of the boundary conditions (4.2) and (4.3) given by (4.19) and the following holds:

$$a \left(V_h, V_h, Y_h\right) + \mathcal{L}_h \left(V_h, V_h, Y_h\right) + \mathcal{P}_h \left(V_h, Y_h\right) = f \left(Y_h\right) + \mathcal{F}_h \left(Y_h\right) \quad (4.23)$$
$$\forall Y_h = \left(\boldsymbol{y}_h, q_h\right) \in \left(\mathcal{Y}_h \times \mathcal{Q}_h\right).$$

For the choice of parameters $\delta_K$, $\tau_K$, extensive theoretical analysis of the problem at hand is necessary. Here we limit ourselves to a choice according to [4], [7] in the following fashion:

$$\delta_K = \delta^* \frac{h_K}{2 \left|\left|\overline{\boldsymbol{w}}\right|\right|_{L^\infty(K)}} \xi \left(Re^{\overline{\boldsymbol{w}}}\right); \ \tau_K = \tau^*, \quad (4.24)$$

where

$$Re^{\overline{w}} = \frac{h_K \, ||\overline{w}||_{L^\infty(K)}}{2\nu} \qquad (4.25)$$

is the local Reynolds number, $h_K$ is the size of element $K$ measured in the direction of $\overline{w}$ and $|| \cdot ||_{L^\infty(K)}$ stands for the norm in the space $L^\infty(K)$:

$$||\overline{w}||_{L^\infty(K)} = \operatorname*{ess\,sup}_{x \in K} |\overline{w}(x)|.$$

The factor $\xi\left(Re^{\overline{w}}\right)$ is defined as

$$\xi\left(Re^{\overline{w}}\right) = \min\left(\frac{Re^{\overline{w}}}{6}, 1\right),$$

and $\delta^*$, $\tau^* \in (0, 1]$ are additional free parameters. For $P_2/P_1$ elements it is possible to set $\tau^* = 1$.

## 4.5 Linearization of the problem - Oseen iterations

In our stabilized discrete problem (4.23) we have two terms which contain the nonlinear convective term $(((v - w) \cdot \nabla) \, v)$. We can define a sequence of linear problems, so that the solutions of these problems will gradually tend to the solution of (4.23). We define the sequence of problems for $j \in \mathbb{N}_0$ in the following way: Find a solution $V_h^{j+1} = \left(v_h^{j+1}, p_h^{j+1}\right)$ such that $v_h^{j+1}$ satisfies the approximations of the boundary conditions (4.2) and (4.3) given by (4.19) and

$$a\left(V_h^j, V_h^{j+1}, Y_h\right) + \mathcal{L}_h\left(V_h^j, V_h^{j+1}, Y_h\right) + \mathcal{P}_h\left(V_h^{j+1}, Y_h\right) = f\left(Y_h\right) + \mathcal{F}_h\left(Y_h\right) \quad (4.26)$$
$$\forall Y_h = \left(y_h, q_h\right) \in \left(\mathcal{Y}_h \times \mathcal{Q}_h\right).$$

The initial approximation $V_h^0$ can be set as the solution from the previous time level:

$$V_h^0 = \left(v_h^0, p_h^0\right) = \left(\tilde{v}_h^n, \tilde{p}_h^n\right),$$

or as an extrapolation from the two previous time levels:

$$V_h^0 = \left(2\tilde{v}_h^n - \tilde{v}_h^{n-1}, 2\tilde{p}_h^n - \tilde{p}_h^{n-1}\right),$$

where we shall remind the notation

$$\tilde{v}^i = v^i \circ \mathcal{A}_{t_i} \circ \mathcal{A}_{t_{n+1}}^{-1}, \quad \tilde{p}^i = p^i \circ \mathcal{A}_{t_i} \circ \mathcal{A}_{t_{n+1}}^{-1} \ i = n - 1, n.$$

Numerical experiments show that only a few iterations of (4.26) have to be computed at each time level.

## 4.6 Algebraic interpretation

Now we want to transform the linearized problem with the approximation of the boundary conditions into the system of algebraic equations. First, let us denote by $\overline{\boldsymbol{v}}_h$ the function from the space $\mathcal{V}_h$, approximating the boundary conditions in the following way:

$$
\begin{aligned}
\overline{\boldsymbol{v}}_h\left(\boldsymbol{z}\right) &= \boldsymbol{v}_D\left(\boldsymbol{z}, t\right),\ \forall\, \boldsymbol{z} \in \mathcal{M}_h^2 \cap \Gamma_D,\ t = t_{n+1}, \\
\overline{\boldsymbol{v}}_h\left(\boldsymbol{z}\right) &= \boldsymbol{w}\left(\boldsymbol{z}\right),\ \forall\, \boldsymbol{z} \in \mathcal{M}_h^2 \cap \Gamma_{W_t}, \\
\overline{\boldsymbol{v}}_h\left(\boldsymbol{z}\right) &= 0,\ \forall\, \boldsymbol{z} \in \mathcal{M}_h^2 \backslash \left(\Gamma_D \cup \Gamma_{W_t}\right).
\end{aligned}
$$

Then, due to the linearity of (4.26), we can write

$$
\boldsymbol{v}_h^{j+1} = \overline{\boldsymbol{v}}_h + \boldsymbol{u}_h^{j+1},\quad \boldsymbol{u}_h^{j+1} \in \mathcal{Y}_h, \tag{4.27}
$$

where $V_h^{j+1} = \left(\boldsymbol{v}_h^{j+1}, p_h^{j+1}\right)$ is the solution of (4.26). Moreover, since we have previously constructed the bases of the spaces $\mathcal{Y}_h$ and $\mathcal{Q}_h$ (see Section 4.3), we can write:

$$
\boldsymbol{u}_h^{j+1} = \sum_{\alpha=1}^{R} u_\alpha \boldsymbol{y}_\alpha^* \tag{4.28}
$$

$$
p_h^{j+1} = \sum_{\beta=1}^{S} p_\beta q_\beta^*, \tag{4.29}
$$

where $R = 2\mathcal{N}_h^2$, $\{\boldsymbol{y}_\alpha^*\}_{\alpha=1}^{R}$ is the basis of $\mathcal{Y}_h$, $S = \mathcal{N}_h^1$ and $\{q_\beta^*\}_{\beta=1}^{S}$ is the basis of $\mathcal{Q}_h$.

Taking into account linearity of the forms appearing in (4.26), we can use solely the basis functions as test functions in (4.26). By substituting the function $Y_h = (\boldsymbol{y}_i^*, q_k^*)$, $i \in \{1, ..., R\}$, $k \in \{1, ..., S\}$ and relations (4.27), (4.28) and (4.29)

into (4.26), we get

$$
\begin{aligned}
a\left(V_h^j, V_h^{j+1}, Y_h\right) &= \frac{3}{2\tau} \sum_{\alpha=1}^{R} \left(\boldsymbol{y}_\alpha^*, \boldsymbol{y}_i^*\right) u_\alpha + \nu \sum_{\alpha=1}^{R} \left(\left(\boldsymbol{y}_\alpha^*, \boldsymbol{y}_i^*\right)\right) u_\alpha + \\
&+ \sum_{\alpha=1}^{R} \left(\left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right) \boldsymbol{y}_\alpha^*, \boldsymbol{y}_i^*\right) u_\alpha - \sum_{\beta=1}^{S} \left(q_\beta, \nabla \cdot \boldsymbol{y}_i^*\right) p_\beta + \\
&+ \sum_{\alpha=1}^{R} \left(\nabla \cdot \boldsymbol{y}_\alpha^*, q_k^*\right) u_\alpha + \frac{3}{2\tau} \left(\overline{\boldsymbol{v}}_h, \boldsymbol{y}_i^*\right) + \nu \left(\left(\overline{\boldsymbol{v}}_h, \boldsymbol{y}_i^*\right)\right) + \\
&+ \left(\left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right) \overline{\boldsymbol{v}}_h, \boldsymbol{y}_i^*\right) + \left(\nabla \cdot \overline{\boldsymbol{v}}_h, q_k^*\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_h\left(V_h^j, V_h^{j+1}, Y_h\right) &= \\
&= \sum_{K \in \mathcal{T}_h} \delta_K \left[\sum_{\alpha=1}^{R} \left(\frac{3}{2\tau}\boldsymbol{y}_\alpha^* - \nu \triangle \boldsymbol{y}_\alpha^* + \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right) \boldsymbol{y}_\alpha^*, \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right) \boldsymbol{y}_i^*\right)_K u_\alpha\right] + \\
&+ \sum_{K \in \mathcal{T}_h} \delta_K \left[\sum_{\beta=1}^{S} \left(\nabla q_\beta^*, \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right) \boldsymbol{y}_i^*\right)_K p_\beta\right] + \\
&+ \sum_{K \in \mathcal{T}_h} \delta_K \left(\frac{3}{2\tau}\overline{\boldsymbol{v}}_h - \nu \triangle \overline{\boldsymbol{v}}_h + \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right) \overline{\boldsymbol{v}}_h, \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right) \boldsymbol{y}_i^*\right)_K,
\end{aligned}
$$

$$
f\left(Y_h\right) = \frac{1}{2\tau} \left(4\tilde{\boldsymbol{v}}^n - \tilde{\boldsymbol{v}}^{n-1}, \boldsymbol{y}_i^*\right) - \int_{\Gamma_O} p_{ref} \left(\boldsymbol{n} \cdot \boldsymbol{y}_i^*\right) \mathrm{d}S,
$$

$$
\mathcal{F}_h\left(Y_h\right) = \sum_{K \in \mathcal{T}_h} \delta_K \left(\frac{1}{2\tau} \left(4\tilde{\boldsymbol{v}}^n - \tilde{\boldsymbol{v}}^{n-1}\right), \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right) \boldsymbol{y}_i^*\right)_K,
$$

$$
\begin{aligned}
\mathcal{P}_h\left(V_h^{j+1}, Y_h\right) &= \sum_{K \in \mathcal{T}_h} \tau_K \left[\sum_{\alpha=1}^{R} \left(\nabla \cdot \boldsymbol{y}_\alpha^*, \nabla \cdot \boldsymbol{y}_i^*\right)_K u_\alpha\right] + \\
&+ \sum_{K \in \mathcal{T}_h} \tau_K \left(\nabla \cdot \overline{\boldsymbol{v}}_h, \nabla \cdot \boldsymbol{y}_i^*\right)_K,
\end{aligned}
$$

where $\overline{\boldsymbol{w}}_h^j = \left(\boldsymbol{v}_h^j - \boldsymbol{w}\right)$ and the symbol $((\boldsymbol{f}, \boldsymbol{g}))$ stands for $(\nabla \boldsymbol{f}, \nabla \boldsymbol{g})$. Now we can set

$$
\begin{aligned}
a_{i\alpha} &= \frac{3}{2\tau}\left(\boldsymbol{y}_\alpha^*, \boldsymbol{y}_i^*\right) + \nu\left(\left(\boldsymbol{y}_\alpha^*, \boldsymbol{y}_i^*\right)\right) + \left(\left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right)\boldsymbol{y}_\alpha^*, \boldsymbol{y}_i^*\right) + \\
&\quad + \sum_{K \in \mathcal{T}_h} \delta_K\left(\frac{3}{2\tau}\boldsymbol{y}_\alpha^* - \nu\triangle\boldsymbol{y}_\alpha^* + \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right)\boldsymbol{y}_\alpha^*, \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right)\boldsymbol{y}_i^*\right)_K + \\
&\quad + \sum_{K \in \mathcal{T}_h} \tau_K\left(\nabla \cdot \boldsymbol{y}_\alpha^*, \nabla \cdot \boldsymbol{y}_i^*\right)_K, \\
b_{i\beta} &= -\left(q_\beta^*, \nabla \cdot \boldsymbol{y}_i^*\right), \\
c_{i\beta} &= \sum_{K \in \mathcal{T}_h} \delta_K\left(\nabla q_\beta^*, \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right)\boldsymbol{y}_i^*\right)_K, \\
F_i &= \frac{1}{2\tau}\left(4\tilde{\boldsymbol{v}}^n - \tilde{\boldsymbol{v}}^{n-1}, \boldsymbol{y}_i^*\right) - \int_{\Gamma_O} p_{ref}\left(\boldsymbol{n} \cdot \boldsymbol{y}_i^*\right)\mathrm{d}S + \\
&\quad + \sum_{K \in \mathcal{T}_h} \delta_K\left(\frac{1}{2\tau}\left(4\tilde{\boldsymbol{v}}^n - \tilde{\boldsymbol{v}}^{n-1}\right), \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right)\boldsymbol{y}_i^*\right)_K - \\
&\quad - \sum_{K \in \mathcal{T}_h} \delta_K\left(\frac{3}{2\tau}\overline{\boldsymbol{v}}_h - \nu\triangle\overline{\boldsymbol{v}}_h + \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right)\overline{\boldsymbol{v}}_h, \left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right)\boldsymbol{y}_i^*\right)_K - \\
&\quad - \frac{3}{2\tau}\left(\overline{\boldsymbol{v}}_h, \boldsymbol{y}_i^*\right) - \nu\left(\left(\overline{\boldsymbol{v}}_h, \boldsymbol{y}_i^*\right)\right) + \\
&\quad - \left(\left(\overline{\boldsymbol{w}}_h^j \cdot \nabla\right)\overline{\boldsymbol{v}}_h, \boldsymbol{y}_i^*\right) - \sum_{K \in \mathcal{T}_h} \tau_K\left(\nabla \cdot \overline{\boldsymbol{v}}_h, \nabla \cdot \boldsymbol{y}_i^*\right)_K, \\
G_m &= \left(\nabla \cdot \overline{\boldsymbol{v}}_h, q_m^*\right).
\end{aligned}
$$

We obtain the following system of linear algebraic equations for the unknown constants $u_\alpha$, $\alpha = 1, ..., R$, $p_\beta$, $\beta = 1, ..., S$, where we split the equation (4.26), derived by summing and manipulating the equations (4.11) and (4.12), to get:

$$
\sum_{\alpha=1}^{R} a_{i\alpha} u_\alpha + \sum_{\beta=1}^{S} \left(b_{i\beta} + c_{i\beta}\right) p_\beta = F_i, \ i = 1, ..., R, \tag{4.30}
$$

$$
\sum_{\alpha=1}^{R} b_{\alpha m} u_\alpha = G_m, \ m = 1, ..., S. \tag{4.31}
$$

We can introduce the notation

$$
\begin{aligned}
U &= (u_1, ..., u_R), \\
P &= (p_1, ..., p_S), \\
\mathbb{A} &= (a_{ij})_{i,j=1}^{R}, \\
\mathbb{B} &= (b_{ij})_{i=1,...,R, \, j=1,...,S}, \\
\mathbb{C} &= (c_{ij})_{i=1,...,R, \, j=1,...,S}, \\
F &= (F_1, ..., F_R), \\
G &= (G_1, ..., G_S),
\end{aligned}
$$

with the aid of which we can rewrite the system of equations (4.30), (4.31) in the matrix form:

$$
\begin{pmatrix} A & B+C \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}. \tag{4.32}
$$

This sparse linear system is then solved in our case by the direct solver UMF-PACK [11] and thus one iteration step of an approximation of the solution of the stabilized discretized problem is obtained.

# Chapter 5

# Description of the numerical experiment

We shall solve the problem by means described in Chapter 4 implemented in the software FEMFLUID [9]. The main goal is to get acquainted with the software, to show the applicability of the developed method and to present a solution of a model example. We shall be concerned with the case when the movement of the modified airfoil NACA0012 is prescribed. The airfoil NACA0012 is modified in the sense of splitting the original airfoil into two parts separated by a slot - see Figure 5.2. Let us remind that the reference domain of the ALE mapping in our case coincides with the initial domain, i.e. $\Omega_{ref} = \Omega_0$. Figure 5.1 shows $\Omega_0$.
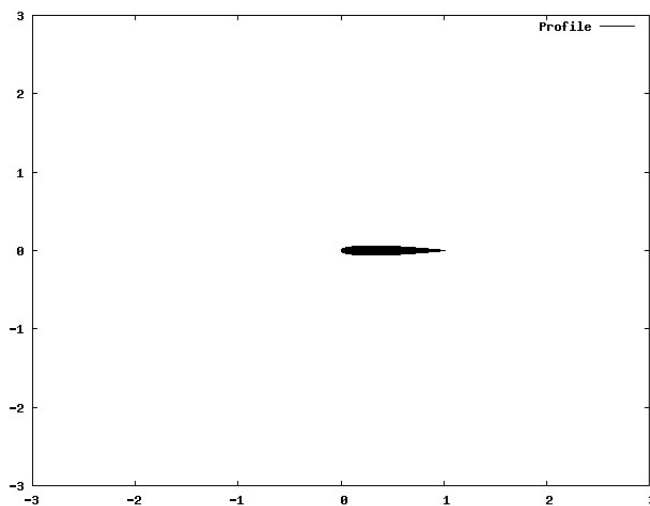


Figure 5.1: Reference domain $\Omega_{ref} = \Omega_0$

We shall describe the position of the airfoil $\Gamma_{W_t}$ at time $t$ by two parameters, *displacement* $H(t)$ and *rotation angle* $\gamma(t)$. We shall set up a point $E(t) = (\hat{x}_{E1}(t),$

$\hat{x}_{E2}(t)$) within the interior of the airfoil $\Gamma_{W_t}$, around which the airfoil rotates and which also has constant $\hat{x}_{E1}(t)$ coordinate in dependence on time. We shall call this point the *elastic axis*. As can be seen from Figure 5.2, we have $\hat{x}_{E1}(t) = 0.2$. We denote $E(0) = (\hat{x}_{E1}(0), \hat{x}_{E2}(0))$ by $E_0 = (\hat{x}_{E01}, \hat{x}_{E02})$. Let us now set $H(t) = \hat{x}_{E2}(t) - \hat{x}_{E02}$. The parameter $\gamma(t)$ represents the angle (measured in positive direction, i.e. counterclockwise) between the axis of symmetry of $\Gamma_{W_t}$ and the axis of symmetry of $\Gamma_{W_0}$ (see Figure 5.2). As can be seen we have $\hat{x}_{E02} = 0$ and the axis of symmetry of $\Gamma_{W_0}$ coincides with a part of the bisector given by $x_2 = 0$.
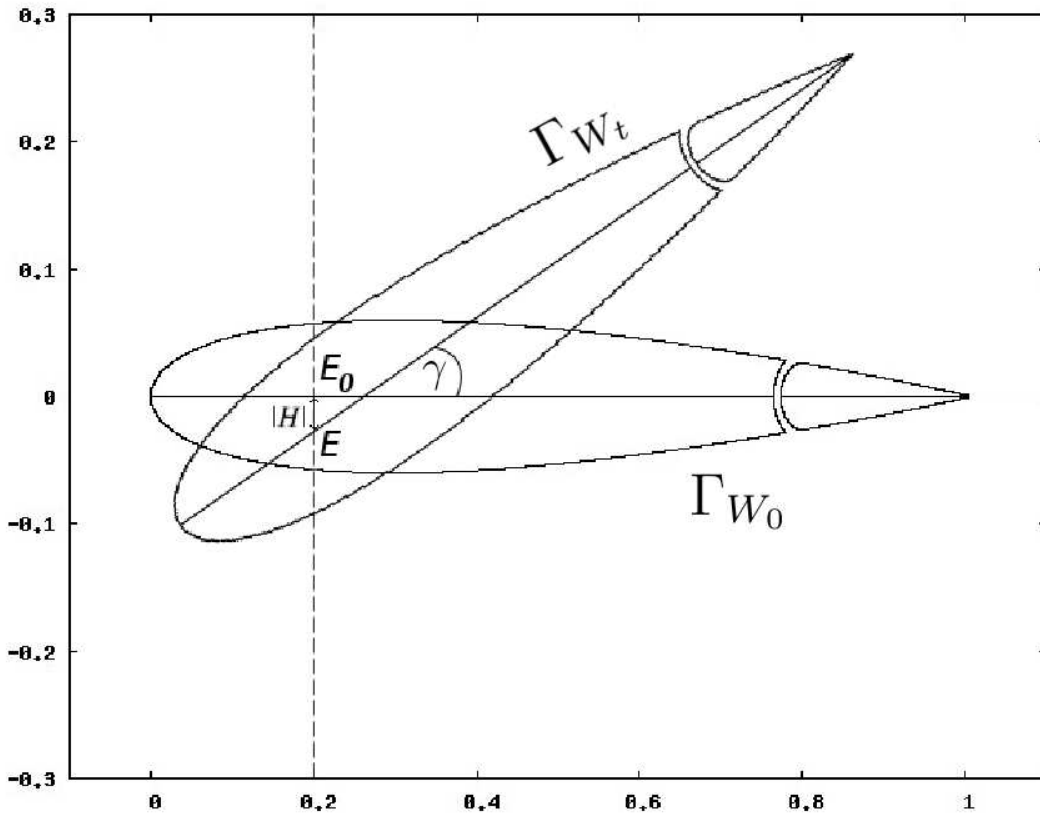


Figure 5.2: Airfoil position at $t = 0$ and shifted and rotated airfoil.

## 5.1 Mesh generation

The basic input for the computational process using FEM is the triangulation $\mathcal{T}_h$ (see Section 4.3) of the reference domain $\Omega_0$. We have to keep in mind that the suitable triangulation should have sufficient amount of elements in order to obtain a good approximation of the solution. On the other hand, with the increasing number of elements the complexity and computational time increases as well.

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There are several approaches to the construction of a triangulation $\mathcal{T}_h$. The idea used for generation of the mesh used in our simulation is to create a simple initial triangulation $\mathcal{T}_{h_0}$ consisting of possibly hundreds of elements. On this initial triangulation $\mathcal{T}_{h_0}$ we can compute or approximate (e.g. from the empirical knowledge of how the fluid behaves) the solution for several time instants very fast. Knowledge of this solution or approximation can be used for the refinement of the triangulation $\mathcal{T}_{h_0}$ in a suitable non-uniform way. In a real implementation this means to add more elements where the solution grows steeply and vice versa. On such refined triangulation, another approximate solution can be computed, the mesh can again be refined where appropriate, etc.

With this iterative process we can continue until the refined triangulation fits our needs, i.e. it is refined sufficiently to capture the solution and does not consist of too many elements. Software used for refining of the triangulation used in our case was ANGENER [8]. Examples of triangulations of the domain $\Omega_0$ with various numbers of elements are shown in Figures 5.3 - 5.6 (output from ANGENER plotted by GNUPLOT).
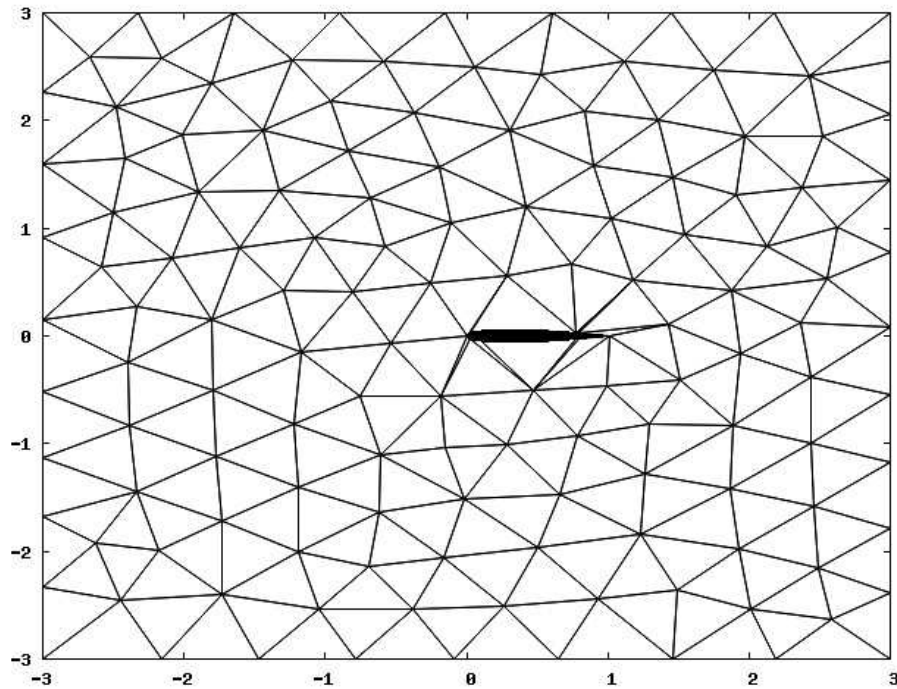


Figure 5.3: Number of elements $\approx$ 200.
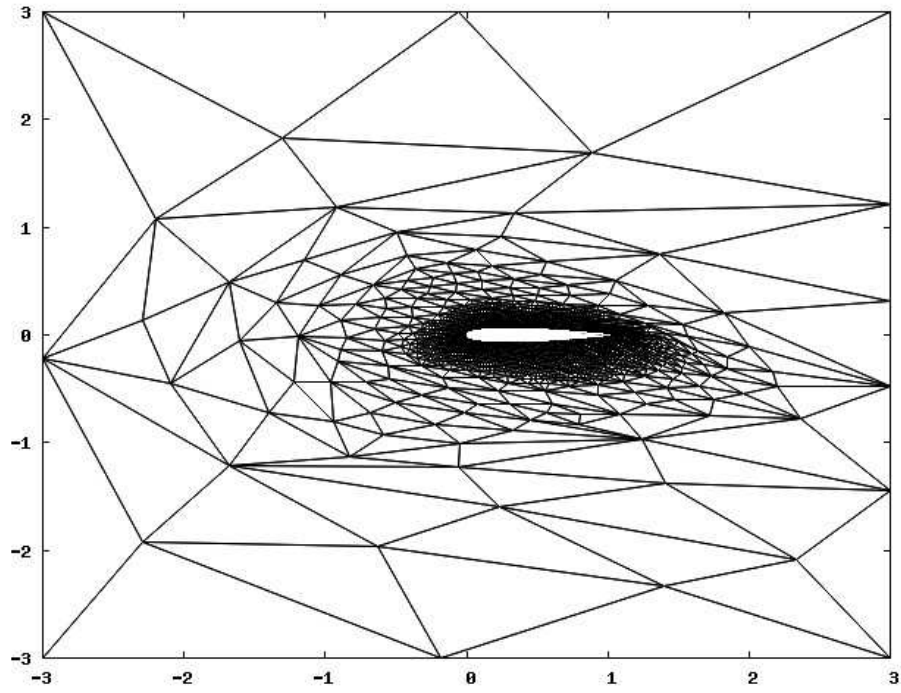
Figure 5.4: Number of elements ≈ 4 000.
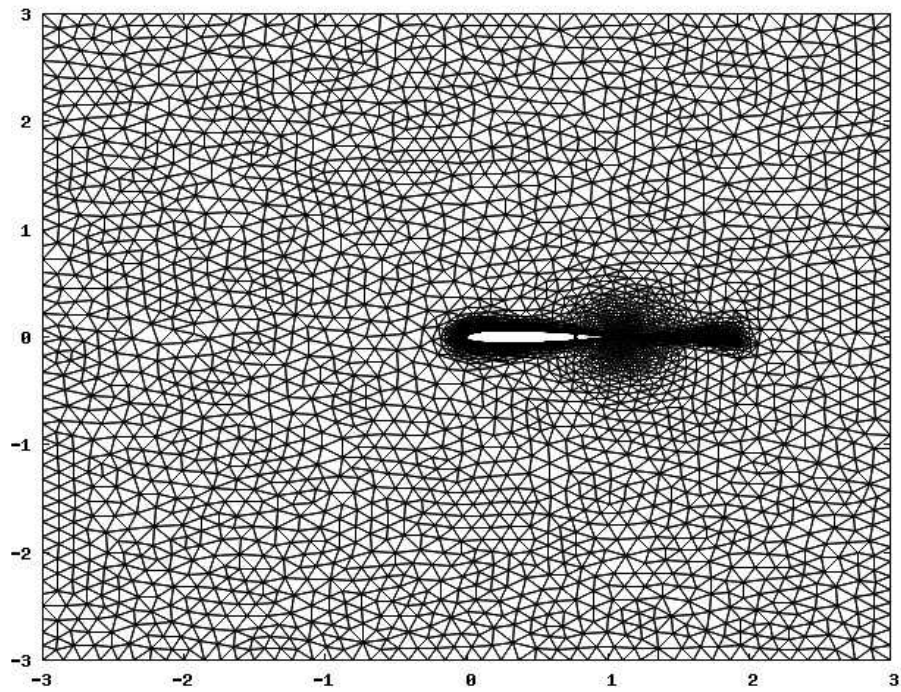


Figure 5.5: Number of elements ≈ 8 000.

46
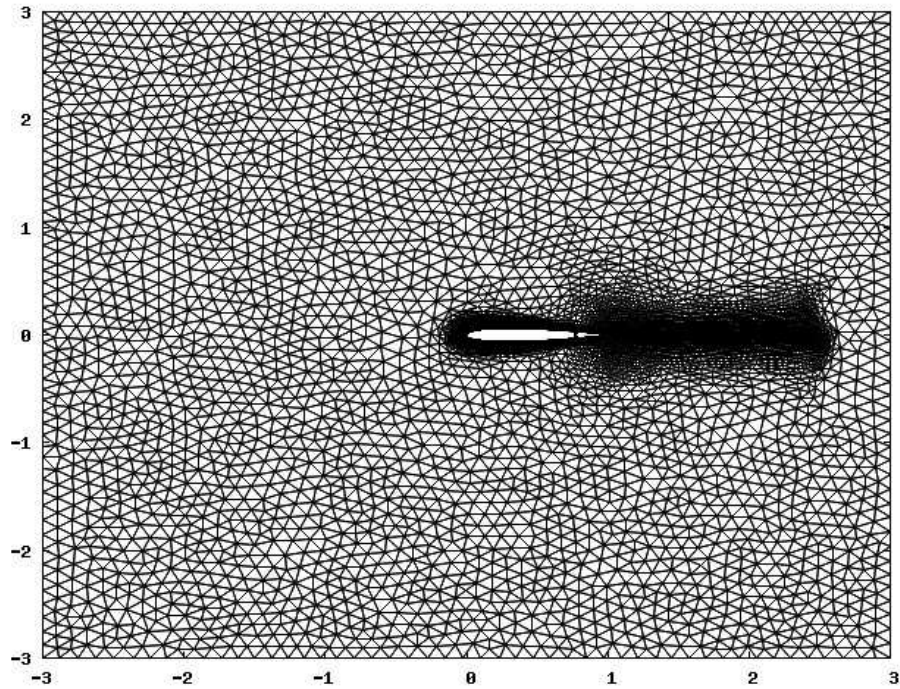
Figure 5.6: Number of elements = 12 248.

In Figure 5.6 we can see the final triangulation that was used for the actual computation. In Figures 5.7 and 5.8 we can see details of the triangulation around the slot between the two parts of the airfoil and behind the airfoil respectively.
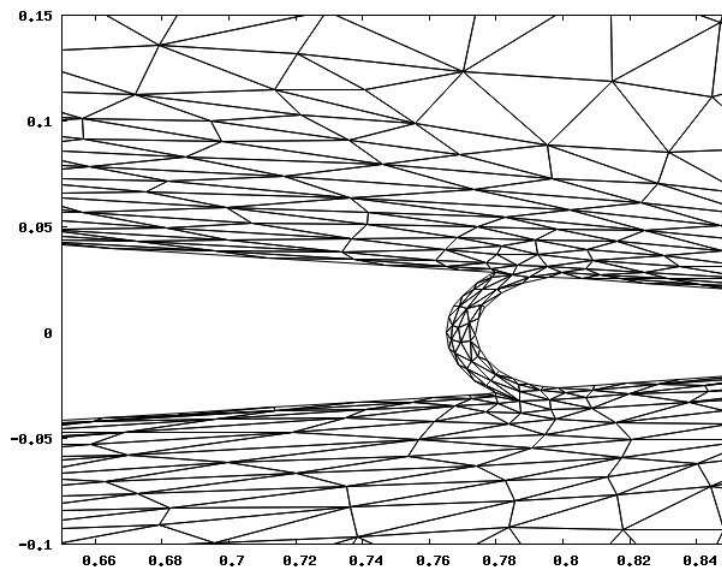


Figure 5.7: Detail of the triangulation in the slot.

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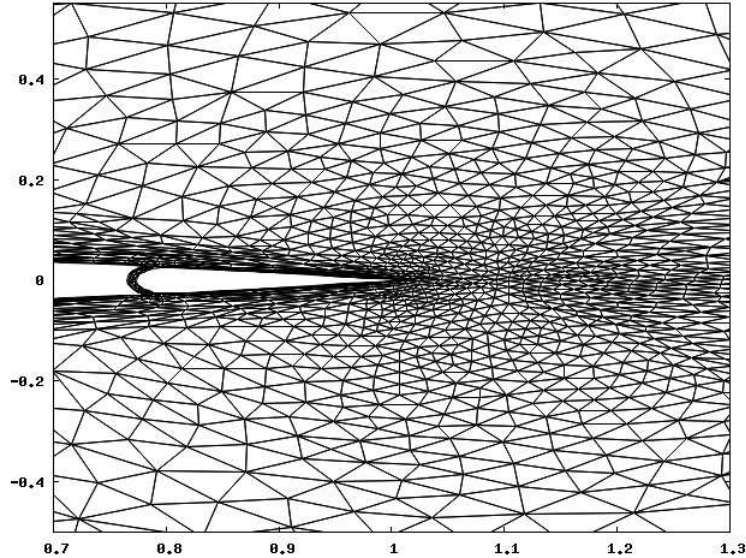
Figure 5.8: Detail of the triangulation behind the profile.

## 5.2 Construction of the ALE mapping

For the construction of the ALE mapping the following approach can be used. Let the displacement $H(t_i)$ and the rotation angle $\gamma(t_i)$ be known for $i = 0, ..., n+1$. We want to construct the ALE mapping $\mathcal{A}_{t_{n+1}} : \Omega_0 \rightarrow \Omega_{t_{n+1}}$. To this end, we consider two circles $C_1$, $C_2 \subset \Omega_0$ with center at the elastic axis and radii $R_1$, $R_2$, $0 < R_1 < R_2$ such that $\Gamma_{W_t}$ lies inside $C_1$. The ALE mapping is constructed in such a way that the inner circle $C_1$ moves together with the airfoil $\Gamma_{W_t}$ as a solid body, whereas the exterior $\Omega_0 \backslash C_2$ of the outer circle $C_2$ is not deformed. First for $t = t_{n+1}$ we introduce the mapping $\mathcal{H}_t(\hat{x}_1, \hat{x}_2)$, $\hat{\boldsymbol{X}} = (\hat{x}_1, \hat{x}_2) \in \Omega_0$, describing the motion of the airfoil:

$$\mathcal{H}_t(\hat{x}_1, \hat{x}_2) = \begin{pmatrix} \cos(\gamma(t)) & \sin(\gamma(t)) \\ -\sin(\gamma(t)) & \cos(\gamma(t)) \end{pmatrix} \cdot \begin{pmatrix} \hat{x}_1 - \hat{x}_{E01} \\ \hat{x}_2 - \hat{x}_{E02} \end{pmatrix} + \begin{pmatrix} \hat{x}_{E01} \\ \hat{x}_{E02} \end{pmatrix} + \begin{pmatrix} 0 \\ H(t) \end{pmatrix}.$$

Denoting by $\mathcal{I}d$ the identical mapping, the ALE mapping can be defined as

$$\mathcal{A}_{t_{n+1}}(\hat{x}_1, \hat{x}_2) = \theta \mathcal{H}_{t_{n+1}}(\hat{x}_1, \hat{x}_2) + (1 - \theta)\mathcal{I}d(\hat{x}_1, \hat{x}_2). \tag{5.1}$$

Here

$$\theta = \theta(\hat{r}) = \frac{\cos(\psi(\hat{r})) + 1}{2}, \tag{5.2}$$

$$\psi(\hat{r}) = \min\left(\max\left(0, \frac{\hat{r} - R_1}{R_2 - R_1}\right), 1\right),$$

and

$$\hat{r} = \hat{r}(\hat{x}_1, \hat{x}_2) = \sqrt{(\hat{x}_1 - \hat{x}_{E01})^2 + (\hat{x}_2 - \hat{x}_{E02})^2}. \qquad (5.3)$$

From this construction one can see that the ALE mapping is smooth in the domain $\Omega_0$. The knowledge of the ALE mapping at time instants $t_{n-1}$, $t_n$, $t_{n+1}$ allows us to approximate the domain velocity $\boldsymbol{w}^{n+1}$ with the aid of the difference scheme previously used for the velocity $\boldsymbol{v}$ as

$$\boldsymbol{w}^{n+1}(\boldsymbol{x}) = \frac{3\boldsymbol{x} - 4\mathcal{A}_{t_n}\left(\mathcal{A}_{t_{n+1}}^{-1}(\boldsymbol{x})\right) + \mathcal{A}_{t_{n-1}}\left(\mathcal{A}_{t_{n+1}}^{-1}(\boldsymbol{x})\right)}{2\tau}, \quad \boldsymbol{x} \in \Omega_{t_{n+1}}. \qquad (5.4)$$

## 5.3 Data

In the computation carried out with the aid of the FEMFLUID software we used the following data for our problem:

$$
\begin{aligned}
\boldsymbol{v}_0(\boldsymbol{x}) &= (v_1(\boldsymbol{x}), v_2(\boldsymbol{x})) = (1, 0), \ \boldsymbol{x} \in \Omega_0, \\
\boldsymbol{v}_D(\boldsymbol{x}) &= (v_1(\boldsymbol{x}), v_2(\boldsymbol{x})) = (1, 0), \ \boldsymbol{x} \in \Gamma_D, \\
p_{ref} &= 0, \\
\tau &= 2 \cdot 10^{-4}\, s, \\
T &= 2\, s, \\
\tilde{T} &= 1.998\, s, \\
\nu &= 10^{-4}, \\
L &= 1\, m, \\
U &= 1\, m/s, \\
H(t) &= H_{max} \sin(2\pi f_H t), \ H_{max} = 0.03\, m, \ f_H = 3\, Hz \\
\gamma(t) &= \gamma_{max} \sin(2\pi f_\gamma t), \ \gamma_{max} = 3°, \ f_\gamma = 3\, Hz.
\end{aligned}
$$

## 5.4 Results

In the numerical experiment, 3-5 Oseen's iterations were computed at each time level. This iterative process at each time level is designed to stop when the difference between two successive solutions drops below a prescribed value. Software used for the visualization of outputs from the FEMFLUID software was ParaView [10]. On the attached DVD many more outputs can be found.

**Velocity field around the flap**
In the following Figures we can observe the development of some characteristic features of the velocity field $\boldsymbol{v}$ around the flap and around the slot between the flap and the main part of the airfoil. Specific to this model of an airfoil is that the

fluid can move 'through' the airfoil. From the following figures one can see that even for such a small amplitudes of the airfoil's motion the distortion of the fluid in the slot is observable.
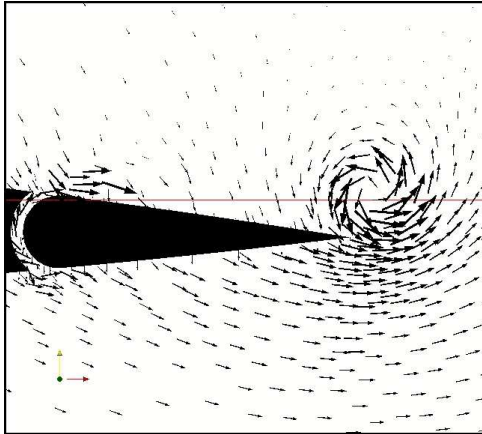


$t = 0.02s$, $\gamma = -1.10°$, $H = -11.04\,mm$.
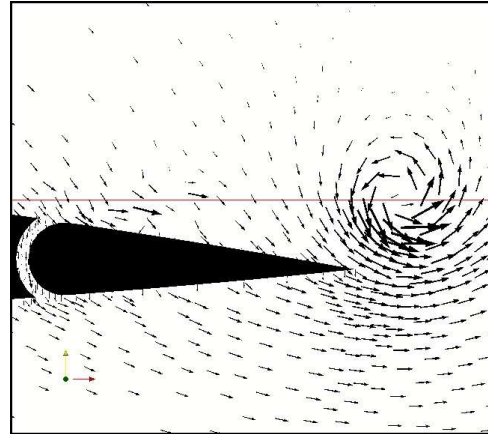


$t = 0.04\,s$, $\gamma = -2.05°$, $H = -20.54\,mm$.
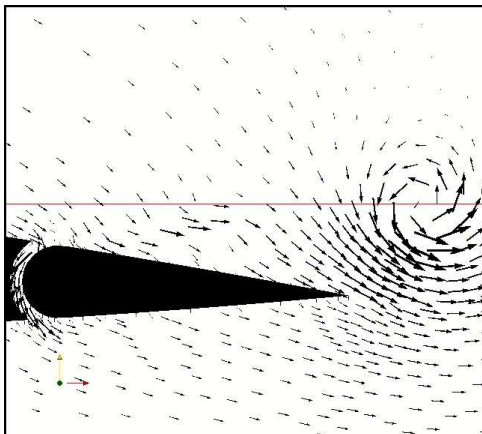


$t = 0.06s$, $\gamma = -2.71°$, $H = -27.14\,mm$.
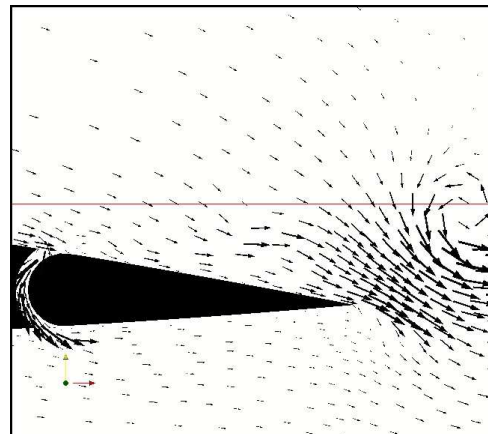


$t = 0.08\,s$, $\gamma = -2.99°$, $H = -29.94\,mm$.
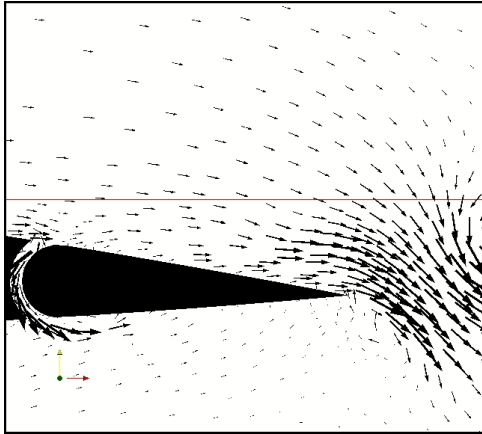
$t = 0.1s$, $\gamma = -2.85°$, $H = -28.53\,mm$.
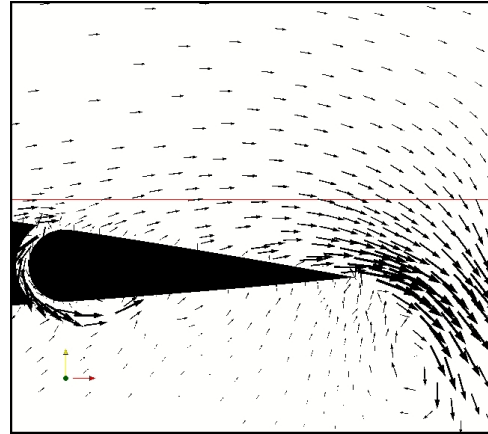


$t = 0.12\,s$, $\gamma = -2.31°$, $H = -23.12\,mm$.
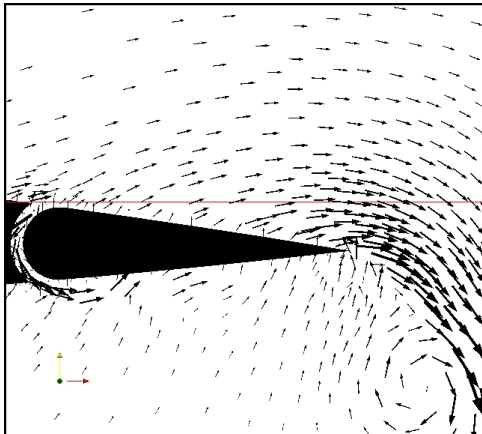


$t = 0.14s$, $\gamma = -1.45°$, $H = -14.45\,mm$.
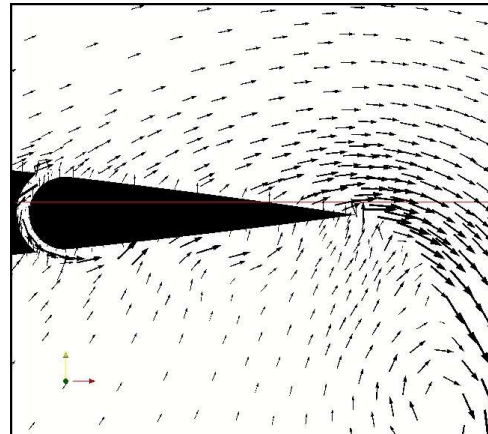


$t = 0.16\,s$, $\gamma = -0.38°$, $H = -3.76\,mm$.
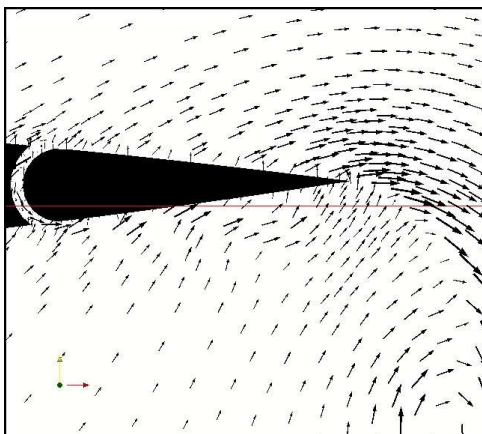


$t = 0.18s$, $\gamma = 0.75°$, $H = 7.46\,mm$.



$t = 0.2\,s$, $\gamma = 1.76°$, $H = 17.63\,mm$.

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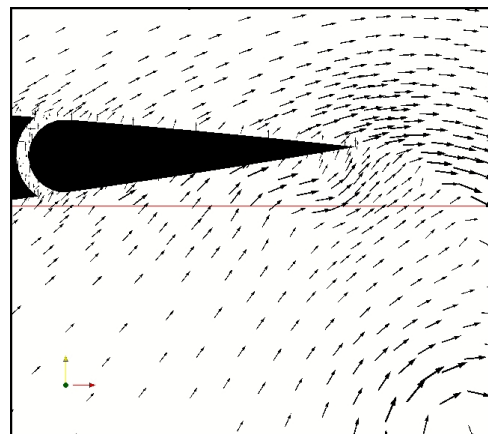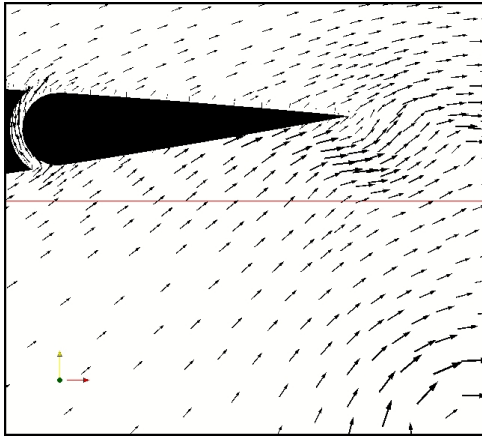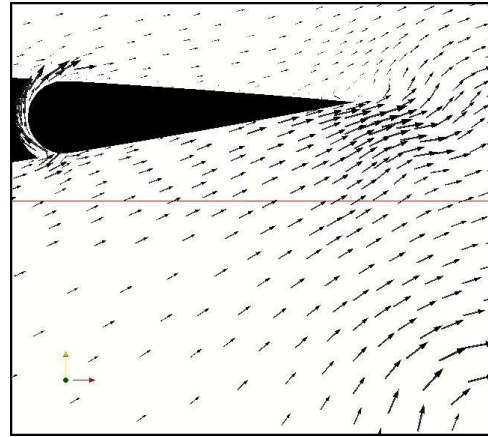
$t = 0.22s,\ \gamma = 2.53°,\ H = 25.33\,mm.$



$t = 0.24\,s,\ \gamma = 2.95°,\ H = 29.47\,mm.$
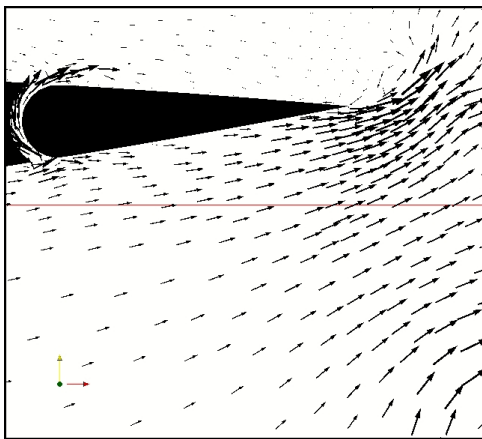


$t = 0.22s,\ \gamma = 2.95°,\ H = 29.47\,mm.$
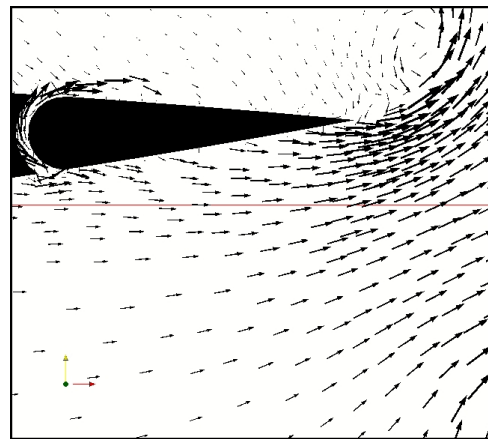


$t = 0.24\,s,\ \gamma = 2.53°,\ H = 25.33\,mm.$
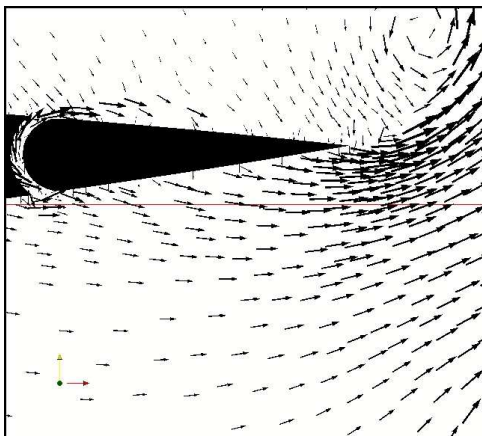


$t = 0.22s,\ \gamma = 1.76°,\ H = 17.63\,mm.$
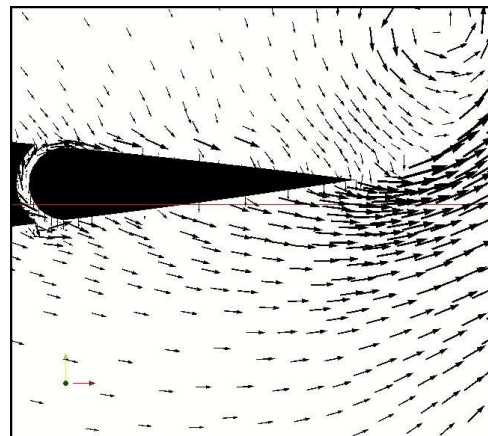


$t = 0.24\,s,\ \gamma = 0.75°,\ H = 7.46\,mm.$

## Isolines of the first component of the velocity

Development of characteristic features of the first component of the velocity in a broader view.
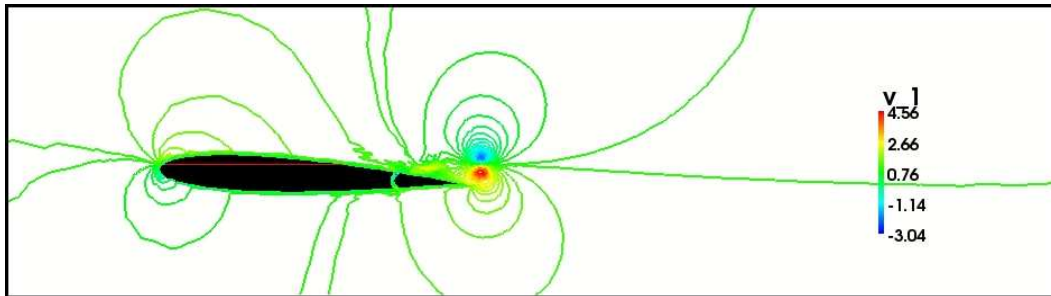


$v_1$ isolines; $t = 0.06\,s$, $\gamma = -2.71°$, $H = -27.14\,mm$.
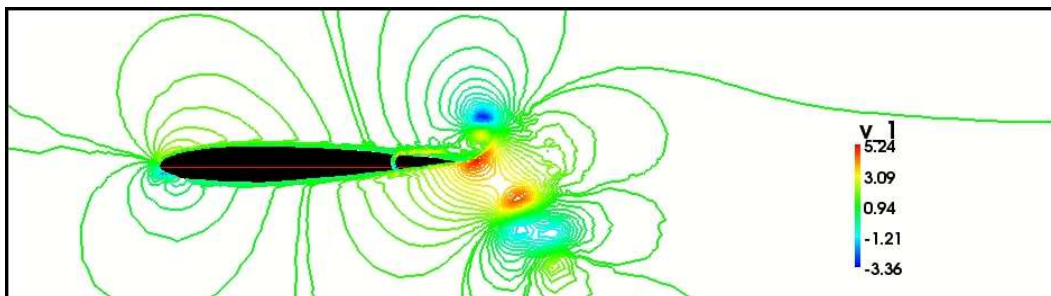


$v_1$ isolines; $t = 0.32\,s$, $\gamma = 0.75°$, $H = 7.46\,mm$.
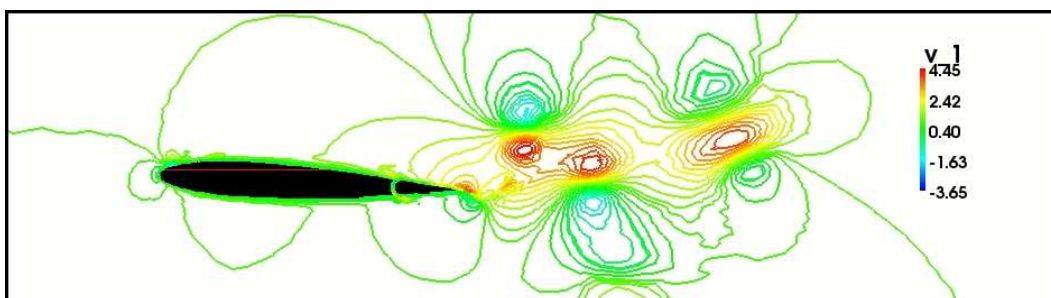


$v_1$ isolines; $t = 0.76\,s$, $\gamma = -2.95°$, $H = -29.47\,mm$.
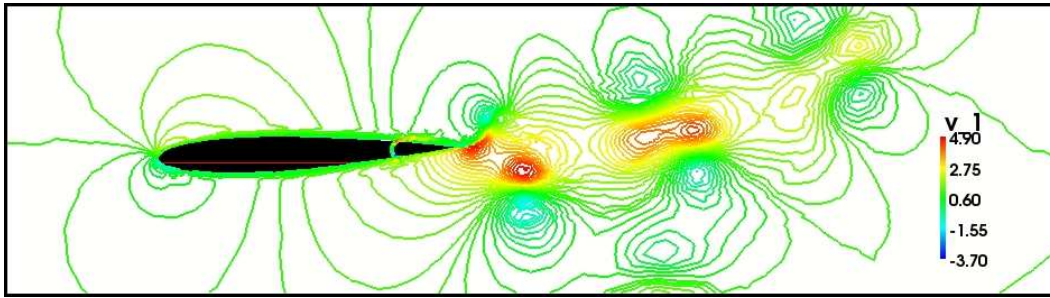
$v_1$ isolines; $t = 0.96\,s$, $\gamma = 2.05°$, $H = 20.54\,mm$.
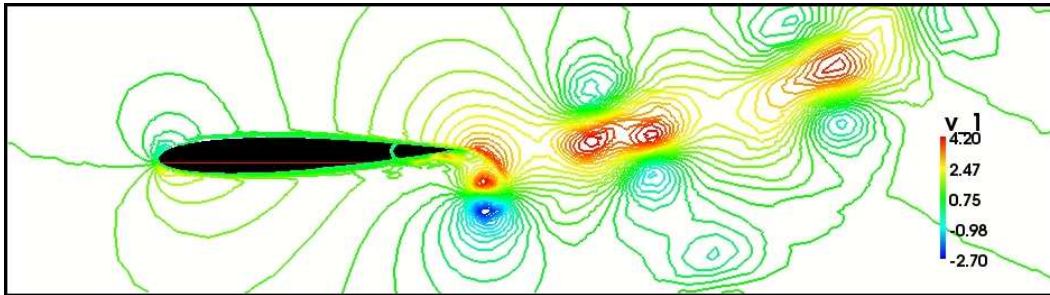


$v_1$ isolines; $t = 1.2\,s$, $\gamma = 1.76°$, $H = 17.63\,mm$.
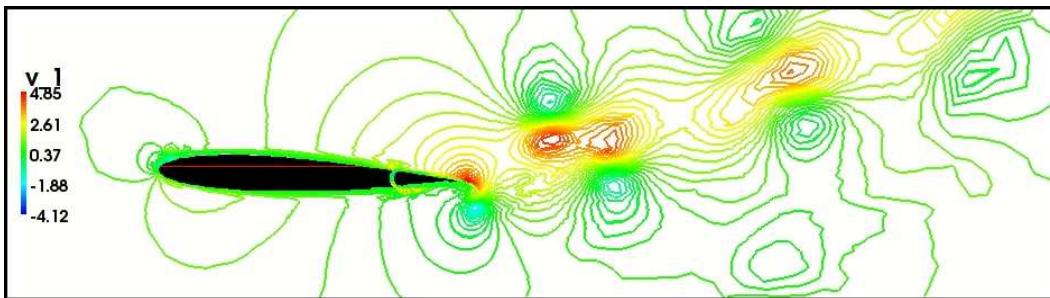


$v_1$ isolines; $t = 1.46\,s$, $\gamma = -2.05°$, $H = -20.54\,mm$.

**Isolines of the second component of the velocity**
As in the previous Figures we can see that an interesting vortex formation appears
in the flow past the moving airfoil.
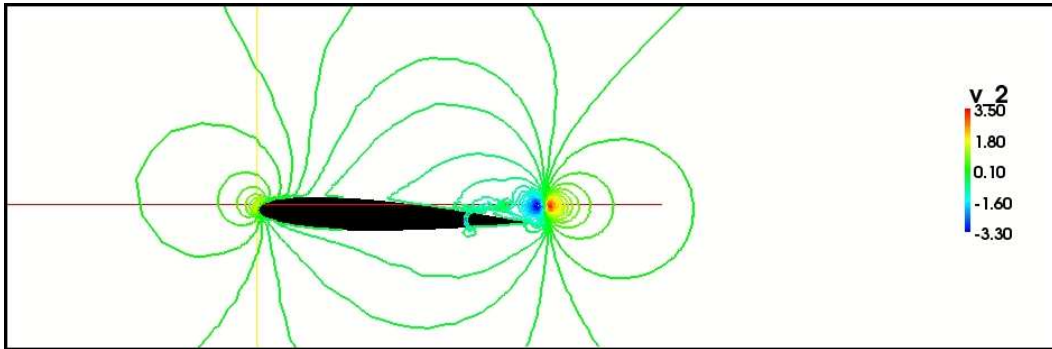
$v_2$ isolines; $t = 0.06\,s$, $\gamma = -2.71°$, $H = -27.14\,mm$.
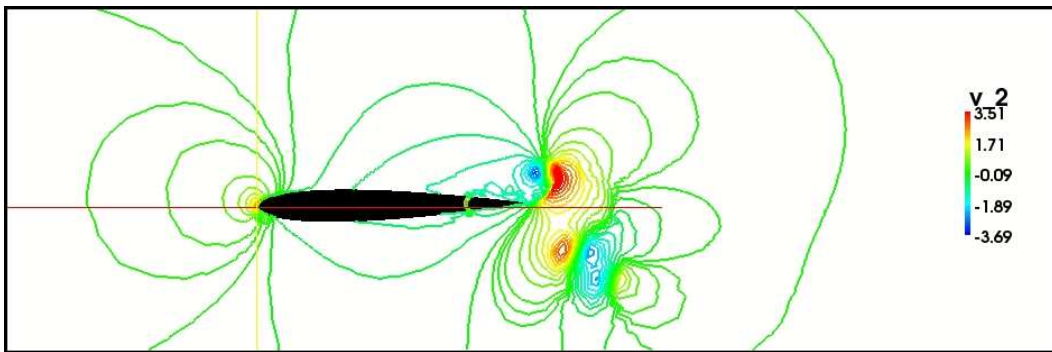


$v_2$ isolines; $t = 0.32\,s$, $\gamma = 0.75°$, $H = 7.46\,mm$.
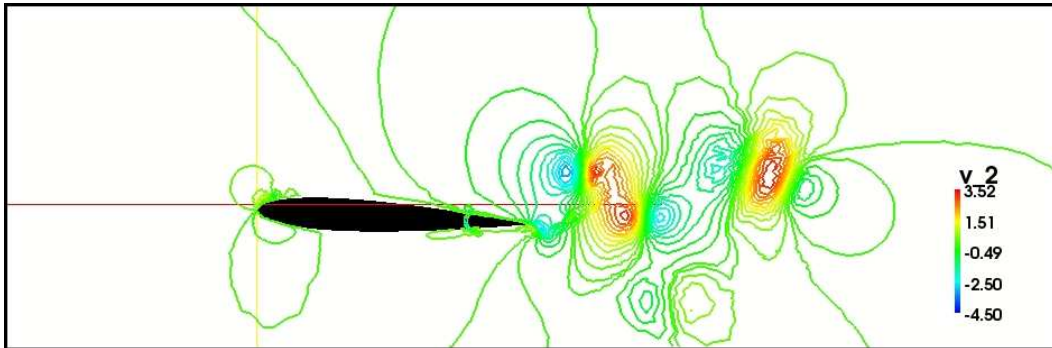


$v_2$ isolines; $t = 0.76\,s$, $\gamma = -2.95°$, $H = -29.47\,mm$.
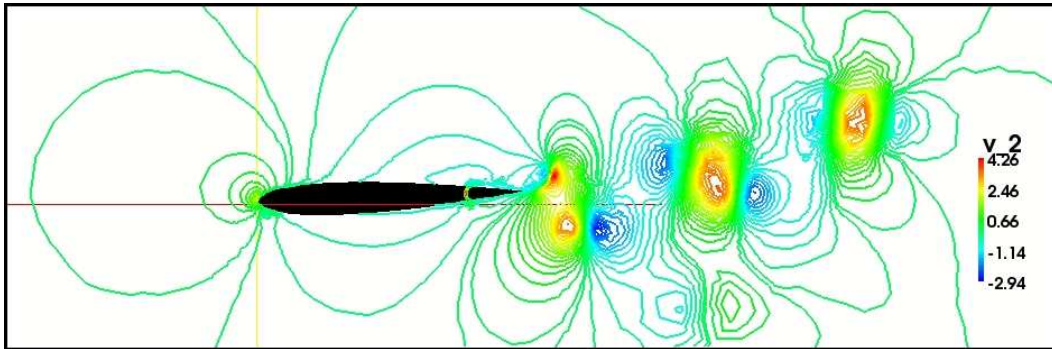
$v_2$ isolines; $t = 0.96\,s$, $\gamma = 2.05°$, $H = 20.54\,mm$.
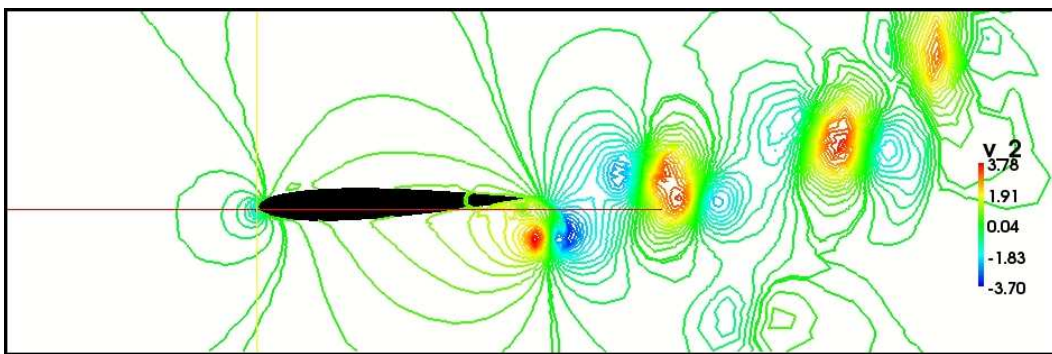


$v_2$ isolines; $t = 1.2\,s$, $\gamma = 1.76°$, $H = 17.63\,mm$.
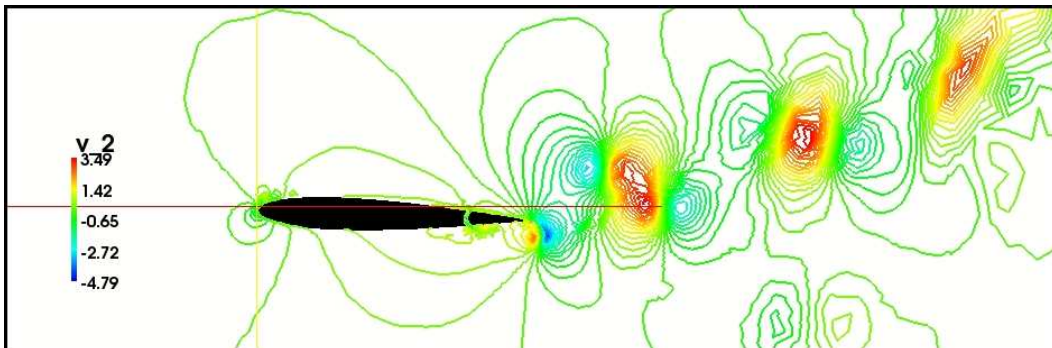


$v_2$ isolines; $t = 1.46\,s$, $\gamma = -2.05°$, $H = -20.54\,mm$.

## Isolines of the pressure

Development of characteristic features of the pressure in a broader view. Here the pressure $p$ is not a thermodynamical quantity, which explains why it attains negative values.
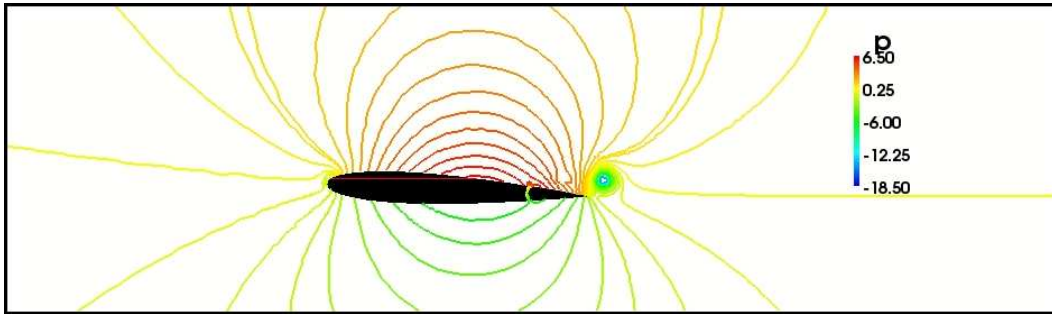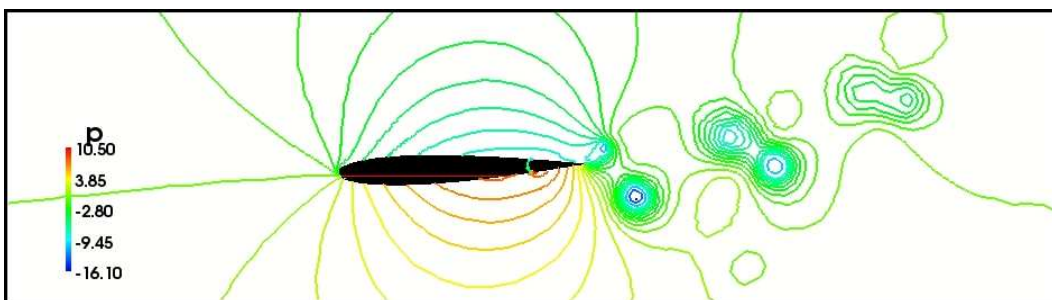
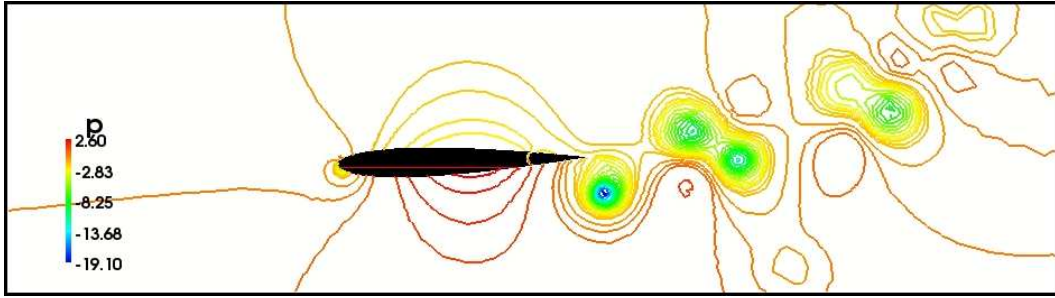$p$ isolines; $t = 0.06\,s$, $\gamma = -2.71°$, $H = -27.14\,mm$.
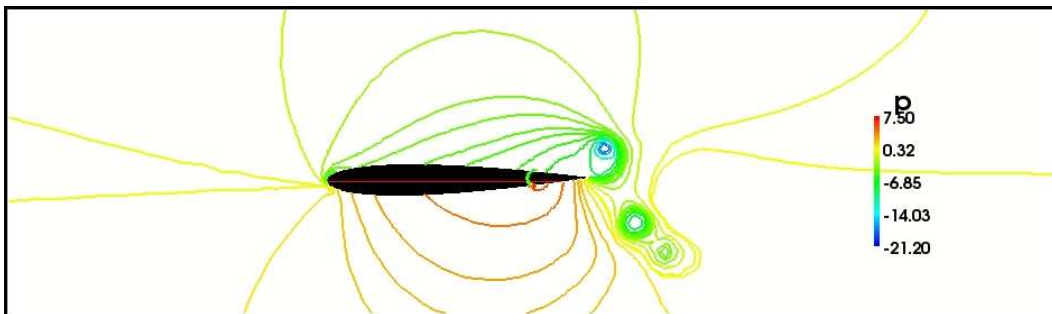


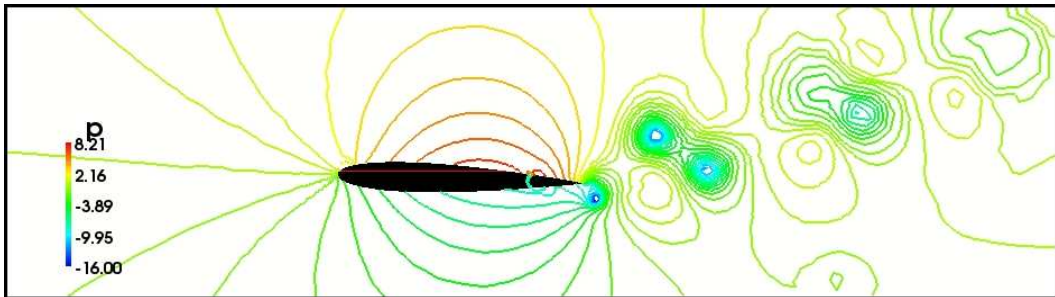$p$ isolines; $t = 0.32\,s$, $\gamma = 0.75°$, $H = 7.46\,mm$.
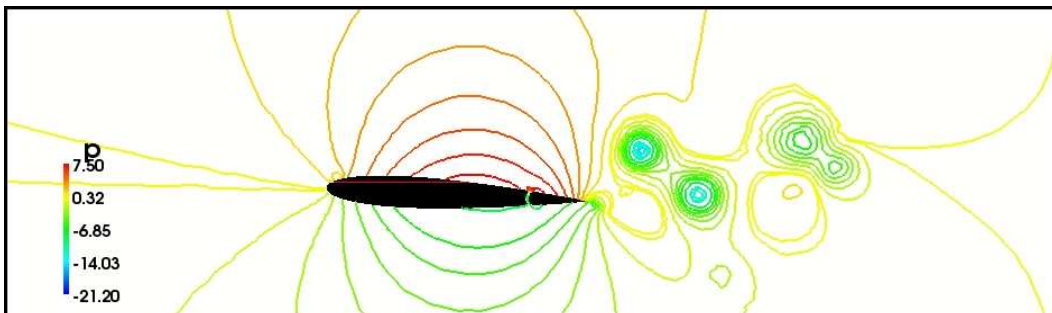


$p$ isolines; $t = 0.76\,s$, $\gamma = -2.95°$, $H = -29.47\,mm$.



$p$ isolines; $t = 0.96\,s$, $\gamma = 2.05°$, $H = 20.54\,mm$.

$p$ isolines; $t = 1.2\,s$, $\gamma = 1.76°$, $H = 17.63\,mm$.



$p$ isolines; $t = 1.46\,s$, $\gamma = -2.05°$, $H = -20.54\,mm$.

# Conclusion

An Overview of equations describing the fluid flow was presented, as well as widely used approach to handle time-dependency of a computational domain using the ALE method. Discretization procedure for simplified model of viscous incompressible flow in time-dependent domain using finite element and finite difference methods (for the space discretization and the time discretization respectively) was described. Possible solutions of problems arising from the mathematical model under consideration (nonlinearity, instability) were described. A numerical simulation with the aid of available software was carried out in order to complement the theoretical matter. The model problem was the moving airfoil with two degrees of freedom. Nevertheless, the experiment described in Chapter 5 serves only for illustrative purposes. Real simulation would require higher Reynolds number as well as a model taking into account effects of turbulency. Another topic is the interaction between the fluid and the moving boundary. All these amendments require greater computational performance and longer computational time.

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