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BACHELOR THESIS

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Light scattering on a moving black hole

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My big gratitude goes to my supervisor, not only for many useful ideas and patience, but also for his leadership which put me on the right way and helped to concentrate my efforts.
There were many other individuals who inspired and supported me in many ways during my work, amongst them my family and close friends.
Without their support, this work could never be possible. . .

I wrote this work by myself and using only the sources reffered to.
I agree with making this thesis publicly available.

Prague May 28, 2009 Peter Greškovič

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Abstract: In this work we consider gravitational bending of light in Schwarzschild metric from coordinate system in which gravitational center moves with constant non-zero velocity. Although we do not provide means to find path of a light ray in this system, in scattering mode we are able to find its outgoing wave vector after the deflection from its incoming wave vector, identify other parameters that are important in this process and assess the impact of the deflection on both photon and moving black hole. We provide formula for dynamical friction of black hole due to the gravitational bending of homogeneous isotropic radiation and point out a possibility of determination of parameters of the system from spectrum of deflected radiation.

Keywords: relativity, scattering of light, gravitational bending of light, wave vector, Schwarzschild metric

Názov práce:	Rozptyl svetla na pohybujúcej sa čiernej diere
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Abstrakt: V tejto práci sa zaoberáme ohybom svetla v gravitačnom poli popísanom Schwarzschildovou metrikou z pohľadu systému súradníc v ktorom sa gravitačné centrum pohybuje rovnomerne a priamočiaro s nenulovour rýchlosťou. Hoci nepodávame súradnicový popis trajektórie svetelného lúča v tomto systéme, v móde rozptylu vieme nájsť výstupný vlnový vektor fotónov po ohybe z ich vstupného vlnového vektora, pri tom identifikujeme ďalšie parametre, ktore sú pri tomto procese dôležité a zhodnotíme vplyv tohto ohybu na fotón i pohybujúcu sa čiernu dieru. Dávame vzťah pre dynamické trenie čiernej diery spôsobené ohybom svetelných lúčov homogénneho a izotropného žiarenia v jej gravtačnom poli a úpozorňujeme na možnosť určenia parametrov systému zo spektra rozptýleného žiarenia.

Kľúčové slová: teórie relativity, rozptyl svetla, ohyb svetla v gravitačnom poli, Schwarzschildova metrika, vlnový vektor

Contents

Introduction

In this work we consider a problem of light scattering on a moving non-rotating and uncharged black hole in an asymptotycaly flat spacetime without cosmological constant. It is important to stress, that we do not deal with gravitational lensing. We only look at changes of direction and frequency of photons at nearly infinite distance from the black hole and discuss the effects of gravitational deflection on both the the photons and the blacke hole. This work consists of three main chapters.

In the first chapter we review some basic ideas of the special and the general theories of relativity. We do not derive or explain any of them in too much detail, but we give the necessary background required to understand them and we refer to appropriate literature for more details. The purpose of this chapter is to introduce the basic terms used later and to provide for some basic results of the theories upon which the following chapters are built. The most important of them are the applications of special relativity and the particular Schwarzschild metric.

Second chapter deals mostly with null geodetics in curved spacetime. The particular application of general relativity, primary concern of which is to find the path of light in gravitational field. Basic procedures this are explained in more detail and we derive the equations of geodetic in Schwarzschild metric. Formulas needed to find the deflection angle of a light ray that comes from and escapes to infinite distance are given.

The third chapter is the main part of this work. It contains our original research that deals with the problem of deflection of light on a moving black hole. For a photon with known wave vector, the wave vector after the deflection is calculated. From the change of direction and frequency of the photons, total dynamical efect on the black hole is calculated. We give formulas for dynamnical friction due to the gravitational deflection of photons and discuss usability of the used method. In the end, we briefly mention one possible interesting application of our results.

Finally, we conclude the results that we find and propose potentialy perspective ways for expansion of this topic.

Chapter 1

Theory of relativity

1.1 Special relativity

In this section we briefly summarize some basic ideas of special relativity. Our aim is to show the coordinate transformation between two inertial coordinate systems and some practical implications of this transformation. Namely, we describe *transformation of speed vector* components and *special relativistic redshift*.

Special relativity points out a class of inertial coordinate systems that move in non-rotating maner and with constant velocity relative to each other. It is postulated that none of these systems is special in any way and that the speed of light is the same in all of them. From these two basic principles one can find *Lorentz transformation* between spatial coordinates and time in two such systems[1] that overlap in time $t = t' = 0$ and move in direction of x-axis with velocity v relative to each other:

$$t' = \gamma(t - \beta x/c) \qquad x' = \gamma(x - vt) \qquad y' = y \qquad z' = z$$
$$\beta \equiv v/c \qquad \gamma \equiv 1\Big/\sqrt{1 - v^2/c^2} \tag{1.1}$$

where primed are the coordinates in the system moving relative to the first one (observer). In matrix notation this can be written as

$$(x^\mu)' = \Lambda^\mu_\nu x^\nu \tag{1.2}$$

where the Λ is the Lorentz transformation matrix

$$\Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.3}$$

[1]See [1] or [3] for details.

Prerequisity for using the Lorentz transformation is that the spacetime where we want to use it is flat. In the other words, the differential distance between two points (events) is given by so called space-time interval.

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{1.4}$$

Now having an object moving with speed $\vec{u}$ in the system of observer, Lorentz transformation leads to the following transformation of the speed vector in the moving system:

$$u'_p = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{u_p - v}{1 - u_p v/c^2} \tag{1.5}$$

$$u'_n = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{\mathrm{d}z'}{\mathrm{d}t'} = \frac{u_n}{\gamma(1 - u_p v/c^2)} \tag{1.6}$$

where u_p is the component of the speed vector parallel to the velocity $\vec{v}$ and u_n are components normal to that direction.

From these relations one can derive formula for relativistic aberation. If α is an angle between direction of velocity $\vec{v}$ and speed of the light in the coordinate system of observer, in the moving system this angle is α'.

$$\cos\alpha' = \frac{(c_x)'}{c} = \frac{\cos\alpha - \beta}{1 - \beta\cos\alpha} \tag{1.7}$$

Transformation of time in (1.1) leads to reciprocal transformation of frequency ($\nu = 1/T$). Hence light with frequency ν', emitted in the moving moving system will be, in the system of observer, observed with another frequency ν. If change of the x coordinate is zero along the line of sight, then $\nu = \nu'/\gamma$ (*transverse redshift*) and

$$\nu = \frac{\nu'}{\gamma\sqrt{1 + \beta\cos\alpha}} \tag{1.8}$$

otherwise. Here the α is an angle between the velocity $\vec{v}$ and the direction to the source of light, measured in the system of observer. Transforming this angle to the system of the source by (1.7) we get

$$\nu = \gamma\left(1 - \beta\cos\alpha'\right)\nu' \tag{1.9}$$

Lorentz transformation also leads to transformation of acceleration $\vec{A}$ and momentum. To preserve the 2nd Newton law in the form $\mathrm{d}\vec{p}/\mathrm{d}t = M.\vec{A}$ in all frames, it may be useful to introduce quantity called *relativistic mass*. If M' is a mass of an object in its rest frame, then the relativistic mass is

$$M = \gamma M' \tag{1.10}$$

1.2 Gravitation theory

In this work we investigate effect of strong gravitational field on light. This effect is described by the general theory relativity, which describes gravitation as curvature of spacetime caused by energy. Following section sums up some basic terms of this theory and introduces conventions used later. We want to write down the Einstein field equations.

Curved spacetime is a manifold[2] described by its *metric*. The metric is a function that defines distance s between some two points on the manifold. In differential form it is defined by the relation

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1.11}$$

where the metric $g_{\mu\nu}$ is a function of place on the manifold and $\mathrm{d}x^\alpha$ are components of infinitesimal displacement expressed in local coordinates.

Marking partial derivative of a vector field x^α with respect to β-th local coordinate as ${x^\alpha}_{,\beta}$ we introduce Christoffel symbol

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}\right) \tag{1.12}$$

where we used $g^{\mu\nu}$ for inverse of the metric ($g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu{}_\nu$). Combination of Christoffel symbols can be used to write down a quantity that is used to describe curvature of spacetime called Riemann tensor:

$$R^\rho{}_{\sigma\mu\nu} = \Gamma^\rho{}_{\nu\sigma,\mu} - \Gamma^\rho{}_{\mu\sigma,\nu} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} \tag{1.13}$$

It is some N-dimensional analogy to reciprocal radius. Contraction of the Riemann tensor $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ is called *Ricci tensor* and second contraction $R = g^{\mu\nu}R_{\mu\nu}$ is *scalar curvature*.

Mass, or any other kind of energy (even pressure), does curve our spacetime (produce gravity). Distribution of energy can be described by so called *stress-energy tensor* $T^{\mu\nu}$. Curvature of the spacetime (gravitation) is linked to the stress-energy tensor via *Einstein field equations*

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu} \tag{1.14}$$

where G is gravitational constant, c is speed of light in vacuum and Λ is the fundamental cosmological constant.

1.2.1 Some solutions to the field equations

Einstein field equations can be looked at as complicated set of ten coupled nonlinear second order partial differential equations for ten indipendent components of the metric. There are not many exact solutions known.

[2]See [1] or [2] for detailed mathematical definitions and derivation of formulas.

One of the simplest solutions is *Schwarzschild metric*. It can be derived as spericaly symmetric static vacuum solution of the field equations with zero cosmological constant (see [6]) and interpreted as the gravitational field of an uncharged non-rotating massive point with mass M. In polar coordinates r, θ, ϕ and time t it takes the following form:

$$\mathrm{d}s^2 = -c^2\left(1-\frac{2GM}{rc^2}\right)\mathrm{d}t^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\cdot\mathrm{d}\phi^2\right) \quad (1.15)$$

This metric is a very good approximation for gravitational field of any uncharged slowly rotating sphericaly symmetric body, like a star or a planet, therefore it is the most important.

There are two singularities in the above metric. The first one occures when $r = r_s = 2GM/c^2$ (so called *event horizont*) and it is just a matter of choice of coordinates. The other one is at $r = 0$ and it is a real physical singularity. Objects below schwarzschild radius inevitably hit the central singularity, since all possible spacetime trajectories point towards it. The metric is usually supposed to be valid for radii much larger than r_s.[3]

Generalization of the Schwarzschild metric for universe with non-zero cosmological constant is called *Schwarzschild-(anti-)de Sitter spacetime*:

$$\begin{aligned}\mathrm{d}s^2 &= -c^2\left(1-\frac{2GM}{rc^2}-\frac{\Lambda}{3}r^2\right)\mathrm{d}t^2 + \\ &\quad +\left(1-\frac{2GM}{rc^2}-\frac{\Lambda}{3}r^2\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\cdot\mathrm{d}\phi^2\right)\end{aligned} \quad (1.16)$$

This metric is slightly different from the previous. Properties depend on mass and value of the cosmological constant. There is some limiting mass of an object that can undergo gravitational collapse. Investigation using method described in the next paragraph shows that there are some new types of motion allowed (see [4] for details).

For a charged body with an electric charge Q, the energy of the electromagnetic field must be taken into account. Corresponging modification of the Schwarzschild metric is known as *Reissner–Nordstrøm geometry*:

$$\begin{aligned}\mathrm{d}s^2 &= -c^2\left(1-\frac{2GM}{rc^2}+\frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2}\right)\mathrm{d}t^2 + \\ &\quad +\left(1-\frac{2GM}{rc^2}+\frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\cdot\mathrm{d}\phi^2\right)\end{aligned} \quad (1.17)$$

Here ε_0 is the electric constant.

[3] Schwarzschild radius for mass of the Earth is less than 9 mm.

Depending on the value of Q there may be either one, two or three singularities in this metric. At the centre there is timelike line. Case where this would be the only (naked) singularity is belived to be unphysical for serious reasons (preservation of principle of causality). Moreover, the mass M would have to be negative. Solution with two singularities (one event horizont) would be highly unstable. Otherwise there are two event horizonts. There are some doubts about stability of this solution too [5].

Another important solution is axialy symmetric *Kerr metric*. It can be used to describe the gravitational field of a rotating sphericaly symetric uncharged body in universe with zero cosmological constant. In standard polar coordinates it takes this rather complicated form:

$$\begin{aligned} \mathrm{d}s^2 &= c^2\left(1-\frac{2GM}{c^2}\cdot\frac{r}{\rho^2}\right)\mathrm{d}t^2-\frac{\rho^2}{\Lambda^2}\mathrm{d}r^2-\rho^2\mathrm{d}\theta^2- \\ &\quad -\left(r^2+\alpha^2+\frac{2GM}{c^2}\cdot\frac{r\alpha^2}{\rho^2}\sin^2\theta\right)\sin^2\theta\mathrm{d}\phi^2+\frac{4GM}{c^2}\cdot\frac{r\alpha\sin^2\theta}{\rho^2}c\mathrm{d}t\mathrm{d}\phi \\ \alpha &= \frac{J}{Mc}, \qquad \rho^2=r^2+\alpha^2\cos^2\theta, \qquad \Lambda^2=r^2-\frac{2GM}{c^2}\cdot r+\alpha^2 \end{aligned} \tag{1.18}$$

This metric is not static. Spacetime around is partially spinning with the body that generates the metric.[4] There are two singular surfaces in this metric. Inner surface is spherical event horizont simmilar to that of Schwarzschild metric. Outer surface is elipsoidal and coincides with the inner surface on poles of the rotation (ergosphere). Within the ergosphere nothing can orbit in direction opposing the rotation.

Further generalization of the above mentioned metrices is possible. *Kerr–Newman metric* would be generated by a rotating charged body. All metrices could also be rewriten in a universe with non-zero cosmological constant. Those are *-de Sitter* ($\Lambda > 0$) and *-anti-de Sitter* ($\Lambda < 0$) metrices.

[4] This is reffered to as *frame dragging* effect.

Chapter 2

Light rays in gravitational field

2.1 Radial movement constraints

Before we proceed to the calculation of trajectories in gravitational field, we should recognize what kinds of motion are physicaly feasible. Below, we demonstrate one method that can be used for this purpose. We limit to Schwarzschild metric. From this point on, we use geometric coordinates in which $c = G = 1$.

Motions of test particles can be studied by means of *effective potential*. Effective potential is a quantity which combines angular momentum and potential energy. It can be interpreted as a measure of how much energy would be required for a test particle to escape from gravitational field of a massive body. The formula for the effective potential can be derived from connection between four-momentum $\vec{p}$ of the particle and its rest mass m.

$$g_{\mu\nu} p^\mu p^\nu + m^2 = 0 \tag{2.1}$$

We can plug Schwarzschild metric into this relation. It is convenient to work in polar coordinates in $\theta = \pi/2$ plane. Since

$$p^t = -g^{tt} p_t = -g^{tt} \frac{\sqrt{|g^{tt}|} p_t}{\sqrt{|g^{tt}|}} = -g^{tt} \frac{p \cdot e_t}{\sqrt{|g^{tt}|}} = -g^{tt} E = \frac{-E}{1 - 2M/r} \tag{2.2}$$

where e_t is dual basis vector of proper reference frame (see [1]) and

$$p^\phi = g^{\phi\phi} p_\phi = \cdots = \frac{L}{r^2} \tag{2.3}$$

relation (2.1) becomes

$$-\frac{E^2}{(1 - 2M/r)} + \frac{m^2}{(1 - 2M/r)} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau} \right)^2 + \frac{L^2}{r^2} + m^2 = 0 \tag{2.4}$$

If we set $\widetilde{E} \equiv E/m$ and $\widetilde{L} \equiv L/m$, the last equation can be written as

$$\begin{aligned}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 &= \widetilde{E}^2 - \underbrace{(1-2M/r)(1+\widetilde{L}^2/r^2)} \\ &= \widetilde{E}^2 - \qquad\qquad \widetilde{V}^2 \end{aligned} \tag{2.5}$$

where $\widetilde{V}$ is the effective potential.

Similar expression can be derived for zero-mass test particle when we introduce so called *impact parameter*. It is defined as angular momentum over linear momentum. Geometricaly it can be interpreted as a distance of a line asymptotic to trajectory from the center of gravitation. Using the well known relation $E^2 = p^2 + m^2$ we can write

$$b = \lim_{m\to 0} \frac{L}{(E^2 - m^2)^{\frac{1}{2}}} = \lim_{m\to 0} \frac{\widetilde{L}}{(\widetilde{E}^2 - 1)^{\frac{1}{2}}} = \frac{\widetilde{L}}{\widetilde{E}} \left(= \frac{L}{E}\right) \tag{2.6}$$

Using this and setting $m = 0$ in equation (2.1) we get

$$\begin{aligned}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 &= \frac{1}{b^2} - \underbrace{\frac{1-2M/r}{r^2}} \\ &= b^{-2} - \quad B^{-2} \end{aligned} \tag{2.7}$$

We interpret B as the effective potential for zero-mass particle.

Usage of the expression for the effective potential is quite simple. From both equations (2.5) and (2.7) it is evident that the test particle reaches a turning point ($\dot{r} = 0$) when either energy or impact parameter is equal to the effective potential.

A massive particle with energy E and angular momentum L, which are constants of motion (by [2]), can have either one (circular orbit) or two (elliptical orbit) turning points depending on the values of E and L. The particle falls into the black hole if either energy is too small for given L or the angular momentum is too small for the given energy.

Radial movement of a massless particle, like photon, on the other hand, depends solely on it's impact parameter. Since both $\widetilde{E}$ and $\widetilde{L}$ are constant, so is b. Effective potential depends on the distance r. Photon reaches a turning point when $B = b$. From relation (2.7) we can see that the photon can reach the turning point only if b is more than minimum value of B, that is $b > b_c$, where

$$b_c = M\sqrt{27} \tag{2.8}$$

Photons with the critical impact parameter b_c remain on a circular orbit at $r = 3M$. Photons with smaller impact parameter coming from outside fall into the singularity.

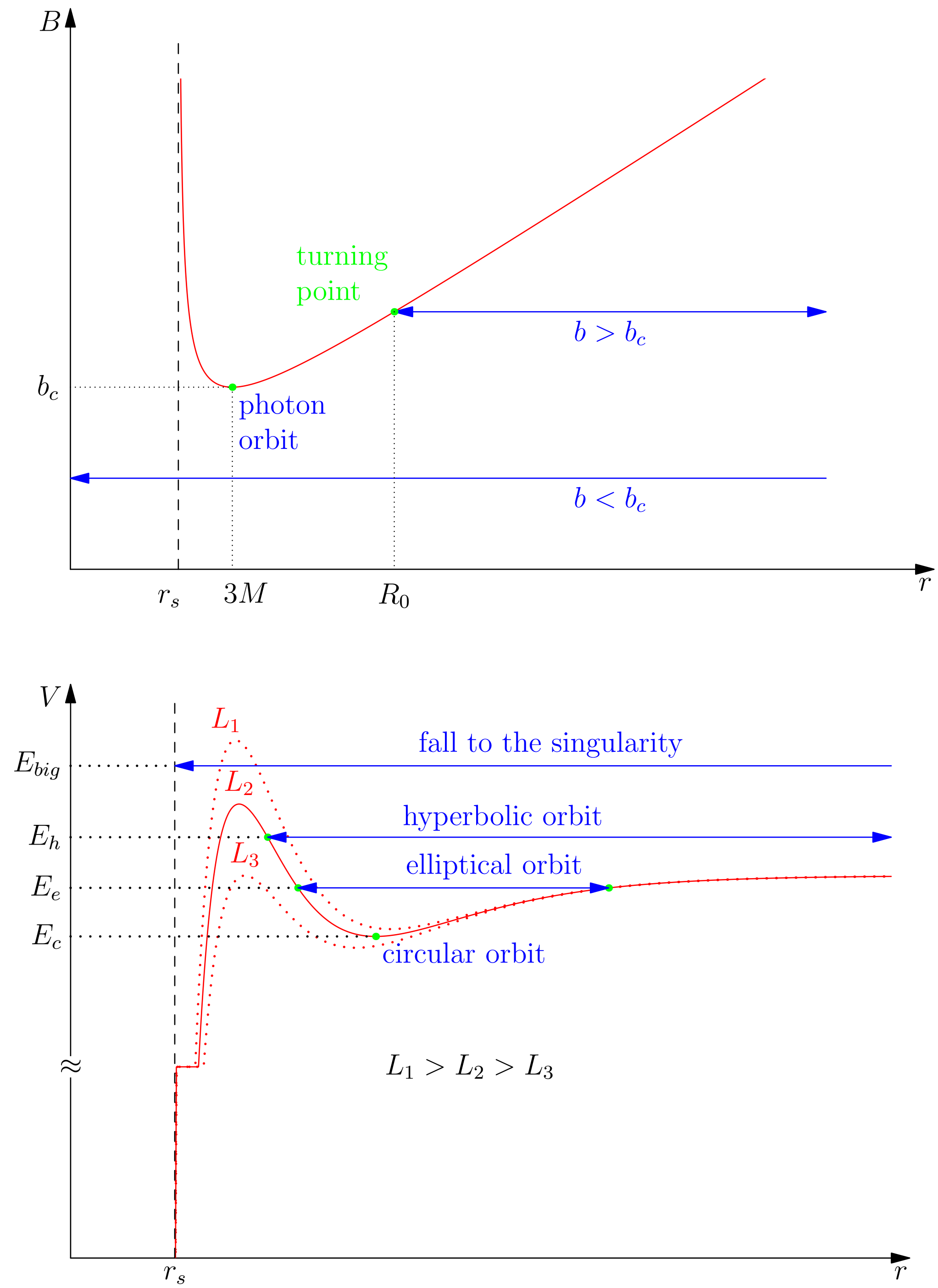

Figure 2.1: The effective potential as a function of radial distance r for a photon (up) and for a massive test particle (down). Turning points are in green color and possible configurations in blue (see description in text).

2.2 Null geodesics in swarzschild spacetime

We take a closer look at the motion of a photon in gravitational field. In the previous paragraph, the basical equations for time developmet of radial coordinate are given. Here we look at them again from different point of view and give formula for *total angular deflection of photons* in Schwarzschild spacetime.

Both massless and free massive particles move along geodesic curves in gravitational field. A point moves along geodesic if

$$\mathrm{d}s = g_{\mu\nu} u^{\mu} u^{\nu} \tag{2.9}$$

where $u^{\alpha} = \mathrm{d}x^{\alpha}/\mathrm{d}\tau$ ($\mathrm{d}\tau$ the local time) are components of its four-velocity. For massive particles the $\mathrm{d}s = 1$ and for photons the $\mathrm{d}s$ has value of zero. To get the null geodesic curve as a function of coordinates and time we simply plug in the metric into the previous equation and use $u^{\alpha} = p^{\alpha}/m$. With Schwarzschild metric in polar coordinates this becomes[1]

$$0 = -\left(1 - \frac{2M}{r}\right)^{-1} \widetilde{E}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{\widetilde{L}^2}{r^2} \tag{2.10}$$

where $\widetilde{E}$ and $\widetilde{L}$ are as above. Solving this for $\dot{r}$ and dividing $\dot{\phi} = p^{\phi}/m$ from (2.3) by the $\dot{r}$ we obtain

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\widetilde{L}}{r^2}\left[\widetilde{E}^2 - \frac{\widetilde{L}^2}{r^3}(r - 2M)\right]^{-1/2} = \left[r^4 b^2 - r(r - 2M)\right]^{-1/2} \tag{2.11}$$

The b must be greater than b_c. If this is the case, the photon reaches the turning point at the distance R_0 where $b = B$:

$$R_0^3 - b^2(R_0 - 2M) = 0 \tag{2.12}$$

Real root to this equation is (by [2])

$$R_0 = \frac{2b}{\sqrt{3}} \cos\left[\frac{1}{3} \arccos\left(-\frac{b_c}{b}\right)\right] \tag{2.13}$$

Hence the total deflection angle for a photon coming from infinity and escaping again to infinity would be

$$\Delta\phi = 2\int_{R_0}^{\infty} \frac{\mathrm{d}\phi}{\mathrm{d}r}\mathrm{d}r = 2\int_{R_0}^{\infty} \frac{\mathrm{d}r}{\left[r^4 b^{-2} - r(r - 2M)\right]^{1/2}} \tag{2.14}$$

[1] Note that we can recover the equation (2.4) immediately.

2.3 Aproximative formulas for deflection angle

A good overview of aproximative methods is given in [7]. Here we list some approximative formulas for the total deflection angle of light that comes from and escapes to infinity in Schwarzschild spacetime.

Incomplete elliptic integral of the first kind is defined as

$$F(p;q) = \int_0^p \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-q^2u^2)}} \tag{2.15}$$

The equation (2.14) can be rewriten in tems of this integral like

$$\Delta\phi = 4\sqrt{\frac{\overline{R}_0}{\Upsilon}}\left[F\left(\frac{\pi}{2};\kappa\right) - F\left(\varpi;\kappa\right)\right] \tag{2.16}$$

where $\overline{R}_0 \equiv R_0/M$ and

$$\Upsilon \equiv \sqrt{\frac{\overline{R}_0 - 2}{\overline{R}_0 + 6}}, \quad \kappa \equiv \sqrt{(\Upsilon - \overline{R}_0 + 6)/2\Upsilon}, \quad \varpi \equiv \arcsin\sqrt{\frac{2 + \Upsilon - \overline{R}_0}{6 + \Upsilon - \overline{R}_0}}$$

This is reffered to as *Darwin deflection formula*.

Although Darwin formula gives precise value of deflection angle $\Delta\phi$, it can be desairable to express it without using special functions.

Many approximative methods are based on expansion of (2.14) or the elliptic integral with respect to various variables. Expansion of $F(p;q)$ for small φ and κ with respect to κ can be rewriten in terms of M^2/b^2 and approximation to the first order gives the well known *Einstein formula*

$$\Delta\phi \approx \frac{4M}{b} + O\left(\frac{M^2}{b^2}\right) \tag{2.17}$$

Second order approximation slightly improves this result to

$$\Delta\phi \approx \frac{4M}{b - 3M} + O\left(\frac{M^2}{b^2}\right) \tag{2.18}$$

Detailed derivation of these formulas is given in [8] (*Mutka-Mähönen*).

Another interesting method was proposed by *Beloborodov* [9]. If Ψ is an escape angle given by integration of (2.11) from R to infinity and α is an angle between radial direction and direction of a given geodetic at radius R ($\tan\alpha = (\mathrm{d}r/R\mathrm{d}\phi)|_R$), then expansion of an expression $1 - \cos\Psi$ in terms of small variable $(1-\cos\alpha)^n$ can be after linear approximation turn into following formula:

$$1 - \cos\alpha \approx (1 - \cos\Psi)\left(1 - \frac{r_s}{R}\right) + O\left[(1-\cos\alpha)^3\right] \tag{2.19}$$

Since $\Psi = \Delta\phi + \pi/2$ and at the turning point $\alpha = \pi/2$ and $R = R_0$, for small $\Delta\phi$ we can find

$$\Delta\phi \approx \frac{4M}{R_0 - 2M} + O\left[(1-\cos\alpha)^3\right] \tag{2.20}$$

All of these approximations approach correct value of $\Delta\phi$ as $b \to \infty$, but fail for small value of impact parameter ($b \to b_c+$).

Integral (2.14) can be solved analyticaly only in three special cases: radial rays, $b \to \infty$ limit and at the photon orbit. Following relation can be found as a small perturbation to the last case

$$\Delta\phi \approx 2 \cdot \left\{ -\log\left(\frac{b}{b_c} - 1\right) + \log\left[216\left(7 - 4\sqrt{3}\right)\right] - \pi \right\} \tag{2.21}$$

In contrast to previous formulas, this approximation given by *Bozza* [10] goes to correct value of $\Delta\phi$ for small values of impact parameter ($b \to b_c$).

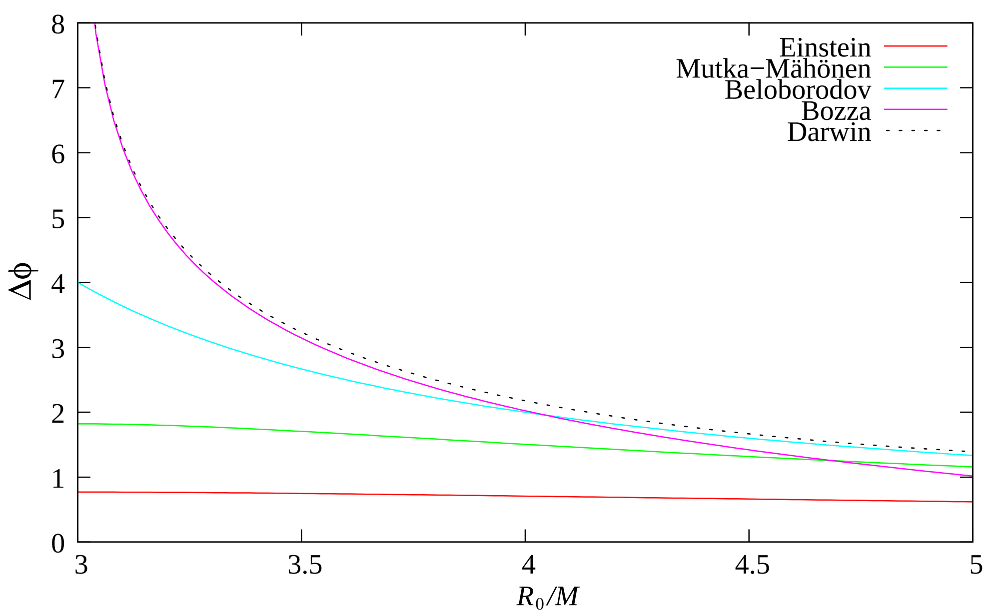

Figure 2.2: Comparison of precision of listed approximative methods for small values of impact parameter. The formula of Bozza approximates the precise value (Darwin) very well in this range, but it fails for bigger values of b very fast. Beloborodov approximation remains surpraisingly precise.

Chapter 3

Fast moving black hole

In coordinate system, which will be refered to as a system of observer, we consider a light bending problem on a Schwarzschild black hole (BH) moving in this system with constant velocity.

First, we describe how the BH affects the passing by light and how this effect is different from the stationary case. Then we consider how the energy and momentum is redistributed in a system with homogeneous isotropic ratiation and a moving BH. Finally, we discuss possible limits of proposed model and estimate magnitude of calculated effect in one particular case.

When we refer to mass of BH in the system of observer, we mean its relativistic mass. All calculations are done under three assumptions:

- The light comes from and escapes to very big distance (*infinity*).
- Spacetime around BH is *asymptoticaly flat*.
- Change in momentum of the deflected light has only small effect on the movement of BH (i.e. BH remains in nearly *constant motion* during deflection).
- Wavelength of the light is much shorter than Schwarzschild radius, so we can use *geometrical optics.*

Under these conditions we can use Lorentz transformation (LT) to transform incoming light rays at infinity (lower index in) from the system of observer (unprimed) to coordinate system of the moving BH (primed). Then we can use formulas for gravitational light bending as as given in previous chapter. Finally, we can use inverse LT to transform them back to the original coordinate system. In Schwarzschild metric, bending of light coming from and escaping to infinity means a simple rotation of its wave vector $\vec{k}$ in the plane of deflection ($R(\Delta\phi')$). Hence, the outgoing wave vector will be

$$\vec{k_{\mathrm{out}}} = \Lambda^{-1} \cdot R(\Delta\phi') \cdot \Lambda \cdot \vec{k_{\mathrm{in}}} \tag{3.1}$$

3.1 Light rays parallel to velocity of black hole

In the simplest case, light rays are parallel to the direction of black hole velocity $\vec{v}$. Let us suppose that they are coming from infinite distance with impact parameter b towards the black hole. We can calculate change in frequency ν and momentum $\vec{p}$ of deflected radiation.

As the first step, we must write down componets of the wave vector of the incoming light $\vec{k_{\text{in}}}$. We choose direction of the x-axis to be in direction of the velocity $\vec{v}$ and the plane of deflection to be xy-plane. Since the $\vec{k_{\text{in}}}$ must be null vector

$$\vec{k_{\text{in}}} = \left(\frac{\omega_{\text{in}}}{c}, \frac{\omega_{\text{in}}}{c}, 0, 0\right) \tag{3.2}$$

where $\omega_{\text{in}} = 2\pi\nu_{\text{in}}$.

Now by applying LT or using the relation (1.9) with angle 0 we find

$$\left(\vec{k_{\text{in}}}\right)' = \left(\frac{\omega'_{\text{in}}}{c}, \frac{\omega'_{\text{in}}}{c}, 0, 0\right) \tag{3.3}$$

where

$$\omega'_{\text{in}} = 2\pi\nu'_{\text{in}} \tag{3.4}$$

and

$$\nu'_{\text{in}} = \nu_{\text{in}} \cdot \sqrt{\frac{1-\beta}{1+\beta}} \tag{3.5}$$

This is just the formula for special relativistic redshift of radiation from a source moving along the line of sight in its usual form.

In coordinate system of the moving BH, the energy of the deflected light must be conserved, so that

$$\nu'_{\text{in}} = \nu'_{\text{out}} \tag{3.6}$$

and $\omega'_{\text{out}} = \omega'_{\text{in}}$. The wave vector of the deflected light that escapes to infinity will have, in the system of BH, components

$$\vec{k_{\text{out}}}' = \left(\frac{\omega'_{\text{in}}}{c}, \frac{\omega'_{\text{in}}}{c}\cos\left(\Delta\phi'\right), \frac{\omega'_{\text{in}}}{c}\sin\left(\Delta\phi'\right), 0\right) \tag{3.7}$$

where $\Delta\phi'$ must be calculated using b' and M'. Impact parameter of the ray is not changed ($b = b'$). See (1.10) on how to get M' from M.

We can now use inverse LT (Λ^{-1}) to find the components of the wave vector in the system of observer. Time component of the wave vector is

$$k^t_{\text{out}} = \left(\Lambda^{-1}\right)^t_{\mu} k^{\mu}_{\text{out}} = \gamma\left(k^t_{\text{out}}\right)' + \gamma\beta\left(k^x_{\text{out}}\right)' \tag{3.8}$$

From the time component of the wave vector we can find ν_{out}.

$$\nu_{\text{out}} = \gamma^2\left(1-\beta\right)\left[1+\beta\cos\left(\Delta\phi'\right)\right] \cdot \nu_{\text{in}} \tag{3.9}$$

We can now see that $\nu_{\text{out}} \neq \nu_{\text{in}}$. The light either drains or supplies kinetic energy of BH (depending on direction). This result may be compared to the equation (3.6) which holds when BH does not move.

To calculate the deflection angle in the system of observer we simply transform $\Delta\phi'$ by (1.7).

$$\cos(\Delta\phi) = \frac{\cos(\Delta\phi') + \beta}{1 + \beta\cos(\Delta\phi')} \tag{3.10}$$

Angle $\Delta\phi'$ should be compared to the deflection angle in stationary case calculated using M and b.

Momentum of a photon is given by

$$\vec{p} = \hbar\vec{k} \tag{3.11}$$

Change in x-component of the momentum of one photon that was initialy parallel to $\vec{v}$ in the system of BH is

$$\Delta p'_x = \frac{h\nu'_{\text{in}}}{c} \cdot \Big[\cos(\Delta\phi') - 1\Big] \tag{3.12}$$

which is a function of $\Delta\phi'$ and hence of b' and the incoming frequency ν_{in}.

Let us think of photons with momentum initially parallel to velocity $\vec{v}$ as if they originated on a plane perpendicular to $\vec{v}$ at infinite distance from BH. Let the number of photons per time $\mathrm{d}t'$ coming from area $\mathrm{d}S'$ on that plane be $\mathcal{F}' = \mathrm{d}N/\mathrm{d}S'\mathrm{d}t'$. We are only interested in those photons that do not fall into BH (and neglect those with $b' < b'_c$). Photons with impact parameter in an interval $\langle b', b' + \mathrm{d}b'\rangle$ come from annular zone with an area

$$\mathrm{d}S' = S(b' + \mathrm{d}b') - S(b') = \pi\left(b'^2 - 2b'\,\mathrm{d}b' + (\mathrm{d}b')^2\right) - \pi b'^2 = 2\pi b'\,\mathrm{d}b' \tag{3.13}$$

They all are deflected by the same angle and due to the symmetry of the problem their total change of momentum will be equal to total change in its x-component. Total change of momentum of photons with b' smaller than some $b'_{\max}$ deflected per time $\mathrm{d}t'$ in the system of BH is

$$F'_{\|}(b'_{\max}, \nu'_{\text{in}}) = \left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)_{\text{total}} = \int_{b'_c}^{b'_{\max}} \Delta p'_x(b')\,\mathcal{F}'\,2\pi b'\,\mathrm{d}b' \tag{3.14}$$

If we know the flux $\mathcal{F}'$ of photons with some frequency ν_{in}, we can find their dragging force F' on BH. This integral diverges for $b_{\max} \to \infty$, what can lead to a controversy which is to be discussed and explained later.

3.2 Light rays coming from arbitrary direction

In the system of observer, it is impossible to adopt the concept of impact parameter for light rays that are not parallel to the velocity of BH. It would be

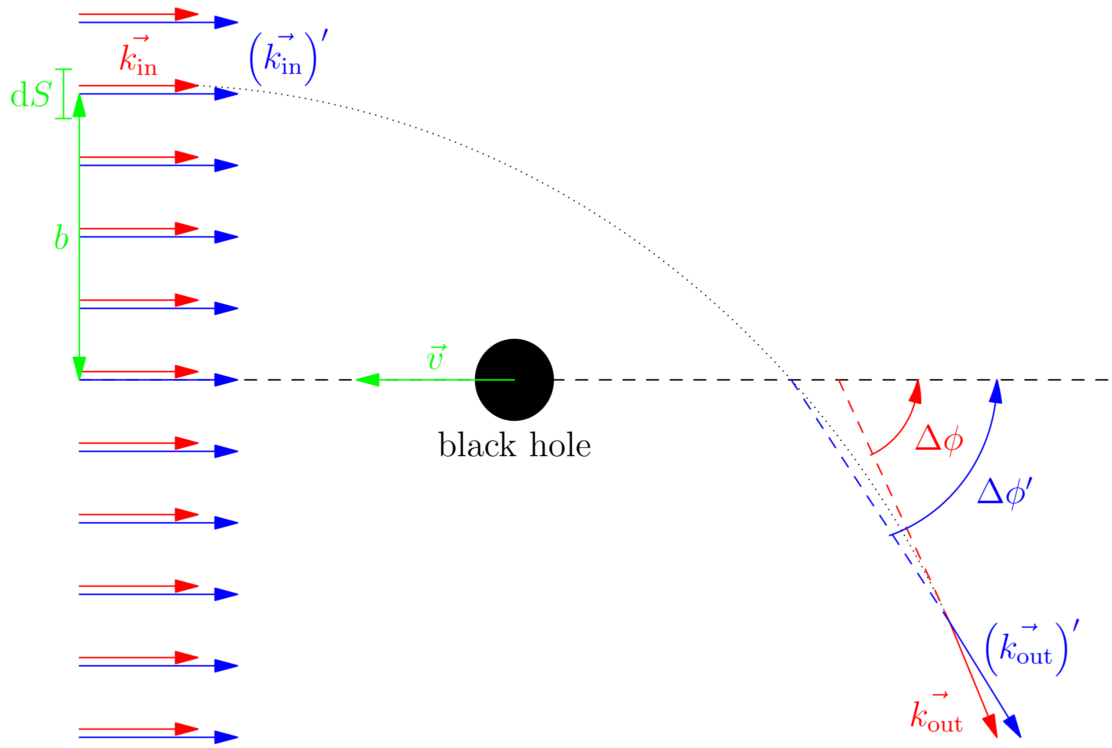

Figure 3.1: Pictorial explanation of the important parameters used in the calculation of dragging force of light rays parallel to the velocity of BH.

some function of position of BH and hence of time and at infinity it cannot be defined at all. However, in the system of BH we can define impact parameter b' as before and calculate the deflection angle. We give the deflection angle and the frequency change in the system of observer as a function of the *plane of deflection* and *initial direction*. We also show a way to trace the movement of a photon with a given incoming direction and impact parameter in the system of BH.

Direction of a given light ray at some point on that ray is an angle ρ between the direction of velocity $\vec{v}$ and wave vector of a photon in the ray at that point. Directions at infinite distance $\rho_{\rm in}$ (ininal direction) and $\rho_{\rm out}$ (final direction) can be recalculated between the coordinate system of observer and BH by (1.7).

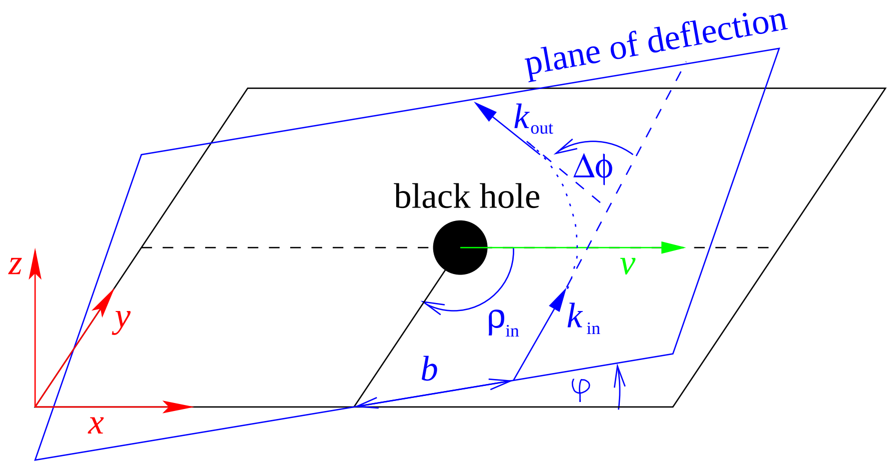

Figure 3.2: Pictorial explanation of the important parameters used in the calculation of the deflection of light rays coming from arbitrary direction.

In the system of observer, we call plane of deflection any plane parallel to both incoming and outgoing wave vectors.[1] An angle φ is an angle between such a plane and a plane of deflection of a photon with the same initial direction ρ_{in}, for which the three vectors $\vec{k}_{\text{in}}$, $\vec{k}_{\text{out}}$ and $\vec{v}$ are in the same plane. This angle can be transformed to the system of BH.

$$\cos\varphi' = \gamma(1 - \beta\cos\rho_{\text{in}})\cos\varphi \tag{3.15}$$

For the sake of definition, in case when given incoming and outgoing wave vectors are parallel ($\Delta\phi' = n.\pi$), we could find the deflection plane in the system of BH (as a plane given by any three points on trajectory) and invert this formula to find φ. Only the angle φ' as a parameter is important and it is always defined.

Since at infinity all rays deflected by BH are radial and the problem is axialy symetric, the important parameters are just ρ'_{in}, φ' and b'.

Next we use the same method as in the previous paragraph. We find outgoing wave vector of a photon with known incoming wave vector. If we set coordinate system so, that velocity $\vec{v}$ is in the direction of x-axis, then the photon at infinity has, in the system of observer, incoming wave vector of the form

$$\vec{k}_{\text{in}} = \left(\frac{\omega_{\text{in}}}{c}, \frac{\omega^x_{\text{in}}}{c}, \frac{\omega^y_{\text{in}}}{c}, \frac{\omega^z_{\text{in}}}{c}\right) \tag{3.16}$$

where $\omega_{\text{in}} = \sqrt{(\omega^x_{\text{in}})^2 + (\omega^y_{\text{in}})^2 + (\omega^z_{\text{in}})^2}$ and in the system of BH we can find the incoming wave vector by LT.

$$(k^\mu_{\text{in}})' = \Lambda^\mu_\nu k^\nu_{\text{in}} \tag{3.17}$$

[1]Even if trajectory of the photon is not planar.

We can use the angle φ' and spatial part of vector $(\vec{k_{\text{in}}})'$ to find the actual deflection plane in the system of BH. It is given by the vector $(\vec{k_{\text{in}}})'$ and the rotation of $\vec{v}$ by $-\varphi'$ around the $(\vec{k_{\text{in}}})'$. Then we can rotate the spatial part of $(\vec{k_{\text{in}}})'$ in that plane by the angle $\Delta\phi'$ to find $(\vec{k_{\text{out}}})'$.

$$\left(k^t_{\text{out}}\right)' = \left(k^t_{\text{in}}\right)' = \frac{\omega'_{\text{in}}}{c} \qquad \text{and} \qquad k^i_{\text{out}} = R^i_j k^j_{\text{in}} \text{ for } i,j \in \{x,y,z\} \tag{3.18}$$

where $\omega'_{\text{in}} = 2\pi\nu'_{\text{in}}$ na ν'_{in} is given by (1.9)[2]

$$\nu'_{\text{in}} = \gamma(1+\beta\cos\rho_{\text{in}})\nu_{\text{in}} \tag{3.19}$$

and

$$\begin{aligned} R \;=\; & \\ \frac{1}{\mathcal{D}^2} \cdot & \begin{bmatrix} \mathcal{A}^2 + (\mathcal{B}^2+\mathcal{C}^2)\cos\Delta\phi', & \mathcal{AB}(1-\cos\Delta\phi') - \mathcal{CD}\sin\Delta\phi', & \mathcal{AC}(1-\cos\Delta\phi') + \mathcal{BD}\sin\Delta\phi' \\ \mathcal{AB}(1-\cos\Delta\phi') + \mathcal{CD}\sin\Delta\phi', & \mathcal{B}^2 + (\mathcal{A}^2+\mathcal{C}^2)\cos\Delta\phi', & \mathcal{BC}(1-\cos\Delta\phi') - \mathcal{AD}\sin\Delta\phi' \\ \mathcal{AC}(1-\cos\Delta\phi') - \mathcal{BD}\sin\Delta\phi', & \mathcal{BC}(1-\cos\Delta\phi') + \mathcal{AD}\sin\Delta\phi', & \mathcal{C}^2 + (\mathcal{A}^2+\mathcal{B}^2)\cos\Delta\phi' \end{bmatrix} \end{aligned} \tag{3.20}$$

$$\begin{aligned}
\mathcal{K}^2 &\equiv (k^x_{\text{in}})'^2 + (k^y_{\text{in}})'^2 + (k^z_{\text{in}})'^2 \\
\mathcal{A} &\equiv \frac{(k^z_{\text{in}})'}{\mathcal{K}^2}\left[(k^x_{\text{in}})'(k^y_{\text{in}})'(1-\cos\varphi') - (k^z_{\text{in}})'\mathcal{K}\sin\varphi'\right] - \\
&\quad \frac{(k^y_{\text{in}})'}{\mathcal{K}^2}\left[(k^x_{\text{in}})'(k^z_{\text{in}})'(1-\cos\varphi') + (k^x_{\text{in}})'\mathcal{K}\sin\varphi'\right] \\
\mathcal{B} &\equiv \frac{(k^z_{\text{in}})'}{\mathcal{K}^2}\left[(k^x_{\text{in}})'^2 + ((k^y_{\text{in}})'^2 + (k^z_{\text{in}})'^2)\cos\varphi'\right] - \\
&\quad \frac{(k^x_{\text{in}})'}{\mathcal{K}^2}\left[(k^x_{\text{in}})'(k^z_{\text{in}})'(1-\cos\varphi') + (k^x_{\text{in}})'\mathcal{K}\sin\varphi'\right] \\
\mathcal{C} &\equiv \frac{(k^y_{\text{in}})'}{\mathcal{K}^2}\left[(k^x_{\text{in}})'^2 + ((k^y_{\text{in}})'^2 + (k^z_{\text{in}})'^2)\cos\varphi'\right] - \\
&\quad \frac{(k^x_{\text{in}})'}{\mathcal{K}^2}\left[(k^x_{\text{in}})'(k^y_{\text{in}})'(1-\cos\varphi') - (k^z_{\text{in}})'\mathcal{K}\sin\varphi'\right] \\
\mathcal{D}^2 &\equiv \mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2
\end{aligned}$$

When we have the wave vector after the deflection in the system of BH, we can find the wave vector in the system of observer by inverse LT.

$$k^\mu_{\text{out}} = \left(\Lambda^{-1}\right)^\mu_{\ \nu} (k^\nu_{\text{out}})' \tag{3.21}$$

From time component of $\vec{k_{\text{out}}}$ we can find outgoing frequency ν_{out} as a function of b', φ' and ρ'_{in}.

$$\nu_{\text{out}} = \frac{c}{2\pi} k^t_{\text{out}} \tag{3.22}$$

[2]Speed is in opposite direction than in formula (1.9).

This formula may be used to calculate change in momentum of a given photon. However, to find total change in momentum of all photons coming from certain direction we use different approach.

Since the spacetime in finite distance around BH is not flat, we cannot use LT to find relation between the wave vector (or four-momentum) and position of a photon in the system of observer and the system of BH. However, if we know the total deflection angle, the initial direction and the deflection plane, equatios above can be used to find the deflection plane and the deflection angle in the system of BH. Below we give a way to find the trajectory of a photon in the system of BH. To make it work we need to know r'_s (or M') of BH.

Let us consider a light ray at some point in distance R' from BH. The angle between wave vector of a photon in the ray and radial direction (away from BH) in local tetrad is α'. Formula (2.19) allows us to calculate the angle by which this photon will be deflected.

$$\cos\Psi'_1 = 1 - \frac{1-\cos\alpha'}{1-r'_s/R'} \tag{3.23}$$

If we set the angle in formula (2.19) to $\alpha' - \pi/2$, we get the angle by which this photon has already been deflected.

$$\cos\Psi'_2 = 1 - \frac{1-\sin\alpha'}{1-r'_s/R'} \tag{3.24}$$

Total deflection angle in the system of BH must be

$$\Delta\phi' = \Psi'_1 + \Psi'_2 \tag{3.25}$$

and obviously $\alpha' \in \langle 0, \pi/2 \rangle$.

If we set polar coordinate system centered on BH so that equatorial plane is the plane of deflection and identify one of polar coordinates with the angle Ψ_1 or Ψ_2, the equations (3.23)–(3.25) together with the equations for $\Delta\phi'$ are paramtric equations for the null geodetic in this system.

3.3 Dynamical friction force due to homogeneous isotropic radiation

Formula (3.22) is complicated function of φ', $\rho'_{\rm in}$ and b'. Fortunately, there is a better way than this formula to find the total change of momentum of all photons coming from certain directions. We use axial symetry to transform the problem for all photons coming from a given direction to the problem of parallel rays, which we already solved, and then we use axial symetry around $\vec{v}$ to determine the total dragging force on BH.

First, we redefine the term initial direction. Here we use it to designate some specific direction vector in the system of BH. We use parameters ϱ', which is again the angle of the direction vector to the direction of velocity $\vec{v}$ and $\vartheta' \in \langle 0, 2\pi)$, which is used to select the plane given by the direction vector, the velocity vector and the position of BH. The direction at infinity, ($\varrho'_{\rm in}$, $\vartheta'_{\rm in}$) in the system of BH, can be transformed to the direction at infinity in the system of observer: $\vartheta'_{\rm in} = \vartheta_{\rm in}$ and $\varrho'_{\rm in}$ is transformed to $\varrho_{\rm in}$ by (1.7).

Coordinate system of observer is set so, that the x-axis is again in the direction of $\vec{v}$ and the xy-plane contains trajectory of BH and is parallel to the incoming wave vector. In this system, the incoming wave vector has components

$$\vec{k_{\rm in}} = \left(\frac{\omega_{\rm in}}{c}, \frac{\omega_{\rm in}}{c} \cos \rho_{\rm in}, \frac{\omega_{\rm in}}{c} \sin \rho_{\rm in}, 0 \right) \tag{3.26}$$

and in the system of BH

$$\vec{k_{\rm in}}' = \left(\frac{\omega'_{\rm in}}{c}, \frac{\omega'_{\rm in}}{c} \cos \varrho'_{\rm in}, \frac{\omega'_{\rm in}}{c} \sin \varrho'_{\rm in}, 0 \right) \tag{3.27}$$

where $\omega_{\rm in} = 2\pi\nu'_{\rm in}$, the $\nu'_{\rm in}$ is given by (3.19) and the $\varrho_{\rm in}$ is given by (1.7).

Photons coming from a given intial direction ($\varrho_{\rm in}$, $\vartheta_{\rm in}$) in the system of observer come from some initial direction ($\varrho'_{\rm in}$, $\vartheta'_{\rm in}$), which is given by LT, in the system of BH. All photons with the same impact parameter b' are deflected by the same angle $\Delta\phi'$ in the system of BH. If we take photons, that come from the same initial direction, due to the axial symetry of the problem, the only component of total change of their momentum that survives integration via all possible deflection planes has the exactly opposite direction as was the initial direction of the photons. We can use the formula (3.14) to find the total dragging force $F'_{\|}(b'_{\rm max}, \nu'_{\rm in})$ of all photons of certain frequency coming from infinity from some direction in the system of BH. If we know the incoming frequency $\nu_{\rm in}$ in the system of observer, frequency $\nu'_{\rm in}$ is a function of direction in the system of BH. We can rewrite $F'_{\|}(b'_{\rm max}, \nu'_{\rm in})$ to $F'_{\|}(b'_{\rm max}, v, \nu_{\rm in}, \rho'_{\rm in})$ easily. The $\nu'_{\rm in}$ is only a function of $\rho_{\rm in}$ component of the initial direction and in the equations (3.12) and (3.14) it is present only as a multiplicative term. Thus we can use (3.19) and simply multiply the force vector by the factor $\gamma(1 + \beta \cos \varrho_{\rm in})$ and go from $\varrho_{\rm in}$ to $\varrho'_{\rm in}$.

$$F'_{\angle}(b'_{\rm max}, v, \nu_{\rm in}, \varrho'_{\rm in}) = \gamma \left(1 + \beta \frac{\cos \varrho'_{\rm in} + \beta}{1 + \beta \cos \varrho'_{\rm in}} \right) \cdot F'_{\|}(b'_{\rm max}, \nu_{\rm in}) \tag{3.28}$$

Inverse LT of the force vector would obviously result in a force of opposite direction to the direction of photons ($\varrho_{\rm in}$, $\vartheta_{\rm in}$) in the system of observer.

We can integrate dragging force of photons coming from all directions in the system of BH and then transform it to the system of observer. Due to axial symetry of the problem, the only component of the force that can survive the integration via all directions must be parallel to the direction of velocity $\vec{v}$. That is why we can integrate only that component.

$$F'_{\parallel\vec{v}}(b'_{\max}, v, \nu_{\text{in}}, \varrho'_{\text{in}}) = F'_{\angle}(b'_{\max}, v, \nu_{\text{in}}, \varrho'_{\text{in}}) \cdot \cos\varrho'_{\text{in}} \tag{3.29}$$

Let the number of photons per time $\mathrm{d}t$ coming towards BH from some area $\mathrm{d}S$ of all planes at infinity that are perpendicular to all directions in some solid angle $\mathrm{d}\Omega$ is

$$\mathcal{J} = \frac{\mathrm{d}N}{\mathrm{d}t\mathrm{d}S\mathrm{d}\Omega} = \frac{\mathcal{F}}{\mathrm{d}\Omega} \tag{3.30}$$

in the system of observer. To get $\mathcal{J}'$ in the system of BH we need to do several transformations. First we review the relations for the value of an angle in the two systems.

$$\cos\alpha = \frac{\cos\alpha' + \beta}{\delta} \qquad \text{or} \qquad \sin\alpha = \frac{\sin\alpha'}{\gamma\delta} \tag{3.31}$$

where the δ is called Doppler factor.

$$\delta = 1 + \beta\cos\alpha \tag{3.32}$$

Then we can write

$$\frac{\mathrm{d}S}{\mathrm{d}S'} = \frac{\sin\alpha'}{\sin\alpha} = \gamma\delta \tag{3.33}$$

where α is an angle between the normal of the area $\mathrm{d}S$ and the velocity $\vec{v}$.

$$\frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{1}{\gamma} \tag{3.34}$$

and

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega'} = \frac{\sin\alpha\ \mathrm{d}\alpha}{\sin\alpha'\ \mathrm{d}\alpha'} = \frac{1}{\delta^2} \tag{3.35}$$

Now we can write

$$\mathcal{J}' = \frac{\mathrm{d}N}{\mathrm{d}t'\mathrm{d}S'\mathrm{d}\Omega'} = \frac{\mathrm{d}t}{\mathrm{d}t'}\frac{\mathrm{d}S}{\mathrm{d}S'}\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega'} \cdot \mathcal{J} = \frac{\mathcal{J}}{\delta} \tag{3.36}$$

The reason for introducing $\mathcal{J}$ is that during the integration of (3.29) we also have to integrate (3.14). The $\mathcal{J}$ is basicaly just $\mathcal{F}$, which appears at that integral, as a function of direction. Now if we are given this $\mathcal{J}$ in the system of observer, we have the required transformation we need to integrate the force in the system of BH. The transformation of $\mathcal{J}$ given above allowes us not to transform the solid angle $\mathrm{d}\Omega$ if we integrate through all directions in both systems.

If we fix ν_{in} and $\mathcal{F}$ to a constant values in the system of observer, it gives us the transformation of $\mathcal{F}$.

$$\mathcal{F}' = \delta\mathcal{F} \tag{3.37}$$

From the equations (3.37) and (3.29) we can see that there are only multiplicative factors that depend on the direction in the integral of total force. After applying all the transformations we can write

$$\begin{aligned} F'(b'_{\max}, v, \nu_{\text{in}}) &= \gamma F'_{\parallel}(b'_{\max}, v, \nu_{\text{in}}) \cdot \\ &\cdot \int_0^{2\pi}\int_0^{\pi} \left(1 + \beta\frac{\cos\varrho + \beta}{1 + \beta\cos\varrho}\right)(1 + \beta\cos\varrho)\cos\varrho \,\mathrm{d}\varrho'\,\mathrm{d}\vartheta \\ &= 2\pi^2\gamma\beta F'_{\parallel}(b'_{\max}, v, \nu_{\text{in}}) \end{aligned} \tag{3.38}$$

This is the dynamical friction force due to the deflection of homogeneous isotropic radiation in the system of BH. This integral approaches $-\infty$ as $v \to c$ and zero value as $v \to 0$. Resulting force accelerates BH in its comoving system with acceleration

$$\vec{A}' = \left(0, \frac{F'(b'_{\max}, v, \nu_{\text{in}})}{M'}, 0, 0\right) \tag{3.39}$$

We can use inverse LT to transform the acceleration A' for the observer.

$$\vec{A} = \left(\frac{\gamma\beta F'(b'_{\max}, v, \nu_{\text{in}})}{M'}, \frac{\gamma F'(b'_{\max}, v, \nu_{\text{in}})}{M'}, 0, 0\right) \tag{3.40}$$

Since the acceleration depends on the velocity via $F'_{\parallel}(b'_{\max}, v, \nu_{\text{in}})$, γ and β and the first one is always negative and the second and the third one are always positive, the acceleration is negative and leads to exponential slowdown of BH. We can find momentairly value of 3D acceleration in the system of observer as the x-component of the vector (3.40).

$$a = \frac{\gamma^2 F'(b'_{\max}, v, \nu_{\text{in}})}{M} \tag{3.41}$$

Last formula yelds the form of the three-force in the system of observer.

$$F = \gamma^2 F'(b'_{\max}, v, \nu_{\text{in}}) \tag{3.42}$$

3.4 Discussion

In this paragraph we take closer look at the method that was used in this chapter. The conditions when the starting assumptions hold are specified in more detail. One particular problem of the used method is discussed and possible solutions to it proposed. In the end we give a very rough estimate of the magnitude of the effect of dynamical friction on a stellar BH and point out one interesting possible application of the results that were obtained in this chapter.

We have already scratched the surface of the problem of the first two conditions. Spacetime metric around non-rotating and uncharged mass point that moves in our reference frame is not known to us. To find the metric, we would have to start with the stress-energy tensor of such point

$$T^{\mu\nu}(t,x,y,z) = \gamma m v^{\mu}(t) v^{\nu}(t) \delta[x - x(t)]\delta[y - y(t)]\delta[z - z(t)] \tag{3.43}$$

$$\vec{v}(t) \equiv \left(1, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)$$

and solve the Einstein field equations for the metric $g^{\mu\nu}$. We can assume that this spacetime is not flat. In the spacetime that is not flat we cannot use LT to transform vectors, so *in close vicinity of a moving BH we cannot make any statements about the behaviour of a photon or its trajectory.* However, as in the case of Schwarzschild metric, we can assume, that at very big distance from BH this metric is asymptoticaly flat. All transformations that we have done using LT were done in this "surrounding" frame at the very big distance from BH. Very big distance is such distance, where any photon that does not move almost radialy is not affected by BH at all.

The condition about the constant motion can be stated like follows: during the time, that a photon needs to come from the very big distance and escape back to the very big distance, the velocity of the BH remains approximately the same. If we take only one photon, unequality

$$h\nu \ll \frac{Mv^2}{2} \tag{3.44}$$

must hold or else the photon has simmilar kinetic energy as BH. In most of the cases, such a photon not only affects the direction and magnitude of the velocity vector of BH, but it would probably also change spacetime metric around itself.

The last condition sets upper limit to wavelength. If it was similar to the r_s, we would have to use wave optics and consider other effects like interference of the wave plane with itself, gravitational redshift and so on. Not to work in this mode, following must hold:

$$\frac{c}{\nu} \gg \frac{2GM}{c^2} \tag{3.45}$$

If all the above conditions are fulfilled, then the results obtained in this chapter could be used to investigate the frequency change of photons and the redistribution of momentum in the system of BH and the surrounding radiation. However, there is one more problem with the method itself. We have already metioned that the integral (3.14) diverges for big $b'_{\max}$. This property can be seen immediately from the linear term. At the same time, we assume that the space around BH is very big, even infinite. From real observations we know that our own universe is filled with photons that come from every direction. If we calculate the reaction of all, or much enough photons in our universe and assume, that they are affected by a gravitating body, like we did in this chapter, then this result inevetably leads to the conclusion that it either cannot have any velocity in any observer frame, which is obviously wrong, or that the assumption about the constant motion is invalidated immediately. This is inherent problem of the method and it is related to the hidden assumption that the gravitational interaction between the photons and BH is instant. That way, the photons that are very far from BH are of big influence, even though they deflected by very small angle, because there is just too much of them. The total effect is the bigger the more distant photons are considered.

The gravitational interaction is not instant. It can propagate at most with the speed of light. That way we could find quite good upper limit for photons that affect BH if we find out how long ago they could have had any influence. Photons that are further than the light could have traveled during the time BH exists could not possibly have affected it. Their effect also must be null if the velocity is zero. The relaxation time (after which BH would stop if the acceleration remained unchanged) is

$$T_{\text{stop}} = v/a \tag{3.46}$$

and the estimate for maximal impact parameter $b'_{\max} = T_{\text{stop}}.c$. Or we could use cosmological limits like hubble radius. No light further than that can reach us, as behind that radius spacetime expansion is faster than the speed of light. All of these are just possible estimates for the $b'_{\max}$ and each of them can give different value. Some of them bring more problems than they solve, for example with the expading universe, the real spacetime might not even be asymptoticaly flat and there most likely is some cosmological constant, so we should not use Schwarzschild metric at all. We should also check every estimate of $b'_{\max}$ against the assumptions before and also keep in mind that it should be big enough to make it possible to neglect the photons that fall to BH ($b' < b'_c$).

The point is, that every different value of maximal impact parameter gives a different result for the acceleration and dynamical friction. In fact, the best way to determine the $b'_{\max}$ value could be to use observed value of acceleration to find it. However, as the result depends on the $b'_{\max}$ very much, for almost any acceleration there could be found the corresponding value of $b'_{\max}$. There is still the question if this effect is even observable.

Let us try to answer this question. We are going to use this model to find change in dynamics of a stelar BH with mass $2M_{\odot}$. Let us suppose it is moving in a galaxy where the only light that can affect it and is affected by it originates in that galaxy and strong interstellar absorption does not allow for any light with $b'_{\max}$ bigger than 3000 ly to be considered. Based on a rough estimate of 0.001 stars per cubic light year, there is about 10^8 stars within that radius. If they are all as bright as our Sun they give out an energy which could be distribuded to a 4π (steradians of directions) times the surface of a circle with radius 3000 ly to make approximately $\mathcal{F} \doteq 4 \times 10^{44}$ photons with wavelength around 555 nm per square light year per second. Distance 3000 ly can be considered very far. Calculation of the integral (3.39) for speed v=300 km.s^{-1} results in dragging force due to the radiation in the order of about $F \doteq 1$ kN. All the assumptions are obviously fulfilled.

The example provided was not very practical. It showed that even in this semi-real situation this effect would probably be smaller than just about anything else that does have potential to change the dynamics of the system. It could still play some role only in cosmological times. But there is something even more usefull that can be done with the results provided in this chapter. If, in the system of BH, we know the deflection angle $\Delta\phi'$ and mass M' of BH, we can determine the impact parameter b' e.g. from (2.18). On the other hand, if we knew the b' and the angle $\Delta\phi'$, we could use the same formula to determine the mass M' of BH. As we have seen in the previous paragraphs, if BH moves with constant velocity $\vec{v}$, there is the change in frequency that depends on the deflection angle and hence on b'. But number of photons coming from certain direction that are deflected to another one depends on the impact parameter b'. Thus, if we know how many photons with a given frequency comes from any direction and with any impact parameter, we can calculate the spectrum of radiation that is deflected to some given direction. In reverse, this opens a new interesting possibility to determine some usefull parameters of the system like mass M' of BH or relative velocity $\vec{v}$ from the spectrum. This is one of the possible ways to continue with this work which certainly desrves more attention.

Conclusions

We have reviewed those aspects of both special and general theories of relativity that were necessary for our goal – the investigation of light scattering on a moving black hole. We gave particular emphasis to equations of geodetic in Schwarzschild spacetime.

The equations for two main effects were provided. First, the change of frequency of a particular photon and its outgoing direction after the deflection on a moving black hole were calculated, provided that the incoming parameters of the photon and the plane of deflection were known. Second, the formulas for total effect of photons of homogeneous isotropic radiation on a black hole moving in the rest frame of observer were given. This effect manifests itself as a kind of dynamical friction.

Limitations of model used to find those results were discussed. We have come to conclusion, that the effect of dynamical friction due to the gravitational bending of radiation has too small magnitude on a small scale to be measured and the problems of proposed model are too big to get relevant results on cosmological scales, where it may play some role.

However, the results for the frequency change of a photon bent from and to certain direction could be at certain cicumstances used in conjunction with known number of photons with certain parameters to find important parameters of the system from spectrum of deflected radiation. Althought the effects are likely to be of very small order, more attention should go to theoretical investigation of this possibility.

References

[1] C. W. Misner, K. S. Thorne, J. A. Wheeler: *Gravitation*, W. H. Freeman and Company, New York, 1973.

[2] R. M. Wald: *General Relativity*, The University of Chicago Press, Chicago, 1984.

[3] D. Mihalas, B. Weibel–Mihalas: *Foudations of Radiation Hydrodynamics*, Dover Publications, Inc., Mineola, New York, 1999.

[4] Z. Stuchlík, S. Hledík: *Some properties of the Schwarzschild-de Sitter and Schwarzschild-anti-de Sitter spacetimes*, Phys. Rev. **D** 60, 044006 (1999).

[5] S. M. Carrol: *Spacetime and Geometry: An Introduction to General Relativity*, Addison–Wesley, San Francisco, 2003.

[6] K. Schwarzschild: *On the Gravitational Field of a Mass Point According to Einstein's Theory*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, Phys.-Math. Klasse, 189-196 (1916).

[7] P. Amore, S. A. Diaz: *Analytical formulas for gravitational lensing*, Phys. Rev. **D** 73, 083004 (2006).

[8] P. T. Mutka, P. H. Mähönen: *Approximation of Light-Ray Deflection Angle and Gravitational Lenses in the Schwarzschild Metric*, ApJL **576**, L107–L112, (2002).

[9] A. M. Beloborodov: *Gravitational Bending of Light Near Compact Objects*, ApJL **566**, L85–L88 (2002).

[10] V. Bozza: *Gravitational lensing in the strong field limit*, Phys. Rev. **D** 66, 103001 (2002).

[11] S. Chandrasekhar: *Mathematical Theory of Black Holes*, Oxford University Press, Oxford, 1998.

The Official Unabashed Scientific Dictionary defines black holes as what you get in black socks.