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DIPLOMOVÁ PRÁCE

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Compactness of operators on function spaces

Katedra matematické analýzy
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Prohlašuji, že jsem svou diplomovou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 15.4.2010 Eva Pernecká

Contents

Název práce: Kompaktnost operátorů na prostorech funkcí
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Abstrakt: Operátory Hardyho typu obsahující suprema se ukázaly být užitečným nástrojem v teorii interpolací, pro odvození nerovností Sobolevova typu, pro odhady nerostoucích přerovnání frakčních maximálních funkcí či pro popis norem vyskytujících se v optimálních Sobolevových vnořeních. Tato práce se zabývá kompaktností těchto operátorů na váhových Banachových prostorech funkcí. Definujeme jistou kategorii párů váhových Banachových prostorů funkcí a vyslovíme a dokážeme kritérium pro kompaktnost operátoru Hardyho typu obsahujícího supremum, který působí mezi dvojicí prostorů náležející do této kategorie. Dále ukážeme, že zmíněná kategorie zahrnuje jisté dvojice váhových Lebesgueových prostorů určené vztahem mezi jejich exponenty. Kromě toho přineseme rozšíření kritéria na všechny váhové Lebesgueovy prostory, přičemž v důkazu využijeme charakterizaci kompaktnosti operátorů s oborem hodnot v kuželu nezáporných nerostoucích funkcí, kterou uvádíme jako samostatný výsledek.
Klíčová slova: operátor, Banachův prostor funkcí, kompaktnost, operátory Hardyho typu obsahující suprema, váhový Lebesgueův prostor

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Abstract: Hardy-type operators involving suprema have turned out to be a useful tool in the theory of interpolation, for deriving Sobolev-type inequalities, for estimates of the non-increasing rearrangements of fractional maximal functions or for the description of norms appearing in optimal Sobolev embeddings. This thesis deals with the compactness of these operators on weighted Banach function spaces. We define a category of pairs of weighted Banach function spaces and formulate and prove a criterion for the compactness of a Hardy-type operator involving supremum which acts between a couple of spaces belonging to this category. Further, we show that the category contains specific pairs of weighted Lebesgue spaces determined by a relation between the exponents. Besides, we bring an extension of the criterion to all weighted Lebesgue spaces, in proof of which we use characterization of the compactness of operators having the range in the cone of non-negative non-increasing functions, introduced as a separate result.
Keywords: operator, Banach function space, compactness, Hardy-type operators involving suprema, weighted Lebesgue space

Introduction

Hardy-type operators involving suprema have recently become an object of increased interest because of their role in the limiting interpolation theory or in the search for optimal pairs of r.i. norms for which a Sobolev-type inequality holds. In addition to this, they are used to characterize the associate norm of an operator-induced norm, which acts as an optimal domain norm in a Sobolev embedding. Also the fact that these operators stand on both ends of a sharp rearrangement inequality for the fractional maximal operator (see [2]) confirms that they are of great importance. The main aim of this thesis is to introduce a criterion for the compactness of two-weighted Hardy type operators involving suprema on weighted Banach function spaces and to pay extra attention to the study of this problem for the special case of weighted Lebesgue spaces.

In Chapter 1 we establish the basic setting for our work drawing from functional analysis.

Chapter 2 is devoted to a brief survey of the theory of Banach function spaces following Bennett and Sharpley [1] with focus on the issues of absolute continuity of norm and associate space. A section about, what we call, weighted Lebesgue spaces is also a part of this chapter.

The main results are presented in Chapter 3, which is structured into three sections. In Section 1 we characterize the compactness of bounded operators with their values in the cone of non-negative non-increasing functions on weighted Banach function spaces. The key tool is the absolute continuity of norm. In Section 2 we establish a necessary and sufficient condition for a two-weighted mapping involving supremum to be a compact operator between a pair of weighted Banach function spaces which belongs to a category determined by assumptions concerning the boundedness of the mapping. The approach is based on the methods in the spirit of those developed by Edmunds, Gurka and Pick when dealing with the compactness of Hardy-type integral operators on weighted Banach function spaces in [3]. In Section 3, using the weighted inequalities for Hardy-type operators involving suprema derived by Gogatishvili, Opic and Pick in [4], we show that the outcome of the second section is applicable to a couple of weighted Lebesgue spaces with the exponent of the domain space less than or equal to the exponent of the target space. Besides, combining the result of the first section with the technique of discretization and antidiscretization, we perform a self-contained proof of characterization of the compactness of two-weighted Hardy type operators involving suprema on a general pair of weighted Lebesgue spaces with any relation between the exponents.

Chapter 1

Preliminaries

To begin, we establish a necessary background for our work.

Definition 1.0.1. A *Banach space* is normed vector space, which is complete in the metric defined by its norm, which means that all Cauchy sequences converge.

Definition 1.0.2. Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively . By an *operator* T from X into Y we understand a well-defined mapping of the space X into the space Y and denote it by $T : X \to Y$.

We say that an operator $T : X \to Y$ is *bounded* if for every $x \in X$,

$$\|Tx\|_Y \leq c(T)\|x\|_X,$$

where $c(T) \geq 0$ is a constant independent of x.

An operator $T : X \to Y$ is *compact* if $\overline{\{Tx; x \in X, \|x\|_X \leq 1\}}$ is a compact set in Y. This is true if and only if the set $\{Tx; x \in X, \|x\|_X \leq 1\}$ is totally bounded, because Y is complete. Another equivalent characterization of the compactness of operator $T : X \to Y$ says that T is compact if and only if every sequence $\{x_n\}$ lying in the closed unit ball of X contains a subsequence $\{x_{n_k}\}$ such that the sequence $\{Tx_{n_k}\}$ is convergent in Y.

An operator $T : X \to Y$ is said to be of *finite rank* if $\{Tx; x \in X\}$ is a subset of a finite-dimensional subspace of Y.

Note that if $T : X \to Y$ is compact and $T(ax) = aT(x)$ for every $x \in X$ and every $a > 0$, then T is bounded.

Proposition 1.0.3. *Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively.*

(i) If $T : X \to Y$ is a bounded operator of finite rank, then T is compact.

(ii) If $T : X \to Y$ is an operator such that for every $\varepsilon > 0$ there exists a compact operator $S : X \to Y$ satisfying

$$\sup\{\|Tx - Sx\|_Y; x \in X, \|x\|_X \leq 1\} < \varepsilon,$$

then T is a compact operator.

Proof. (i) This is an immediate consequence of the fact that the closed unit ball of a finite-dimensional Banach space is compact.

(ii) For $\{x_n\} \subset \{x \in X; \|x\|_X \leq 1\}$ find a subsequence $\{x_{n_k}\}$ such that $\{Sx_{n_k}\}$ is convergent in Y. Because

$$\|Tx_{n_k} - Tx_{n_l}\|_Y \leq \|Tx_{n_k} - Sx_{n_k}\|_Y + \|Sx_{n_k} - Sx_{n_l}\|_Y + \|Sx_{n_l} - Tx_{n_l}\|_Y,$$

$\{Tx_{n_k}\}$ is Cauchy, thus convergent in Y.

□

Notation 1.0.4. We adopt convention that $c(\cdot)$ denotes a constant depending only on the parameters enumerated in the parentheses. The value of the constant may change even within one string of (in)equalities.

We use the symbol λ to denote the one-dimensional Lebesgue measure on $\mathbb{R}$.

Chapter 2

Banach function spaces

2.1 The fundamentals of Banach function spaces

The common features of various classes of Banach spaces consisting of measurable functions gave rise to the abstract theory of the so-called Banach function spaces. These are Banach spaces of measurable functions possessing a norm related to the underlying measure. Furthermore, they are enriched by a natural order structure given by a pointwise comparison of functions. The nature of the Banach function spaces provides thus scope for an interesting interplay between functional analysis, measure theory and theory of lattices. The essential, and for our further purposes also the most significant, examples of Banach function spaces are the Lebesgue spaces. Because of the role of weighted Lebesgue spaces in our work, we shall take a closer look at them in an individual section thereinafter. To continue, among important Banach function spaces we can include the Lorentz spaces or the Orlicz spaces, for instance. The origin of most of the information to be presented in this section is in the book by Bennett and Sharpley [1].

Let (Ω, μ) be a totally σ-finite measure space, $\mathcal{M}(\Omega, \mu)$ the collection of all μ-measurable functions on Ω whose values lie in $[-\infty, \infty]$, $\mathcal{M}^+(\Omega, \mu)$ the cone of all functions from $\mathcal{M}(\Omega, \mu)$ with their values in $[0, \infty]$ and $\mathcal{M}_0(\Omega, \mu)$ the class of functions from $\mathcal{M}(\Omega, \mu)$ which are finite μ-a.e. on Ω . The characteristic function of a μ-measurable set E is denoted by χ_E. By a simple function we understand a finite sum of functions, each of which is defined as a finite real multiple of a characteristic function of a set having finite measure.

Definition 2.1.1. A mapping $\rho : \mathcal{M}^+(\Omega, \mu) \to [0, \infty]$ is called a *Banach function norm* if, for all f, g, f_n, $(n = 1, 2, 3, \ldots)$, in $\mathcal{M}^+(\Omega, \mu)$, for all constants $a \geq 0$, and for all μ-measurable subsets E of Ω, the following properties hold:

(P1) $\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.;

(P2) $\rho(af) = a\rho(f)$;

(P3) $\rho(f + g) \leq \rho(f) + \rho(g)$;

(P4) $g \leq f$ μ-a.e. $\Rightarrow \rho(g) \leq \rho(f)$;

(P5) $f_n \uparrow f$ μ-a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$;

(P6) $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$;

(P7) $\mu(E) < \infty \Rightarrow \int_E f\, d\mu \leq C_E \rho(f)$,
for some constant $C_E \in (0, \infty)$ depending on E and ρ but independent of f.

Definition 2.1.2. For a Banach function norm $\rho : \mathcal{M}^+(\Omega, \mu) \to [0, \infty]$, we call a *Banach function space* the collection of all functions[1] f in $\mathcal{M}(\Omega, \mu)$ for which $\rho(|f|) < \infty$. We denote it by (X, ρ), or shortly X. For each $f \in X$, we define $\|f\|_X = \rho(|f|)$.

We state the basic properties of just defined Banach function spaces in the theorem below. Their proofs can be found in [1], Chapter 1, Section 1.

Theorem 2.1.3. *Let $\rho : \mathcal{M}^+(\Omega, \mu) \to [0, \infty]$ be a Banach function norm. Then the Banach function space $X = (X, \rho)$ is a vector space (under the multiplication by scalars and sum of the functions) and $\|\cdot\|_X$ is a norm on X. The vector space X equipped with the norm $\|\cdot\|_X$ is a Banach space and the following properties hold for all f, g, f_n, $(n = 1, 2, 3, \dots)$, in $\mathcal{M}(\Omega, \mu)$ and all μ-measurable subsets E of Ω:*

(i) *(the lattice property) If $|g| \leq |f|$ μ-a.e. and $f \in X$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$.*

(ii) *(the Fatou property) Suppose $f_n \in X, f_n \geq 0, (n = 1, 2, \dots)$, and $f_n \uparrow f$ μ-a.e. Then either $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$ or $f \notin X$ and $\|f_n\|_X \uparrow \infty$.*

(iii) *(Fatou's lemma) Assume that $f_n \in X, (n = 1, 2, \dots), f_n \to f$ μ-a.e., and $\liminf_{n\to\infty} \|f_n\|_X < \infty$. Then $f \in X$ and $\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X$.*

(iv) *Every simple function belongs to X.*

(v) *If $\mu(E) < \infty$, then there is a constant $C_E \in (0, \infty)$ such that $\int_E |f|\, d\mu \leq C_E \|f\|_X$ for all $f \in X$.*

(vi) *If $f_n \to f$ in X, then $f_n \to f$ in measure on every set of finite measure. In particular, there exists a subsequence of $\{f_n\}$ converging pointwise μ-a.e. to f.*

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[1] Any two functions coinciding μ-a.e. are identified.

Definition 2.1.4. A function f belonging to a Banach function space X has *absolutely continuous norm* in X if $\lim_{n\to\infty} \|f\chi_{E_n}\|_X = 0$ for every sequence $\{E_n\}_{n=1}^{\infty}$ of μ-measurable subsets of Ω such that $\chi_{E_n} \to \chi_{\emptyset}$ μ-a.e. on Ω. The set of all functions in X with absolutely continuous norm is denoted by X_a. Provided X_a coincides with X, the space X itself is said to have absolutely continuous norm.

Definition 2.1.5. In a Banach function space X a subset Y of X_a is of *uniformly absolutely continuous norm* if, for every sequence $\{E_n\}_{n=1}^{\infty}$ of μ-measurable subsets of Ω, such that $\chi_{E_n} \to \chi_{\emptyset}$ μ-a.e. on Ω, and each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ satisfying

$$f \in Y,\ n \geq n_0 \ \Rightarrow \|f\chi_{E_n}\|_X < \varepsilon.$$

We state one useful, still quite simple, observation concerning elements of X_a, whose proof is in Chapter 1, Section 3 of [1].

Proposition 2.1.6. *If $f \in X$ has absolutely continuous norm, then to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that for every μ-measurable set $E \subset \Omega$ with $\mu(E) < \delta$ we have $\|f\chi_E\|_X < \varepsilon$.*

With respect to our later needs, we are interested in the question of the absolute continuity of norm of characteristic functions.

Definition 2.1.7. For a Banach function space X define X_b to be the closure of the set of simple functions in X in the topology given by the norm $\|\cdot\|_X$.

Theorem 2.1.8. *Let X be a Banach function space. Then $X_a \subset X_b$. The subspaces X_a and X_b coincide if and only if for every set E of finite measure, the characteristic function χ_E has absolutely continuous norm.*

For proof cf. [1], Chapter 1, Section 3.

Lemma 2.1.9. *Let X and Y be Banach function spaces equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. If $R : X \to Y$ and $S : X \to Y$ are compact operators, then the mapping T defined by*

$$(Tf)(t) = \max\{(Rf)(t), (Sf)(t)\}, \quad \forall f \in X, \forall t \in \Omega,$$

is a compact operator from X to Y.

Proof. First, $T : X \to Y$. Indeed, for each $f \in X$ and $t \in \Omega$,

$$|Tf(t)| \leq |Rf(t)| + |Sf(t)|,$$

so by the lattice property of Y and the fact that both operators R and S map X into Y, T is an operator from X to Y. Take a sequence $\{f_n\}$ lying in $\{f \in X; \|f\|_X \leq 1\}$. Due to the assumption, there is a subsequence $\{f_{n_k}\}$

such that both $\{Rf_{n_k}\}$ and $\{Sf_{n_k}\}$ are convergent, thus Cauchy, sequences in Y. Fix $t \in \Omega$. Then

$$\begin{aligned}(Tf_{n_k})(t) - (Tf_{n_l})(t) &= \max\{(Rf_{n_k})(t) - \max\{(Rf_{n_l})(t), (Sf_{n_l})(t)\},\\ &\qquad (Sf_{n_k})(t) - \max\{(Rf_{n_l})(t), (Sf_{n_l})(t)\}\}\\ &\leq \max\{(Rf_{n_k})(t) - (Rf_{n_l})(t), (Sf_{n_k})(t) - (Sf_{n_l})(t)\}\\ &\leq |(Rf_{n_k})(t) - (Rf_{n_l})(t)| + |(Sf_{n_k})(t) - (Sf_{n_l})(t)|.\end{aligned}$$

By symmetry,

$$|(Tf_{n_k})(t) - (Tf_{n_l})(t)| \leq |(Rf_{n_k})(t) - (Rf_{n_l})(t)| + |(Sf_{n_k})(t) - (Sf_{n_l})(t)|.$$

Hence
$$\|Tf_{n_k} - Tf_{n_l}\|_Y \leq \|Rf_{n_k} - Rf_{n_l}\|_Y + \|Sf_{n_k} - Sf_{n_l}\|_Y$$
and we see that also $\{Tf_{n_k}\}$ is Cauchy, consequently convergent, in Y. □

Definition 2.1.10. If ρ is a Banach function norm, we define its *associate norm* ρ' at $g \in \mathcal{M}^+(\Omega, \mu)$ by

$$\rho'(g) = \sup\left\{\int_\Omega fg\, d\mu; f \in \mathcal{M}^+(\Omega, \mu), \rho(f) \leq 1\right\}.$$

Theorem 2.1.11. *If ρ is a Banach function norm, then its associate norm ρ' is a Banach function norm as well.*

Definition 2.1.12. Let ρ be a Banach function norm, $X = (X, \rho)$ be the Banach function space determined by ρ and ρ' be the associate norm of ρ. The Banach function space $X' = (X', \rho')$ determined by ρ' is called the *associate space* of X.

The definitions of "associate notions" imply that for a function g belonging to the associate space X',

$$\|g\|_{X'} = \sup\left\{\int_\Omega |fg| d\mu; f \in X, \|f\|_X \leq 1\right\},$$

where $\|g\|_{X'} = \rho'(|g|)$ by definition.

Theorem 2.1.13 (Hölder's inequality)**.** *Let X be a Banach function space and X' be its associate space. Provided $f \in X$ and $g \in X'$, function fg is integrable and*
$$\int_\Omega |fg| d\mu \leq \|f\|_X \|g\|_{X'}.$$

For more details about associate norms and spaces including omitted proofs see Chapter 1, Section 2 of [1].

Definition 2.1.14. Function v is a *weight* if it is Lebesgue-measurable, positive and finite λ-a.e. on $(0,\infty)$ and if to each $x \in (0,\infty)$ there corresponds $\delta > 0$ such that $\int_{x-\delta}^{x+\delta} v(t)dt < \infty$.

Remark 2.1.15. Let v be a weight. In a special case when the underlying measure space is the interval $(0,\infty)$ endowed with a measure ν given by $\nu(E) = \int_E v(t)dt$ for every Lebesgue-measurable subset E of $(0,\infty)$, we denote a Banach function space X built upon this setting by $X(v)$ and call it a *weighted Banach function space*. Note that from the definition of a weight follows that compact sets have finite measure and hence their characteristic functions are elements of space $X(v)$. In what follows, we shall work solely with weighted Banach function spaces.

What we have presented here is just a brief overview of objects and some of their properties, that will occur in the subsequent sections, and was by no means intended as a comprehensive survey of the theory of Banach function spaces. This area is much richer and contains plenty of issues to study. For instance, one can focus on other properties of Banach function spaces from functional-analytic point of view, such as duality or reflexivity. There is also widely developed theory of a considerable subclass of Banach function spaces called rearrangement-invariant Banach function spaces and theory of interpolation of operators on Banach function spaces.

2.2 Weighted Lebesgue spaces

Let (Ω,μ) be a totally σ-finite measure space. The Lebesgue spaces $L^p((\Omega,\mu))$ constructed upon the measure space (Ω,μ), with which we are familiar from measure theory, can be regarded as the Banach function spaces derived from the Banach function norms defined for $f \in \mathcal{M}^+(\Omega,\mu)$ by

$$\rho_p(f) = \begin{cases} \left(\int_\Omega f^p d\mu\right)^{\frac{1}{p}} & \text{when } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{t\in\Omega} f(t) & \text{when } p = \infty. \end{cases}$$

From this point of view, especially, for the measure space $((0,\infty),\nu)$ described in Remark 2.1.15, we obtain an example of weighted Banach function spaces, the so-called weighted Lebesgue spaces $L^p(v)$. Their exact definition as well as some of their properties, selected with our further intentions in mind, are presented in this section. Although most of the results listed below can be extended for general Lebesgue spaces, we decided to formulate them only for the special class of weighted Lebesgue spaces, as that is the form in which we shall apply them later. Some statements are followed by a short note in which we discuss the general case of Lebesgue spaces. Majority of these items can be found in [1] or [6].

If $1 \le p < \infty$, the conjugate number p' is given by

$$p' = \begin{cases} \frac{p}{p-1} & \text{when } 1 < p < \infty, \\ \infty & \text{when } p = 1. \end{cases}$$

Definition 2.2.1. For $p \in [1, \infty]$ and a weight v introduced in Definition 2.1.14, we define the *weighted Lebesgue space* $L^p(v)$ as the set of all Lebesgue-measurable functions[2] f on $(0, \infty)$, for which the inequality $\|f\|_{p,v} < \infty$ holds, where

$$\|f\|_{p,v} = \begin{cases} \left(\int_0^\infty |f(t)|^p v(t) dt\right)^{\frac{1}{p}} & \text{when } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{0<t<\infty} |f(t)| & \text{when } p = \infty. \end{cases}$$

Theorem 2.2.2. *Let $1 \le p \le \infty$ and let v be a weight. Then the weighted Lebesgue space $L^p(v)$ is a Banach function space. Moreover, on condition that $1 \le p < \infty$, $L^p(v)$ has absolutely continuous norm.*

Proof. It is enough to verify that $\|\cdot\|_{p,v}$ restricted to $\mathcal{M}^+((0,\infty),\lambda)$ is a Banach function norm, i.e. (P1)-(P7) is true, because then clearly $L^p(v)$ will be the Banach function space determined by the Banach function norm defined as the restriction of $\|\cdot\|_{p,v}$ to $\mathcal{M}^+((0,\infty),\lambda)$. Conditions (P1), (P2), (P4) and (P6) are obvious. Item of (P3) is in classical measure theory known as the Minkowski inequality. (P5) is a consequence of the monotone convergence theorem and (P6) of Hölder's inequality. Absolute continuity of norm follows simply from the Lebesgue dominated convergence theorem. $\square$

Theorem 2.2.2 is valid even for more general Lebesgue spaces $L^p(\Omega, \mu)$ over a totally σ-finite measure space (Ω, μ). However, in case $p = \infty$, the underlying measure space affects answer to the question of absolute continuity of norm. For instance, if the measure μ is continuous, then $L^\infty(\Omega, \mu)_a = \{0\}$, while for a discrete measure μ, like in case of the space l^∞ built over natural numbers with the counting measure, $l^\infty_a = c_0$.

By virtue of Theorem 2.2.2, from this moment on by a weighted Lebesgue space $L^p(v)$ we understand the corresponding Banach function space

$$(L^p(v), \|\cdot\|_{p,v} \text{ restricted to } \mathcal{M}^+((0,\infty),\lambda)).$$

We see that $\|\cdot\|_{L^p(v)} = \|\cdot\|_{p,v}$ on $L^p(v)$.

Theorem 2.2.3. *Each function $f \in L^p(v)$ is p-mean continuous, which means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $h \in \mathbb{R}$ with $|h| < \delta$ we have*

$$\int_0^\infty |f(t+h) - f(t)|^p v(t) dt < \varepsilon^p,$$

where f is defined by 0 outside the interval $(0, \infty)$.

[2]Two functions being identified if they coincide λ-a.e. on $(0, \infty)$.

Sketch of the proof. We first find an open bounded subinterval I of $(0,\infty)$, such that $\left\|f\chi_{(0,\infty)\setminus I}\right\|_{p,v}$ is small enough. Since v as a weight is locally integrable, the rest of the proof can be carried out in the same way as in [5, Theorem 2.4.2], where the assertion is formulated for a Lebesgue space constructed upon a nonempty bounded open subset of $\mathbb{R}^n$ endowed with the n-dimensional Lebesgue measure. □

The study of compactness of operators on Banach spaces goes hand in hand with the theory of compact sets in corresponding Banach spaces. Here is a consequence of the compactness of a set in a weighted Lebesgue space.

Theorem 2.2.4. *Let $1 \le p < \infty$. A compact set $A \subset L^p(v)$ is p-mean equicontinuous, i.e.*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall f \in A:\ |h| < \delta \Rightarrow \int_0^\infty |f(t+h) - f(t)|^p v(t)dt < \varepsilon^p,$$

where f is defined by 0 outside the interval $(0,\infty)$.

Proof. Define $f^h(t) = f(t+h)$ for every $f \in L^p(v)$, $h \in \mathbb{R}$ and $t \in (0,\infty)$. For a given $\varepsilon > 0$ find $f_1,\dots,f_k \in A$ such that for every $g \in A$ there is $i \in \{1,\dots,k\}$ for which $\|g - f_i\|_{p,v} < \frac{\varepsilon}{3}$. According to Theorem 2.2.3, each of the functions $f_1,\dots,f_k$ is p-mean continuous. Fix $\delta > 0$ satisfying $\left\|f_i^h - f_i\right\|_{p,v} < \frac{\varepsilon}{3}$ for every $h \in \mathbb{R}$ with $|h| < \delta$ and every $i \in \{1,\dots,k\}$. Then for $h \in (-\delta,\delta)$ and $f \in A$ we have

$$\left\|f^h - f\right\|_{p,v} \le \left\|f^h - f_i^h\right\|_{p,v} + \left\|f_i^h - f_i\right\|_{p,v} + \|f_i - f\|_{p,v} < \varepsilon,$$

where i is such that $\|f - f_i\|_{p,v} < \frac{\varepsilon}{3}$. □

Theorem 2.2.5. *$L^{p'}(v)$ is the associate space of $L^p(v)$.*

This statement is true for any Lebesgue space $L^p((\Omega,\mu))$ and is proved in Chapter 1, Section 2 of [1].

Chapter 3

The main results

3.1 Compactness of operators having range in non-negative and non-increasing functions on weighted Banach function spaces

Our first result brings in characterization of compact bounded operators with range in the cone of non-negative non-increasing functions on weighted Banach function spaces in terms of uniform absolute continuity of norm. This class of operators contains among others operators involving suprema, on which we shall focus in the remainder of the chapter. Analogical outcome was presented by Luxemburg and Zaanen in [7], however for integral operators with kernels. This does not cover our case, e.g. because of the linearity of mentioned integral operators, which is not necessary true in our setting. On the other hand, neither the following theorem is a generalization of that due to Luxemburg and Zaanen, since the integral operator does not have to produce non-increasing functions, which is what we require.

Theorem 3.1.1. *Let $X = X(v)$ and $Y = Y(w)$ be weighted Banach function spaces equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Assume that $Y_a = Y_b$. For a bounded operator R from X to Y, such that Rf is non-negative non-increasing function for each $f \in X$ and $\{Rf; f \in X, \|f\|_X \leq 1\} \subset Y_a$, the following two statements are equivalent:*

(i) *The operator R is compact from X to Y.*

(ii) *The set $\{Rf; f \in X, \|f\|_X \leq 1\}$ is of uniformly absolutely continuous norm in Y.*

Proof. Assume first that R is compact. Consider $\varepsilon > 0$ and a sequence $\{E_n\}$ of λ-measurable subsets of $(0,\infty)$, such that $\chi_{E_n} \to \chi_\emptyset$ λ-a.e. on

$(0,\infty)$[1]. Since $\overline{\{Rf; f \in X, \|f\|_X \le 1\}}$ is compact, there exist $k \in \mathbb{N}$ and a set $\{g_1, \ldots, g_k\} \subset \overline{\{Rf; f \in X, \|f\|_X \le 1\}}$ with the following property:

$$\forall g \in \overline{\{Rf; f \in X, \|f\|_X \le 1\}}\ \exists i \in \{1, \ldots, k\} : \|g - g_i\|_Y < \frac{\varepsilon}{2}.$$

According to the assumption, all functions in $\overline{\{Rf; f \in X, \|f\|_X \le 1\}}$ have absolutely continuous norms. Therefore, there is an $n_0 \in \mathbb{N}$ satisfying that whenever $n \ge n_0$, the inequality $\|g_i \chi_{E_n}\|_Y < \frac{\varepsilon}{2}$ holds for every $i = 1, \ldots, k$. Thus,

$$\|(Rf)\chi_{E_n}\|_Y \le \|(Rf - g_i)\chi_{E_n}\|_Y + \|g_i \chi_{E_n}\|_Y < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where $i \in \{1, \ldots, k\}$ is chosen to satisfy $\|Rf - g_i\|_Y < \frac{\varepsilon}{2}$. Hence the set $\{Rf; f \in X, \|f\|_X \le 1\}$ is of uniformly absolutely continuous norm in Y.

Conversely, suppose that the set $\{Rf; f \in X, \|f\|_X \le 1\}$ is of uniformly absolutely continuous norm in Y. Then for any $\eta > 0$, there exist $0 < a < b < \infty$, such that $\left\|(Rf)\chi_{(0,a)}\right\|_Y < \frac{\eta}{2}$ and $\left\|(Rf)\chi_{(b,\infty)}\right\|_Y < \frac{\eta}{2}$ for each $f \in X$ with $\|f\|_X \le 1$. Hence

$$\sup_{\|f\|_X \le 1} \left\|(Rf)\chi_{(0,a)}\right\|_Y + \sup_{\|f\|_X \le 1} \left\|(Rf)\chi_{(b,\infty)}\right\|_Y \le \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

We can write

$$(Rf)(t) = (Rf)(t)\chi_{(0,a)}(t) + (Rf)(t)\chi_{[a,b]}(t) + (Rf)(t)\chi_{(b,\infty)}(t)$$

for each $f \in X$ and each $t \in (0,\infty)$. Set $(Tf)(t) = (Rf)(t)\chi_{[a,b]}(t)$ for any $f \in X$ and $t \in (0,\infty)$. In view of Proposition 1.0.3, it is enough to show that T is a compact operator in order to obtain the compactness of R. Take an arbitrary $\varepsilon > 0$. By Proposition 2.1.6 applied to the function $\chi_{[a,b]} \in Y_b = Y_a$, we find[2] $\delta > 0$ for which $\left\|\chi_{[c,d]}\right\|_Y < \varepsilon$ for every $a \le c < d \le b$ such that $d - c < \delta$. Consider a partition $a = \alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \alpha_n = b$ of the interval $[a,b]$ with $\alpha_i - \alpha_{i-1} < \delta$ for all $i \in \{1, \ldots, n\}$. Denote $I_i = [\alpha_{i-1}, \alpha_i)$ for $i \in \{1, \ldots, n-1\}$ and $I_n = [\alpha_{n-1}, \alpha_n]$. Define mapping S by

$$(Sf)(t) = \sum_{i=1}^{n} (Rf)(\alpha_i)\chi_{I_i}(t), \quad f \in X,\ t \in (0,\infty).$$

By virtue of the properties of R, namely that Rf is non-negative and non-increasing function for each $f \in X$ and R is a bounded operator, for arbitrary

[1]Thanks to the properties of weights, namely that they are positive λ-a.e. on $(0,\infty)$, this is equivalent to the pointwise convergence on a set $A \subset (0,\infty)$ for which $\int_{(0,\infty)\setminus A} w(t)dt = 0$.

[2]Here it is important that $\int_a^b w(t)dt < \infty$ and therefore to each $\eta > 0$ there corresponds a $\delta > 0$ such that $\int_E w(t)dt < \eta$ for any λ-measurable set $E \subset [a,b]$ with $\lambda(E) < \delta$.

$x \in (0, \infty)$ and $f \in X$ we have

$$\begin{aligned}(Rf)(x) &= \left\|\chi_{(y,x]}\right\|_Y^{-1} \left\|(Rf)(x)\chi_{(y,x]}\right\|_Y \\ &\leq \left\|\chi_{(y,x]}\right\|_Y^{-1} \left\|Rf\chi_{(y,x]}\right\|_Y \\ &\leq \left\|\chi_{(y,x]}\right\|_Y^{-1} \|Rf\|_Y \\ &\leq \left\|\chi_{(y,x]}\right\|_Y^{-1} c(R)\|f\|_X,\end{aligned}$$

where y is a point in the interval $(0, x)$ and the constant $c(R)$ satisfies $\|Rf\|_Y \leq c(R)\|f\|_X$ for all $f \in X$. Application to $x = \alpha_i$ for any $i \in \{0, \ldots, n\}$ gives

$$(Rf)(\alpha_i) \leq \left\|\chi_{(\frac{a}{2},\alpha_i]}\right\|_Y^{-1} c(R)\|f\|_X \leq \left\|\chi_{(\frac{a}{2},a]}\right\|_Y^{-1} c(R)\|f\|_X.$$

Moreover, $\chi_{I_i} \in Y$ because $I_i \subset [a, b]$. From the preceding it follows that S is a bounded finite rank, by Proposition 1.0.3 consequently compact, operator from X to Y. For $f \in X$ we get

$$\begin{aligned}\|Tf - Sf\|_Y &= \left\|Rf\chi_{[a,b]} - \sum_{i=1}^{n}(Rf)(\alpha_i)\chi_{I_i}\right\|_Y \\ &= \left\|\sum_{i=1}^{n}\left[Rf - (Rf)(\alpha_i)\right]\chi_{I_i}\right\|_Y \\ &\leq \sum_{i=1}^{n}\left\|\left[Rf - (Rf)(\alpha_i)\right]\chi_{I_i}\right\|_Y \\ &\leq \sum_{i=1}^{n}\left\|\left[(Rf)(\alpha_{i-1}) - (Rf)(\alpha_i)\right]\chi_{I_i}\right\|_Y \\ &= \sum_{i=1}^{n}\left[(Rf)(\alpha_{i-1}) - (Rf)(\alpha_i)\right]\|\chi_{I_i}\|_Y \\ &< \varepsilon\sum_{i=1}^{n}\left[(Rf)(\alpha_{i-1}) - (Rf)(\alpha_i)\right] \\ &= \varepsilon\left[(Rf)(a) - (Rf)(b)\right] \\ &\leq \varepsilon(Rf)(a) \leq \varepsilon\left\|\chi_{(\frac{a}{2},a]}\right\|_Y^{-1} c(R)\|f\|_X,\end{aligned} \tag{3.1}$$

by using the fact that Rf is non-increasing and the estimate for $(Rf)(a)$ carried out above. This yields

$$\sup_{\|f\|_X \leq 1}\|Tf - Sf\|_Y < c(a, R)\varepsilon,$$

where $c(a, R)$ is a constant depending only on a and $c(R)$. Thanks to Proposition 1.0.3 again, we arrive at the compactness of the operator T and so finally at the compactness of the operator R. $\square$

Remark 3.1.2. Let us present an example showing that the assumption that R maps all functions from X to the class of non-increasing functions is indispensable.

Consider weights v, w, such that $v(t) = 1$ for all $t \in (0,\infty)$ and $\int_0^\infty w(t)dt < \infty$. The spaces $L^\infty(v)$ and $L^1(w)$ are thus Banach function spaces. In addition, $L^1(w)$ has absolutely continuous norm by virtue of the Lebesgue dominated convergence theorem. For any $f \in L^\infty(v)$ we have

$$\int_0^\infty |f(t)|w(t)dt \le \int_0^\infty w(t)dt\|f\|_{\infty,v}.$$

Set

$$Rf = |f|, \quad f \in L^\infty(v).$$

Then R is a well defined bounded operator from $L^\infty(v)$ to $L^1(w)$, which assigns a non-negative but not necessarily non-increasing function from $L^1(w)$ to each function from $L^\infty(v)$. We assert that $\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}$ is of uniformly absolutely continuous norm, however R is not compact. Indeed, take a sequence $\{E_n\}$ of λ-measurable subsets of $(0,\infty)$, such that $\chi_{E_n} \to \chi_\emptyset$ λ-a.e. on $(0,\infty)$ and $\varepsilon > 0$. Using the Lebesgue dominated convergence theorem, we find $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and all $f \in L^\infty(v)$ with $\|f\|_{\infty,v} \le 1$,

$$\|\chi_{E_n} Rf\|_{1,w} = \int_0^\infty \chi_{E_n}(t)|f(t)|w(t)dt \le \int_0^\infty \chi_{E_n}(t)w(t)dt < \varepsilon.$$

Hence, $\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}$ is of uniformly absolutely continuous norm. By Theorem 2.2.4, if $\overline{\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}}$ was a compact set in $L^1(w)$, then it would be p-mean equicontinuous, which means that it would satisfy

$$\begin{aligned}&\forall \varepsilon > 0\, \exists \delta > 0\, \forall g \in \overline{\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}} : \\ &\qquad\qquad |h| < \delta \Rightarrow \int_0^\infty |g(t+h) - g(t)|w(t)dt < \varepsilon,\end{aligned}$$

considering g defined by 0 outside the interval $(0,\infty)$. To show that $\overline{\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}}$ is not p-mean equicontinuous, take $\varepsilon = \frac{1}{2}\int_0^\infty w(t)dt$ and for each $\delta > 0$ put

$$f_\delta(t) = \chi_{\bigcup\limits_{k\in\mathbb{N}\cup\{0\}}\left((2k+1)\frac{\delta}{2},(2k+2)\frac{\delta}{2}\right]}(t), \quad t \in (0,\infty).$$

Then $f_\delta \in \overline{\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}}$, because $f_\delta \in L^\infty(v)$ with $\|f_\delta\|_{\infty,v} \le 1$ and $Rf_\delta = f_\delta$, and for $h = \frac{\delta}{2}$ we get

$$\int_0^\infty |f_\delta(t+h) - f_\delta(t)|w(t)dt = \int_0^\infty w(t)dt > \varepsilon.$$

Therefore, the set $\overline{\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}}$ and consequently the operator R are not compact.

3.2 Compactness of operators involving suprema on weighted Banach function spaces

From this moment on, we confine ourselves to the operators involving suprema described below. The result to be introduced was inspired by the work of Edmunds, Gurka and Pick in [3] dealing with Hardy-type integral operators. They formulated a criterion for the compactness of a generalized Hardy operator between two spaces falling into the category of pairs of spaces, for which the Muckenhoupt-type condition is equivalent to the boundedness of considered operator. Our aim is alike. For a given mapping involving supremum we determine a class of pairs of spaces, for which we can prove a general necessary and sufficient condition for the mapping to be a compact operator. Similarly to [3], the class of couples of spaces is related to the boundedness of the operator under consideration and the characterization of the compactness of the operator is expressed in terms of the norms of weights figuring in the definitions of the spaces and the operator.

Before we will come to the principal theorem, we need to establish some notation, definitions and auxiliary assertions.

Notation 3.2.1. In keeping with notation in Section 2.1, $\mathcal{M}((0,\infty),\lambda)$ denotes the set of all Lebesgue-measurable functions on $(0,\infty)$.

For a weight h we put $H(t) = \int_0^t h(s)ds$, $t \in (0,\infty)$.

Let u, h be weights and let $I \subset (0,\infty)$ be an interval. We define

$$\bar{u}_I(t) = H(t) \sup_{t \le \tau < \infty} \frac{u(\tau)\chi_I(\tau)}{H(\tau)}, \quad t \in (0,\infty).$$

It is obvious that $\bar{u}_I(t) \ge u(t)\chi_I(t)$ for every $t \in (0,\infty)$ and that the function $\frac{\bar{u}_I}{H}$ is non-increasing. We abbreviate

$$\bar{u}(t) = H(t) \sup_{t \le \tau < \infty} \frac{u(\tau)\chi_{(0,\infty)}(\tau)}{H(\tau)}, \quad t \in (0,\infty).$$

We use the symbol T^I to denote the mapping given at a function f by $\chi_I Tf$, where T is some mapping defined at f and $I \subset (0,\infty)$ is an interval.

Definition 3.2.2. For a weight h satisfying $H(t) < \infty$ for every $t \in (0,\infty)$ and a weight u we define the mapping $T_{u,h}$ at $f \in \mathcal{M}((0,\infty),\lambda)$ by

$$(T_{u,h}f)(t) = \sup_{t \le \tau < \infty} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds, \quad t \in (0,\infty).$$

Let $I \subset (0,\infty)$ be an interval. For $f \in \mathcal{M}((0,\infty),\lambda)$ we set

$$(T_{u,h,I}f)(t) = \sup_{t \le \tau < \infty} \frac{u(\tau)\chi_I(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)\chi_I(s)ds, \quad t \in (0,\infty).$$

One can easily see that $T_{u,h}f$ is non-negative non-increasing function for each $f \in \mathcal{M}((0,\infty),\lambda)$.

Let's have a look at a consequence of the boundedness of an operator $T^I_{u,h,I} : X(v) \to Y(w)$.

Lemma 3.2.3. *Let $X = X(v)$ and $Y = Y(w)$ be weighted Banach function spaces equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and let $I \subset (0,\infty)$ be an interval. If the operator $T^I_{u,h,I} : X \to Y$ is bounded, then*

$$\sup_{x\in I} \left\| \frac{\bar{u}_I(x)}{H(x)} \chi_{(0,x)}\chi_I + \frac{\bar{u}_I}{H}\chi_{[x,\infty)}\chi_I \right\|_Y \left\| \frac{h}{v}\chi_{(0,x)}\chi_I \right\|_{X'} < \infty. \tag{3.2}$$

Proof. Since $T^I_{u,h,I}$ is bounded, there exists a constant $c(T^I_{u,h,I}) > 0$ such that

$$\left\| T^I_{u,h,I} f \right\|_Y \le c(T^I_{u,h,I}) \|f\|_X \quad \forall f \in X.$$

Take $f \in X$ with $\|f\|_X \le 1$ and $x \in I$. Then for $t \in (0,x)\cap I$ we have

$$\begin{aligned}
(T_{u,h,I}f)(t) &= \sup_{t\le\tau<\infty} \frac{u(\tau)\chi_I(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)\chi_I(s)ds \\
&\ge \sup_{x\le\tau<\infty} \frac{u(\tau)\chi_I(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)\chi_I(s)ds \\
&\ge \sup_{x\le\tau<\infty} \frac{u(\tau)\chi_I(\tau)}{H(\tau)} \int_0^x |f(s)|h(s)\chi_I(s)ds \\
&= \frac{\bar{u}_I(x)}{H(x)} \int_0^x |f(s)|h(s)\chi_I(s)ds,
\end{aligned}$$

while for $t \in [x,\infty)\cap I$ we have

$$\begin{aligned}
(T_{u,h,I}f)(t) &= \sup_{t\le\tau<\infty} \frac{u(\tau)\chi_I(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)\chi_I(s)ds \\
&\ge \sup_{t\le\tau<\infty} \frac{u(\tau)\chi_I(\tau)}{H(\tau)} \int_0^x |f(s)|h(s)\chi_I(s)ds \\
&= \frac{\bar{u}_I(t)}{H(t)} \int_0^x |f(s)|h(s)\chi_I(s)ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
c(T^I_{u,h,I}) \ge c(T^I_{u,h,I})\|f\|_X \ge \left\|T^I_{u,h,I}f\right\|_Y &= \left\|\chi_{(0,x)}T^I_{u,h,I}f + \chi_{[x,\infty)}T^I_{u,h,I}f\right\|_Y \\
&= \left\| \chi_{(0,x)}\chi_I T_{u,h,I}f + \chi_{[x,\infty)}\chi_I T_{u,h,I}f \right\|_Y \\
&\ge \left\| \frac{\bar{u}_I(x)}{H(x)}\chi_{(0,x)}\chi_I + \frac{\bar{u}_I}{H}\chi_{[x,\infty)}\chi_I \right\|_Y \int_0^x |f(s)| \frac{h(s)}{v(s)} \chi_I(s) v(s) ds.
\end{aligned}$$

By the definition of the associate norm, passing to the supremum over all $f \in X$ with $\|f\|_X \le 1$ gives

$$\left\| \frac{\bar{u}_I(x)}{H(x)} \chi_{(0,x)} \chi_I + \frac{\bar{u}_I}{H} \chi_{[x,\infty)} \chi_I \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \chi_I \right\|_{X'} \le c(T^I_{u,h,I}).$$

In conclusion, we take the supremum over all $x \in I$ to obtain (3.2). □

Lemma 3.2.3 shows that (3.2) is always necessary for the boundedness of $T^I_{u,h,I} : X(v) \to Y(w)$. It turns out that for some spaces it is also sufficient, while for the other spaces it is not. This justifies our following definition.

Definition 3.2.4. We say that a pair of weighted Banach function spaces $(X(v), Y(w))$ belongs to the category $\mathbb{M}(T_{u,h})$, write $(X(v), Y(w)) \in \mathbb{M}(T_{u,h})$, if for each interval $I \subset (0,\infty)$ the condition (3.2) implies that the mapping $T^I_{u,h,I}$ is a bounded operator from X to Y and

$$\begin{aligned} \sup_{x\in I} & \left\| \frac{\bar{u}_I(x)}{H(x)} \chi_I \chi_{(0,x)} + \frac{\bar{u}_I}{H} \chi_I \chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_I \chi_{(0,x)} \right\|_{X'} \\ & \le \sup\{ \| T^I_{u,h,I} f \|_Y ; f \in X, \|f\|_X \le 1 \} \\ & \le K \sup_{x\in I} \left\| \frac{\bar{u}_I(x)}{H(x)} \chi_I \chi_{(0,x)} + \frac{\bar{u}_I}{H} \chi_I \chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_I \chi_{(0,x)} \right\|_{X'}, \end{aligned} \tag{3.3}$$

where $K \ge 1$ is a constant independent of v, w, u, h and I.

This quite technical lemma turns out to be crucial in the proofs of both the following theorems.

Lemma 3.2.5. *Let u, h be weights and $X = X(v)$, $Y = Y(w)$ be weighted Banach function spaces endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. On X', the associate space of X, consider the norm $\|\cdot\|_{X'}$. Suppose that*

$$\lim_{a\to 0_+} \sup_{0<x<a} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,x)} + \frac{\bar{u}}{H} \chi_{[x,a)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} = 0 \tag{3.4}$$

and

$$\lim_{b\to \infty} \sup_{b<x<\infty} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(b,x)} + \frac{\bar{u}}{H} \chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} = 0. \tag{3.5}$$

Then

$$\sup_{0<x<\infty} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,x)} + \frac{\bar{u}}{H} \chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} < \infty. \tag{3.6}$$

Proof. Conditions (3.4) and (3.5) give existence of $a \in (0,\infty)$ and $b \in (0,\infty)$, such that $a < b$ and

$$\sup_{0<x\le a} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,x)} + \frac{\bar{u}}{H} \chi_{[x,a)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} \le 1 \tag{3.7}$$

and

$$\sup_{b\le x<\infty}\left\|\frac{\bar u(x)}{H(x)}\chi_{(b,x)}+\frac{\bar u}{H}\chi_{[x,\infty)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\le 1. \tag{3.8}$$

Denote

$$\Psi(x)=\rho_Y\left(\frac{\bar u(x)}{H(x)}\chi_{(0,x)}+\frac{\bar u}{H}\chi_{[x,\infty)}\right)\rho_{X'}\left(\frac{h}{v}\chi_{(0,x)}\right),$$

where ρ_Y and $\rho_{X'}$ are the Banach function norms determining the spaces Y and X' respectively. If the value $\rho_Y(f)$ or $\rho_{X'}(f)$ is finite for some function f, then it can be replaced by the norm of f in the corresponding Banach space. Writing norms throughout each of the future calculations is excused by the fact that in each one we arrive at a finite upper bound. Since

$$\sup_{0<x<\infty}\Psi(x)=\max\left\{\sup_{0<x<a}\Psi(x),\ \sup_{a\le x\le b}\Psi(x),\ \sup_{b<x<\infty}\Psi(x)\right\},$$

we estimate the supremum of the function Ψ over each of the intervals $(0,a)$, $[a,b]$ and (b,∞) separately.

For the $a>0$ we have

$$\begin{aligned}
\sup_{0<x<a}\Psi(x) &= \sup_{0<x<a}\left\|\frac{\bar u(x)}{H(x)}\chi_{(0,x)}+\frac{\bar u}{H}\chi_{[x,a)}+\frac{\bar u}{H}\chi_{[a,b]}+\frac{\bar u}{H}\chi_{(b,\infty)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\
&\le \sup_{0<x<a}\left\|\frac{\bar u(x)}{H(x)}\chi_{(0,x)}+\frac{\bar u}{H}\chi_{[x,a)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\
&\quad+\sup_{0<x<a}\left\|\frac{\bar u}{H}\chi_{[a,b]}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\
&\quad+\sup_{0<x<a}\left\|\frac{\bar u}{H}\chi_{(b,\infty)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\
&\le \sup_{0<x<a}\left\|\frac{\bar u(x)}{H(x)}\chi_{(0,x)}+\frac{\bar u}{H}\chi_{[x,a)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\
&\quad+\left\|\frac{\bar u}{H}\chi_{[a,b]}\right\|_Y\left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'}\\
&\quad+\left\|\frac{\bar u}{H}\chi_{(b,\infty)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'}\\
&\le \sup_{0<x<a}\left\|\frac{\bar u(x)}{H(x)}\chi_{(0,x)}+\frac{\bar u}{H}\chi_{[x,a)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\
&\quad+\left\|\frac{\bar u}{H}\chi_{[a,b]}\right\|_Y\left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'}\\
&\quad+\left\|\frac{\bar u}{H}\chi_{(b,\infty)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,b)}\right\|_{X'}.
\end{aligned}$$

Due to (3.7), the first summand is less than or equal to one and due to (3.8) evaluated at $x=b$, also the last summand is less than or equal to one. As $\frac{\bar u}{H}$

is non-increasing, the middle term can be treated as

$$\left\|\frac{\bar{u}}{H}\chi_{[a,b]}\right\|_Y \left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'} \leq \frac{\bar{u}(a)}{H(a)}\left\|\chi_{[a,b]}\right\|_Y \left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'} \leq \frac{\left\|\chi_{[a,b]}\right\|_Y}{\left\|\chi_{(0,a)}\right\|_Y} < \infty,$$

where the last but one estimate follows from (3.7) evaluated at $x = a$ and the last one then from the properties of w, namely that w is locally integrable and positive λ-a.e. on $(0, \infty)$. Thus $\sup_{0<x<a} \Psi(x)$ is finite.

Regarding the interval $[a, b]$, write

$$\begin{aligned}
\sup_{a\leq x\leq b} \Psi(x) &= \sup_{a\leq x\leq b} \left\|\frac{\bar{u}(x)}{H(x)}\chi_{(0,a)} + \frac{\bar{u}(x)}{H(x)}\chi_{[a,x)} + \frac{\bar{u}}{H}\chi_{[x,b]} + \frac{\bar{u}}{H}\chi_{(b,\infty)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'} \\
&\leq \sup_{a\leq x\leq b} \left\|\frac{\bar{u}(a)}{H(a)}\chi_{(0,a)} + \frac{\bar{u}}{H}\chi_{[a,b]} + \frac{\bar{u}}{H}\chi_{(b,\infty)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'} \\
&\leq \sup_{a\leq x\leq b} \left\|\frac{\bar{u}(a)}{H(a)}\chi_{(0,a)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'} + \sup_{a\leq x\leq b} \left\|\frac{\bar{u}}{H}\chi_{[a,b]}\right\|_Y \left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'} \\
&\quad + \sup_{a\leq x\leq b} \left\|\frac{\bar{u}}{H}\chi_{(b,\infty)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'} \\
&\leq \left\|\frac{\bar{u}(a)}{H(a)}\chi_{(0,a)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,b)}\right\|_{X'} + \frac{\bar{u}(a)}{H(a)}\left\|\chi_{[a,b]}\right\|_Y \left\|\frac{h}{v}\chi_{(0,b)}\right\|_{X'} \\
&\quad + \left\|\frac{\bar{u}}{H}\chi_{(b,\infty)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,b)}\right\|_{X'},
\end{aligned}$$

where we again used that $\frac{\bar{u}}{H}$ is non-increasing. Thanks to (3.8), the last term is less than or equal to one. Inequalities (3.7) and (3.8), respectively, yield

$$\frac{\bar{u}(a)}{H(a)} \leq \left\|\chi_{(0,a)}\right\|_Y^{-1} \left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'}^{-1}$$

and

$$\left\|\frac{h}{v}\chi_{(0,b)}\right\|_{X'} \leq \left\|\frac{\bar{u}}{H}\chi_{[b,\infty)}\right\|_Y^{-1}.$$

Hence,

$$\begin{aligned}
&\sup_{a\leq x\leq b} \Psi(x) \\
&\leq \left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'}^{-1} \left\|\frac{\bar{u}}{H}\chi_{[b,\infty)}\right\|_Y^{-1} + \left\|\chi_{(0,a)}\right\|_Y^{-1} \left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'}^{-1} \left\|\chi_{[a,b]}\right\|_Y \left\|\frac{\bar{u}}{H}\chi_{[b,\infty)}\right\|_Y^{-1} + 1.
\end{aligned}$$

Because all of the weights are positive λ-a.e. on $(0, \infty)$ and locally integrable, the expression on the right hand side of the above inequality is finite.

Concerning the interval (b,∞), we proceed as follows.

$$\begin{aligned}\sup_{b<x<\infty}\Psi(x) &= \sup_{b<x<\infty}\left\|\frac{\bar u(x)}{H(x)}\chi_{(0,b]}+\frac{\bar u(x)}{H(x)}\chi_{(b,x)}+\frac{\bar u}{H}\chi_{[x,\infty)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\ &\le \sup_{b<x<\infty}\frac{\bar u(x)}{H(x)}\left\|\chi_{(0,b]}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\ &\quad+\sup_{b<x<\infty}\left\|\frac{\bar u(x)}{H(x)}\chi_{(b,x)}+\frac{\bar u}{H}\chi_{[x,\infty)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}.\end{aligned}$$

The expression $\left\|\chi_{(0,b]}\right\|_Y$ makes sense, because $\chi_{(0,b]}=\chi_{(0,a)}+\chi_{[a,b]}$ and $\chi_{(0,a)}\in Y$ according to (3.7) and $\chi_{[a,b]}\in Y$ as $[a,b]$ is of finite measure. The latter term of the above estimate is exactly the formula from the left hand side of (3.8), therefore it is less than or equal to one. To deal with the first summand, pick an arbitrary $c\in(b,\infty)$. Since w is positive λ-a.e. on $(0,\infty)$ and locally integrable, there is a constant $0<L<\infty$, such that

$$\left\|\chi_{(0,b]}\right\|_Y\le L\left\|\chi_{(b,c)}\right\|_Y.$$

Using this, we arrive at

$$\begin{aligned}&\sup_{b<x<\infty}\frac{\bar u(x)}{H(x)}\left\|\chi_{(0,b]}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\ &=\max\left\{\sup_{b<x<c}\frac{\bar u(x)}{H(x)}\left\|\chi_{(0,b]}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'},\ \sup_{c\le x<\infty}\frac{\bar u(x)}{H(x)}\left\|\chi_{(0,b]}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\right\}\\ &\le\max\left\{\frac{\bar u(b)}{H(b)}\left\|\chi_{(0,b]}\right\|_Y\left\|\frac{h}{v}\chi_{(0,c)}\right\|_{X'},\ L\sup_{c\le x<\infty}\frac{\bar u(x)}{H(x)}\left\|\chi_{(b,c)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\right\}\\ &\le\max\left\{\frac{\bar u(b)}{H(b)}\left\|\chi_{(0,b]}\right\|_Y\left\|\frac{h}{v}\chi_{(0,c)}\right\|_{X'},\ L\sup_{c\le x<\infty}\frac{\bar u(x)}{H(x)}\left\|\chi_{(b,x)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\right\}\\ &\le\max\left\{\frac{\bar u(b)}{H(b)}\left\|\chi_{(0,b]}\right\|_Y\left\|\frac{h}{v}\chi_{(0,c)}\right\|_{X'},\ L\sup_{b\le x<\infty}\frac{\bar u(x)}{H(x)}\left\|\chi_{(b,x)}\right\|_Y\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\right\}\\ &\le\max\left\{\left\|\chi_{(0,a)}\right\|_Y^{-1}\left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'}^{-1}\left\|\chi_{(0,b]}\right\|_Y\left\|\frac{\bar u(c)}{H(c)}\chi_{(b,c)}+\frac{\bar u}{H}\chi_{[c,\infty)}\right\|_Y^{-1},\ L\right\},\end{aligned}$$

where the last inequality is derived from (3.8), the monotonicity of $\frac{\bar u}{H}$ and the estimate for $\frac{\bar u(a)}{H(a)}$ carried out above. Again, as we suppose that weights are positive λ-a.e. on $(0,\infty)$ and $\chi_{(0,b]}\in Y$, the maximum, which we focus on, is finite. Thus also $\sup_{b<x<\infty}\Psi(x)<\infty$.

Finally, (3.6) is true. $\square$

And now, we are in the position to present a theorem bringing characterization of the compactness of $T_{u,h}$ from X to Y for a couple (X,Y) picked from the category $\mathbb{M}(T_{u,h})$.

Theorem 3.2.6. *Let $X = X(v)$ and $Y = Y(w)$ be weighted Banach function spaces, such that $(X(v), Y(w)) \in \mathbb{M}(T_{u,h})$ and $Y = Y_a$, and let them be equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Then $T_{u,h}$ is a compact operator from $X(v)$ into $Y(w)$ if and only if both of the following conditions are satisfied:*

$$\lim_{a\to 0+} \sup_{0<x<a} \left\| \frac{\bar{u}(x)}{H(x)}\chi_{(0,x)} + \frac{\bar{u}}{H}\chi_{[x,a)} \right\|_Y \left\| \frac{h}{v}\chi_{(0,x)} \right\|_{X'} = 0 \tag{3.9}$$

and

$$\lim_{b\to \infty} \sup_{b<x<\infty} \left\| \frac{\bar{u}(x)}{H(x)}\chi_{(b,x)} + \frac{\bar{u}}{H}\chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v}\chi_{(0,x)} \right\|_{X'} = 0. \tag{3.10}$$

Proof. Necessity: For contradiction, suppose that $T_{u,h}$ is a compact operator from X to Y, yet the negation of (3.9) is true. Then there exist $\varepsilon > 0$, a decreasing sequence $\{a_n\} \subset (0,\infty)$ with $\lim_{n\to\infty} a_n = 0$ and points $x_n \in (0, a_n)$, such that

$$\left\| \frac{\bar{u}(x_n)}{H(x_n)}\chi_{(0,x_n)} + \frac{\bar{u}}{H}\chi_{[x_n,a_n)} \right\|_Y \left\| \frac{h}{v}\chi_{(0,x_n)} \right\|_{X'} > \varepsilon.$$

From the definition of the associate norm and by the absolute continuity of integral, there are a sequence $\{f_n\} \subset X$ with $\|f_n\|_X \le 1$ and numbers $\beta_n \in (0, x_n)$ satisfying

$$\int_{\beta_n}^{x_n} |f_n(s)|h(s)ds > \frac{1}{2}\int_0^{x_n} |f_n(s)|h(s)ds > \frac{1}{4}\left\| \frac{h}{v}\chi_{(0,x_n)} \right\|_{X'}.$$

Define functions $F_n = f_n\chi_{(\beta_n,x_n)}$. Clearly, the lattice property of X gives that these functions lie in the closed unit ball of X. Since $\{a_n\}$ is decreasing, for every $n \in \mathbb{N}$ we can find $m_0 \in \mathbb{N}$, such that for every $m \ge m_0$ the inequalities $a_m < \beta_n$ and $\|\chi_{(0,a_m)}T_{u,h}F_n\|_Y < \frac{1}{8}\varepsilon$ hold. The latter inequality is guaranteed by the absolute continuity of the norm of the function $T_{u,h}F_n$, which follows from the assumptions that $T_{u,h} : X \to Y$ and $Y = Y_a$. Now, for $m \ge m_0$ and $t \ge x_m$, we get

$$\begin{aligned}
(T_{u,h}F_m)(t) &= \sup_{t\le\tau<\infty} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f_m(s)|\chi_{(\beta_m,x_m)}(s)h(s)ds \\
&= \sup_{t\le\tau<\infty} \frac{u(\tau)}{H(\tau)} \int_{\beta_m}^{x_m} |f_m(s)|(s)h(s)ds \\
&= \frac{\bar{u}(t)}{H(t)} \int_{\beta_m}^{x_m} |f_m(s)|h(s)ds.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\|T_{u,h}F_m - T_{u,h}F_n\|_Y \\
&\quad \geq \left\|\chi_{(0,a_m)}(T_{u,h}F_m - T_{u,h}F_n)\right\|_Y \\
&\quad \geq \left\|\chi_{(0,a_m)}T_{u,h}F_m\right\|_Y - \left\|\chi_{(0,a_m)}T_{u,h}F_n\right\|_Y \\
&\quad \geq \left\|\chi_{(0,x_m)}T_{u,h}F_m(x_m) + \chi_{[x_m,a_m)}T_{u,h}F_m\right\|_Y \\
&\qquad - \left\|\chi_{(0,a_m)}T_{u,h}F_n\right\|_Y \\
&\quad = \left\|\chi_{(0,x_m)}\frac{\bar{u}(x_m)}{H(x_m)}\int_{\beta_m}^{x_m}|f_m(s)|h(s)ds + \chi_{[x_m,a_m)}\frac{\bar{u}}{H}\int_{\beta_m}^{x_m}|f_m(s)|h(s)ds\right\|_Y \\
&\qquad - \left\|\chi_{(0,a_m)}T_{u,h}F_n\right\|_Y \\
&\quad \geq \frac{1}{4}\left\|\frac{\bar{u}(x_m)}{H(x_m)}\chi_{(0,x_m)} + \frac{\bar{u}}{H}\chi_{[x_m,a_m)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,x_m)}\right\|_{X'} - \frac{1}{8}\varepsilon \\
&\quad \geq \frac{1}{8}\varepsilon > 0.
\end{aligned}
$$

To derive the (in)equalities we used that $T_{u,h}$ maps X into the class of non-increasing functions and the definition of $\{F_n\}$, a_m and x_m. So, we have found the sequence $\{F_n\} \subset \{f \in X; \|f\|_X \leq 1\}$ such that none of its subsequences can be Cauchy, thus neither convergent in Y. This is a contradiction with the compactness of $T_{u,h}$.

As for the necessity of (3.10), although we proceed similarly, we state the whole proof for the sake of completeness. Suppose again that the negation of this statement is true and $T_{u,h}$ is a compact operator. Then, there exist $\varepsilon > 0$, an increasing sequence $\{b_n\} \subset (0,\infty)$ with $\lim_{n\to\infty} b_n = \infty$, points $x_n \in (b_n,\infty)$, a sequence $\{f_n\} \subset X$ with $\|f_n\|_X \leq 1$ and numbers $\beta_n \in (0,x_n)$ satisfying

$$
\left\|\frac{\bar{u}(x_n)}{H(x_n)}\chi_{(b_n,x_n)} + \frac{\bar{u}}{H}\chi_{[x_n,\infty)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,x_n)}\right\|_{X'} > \varepsilon
$$

and

$$
\int_{\beta_n}^{x_n}|f_n(s)|h(s)ds > \frac{1}{2}\int_0^{x_n}|f_n(s)|h(s)ds > \frac{1}{4}\left\|\frac{h}{v}\chi_{(0,x_n)}\right\|_{X'}.
$$

Put $F_n = f_n\chi_{(\beta_n,x_n)}$. For every $n \in \mathbb{N}$ find $m_0 \in \mathbb{N}$ such that $x_n < b_m$ and $\|\chi_{(b_m,\infty)}T_{u,h}F_n\|_Y < \frac{1}{8}\varepsilon$ for every $m \geq m_0$.

Then,

$$\begin{aligned}
&\|T_{u,h}F_m - T_{u,h}F_n\|_Y \\
&\quad\geq \left\|\chi_{(b_m,\infty)}(T_{u,h}F_m - T_{u,h}F_n)\right\|_Y \\
&\quad\geq \left\|\chi_{(b_m,\infty)}T_{u,h}F_m\right\|_Y - \left\|\chi_{(b_m,\infty)}T_{u,h}F_n\right\|_Y \\
&\quad\geq \left\|\chi_{(b_m,x_m)}T_{u,h}F_m(x_m) + \chi_{[x_m,\infty)}T_{u,h}F_m\right\|_Y \\
&\qquad - \left\|\chi_{(b_m,\infty)}T_{u,h}F_n\right\|_Y \\
&\quad= \left\|\chi_{(b_m,x_m)}\frac{\bar{u}(x_m)}{H(x_m)}\int_{\beta_m}^{x_m}|f_m(s)|h(s)ds + \chi_{[x_m,\infty)}\frac{\bar{u}}{H}\int_{\beta_m}^{x_m}|f_m(s)|h(s)ds\right\|_Y \\
&\qquad - \left\|\chi_{(b_m,\infty)}T_{u,h}F_n\right\|_Y \\
&\quad\geq \frac{1}{4}\left\|\frac{\bar{u}(x_m)}{H(x_m)}\chi_{(b_m,x_m)} + \frac{\bar{u}}{H}\chi_{[x_m,\infty)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,x_m)}\right\|_{X'} - \frac{1}{8}\varepsilon \\
&\quad\geq \frac{1}{8}\varepsilon > 0.
\end{aligned}$$

Thus, with the same explanation as before, the operator $T_{u,h}$ cannot be compact, which is a contradiction.

Sufficiency: Given an interval $I \subset (0,\infty)$, set

$$(T_{u_I,h}f)(t) = \sup_{t\leq\tau<\infty}\frac{u(\tau)\chi_I(\tau)}{H(\tau)}\int_0^\tau |f(s)|h(s)ds, \quad f \in X, t \in (0,\infty).$$

Observe that for $0 < a < b < \infty$, $f \in X$ with $\|f\|_X \leq 1$ and $t \in (0,\infty)$ we have

$$\begin{aligned}
(T^{(0,a)}_{u_{[a,\infty)},h}f)(t) + (T^{[a,b]}_{u,h}f)(t) \leq (T_{u,h}f)(t) &\leq (T^{(0,a)}_{u_{(0,a)},h}f)(t) + (T^{(0,a)}_{u_{[a,\infty)},h}f)(t) \\
&\quad + (T^{[a,b]}_{u,h}f)(t) + (T^{(b,\infty)}_{u,h}f)(t).
\end{aligned}$$

So,

$$0 \leq T_{u,h}f - T^{(0,a)}_{u_{[a,\infty)},h}f - T^{[a,b]}_{u,h}f \leq T^{(0,a)}_{u_{(0,a)},h}f + T^{(b,\infty)}_{u,h}f$$

meant pointwise. After we prove that for a proper choice of a and b, the function on the right hand side of the inequality lies in Y and has small norm and after we show that the mapping giving the function subtracted on the left hand side is under our assumptions a compact operator from X to Y, we shall refer to Proposition 1.0.3 to establish the compactness of the operator $T_{u,h}$.

Condition (3.9) guarantees for each $\varepsilon > 0$ the existence of $a \in (0,\infty)$ such that

$$\sup_{0<x\leq a}\left\|\frac{\bar{u}(x)}{H(x)}\chi_{(0,x)} + \frac{\bar{u}}{H}\chi_{[x,a)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'} < \varepsilon. \tag{3.11}$$

Hence,

$$\sup_{0<x\leq a}\left\|\frac{\bar{u}_{(0,a)}(x)}{H(x)}\chi_{(0,x)} + \frac{\bar{u}_{(0,a)}}{H}\chi_{[x,a)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'} < \varepsilon,$$

since the function in the norm of space Y is at each point less than or equal to the function standing ibidem in (3.11). Because the pair $((X,v),(Y,w))$ belongs to the category $\mathbb{M}(T_{u,h})$, the operator $T^{(0,a)}_{u_{(0,a)},h} = T^{(0,a)}_{u,h,(0,a)} : X \to Y$ is bounded and $\sup\{\|T^{(0,a)}_{u,h,(0,a)}f\|_Y; f \in X, \|f\|_X \le 1\} \le K\varepsilon$, where $K \ge 1$ is a constant independent of v, w, u, h and a. Thus, $T^{(0,a)}_{u_{(0,a)},h}f \in Y$ and $\left\|T^{(0,a)}_{u_{(0,a)},h}f\right\|_Y \le K\varepsilon\|f\|_X$ for every $f \in X$.

From assumption (3.10), for a given $\varepsilon > 0$ we find $b \in (a,\infty)$ such that

$$\sup_{b \le x < \infty} \left\| \frac{\bar{u}(x)}{H(x)}\chi_{(b,x)} + \frac{\bar{u}}{H}\chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v}\chi_{(0,x)} \right\|_{X'} < \varepsilon, \tag{3.12}$$

where a is from the previous paragraph and corresponds to ε. Then also

$$\sup_{b \le x < \infty} \left\| \frac{\bar{u}_{(b,\infty)}(x)}{H(x)}\chi_{(b,x)} + \frac{\bar{u}_{(b,\infty)}}{H}\chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v}\chi_{(b,x)} \right\|_{X'} < \varepsilon. \tag{3.13}$$

Therefore $T^{(b,\infty)}_{u,h,(b,\infty)} : X \to Y$ is a bounded operator from X to Y and $\sup\{\|T^{(b,\infty)}_{u,h,(b,\infty)}f\|_Y; f \in X, \|f\|_X \le 1\} \le K\varepsilon$, where $K \ge 1$ is a constant independent of v, w, u, h and b. For $f \in X$ with $\|f\|_X \le 1$ and $t \in (0,\infty)$, we estimate

$$\begin{aligned} (T^{(b,\infty)}_{u,h}f)(t) &\le \chi_{(b,\infty)}(t)\frac{\bar{u}(t)}{H(t)}\int_0^b |f(s)|h(s)ds + (T^{(b,\infty)}_{u,h,(b,\infty)}f)(t) \\ &\le \chi_{(b,\infty)}(t)\frac{\bar{u}(t)}{H(t)}\left\|\frac{h}{v}\chi_{(0,b)}\right\|_{X'} + (T^{(b,\infty)}_{u,h,(b,\infty)}f)(t). \end{aligned}$$

With reference to (3.12) and (3.13), the function given at $t \in (0,\infty)$ by the expression on the right hand side is an element of Y. In agreement with the lattice property of Y, so is the function defined on the left and

$$\left\|T^{(b,\infty)}_{u,h}f\right\|_Y \le (1+K)\varepsilon.$$

To summarize our achievement so far, we have managed to find $a \in (0,\infty)$ and $b \in (0,\infty)$ corresponding to a given $\varepsilon > 0$, such that $a < b$ and, for any $f \in X$ with $\|f\|_X \le 1$, the function

$$T_{u,h}f - T^{(0,a)}_{u_{[a,\infty)},h}f - T^{[a,b]}_{u,h}f$$

falls into Y and

$$\left\|T_{u,h}f - T^{(0,a)}_{u_{[a,\infty)},h}f - T^{[a,b]}_{u,h}f\right\|_Y < C\varepsilon,$$

where $C > 0$ is a constant independent of v, w, u, h, a and b.

Now, we are left with the proof of the statement that the mapping, which assigns

$$T^{(0,a)}_{u_{[a,\infty)},h} f + T^{[a,b]}_{u,h} f$$

to a function $f \in X$, is a compact operator from X to Y.

The function $\chi_{(0,a)}$ is in Y, because according to (3.11) its nonzero multiple, concretely $\frac{\bar{u}(a)}{H(a)}\chi_{(0,a)}$, is in Y. For any $f \in X$ and $t \in (0,a)$, we can write

$$\begin{aligned}(T_{u_{[a,\infty)},h} f)(t) &= (T_{u,h} f)(a) \leq \left\|\chi_{(0,a)}\right\|_Y^{-1} \left\|\chi_{(0,a)}(T_{u,h} f)(a)\right\|_Y \\ &\leq \left\|\chi_{(0,a)}\right\|_Y^{-1} \|T_{u,h} f\|_Y \leq \left\|\chi_{(0,a)}\right\|_Y^{-1} c(T_{u,h}) \|f\|_X.\end{aligned}$$

Here, $c(T_{u,h}) > 0$ is a constant satisfying $\|T_{u,h} f\|_Y \leq c(T_{u,h})\|f\|_X$ for every $f \in X$. This outcome is based on the monotonicity of the function $T_{u,h} f$ and the boundedness of the operator $T_{u,h} : X \to Y$ following from the assumption that $(X, Y) \in \mathbb{M}(T_{u,h})$ and from Lemma 3.2.5 in combination with (3.9) and (3.10). Clearly, the expression standing before $\|f\|_X$ at the end of the formula is, due to the fact that w is positive λ-a.e., a positive and finite constant independent of f. We obtained that the mapping $T^{(0,a)}_{u_{[a,\infty)},h}$ is a bounded finite rank operator from X to Y. By virtue of Proposition 1.0.3, this operator is compact.

Since the interval $[a, b]$, as a compact set, has finite measure, the function $\chi_{[a,b]}$ belongs to Y. Further, use the boundedness of the operator $T_{u,h} : X \to Y$ again and the lattice property of Y to arrive at the observation that $T^{[a,b]}_{u,h}$ is a bounded operator from X to Y. Obviously, the image of each function f is non-negative on $(0, \infty)$ and non-increasing on $[a, b]$. Since the features of the operator $T^{[a,b]}_{u,h}$ meet the pivotal requirements imposed on operators in formulation of Theorem 3.1.1, to show the compactness of this operator, we can apply the method which we used in the proof of Theorem 3.1.1 after we had restricted the problem to an interval $[a, b]$. To be concrete, thanks to the assumption that $Y = Y_a$, for an arbitrary $\eta > 0$ we find a decomposition $a = \alpha_0 < \alpha_1 \ldots < \alpha_n = b$ such that $\|\chi_{[\alpha_{i-1},\alpha_i]}\|_Y < \eta$ for each $i \in \{1, \ldots, n\}$. Set $I_i = [\alpha_{i-1}, \alpha_i)$ for $i \in \{1, \ldots, n-1\}$ and $I_n = [\alpha_{n-1}, \alpha_n]$. Define

$$(Sf)(t) = \sum_{i=1}^{n} (T_{u,h} f)(\alpha_i)\chi_{I_i}(t), \quad f \in X,\, t \in (0, \infty).$$

Then $S : X \to Y$ is a compact operator (for more details see the proof of Theorem 3.1.1) and via the same process as in (3.1), used for appropriate operators, we obtain

$$\sup_{\|f\|_X \leq 1} \left\|T^{[a,b]}_{u,h} f - Sf\right\|_Y \leq \eta (T_{u,h} f)(a) \leq \eta \left\|\chi_{(0,a)}\right\|_Y^{-1} c(T_{u,h}).$$

where the constant $c(T_{u,h}) > 0$ satisfies $\|T_{u,h}f\|_Y \leq c(T_{u,h})\|f\|_X$ for every $f \in X$. So, with reference to Proposition 1.0.3, the operator $T^{[a,b]}_{u,h}$ is compact from X to Y.

We have shown that the mapping $T^{(0,a)}_{u_{[a,\infty)},h} + T^{[a,b]}_{u,h}$ is a compact operator from X to Y. To conclude, we apply Proposition 1.0.3 to get the compactness of the operator $T_{u,h}$, as desired. □

Remark 3.2.7. The statement of Theorem 3.2.6 remains true if we replace the condition $Y = Y_a$ with either one of the following assumptions:

(a) $T_{u,h}(X) \subset Y_a$;

(b) w satisfies $\int_0^x w(s)ds < \infty$ for each $x \in (0,\infty)$, $Y_a = Y_b$ and $\lim_{x\to\infty} \left\| \frac{\bar{u}}{H}\chi_{[x,\infty)} \right\|_Y = 0$.

The proof can be carried out along the same lines as that of Theorem 3.2.6, therefore it is omitted.

3.3 Compactness of operators involving suprema on weighted Lebesgue spaces

In the end, we solve the same problem, i.e. the characterization of weights v and w for which $T_{u,h}$, defined in 3.2.2, is a compact operator from $X(v)$ to $Y(w)$, but with concern only in weighted Lebesgue spaces and for a special case of weights. Namely, we assume that v and w do not have a singularity at zero, which means $\int_0^x v(t)dt < \infty$ and $\int_0^x w(t)dt < \infty$ for every $x \in (0,\infty)$. Besides, we want u to be continuous. Under such circumstances, Gogatishvili, Opic and Pick in [4] studied the boundedness of $T_{u,h}$ from $L^p(v)$, $1 \leq p < \infty$, into $L^q(w)$, $0 < q < \infty$. In [4, Theorem 4.2], they showed that for $p \leq q$, a mapping $T_{u,h}$ is a bounded operator from $L^p(v)$ to $L^q(w)$ if and only if

$$\sup_{0<x<\infty} \left\| \frac{\bar{u}(x)}{H(x)}\chi_{(0,x)} + \frac{\bar{u}}{H}\chi_{[x,\infty)} \right\|_{q,w} \left\| \frac{h}{v}\chi_{(0,x)} \right\|_{p',v} < \infty,$$

and that

$$\begin{aligned} &\sup_{0<x<\infty} \left\| \frac{\bar{u}(x)}{H(x)}\chi_{(0,x)} + \frac{\bar{u}}{H}\chi_{[x,\infty)} \right\|_{q,w} \left\| \frac{h}{v}\chi_{(0,x)} \right\|_{p',v} \\ &\leq \sup\{\|T_{u,h}f\|_{q,w}\,; f \in L^p(v), \|f\|_{p,v} \leq 1\} \\ &\leq c(p,q) \sup_{0<x<\infty} \left\| \frac{\bar{u}(x)}{H(x)}\chi_{(0,x)} + \frac{\bar{u}}{H}\chi_{[x,\infty)} \right\|_{q,w} \left\| \frac{h}{v}\chi_{(0,x)} \right\|_{p',v}. \end{aligned}$$

The method of the proof however works equally well for the mapping $T^I_{u,h,I}$, where $I \subset (0,\infty)$ is any open interval, and gives that (3.2) is equivalent to the boundedness of the operator $T^I_{u,h,I} : L^p(v) \to L^q(w)$ and that (3.3) is satisfied. This is due to the facts that we have no requirements about the integrability of weights over the whole interval $(0, \infty)$ and that a weight is surely positive and finite λ-a.e. on I and integrable at the left endpoint of I. If we have an interval $I \subset (0, \infty)$ such that one or both of the endpoints of I belong to I, both equivalence (3.2) with the boundedness of $T^I_{u,h,I} : L^p(v) \to L^q(w)$ and inequality (3.3) follow from the continuity of integral and continuity of the weight u, because it allows us to pass to the interior of I. So, we can see that $(L^p(v), L^q(w)) \in \mathbb{M}(T_{u,h})$ for $p \le q$ where $1 \le p, q < \infty$. Hence, the case $p \le q$ in the following theorem is covered by Theorem 3.2.6. Nevertheless, Theorem 3.2.6 does not answer the question for $q < p$. Here we bring complete characterization of the compactness of $T_{u,h}$ from $L^p(v)$ to $L^q(w)$ for any $1 \le p, q < \infty$, including an alternative direct proof provided $p \le q$. The point of departure is the results about bondedness introduced in [4].

Definition 3.3.1. Let $I \in \mathbb{Z} \cup \{-\infty\}$ and $J \in \mathbb{Z} \cup \{\infty\}$. An increasing sequence $\{x_k\}_{k=I}^{k=J} \subset [0, \infty]$ is called a *covering sequence* if $\lim_{k\to-\infty} x_k = 0$ for $I = -\infty$, $x_I = 0$ for $I \in \mathbb{Z}$, $\lim_{k\to\infty} x_k = \infty$ for $J = \infty$ and $x_J = \infty$ for $J \in \mathbb{Z}$.

Consider $a \in (0, \infty)$, $I \in \mathbb{N} \cup \{0\}$ and $J \in \mathbb{N} \cup \{\infty\}$, $J \ge I$. We say that $\{x_k\}_{k=I}^{k=J} \subset [0, a]$ is a *sequence convenient for the interval* $[0, a]$ if it is a decreasing sequence satisfying $x_I = a$, $x_J = 0$ for $J \in \mathbb{N}$ and $\lim_{k\to\infty} x_k = 0$ for $J = \infty$.

Notation 3.3.2. For $1 \le p < \infty$, $0 \le \alpha < \beta \le \infty$ and weights h, v, we denote

$$\sigma_{p,h}(\alpha, \beta) = \begin{cases} \left(\int_\alpha^\beta [v(s)]^{1-p'} [h(s)]^{p'} ds \right)^{\frac{1}{p'}} & \text{when } 1 < p < \infty, \\ \operatorname{ess\,sup}_{\alpha < s < \beta} \frac{h(s)}{v(s)} & \text{when } p = 1. \end{cases}$$

The symbol $\sigma_{p,h}(\alpha, \beta)$ does not reflect dependence on v, but we shall use it only in context with fixed v, where no confusion should occur.

If not otherwise stated, we stick to definitions and notation from Section 3.2.

Lemma 3.3.3. *Let $1 \le p, q < \infty$, $q < p$, and let u, h, v, w be weights, such that u is continuous on $(0,\infty)$ and $\int_0^x h(t)dt < \infty$, $\int_0^x v(t)dt < \infty$, $\int_0^x w(t)dt < \infty$ for every $x \in (0,\infty)$. Define r by $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Suppose*

$$\sup_{\{x_k\}} \left(\sum_k \left(\int_{x_{k-1}}^{x_{k+1}} \min\left\{ \frac{\bar{u}(x_k)}{H(x_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} [\sigma_{p,h}(x_{k-1}, x_k)]^r \right)^{\frac{1}{r}} < \infty, \tag{3.14}$$

where the supremum is taken over all covering sequences $\{x_k\}$. Then for every $\varepsilon > 0$ there exist $a \in (0,\infty)$ and $b \in (0,\infty)$, such that $a < b$ and

$$\sup_{\{x_k\}} \left(\sum_k \left(\int_{x_{k+1}}^{x_{k-1}} \min\left\{ \frac{\bar{u}(x_k)}{H(x_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} [\sigma_{p,h}(x_{k+1}, x_k)]^r \right)^{\frac{1}{r}} < \varepsilon, \tag{3.15}$$

where the supremum is taken over all sequences $\{x_k\}$ convenient for the interval $[0,a]$, and

$$\sup_{\{x_k\}} \left(\sum_k \left(\int_{x_{k-1}}^{x_{k+1}} \min\left\{ \frac{\bar{u}(x_k)}{H(x_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} [\sigma_{p,h}(x_{k-1}, x_k)]^r \right)^{\frac{1}{r}} < \varepsilon, \tag{3.16}$$

where the supremum is taken over all increasing sequences $\{x_k\}_{k=I}^{k=J}$ with $x_I = b$ for $I \in \mathbb{Z}$ or $\lim_{k\to-\infty} x_k = b$ for $I = -\infty$ and $x_J = \infty$ for $J \in \mathbb{N}$ or $\lim_{k\to\infty} x_k = \infty$ for $J = \infty$.

Proof. If (3.15) was not true, there would exist an $\tilde{\varepsilon} > 0$ such that for each $0 < x < \infty$

$$\sup_{\{x_k\}} \left(\sum_k \left(\int_{x_{k+1}}^{x_{k-1}} \min\left\{ \frac{\bar{u}(x_k)}{H(x_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} [\sigma_{p,h}(x_{k+1}, x_k)]^r \right)^{\frac{1}{r}} \ge \tilde{\varepsilon}, \tag{3.17}$$

where the supremum would be taken over all sequences $\{x_k\}$ convenient for the interval $[0,x]$. But then we would find a covering sequence $\{y_k\}$ for which

$$\sum_k \left(\int_{y_{k-1}}^{y_{k+1}} \min\left\{ \frac{\bar{u}(y_k)}{H(y_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} [\sigma_{p,h}(y_{k-1}, y_k)]^r = \infty,$$

and that would lead to a contradiction with (3.14). Indeed, here we perform the construction of such a covering sequence $\{y_k\}$. Set $x_1 = 1$. According to (3.17), there is a sequence $\{x_k^1\}_{k=1}^{k=J_1}$ convenient for the interval $[0, x_1]$ with the property

$$\sum_k \left(\int_{x_{k+1}^1}^{x_{k-1}^1} \min\left\{ \frac{\bar{u}(x_k^1)}{H(x_k^1)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_{k+1}^1, x_k^1)\right]^r \ge \left(\frac{\tilde{\varepsilon}}{2}\right)^r.$$

Take the smallest possible $K_1 \in \mathbb{N}$ such that $x^1_{K_1} < \frac{1}{2}$ and

$$\sum_{k=2}^{K_1} \left(\int_{x^1_{k+1}}^{x^1_{k-1}} \min\left\{ \frac{\bar{u}(x^1_k)}{H(x^1_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x^1_{k+1}, x^1_k)\right]^r > \frac{\tilde{\varepsilon}^r}{2^{r+1}}.$$

If $K_1 < J_1$, put $x_2 = x^1_{K_1}$. In the situation when $K_1 = J_1$, thus $x^1_{K_1} = 0$, there must be some $x_2 \in (x^1_{K_1}, \min\{\frac{1}{2}, x^1_{K_1-1}\})$, for which the inequality

$$\sum_{k=2}^{K_1-2} \left(\int_{x^1_{k+1}}^{x^1_{k-1}} \min\left\{ \frac{\bar{u}(x^1_k)}{H(x^1_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x^1_{k+1}, x^1_k)\right]^r$$
$$+ \left(\int_{x_2}^{x^1_{K_1-2}} \min\left\{ \frac{\bar{u}(x^1_{K_1-1})}{H(x^1_{K_1-1})}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_2, x^1_{K_1-1})\right]^r > \frac{\tilde{\varepsilon}^r}{2^{r+1}}$$

holds. Define $y_0 = \infty$ and $y_k = x^1_{-k}$ for $k = -K_1 + 1, \ldots, -1$. Assume that we have already built a sequence $\{y_k\}^0_{k=n-\sum_{j=1}^n K_j}$ and that we know a point $x_{n+1} \in (0, \min\{\frac{1}{n+1}, x^n_{K_n-1}\})$. Like in the case of $n = 1$, we find a sequence $\{x^{n+1}_k\}^{k=J_{n+1}}_{k=1}$ convenient for the interval $[0, x_{n+1}]$, a natural number K_{n+1} and a point $x_{n+2} \in (0, \min\{\frac{1}{n+2}, x^{n+1}_{K_{n+1}-1}\})$ satisfying

$$\sum_{k=2}^{K_{n+1}-2} \left(\int_{x^{n+1}_{k+1}}^{x^{n+1}_{k-1}} \min\left\{ \frac{\bar{u}(x^{n+1}_k)}{H(x^{n+1}_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x^{n+1}_{k+1}, x^{n+1}_k)\right]^r$$
$$+ \left(\int_{x_{n+2}}^{x^{n+1}_{K_{n+1}-2}} \min\left\{ \frac{\bar{u}(x^{n+1}_{K_{n+1}-1})}{H(x^{n+1}_{K_{n+1}-1})}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}}$$
$$\times \left[\sigma_{p,h}(x_{n+2}, x^{n+1}_{K_{n+1}-1})\right]^r > \frac{\tilde{\varepsilon}^r}{2^{r+1}}.$$

Continue with definition of required covering sequence by $y_{k+n-\sum_{j=1}^n K_j} = x^{n+1}_{-k}$ for $k = -K_{n+1}+1, \ldots, -1$. This way we obtain a sequence $\{y_k\}^0_{k=-\infty} \subset (0, \infty]$, which is increasing with $y_0 = \infty$ and $\lim_{k\to-\infty} y_k = 0$. Furthermore,

$$\sum_k \left(\int_{y_{k-1}}^{y_{k+1}} \min\left\{ \frac{\bar{u}(y_k)}{H(y_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(y_{k-1}, y_k)\right]^r = \infty.$$

So, the described sequence $\{y_k\}$ is a covering sequence implementing a contradiction with (3.14). Analogical reasoning proves (3.16). $\square$

Theorem 3.3.4. *Let $1 \le p, q < \infty$ and let u, h, v, w be weights, such that u is continuous on $(0,\infty)$ and $\int_0^x h(t)dt < \infty$, $\int_0^x v(t)dt < \infty$, $\int_0^x w(t)dt < \infty$ for every $x \in (0,\infty)$.*

(i) *Let $p \le q$. Then $T_{u,h}$ is a compact operator from $L^p(v)$ to $L^q(w)$ if and only if both of the following conditions are satisfied:*

$$\lim_{a\to 0_+} \sup_{0<x<a} \left(\left(\frac{\bar{u}(x)}{H(x)}\right)^q \int_0^x w(t)dt + \int_x^a \left(\frac{\bar{u}(t)}{H(t)}\right)^q w(t)dt \right)^{\frac{1}{q}} \sigma_{p,h}(0,x) = 0 \tag{3.18}$$

and

$$\lim_{b\to \infty} \sup_{b<x<\infty} \left(\left(\frac{\bar{u}(x)}{H(x)}\right)^q \int_b^x w(t)dt + \int_x^\infty \left(\frac{\bar{u}(t)}{H(t)}\right)^q w(t)dt \right)^{\frac{1}{q}} \sigma_{p,h}(0,x) = 0. \tag{3.19}$$

(ii) *Let $q < p$. Define r by $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then $T_{u,h}$ is a compact operator from $L^p(v)$ to $L^q(w)$ if and only if*

$$\sup_{\{x_k\}} \left(\sum_k \left(\int_{x_{k-1}}^{x_{k+1}} \min\left\{ \frac{\bar{u}(x_k)}{H(x_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} [\sigma_{p,h}(x_{k-1},x_k)]^r \right)^{\frac{1}{r}} < \infty, \tag{3.20}$$

where the supremum is taken over all covering sequences $\{x_k\}$.

Proof. Necessity: Since $T_{u,h}$ as a compact operator is also bounded and formula (3.20) coincides with the condition equivalent to the boundedness of the operator $T_{u,h}$ (see [4, Theorem 4.2]), the necessity of condition (3.20) for the compactness of the operator $T_{u,h}$ in case of $q < p$ is obvious.

Thus to establish necessity, we are left with the proof for $p \le q$. Take $\varepsilon > 0$. We want to find $a \in (0,\infty)$ and $b \in (0,\infty)$, for which

$$\sup_{0<x<a} \left(\left(\frac{\bar{u}(x)}{H(x)}\right)^q \int_0^x w(t)dt + \int_x^a \left(\frac{\bar{u}(t)}{H(t)}\right)^q w(t)dt \right)^{\frac{1}{q}} \sigma_{p,h}(0,x) \le \varepsilon \tag{3.21}$$

and

$$\sup_{b<x<\infty} \left(\left(\frac{\bar{u}(x)}{H(x)}\right)^q \int_b^x w(t)dt + \int_x^\infty \left(\frac{\bar{u}(t)}{H(t)}\right)^q w(t)dt \right)^{\frac{1}{q}} \sigma_{p,h}(0,x) \le \varepsilon. \tag{3.22}$$

According to Theorem 3.1.1 and the assumptions, the set $\{T_{u,h}f; f \in L^p(v), \|f\|_{p,v} \le 1\}$ is of uniformly absolutely continuous norm in $L^q(w)$. Therefore, for the given ε, there exist $0 < a < b < \infty$ that satisfy $\left\|\chi_{(0,a)}T_{u,h}f\right\|_{q,w} < \varepsilon$ and $\left\|\chi_{(b,\infty)}T_{u,h}f\right\|_{q,w} < \varepsilon$ for every $f \in L^p(v)$ with $\|f\|_{p,v} \le 1$. We verify that this a and this b are the ones we search for.

Let's start with inequality (3.21). Fix $x \in (0,a)$. We have

$$\left(\left(\frac{\bar{u}(x)}{H(x)}\right)^q \int_0^x w(t)dt + \int_x^a \left(\frac{\bar{u}(t)}{H(t)}\right)^q w(t)dt\right)^{\frac{1}{q}} \sigma_{p,h}(0,x)$$
$$= \left(\int_0^a \left[\min\left\{\frac{\bar{u}(x)}{H(x)}, \frac{\bar{u}(t)}{H(t)}\right\} \sigma_{p,h}(0,x)\right]^q w(t)dt\right)^{\frac{1}{q}}.$$

Put

$$\Phi_p(t) = \min\left\{\frac{\bar{u}(x)}{H(x)}, \frac{\bar{u}(t)}{H(t)}\right\} \sigma_{p,h}(0,x).$$

First, consider $p > 1$. Define $E_n = \{y \in (0,x); [v(y)]^{1-p'}[h(y)]^{p'} \leq n\}$ and

$$\phi_n(t) = \min\left\{\frac{\bar{u}(x)}{H(x)}, \frac{\bar{u}(t)}{H(t)}\right\} \left(\int_{E_n} [v(s)]^{1-p'}[h(s)]^{p'} ds\right)^{\frac{1}{p'}}$$

for all $n \geq n_0$, where $n_0 \in \mathbb{N}$ is the smallest one satisfying $\lambda(E_{n_0}) > 0$. The sequence $\{\phi_n(t)\}_{n_0}^{\infty}$ increases monotonically to $\Phi_p(t)$ for each $t \in (0,\infty)$. Now, if for each $n \geq n_0$ we construct a function f_n such that $f_n \in L^p(v)$, $\|f_n\|_{p,v} \leq 1$ and $\phi_n(t) \leq T_{u,h}f_n(t)$ for all $t \in (0,\infty)$, we will obtain

$$\left(\int_0^a \left[\min\left\{\frac{\bar{u}(x)}{H(x)}, \frac{\bar{u}(t)}{H(t)}\right\} \left(\int_{E_n} [v(s)]^{1-p'}[h(s)]^{p'} ds\right)^{\frac{1}{p'}}\right]^q w(t)dt\right)^{\frac{1}{q}}$$
$$\leq \left\|\chi_{(0,a)} T_{u,h} f_n\right\|_{q,w} < \varepsilon.$$

Afterwards, letting n go to infinity will yield

$$\left(\left(\frac{\bar{u}(x)}{H(x)}\right)^q \int_0^x w(t)dt + \int_x^a \left(\frac{\bar{u}(t)}{H(t)}\right)^q w(t)dt\right)^{\frac{1}{q}} \sigma_{p,h}(0,x) \leq \varepsilon.$$

Passing to the supremum over all $x \in (0,a)$ will then give desired inequality (3.21). Well, for $n \geq n_0$, set

$$f_n(y) = \chi_{E_n}(y)[v(y)]^{1-p'}[h(y)]^{p'-1} \left(\int_{E_n} [v(z)]^{1-p'}[h(z)]^{p'} dz\right)^{-\frac{1}{p}}, \quad y \in (0,\infty).$$

Then f_n is well-defined, Lebesgue-measurable and

$$\left(\int_0^\infty [f_n(y)]^p v(y)dy\right)^{\frac{1}{p}}$$
$$= \left(\int_{E_n} [v(y)]^{1-p'}[h(y)]^{p'} \left(\int_{E_n} [v(z)]^{1-p'}[h(z)]^{p'} dz\right)^{-1} dy\right)^{\frac{1}{p}} = 1.$$

This shows that $f_n \in L^p(v)$ and $\|f_n\|_{p,v} \leq 1$. Moreover, since $\frac{\bar{u}}{H}$ is non-increasing and $\operatorname{supp} f_n \subset (0,x)$, taking $t \leq x$ we get

$$\begin{aligned}
\phi_n(t) &= \frac{\bar{u}(x)}{H(x)}\left(\int_{E_n}[v(s)]^{1-p'}[h(s)]^{p'}ds\right)^{\frac{1}{p'}} \\
&= \frac{\bar{u}(x)}{H(x)}\left(\int_{E_n}[v(s)]^{1-p'}[h(s)]^{p'}ds\right)^{1-\frac{1}{p}} \\
&= \frac{\bar{u}(x)}{H(x)}\int_0^x f_n(s)h(s)ds \\
&= \sup_{x\leq\tau<\infty}\frac{u(\tau)}{H(\tau)}\int_0^x f_n(s)h(s)ds \\
&\leq \sup_{x\leq\tau<\infty}\frac{u(\tau)}{H(\tau)}\int_0^\tau f_n(s)h(s)ds \\
&\leq \sup_{t\leq\tau<\infty}\frac{u(\tau)}{H(\tau)}\int_0^\tau f_n(s)h(s)ds \\
&= T_{u,h}f_n(t).
\end{aligned}$$

Again, using that $\frac{\bar{u}}{H}$ is non-increasing and $\operatorname{supp} f_n \subset (0,x)$, for $t > x$ we have

$$\begin{aligned}
\phi_n(t) &= \frac{\bar{u}(t)}{H(t)}\left(\int_{E_n}[v(s)]^{1-p'}[h(s)]^{p'}ds\right)^{\frac{1}{p'}} \\
&= \frac{\bar{u}(t)}{H(t)}\left(\int_{E_n}[v(s)]^{1-p'}[h(s)]^{p'}ds\right)^{1-\frac{1}{p}} \\
&= \frac{\bar{u}(t)}{H(t)}\int_0^t f_n(s)h(s)ds \\
&= \sup_{t\leq\tau<\infty}\frac{u(\tau)}{H(\tau)}\int_0^t f_n(s)h(s)ds \\
&\leq \sup_{t\leq\tau<\infty}\frac{u(\tau)}{H(\tau)}\int_0^\tau f_n(s)h(s)ds \\
&= T_{u,h}f_n(t).
\end{aligned}$$

So, f_n has all the properties we asked for, and (3.21) is true.

In case of $p = 1$, define $\mathcal{E} = \{E \subset (0,x); E \text{ is } \lambda - \text{measurable}, \lambda(E) > 0\}$. For each $E \in \mathcal{E}$, put

$$\phi_E(t) = \min\left\{\frac{\bar{u}(x)}{H(x)}, \frac{\bar{u}(t)}{H(t)}\right\}(\lambda(E))^{-1}\int_E \frac{h(s)}{v(s)}ds.$$

Then

$$\Phi_1(t) = \sup_{E\in\mathcal{E}}\phi_E(t)$$

for every $t \in (0,\infty)$. Similarly to the case when $p > 1$, for any $E \in \mathcal{E}$ we want to find a function f_E such that $f_E \in L^p(v)$, $\|f_E\|_{p,v} \le 1$ and $\phi_E(t) \le T_{u,h} f_E(t)$ for all $t \in (0,\infty)$. If we succeed, we will get

$$\left(\int_0^a \left[\min\left\{\frac{\bar{u}(x)}{H(x)}, \frac{\bar{u}(t)}{H(t)}\right\}(\lambda(E))^{-1}\int_E \frac{h(s)}{v(s)}ds\right]^q w(t)dt\right)^{\frac{1}{q}} \\ \le \left\|\chi_{(0,a)} T_{u,h} f_E\right\|_{q,w} < \varepsilon.$$

Hence, taking the supremum over $E \in \mathcal{E}$ will then give

$$\left(\left(\frac{\bar{u}(x)}{H(x)}\right)^q \int_0^x w(t)dt + \int_x^a \left(\frac{\bar{u}(x)}{H(x)}\right)^q w(t)dt\right)^{\frac{1}{q}} \sigma_{p,h}(0,x) \le \varepsilon.$$

Finally, to obtain inequality (3.21), we will pass to the supremum over $x \in (0,a)$. For $E \in \mathcal{E}$, such a suitable f_E can be defined by

$$f_E(y) = \frac{\chi_E(y)}{\lambda(E)v(y)}, \quad y \in (0,\infty).$$

Indeed, f_E is Lebesgue-measurable on $(0,\infty)$ and

$$\int_0^\infty f_E(y)v(y)dy = (\lambda(E))^{-1}\int_E \frac{v(y)}{v(y)}dy = 1,$$

whence $f_E \in L^p(v)$ and $\|f_E\|_{p,v} = 1$. In addition, for $t \le x$,

$$\begin{aligned}
\phi_E(t) &= \frac{\bar{u}(x)}{H(x)}(\lambda(E))^{-1}\int_E \frac{h(s)}{v(s)}ds \\
&= \frac{\bar{u}(x)}{H(x)}\int_E f_E(s)h(s)ds \\
&= \frac{\bar{u}(x)}{H(x)}\int_0^x f_E(s)h(s)ds \\
&= \sup_{x\le\tau<\infty} \frac{u(\tau)}{H(\tau)}\int_0^x f_E(s)h(s)ds \\
&\le \sup_{x\le\tau<\infty} \frac{u(\tau)}{H(\tau)}\int_0^\tau f_E(s)h(s)ds \\
&\le \sup_{t\le\tau<\infty} \frac{u(\tau)}{H(\tau)}\int_0^\tau f_E(s)h(s)ds \\
&= T_{u,h} f_E(t),
\end{aligned}$$

thanks to the fact that $\frac{\bar{u}}{H}$ is non-increasing and $\operatorname{supp} f_E \subset (0,x)$. When $t > x$, the same facts give

$$\begin{aligned}\phi_E(t) &= \frac{\bar{u}(t)}{H(t)}(\lambda(E))^{-1}\int_E \frac{h(s)}{v(s)}ds \\ &= \frac{\bar{u}(t)}{H(t)}\int_E f_E(s)h(s)ds \\ &= \frac{\bar{u}(t)}{H(t)}\int_0^t f_E(s)h(s)ds \\ &= \sup_{t\le\tau<\infty}\frac{u(\tau)}{H(\tau)}\int_0^t f_E(s)h(s)ds \\ &\le \sup_{t\le\tau<\infty}\frac{u(\tau)}{H(\tau)}\int_0^\tau f_E(s)h(s)ds \\ &= T_{u,h}f_E(t),\end{aligned}$$

as desired. Hence, f_E meets all our requirements and (3.21) holds.

To prove inequality (3.22), we proceed the same way as for (3.21), with the difference that we work with (b,∞) instead of $(0,a)$.

Sufficiency: Also in this part of proof, we would like to use Theorem 3.1.1. For that purpose, we need to know that assumptions (3.18) and (3.19) for $p \le q$, or assumption (3.20) for $q < p$, imply that $T_{u,h}$ is a bounded operator from $L^p(v)$ to $L^q(w)$. The latter one was directly shown in [4, Theorem 4.2]. According to the same source, the statement for $p \le q$ is equivalent to the result of Lemma 3.2.5 applied to the spaces $L^p(v)$ and $L^q(w)$.

We divide the remainder of the proof into three steps. In the first one we consider a slightly modified operator and show, roughly speaking, that all functions from the image of the unit ball of $L^p(v)$ are small near 0 in the sense of norm in $L^q(w)$. The second step is devoted to an analogy, in which 0 is replaced by ∞. And in the third step we derive the conclusion.

Step 1: Let $\varepsilon > 0$. If $p \le q$, we use (3.18) to find $a \in (0,\infty)$ for which

$$\sup_{0<x\le a}\left(\left(\frac{\bar{u}(x)}{H(x)}\right)^q\int_0^x w(t)dt + \int_x^a\left(\frac{\bar{u}(t)}{H(t)}\right)^q w(t)dt\right)^{\frac{1}{q}}\sigma_{p,h}(0,x) < \varepsilon. \tag{3.23}$$

Provided $q < p$, by virtue of condition (3.20) and Lemma 3.3.3 there must exist $a \in (0,\infty)$ satisfying

$$\sup_{\{x_k\}}\left(\sum_k\left(\int_{x_{k+1}}^{x_{k-1}}\min\left\{\frac{\bar{u}(x_k)}{H(x_k)},\frac{\bar{u}(t)}{H(t)}\right\}^q w(t)dt\right)^{\frac{r}{q}}[\sigma_{p,h}(x_{k+1},x_k)]^r\right)^{\frac{1}{r}} < \varepsilon, \tag{3.24}$$

where the supremum is taken over all sequences $\{x_k\}$ convenient for the interval $[0,a]$. Because $\int_0^a w(t)dt < \infty$ and integral is continuous, there exists a sequence $\{x_k\}_{k=1}^\infty$ convenient for the interval $[0,a]$ such that $\int_{x_{k+1}}^{x_k} w(t)dt = 2^{-k}\int_0^a w(t)dt$ for each $k \in \mathbb{N}$.

Taking $f \in L^p(v)$, we have

$$\begin{aligned}
\left\| T^{(0,a)}_{u,h,(0,a)} f \right\|^q_{q,w} &= \int_0^a \left[\sup_{t \le \tau < a} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q w(t) dt \\
&= \sum_{k=1}^\infty \int_{x_{k+1}}^{x_k} \left[\sup_{t \le \tau < a} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q w(t) dt \\
&\le \sum_{k=1}^\infty \int_{x_{k+1}}^{x_k} \left[\sup_{x_{k+1} \le \tau < a} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q w(t) dt \\
&= \sum_{k=1}^\infty \max_{2 \le i \le k+1} \left\{ \left[\sup_{x_i \le \tau < x_{i-1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q \right\} \\
&\qquad \times 2^{-k} \int_0^a w(t) dt \\
&\le \sum_{k=1}^\infty \sum_{i=2}^{k+1} \left[\sup_{x_i \le \tau < x_{i-1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q 2^{-k} \int_0^a w(t) dt \\
&= \sum_{i=2}^\infty \left[\sup_{x_i \le \tau < x_{i-1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q \sum_{k=i-1}^\infty 2^{-k} \int_0^a w(t) dt \\
&= \sum_{i=2}^\infty \left[\sup_{x_i \le \tau < x_{i-1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q 2^{-i+2} \int_0^a w(t) dt \\
&= 4 \sum_{i=2}^\infty \left[\sup_{x_i \le \tau < x_{i-1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q \int_{x_{i+1}}^{x_i} w(t) dt \\
&\le 8 \sum_{i=2}^\infty \left[\frac{u(z_i)}{H(z_i)} \int_0^{z_i} |f(s)| h(s) ds \right]^q \int_{z_{i+2}}^{z_i} w(t) dt \\
&\le c(q) \sum_{i=2}^\infty \left[\frac{u(z_i)}{H(z_i)} \int_{z_{i+2}}^{z_i} |f(s)| h(s) ds \right]^q \int_{z_{i+2}}^{z_i} w(t) dt \\
&\quad + c(q) \sum_{i=2}^\infty \left[\frac{u(z_i)}{H(z_i)} \int_0^{z_{i+2}} |f(s)| h(s) ds \right]^q \int_{z_{i+2}}^{z_i} w(t) dt \\
&=: S_1^a + S_2^a,
\end{aligned}$$

where $z_i \in [x_i, x_{i-1})$ and

$$\frac{u(z_i)}{H(z_i)} \int_0^{z_i} |f(s)| h(s) ds > \frac{1}{2} \sup_{x_i \le \tau < x_{i-1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds,$$

for each $i \ge 2$.

Hölder's inequality yields

$$\int_{z_{i+2}}^{z_i} |f(s)| h(s) ds \le \sigma_{p,h}(z_{i+2}, z_i) \left(\int_{z_{i+2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{1}{p}}.$$

In view of this,

$$S_1^a \leq c(q) \sum_{i=2}^{\infty} \left[\frac{u(z_i)}{H(z_i)} \sigma_{p,h}(z_{i+2}, z_i) \left(\int_{z_{i+2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{1}{p}} \right]^q \int_{z_{i+2}}^{z_i} w(t) dt.$$

A few final steps of estimating S_1^a differ with respect to the relation between p and q. First, consider $p \leq q$.

$$\begin{aligned} S_1^a &\leq c(q) \sum_{i=2}^{\infty} \left(\frac{u(z_i)}{H(z_i)} \right)^q \int_0^{z_i} w(t) dt \, (\sigma_{p,h}(0, z_i))^q \left(\int_{z_{i+2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{q}{p}} \\ &\leq c(q) \varepsilon^q \left(\sum_{i=2}^{\infty} \int_{z_{i+2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{q}{p}} \\ &\leq c(q) \varepsilon^q \|f\|_{p,v}^q . \end{aligned}$$

To check the last but one inequality, we recall that $\frac{q}{p} \geq 1$, $u(z_i) \leq \bar{u}(z_i)$ and refer to (3.23). Continue with case $q < p$. Apply Hölder's inequality for sums with the exponents $\frac{p}{q}$ and $\frac{r}{q}$ to obtain

$$\begin{aligned} S_1^a &\leq c(q) \left(\sum_{i=2}^{\infty} \left(\frac{u(z_i)}{H(z_i)} \right)^r \left(\int_{z_{i+2}}^{z_i} w(t) dt \right)^{\frac{r}{q}} (\sigma_{p,h}(z_{i+2}, z_i))^r \right)^{\frac{q}{r}} \\ &\quad \times \left(\sum_{i=2}^{\infty} \int_{z_{i+2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{q}{p}} \\ &= c(q) \left(\sum_{i=2}^{\infty} \left(\int_{z_{i+2}}^{z_i} \left(\frac{u(z_i)}{H(z_i)} \right)^q w(t) dt \right)^{\frac{r}{q}} (\sigma_{p,h}(z_{i+2}, z_i))^r \right)^{\frac{q}{r}} \\ &\quad \times \left(\sum_{i=2}^{\infty} \int_{z_{i+2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{q}{p}} . \end{aligned} \tag{3.25}$$

We can rewrite the first sum on the right hand side as follows,

$$\begin{aligned} &\sum_{i=2}^{\infty} \left(\int_{z_{i+2}}^{z_i} \left(\frac{u(z_i)}{H(z_i)} \right)^q w(t) dt \right)^{\frac{r}{q}} (\sigma_{p,h}(z_{i+2}, z_i))^r \\ &\qquad = \sum_{i=1}^{\infty} \left(\int_{z_{2i+2}}^{z_{2i}} \left(\frac{u(z_{2i})}{H(z_{2i})} \right)^q w(t) dt \right)^{\frac{r}{q}} (\sigma_{p,h}(z_{2i+2}, z_{2i}))^r \\ &\qquad\quad + \sum_{i=1}^{\infty} \left(\int_{z_{2i+3}}^{z_{2i+1}} \left(\frac{u(z_{2i+1})}{H(z_{2i+1})} \right)^q w(t) dt \right)^{\frac{r}{q}} (\sigma_{p,h}(z_{2i+3}, z_{2i+1}))^r . \end{aligned}$$

After we set $z_0 = a$ and $z_1 = a$, both the sequences $\{z_{2i}\}_{i=0}^{\infty}$ and $\{z_{2i+1}\}_{i=0}^{\infty}$ will become convenient for the interval $[0, a]$. In addition to this,

$$\sum_{i=1}^{\infty}\left(\int_{z_{2i+2}}^{z_{2i}}\left(\frac{u(z_{2i})}{H(z_{2i})}\right)^{q} w(t)dt\right)^{\frac{r}{q}} (\sigma_{p,h}(z_{2i+2}, z_{2i}))^{r}$$
$$\leq \sum_{i=1}^{\infty}\left(\int_{z_{2i+2}}^{z_{2i-2}} \min\left\{\frac{\bar{u}(z_{2i})}{H(z_{2i})}, \frac{\bar{u}(t)}{H(t)}\right\}^{q} w(t)dt\right)^{\frac{r}{q}} (\sigma_{p,h}(z_{2i+2}, z_{2i}))^{r}$$

and

$$\sum_{i=1}^{\infty}\left(\int_{z_{2i+3}}^{z_{2i+1}}\left(\frac{u(z_{2i+1})}{H(z_{2i+1})}\right)^{q} w(t)dt\right)^{\frac{r}{q}} (\sigma_{p,h}(z_{2i+3}, z_{2i+1}))^{r}$$
$$\leq \sum_{i=1}^{\infty}\left(\int_{z_{2i+3}}^{z_{2i-1}} \min\left\{\frac{\bar{u}(z_{2i+1})}{H(z_{2i+1})}, \frac{\bar{u}(t)}{H(t)}\right\}^{q} w(t)dt\right)^{\frac{r}{q}} (\sigma_{p,h}(z_{2i+3}, z_{2i+1}))^{r},$$

where the inequalities are implied by relation $u(t) \leq \bar{u}(t)$ for all $t \in (0, \infty)$ combined with the fact that function $\frac{\bar{u}}{H}$ is non-increasing. In view of the above, we return to (3.25) and use (3.24). We arrive at

$$S_1^a \leq c(p, q)\varepsilon^q \left\| f \right\|_{p,v}^q .$$

As for S_2^a goes, observe that

$$\begin{aligned} S_2^a &\leq c(q)\sum_{i=2}^{\infty}\left[\frac{\bar{u}(z_i)}{H(z_i)}\int_0^{z_{i+2}} |f(s)|h(s)ds\right]^q \int_{z_{i+2}}^{z_i} w(t)dt \\ &= c(q)\sum_{i=2}^{\infty}\int_{z_{i+2}}^{z_i}\left[\frac{\bar{u}(z_i)}{H(z_i)}\int_0^{z_{i+2}} |f(s)|h(s)ds\right]^q w(t)dt \\ &\leq c(q)\sum_{i=2}^{\infty}\int_{z_{i+2}}^{z_i}\left[\frac{\bar{u}(t)}{H(t)}\int_0^{t} |f(s)|h(s)ds\right]^q w(t)dt \\ &\leq c(q)\int_0^a\left[\frac{\bar{u}(t)}{H(t)}\int_0^{t} |f(s)|h(s)ds\right]^q w(t)dt. \end{aligned}$$

Using the assertion of Kufner and Opic in [8, Theorem 1.14] for $p \leq q$ or the one of Sawyer in [9, Theorem 3] for $q < p$, (3.23) or (3.24), respectively, imply that there is a constant $c(p, q)$ such that

$$\begin{aligned} S_2^a &\leq c(p, q)\varepsilon^q \left(\int_0^a |f(t)|^p v(t)dt\right)^{\frac{q}{p}} \\ &\leq c(p, q)\varepsilon^q \left\| f \right\|_{p,v}^q . \end{aligned}$$

No matter what the relation between p and q is, when we combine the estimates for S_1^a and S_2^a, we will arrive at

$$\left\| T^{(0,a)}_{u,h,(0,a)} f \right\|_{q,w} \leq c(p,q)\varepsilon \left\| f \right\|_{p,v}.$$

STEP 2: Let's start the study of the situation near ∞ provided that the weight w satisfies $\int_0^\infty w(t)dt < \infty$. In this case, for an arbitrary $\varepsilon > 0$ we can find $b \in (a,\infty)$, where a is from Step 1 and corresponds to ε, such that $\int_b^\infty w(t)dt < \varepsilon^q$ and $\int_0^b w(t)dt \geq 2^{-1}\int_0^\infty w(t)dt$. For $f \in L^p(v)$ then

$$\begin{aligned}
\left\| T^{(b,\infty)}_{u,h} f \right\|_{q,w} &= \left(\int_b^\infty \left[T_{u,h}f(t)\right]^q w(t)dt \right)^{\frac{1}{q}} \\
&\leq \left(\int_b^\infty \left[T_{u,h}f(b)\right]^q w(t)dt \right)^{\frac{1}{q}} \\
&= \left(\int_b^\infty w(t)dt \right)^{\frac{1}{q}} \left(\int_0^b w(t)dt \right)^{-\frac{1}{q}} \left(\int_0^b \left[T_{u,h}f(b)\right]^q w(t)dt \right)^{\frac{1}{q}} \\
&\leq \varepsilon 2^{\frac{1}{q}} \left(\int_0^\infty w(t)dt \right)^{-\frac{1}{q}} \left(\int_0^b \left[T_{u,h}f(t)\right]^q w(t)dt \right)^{\frac{1}{q}} \\
&\leq \varepsilon 2^{\frac{1}{q}} \left(\int_0^\infty w(t)dt \right)^{-\frac{1}{q}} \left\| T_{u,h} f \right\|_{q,w}.
\end{aligned}$$

Since by Lemma 3.2.5 conditions (3.18) and (3.19) imply the boundedness of the operator $T_{u,h}$, we have

$$\left\| T^{(b,\infty)}_{u,h} f \right\|_{q,w} \leq c(q,w,T_{u,h})\varepsilon \left\| f \right\|_{p,v},$$

for some constant $c(q,w,T_{u,h})$, which depends only on q, L^1-norm of the weight w and $c(T_{u,h})$. The latter one being a constant from formula

$$\left\| T_{u,h} f \right\|_{q,w} \leq c(T_{u,h}) \left\| f \right\|_{p,v}, \quad f \in L^p(v).$$

Now, suppose that $\int_0^\infty w(t)dt = \infty$ for the weight w. Take an arbitrary $\varepsilon > 0$ and fix a corresponding a from Step 1. Due to (3.19), for $p \leq q$ we find $b \in (a,\infty)$ that satisfies

$$\sup_{b \leq x < \infty} \left(\left(\frac{\bar{u}(x)}{H(x)} \right)^q \int_b^x w(t)dt + \int_x^\infty \left(\frac{\bar{u}(t)}{H(t)} \right)^q w(t)dt \right)^{\frac{1}{q}} \sigma_{p,h}(0,x) < \varepsilon. \quad (3.26)$$

For $q < p$ we use (3.20) and Lemma 3.3.3 to get the existence of $b \in (a,\infty)$ such that

$$\sup_{\{x_k\}} \left(\sum_k \left(\int_{x_{k-1}}^{x_{k+1}} \min\left\{ \frac{\bar{u}(x_k)}{H(x_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_{k-1},x_k)\right]^r \right)^{\frac{1}{r}} < \varepsilon, \quad (3.27)$$

where the supremum is taken over all increasing sequences $\{x_k\}_{k=I}^{k=J}$ with $x_I = b$ for $I \in \mathbb{Z}$ or $\lim_{k\to-\infty} x_k = b$ for $I = \infty$ and $x_J = \infty$ for $J \in \mathbb{N}$ or $\lim_{k\to\infty} x_k = \infty$ for $J = \infty$. There exists an increasing sequence $\{x_k\}_{k=-\infty}^{\infty}$ lying in the interval (b, ∞), such that $\lim_{k\to\infty} x_k = \infty$, $\lim_{k\to-\infty} x_k = b$ and $\int_{x_k}^{x_{k+1}} w(t)dt = 2^k$ for every $k \in \mathbb{Z}$.

Similarly to the interval $(0, a)$, for $f \in L^p(v)$ we obtain

$$\begin{aligned}
\left\|T_{u,h}^{(b,\infty)} f\right\|_{q,w}^q &= \int_b^\infty \left[\sup_{t\le\tau<\infty} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds\right]^q w(t)dt \\
&= \sum_{k=-\infty}^{\infty} \int_{x_k}^{x_{k+1}} \left[\sup_{t\le\tau<\infty} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds\right]^q w(t)dt \\
&\le \sum_{k=-\infty}^{\infty} \int_{x_k}^{x_{k+1}} \left[\sup_{x_k\le\tau<\infty} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds\right]^q w(t)dt \\
&= \sum_{k=-\infty}^{\infty} \sup_{k\le i<\infty} \left\{\left[\sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds\right]^q\right\} 2^k \\
&\le \sum_{k=-\infty}^{\infty} \sum_{i=k}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds\right]^q 2^k \\
&= \sum_{i=-\infty}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds\right]^q \sum_{k=-\infty}^{i} 2^k \\
&= \sum_{i=-\infty}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds\right]^q 2^{i+1} \\
&= 4 \sum_{i=-\infty}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds\right]^q \int_{x_{i-1}}^{x_i} w(t)dt.
\end{aligned}$$

Again, take $\{z_i\}_{i=-\infty}^{\infty}$, for which $z_i \in [x_i, x_{i+1})$ and

$$\frac{u(z_i)}{H(z_i)} \int_0^{z_i} |f(s)|h(s)ds > \frac{1}{2} \sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds.$$

Then,

$$\begin{aligned}
\left\|T_{u,h}^{(b,\infty)} f\right\|_{q,w}^q &\le 8 \sum_{i=-\infty}^{\infty} \left[\frac{u(z_i)}{H(z_i)} \int_0^{z_i} |f(s)|h(s)ds\right]^q \int_{z_{i-2}}^{z_i} w(t)dt \\
&\le c(q) \sum_{i=-\infty}^{\infty} \left[\frac{u(z_i)}{H(z_i)} \int_{z_{i-2}}^{z_i} |f(s)|h(s)ds\right]^q \int_{z_{i-2}}^{z_i} w(t)dt \\
&\quad + c(q) \sum_{i=-\infty}^{\infty} \left[\frac{u(z_i)}{H(z_i)} \int_0^{z_{i-2}} |f(s)|h(s)ds\right]^q \int_{z_{i-2}}^{z_i} w(t)dt =: S_1^b + S_2^b.
\end{aligned}$$

Concerning the first term in case $p \leq q$, Hölder's inequality, $\frac{q}{p} \geq 1$, $u(z_i) \leq \bar{u}(z_i)$ and (3.26) yield

$$\begin{aligned} S_1^b &\leq c(q) \sum_{i=-\infty}^{\infty} \left[\frac{u(z_i)}{H(z_i)} \sigma_{p,h}(z_{i-2}, z_i) \left(\int_{z_{i-2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{1}{p}} \right]^q \int_b^{z_i} w(t) dt \\ &\leq c(q) \sum_{i=-\infty}^{\infty} \left(\frac{u(z_i)}{H(z_i)} \right)^q \int_b^{z_i} w(t) dt \, (\sigma_{p,h}(0, z_i))^q \left(\int_{z_{i-2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{q}{p}} \\ &\leq c(q) \varepsilon^q \left(\sum_{i=-\infty}^{\infty} \int_{z_{i-2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{q}{p}} \leq c(q) \varepsilon^q \left\| f \right\|_{p,v}^q . \end{aligned}$$

Provided $q < p$, by Hölder's inequality applied two times and (3.27) we get

$$\begin{aligned} S_1^b &\leq c(q) \sum_{i=-\infty}^{\infty} \left[\frac{u(z_i)}{H(z_i)} \sigma_{p,h}(z_{i-2}, z_i) \left(\int_{z_{i-2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{1}{p}} \right]^q \int_{z_{i-2}}^{z_i} w(t) dt \\ &\leq c(q) \left(\sum_{i=-\infty}^{\infty} \left(\int_{z_{i-2}}^{z_i} \left(\frac{u(z_i)}{H(z_i)} \right)^q w(t) dt \right)^{\frac{r}{q}} (\sigma_{p,h}(z_{i-2}, z_i))^r \right)^{\frac{q}{r}} \\ &\quad \times \left(\sum_{i=-\infty}^{\infty} \int_{z_{i-2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{q}{p}} \\ &= c(q) \Bigg(\sum_{i=-\infty}^{\infty} \left(\int_{z_{2i-2}}^{z_{2i}} \left(\frac{u(z_{2i})}{H(z_{2i})} \right)^q w(t) dt \right)^{\frac{r}{q}} (\sigma_{p,h}(z_{2i-2}, z_{2i}))^r \\ &\qquad + \sum_{i=-\infty}^{\infty} \left(\int_{z_{2i-3}}^{z_{2i-1}} \left(\frac{u(z_{2i-1})}{H(z_{2i-1})} \right)^q w(t) dt \right)^{\frac{r}{q}} (\sigma_{p,h}(z_{2i-3}, z_{2i-1}))^r \Bigg)^{\frac{q}{r}} \\ &\quad \times \left(\sum_{i=-\infty}^{\infty} \int_{z_{i-2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{q}{p}} \\ &\leq c(q) \Bigg(\sum_{i=-\infty}^{\infty} \left(\int_{z_{2i-2}}^{z_{2i+2}} \min \left\{ \frac{\bar{u}(z_{2i})}{H(z_{2i})}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t) dt \right)^{\frac{r}{q}} (\sigma_{p,h}(z_{2i-2}, z_{2i}))^r \\ &\qquad + \sum_{i=-\infty}^{\infty} \left(\int_{z_{2i-3}}^{z_{2i+1}} \min \left\{ \frac{\bar{u}(z_{2i-1})}{H(z_{2i-1})}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t) dt \right)^{\frac{r}{q}} \\ &\qquad\qquad \times (\sigma_{p,h}(z_{2i-3}, z_{2i-1}))^r \Bigg)^{\frac{q}{r}} \left(\sum_{i=-\infty}^{\infty} \int_{z_{i-2}}^{z_i} |f(s)|^p v(s) ds \right)^{\frac{q}{p}} \\ &\leq c(p,q) \varepsilon^q \left\| f \right\|_{p,v}^q , \end{aligned}$$

because $\{z_{2i}\}_{i=-\infty}^{\infty}$ and $\{z_{2i+1}\}_{i=-\infty}^{\infty}$ satisfy the demands on sequences for which (3.27) is formulated.

Referring to [8, Theorem 1.14] for $p \le q$ and [9, Theorem 3] for $q < p$ as before, from (3.26) and (3.27), respectively, we obtain

$$\begin{aligned}
S_2^b &\le c(q) \sum_{i=-\infty}^{\infty} \left[\frac{\bar{u}(z_i)}{H(z_i)} \int_0^{z_{i-2}} |f(s)| h(s) ds \right]^q \int_{z_{i-2}}^{z_i} w(t) dt \\
&= c(q) \sum_{i=-\infty}^{\infty} \int_{z_{i-2}}^{z_i} \left[\frac{\bar{u}(z_i)}{H(z_i)} \int_0^{z_{i-2}} |f(s)| h(s) ds \right]^q w(t) dt \\
&\le c(q) \sum_{i=-\infty}^{\infty} \int_{z_{i-2}}^{z_i} \left[\frac{\bar{u}(t)}{H(t)} \int_0^{t} |f(s)| h(s) ds \right]^q w(t) dt \\
&\le c(q) \int_b^{\infty} \left[\frac{\bar{u}(t)}{H(t)} \int_0^{t} |f(s)| h(s) ds \right]^q w(t) dt \\
&\le c(p,q) \varepsilon^q \left(\int_b^{\infty} |f(t)|^p v(t) dt \right)^{\frac{q}{p}} \\
&\le c(p,q) \varepsilon^q \left\| f \right\|_{p,v}^q .
\end{aligned}$$

The above together with the estimates of S_1^b gives

$$\left\| \chi_{(b,\infty)} T_{u,h} f \right\|_{q,w} \le c(p,q) \varepsilon \left\| f \right\|_{p,v} .$$

Step 3: Finally, we are in the position to verify the compactness of the operator $T_{u,h}$. Actually, we shall show that the set $\{T_{u,h} f; f \in L^p(v), \|f\|_{p,v} \le 1\}$ is of uniformly absolutely continuous norm in $L^q(w)$ and apply Theorem 3.1.1. Note that the operator $T_{u,h}$ and the spaces $L^p(v)$ and $L^q(w)$ fall into the setting of Theorem 3.1.1, due to the properties of $T_{u,h}$ and Theorem 2.2.2.

Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of λ-measurable subsets of $(0,\infty)$ such that $\chi_{E_n} \to \chi_{\emptyset}$ λ-a.e. on $(0,\infty)$. Consider an arbitrary $\varepsilon > 0$. By the previous two steps, we are able to find $0 < a < b < \infty$ satisfying

$$\left\| T_{u,h,(0,a)}^{(0,a)} f \right\|_{q,w} \le \frac{\varepsilon}{4} \left\| f \right\|_{p,v}$$

and

$$\left\| T_{u,h}^{(b,\infty)} f \right\|_{q,w} \le \frac{\varepsilon}{4} \left\| f \right\|_{p,v} .$$

As $\int_0^b w(t) dt < \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have

$$\left(\int_0^b \chi_{E_n}(t) w(t) dt \right)^{\frac{1}{q}} < \min \left\{ \frac{\varepsilon}{4}, \frac{\varepsilon}{4c(T_{u,h})} \left(\int_0^a w(t) dt \right)^{\frac{1}{q}} \right\},$$

where $c(T_{u,h})$ denotes a constant from the inequality describing the boundedness of $T_{u,h}$, i.e.

$$\left\| T_{u,h} f \right\|_{q,w} \le c(T_{u,h}) \left\| f \right\|_{p,v} , \quad f \in L^p(v).$$

For $n \geq n_0$ and $f \in L^p(v)$ with $\|f\|_{p,v} \leq 1$ we can write

$$\begin{aligned}\|\chi_{E_n} T_{u,h} f\|_{q,w} &\leq \left\|\chi_{E_n} T_{u,h}^{(0,a)} f\right\|_{q,w} + \left\|\chi_{E_n} T_{u,h}^{[a,b]} f\right\|_{q,w} + \left\|\chi_{E_n} T_{u,h}^{(b,\infty)} f\right\|_{q,w} \\ &=: N_1 + N_2 + N_3.\end{aligned}$$

Now,

$$\begin{aligned} N_1 &\leq \left\|T_{u,h,a}^{(0,a)} f\right\|_{q,w} + \left\|\chi_{E_n}\chi_{(0,a)} T_{u,h} f(a)\right\|_{q,w} \\ &\leq \frac{\varepsilon}{4} + \left(\int_0^a [T_{u,h} f(a)]^q \, w(t)dt\right)^{\frac{1}{q}} \left(\int_0^a w(t)dt\right)^{-\frac{1}{q}} \left(\int_0^a \chi_{E_n} w(t)dt\right)^{\frac{1}{q}} \\ &\leq \frac{\varepsilon}{4} + \|T_{u,h} f\|_{q,w} \left(\int_0^a w(t)dt\right)^{-\frac{1}{q}} \left(\int_0^a \chi_{E_n} w(t)dt\right)^{\frac{1}{q}} \\ &\leq \frac{\varepsilon}{4} + c(T_{u,h}) \, \|f\|_{p,v} \left(\int_0^a w(t)dt\right)^{-\frac{1}{q}} \left(\int_0^a \chi_{E_n} w(t)dt\right)^{\frac{1}{q}} \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Concerning N_2, we use the monotonicity of the function $T_{u,h} f$ and the same estimate for $T_{u,h} f(a)$ as we used while treating the second term of the previous calculation and arrive at

$$\begin{aligned} N_2 &= \left(\int_a^b [T_{u,h} f(t)]^q \, \chi_{E_n}(t) w(t)dt\right)^{\frac{1}{q}} \\ &\leq \left(\int_a^b [T_{u,h} f(a)]^q \, \chi_{E_n}(t) w(t)dt\right)^{\frac{1}{q}} \\ &\leq c(T_{u,h}) \left(\int_0^a w(t)dt\right)^{-\frac{1}{q}} \left(\int_a^b \chi_{E_n} w(t)dt\right)^{\frac{1}{q}} \\ &< \frac{\varepsilon}{4}. \end{aligned}$$

And obviously,

$$N_3 \leq \left\|\chi_{(b,\infty)} T_{u,h} f\right\|_{q,w} \leq \frac{\varepsilon}{4}.$$

All in all,

$$\|\chi_{E_n} T_{u,h} f\|_{q,w} < \varepsilon.$$

This finishes the proof of the fact that the set $\{T_{u,h} f; f \in L^P(v), \|f\|_{p,v} \leq 1\}$ is of uniformly absolutely continuous norm in $L^q(w)$ and thus completes the whole proof of Theorem 3.3.4. $\square$

Bibliography

[1] Bennett C., Sharpley R.: *Interpolation of Operators*, Academic Press, Orlando, 1988.

[2] Cianchi A., Kerman R., Opic B., Pick L.: *A sharp rearrangement inequality for the fractional maximal operator*, Studia Math. 138 (3) (2000), 277–284.

[3] Edmunds D. E., Gurka P., Pick L.: *Compactness of Hardy-type integral operators in weighted Banach function spaces*, Studia Math. 109 (1994), 73–90.

[4] Gogatishvili A., Opic B., Pick L.: *Weighted inequalities for Hardy-type operators involving suprema*, Collect. Math. **57**, 3 (2006), 227–255.

[5] Kufner A., John O., Fučík S.: *Function spaces*, Academia, Praha, 1977.

[6] Lukeš J., Malý J.: *Míra a integrál*, Karolinum, Praha, 2002.

[7] Luxemburg W. A. J., Zaanen A. C.: *Compactness of Integral Operators in Banach Function Spaces*, Math. Ann. 149 (1963), 150–180.

[8] Opic B., Kufner A.: *Hardy-type Inequalities*, Longman Scientific and Technical, Harlow, 1990.

[9] Sawyer E.: *Weighted Lebesgue and Lorentz norm inequalities for the Hardy operator*, Trans. Amer. Math. Soc. **281** (1984), 329–337.