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DIPLOMOVÁ PRÁCE

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Slabá řešení stochastických diferenciálních rovnic

Katedra pravděpodobnosti a matematické statistiky

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Prohlašuji, že jsem svou diplomovou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne Martina Hofmanová

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Abstrakt: Hlavním výsledkem předložené práce je důkaz existence slabého řešení stochastické diferenciální rovnice s koeficienty spojitými v proměnné x a majícími v této proměnné nejvýše lineární růst. Standardní metody důkazu tohoto tvrzení (ať založené na konceptu slabého řešení či na řešení martingalového problému) využívají větu o integrální reprezentaci martingalů, jejíž důkaz je sám o sobě dosti komplikovaný, pokud je dimenze prostoru větší než jedna. Jednoduchá modifikace běžného postupu však dovoluje identifikovat slabé řešení elementárním způsobem, bez nutnosti aplikace zmiňované věty. V úvodních kapitolách jsou shrnuty důležité pomocné výsledky. Jedná se především o charakterizaci prostoru spojitých funkcí coby prostoru trajektorií a dále o důležitou větu umožňujíci aproximovat spojité funkce lipschitzovskými.
Klíčová slova: slabé řešení, silné řešení, martingal, Brownův pohyb, lipschitzovské funkce

Title: Weak solutions to stochastic differential equations
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Abstract: In the present work we study a stochastic differential equation with coefficients continuous in x having in this variable linear growth. As a main result we show that there exists a weak solution to this equation by a new, more elementary method. Standard methods are based either on the concept of the weak solution or equivalently on solving a martingale problem. However, both approaches employ the integral representation theorem for martingales, whose proof becomes rather complicated in dimension greater than one. By a simple modification of the usual procedure, one can identify the weak solution elementary, with no need to apply the above mentioned theorem. In the preliminaries we summarize some auxiliary results: namely, some properties of the space of continuous functions as the space of trajectories are established and an important theorem which allows us to approximate continuous function by functions Lipschitz continuous is proved.
Keywords: weak solution, strong solution, martingale, Brownian motion, Lipschitz continuous functions

Contents

Chapter 1

Introduction

Kiyosi Itô in his seminal papers (see e.g. [It46], [It51]) showed that a stochastic differential equation

$$\mathrm{d}X = b(X)\mathrm{d}t + \sigma(X)\mathrm{d}W \tag{1.1}$$

$$X(0) = \psi \tag{1.2}$$

driven by an k-dimensional Brownian motion W has a unique solution, provided that $b : \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}^m \to \mathbb{M}_{m\times k}$ are Lipschitz continuous functions. (In this Introduction, we shall discuss only autonomous equations for notational simplicity.) A next important step was taken by A. Skorokhod ([Sk61], [Sk62]), who proved that there exists a solution to (1.1), (1.2) if coefficients b and σ are continuous functions of a linear growth,

$$\exists K > 0 \quad \forall x \in \mathbb{R}^m \quad \|b(x)\| \vee \|\sigma(x)\| \le K(1 + \|x\|).$$

It was realized only later that two different concepts of a solution are involved: for Lipschitzian coefficients, there exists an $(\mathscr{F}_t)$-progressively measurable process in $\mathbb{R}^m$ solving (1.1) and satisfying initial condition (1.2), whenever $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis carrying an k-dimensional $(\mathscr{F}_t)$-Brownian motion and ψ is an $\mathscr{F}_0$-measurable function. (We say that X is a strong solution of (1.1), (1.2).) On the other hand, for continuous coefficients may be found a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, an k-dimensional $(\mathscr{F}_t)$-Brownian motion W and an $(\mathscr{F}_t)$-progressively measurable process X such that X solves (1.1) and $X(0) \overset{d}{\sim} \psi$. (Such X is called a weak solution.) The difference is substantial in general: M. T. Barlow found a (uniformly positive) continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} \mathrm{d}X &= g(X)\mathrm{d}W \\ X(0) &= x, \quad x \in \mathbb{R} \end{aligned}$$

has no strong solution (see [Ba82]).

Skorokhod's existence theorem is remarkable not only by itself, but also because the method of its proof. Let

$$\mathrm{d}\tilde{X}_n = b_n(\tilde{X}_n)\mathrm{d}t + \sigma_n(\tilde{X}_n)\mathrm{d}\tilde{W}, \quad \tilde{X}_n(0) = \psi, \qquad n \in \mathbb{N}$$

be a sequence of equations which have strong solutions and approximate (1.1) in a suitable sense. (We shall approximate b and σ by Lipschitz continuous functions having the same linear growth as b and σ have, but likewise it is possible to use e.g. finite difference approximations.) The linear growth hypothesis makes it possible to prove that the laws of $\{\tilde{X}_n;\ n \in \mathbb{N}\}$ are tight, that is, form a relatively weakly compact set of measures on the space of continuous trajectories. Then another theorem of Skorokhod may be invoked, which yields a subsequence $\{\tilde{X}_{n_l}\}$ of $\{\tilde{X}_n\}$, a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and sequences $\{X_l;\ l \in \mathbb{N}_0\}$, $\{W_l;\ l \in \mathbb{N}_0\}$ such that

$$\begin{aligned} (X_l, W_l) &\overset{d}{\sim} (\tilde{X}_{n_l}, \tilde{W}), \quad l \in \mathbb{N}; \\ (X_l, W_l) &\longrightarrow (X_0, W_0), \quad l \to \infty, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{1.3}$$

It is claimed that X_0 is the desired (weak) solution. Skorokhod's papers are written in a very concise way and essentially no argument supporting the claim is offered. Nowadays, standard version of Skorokhod's proofs is as follows (see [SV79, Theorem 6.1.6], [IW89, Theorem IV.2.2], [KS88, Theorem 5.4.22]): under a suitable integrability assumption upon the initial condition, for all $l \in \mathbb{N}$,

$$\tilde{M}_l = \tilde{X}_{n_l} - \tilde{X}_{n_l}(0) - \int_0^\cdot b_{n_l}\big(\tilde{X}_{n_l}(s)\big)\mathrm{d}s$$

is a martingale with a (tensor) quadratic variation

$$\langle\!\langle \tilde{M}_l \rangle\!\rangle = \int_0^\cdot \sigma_{n_l}\big(\tilde{X}_{n_l}(s)\big)\sigma_{n_l}^*\big(\tilde{X}_{n_l}(s)\big)\mathrm{d}s.$$

Equality in law implies that also

$$M_l = X_l - X_l(0) - \int_0^\cdot b_{n_l}\big(X_l(s)\big)\mathrm{d}s$$

are martingales with quadratic variations

$$\langle\!\langle M_l \rangle\!\rangle = \int_0^\cdot \sigma_{n_l}\big(X_l(s)\big)\sigma_{n_l}^*\big(X_l(s)\big)\mathrm{d}s.$$

Using convergence $\mathbb{P}$-almost everywhere, it is possible to show

$$M_0 = X_0 - X_0(0) - \int_0^\cdot b\big(X_0(s)\big)\mathrm{d}s$$

is a martingale with a quadratic variation

$$\langle\!\langle M_0 \rangle\!\rangle = \int_0^\cdot \sigma\big(X_0(s)\big)\sigma^*\big(X_0(s)\big)\mathrm{d}s.$$

By the integral representation theorem for martingales with an absolutely continuous quadratic variation, there exists a Brownian motion $\hat{W}$ (generally on an extended probability space) satisfying

$$M_0 = \int_0^{\cdot} \sigma\big(X_0(s)\big)\mathrm{d}\hat{W}(s).$$

Therefore, X_0 is a weak solution to (1.1), (1.2). (In the cited books, martingale problems are used instead of weak solutions. But the integral representation theorem is hidden in the construction of a weak solution from a solution to the martingale problem, so a complete proof is essentially again the one sketched above.)

The proof of the integral representation theorem, although based on a simple and beautiful idea, becomes rather technical if the space dimension is greater than one. Therefore, we aim at finding an alternative, more elementary approach. We start again with a sequence $\{(X_l, W_l)\}$ such that (1.3) holds true. Using the almost sure convergence, we show in a straightforward manner that

$$\|M_0\|^2 - \int_0^{\cdot} \|\sigma\big(X_0(s)\big)\|^2\mathrm{d}s, \quad M_0 \otimes W_0 - \int_0^{\cdot} \sigma\big(X_0(s)\big)\mathrm{d}s$$

are martingales, where we set $M_0 \otimes W_0 = \big(M_0^i W_0^j\big)_{i,j}$. In other words,

$$\left\langle\!\!\left\langle M_0 - \int_0^{\cdot} \sigma(X_0)\mathrm{d}W_0(s) \right\rangle\!\!\right\rangle = 0,$$

hence we conclude that X_0 is a weak solution. If the additional integrability hypothesis on ψ is not satisfied, the proof remains almost the same, only a suitable cut-off procedure must be amended.

This method of identifying the limit was proposed by Martin Ondreját (see [BO07], [On]) for studying infinitely dimensional stochastic equations, namely, stochastic wave maps between manifolds, where integral representation theorems for martingales are no longer available. We believe, however, that the method has its own merits for finite dimensional equations as well, since it refers only to basic properties of martingales and stochastic integrals, and that it is worth being presented in detail, to provide an illuminating comparison with other available approaches.

The thesis is organized as follows. Initially, the space of trajectories is defined as the space of all $\mathbb{R}^m$-valued continuous functions on $[0, \infty)$. A convenient metrization of this space can be done in order to obtain the Polish space property (separability and completeness). But it is not the only nice quality of this metric space. In subsequent sections, we prove that its metric describes the local uniform convergence; the Borel σ-algebra is generated by the system of projections (thus it is the restriction of the natural σ-algebra on the space of all mappings). Afterwards, we prove a criteria for relative compactness in the space of trajectories, based on a generalized Arzelà-Ascoli type theorem, which leads us to a characterization of tight collections of Borel probability measures. In the next chapter, we introduce the approximation of a continuous function by a sequence of Lipschitz continuous functions.

The main result – the proof of the existence of a weak solution – is contained in the fourth chapter. Firstly, Theorem 4.2.2 establishes the existence under an additional integrability assumption upon the initial condition and finally, in Theorem 4.2.8, this hypothesis is avoided.

Chapter 2

Space of continuous functions

Let us introduce the notation used throughout the work. From now on, $\mathscr{C}_m$ stands for the space of all continuous functions defined on $[0,\infty)$ with values in $\mathbb{R}^m$, $m \in \mathbb{N}$, i.e.

$$\mathscr{C}_m \equiv \mathscr{C}(\mathbb{R}_+, \mathbb{R}^m) = \{f : [0,\infty) \to \mathbb{R}^m;\ f \text{ continuous}\}.$$

For $T > 0$, f, $g \in \mathscr{C}_m$ we denote by

$$|||f - g|||_T = \sup_{0 \le t \le T} \|f(t) - g(t)\|$$

the supremum metric on $[0,T]$ and define

$$\varrho(f,g) = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{|||f - g|||_N}{1 + |||f - g|||_N}.$$

As will be seen immediately, the space $(\mathscr{C}_m, \varrho)$ is a metric space having fundamental properties such as separability and completeness. Therefore we are allowed to employ several nontrivial theorems which assume the space under investigation to be "nice" in such fashion. Moreover, the corresponding convergence leads to a very simple structure of the Borel σ-algebra.

Lemma 2.0.1. *The above defined function ϱ is a metric on $\mathscr{C}_m$.*

Proof. Obviously, $\varrho(f,g) \ge 0$ for any f, $g \in \mathscr{C}_m$ and $\varrho(f,g) = 0$ if and only if $f = g$. The symmetry of ϱ is a clear conclusion of symmetry of the supremum metric on $[0,N]$, $N \in \mathbb{N}$. In order to prove the triangule inequality, it suffices to show for any f, g, $h \in \mathscr{C}_m$ and $N \in \mathbb{N}$ that

$$\frac{|||f - g|||_N}{1 + |||f - g|||_N} \le \frac{|||f - h|||_N}{1 + |||f - h|||_N} + \frac{|||h - g|||_N}{1 + |||h - g|||_N}.$$

Since the function $\varphi(u) = \frac{u}{1+u}$, $u \in [0,\infty)$ is increasing and the supremum metric satisfies the triangule inequality, it follows that $\varphi(|||f - g|||_N) \le \varphi(|||f - h|||_N +$

$\|h-g\|_N)$ and therefore

$$\begin{aligned}\frac{\|f-g\|_N}{1+\|f-g\|_N} &\le \frac{\|f-h\|_N+\|h-g\|_N}{1+\|f-h\|_N+\|h-g\|_N} = \\ &= \frac{\|f-h\|_N}{1+\|f-h\|_N+\|h-g\|_N} + \frac{\|h-g\|_N}{1+\|f-h\|_N+\|h-g\|_N} \le \\ &\le \frac{\|f-h\|_N}{1+\|f-h\|_N} + \frac{\|h-g\|_N}{1+\|h-g\|_N}.\end{aligned}$$

□

Proposition 2.0.2. *The local uniform convergence of functions in $\mathscr{C}_m$ is metrizable by ϱ.*

Proof. Assume that $\{f_n\} \subset \mathscr{C}_m$, $f \in \mathscr{C}_m$. We need to prove that this sequence converge to f due to the metric ϱ if and only if it converge to f locally uniformly on $[0, \infty)$.

$\Leftarrow$ If $f_n \rightrightarrows_{loc} f$ then $\|f_n - f\|_N \to 0$ for all $N \in \mathbb{N}$. We now recall the function φ defined in the previous proof. By its right-continuity at the point 0, we conclude

$$\frac{\|f_n-f\|_N}{1+\|f_n-f\|_N} \longrightarrow 0, \quad n\to\infty, \quad \forall N \in \mathbb{N}$$

and

$$\lim_{n\to\infty} \varrho(f_n,f) = \lim_{n\to\infty} \sum_{N=1}^{\infty} \frac{1}{2^N}\frac{\|f_n-f\|_N}{1+\|f_n-f\|_N} = \sum_{N=1}^{\infty} \frac{1}{2^N} \lim_{n\to\infty} \frac{\|f_n-f\|_N}{1+\|f_n-f\|_N} = 0$$

where the interchange of limit and sum in the second equality is based on the Lebesque dominated convergence theorem, since the integrand has the integrable dominating function

$$\left|\frac{1}{2^N}\frac{\|f_n-f\|_N}{1+\|f_n-f\|_N}\right| \le \frac{1}{2^N}.$$

$\Rightarrow$ If $\varrho(f_n, f) \to 0$ then

$$0 = \lim_{n\to\infty} \sum_{N=1}^{\infty} \frac{1}{2^N}\frac{\|f_n-f\|_N}{1+\|f_n-f\|_N} = \sum_{N=1}^{\infty} \frac{1}{2^N} \lim_{n\to\infty} \frac{\|f_n-f\|_N}{1+\|f_n-f\|_N}$$

which implies that $\|f_n - f\|_N \to 0$ for any $N \in \mathbb{N}$ considering that the function φ is injective. It remains to show that this leads to the local uniform convergence $f_n \rightrightarrows_{loc} f$ on $[0, \infty)$. Fix $[a, b] \subset [0, \infty)$ and denote $N = \lceil b \rceil$. Consequently

$$\sup_{a\le t\le b} \|f_n(t) - f(t)\| \le \sup_{0\le t\le N} \|f_n(t)-f(t)\| = \|f_n - f\|_N \longrightarrow 0, \quad n \to \infty.$$

□

Note that we have actually proved more, namely

$$\varrho(f_n,f)\to 0 \quad \Longleftrightarrow \quad \|f_n-f\|_N \to 0 \quad \forall N\in\mathbb{N} \quad \Longleftrightarrow \quad f_n \rightrightarrows_{loc} f.$$

2.1 Separability and completeness

Both theorems included in this section transfer the appropriate properties from $(\mathscr{C}_m(K), \|\,.\,\|_\infty)$, K compact[1], by taking advantage of the convergence on $\mathscr{C}_m$.

Theorem 2.1.1. *The space $(\mathscr{C}_m, \varrho)$ is separable.*

Proof. Let us first prove the theorem in special case, for $m = 1$. By definition, it is needed to find a countable dense set $A \subset \mathscr{C}_1$. According to the Stone-Weierstrass theorem, the space $(\mathscr{C}_1(K), \|\,.\,\|_\infty)$, K compact, is separable. So in our notation this means that $(\mathscr{C}_1([0, N]), \|\!|\,.\,\|\!|_N)$ is separable $\forall N \in \mathbb{N}$, which gives the existence of a countable dense set $A_N \subset \mathscr{C}_1([0, N])$, $N \in \mathbb{N}$.

Fix $f \in \mathscr{C}_1$ arbitrary, then $\forall N \in \mathbb{N}$ there exists a sequence of functions $\{f_n^{(N)}\} \subset A_N$ so that

$$\|\!|f - f_n^{(N)}\|\!|_N \to 0,\ n \to \infty.$$

Because each function $f_n^{(N)}$ is defined only on $[0, N]$, it is convenient to extend its domain on $[N, \infty)$ constantly by the value $f_n^{(N)}(N)$. As a result we obtain $\forall N \in \mathbb{N}$ a sequence of functions in $\mathscr{C}_1$, which converges uniformly to f on interval $[0, N]$.

The task is now to construct the final sequence for which $f_n \rightrightarrows_{loc} f$.[2]

- Let $\{\varepsilon_N\} \subset (0, \infty)$ be such a sequence that $\varepsilon_N \searrow 0, \quad N \to \infty$.
- $N = 1$: in the sequence $\{f_n^{(1)}\}$ we find the smallest index n_1 so that

  $$\|\!|f - f_{n_1}^{(1)}\|\!|_1 < \varepsilon_1$$

  and set $f_1 = f_{n_1}^{(1)}$.
- ...
- $N = k$: in the sequence $\{f_n^{(k)}\}$ we find the smallest index n_k so that

  $$\|\!|f - f_{n_k}^{(k)}\|\!|_k < \varepsilon_k$$

  and set $f_k = f_{n_k}^{(k)}$.
- ...

Application of the above induction yields $\{f_n\} \subset A = \bigcup_{N\in\mathbb{N}} A_N$, therefore we have $A \subset \mathscr{C}_1$ which is countable and it remains to prove the local uniform convergence f_n to f on $[0, \infty)$:

Suppose that $K \subset [0, \infty)$ is compact. We show, that $f_n \rightrightarrows f$ on K, i.e.

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \sup_{t\in K} |f_n(t) - f(t)| < \varepsilon.$$

[1] $\mathscr{C}_m(K) = \{f : K \to \mathbb{R}^m;\ f \text{ continuous}\}$

[2] using diagonal method

Let $\varepsilon > 0$ be given. We find $N_1 \in \mathbb{N}$ so that $\varepsilon > \varepsilon_{N_1}$, then $N_2 \in \mathbb{N}$ so that $K \subset [0, N_2]$ and set $N = \max\{N_1, N_2\}$. By the construction of the sequence $\{f_n\}$, it follows

$$\forall n \geq N \quad |||f_n - f|||_N < \varepsilon_N < \varepsilon,$$

which gives the claim.

As the next part of the proof we consider the case $m > 1$ and observe

$$\mathscr{C}_m = \mathscr{C}_1 \times \cdots \times \mathscr{C}_1.$$

Indeed, every function $f \in \mathscr{C}_m$ can be rewritten as $f = (f_1, \ldots, f_m)$ where $f_i : [0, \infty) \to \mathbb{R}$, $i = 1, \ldots m$. According to [En89, Proposition 2.3.6] f is continuous if and only if all f_i are continuous.

Finally, let $f = (f_1, \ldots, f_m) \in \mathscr{C}_m$ be any function, then for all $i \in \{1, \ldots, m\}$ there exists a sequence $\{f_n^{(i)}\} \subset A$ so that $f_n^{(i)} \rightrightarrows_{loc} f_i$ on $[0, \infty)$. Hence

$$f_n = (f_n^{(1)}, \ldots, f_n^{(m)}) \rightrightarrows_{loc} f = (f_1, \ldots, f_m) \quad \text{on } [0, \infty),$$

consequently $\varrho(f_n, f) \to 0 \quad n \to \infty$ and $\{f_n\} \subset A \times \cdots \times A$, which is thus countable dense subset of $\mathscr{C}_m$.

□

Theorem 2.1.2. *The space $(\mathscr{C}_m, \varrho)$ is complete.*

Proof. Fix $T > 0$ and let us show that the space $\left(\mathscr{C}_m([0, T]), |||\,.\,|||_T\right)$ is complete. Suppose that $\{f_n\} \subset \mathscr{C}_m([0, T])$ is a cauchy sequence so

$$\forall \varepsilon > 0 \quad \exists n \in \mathbb{N} \quad \forall k, m \geq n \quad |||f_m - f_k|||_T < \varepsilon.$$

From this we can conclude the cauchy property of the sequence $\{f_n(t)\} \subset \mathbb{R}^m$ for any fixed $t \in [0, T]$. The completenes of $\mathbb{R}^m$ leads to the existence of a point in $\mathbb{R}^m$, denoted by $f(t)$, satisfying $f_n(t) \to f(t)$. The continuity of f is apparent from the following estimate. Fix $t, s \in [0, T]$,

$$\|f(t) - f(s)\| \leq \|f(t) - f_n(t)\| + \|f_n(t) - f_n(s)\| + \|f_n(s) - f(s)\|.$$

Now, suppose that $\{f_n\} \subset \mathscr{C}_m$ is a cauchy sequence. From the definition of the metric ϱ we easily obtain that for all $N \in \mathbb{N}$ the sequence $\{f_n|_{[0,N]}\}$ is cauchy. Hence for all $N \in \mathbb{N}$ there exists such a function $f^{(N)} \in \mathscr{C}_m([0, N])$ that $f_n \rightrightarrows f^{(N)}$ on $[0, N]$. Obviously, if $M, N \in \mathbb{N}$, $M > N$ then $f^{(M)}|_{[0,N]} = f^{(N)}$ and therefore it is correct to define $f = f^{(N)}$ on $[0, N]$, $N \in \mathbb{N}$. This function satisfy the condition $f_n \rightrightarrows_{loc} f$ on $[0, \infty)$ and so $\varrho(f_n, f) \to 0$.

□

The latter has a terminological corrolary.

Corrolary 2.1.3. *The space $(\mathscr{C}_m, \varrho)$ is a Polish space.*

2.2 Borel sets

Let $\mathscr{B}(\mathscr{C}_m)$ denote the family of Borel sets in $(\mathscr{C}_m, \varrho)$. The task of this section is to describe these sets. Here and subsequently, π_t, $t \geq 0$ stands for the canonical projections

$$\begin{aligned} \pi_t : \mathscr{C}_m &\longrightarrow \mathbb{R}^m \\ f &\longmapsto f(t), \quad t \geq 0. \end{aligned}$$

First, let us make two general observations.

Lemma 2.2.1. *Let (X, d) be a separable metric space. Then any open subset $U \subset X$ is a countable union of balls.*

Proof. Since X is separable, there exists a countable dense subset $A \subset X$, which implies:

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists y \in A \quad \text{so that} \quad d(x, y) < \varepsilon.$$

Let $U \subset X$ be any open subset, then by definition

$$\forall x \in U \quad \exists r > 0 \quad \text{so that} \quad \mathcal{U}(x, r) \subset U.$$

Denote $A_U = A \cap U$ and define

$$\mathcal{A} = \{\mathcal{U}(y, q);\ y \in A_U,\ q \in \mathbb{Q}, \mathcal{U}(y, q) \subset U\},$$

which is a countable system of balls. The proof is completed by showing that $\bigcup \mathcal{A} = U$.[3]

Obviously $\bigcup \mathcal{A} \subset U$.

To see the other inclusion, let $x \in U$ be given. First, we find $\varepsilon > 0$ so that $\mathcal{U}(x, \varepsilon) \subset U$, then there exists $y \in A_U$ satisfying the condition $y \in \mathcal{U}(x, \frac{\varepsilon}{3})$, i.e. $d(x, y) < \frac{\varepsilon}{3}$. Next we find $q \in \mathbb{Q}$ so that $\frac{\varepsilon}{3} < q \leq \frac{\varepsilon}{2}$. This clearly implies $x \in \mathcal{U}(y, q)$ and furthermore we conclude $\mathcal{U}(y, q) \subset \mathcal{U}(x, \varepsilon) \subset U$, and due to this fact $x \in \bigcup \mathcal{A}$. It remains to prove the first inclusion. On the contrary, let us assume the existence of $z \in \mathcal{U}(y, q)$ so that $z \notin \mathcal{U}(x, \varepsilon)$. Then, by the above

$$d(y, z) < q, \quad d(x, z) \geq \varepsilon, \quad d(x, y) < \frac{\varepsilon}{3}.$$

Since d is a metric, it satisfied the triangule inequality:

$$d(x, z) \leq d(x, y) + d(y, z) < \frac{\varepsilon}{3} + q \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{2} = \frac{5\,\varepsilon}{6} < \varepsilon,$$

which contradicts the fact that $d(x, z) \geq \varepsilon$.

□

[3] $\bigcup \mathcal{A}$ means the union of all subsets in $\mathcal{A}$

The latter has a straightforward consequence stated in the next corrolary which does not hold in general topological spaces. Spaces having this property are called second-countable spaces or spaces satisfying the second axiom of countability. In fact, second-countability implies separability but the reverse implication is not always true. However, in metrizable spaces we deal with equivalent properties.[4]

Corrolary 2.2.2. *Any separable metric space (X, d) has a countable base of topology.*

Proof. Let A be a countable dense subset in X. According to the previous proof, the family

$$\mathcal{B} = \{\mathcal{U}(y, q);\ y \in A,\ q \in \mathbb{Q}\}$$

can be considered as a base and its cardinality is clearly $\aleph_0$.

□

Theorem 2.2.3. *The Borel σ-algebra on $\mathscr{C}_m$ is generated by the collection of canonical projections π_t, $t \geq 0$, i.e. $\mathscr{B}(\mathscr{C}_m) = \sigma(\pi_t;\ t \geq 0)$.*

Proof. To prove the statement we need to show two inclusions.

$\sigma(\pi_t,\ t \geq 0) \subset \mathscr{B}(\mathscr{C}_m)$: it is sufficient to show that for all $t \geq 0$ the projection π_t is a measurable function, i.e.

$$\pi_t : (\mathscr{C}_m, \mathscr{B}(\mathscr{C}_m)) \to (\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m)).$$

Since both these σ-algebras are Borel it is enough to prove that π_t are continuous functions for all $t \geq 0$. We use the Heine definition of continuity: let $\{f_n\} \subset \mathscr{C}_m$ be a given sequence so that $\varrho(f_n, f) \to 0$ for some $f \in \mathscr{C}_m$. Then $f_n \rightrightarrows_{loc} f$ on $[0, \infty)$, the pointwise convergence follows

$$\pi_t(f_n) = f_n(t) \to f(t) = \pi_t(f) \quad n \to \infty, \quad t \geq 0$$

and therefore all π_t are continuous.

$\mathscr{B}(\mathscr{C}_m) \subset \sigma(\pi_t;\ t \geq 0)$: since we deal with a separable metric space $\mathscr{C}_m$, by 2.2.1, it is sufficient to show that any open ball $\mathcal{U}(g, \delta) = \{f \in \mathscr{C}_m;\ \varrho(f, g) < \delta\}$ belongs to $\sigma(\pi_t;\ t \geq 0)$. Function $f \longmapsto \|\!|f - g|\!\|_N$ is $\sigma(\pi_t;\ t \geq 0)$-measurable for all $N \in \mathbb{N}$, indeed

$$\begin{aligned}\big\{f \in \mathscr{C}_m;\ \|\!|f - g|\!\|_N < \varepsilon\big\} &= \bigcup_{n\in\mathbb{N}} \bigcap_{\substack{0\leq r\leq N\\ r\in\mathbb{Q}}} \Big\{f \in \mathscr{C}_m;\ \|f(r) - g(r)\| \leq \varepsilon\Big(1 - \frac{1}{n}\Big)\Big\} = \\ &= \bigcup_{n\in\mathbb{N}} \bigcap_{\substack{0\leq r\leq N\\ r\in\mathbb{Q}}} \pi_r^{-1}\big[(-\infty, \varepsilon)\big]\end{aligned}$$

[4]see [En89, Corrolary 4.1.16]

and similarly for the set $\{f \in \mathscr{C}_m;\ \|f - g\|_N > \varepsilon\}$. This is enough since $\{(p,\infty), (-\infty, q);\ p, q \in \mathbb{Q}\}$ forms a subbase of the Euclidean topology on $\mathbb{R}$. Then $\varrho(\,.\,, g)$ can be rewritten as a convergent sum of $\sigma(\pi_t;\ t \geq 0)$-measurable functions hence it is $\sigma(\pi_t;\ t \geq 0)$-measurable and the claim is proved. □

Remark 2.2.4. It is possible to show the last inclusion

$$\mathscr{B}(\mathscr{C}_m) \subset \sigma(\pi_t;\ t \geq 0)$$

in another way, employing more advanced concept from general topology. While the basic idea of the first proof was to determine the topology on $\mathscr{C}_m$ by the metric ϱ, in the second proof, the topology is described as the compact-open topology[5] on $\mathscr{C}_m$. Subsequent steps are analogous, the main work is done only for elements of the base (respectively subbase) of these topologies. Altough the demonstration of this method is performed in the following, it serves only as an illustration. In comparison with the other approach, this is more complicated.

Proof. Since $\mathscr{B}(\mathscr{C}_m)$ is generated by topology on $\mathscr{C}_m$, it sufficies to show that if $\mathscr{S}$ denotes a subbase of topology on $\mathscr{C}_m$ and whenever $S \in \mathscr{S}$ then also $S \in \sigma(\pi_t,\ t \geq 0)$. The topology considered on $(\mathscr{C}_m, \varrho)$ is the so called compact-open topology which is generated by the subbase

$$\mathscr{S} = \big\{\{f \in \mathscr{C}_m;\ f(K) \subset U\};\ K \subset [0,\infty) \text{ compact},\ U \subset \mathbb{R}^m \text{ open}\big\}.$$

By [En89, lemma 3.4.6], it is sufficient to consider only $U \in \mathscr{U}$, where $\mathscr{U}$ is any subbase of the Euclidean topology on $\mathbb{R}^m$. Clearly, we can choose

$$\mathscr{U} = \left\{ \prod_{i=1}^m (p_i, \infty),\ \prod_{i=1}^m (-\infty, q_i);\ p_i, q_i \in \mathbb{Q} \right\}.$$

Fix $K \subset [0,\infty)$ compact, any $U = \prod_{i=1}^m (p_i,\infty)$ and $V = \prod_{i=1}^m (-\infty, q_i)$. There exist such sequences

$$\big\{p_n^{(i)}\big\} \subset (p_i,\infty),\ \big\{q_n^{(i)}\big\} \subset (-\infty, q_i) \quad \text{that} \quad p_n^{(i)} \searrow p_i,\ q_n^{(i)} \nearrow q_i,\ \ i = 1,\dots,m.$$

Next, we define for $n \in \mathbb{N}$ the sets

$$U_n = \prod_{i=1}^m \big[p_n^{(i)}, \infty\big), \quad V_n = \prod_{i=1}^m \big(-\infty, q_n^{(i)}\big]$$

for which $U = \cup_{n\in\mathbb{N}} U_n$, $V = \cup_{n\in\mathbb{N}} V_n$. Every compact subset in $\mathbb{R}$ is a G_δ set, i.e. it can be rewritten as a countable intersection of open sets, indeed, denoting $G_l = \{x \in \mathbb{R};\ \mathrm{dist}(x, K) < \frac{1}{l}\}$, $l \in \mathbb{N}$, yields $K = \cap_{l\in\mathbb{N}} G_l$. Furthermore, each G_l is a countable union of open intervals $I_{l,k}$, $k \in \mathbb{N}$ hence $K \subset \cup_{k\in\mathbb{N}} I_{l,k}$, $l \in \mathbb{N}$ and

[5] in metric spaces, compact-open topology corresponds to local uniform convergence, see [En89] for more details

by compactness, for all $l \in \mathbb{N}$ there exists a finite subcover $I_{l,1}, \dots, I_{l,k_l}$. Then also $K \subset \operatorname{cl} I_{l,1} \cup \dots \cup \operatorname{cl} I_{l,k_l}$ and therefore $K = \bigcap_{l\in\mathbb{N}} \bigcup_{i=1}^{k_l} \operatorname{cl} I_{l,i}$. This leads to

$$\begin{aligned}\big\{f \in \mathscr{C}_m;\ f[K] \subset U\big\} &= \bigcup_{l\in\mathbb{N}} \bigcap_{i=1}^{k_l} \big\{f \in \mathscr{C}_m;\ f\big[\operatorname{cl} I_{l,i}\big] \subset U\big\} = \\ &= \bigcup_{l\in\mathbb{N}} \bigcap_{i=1}^{k_l} \bigcap_{x\in \operatorname{cl} I_{l,i}} \big\{f \in \mathscr{C}_m;\ f(x) \in U\big\} = \\ &= \bigcup_{l\in\mathbb{N}} \bigcap_{i=1}^{k_l} \bigcup_{n\in\mathbb{N}} \bigcap_{\substack{r\in \operatorname{cl} I_{l,i} \\ r\in\mathbb{Q}}} \big\{f \in \mathscr{C}_m;\ f(r) \in U_n\big\} = \\ &= \bigcup_{l\in\mathbb{N}} \bigcap_{i=1}^{k_l} \bigcup_{n\in\mathbb{N}} \bigcap_{\substack{r\in \operatorname{cl} I_{l,i} \\ r\in\mathbb{Q}}} \pi_r^{-1}\big[U_n\big]\end{aligned}$$

and similarly

$$\big\{f \in \mathscr{C}_m;\ f[K] \subset V\big\} = \bigcup_{l\in\mathbb{N}} \bigcap_{i=1}^{k_l} \bigcup_{n\in\mathbb{N}} \bigcap_{\substack{r\in \operatorname{cl} I_{l,i} \\ r\in\mathbb{Q}}} \pi_r^{-1}\big[V_n\big].$$

The proof is now finished since we have used only countable intersections and countable unions.

□

For better comprehension of this issue, let us denote $(\mathbb{R}^m)^{[0,\infty)}$ the space of all mappings from $[0,\infty)$ to $\mathbb{R}^m$. It is considered as the cartesian product

$$(\mathbb{R}^m)^{[0,\infty)} = \prod_{t\geq 0} \mathbb{R}^m$$

which can be translated as $f = \prod_{t\geq 0} f(t)$, $f \in (\mathbb{R}^m)^{[0,\infty)}$. Topology of cartesian product is the so called Tychonoff topology[6] which is induced by canonical projections

$$\begin{aligned}\varphi_t : \prod_{v\geq 0} \mathbb{R}^m &\longrightarrow \mathbb{R}^m \\ f &\longmapsto f(t), \quad t \geq 0.\end{aligned}$$

Since the family of Borel sets is generated by topology and $\pi_t = \varphi_t|_{\mathscr{C}_m}$, it follows that $\mathscr{B}(\mathscr{C}_m) = \otimes_{t\geq 0}\mathscr{B}(\mathbb{R}^m) \cap \mathscr{C}_m$. Needful to emphasize that we cannot say that the usual topology on $\mathscr{C}_m$ (induced by its metric) is a topology of subspace of $(\mathbb{R}^m)^{[0,\infty)}$.

[6]see [En89]

2.3 Compact sets

In this section, we first summarize important concepts related to compactness and afterwards we pronounce a characterization of relatively compact subsets in $\mathscr{C}_m$. This result generalizes the original Arzelà-Ascoli theorem A.4 which in our notation describes relatively compact sets in $\mathscr{C}_m(K)$, $K \subset \mathbb{R}$ compact.

The following definition can be adequately formulated in topological spaces, we only require the space to be Hausdorff.[7] Since all metrizable spaces are Hausdorff, this condition is not at all restrictive for us.

Definition 2.3.1. Let X be a metric space. A set $K \subset X$ is said to be

- compact if every open cover has a finite subcover,
- relatively compact if its closure is compact,
- sequentially compact if every sequence in K has a subsequence convergent in K.

Lemma 2.3.2. *Let (X, d) be a metric space. A set $K \subset X$ is compact if and only if it is sequentially compact.*

Proof. see [En89, Theorem 4.1.17] □

Lemma 2.3.3. *Let (X, d) be a metric space and $K \subset X$. Then K is relatively compact if and only if for any sequence in K there exists a convergent subsequence.*[8]

Proof. ⇒ Let $\{f_n\} \subset K$ be a given sequence. Then also $\{f_n\} \subset \operatorname{cl} K$ and since K is relatively compact, i.e. $\operatorname{cl} K$ is compact and sequentially compact, there exits a subsequence which converges in $\operatorname{cl} K$.

⇐ It suffices to prove that $\operatorname{cl} K$ is sequentially compact so fix a sequence $\{f_n\} \subset \operatorname{cl} K$. As for all $f \in \operatorname{cl} K$ there exists such a sequence $\{h_k\} \subset K$ that

$$d(h_k, f) \to 0 \quad k \to \infty,$$

we can find a sequence $\{g_n\} \subset K$ so that

$$d(f_n, g_n) < \frac{1}{2^n} \quad \forall n \in \mathbb{N}.$$

According to the assumption, there exists a subsequence $\{g_{n_k}\} \subset \{g_n\}$ and $g \in \operatorname{cl} K$ so that

$$d(g_{n_k}, g) \to 0 \quad k \to \infty.$$

[7] or T_2-space, all distinct points have disjoint neighbourhoods
[8] the limit need not be in K

The proof is finished when we show that also $d(f_{n_k}, g) \to 0$, $k \to \infty$. Let $\varepsilon > 0$ be given. Find $k_0 \in \mathbb{N}$ satisfying

$$d(g_{n_k}, g) < \frac{\varepsilon}{2} \quad \forall k \geq k_0 \qquad \text{and} \qquad \frac{1}{2^{n_{k_0}}} < \frac{\varepsilon}{2}.$$

Hence for all $k \geq k_0$

$$d(f_{n_k}, g) \leq d(f_{n_k}, g_{n_k}) + d(g_{n_k}, g) < \varepsilon,$$

and the claim follows. □

The following theorem is the above mentioned generalization of the Arzelà-Ascoli theorem A.4.

Theorem 2.3.4. *A set $K \subset \mathscr{C}_m$ is relatively compact if and only if for all $N \in \mathbb{N}$ the set $\{f|_{[0,N]};\ f \in K\}$ is equicontinuous and bounded.*[9]

Proof. By the classical Arzelà-Ascoli theorem it suffices to prove the following equivalency:

$K \subset \mathscr{C}_m$ is relatively compact if and only if for all $N \in \mathbb{N}$ the set $\{f|_{[0,N]};\ f \in K\}$ is relatively compact.

With respect to the preceding lemma we will show:

Let $\{f_n\} \subset K$ be a given sequence. Then there exists a subsequence convergent in the metric ϱ if and only if for all $N \in \mathbb{N}$ there exists a subsequence of $\{f_n|_{[0,N]}\}$ convergent uniformly on $[0, N]$.

$\Rightarrow$ By the assumption, there exists $\{f_{n_k}\}$ a subsequence of $\{f_n\}$ which converges in the metric ϱ. Hence it converges locally uniformly on $[0, \infty)$ and the restrictions of these functions $\{f_{n_k}|_{[0,N]}\}$ converge uniformly on $[0, N]$.

$\Leftarrow$ Since $\forall N \in \mathbb{N}$ there exists a subsequence of the corresponding sequence $\{f_n|_{[0,N]}\}$ which converges uniformly, we will proceed step by step:

- $N = 1$: from $\{f_n|_{[0,1]}\}$ we choose a subsequence convergent uniformly on $[0, 1]$ and denote by $\{f_n^{(1)}\}$,
- $N = 2$: from the subsequence $\{f_n^{(1)}\}$ we choose another subsequence so that it converges uniformly on $[0, 2]$ and denote it by $\{f_n^{(2)}\}$,
- ...
- $N = k$: from the previous subsequence we choose a subsequence convergent uniformly on $[0, k]$, denote by $\{f_n^{(k)}\}$,
- ...

[9] it is bounded in the Banach space $\left(\mathscr{C}_m([0, N]), \|\cdot\|_N\right)$

As the final sequence we choose the diagonal subsequence from our system: $\{f_{n_k}\} = \{f_k^{(k)}\}$, i.e. f_{n_k} is the k^{th} element of the k^{th} sequence. One can easily imagine that this is really a subsequence of $\{f_n\}$, namely that its elements were in the original sequence in the same order.

What remains now is to show that $\{f_{n_k}\}$ converges in the space $(\mathscr{C}_m, \varrho)$. Denote by $f \in \operatorname{cl} K$ the function for which

$$f|_{[0,N]} = \lim_{n\to\infty} f_n^{(N)} \quad \forall N \in \mathbb{N}.$$

This function is well defined because of the uniqueness of the uniform limit. Let $A \subset [0, \infty)$ be a given compact subset and $\varepsilon > 0$. Find $N \in \mathbb{N}$ so that $A \subset [0, N]$ and consider the sequence $\{f_n^{(N)}\}$. It converges uniformly on the interval $[0, N]$ hence

$$\exists k \in \mathbb{N} \quad \forall n \geq k \quad \|\!|f_n^{(N)} - f|\!\|_N < \varepsilon.$$

Finding the smallest index $l \geq k$ so that the function $f_l^{(N)}$ is an element of the diagonal sequence, i.e. $f_l^{(N)} = f_{n_m}$ for some $m \geq N$, leads to

$$\forall k \geq m \quad \|\!|f_{n_k} - f|\!\|_N < \varepsilon$$

therefore

$$\forall k \geq m \quad \sup_{t\in A} \|f_{n_k}(t) - f(t)\| < \varepsilon$$

and as a consequence we get the local uniform convergence. □

2.4 Tightness

In the following section we introduce the modulus of continuity used for the functions in $\mathscr{C}_m$. Its properties allow us to state the analogue of the Arzelà-Ascoli theorem 2.3.4 which will be more convenient for the main result of this section, the tightness characterization of collections of Borel probability measures on $\mathscr{C}_m$.

Definition 2.4.1. Suppose that $f \in \mathscr{C}_m$. For $T > 0$, $\delta > 0$ we define

$$\omega_T(f, \delta) = \sup_{\substack{0\leq s,t\leq T\\ |t-s|<\delta}} \|f(t) - f(s)\|$$

and for any fixed $T > 0$ we call $\omega_T(f, .)$ the modulus of continuity of a function f on $[0, T]$.

Now, we explain the main properties of the above defined ω as a function of f and as a function of δ.

Lemma 2.4.2. *For all $T > 0$, $\delta > 0$ the function $f \longmapsto \omega_T(f, \delta)$ is continuous on $\mathscr{C}_m$.*

Proof. Let $T > 0,\ \delta > 0$ be given. By the Heine definition of continuity, we prove that for any $f \in \mathscr{C}_m$ the function $\omega_T(\,.\,,\delta)$ is continuous in f. Consider such a sequence $\{f_n\} \subset \mathscr{C}_m$ that $f_n \to f$ in $(\mathscr{C}_m, \varrho)$ and show

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad |\omega_T(f_n, \delta) - \omega_T(f, \delta)| < \varepsilon.$$

Let $\varepsilon > 0$ be given and denote $K = [0, T]$. Since $f_n \rightrightarrows_{loc} f$ on $[0, \infty)$ it holds that $f_n \rightrightarrows f$ on K. Hence we find $n_0 \in \mathbb{N}$ for which

$$\forall n \geq n_0 \quad \forall t \in K \quad \|f_n(t) - f(t)\| < \frac{\varepsilon}{3}$$

and from the triangule inequality for all $t,\ s \in K$

$$\|f(t) - f(s)\| \leq \|f(t) - f_n(t)\| + \|f_n(t) - f_n(s)\| + \|f_n(s) - f(s)\|$$

therefore

$$\|f(t) - f(s)\| - \|f_n(t) - f_n(s)\| \leq \|f(t) - f_n(t)\| + \|f_n(s) - f(s)\| < \frac{2\varepsilon}{3}.$$

As the previous inequality is satisfied for all $t,\ s \in K$ it is satisfied also for their supremum

$$\begin{aligned}\varepsilon > \frac{2\varepsilon}{3} &\geq \sup_{t,s \in K} \Big(\|f(t) - f(s)\| - \|f_n(t) - f_n(s)\| \Big) \\ &\geq \sup_{\substack{t,s \in K \\ |t-s|<\delta}} \Big(\|f(t) - f(s)\| - \|f_n(t) - f_n(s)\| \Big) \geq \\ &\geq \sup_{\substack{t,s \in K \\ |t-s|<\delta}} \Big(\|f(t) - f(s)\| - \sup_{\substack{t,s \in K \\ |t-s|<\delta}} \|f_n(t) - f_n(s)\| \Big) \geq \\ &\geq \sup_{\substack{t,s \in K \\ |t-s|<\delta}} \|f(t) - f(s)\| - \sup_{\substack{t,s \in K \\ |t-s|<\delta}} \|f_n(t) - f_n(s)\| = \omega_T(f, \delta) - \omega_T(f_n, \delta).\end{aligned}$$

If we apply the triangule inequality in the following way

$$\|f_n(t) - f_n(s)\| \leq \|f_n(t) - f(t)\| + \|f(t) - f(s)\| + \|f(s) - f_n(s)\|,$$

we obtain

$$\|f_n(t) - f_n(s)\| - \|f(t) - f(s)\| \leq \|f_n(t) - f(t)\| + \|f(s) - f_n(s)\| < \frac{2\varepsilon}{3}$$

and analogically

$$\varepsilon > \frac{2\varepsilon}{3} \geq \omega_T(f_n, \delta) - \omega_T(f, \delta),$$

hence

$$|\omega_T(f_n, \delta) - \omega_T(f, \delta)| < \varepsilon.$$

□

Lemma 2.4.3. *For all* $f \in \mathscr{C}_m$, $T > 0$ *the function* $\delta \longmapsto \omega_T(f, \delta)$ *is nondecreasing on* $(0, \infty)$.

Proof. Let $f \in \mathscr{C}_m$, $T > 0$ be given. Fix $\delta_1, \delta_2 > 0$, $\delta_1 < \delta_2$ and show that $\omega_T(f, \delta_1) \leq \omega_T(f, \delta_2)$. Obviously

$$\omega_T(f, \delta_1) = \sup_{\substack{0 \leq s,t \leq T \\ |t-s| < \delta_1}} \|f(t) - f(s)\| \leq \sup_{\substack{0 \leq s,t \leq T \\ |t-s| < \delta_2}} \|f(t) - f(s)\| = \omega_T(f, \delta_2).$$

□

Lemma 2.4.4. *For all* $f \in \mathscr{C}_m$, $T > 0$ *it stands that* $\lim_{\delta \to 0_+} \omega_T(f, \delta) = 0$.

Proof. Let $f \in \mathscr{C}_m$, $T > 0$ be given. We need to prove

$$\forall \varepsilon > 0 \quad \exists \Delta > 0 \quad \forall \delta \in (0, \Delta) \quad \omega_T(f, \delta) < \varepsilon.$$

Let $\varepsilon > 0$ be given. Function f is continuous on $\mathbb{R}_+$ hence its restriction on $[0, T]$ is uniformly continuous so

$$\exists \Delta > 0 \quad \forall s, t \in [0, T], \ |t - s| < \Delta \quad \|f(t) - f(s)\| < \frac{\varepsilon}{2}$$

and for the supremum

$$\sup_{\substack{0 \leq s,t \leq T \\ |t-s| < \Delta}} \|f(t) - f(s)\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Finally, the definition of the function $\omega_T(f, \, . \,)$ together with the fact that it is nondecreasing on $(0, \infty)$ implies

$$\forall \delta \in (0, \Delta) \quad \omega_T(f, \delta) < \varepsilon.$$

□

Next, it comes the second generalization of the Arzelà-Ascoli theorem.

Theorem 2.4.5. *A set* $K \subset \mathscr{C}_m$ *is relatively compact if and only if*

$$\sup_{f \in K} \|f(0)\| < \infty, \qquad \lim_{\delta \to 0_+} \sup_{f \in K} \omega_T(f, \delta) = 0 \quad \forall T > 0.$$

Proof. With respect to Theorem 2.3.4 proved in the previous section it is sufficient to show the equivalency of the following conditions:

(i) $\sup_{f \in K} \|f(0)\| < \infty, \quad \lim_{\delta \to 0_+} \sup_{f \in K} \omega_T(f, \delta) = 0 \quad \forall T > 0,$

(ii) for all $N \in \mathbb{N}$ the set $\{f|_{[0,N]};\, f \in K\}$ is equicontinuous and bounded.

(i)⇒(ii) Let $N \in \mathbb{N}$ be given and show the equicontinuity. Set $T = N$, then according to the assumptions

$$\lim_{\delta\to 0_+} \sup_{f\in K} \sup_{\substack{0\le s,t\le N\\ |t-s|<\delta}} \|f(t)-f(s)\| = 0. \tag{2.1}$$

It is required to prove: if $s \in [0, N]$ then

$$\forall\varepsilon>0 \quad \exists\delta>0 \quad \forall t\in[0,N],\ |t-s|<\delta \quad \|f(t)-f(s)\|<\varepsilon \quad \forall f\in K.$$

From (2.1) we obtain

$$\forall\varepsilon>0 \quad \exists\Delta>0 \quad \forall\delta\in(0,\Delta) \quad \sup_{f\in K} \sup_{\substack{0\le s,t\le N\\ |t-s|<\delta}} \|f(t)-f(s)\|<\varepsilon$$

or in other words

$$\forall\varepsilon>0 \quad \exists\Delta>0 \quad \forall\delta\in(0,\Delta) \quad \forall s,t\in[0,N],\ |t-s|<\delta \\ \|f(t)-f(s)\|<\varepsilon \quad \forall f\in K,$$

and we are done.

As the next step, let us prove the boundedness. Fix $t \in [0, N]$. We need to show

$$\sup_{f\in K}\|f(t)\|<\infty.$$

Evidently, it stands

$$\sup_{f\in K}\|f(t)\| \le \sup_{f\in K}\Big(\|f(0)\|+\|f(t)-f(0)\|\Big) \le \sup_{f\in K}\|f(0)\|+\sup_{f\in K}\|f(t)-f(0)\|,$$

where the first summand on the right hand side is finite by the assumptions. To estimate the second one, fix $\varepsilon > 0$. It again follows from the assumptions that there exists $\delta > 0$ so that

$$\sup_{f\in K} \sup_{\substack{0\le s,t\le N\\ |t-s|<\delta}} \|f(t)-f(s)\|<\varepsilon.$$

Let us find such a partition D of the interval $[0, t]$ that its norm[10] satisfies $\|D\| < \delta$ and denote $D = \{0 = t_0 < t_1 < \cdots < t_n = t\}$. Then we have

$$\begin{aligned}\sup_{f\in K}\|f(t)-f(0)\| &\le \sup_{f\in K}\Big(\|f(t_0)-f(t_1)\|+\cdots+\|f(t_{n-1})-f(t_n)\|\Big) \le \\ &\le \sup_{f\in K}\|f(t_0)-f(t_1)\|+\cdots+\sup_{f\in K}\|f(t_{n-1})-f(t_n)\| < \\ &< n\varepsilon<\infty.\end{aligned}$$

[10]If $D = \{0 = t_0 < t_1 < \cdots < t_n = t\}$, then its norm is defined by $\|D\| = \max\{|t_i - t_{i-1}|;\ i = 1, \ldots, n\}$

(ii)⇒(i) The condition $\sup_{f\in K}\|f(0)\| < \infty$ is a straightforward consequence of the boundedness of the set $\{f|_{[0,N]};\ f \in K\}$ for some $N \in \mathbb{N}$.

Let $T > 0$ be given and find $N \in \mathbb{N}$ so that $N \geq T$. Since the set $\{f|_{[0,N]};\ f \in K\}$ is equicontinuous, we have for $s \in [0, N]$

$$\forall\varepsilon > 0 \quad \exists\Delta > 0 \quad \forall t \in [0,N],\ |t-s| < \Delta \quad \|f(t) - f(s)\| < \frac{\varepsilon}{2} \quad \forall f \in K.$$

Moreover, each function $f|_{[0,N]}$ is uniformly continuous on $[0, N]$ and therefore

$$\forall\varepsilon > 0 \quad \exists\Delta > 0 \quad \forall s,t \in [0,N],\ |t-s| < \Delta \quad \|f(t) - f(s)\| < \frac{\varepsilon}{2} \quad \forall f \in K$$

so

$$\forall\varepsilon > 0 \quad \exists\Delta > 0 \quad \sup_{\substack{0\leq s,t\leq N \\ |t-s|<\Delta}} \|f(t) - f(s)\| \leq \frac{\varepsilon}{2} < \varepsilon \quad \forall f \in K.$$

We deduce that

$$\forall\varepsilon > 0 \quad \exists\Delta > 0 \quad \sup_{f\in K} \omega_T(f,\Delta) < \varepsilon$$

and as $\omega_T(f, \,.\,)$ is nondecreasing for all $T > 0$, $f \in \mathscr{C}_m$ it follows that

$$\forall\varepsilon > 0 \quad \exists\Delta > 0 \quad \forall\delta \in (0,\Delta) \quad \sup_{f\in K} \omega_T(f,\delta) < \varepsilon.$$

□

We present the general definition of tightness in topological spaces. However, we will use only tightness in metric spaces and consider only probability measures.

Definition 2.4.6. Suppose that X is a topological space and that $\mathcal{A}$ is a σ-algebra containing the topology.[11] We say that a collection $\mathcal{M}$ of measures on $\mathcal{A}$ is tight if for any given $\varepsilon > 0$ there exists $K_\varepsilon \subset X$, compact, such that for all $\mu \in \mathcal{M}$

$$\mu(X \setminus K_\varepsilon) < \varepsilon. \tag{2.2}$$

Remark 2.4.7. For $\mathcal{M}$ being a suitable family of probability measures, (2.2) can be equivalently rewritten as

$$\mu(K_\varepsilon) > 1 - \varepsilon. \tag{2.3}$$

Theorem 2.4.8. *A sequence $\{\mu_n;\ n \in \mathbb{N}\}$ of Borel probability measures on $\mathscr{C}_m$ is tight if and only if the following conditions are satisfied*

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \mu_n\big\{f \in \mathscr{C}_m;\ \|f(0)\| \geq R\big\} = 0, \tag{2.4}$$

$$\lim_{\delta\to 0+} \sup_{n\in\mathbb{N}} \mu_n\big\{f \in \mathscr{C}_m;\ \omega_T(f,\delta) > \eta\big\} = 0, \quad \forall T > 0,\ \forall\eta > 0. \tag{2.5}$$

[11] i.e. $\mathscr{B}(X) \subset \mathcal{A}$

Proof. $\Rightarrow$ Assume that $\{\mu_n\}$ is tight, i.e. for all $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset \mathscr{C}_m$ such that $\mu_n(K_\varepsilon) > 1 - \varepsilon$ holds for all $n \in \mathbb{N}$. Since K_ε is compact so relatively compact, the Arzelà-Ascoli theorem 2.4.5 gives

$$\sup_{f \in K_\varepsilon} \|f(0)\| < \infty, \quad \lim_{\delta \to 0_+} \sup_{f \in K_\varepsilon} \omega_T(f, \delta) = 0 \quad \forall T > 0.$$

To prove (2.4), it is needed to verify

$$\forall \varepsilon > 0 \quad \exists r > 0 \quad \forall R \geq r \quad \sup_{n \in \mathbb{N}} \mu_n \big\{ f \in \mathscr{C}_m;\ \|f(0)\| \geq R \big\} < \varepsilon.$$

Let $\varepsilon > 0$ be given and fix arbitrary $\varepsilon' \in (0, \varepsilon)$. We employ tightness of the collection $\{\mu_n\}$ to obtain a compact subset $K_{\varepsilon'} \subset \mathscr{C}_m$ that satisfies (2.3) and

$$\sup_{f \in K_{\varepsilon'}} \|f(0)\| = M < \infty.$$

Setting $r = M$ implies

$$\forall R \geq r \quad \big\{ f \in \mathscr{C}_m;\ \|f(0)\| \geq R \big\} \subset \mathscr{C}_m \setminus K_{\varepsilon'}$$

and therefore

$$\forall R \geq r \quad \forall n \in \mathbb{N} \quad \mu_n \big\{ f \in \mathscr{C}_m;\ \|f(0)\| \geq R \big\} \leq \varepsilon',$$

i.e.

$$\forall R \geq r \quad \sup_{n \in \mathbb{N}} \mu_n \big\{ f \in \mathscr{C}_m;\ \|f(0)\| \geq R \big\} \leq \varepsilon' < \varepsilon.$$

To prove (2.5) it is necessary to show for all $T > 0$, $\eta > 0$

$$\forall \varepsilon > 0 \quad \exists \Delta > 0 \quad \forall \delta \in (0, \Delta) \quad \sup_{n \in \mathbb{N}} \mu_n \big\{ f \in \mathscr{C}_m;\ \omega_T(f, \delta) > \eta \big\} < \varepsilon.$$

Let $T > 0$, $\eta > 0$, $\varepsilon > 0$ be given and choose again $\varepsilon' \in (0, \varepsilon)$. Application of tightness leads to the existence of $K_{\varepsilon'} \subset \mathscr{C}_m$, compact, satisfying (2.3) and by the Arzelà-Ascoli theorem 2.4.5 also $\lim_{\delta \to 0_+} \sup_{f \in K_{\varepsilon'}} \omega_T(f, \delta) = 0$, which means

$$\exists \Delta > 0 \quad \forall \delta \in (0, \Delta) \quad \sup_{f \in K_{\varepsilon'}} \omega_T(f, \delta) \leq \eta.$$

For the rest we use similar approach, namely we have

$$\forall \delta \in (0, \Delta) \quad \big\{ f \in \mathscr{C}_m;\ \omega_T(f, \delta) > \eta \big\} \subset \mathscr{C}_m \setminus K_{\varepsilon'}$$

and as a result

$$\forall \delta \in (0, \Delta) \quad \sup_{n \in \mathbb{N}} \mu_n \big\{ f \in \mathscr{C}_m;\ \omega_T(f, \delta) > \eta \big\} \leq \varepsilon' < \varepsilon.$$

$\Leftarrow$ Our task here is to show for any $\varepsilon > 0$ the existence of a compact subset $K_\varepsilon \subset \mathscr{C}_m$ that for all $n \in \mathbb{N}$ the (2.3) is satisfied. Let $\varepsilon > 0$ be given and fix such a sequence of positive numbers $\{\varepsilon_{m,k}\}_{m,k=1}^{\infty}$ that

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\varepsilon_{m,k} < \frac{\varepsilon}{2}.$$

Denote $T_k = k,\ k \in \mathbb{N}_0$. According to the assumptions (2.4), (2.5), it stands

$$\exists r > 0 \quad \forall R \geq r \quad \sup_{n\in\mathbb{N}} \mu_n\big\{f \in \mathscr{C}_m;\ \|f(0)\| \geq R\big\} < \frac{\varepsilon}{2},$$

$$\forall k \in \mathbb{N}\ \ \forall m \in \mathbb{N}\ \ \exists \Delta > 0\ \ \forall \delta \in (0,\Delta)\ \ \sup_{n\in\mathbb{N}} \mu_n\Big\{f \in \mathscr{C}_m;\ \omega_{T_k}(f,\delta) > \frac{1}{m}\Big\} < \varepsilon_{m,k}.$$

Next, we define sets

$$\begin{aligned} S_\varepsilon^0 &= \big\{f \in \mathscr{C}_m;\ \|f(0)\| < r\big\}, \\ S_\varepsilon^{m,k} &= \Big\{f \in \mathscr{C}_m;\ \omega_{T_k}(f,\Delta) \leq \frac{1}{m}\Big\}, \quad m,k \in \mathbb{N} \end{aligned}$$

and finally

$$S_\varepsilon = S_\varepsilon^0 \cap \bigcap_{k=1}^{\infty}\bigcap_{m=1}^{\infty} S_\varepsilon^{m,k}.$$

This set is inevitably nonempty, since it contains for example any constant function $f(t) = c$, where $\|c\| < r$. Moreover, it is relatively compact as will be proved immediately by the Arzelà-Ascoli theorem 2.4.5.

The first property is a clear consequence of its construction:

$$\sup_{f\in S_\varepsilon}\|f(0)\| \leq \sup_{f\in S_\varepsilon^0}\|f(0)\| \leq r.$$

To show the second property it is needed to verify

$$\forall T > 0 \quad \forall \xi > 0 \quad \exists \Delta > 0 \quad \forall \delta \in (0,\Delta) \quad \sup_{f\in S_\varepsilon}\omega_T(f,\delta) < \xi.$$

Let $T > 0$, $\xi > 0$ be given and take the same Δ as in the construction of $S_\varepsilon^{m,k}$. Since we are interested particularly in small ξ, without loss of generality, we can assume that $\xi \in (0,1)$. Next, find numbers

$$\begin{aligned} k_0 &\in \mathbb{N} & &\text{such that} & &T_{k_0-1} < T \leq T_{k_0}, \\ m_0 &\in \mathbb{N}\setminus\{1\} & &\text{such that} & &\frac{1}{m_0} < \xi \leq \frac{1}{m_0-1}. \end{aligned}$$

As any function $f \in S_\varepsilon$ belongs also to $S_\varepsilon^{m_0,k_0}$, it satisfies

$$\omega_T(f,\Delta) \leq \omega_{T_{k_0}}(f,\Delta) \leq \frac{1}{m_0} < \xi$$

which holds also for the supremum

$$\sup_{f\in S_\varepsilon}\omega_T(f,\Delta)\le\frac{1}{m_0}<\xi$$

and by the monotony of $\omega_T(f,\,.\,)$ we conclude

$$\forall\delta\in(0,\Delta)\quad\sup_{f\in S_\varepsilon}\omega_T(f,\delta)\le\sup_{f\in S_\varepsilon}\omega_T(f,\Delta)<\xi$$

hence S_ε is relatively compact.

Let us notice that the sets S_ε^0, $S_\varepsilon^{m,k}$, $m\in\mathbb{N}$, $k\in\mathbb{N}$ are Borel hence S_ε is Borel too. Denote $K_\varepsilon=\operatorname{cl}S_\varepsilon$ the closure of S_ε which is obviously a compact set. Fix arbitrary $n\in\mathbb{N}$, then

$$\begin{aligned}\mu_n(\mathscr{C}_m\setminus K_\varepsilon)\le\mu_n(\mathscr{C}_m\setminus S_\varepsilon)&=\mu_n\left(\mathscr{C}_m\setminus\left(S_\varepsilon^0\cap\bigcap_{k=1}^\infty\bigcap_{m=1}^\infty S_\varepsilon^{m,k}\right)\right)=\\&=\mu_n\left(\left(\mathscr{C}_m\setminus S_\varepsilon^0\right)\cup\left(\mathscr{C}_m\setminus\bigcap_{k=1}^\infty\bigcap_{m=1}^\infty S_\varepsilon^{m,k}\right)\right)\le\\&\le\mu_n(\mathscr{C}_m\setminus S_\varepsilon^0)+\mu_n\left(\mathscr{C}_m\setminus\bigcap_{k=1}^\infty\bigcap_{m=1}^\infty S_\varepsilon^{m,k}\right)\le\\&\le\mu_n(\mathscr{C}_m\setminus S_\varepsilon^0)+\mu_n\left(\bigcup_{k=1}^\infty\bigcup_{m=1}^\infty\mathscr{C}_m\setminus S_\varepsilon^{m,k}\right)\le\\&\le\mu_n(\mathscr{C}_m\setminus S_\varepsilon^0)+\sum_{k=1}^\infty\sum_{m=1}^\infty\mu_n(\mathscr{C}_m\setminus S_\varepsilon^{m,k})<\\&<\frac{\varepsilon}{2}+\sum_{k=1}^\infty\sum_{m=1}^\infty\varepsilon_{m,k}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

So the family $\{\mu_n\}$ is tight.

□

Theorem 2.4.9. *A sequence $\{\mu_n;\,n\in\mathbb{N}\}$ of Borel probability measures on $\mathscr{C}_m$ is tight if* (2.4) *and the condition*

$$\forall T>0\quad\exists\beta>0\qquad\lim_{R\to\infty}\sup_{n\in\mathbb{N}}\mu_n\left\{f\in\mathscr{C}_m;\ \sup_{0\le s,t\le T}\frac{\|f(t)-f(s)\|}{|t-s|^\beta}\ge R\right\}=0\tag{2.6}$$

are satisfied.

Proof. With respect to the preceding theorem, it suffices to prove that (2.6) implies (2.5). Let $T > 0$, $\eta > 0$ be given, then it holds for $\delta > 0$

$$\begin{aligned}
\big\{f \in \mathscr{C}_m;\, \omega_T(f,\delta) > \eta\big\} &= \Big\{f \in \mathscr{C}_m;\, \sup_{\substack{0\le s,t\le T\\ |t-s|<\delta}} \|f(t)-f(s)\| > \eta\Big\} = \\
&= \Big\{f \in \mathscr{C}_m;\, \sup_{\substack{0\le s,t\le T\\ |t-s|<\delta}} \frac{\|f(t)-f(s)\|}{|t-s|^\beta} > \frac{\eta}{|t-s|^\beta}\Big\} \subset \\
&\subset \Big\{f \in \mathscr{C}_m;\, \sup_{\substack{0\le s,t\le T\\ |t-s|<\delta}} \frac{\|f(t)-f(s)\|}{|t-s|^\beta} > \frac{\eta}{\delta^\beta}\Big\} \subset \\
&\subset \Big\{f \in \mathscr{C}_m;\, \sup_{0\le s,t\le T} \frac{\|f(t)-f(s)\|}{|t-s|^\beta} \ge \frac{\eta}{\delta^\beta}\Big\}.
\end{aligned}$$

Accordingly, we obtain

$$\begin{aligned}
&\limsup_{\delta\to 0_+} \sup_{n\in\mathbb{N}} \mu_n\big\{f \in \mathscr{C}_m;\, \omega_T(f,\delta) > \eta\big\} \le \\
&\qquad\le \limsup_{\delta\to 0_+} \sup_{n\in\mathbb{N}} \mu_n\Big\{f \in \mathscr{C}_m;\, \sup_{0\le s,t\le T} \frac{\|f(t)-f(s)\|}{|t-s|^\beta} \ge \frac{\eta}{\delta^\beta}\Big\}
\end{aligned}$$

whereas even the limits exist and their values are zero. Indeed, let $\varepsilon > 0$ be given and by the assumption (2.6) there exists $r > 0$ such that

$$\forall R \ge r \quad \sup_{n\in\mathbb{N}} \mu_n\Big\{f \in \mathscr{C}_m;\, \sup_{0\le s,t\le T} \frac{\|f(t)-f(s)\|}{|t-s|^\beta} \ge R\Big\} < \varepsilon.$$

Find $\Delta > 0$ with property $\frac{\eta}{\Delta^\beta} \ge r$ and consequently, for all $\delta \in (0,\Delta)$

$$\frac{\eta}{\delta^\beta} > \frac{\eta}{\Delta^\beta} \ge r$$

hence

$$\sup_{n\in\mathbb{N}} \mu_n\Big\{f \in \mathscr{C}_m;\, \sup_{0\le s,t\le T} \frac{\|f(t)-f(s)\|}{|t-s|^\beta} \ge \frac{\eta}{\delta^\beta}\Big\} < \varepsilon$$

and the statement is proved.

□

Chapter 3

Approximation by Lipschitz continuous functions

Definition 3.0.10. We call a function $f : \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{R}^K$ continuous in x if for any fixed $t \geq 0$ the function $f(t, .)$ is continuous, i.e. for all $x \in \mathbb{R}^M$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall y \in \mathbb{R}^M,\ \|x - y\| < \delta \qquad \|f(t,x) - f(t,y)\| < \varepsilon.$$

Lemma 3.0.11. *Let $f : \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{R}^K$ be a function continuous in x and let $K \subset \mathbb{R}^M$ be a compact subset. Then for any $t \geq 0$ the function $f(t, .)$ is uniformly continuous on K.*

Proof. Fix $t \geq 0$, let $\varepsilon > 0$ be a given number. By definition 3.0.10, for all $x \in \mathbb{R}^M$ there exists $\delta_x > 0$ so that

$$\forall y \in \mathbb{R}^M,\ \|x - y\| < \delta_x \qquad \|f(t,x) - f(t,y)\| < \frac{\varepsilon}{2}.$$

System

$$\Big\{B\Big(x, \frac{\delta_x}{2}\Big);\ x \in K\Big\}$$

forms an open cover of K and by its compactness, there exists a finite subcover, let us say

$$\Big\{B\Big(x_1, \frac{\delta_{x_1}}{2}\Big), \ldots, B\Big(x_n, \frac{\delta_{x_n}}{2}\Big)\Big\}.$$

Denote $\delta = \frac{1}{2}\min\{\delta_{x_1}, \ldots, \delta_{x_n}\}$. Then whenever $x, y \in K$, $\|x - y\| < \delta$, there exists $i \in \{1, \ldots, n\}$ so that

$$\|x - x_i\| < \delta_{x_i}, \quad \|y - x_i\| < \delta_{x_i}.$$

Therefore for all $t \geq 0$

$$\|f(t,x) - f(t,y)\| \leq \|f(t,x) - f(t,x_i)\| + \|f(t,x_i) - f(t,y)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Definition 3.0.12. For functions $f, g \in \mathscr{L}^1(\mathbb{R}^M)$ we define convolution $f * g$ by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\mathrm{d}y, \quad x \in \mathbb{R}^M.$$

The convolution was defined for functions in $\mathscr{L}^1(\mathbb{R}^M)$ since then by the Young inequality[12] the result is also in $\mathscr{L}^1$. However, the convolution makes sense also in other situations as can be seen from the general form of the Young inequality. Another example is described in the following lemma.

Lemma 3.0.13. *Suppose that $f \in \mathscr{L}^1_{loc}(\mathbb{R}^M)$ and $g \in \mathscr{D}(\mathbb{R}^M)$. Then $f * g \in \mathscr{C}^\infty(\mathbb{R}^M)$.*

Proof. We need only to consider the case $M = 1$ because if $M > 1$ we apply the same approach repeatedly. Fix $x \in \mathbb{R}$. We proceed by induction to show the formula

$$(f * g)^{(n)}(x) = \int_{\mathbb{R}} f(y)g^{(n)}(x-y)\,\mathrm{d}y.$$

It is well known that $g^{(n)} \in \mathscr{D}(\mathbb{R})$, $\forall n \in \mathbb{N}$. For $n = 0$ the statement follows directly from the definition of convolution 3.0.12. Assume thus that it is true for some $n \geq 0$ and conclude its validity for $n+1$. It holds

$$|f(y)g^{(n+1)}(x-y)| \leq \sup_{z\in\mathbb{R}} |g^{(n+1)}(z)|\,|f(y)| \tag{3.1}$$

and the function on the right hand side is an integrable dominating function on the set $\{y \in \mathbb{R};\ g^{(n+1)}(x-y) \neq 0\}$. On the set $\{y \in \mathbb{R}^M;\ g^{(n+1)}(x-y) = 0\}$ we can choose zero function. Application of the derivative of integral dependent on a parameter theorem yields

$$(f * g)^{(n+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} f(y)g^{(n)}(x-y)\,\mathrm{d}y = \int_{\mathbb{R}} f(y)g^{(n+1)}(x-y)\,\mathrm{d}y.$$

For continuity of $(f*g)^{(n+1)}$ it suffices to employ again the integrable dominating function (3.1) and the continuity of integral dependent on a parameter theorem. □

Lemma 3.0.14. *Suppose that $f \in \mathscr{C}^1(\mathbb{R}^M;\mathbb{R}^K)$, then f is locally Lipschitz continuous on $\mathbb{R}^M$.*

Proof. Let $N \in \mathbb{N}$ be given and assume that $x,\ y \in \mathbb{R}^M$, $\|x\| \vee \|y\| \leq N$. We define a function $g : [0,1] \to \mathbb{R}^K$ by $g(t) = f\big(x + t(y-x)\big)$. Since g is a composition of the continuously differentiable function $t \longmapsto x + t(y-x)$, $t \in [0,1]$, and the function $f \in \mathscr{C}^1(\mathbb{R}^M;\mathbb{R}^K)$, it stands that $g \in \mathscr{C}^1([0,1];\mathbb{R}^K)$. So for $t \in (0,1)$

$$\begin{aligned} g'(t) &= \lim_{h\to 0}\frac{g(t+h)-g(t)}{h} = \lim_{h\to 0}\frac{f\big(x+(t+h)(y-x)\big) - f\big(x+t(y-x)\big)}{h} = \\ &= \lim_{h\to 0}\frac{f\big(x+t(y-x)+h(y-x)\big) - f\big(x+t(y-x)\big)}{h} = \\ &= \mathrm{D}_{y-x}f\big(x+t(y-x)\big), \end{aligned}$$

[12] see [Yo12]

where the limit is computed componentwise and by D_{y-x} we denote the directional derivative in the direction $y-x$. So

$$\begin{aligned}\|f(y)-f(x)\| = \|g(1)-g(0)\| &= \Big\| \int_0^1 g'(t)\mathrm{d}t \Big\| = \\ &= \Big\| \int_0^1 \mathrm{D}_{y-x} f\big(x+t(y-x)\big)\,\mathrm{d}t \Big\| = \\ &= \Big\| \int_0^1 \mathrm{D}f\big(x+t(y-x)\big)(y-x)\,\mathrm{d}t \Big\| \leq \\ &\leq \int_0^1 \|\mathrm{D}f\big(x+t(y-x)\big)\|\,\|y-x\|\,\mathrm{d}t \leq \sup_{z;\|z\|\leq N} \|\mathrm{D}f(z)\|\,\|y-x\|\end{aligned}$$

Function $z \longmapsto \mathrm{D}f(z)$, where $\mathrm{D}f$ denotes the Fréchet derivative[13] of f, is continuous thus locally bounded. Consequently, there exists such a constant $K > 0$ that

$$\sup_{z;\|z\|\leq N} \|\mathrm{D}f(z)\| \leq K$$

and the local Lipschitz continuity of f follows.

□

Lemma 3.0.15. *Let $f, g : \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{R}^K$ be functions Lipschitz continuous in x uniformly in t and bounded. Then also the function fg is Lipschitz continuous in x uniformly in t.*

Proof. By the assumptions, we obtain the existence of a constant $K \in (0, \infty)$ so that

$$\forall x,\, y \in \mathbb{R}^M \quad \forall t \geq 0 \quad \|f(t,x)-f(t,y)\| \vee \|g(t,x)-g(t,y)\| \leq K\|x-y\|.$$

The rest is a clear consequence of the following computation

$$\begin{aligned}&\|f(t,x)g(t,x)-f(t,y)g(t,y)\| = \\ &\qquad = \|f(t,x)g(t,x)-f(t,x)g(t,y)+f(t,x)g(t,y)-f(t,y)g(t,y)\| \leq \\ &\qquad \leq \|f(t,x)\|\,\|g(t,x)-g(t,y)\| + \|g(t,y)\|\,\|f(t,x)-f(t,y)\| \leq \\ &\qquad \leq K^2 \sup_{\mathbb{R}_+\times\mathbb{R}^M} \|f(t,x)\| \sup_{\mathbb{R}_+\times\mathbb{R}^M} \|g(t,x)\|\,\|x-y\|.\end{aligned}$$

□

Theorem 3.0.16. *Suppose that $F : \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{R}^K$ is a Borel function which is continuous in x having linear growth in x, i.e.*

$$\exists K^* < \infty \quad \forall x \in \mathbb{R}^M \quad \forall t \geq 0 \quad \|F(t,x)\| \leq K^*\big(1+\|x\|\big).$$

[13] generally, a derivative of a function defined on a Banach space, in finite-dimensional spaces it agrees with the total differential

Then there exists a sequence of functions $F_n : \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{R}^K$, $n \in \mathbb{N}$, *which are globally Lipschitz continuous in* x *uniformly in* t[14]

$$\forall n \in \mathbb{N} \quad \exists K_n < \infty \quad \forall x,\, y \in \mathbb{R}^M \quad \forall t \geq 0 \quad \|F_n(t,x) - F_n(t,y)\| \leq K_n \|x - y\|,$$

have the same linear growth in x

$$\forall t \geq 0 \quad \sup_{n\in\mathbb{N}} \|F_n(t,x)\| \leq K^*\big(2 + \|x\|\big), \quad x \in \mathbb{R}^M \tag{3.2}$$

and furhermore

$$F_n(t,\,.\,) \rightrightarrows_{loc} F(t,\,.\,), \quad n \to \infty \quad \forall t \geq 0.$$

Proof. First of all, fix such a function $\varrho \in \mathscr{C}^\infty(\mathbb{R}^M)$ that

$$\varrho \geq 0, \quad \operatorname{supt} \varrho \subset B(0,1), \quad \int_{\mathbb{R}^M} \varrho \,\mathrm{d}x = 1,$$

in other words, $\varrho \in \mathscr{D}(\mathbb{R}^M)$ where by $\mathscr{D}(\mathbb{R}^M)$ we denote the test function space on $\mathbb{R}^M$.[15] Such a function evidently exists since the test function space on $\mathbb{R}^M$ is nonempty and we can choose for example

$$\varrho = \begin{cases} c\exp\left\{-\frac{1}{1-\|x\|^2}\right\}, & \|x\| \leq 1,\\ 0, & \text{otherwise,}\end{cases}$$

where the constant c is chosen in order to satisfy the condition $\int_{\mathbb{R}^M} \varrho \,\mathrm{d}x = 1$. Next, for $n \in \mathbb{N}$ we define function G_n as the convolution of functions $F(t,\,.\,)$ and $n^M \varrho(n(\,.\,))$ (applied componentwise):

$$G_n(t,x) = n^M \int_{\mathbb{R}^M} F(t,y)\varrho(n(x-y))\,\mathrm{d}y = \left(n^M \int_{\mathbb{R}^M} F^i(t,y)\varrho(n(x-y))\,\mathrm{d}y\right)_{i=1}^{K},$$

$$x \in \mathbb{R}^M, \quad t \geq 0.$$

For all $n \in \mathbb{N}$ the function $\varrho(n(\,.\,))$ has a compact support and functions $F^i(t,\,.\,)$ are continuous so bounded on this compact set. Hence these integrals converge and functions G_n are well defined. In addition, they have the same linear growth in x:

$$\begin{aligned}
\|G_n(t,x)\| &= \|n^M \int_{\mathbb{R}^M} F(t,y)\varrho(n(x-y))\,\mathrm{d}y\| = \\
&= \Big\| \int_{\mathbb{R}^M} F\Big(t, x - \frac{\xi}{n}\Big)\varrho(\xi)\,\mathrm{d}\xi \Big\| \leq \int_{\mathbb{R}^M} \Big\| F\Big(t, x - \frac{\xi}{n}\Big) \Big\| \varrho(\xi)\,\mathrm{d}\xi = \\
&= \int_{B(0,1)} \Big\| F\Big(t, x - \frac{\xi}{n}\Big) \Big\| \varrho(\xi)\,\mathrm{d}\xi \leq K^* \int_{B(0,1)} \left(1 + \Big\| x - \frac{\xi}{n} \Big\|\right) \varrho(\xi)\,\mathrm{d}\xi \leq \\
&\leq K^* \int_{B(0,1)} \left(1 + \|x\| + \Big\| \frac{\xi}{n} \Big\|\right) \varrho(\xi)\,\mathrm{d}\xi \leq K^* \left(1 + \frac{1}{n} + \|x\|\right),
\end{aligned} \tag{3.3}$$

[14] global Lipschitz continuity in x uniform in t implies continuity in x uniform in t

[15] for further information see [Ru91]

for the second expression we used the change of variables formula with diffeomorphism $(y_1, \dots, y_M) \longmapsto n(x_1 - y_1, \dots, x_M - y_M)$, whose Jacobian determinant is n^M. So

$$\forall t \geq 0 \quad \sup_{n\in\mathbb{N}} \|G_n(t,x)\| \leq K^*(2 + \|x\|), \quad x \in \mathbb{R}^M.$$

Fix $n \in \mathbb{N}$. According to Lemmas 3.0.13 and 3.0.14, for any fixed $t \geq 0$, the function $G_n(t, \,.\,)$ is locally Lipschitz continuous. Fix $t \geq 0$, $N \in \mathbb{N}$ and $y,\, z \in \mathbb{R}^M$, $\|y\| \vee \|z\| \leq N$, then

$$\|G_n(t,y) - G_n(t,z)\| \leq \sup_{x; \|x\| \leq N} \|\mathrm{D}G_n(t,x)\| \, \|y - z\|,$$

where by $\mathrm{D}G_n$ we denote the Fréchet derivative of G_n. In order to show that G_n is locally Lipschitz continuous independently of t, it is needed to estimate the supremum in the previous formula by a constant independent of t. Since on the space $\mathbb{M}_{M\times K}$ we consider the Hilbert-Schmidt norm[16] and therefore

$$\|\mathrm{D}G_n(t,x)\| = \left(\sum_{j=1}^{M} \sum_{i=1}^{K} \left| \frac{\partial G_n^i}{\partial x_j}(t,x) \right|^2 \right)^{\frac{1}{2}},$$

it is sufficient to find an estimate of the term

$$\left| \frac{\partial G_n^i}{\partial x_j}(t,x) \right|$$

independently of i, j, t. We have[17]

$$\begin{aligned} \left| \frac{\partial G_n^i}{\partial x_j}(t,x) \right| &= \left| n^M \int_{\mathbb{R}^M} F^i(t,y) \frac{\partial}{\partial x_j} \big(\varrho(n(x-y))\big) \mathrm{d}y \right| = \\ &= \left| n^{M+1} \int_{\mathbb{R}^M} F^i(t,y) \left(\frac{\partial \varrho}{\partial x_j} \right) (n(x-y))\, \mathrm{d}y \right| = \\ &= \left| n \int_{B(0,1)} F^i\Big(t, x - \frac{\xi}{n}\Big) \frac{\partial \varrho}{\partial x_j}(\xi)\, \mathrm{d}\xi \right| \leq \\ &\leq n \int_{B(0,1)} \left| F^i\Big(t, x - \frac{\xi}{n}\Big) \right| \, \left| \frac{\partial \varrho}{\partial x_j}(\xi) \right| \mathrm{d}\xi. \end{aligned}$$

Apparently, from the linear growth property of F, it follows for any $i \in \{1, \dots, K\}$ that $|F^i(t,x)| \leq K^*(1 + \|x\|)$. Denote

$$L = \sup \left\{ \left| \frac{\partial \varrho}{\partial x_j}(\xi) \right|; \; j \in \{1, \dots, M\}, \, \xi \in B(0,1) \right\},$$

[16] generally $\|A\|_{HS} = \sum_{i\in I} \|Ae_i\|$, where A is an operator on Hilbert space H and $\{e_i;\, i \in I\}$ is an orthonormal base of H

[17] by $\frac{\partial}{\partial x_j}\big(\varrho(n(x-y))\big)$ the partial derivative of composed function $\varrho(n(\,.\, - y))$ is denoted, whereas $\left(\frac{\partial \varrho}{\partial x_j}\right)(n(x-y))$ denotes the partial derivative of ϱ at point $n(x-y)$

which is a finite number since all partial derivatives of ϱ are elements of the test function space. We get

$$\left|\frac{\partial G_n^i}{\partial x_j}(t,x)\right| \leq nLK^*\omega_M\Big(1+\frac{1}{n}+\|x\|\Big)$$

where ω_M represents the volume of a unit ball in $\mathbb{R}^M$.[18] As a consequence we obtain the desired estimate independent of t.

Fix $t \geq 0$. To prove the local uniform convergence $G_n(t,\,.\,)$ to $F(t,\,.\,)$ we apply first several steps as in (3.3) which yields

$$\|G_n(t,x)-F(t,x)\| \leq \int_{B(0,1)} \Big\|F\Big(t,x-\frac{\xi}{n}\Big)-F(t,x)\Big\|\,\varrho(\xi)\,\mathrm{d}\xi.$$

Let $\varepsilon > 0$ and $K \subset \mathbb{R}^M$, compact, be given. By the assumptions, the function F is continuous in x hence by 3.0.11, the continuity of $F(t,\,.\,)$ in x is even uniform on K. Denoting by $K' = \{y \in \mathbb{R}^M;\ \mathrm{dist}(y,K) \leq 1\}$ the closed neighbourhood of K, which is also compact, we deduce

$$\exists n_0 \in \mathbb{N}\quad \forall n \geq n_0\quad \forall x \in K\quad \forall \xi \in B(0,1)\quad \Big\|F\Big(t,x-\frac{\xi}{n}\Big)-F(t,x)\Big\| < \varepsilon,$$

therefore

$$\|G_n(t,x)-F(t,x)\| < \varepsilon\int_{B(0,1)} \varrho(\xi)\,\mathrm{d}\xi = \varepsilon.$$

To finish the proof, let us define for all $t \geq 0$ functions $F_n(t,\,.\,)$:

$$F_n(t,x) = \begin{cases} G_n(t,x), & x \in nB(0,1),\\ 0, & x \in \mathbb{R}^M \setminus 2nB(0,1),\\ \Big(2-\frac{\|x\|}{n}\Big)\,G_n(t,x), & \text{otherwise.}\end{cases}$$

We show that the functions F_n are Lipschitz continuous in x uniformly in t. For $n \in \mathbb{N}$ the function $2-\frac{\|x\|}{n}$ obviously satisfies this condition and by Lemma 3.0.15, locally does so also the function $\big(2-\frac{\|x\|}{n}\big)G_n(t,x)$. This implies that F_n is also locally Lipschitz continuous in $x \in \mathbb{R}^M$ uniformly in t. Moreover, $\operatorname{supt} F_n(t,\,.\,) = 2nB(0,1)$ is a compact set and therefore the claim follows.

Since

$$\|F_n(t,x)\| \leq \|G_n(t,x)\| \quad \forall x \in \mathbb{R}^M \quad \forall n \in \mathbb{N},$$

the linear growth property remains also for F_n.

We observe that $\lim_{n\to\infty} F_n(t,x) = \lim_{n\to\infty} G_n(t,x) = F(t,x)$ because

$$\forall x \in \mathbb{R}^M\quad \exists n_0 \in \mathbb{N}\quad \forall n \geq n_0\quad \forall t \geq 0\quad F_n(t,x) = G_n(t,x).$$

[18]since Euclidean metric is used, $\omega_M = \lambda^M\big(B(0,1)\big) = \frac{\pi^{\frac{M}{2}}}{\Gamma\left(\frac{M}{2}+1\right)}$

Even the local uniform convergence holds. Fix $t \geq 0$ and let $\varepsilon > 0$ and $K \subset \mathbb{R}^M$, compact, be given. Denote

$$n_1 = \lceil \sup_{x \in K} \|x\| \rceil,$$

where $\lceil\, . \,\rceil$ stands for the ceiling function. From the definition of the functions F_n we have

$$\forall n \geq n_1 \quad \forall x \in K \quad F_n(t,x) = G_n(t,x).$$

$G_n(t,\, .\,)$ converge to $F(t,\, .\,)$ uniformly on K and therefore

$$\exists n_2 \quad \forall n \geq n_2 \quad \forall x \in K \quad \|G_n(t,x) - F(t,x)\| < \varepsilon.$$

Now, setting $n_0 = \max\{n_1, n_2\}$ yields

$$\forall n \geq n_0 \quad \forall x \in K \quad \|F_n(t,x) - F(t,x)\| = \|G_n(t,x) - F(t,x)\| < \varepsilon.$$

□

Remark 3.0.17. Actually, the proof of 3.0.16 gives more, namely, the functions F_n are bounded.

Proof. Fix $n \in \mathbb{N}$. It follows from the definition of F_n

$$\begin{aligned} &x \in \mathbb{R}^M,\ \|x\| > 2n \quad \text{then} \quad F_n(t,x) = 0 \quad \forall t \geq 0, \\ &x \in \mathbb{R}^M,\ \|x\| \leq 2n \quad \text{then} \quad \|F_n(t,x)\| \leq K^*(2 + \|x\|) \leq K^*(2+2n) \quad \forall t \geq 0. \end{aligned}$$

□

Proposition 3.0.18. *Statement of Theorem* 3.0.16 *remains valid also for any continuous function* $F : \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{M}_{L\times K}$ *of a linear growth.*

Proof. The proof follows from the fact that the space $\mathbb{M}_{L\times K}$ equipped with the Hilbert-Schmidt norm is isometrically isomorphic to $\mathbb{R}^{LK}$ having the Euclidean norm.

□

Chapter 4

Stochastic differential equations

Let $b : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{M}_{m\times k}$ be Borel functions and let W be a k-dimensional Brownian motion. We study the stochastic differential equation

$$\mathrm{d}X = b(t, X)\mathrm{d}t + \sigma(t, X)\mathrm{d}W, \tag{4.1}$$

which can be equivalently rewritten componentwise

$$\mathrm{d}X^i = b^i(t, X)\mathrm{d}t + \sum_{j=1}^{k} \sigma^{ij}(t, X)\mathrm{d}W^j, \quad i = 1, \dots, m.$$

First, we introduce the solution of (4.1) in the strong sense. We summarize theorems useful in the subsequent section, which discusses the solution in other point of view - in the weak sense. The standard reference here is the textbook [KS88] where the detailed discussion of both concepts can be found.

4.1 Strong solution

Throughout this section we fix a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ satisfying the usual conditions, i.e. the filtration $(\mathscr{F}_t)$ is right-continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-negligible sets $\{N \in \mathscr{F};\ \mathbb{P}(N) = 0\}$.

Theorem 4.1.1 (Lévy)**.** *Suppose that X is a continous stochastic process starting at 0 and having values in $\mathbb{R}^m$. Then X is an $(\mathscr{F}_t)$-Brownian motion if and only if X is an $(\mathscr{F}_t)$-local martingale with a tensor variation $\langle\!\langle X \rangle\!\rangle_t = tI$, $t \geq 0$.*[19]

Theorem 4.1.2 (Burkholder-Davis-Gundy)**.** *For all $p \in (0, \infty)$ there exist constants $c_p, C_p \in (0, \infty)$ so that for any m-dimensional continuous local martingale X, $X(0) = 0$, and for any stopping time τ it holds*

$$c_p\, \mathbb{E}\langle X \rangle_\tau^{p/2} \leq \mathbb{E} \sup_{t\geq 0} \|X_{t\wedge\tau}\|^p \leq C_p\, \mathbb{E}\langle X \rangle_\tau^{p/2}.$$

[19] I denotes identity matrix of proper dimension

Definition 4.1.3. Let φ be a random variable with values in $\mathbb{R}^m$. A process X is said to be a strong solution to the equation (4.1) with initial condition φ if

$$\int_0^t \big(\|b(s,X_s)\| + \|\sigma(s,X_s)\|^2\big)\mathrm{d}s < \infty, \quad \forall t \geq 0 \quad \mathbb{P}\text{-a.s.},$$

$$X(t) = \varphi + \int_0^t b(s,X_s)\mathrm{d}s + \int_0^t \sigma(s,X_s)\mathrm{d}W_s, \quad \forall t \geq 0 \quad \mathbb{P}\text{-a.s.}.$$

Theorem 4.1.4. *Suppose that coefficients of the equation* (4.1) *satisfy the linear growth condition*

$$\exists K^* < \infty \quad \forall t \geq 0 \quad \forall x \in \mathbb{R}^m \quad \|b(t,x)\| \vee \|\sigma(t,x)\| \leq K^*(1 + \|x\|)$$

and are Lipschitz continous

$$\exists K < \infty \quad \forall t \geq 0 \quad \forall x,y \in \mathbb{R}^m \quad \|b(t,x) - b(t,y)\| \vee \|\sigma(t,x) - \sigma(t,y)\| \leq K\|x - y\|.$$

Then for any $\mathscr{F}_0$-measurable random variable $\varphi : \Omega \to \mathbb{R}^m$ with $\mathbb{E}\|\varphi\|^2 < \infty$, there exists a unique strong solution of (4.1) *with the initial condition φ. Moreover, for any $p \in [2,\infty)$, $T > 0$, there exists a constant $C^* = C^*(p, K^*, T)$ so that*

$$\mathbb{E} \sup_{0 \leq t \leq T} \|X(t)\|^p \leq C^*\big(1 + \mathbb{E}\|\varphi\|^p\big).$$

A unique strong solution exists even without the assumptions upon the initial condition. However, in that case we don't have any control of its moments.

Theorem 4.1.5. *Suppose that coefficients of the equation* (4.1) *satisfy the linear growth condition*

$$\exists K^* < \infty \quad \forall t \geq 0 \quad \forall x \in \mathbb{R}^m \quad \|b(t,x)\| \vee \|\sigma(t,x)\| \leq K^*(1 + \|x\|)$$

and are Lipschitz continous

$$\exists K < \infty \quad \forall t \geq 0 \quad \forall x,y \in \mathbb{R}^m \quad \|b(t,x) - b(t,y)\| \vee \|\sigma(t,x) - \sigma(t,y)\| \leq K\|x - y\|.$$

Then for any $\mathscr{F}_0$-measurable random variable $\varphi : \Omega \to \mathbb{R}^m$ there exists a unique strong solution of (4.1) *with the initial condition φ.*

4.2 Weak solution

In this section, the main result of our thesis is presented. It deals with the stochastic differential equation (4.1), where b, σ are functions continuous in the space variable x of a linear growth in x. First, in Theorem 4.2.2, we prove the existence of a weak solution for an initial distribution μ satifying an additional integrability assumption whereas in Theorem 4.2.8 we avoid this hypothesis.

Definition 4.2.1. A triple $\mathcal{X} = \big((\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), W, X\big)$ is said to be a weak solution of the equation (4.1) if

(i) $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis,

(ii) W is a k-dimensional $(\mathscr{F}_t)$-Brownian motion,

(iii) X is an m-dimensional $(\mathscr{F}_t)$-progressively measurable process which satisfies the following conditions:

$$\int_0^t \big(\|b(s, X_s)\| + \|\sigma(s, X_s)\|^2\big)\mathrm{d}s < \infty, \quad \forall t \geq 0 \quad \mathbb{P}\text{-a.s.},$$

$$X(t) = X(0) + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s, \quad \forall t \geq 0 \quad \mathbb{P}\text{-a.s.}.$$

Let μ be a Borel probability measure on $\mathbb{R}^m$. We call $\mathcal{X}$ a weak solution with the initial distribution μ if $\mathbb{P} \circ X(0)^{-1} = \mu$.

Proof of the theorem will be divided into several propositions.

Theorem 4.2.2. *Let* $b : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{M}_{m \times k}$ *be Borel functions which are continuous in* x *having linear growth in* x*:*

$$\exists K^* > 0 \quad \forall x \in \mathbb{R}^m \quad \forall t \geq 0 \quad \|b(t, x)\| \vee \|\sigma(t, x)\| \leq K^*(1 + \|x\|).$$

Suppose μ *is such a Borel propability measure on* $\mathbb{R}^m$ *that there exists* $p > 2$ *with property*

$$\int_{\mathbb{R}^m} \|x\|^p \mu(\mathrm{d}x) < \infty.$$

Then there exists a weak solution of the equation (4.1) *with the initial distribution* μ*.*

Proof. According to Theorem 3.0.16 and Proposition 3.0.18, there exist functions $b_n : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma_n : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{M}_{m \times k}$ which are globally Lipschitz continuous in x uniformly in t:

$$\forall n \in \mathbb{N} \quad \exists K_n > 0 \quad \forall x, y \in \mathbb{R}^m \quad \forall t \geq 0$$
$$\|b_n(t, x) - b_n(t, y)\| \vee \|\sigma_n(t, x) - \sigma_n(t, y)\| \leq K_n\|x - y\|,$$

have the same linear growth in x:

$$\forall x \in \mathbb{R}^m \quad \forall t \geq 0 \quad \sup_{n \in \mathbb{N}} \big(\|b_n(t, x)\| \vee \|\sigma_n(t, x)\|\big) \leq K^*(2 + \|x\|)$$

and furthermore:

$$\forall t \geq 0 \qquad b_n(t, .) \rightrightarrows_{loc} b(t, .), \quad \sigma_n(t, .) \rightrightarrows_{loc} \sigma(t, .), \quad n \to \infty.$$

Fix arbitrary stochastic basis $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}})$ which carries a k-dimensional $(\tilde{\mathscr{F}}_t)$-Brownian motion $\tilde{W}$ as well as a $\tilde{\mathscr{F}}_0$-measurable random variable $\psi : \tilde{\Omega} \to \mathbb{R}^m$ with distribution μ. Without loss of generality we assume that $(\tilde{\mathscr{F}}_t)$ satisfies the usual conditions. We apply Theorem 4.1.4 and deduce for all $n \in \mathbb{N}$ that there exists a process $\tilde{X}_n$ which solves the equation

$$\mathrm{d}\tilde{X}_n = b_n(t, \tilde{X}_n)\mathrm{d}t + \sigma_n(t, \tilde{X}_n)\mathrm{d}\tilde{W}, \quad \tilde{X}_n(0) = \psi, \tag{4.2}$$

Since these solutions are stochastic processes with continuous paths, in what follows, they will be understood as random variables with values in the space of all continuous functions. As the next step we proceed with a proposition.

Proposition 4.2.3. *Under conditions stated above,* $\{\tilde{\mathbb{P}} \circ \tilde{X}_n^{-1};\ n \in \mathbb{N}\}$ *is a tight set of measures on* $\mathscr{C}_m$.

Proof. According to Theorem 2.4.9, it is enough to prove the following conditions

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \big(\tilde{\mathbb{P}} \circ \tilde{X}_n^{-1}\big)\big(f \in \mathscr{C}_m;\ \|f(0)\| \geq R\big) = 0, \tag{4.3}$$

$$\forall T > 0\ \ \exists \beta > 0\ \ \lim_{R\to\infty} \sup_{n\in\mathbb{N}} \big(\tilde{\mathbb{P}} \circ \tilde{X}_n^{-1}\big)\left(f \in \mathscr{C}_m;\ \sup_{0\leq s,t\leq T} \frac{\|f(t) - f(s)\|}{|t-s|^\beta} \geq R\right) = 0. \tag{4.4}$$

Let us begin with (4.3). Definition of the measure $\tilde{\mathbb{P}} \circ \tilde{X}_n^{-1}$ as the distribution of the random variable $\tilde{X}_n$ on the space of all continuous paths $(\mathscr{C}_m, \mathscr{B}(\mathscr{C}_m))$ implies

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \big(\tilde{\mathbb{P}} \circ \tilde{X}_n^{-1}\big)\big(f \in \mathscr{C}_m;\ \|f(0)\| \geq R\big) = \\ = \lim_{R\to\infty} \sup_{n\in\mathbb{N}} \tilde{\mathbb{P}}\big(\omega \in \tilde{\Omega};\ \|\tilde{X}_n(\omega, 0)\| \geq R\big).$$

Since for all $n \in \mathbb{N}$ the equation (4.2) has the same initial condition with finite moments up to order $p > 2$ it holds

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \tilde{\mathbb{P}}\big(\|\tilde{X}_n(\omega, 0)\| \geq R\big) = \lim_{R\to\infty} \tilde{\mathbb{P}}\big(\|\tilde{X}_1(0)\| \geq R\big) = 0.$$

Fix $T > 0$. We now prove (4.4) by using Kolmogorov-Chentsov theorem A.1 for processes $(\tilde{X}_n(t);\ 0 \leq t \leq T)$. First of all, it is needed to verify its assumptions thus to find constants $a, b, c > 0$ so that

$$\tilde{\mathbb{E}}\|\tilde{X}_n(t) - \tilde{X}_n(s)\|^a \leq c\,|t-s|^{1+b}, \quad 0 \leq s, t \leq T.$$

Choose arbitrary $n \in \mathbb{N}$ and $s,\ t \in [0, T]$, $s < t$. Process $\tilde{X}_n$ solves the equation (4.2) and therefore

$$\tilde{X}_n(t) = \psi + \int_0^t b_n\big(r, \tilde{X}_n(r)\big)\,\mathrm{d}r + \int_0^t \sigma_n\big(r, \tilde{X}_n(r)\big)\,\mathrm{d}\tilde{W}_r,$$

in consequence we observe the representation of increment

$$\tilde{X}_n(t) - \tilde{X}_n(s) = \int_s^t b_n\big(r, \tilde{X}_n(r)\big)\,\mathrm{d}r + \int_s^t \sigma_n\big(r, \tilde{X}_n(r)\big)\,\mathrm{d}\tilde{W}_r.$$

We shall first suppose that $a \geq 1$ and proceed with estimate

$$\begin{aligned}\tilde{\mathbb{E}}\,\big\|\tilde{X}_n(t)-\tilde{X}_n(s)\big\|^a &= \tilde{\mathbb{E}}\left\| \int_s^t b_n\big(r, \tilde{X}_n(r)\big)\,\mathrm{d}r + \int_s^t \sigma_n\big(r, \tilde{X}_n(r)\big)\,\mathrm{d}\tilde{W}_r\right\|^a \leq \\ &\leq 2^{a-1}\left(\tilde{\mathbb{E}}\left\|\int_s^t b_n\big(r, \tilde{X}_n(r)\big)\,\mathrm{d}r\right\|^a + \tilde{\mathbb{E}}\left\|\int_s^t \sigma_n\big(r, \tilde{X}_n(r)\big)\,\mathrm{d}\tilde{W}_r\right\|^a\right) = \\ &= 2^{a-1}(I_1 + I_2).\end{aligned}$$

Now we consider each term separately.

$$I_1 \;=\; \tilde{\mathbb{E}}\left\|\int_s^t b_n\big(r, \tilde{X}_n(r)\big)\,\mathrm{d}r\right\|^a \leq \tilde{\mathbb{E}}\left(\int_s^t \big\|b_n\big(r, \tilde{X}_n(r)\big)\big\|\,\mathrm{d}r\right)^a$$

By Hölder inequality applied to the inner integral, it follows

$$\begin{aligned}I_1 &\leq |t-s|^{a-1}\,\tilde{\mathbb{E}}\int_s^t \big\|b_n\big(r, \tilde{X}_n(r)\big)\big\|^a\,\mathrm{d}r \leq \\ &\leq |t-s|^{a-1}(K^*)^a \int_s^t \tilde{\mathbb{E}}\big(2 + \big\|\tilde{X}_n(r)\big\|\big)^a\,\mathrm{d}r \leq \\ &\leq |t-s|^{a-1}(K^*)^a\,2^{a-1}\int_s^t \big(2^a + \tilde{\mathbb{E}}\,\big\|\tilde{X}_n(r)\big\|^a\big)\,\mathrm{d}r.\end{aligned}$$

According to Theorem 4.1.4, we see

$$\forall a \geq 2 \quad \exists C^* = C^*(a, K^*, T) \quad \text{ so that } \quad \tilde{\mathbb{E}}\sup_{0\leq r\leq T}\|\tilde{X}_n(r)\|^a \leq C^*(1 + \tilde{\mathbb{E}}\|\psi\|^a),$$

which for $a \geq 2$ leads to the final estimate of I_1

$$I_1 \leq |t-s|^a(K^*)^a\,2^{a-1}\big(2^a + C^*\big(1 + \tilde{\mathbb{E}}\|\psi\|^a\big)\big).$$

To obtain the estimate of I_2, firstly 4.1.2 and afterwards the similar approach as in the case of I_1 are employed.

$$\begin{aligned}I_2 = \tilde{\mathbb{E}}&\left\|\int_s^t \sigma_n\big(r, \tilde{X}_n(r)\big)\,\mathrm{d}\tilde{W}_r\right\|^a \leq C_a\,\tilde{\mathbb{E}}\left(\int_s^t \big\|\sigma_n\big(r, \tilde{X}_n(r)\big)\big\|^2\,\mathrm{d}r\right)^{\frac{a}{2}} \leq \\ &\leq |t-s|^{\frac{a}{2}-1}C_a\,\tilde{\mathbb{E}}\int_s^t \big\|\sigma_n\big(r, \tilde{X}_n(r)\big)\big\|^a\,\mathrm{d}r \leq \\ &\leq |t-s|^{\frac{a}{2}-1}C_a(K^*)^a\int_s^t \tilde{\mathbb{E}}\big(2 + \big\|\tilde{X}_n(r)\big\|\big)^a\,\mathrm{d}r \leq \\ &\leq |t-s|^{\frac{a}{2}-1}C_a(K^*)^a\,2^{a-1}\int_s^t \big(2^a + \tilde{\mathbb{E}}\big\|\tilde{X}_n(r)\big\|^a\big)\,\mathrm{d}r \leq \\ &\leq |t-s|^{\frac{a}{2}}C_a(K^*)^a\,2^{a-1}\big(2^a + C^*\big(1 + \tilde{\mathbb{E}}\|\psi\|^a\big)\big)\end{aligned}$$

Consequently, for $a \geq 2$ we conclude

$$\tilde{\mathbb{E}}\left\|\tilde{X}_n(t)-\tilde{X}_n(s)\right\|^a \leq$$
$$\leq 2^{2a-2}(K^*)^a\big(2^a+C^*\big(1+\tilde{\mathbb{E}}\|\psi\|^a\big)\big)\Big(|t-s|^a+C_a|t-s|^{\frac{a}{2}}\Big) \leq$$
$$\leq 2^{2a-2}(K^*)^a\big(2^a+C^*\big(1+\tilde{\mathbb{E}}\|\psi\|^a\big)\big)\big(T^{\frac{a}{2}}+C_a\big)|t-s|^{\frac{a}{2}}.$$

Setting

$$a>2, \quad b=\frac{a}{2}-1, \quad c=2^{2a-2}(K^*)^a\big(2^a+C^*\big(1+\tilde{\mathbb{E}}\|\psi\|^a\big)\big)(T^{\frac{a}{2}}+C_a)$$

we see that A.1 is applicable and gives that the paths of $\tilde{X}_n$ are locally Hölder continuous with exponent β, where

$$\beta \in \Big[0,\frac{b}{a}\Big)=\Big[0,\frac{a-2}{2a}\Big).$$

The local Hölder continuity is always true independently on n. Finally, we are able to prove the condition (4.4). We have

$$\lim_{R\to\infty}\sup_{n\in\mathbb{N}}\big(\tilde{\mathbb{P}}\circ\tilde{X}_n^{-1}\big)\bigg\{f\in\mathscr{C}_m;\ \sup_{0\leq s,t\leq T}\frac{\|f(t)-f(s)\|}{|t-s|^\beta}\geq R\bigg\}=$$
$$=\lim_{R\to\infty}\sup_{n\in\mathbb{N}}\tilde{\mathbb{P}}\bigg(\sup_{0\leq s,t\leq T}\frac{\|\tilde{X}_n(t)-\tilde{X}_n(s)\|}{|t-s|^\beta}\geq R\bigg).$$

Moreover, by local β-Hölder continuity of the paths, it holds for all $n\in\mathbb{N}$

$$\tilde{\mathbb{P}}\bigg(\sup_{0\leq s,t\leq T}\frac{\|\tilde{X}_n(t)-\tilde{X}_n(s)\|}{|t-s|^\beta}<\infty\bigg)=1,$$

which implies

$$\tilde{\mathbb{P}}\bigg(\sup_{0\leq s,t\leq T}\frac{\|\tilde{X}_n(t)-\tilde{X}_n(s)\|}{|t-s|^\beta}\geq R\bigg)\longrightarrow 0, \quad R\to\infty, \quad \forall n\in\mathbb{N}.$$

The convergence is even uniform in n as can be seen in the proof of A.1 (see [KS88, Theorem 2.2.8] for a deeper discussion).

□

It follows immediately that $\{\tilde{\mathbb{P}}\circ(\tilde{X}_n,\tilde{W})^{-1},\ n\in\mathbb{N}\}$ is a tight set of measures on $\mathscr{C}_m\times\mathscr{C}_k$. Indeed, since all probability measures on the σ-algebra of Borel sets of any Polish space are Radon measures[20], in particular, they are inner regular or tight, we have two tight collections of measures

$$\{\tilde{\mathbb{P}}\circ\tilde{X}_n^{-1},\ n\in\mathbb{N}\},\ \{\tilde{\mathbb{P}}\circ\tilde{W}\}$$

[20]see [LM05]

and so for any $\varepsilon > 0$ there exists $K_1 \subset \mathscr{C}_m, K_2 \subset \mathscr{C}_k$ compact so that

$$\inf_{n\in\mathbb{N}} \big(\tilde{\mathbb{P}} \circ \tilde{X}_n^{-1}\big)(K_1) \geq 1 - \frac{\varepsilon}{2}, \quad \big(\tilde{\mathbb{P}} \circ \tilde{W}\big)(K_2) \geq 1 - \frac{\varepsilon}{2}.$$

De Morgan's law $A \cap B = (A^C \cup B^C)^C$ yields

$$\begin{aligned}\big(\tilde{\mathbb{P}} \circ (\tilde{X}_n, \tilde{W})^{-1}\big)(K_1 \times K_2) = \tilde{\mathbb{P}}\big(\tilde{X}_n \in K_1, \tilde{W} \in K_2\big) &= \\ = 1 - \tilde{\mathbb{P}}\big(\tilde{X}_n \notin K_1 \cup \tilde{W} \notin K_2\big) &\geq \\ \geq 1 - \Big(\tilde{\mathbb{P}}\big(\tilde{X}_n \notin K_1\big) + \tilde{\mathbb{P}}\big(\tilde{W} \notin K_2\big)\Big) &\geq 1 - \varepsilon, \quad n \in \mathbb{N}.\end{aligned}$$

As $K_1 \times K_2$ is compact in $\mathscr{C}_m \times \mathscr{C}_k$ we have showed the claim.

As the next step we need to recall that $\mathscr{C}_m \times \mathscr{C}_k = \mathscr{C}_{m+k}$ and that on any separable metric space, the Cartesian product of Borel σ-algebras is the Borel σ-algebra on Cartesian product, i.e.

$$\big(\mathscr{C}_m \times \mathscr{C}_k, \mathscr{B}(\mathscr{C}_m) \otimes \mathscr{B}(\mathscr{C}_k)\big) = \big(\mathscr{C}_{m+k}, \mathscr{B}(\mathscr{C}_{m+k})\big).$$

If we summarize the situation, there is a Polish space $\mathscr{C}_{m+k}$ and a tight collection of probability measures $\{\tilde{\mathbb{P}} \circ (\tilde{X}_n, \tilde{W})^{-1},\ n \in \mathbb{N}\}$ defined on its Borel σ-algebra $\mathscr{B}(\mathscr{C}_{m+k})$. By the Prochorov theorem A.2, this set of measures is relatively weakly compact and therefore there exists a subsequence which converge weakly. Without loss of generality we can assume that the original sequence has already converged weakly. Then according to the Skorokhod theorem A.3, there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and random variables $(X_n, W) : (\Omega, \mathscr{F}) \to (\mathscr{C}_{m+k}, \mathscr{B}(\mathscr{C}_{m+k}))$, $n \in \mathbb{N}_0$, satisfying

$$(X_n, W_n) \overset{d}{\sim} (\tilde{X}_n, \tilde{W}), \quad n \in \mathbb{N},$$

$$(X_n, W_n) \longrightarrow (X_0, W_0), \quad n \to \infty, \quad \mathbb{P}\text{-a.s.}.$$

Let us define

$$\begin{aligned}\mathscr{F}_t &= \sigma\big(\sigma(X_0(s), W_0(s); 0 \leq s \leq t) \cup \{N;\ \mathbb{P}(N) = 0\}\big), \quad t \geq 0,\\ \tilde{M}_n(t) &= \tilde{X}_n(t) - \psi - \int_0^t b_n\big(r, \tilde{X}_n(r)\big)\mathrm{d}r, \quad n \in \mathbb{N},\\ M_n(t) &= X_n(t) - X_n(0) - \int_0^t b_n\big(r, X_n(r)\big)\mathrm{d}r, \quad n \in \mathbb{N}_0,\end{aligned}$$

where $b_0 = b$, $\sigma_0 = \sigma$.

In the following proposition we will establish a procedure of proving that under the above conditions a process is an $(\mathscr{F}_t)$-martingale. This approach will be first applied to the process $\tilde{M}_n$ and afterwards to other processes in order to finish the proof of the theorem. It is based on the idea of transferring the martingale property from processes on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ (processes marked by $\tilde{}$ and n, e.g. $\tilde{M}_n$) to corresponding processes on $(\Omega, \mathscr{F}, \mathbb{P})$ (e.g. M_n) and finally by limiting to processes with index 0 (e.g. M_0).

The proof of this proposition is understood as a template for this procedure and therefore it will be performed in detail. On the other hand, in the proofs of sequential propositions using this technique we only emphasize the important steps.

Here and subsequently, fix $N \in \mathbb{N}$, times $0 \le s_1 < \cdots < s_N \le s < t$ and let g represent any continuous function

$$g : \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{N\text{-times}} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{N\text{-times}} \to [0,1].$$

Proposition 4.2.4. *Process M_0 is a square-integrable $(\mathscr{F}_v)$-martingale.*

Proof. Since the function g is continuous thus Borel and the processes $\tilde{X}_n$ and $\tilde{W}$ are $(\tilde{\mathscr{F}}_v)$-adapted, the random variable

$$g\big(\tilde{X}_n(s_1), \dots, \tilde{X}_n(s_N), \tilde{W}(s_1), \dots, \tilde{W}(s_N)\big)$$

is $\tilde{\mathscr{F}}_s$-measurable. Then we observe

$$\tilde{M}_n(v) = \tilde{X}_n(v) - \psi - \int_0^v b_n\big(r, \tilde{X}_n(r)\big) \mathrm{d}r = \int_0^v \sigma_n\big(r, \tilde{X}_n(r)\big) \mathrm{d}\tilde{W}(r)$$

is a continuous local $(\tilde{\mathscr{F}}_v)$-martingale. It is even a square-integrable continuous $(\tilde{\mathscr{F}}_v)$-martingal, which follows from the estimate

$$\begin{aligned}
\tilde{\mathbb{E}}\langle \tilde{M}_n \rangle_v = \tilde{\mathbb{E}} \int_0^v \big\|\sigma_n\big(r, \tilde{X}_n(r)\big)\big\|^2 \mathrm{d}r \le \tilde{\mathbb{E}} \int_0^v (K^*)^2 \big(2 + \big\|\tilde{X}_n(r)\big\|\big)^2 \mathrm{d}r \le \\
\le 2(K^*)^2 \int_0^v \big(4 + \tilde{\mathbb{E}}\big\|\tilde{X}_n(r)\big\|^2\big) \mathrm{d}r \le \\
\le 2(K^*)^2 \big(4 + \tilde{\mathbb{E}} \sup_{0 \le r \le v} \big\|\tilde{X}_n(r)\big\|^2\big) v \le \\
\le 2(K^*)^2 \big(4 + C_*\big(1 + \tilde{\mathbb{E}}\|\psi\|^2\big)\big) v,
\end{aligned} \tag{4.5}$$

where the last estimate is a consequence of Theorem 4.1.4 and is finite for all $v \ge 0$. This gives

$$\begin{aligned}
&\tilde{\mathbb{E}}\big[g\big(\tilde{X}_n(s_1), \dots, \tilde{X}_n(s_N), \tilde{W}(s_1), \dots, \tilde{W}(s_N)\big) \tilde{M}_n(t)\big] = \\
&\quad = \tilde{\mathbb{E}}\big[\tilde{\mathbb{E}}\big[g\big(\tilde{X}_n(s_1), \dots, \tilde{X}_n(s_N), \tilde{W}(s_1), \dots, \tilde{W}(s_N)\big) \tilde{M}_n(t) \big| \tilde{\mathscr{F}}_s\big]\big] = \\
&\quad = \tilde{\mathbb{E}}\big[g\big(\tilde{X}_n(s_1), \dots, \tilde{X}_n(s_N), \tilde{W}(s_1), \dots, \tilde{W}(s_N)\big) \tilde{\mathbb{E}}\big[\tilde{M}_n(t) \big| \tilde{\mathscr{F}}_s\big]\big] = \\
&\quad = \tilde{\mathbb{E}}\big[g\big(\tilde{X}_n(s_1), \dots, \tilde{X}_n(s_N), \tilde{W}(s_1), \dots, \tilde{W}(s_N)\big) \tilde{M}_n(s)\big].
\end{aligned} \tag{4.6}$$

We recall that $(X_n, W_n) \overset{d}{\sim} (\tilde{X}_n, \tilde{W})$ which implies the identity of marginal distributions

$$X_n \overset{d}{\sim} \tilde{X}_n, \quad W_n \overset{d}{\sim} \tilde{W}.$$

Now we need to show that for $n \in \mathbb{N}$, $v \geq 0$ and $x \in \mathbb{R}^m$ the mapping

$$h \longmapsto h(v) - x - \int_0^v b_n(r, h(r))\mathrm{d}r$$

is continuous on $\mathscr{C}_m$. Fix $n \in \mathbb{N}$, $v \geq 0$, $h \in \mathscr{C}_m$ and let us show the continuity at h. Let $\varepsilon > 0$ be a given number, then for $f \in \mathscr{C}_m$

$$\begin{aligned}
\left\| h(v) - x - \int_0^v b_n(r, h(r))\mathrm{d}r - \left(f(v) - x - \int_0^v b_n(r, f(r))\mathrm{d}r \right) \right\| = \\
&= \left\| h(v) - f(v) - \int_0^v b_n(r, h(r))\mathrm{d}r + \int_0^v b_n(r, f(r))\mathrm{d}r \right\| \leq \\
&\leq \|h(v) - f(v)\| + \left\| \int_0^v \big(b_n(r, h(r)) - b_n(r, f(r)) \big)\mathrm{d}r \right\| \leq \\
&\leq \|h(v) - f(v)\| + \int_0^v \big\| b_n(r, h(r)) - b_n(r, f(r)) \big\| \mathrm{d}r \leq \\
&\leq \|h(v) - f(v)\| + K_n \int_0^v \|h(r) - f(r)\|\mathrm{d}r \leq \\
&\leq \sup_{0 \leq r \leq v} \|h(r) - f(r)\|(1 + K_n v).
\end{aligned}$$

If we find $L \in \mathbb{N},\ L \geq v$, what remains is to find $\delta > 0$ so that for all $f \in \mathscr{C}_m$ satisfying $\varrho(h, f) < \delta$ it holds

$$|||h - f|||_L < \frac{\varepsilon}{1 + K_n v}$$

and we are done.

The continutity of this mapping leads to its Borel measurability and for that reason $\tilde{M}_n$ and M_n are random variables, which have clearly continuous paths and so their values in $\mathscr{C}_m$. Moreover, they have identical distributions, because they are continuous functions od random variables with the same distribution. Analogically

$$\begin{aligned}
g\big(X_n(s_1), \ldots, X_n(s_N), W_n(s_1), \ldots, W_n(s_N)\big) \overset{d}{\sim} \\
\overset{d}{\sim} g\big(\tilde{X}_n(s_1), \ldots, \tilde{X}_n(s_N), \tilde{W}(s_1), \ldots, \tilde{W}(s_N)\big)
\end{aligned}$$

so

$$\begin{aligned}
g\big(X_n(s_1), \ldots, X_n(s_N), W_n(s_1), \ldots, W_n(s_N)\big) M_n(t) \overset{d}{\sim} \\
\overset{d}{\sim} g\big(\tilde{X}_n(s_1), \ldots, \tilde{X}_n(s_N), \tilde{W}(s_1), \ldots, \tilde{W}(s_N)\big) \tilde{M}_n(t),
\end{aligned}$$

$$\begin{aligned}
g\big(X_n(s_1), \ldots, X_n(s_N), W_n(s_1), \ldots, W_n(s_N)\big) M_n(s) \overset{d}{\sim} \\
\overset{d}{\sim} g\big(\tilde{X}_n(s_1), \ldots, \tilde{X}_n(s_N), \tilde{W}(s_1), \ldots, \tilde{W}(s_N)\big) \tilde{M}_n(s).
\end{aligned}$$

The identity of distributions particularly means the same expected values, hence from the first and the last part in the formula (4.6) we obtain

$$\begin{aligned}\mathbb{E}\big[g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)M_n(t)\big] &= \\ = \mathbb{E}\big[g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)M_n(s)\big]. \quad (4.7)\end{aligned}$$

Our aim here consists in verification similar equality as the latter using X_0, W_0, M_0 instead. First of all, we prove that $M_n(v) \to M_0(v)$ $\mathbb{P}$-a.s. for all $v \geq 0$. Fix $v \geq 0$. According to the construction, $(X_n, W_n) \to (X_0, W_0)$ $\mathbb{P}$-a.s., which leads to

$$\exists \Omega^* \in \mathscr{F},\ \mathbb{P}(\Omega^*) = 1 \quad \text{so that} \quad \forall \omega \in \Omega^* \quad X_n(\omega) \to X_0(\omega) \quad \text{in } \mathscr{C}_m,$$

so $X_n(\omega, v) \to X_0(\omega, v)$. Fix $\omega \in \Omega^*$ and show that $M_n(\omega, v) \to M_0(\omega, v)$, then the proof of this claim will be completed. Since

$$\begin{aligned}&\|M_n(\omega,v) - M_0(\omega,v)\| \leq \\ &\quad \leq \|X_n(\omega,v) - X_0(\omega,v)\| + \left\| \int_0^v \big(b_n\big(r, X_n(\omega,r)\big) - b_0\big(r, X_0(\omega,r)\big)\big)\mathrm{d}r \right\|,\end{aligned}$$

and the first summand clearly goes to zero, we only deal with the second one. We see that

$$\begin{aligned}&\left\| \int_0^v \big(b_n\big(r, X_n(\omega,r)\big) - b_0\big(r, X_0(\omega,r)\big)\big)\mathrm{d}r \right\| \leq \\ &\qquad \leq \int_0^v \Big(\big\|b_n\big(r, X_n(\omega,r)\big) - b_0\big(r, X_n(\omega,r)\big)\big\| + \\ &\qquad\qquad + \big\|b_0\big(r, X_n(\omega,r)\big) - b_0\big(r, X_0(\omega,r)\big)\big\| \Big)\mathrm{d}r.\end{aligned}$$

The first term in the integral converges to zero thanks to the local uniform convergence $b_n(r,\,.\,)$ to $b_0(r,\,.\,)$. Convergence of the second one to zero is a result of continuity of b_0 at x and the convergence $X_n(\omega, r) \to X_0(\omega, r)$. What remains now is to verify the interchange of limit and integral. As all the functions $b_n,\ n \in \mathbb{N}$ have the same linear growth in x and also b_0 does so, this task is reduced to find a function integrable on $[0, v]$ which dominates $\|X_n(\omega,\,.\,)\|$, indeed, let us continue with the previous estimate:

$$\begin{aligned}&\leq \int_0^v \Big(\big\|b_n\big(r, X_n(\omega,r)\big)\big\| + 2\big\|b_0\big(r, X_n(\omega,r)\big)\big\| + \big\|b_0\big(r, X_0(\omega,r)\big)\big\| \Big)\mathrm{d}r \leq \\ &\leq L \int_0^v \Big(\big\|X_n(\omega,r)\big\| + \big\|X_0(\omega,r)\big\| \Big)\mathrm{d}r,\end{aligned} \tag{4.8}$$

where L denotes a suitable constant. Again we use the convergence $X_n(\omega, r) \to X_0(\omega, r)$ and the fact that $\|X_0(\omega,\,.\,)\|$ is a continuous function so locally integrable. It is continuous for almost all $\omega \in \Omega^*$ but excluding a set of probability zero doesn't make any change for us. However, from now on, we use $\hat{\Omega}$ instead of Ω^*, where

$$\hat{\Omega} = \Omega^* \setminus \big\{\omega \in \Omega^*;\ X_0(\omega, \,.\,) \text{ is continuous}\big\}.$$

Thus, as the dominating function we can choose $L^*\|X_0(\omega, \,.\,)\|$, with L^* being a suitable constant.

In order to verify limits on both sides of (4.7), the uniform integrability of relevant systems of random variables is proved. For all $\omega \in \hat{\Omega}$ and for all $v \geq 0$ we have

$$X_n(\omega, v) \to X_0(\omega, v), \quad W_n(\omega, v) \to W_0(\omega, v), \quad M_n(\omega, v) \to M_0(\omega, v), \qquad n \to \infty,$$

which gives

$$g\big(X_n(\omega, s_1), \dots, X_n(\omega, s_N), W_n(\omega, s_1), \dots, W_n(\omega, s_N)\big) M_n(\omega, t) \longrightarrow$$
$$\longrightarrow g\big(X_0(\omega, s_1), \dots, X_0(\omega, s_N), W_0(\omega, s_1), \dots, W_0(\omega, s_N)\big) M_0(\omega, t), \qquad n \to \infty,$$

$$g\big(X_n(\omega, s_1), \dots, X_n(\omega, s_N), W_n(\omega, s_1), \dots, W_n(\omega, s_N)\big) M_n(\omega, s) \longrightarrow$$
$$\longrightarrow g\big(X_0(\omega, s_1), \dots, X_0(\omega, s_N), W_0(\omega, s_1), \dots, W_0(\omega, s_N)\big) M_0(\omega, s), \qquad n \to \infty,$$

and

$$g\big(X_n(s_1), \dots, X_n(s_N), W_n(s_1), \dots, W_n(s_N)\big) M_n(t) \longrightarrow$$
$$\longrightarrow g\big(X_0(s_1), \dots, X_0(s_N), W_0(s_1), \dots, W_0(s_N)\big) M_0(t), \quad \mathbb{P}\text{-a.s.}, \tag{4.9}$$

$$g\big(X_n(s_1), \dots, X_n(s_N), W_n(s_1), \dots, W_n(s_N)\big) M_n(s) \longrightarrow$$
$$\longrightarrow g\big(X_0(s_1), \dots, X_0(s_N), W_0(s_1), \dots, W_0(s_N)\big) M_0(s), \quad \mathbb{P}\text{-a.s.}. \tag{4.10}$$

Let us show that collections of random variables

$$\big\{\|Y_n\|^2 = \|g\big(X_n(s_1), \dots, X_n(s_N), W_n(s_1), \dots, W_n(s_N)\big) M_n(t)\|^2;\ n \in \mathbb{N}\big\}, \tag{4.11}$$

$$\big\{\|Z_n\|^2 = \|g\big(X_n(s_1), \dots, X_n(s_N), W_n(s_1), \dots, W_n(s_N)\big) M_n(s)\|^2;\ n \in \mathbb{N}\big\} \tag{4.12}$$

are uniformly integrable. For that purpose we prepare an estimate which makes us see that the set

$$\{\|Y_n\|,\ \|Z_n\|;\ n \in \mathbb{N}\}$$

is bounded in $\mathscr{L}^p$, $p > 2$. Furthermore, using a smaller p allows us to weaken the assumption about moments of the initial condition, as will be noticeable from the following. For the estimates, we will use the Burkholder-Davis-Gundy inequality 4.1.2 and the Hölder inequality.

$$\begin{aligned}\mathbb{E}\|Y_n\|^p &= \mathbb{E}\big\|g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)M_n(t)\big\|^p \le \\ &\le \mathbb{E}\|M_n(t)\|^p = \tilde{\mathbb{E}}\|\tilde{M}_n(t)\|^p = \tilde{\mathbb{E}}\Big\|\int_0^t \sigma_n\big(r,\tilde{X}_n(r)\big)\mathrm{d}\tilde{W}_r\Big\|^p \le \\ &\le C_p\,\tilde{\mathbb{E}}\Big(\int_0^t \big\|\sigma_n\big(r,\tilde{X}_n(r)\big)\big\|^2\mathrm{d}r\Big)^{\frac{p}{2}} \le C_p\,t^{p-1}\tilde{\mathbb{E}}\int_0^t \big\|\sigma_n\big(r,\tilde{X}_n(r)\big)\big\|^p\mathrm{d}r \le \\ &\le C_p\,t^{p-1}(K^*)^p\int_0^t \tilde{\mathbb{E}}\big(2+\big\|\tilde{X}_n(r)\big)\big\|\big)^p\mathrm{d}r \le \\ &\le 2^{p-1}C_p\,t^{p-1}(K^*)^p\int_0^t \big(2^p+\tilde{\mathbb{E}}\big\|\tilde{X}_n(r)\big\|^p\big)\mathrm{d}r \le \\ &\le 2^{p-1}C_p\,t^p(K^*)^p\big(2^p+C_*\big(1+\tilde{\mathbb{E}}\|\psi\|^p\big)\big),\end{aligned}$$

By the same computation we find out that

$$\begin{aligned}\mathbb{E}\|Z_n\|^2 &= \mathbb{E}\big\|g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)M_n(s)\big\|^2 \le \\ &\le \mathbb{E}\|M_n(s)\|^2 = \tilde{\mathbb{E}}\|\tilde{M}_n(s)\|^2 \le 2^{p-1}C_p\,s^p(K^*)^p\big(2^p+C_*\big(1+\tilde{\mathbb{E}}\|\psi\|^p\big)\big).\end{aligned}$$

Obviously, this implies the uniform integrability of system (4.11), set $\varepsilon = p-2$:

$$\begin{aligned}\sup_{n\in\mathbb{N}}\mathbb{E}\big[\|Y_n\|^2\,\mathbb{I}_{[\|Y_n\|\ge c]}\big] &= \sup_{n\in\mathbb{N}}\mathbb{E}\Big[\frac{\|Y_n\|^{2+\varepsilon}}{\|Y_n\|^\varepsilon}\,\mathbb{I}_{[\|Y_n\|\ge c]}\Big] \le \\ &\le \frac{1}{c^\varepsilon}\sup_{n\in\mathbb{N}}\mathbb{E}\big[\|Y_n\|^p\,\mathbb{I}_{[\|Y_n\|\ge c]}\big] \le \frac{1}{c^\varepsilon}\sup_{n\in\mathbb{N}}\mathbb{E}\|Y_n\|^2 \longrightarrow 0, \quad c\to\infty.\end{aligned}$$

In the case of system (4.12), the analogous approach gives the statement. So we have uniformly integrable sets of random variables (4.11),(4.12). Hence from the $\mathbb{P}$-a.s. convergence in (4.9), (4.10) we conclude the convergence in $\mathscr{L}^2$ and consequently in $\mathscr{L}^1$ so

$$\begin{aligned}&\mathbb{E}\big[g\big(X_0(s_1),\dots,X_0(s_N),W_0(s_1),\dots,W_0(s_N)\big)M_0(t)\big] = \\ &\qquad = \mathbb{E}\big[g\big(X_0(s_1),\dots,X_0(s_N),W_0(s_1),\dots,W_0(s_N)\big)M_0(s)\big]. \quad (4.13)\end{aligned}$$

Adaptability of M_0 to the filtration $(\mathscr{F}_v)$ is a straightforward consequence of the fact that $M_0(v)$ is a Borel function of the process X_0 up to time v. Setting $g\equiv 1$ in the previous part of the proof we obtain for all $v\ge 0$

$$M_n(v) \longrightarrow M_0(v) \quad \text{in } \mathscr{L}^2,$$

hence $M_0(v)\in\mathscr{L}^2$. To finish the proof, it is needed to show for all $A\in\mathscr{F}_s$

$$\mathbb{E}[\mathbb{I}_A M_0(t)] = \mathbb{E}[\mathbb{I}_A M_0(s)]. \quad (4.14)$$

The collection

$$\begin{aligned}\big\{C\subset\mathbb{R}^{2N};\ &\mathbb{E}\big[\mathbb{I}_C\big(X_0(s_1),\dots,X_0(s_N),W_0(s_1),\dots,W_0(s_N)\big)M_0(t)\big] = \\ &= \mathbb{E}\big[\mathbb{I}_C\big(X_0(s_1),\dots,X_0(s_N),W_0(s_1),\dots,W_0(s_N)\big)M_0(s)\big]\big\}\end{aligned}$$

forms a Dynkin system and contains the usual topology on $\mathbb{R}^m$. Indeed, fix $G \subset \mathbb{R}^{2N}$ open, then there exists a sequence of continuous functions

$$f_n : \mathbb{R}^{2N} \to [0,1], \quad n \in \mathbb{N},$$

so that $f_n \nearrow \mathbb{I}_G,\ n \to \infty$. One possible choice is

$$f_n(x) = \min\big\{n \operatorname{dist}(x, G^C), 1\big\}, \quad n \in \mathbb{N},$$

where $\operatorname{dist}(x, G^C)$ denotes the Euclidean distance between the point x and the complement of the set G. From the formula (4.13) it follows due to the Lévy monotone convergence theorem

$$\mathbb{E}\big[\mathbb{I}_G\big(X_0(s_1),\dots,X_0(s_N),W_0(s_1),\dots,W_0(s_N)\big)M_0(t)\big] = \\ = \mathbb{E}\big[\mathbb{I}_G\big(X_0(s_1),\dots,X_0(s_N),W_0(s_1),\dots,W_0(s_N)\big)M_0(s)\big].$$

Since any topology is closed to finite interesections and generates the Borel σ-algebra, using the Dynkin lemma we get

$$\mathbb{E}\big[\mathbb{I}_B\big(X_0(s_1),\dots,X_0(s_N),W_0(s_1),\dots,W_0(s_N)\big)M_0(t)\big] = \\ = \mathbb{E}\big[\mathbb{I}_B\big(X_0(s_1),\dots,X_0(s_N),W_0(s_1),\dots,W_0(s_N)\big)M_0(s)\big], \quad \forall B \in \mathscr{B}(\mathbb{R}^{2N})$$

which gives $\mathbb{E}[\mathbb{I}_A M_0(t)] = \mathbb{E}[\mathbb{I}_A M_0(s)]$ for all sets having the form

$$A = \big\{\big(X_0(s_1),\dots,X_0(s_N),W_0(s_1),\dots,W_0(s_N)\big) \in B\big\}, \quad B \in \mathscr{B}(\mathbb{R}^{2N}).$$

System of these sets for all $N \in \mathbb{N}$ is closed to finite intersections and is included in a Dynkin system of sets for which (4.14) is true. Because it also generates the σ-algebra $\sigma\big(X_0(r), W_0(r);\ r \le s\big)$, which differs from $\mathscr{F}_s$ only in $\mathbb{P}$-negligible sets, repeated application of the Dynkin lemma gives (4.14).

□

Proposition 4.2.5. *Process W_0 is an $(\mathscr{F}_v)$-Brownian motion.*

Proof. The proof falls naturally into two parts. In the first one we use the technique established in the previous proposition to show that W_0 is an $(\mathscr{F}_v)$-martingale. Afterwards, we apply Lévy's martingale characterization of Brownian motion 4.1.1 and finish the proof.

Process $\tilde{W}$ is a $(\tilde{\mathscr{F}}_v)$-Brownian motion thus a continuous $(\tilde{\mathscr{F}}_v)$-martingale and therefore

$$\tilde{\mathbb{E}}\big[g\big(\tilde{X}_n(s_1),\dots,\tilde{X}_n(s_N),\tilde{W}(s_1),\dots,\tilde{W}(s_N)\big)\tilde{W}(t)\big] = \\ = \tilde{\mathbb{E}}\big[g\big(\tilde{X}_n(s_1),\dots,\tilde{X}_n(s_N),\tilde{W}(s_1),\dots,\tilde{W}(s_N)\big)\tilde{W}(s)\big].$$

We recall that $X_n \overset{d}{\sim} \tilde{X}_n,\ W_n \overset{d}{\sim} \tilde{W}$ so

$$\mathbb{E}\big[g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)W_n(t)\big] = \\ = \mathbb{E}\big[g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)W_n(s)\big].$$

Since $W_n \to W_0$ $\mathbb{P}$-a.s. in $\mathscr{C}_k$, we need to prove the boundedness of

$$\big\{\big\|g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)W_n(t)\big\|;\ n\in\mathbb{N}\big\},$$
$$\big\{\big\|g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)W_n(s)\big\|;\ n\in\mathbb{N}\big\}$$

in $\mathscr{L}^2$. However, it is easily seen from

$$\mathbb{E}\big\|g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)W_n(t)\big\|^2 \leq \\ \leq \mathbb{E}\|W_n(t)\|^2 = \tilde{\mathbb{E}}\|\tilde{W}(t)\|^2 = kt,$$

$$\mathbb{E}\big\|g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)W_n(s)\big\|^2 \leq \\ \leq \mathbb{E}\|W_n(s)\|^2 = \tilde{\mathbb{E}}\|\tilde{W}(s)\|^2 = ks$$

and implies

$$\mathbb{E}\big[g\big(X_0(s_1),\dots,X_0(s_N),W_n(s_1),\dots,W_0(s_N)\big)W_0(t)\big] = \\ = \mathbb{E}\big[g\big(X_0(s_1),\dots,X_0(s_N),W_n(s_1),\dots,W_0(s_N)\big)W_0(s)\big].$$

Process W_0 is $(\mathscr{F}_v)$-adapted directly from definition of that filtration and again by choice $g\equiv 1$ we obtain for all $v\geq 0$ the integrability of $W_0(v)$. The martingale property is a consequence of the Dynkin lemma in the same manner as in the case of M_0.

We have already known that $\tilde{W}$ is a $(\tilde{\mathscr{F}}_v)$-Brownian motion, in addition $W_n \overset{d}{\sim} \tilde{W}$, $n\in\mathbb{N}$ which means that W_n, $n\in\mathbb{N}$ are Brownian motions. A clear consequence of convergence $W_n \to W_0$ $\mathbb{P}$-a.s. in $\mathscr{C}_k$ is convergence $W_n \to W_0$ in distribution. We deal with a constant sequence of probability measures so we get $W_0 \overset{d}{\sim} \tilde{W}$ and therefore W_0 is also a Brownian motion. Thus the tensor variation of W_0 is $\langle\!\langle W_0\rangle\!\rangle_v = vI$ which together with the claim that W_0 is an $(\mathscr{F}_v)$-martingale and the Lévy characterization of Brownian motion proves the proposition.

□

Proposition 4.2.6. *Processes*

$$(M_0^i)^2 - \int_0^\cdot \big\|\sigma_0^i\big(r,X_0(r)\big)\big\|^2 \mathrm{d}r, \quad i=1,\dots,m$$

are $(\mathscr{F}_v)$-martingales.[21]

Proof. Fix $i\in\{1,\dots,m\}$. Computation (4.5) implies that $\tilde{M}_n$ is a continuous $\mathscr{L}^2$-martingale with respect to the filtration $(\tilde{\mathscr{F}}_v)$, hence $(\tilde{M}_n^i)^2 - \langle\tilde{M}_n^i\rangle$ is a continuous $(\tilde{\mathscr{F}}_v)$-martingale and

$$\tilde{\mathbb{E}}\big[g\big(\tilde{X}_n(s_1),\dots,\tilde{X}_n(s_N),\tilde{W}(s_1),\dots,\tilde{W}(s_N)\big)\big((\tilde{M}_n^i)^2 - \langle\tilde{M}_n^i\rangle\big)(t)\big] = \\ = \tilde{\mathbb{E}}\big[g\big(\tilde{X}_n(s_1),\dots,\tilde{X}_n(s_N),\tilde{W}(s_1),\dots,\tilde{W}(s_N)\big)\big((\tilde{M}_n^i)^2 - \langle\tilde{M}_n^i\rangle\big)(s)\big].$$

[21] σ_n^i denotes i^{th} row of the matrix σ_n, $n\in\mathbb{N}_0$

Let us show that mapping

$$h \longmapsto \int_0^v \left\|\sigma_n^i(r,h(r))\right\|^2 \mathrm{d}r = \sum_{j=1}^k \int_0^v \left(\sigma_n^{ij}(r,h(r))\right)^2 \mathrm{d}r$$

is for every $n \in \mathbb{N}$ and $v \geq 0$ continuous on $\mathscr{C}_m$. Fix $h \in \mathscr{C}_m$, $n \in \mathbb{N}$ and let $\varepsilon > 0$ be given. Function σ_n is globally Lipschitz continuous in x uniformly in v and bounded, which is obvious from its construction in 3.0.18. Clearly this remains true also for its components σ_n^{ij}, $j = 1, \dots, k$. By 3.0.15 and 3.0.17, we have that $\left(\sigma_n^{ij}\right)^2$ is Lipschitz continuous in x uniformly in v and let us denote by L_n^{ij} its Lipschitz constant.

$$\begin{aligned}\left\|\sum_{j=1}^k \int_0^v \left(\sigma_n^{ij}(r,h(r))\right)^2 \mathrm{d}r - \sum_{j=1}^k \int_0^v \left(\sigma_n^{ij}(r,f(r))\right)^2 \mathrm{d}r\right\| &\leq \\ \leq \sum_{j=1}^k \int_0^v \left\|\left(\sigma_n^{ij}(r,h(r))\right)^2 - \left(\sigma_n^{ij}(r,f(r))\right)^2\right\| \mathrm{d}r &\leq \\ \leq v \sum_{j=1}^k L_n^{ij} \sup_{0\leq r\leq t} \|h(r) - f(r)\| &\end{aligned} \tag{4.15}$$

Fix $N \in \mathbb{N}$, $N \geq t$, find $\delta > 0$ so that for all $f \in \mathscr{C}_m$, $\varrho(h,f) < \delta$ the following holds

$$|\!|\!|h - f|\!|\!|_N < \frac{\varepsilon}{v \sum_{j=1}^k L_n^{ij}}$$

and the continuity at h is proved. With respect to this and the fact that $X_n \overset{d}{\sim} \tilde{X}_n$ we conclude

$$\langle \tilde{M}_n^i \rangle_\cdot = \int_0^\cdot \left\|\sigma_n^i\big(r,\tilde{X}_n(r)\big)\right\|^2 \mathrm{d}r \;\overset{d}{\sim}\; \int_0^\cdot \left\|\sigma_n^i\big(r,X_n(r)\big)\right\|^2 \mathrm{d}r.$$

Moreover, from $M_n \overset{d}{\sim} \tilde{M}_n$ it follows

$$(\tilde{M}_n^i)^2 - \langle \tilde{M}_n^i \rangle_\cdot \;\overset{d}{\sim}\; (M_n^i)^2 - \int_0^\cdot \left\|\sigma_n^i\big(r,X_n(r)\big)\right\|^2 \mathrm{d}r$$

so

$$\begin{aligned}&\mathbb{E}\Big[g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)\Big((M_n^i)_t^2 - \int_0^t \left\|\sigma_n^i\big(r,X_n(r)\big)\right\|^2 \mathrm{d}r\Big)\Big] = \\ &= \mathbb{E}\Big[g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)\Big((M_n^i)_s^2 - \int_0^s \left\|\sigma_n^i\big(r,X_n(r)\big)\right\|^2 \mathrm{d}r\Big)\Big].\end{aligned} \tag{4.16}$$

As the next step we prove that for all $v \geq 0$

$$(M_n^i)_v^2 - \int_0^v \left\|\sigma_n^i\big(r,X_n(r)\big)\right\|^2 \mathrm{d}r \longrightarrow (M_0^i)_v^2 - \int_0^v \left\|\sigma_0^i\big(r,X_0(r)\big)\right\|^2 \mathrm{d}r \quad \mathbb{P}\text{-a.s.}.$$

Fix $v \geq 0$. Previous explanation leads to $M_n(v) \to M_0(v)$ $\mathbb{P}$-a.s., from which we conclude $(M_n^i)_v^2 \to (M_0^i)_v^2$ $\mathbb{P}$-a.s.. So

$$\Big\|(M_n^i)_v^2 - \int_0^v \big\|\sigma_n^i\big(r, X_n(r)\big)\big\|^2 \mathrm{d}r - \Big((M_0^i)_v^2 - \int_0^v \big\|\sigma_0^i\big(r, X_0(r)\big)\big\|^2 \mathrm{d}r\Big)\Big\| \leq$$
$$\leq \big\|(M_n^i)_v^2 - (M_0^i)_v^2\big\| + \Big\|\int_0^v \big\|\sigma_n^i\big(r, X_n(r)\big)\big\|^2 - \big\|\sigma_0^i\big(r, X_0(r)\big)\big\|^2 \mathrm{d}r\Big\|.$$

Convergence of the integrand to zero as well as the existence of an integrable dominating function can be proved in the same manner as in (4.8).

In order to show

$$\mathbb{E}\Big[g\big(X_0(s_1), \dots, X_0(s_N), W_0(s_1), \dots, W_0(s_N)\big)\Big((M_0^i)_t^2 - \int_0^t \big\|\sigma_0^i\big(r, X_0(r)\big)\big\|^2 \mathrm{d}r\Big)\Big] =$$
$$= \mathbb{E}\Big[g\big(X_0(s_1), \dots, X_0(s_N), W_0(s_1), \dots, W_0(s_N)\big)\Big((M_0^i)_s^2 - \int_0^s \big\|\sigma_0^i\big(r, X_0(r)\big)\big\|^2 \mathrm{d}r\Big)\Big] \tag{4.17}$$

we again use the uniform integrability of terms inside the expected values in (4.16). It is enough to verify the boundedness of systems

$$\Big\{\Big\|g\big(X_n(s_1), \dots, X_n(s_N), W_n(s_1), \dots, W_n(s_N)\big)\Big((M_n^i)_t^2 - \int_0^t \big\|\sigma_n^i\big(r, X_n(r)\big)\big\|^2 \mathrm{d}r\Big)\Big\|; \quad n \in \mathbb{N}\Big\},$$

$$\Big\{\Big\|g\big(X_n(s_1), \dots, X_n(s_N), W_n(s_1), \dots, W_n(s_N)\big)\Big((M_n^i)_s^2 - \int_0^s \big\|\sigma_n^i\big(r, X_n(r)\big)\big\|^2 \mathrm{d}r\Big)\Big\|; \quad n \in \mathbb{N}\Big\}$$

in $\mathscr{L}^p$ for any $p > 1$.

$$\mathbb{E}\Big\|g\big(X_n(s_1), \dots, X_n(s_N), W_n(s_1), \dots, W_n(s_N)\big)\Big((M_n^i)_t^2 - \int_0^t \big\|\sigma_n^i\big(r, X_n(r)\big)\big\|^2 \mathrm{d}r\Big)\Big\|^p \leq$$
$$\leq \mathbb{E}\Big\|(M_n^i)_t^2 - \int_0^t \big\|\sigma_n^i\big(r, X_n(r)\big)\big\|^2 \mathrm{d}r\Big\|^p =$$
$$= \tilde{\mathbb{E}}\Big\|(\tilde{M}_n^i)_t^2 - \int_0^t \big\|\sigma_n^i\big(r, \tilde{X}_n(r)\big)\big\|^2 \mathrm{d}r\Big\|^p \leq$$
$$\leq 2^{p-1}\Big(\tilde{\mathbb{E}}\Big\|\int_0^t \sigma_n\big(r, \tilde{X}_n(r)\big)\mathrm{d}\tilde{W}(r)\Big\|^{2p} + \tilde{\mathbb{E}}\Big(\int_0^t \big\|\sigma_n\big(r, \tilde{X}_n(r)\big)\big\|^2 \mathrm{d}r\Big)^p\Big) \leq$$
$$\leq 2^{p-1}(C_{2p} + 1)\, t^{p-1}\, \tilde{\mathbb{E}} \int_0^t \big\|\sigma_n\big(r, \tilde{X}_n(r)\big)\big\|^{2p} \mathrm{d}r \leq$$
$$\leq 2^{p-1}(C_{2p} + 1)\, t^{p-1}\, (K^*)^{2p} \int_0^t \tilde{\mathbb{E}}\big(2 + \|\tilde{X}_n(r)\|\big)^{2p} \mathrm{d}r \leq$$
$$\leq 2^{3p-2}(C_{2p} + 1)\, t^{p-1}\, (K^*)^{2p} \int_0^t \Big(2^{2p} + \tilde{\mathbb{E}}\|\tilde{X}_n(r)\|^{2p}\Big) \mathrm{d}r \leq$$
$$\leq 2^{3p-2}(C_{2p} + 1)\, t^p\, (K^*)^{2p}\Big(2^{2p} + C_*\left(1 + \tilde{\mathbb{E}}\|\psi\|^{2p}\right)\Big),$$

$$\mathbb{E}\Big\|g\big(X_n(s_1),\ldots,X_n(s_N),W_n(s_1),\ldots,W_n(s_N)\big)\Big((M_n^i)_s^2-\int_0^s\big\|\sigma_n^i\big(r,X_n(r)\big)\big\|^2\mathrm{d}r\Big)\Big\|^p\le$$
$$\le 2^{3p-2}(C_{2p}+1)\,s^p\,(K^*)^{2p}\Big(2^{2p}+C_*\left(1+\tilde{\mathbb{E}}\|\psi\|^{2p}\right)\Big).$$

The formula (4.17) is a consequence of limiting in (4.16). What remains now is to show the adaptability, the integrability and the martingale property of

$$(M_0^i)^2-\int_0^{\cdot}\big\|\sigma_0^i\big(r,X_0(r)\big)\big\|^2\mathrm{d}r.$$

But all can be proved in the same manner as in the case of M_0 in 4.2.4.

□

Proposition 4.2.7. *Processes*

$$M_0^iW_0^j-\int_0^{\cdot}\sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}r,\quad i=1,\ldots,m,\ j=1,\ldots,k$$

are $(\mathscr{F}_v)$*-martingales.*

Proof. Fix $i\in\{1,\ldots,m\}$, $j\in\{1,\ldots,k\}$. Since $\tilde{M}_n$ and $\tilde{W}$ are $(\tilde{\mathscr{F}}_v)$-martingales, also $\tilde{M}_n^i\tilde{W}^j-\langle\tilde{M}_n^i,\tilde{W}^j\rangle$ does so and

$$\tilde{\mathbb{E}}\big[g\big(\tilde{X}_n(s_1),\ldots,\tilde{X}_n(s_N),\tilde{W}(s_1),\ldots,\tilde{W}(s_N)\big)\big(\tilde{M}_n^i\tilde{W}^j-\langle\tilde{M}_n^i,\tilde{W}^j\rangle\big)(t)\big]=$$
$$=\tilde{\mathbb{E}}\big[g\big(\tilde{X}_n(s_1),\ldots,\tilde{X}_n(s_N),\tilde{W}(s_1),\ldots,\tilde{W}(s_N)\big)\big(\tilde{M}_n^i\tilde{W}^j-\langle\tilde{M}_n^i,\tilde{W}^j\rangle\big)(s)\big].$$

Cross variation of the processes $\tilde{M}_n^i$, $\tilde{W}^j$ is

$$\langle\tilde{M}_n^i,\tilde{W}^j\rangle_v=\left\langle\sum_{l=1}^k\int_0^{\cdot}\sigma_n^{il}\big(r,\tilde{X}_n(r)\big)\mathrm{d}\tilde{W}^l(r),\tilde{W}^j\right\rangle_v=$$
$$=\sum_{l=1}^k\left\langle\int_0^{\cdot}\sigma_n^{il}\big(r,\tilde{X}_n(r)\big)\mathrm{d}\tilde{W}^l(r),\tilde{W}^j\right\rangle_v=\int_0^v\sigma_n^{ij}\big(r,\tilde{X}_n(r)\big)\mathrm{d}r.$$

By a similar method as the one used in (4.15) applied to function σ_n^{ij} [22], it is possible to show that for all $v\ge0$ the mapping

$$h\longmapsto\int_0^v\sigma_n^{ij}(r,h(r))\mathrm{d}r$$

is continuous on $\mathscr{C}_m$. So

$$M_n^iW_n^j-\int_0^{\cdot}\sigma_n^{ij}\big(r,X_n(r)\big)\mathrm{d}r\ \overset{d}{\sim}\ \tilde{M}_n^i\tilde{W}^j-\langle\tilde{M}_n^i,\tilde{W}^j\rangle$$

[22] globally Lipschitz continuous in x uniformly in v

and

$$\mathbb{E}\Big[g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)\Big(M_n^i W_n^j - \int_0^\cdot \sigma_n^{ij}\big(r,X_n(r)\big)\mathrm{d}r\Big)(t)\Big] =$$
$$= \mathbb{E}\Big[g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)\Big(M_n^i W_n^j - \int_0^\cdot \sigma_n^{ij}\big(r,X_n(r)\big)\mathrm{d}r\Big)(s)\Big]. \tag{4.18}$$

Fix $v \geq 0$. As $M_n(v) \to M_0(v)$ $\mathbb{P}$-a.s. and $W_n(v) \to W_0(v)$ $\mathbb{P}$-a.s., it follows that $M_n^i(v)W_n^j(v) \to M_0^i(v)W_0^j(v)$ $\mathbb{P}$-a.s.. Similar approach as in (4.8) gives convergence

$$\int_0^v \sigma_n^{ij}\big(r,X_n(r)\big)\mathrm{d}r \longrightarrow \int_0^v \sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}r \quad \mathbb{P}\text{-s.j.},$$

which leads to

$$M_n^i(v)W_n^j(v) - \int_0^v \sigma_n^{ij}\big(r,X_n(r)\big)\mathrm{d}r \longrightarrow M_0^i(v)W_0^j(v) - \int_0^v \sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}r \quad \mathbb{P}\text{-a.s..}$$

To prove the formula

$$\mathbb{E}\Big[g\big(X_0(s_1),\dots,X_0(s_N),W_0(s_1),\dots,W_0(s_N)\big)\Big(M_0^i W_0^j - \int_0^\cdot \sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}r\Big)(t)\Big] =$$
$$= \mathbb{E}\Big[g\big(X_0(s_1),\dots,X_0(s_N),W_0(s_1),\dots,W_0(s_N)\big)\Big(M_0^i W_0^j - \int_0^\cdot \sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}r\Big)(s)\Big] \tag{4.19}$$

it is needed to show boundedness of corresponding systems in $\mathscr{L}^p$ for any $p > 1$. We have

$$\begin{aligned}
\mathbb{E}\Big\|g\big(X_n(s_1),\dots,X_n(s_N),W_n(s_1),\dots,W_n(s_N)\big)\Big(M_n^i W_n^j - \int_0^\cdot \sigma_n^{ij}\big(r,X_n(r)\big)\mathrm{d}r\Big)(t)\Big\|^p &\leq \\
\leq \mathbb{E}\Big\|M_n^i(t)W_n^j(t) - \int_0^t \sigma_n^{ij}\big(r,X_n(r)\big)\mathrm{d}r\Big\|^p &= \\
= \tilde{\mathbb{E}}\Big\|\tilde{M}_n^i(t)\tilde{W}^j(t) - \int_0^t \sigma_n^{ij}\big(r,\tilde{X}_n(r)\big)\mathrm{d}r\Big\|^p &\leq \\
\leq 2^{p-1}\Big(\tilde{\mathbb{E}}\big\|\tilde{M}_n^i(t)\tilde{W}^j(t)\big\|^p + \tilde{\mathbb{E}}\Big\|\int_0^t \sigma_n^{ij}\big(r,\tilde{X}_n(r)\big)\mathrm{d}r\Big\|^p\Big) &= \\
= 2^{p-1}(I_1 + I_2).&
\end{aligned}$$

For the expression I_1 we use the Hölder inequality with the choice $\alpha = \frac{2}{p}$, $\beta = \frac{2}{2-p}$

$$I_1 \leq \Big(\tilde{\mathbb{E}}\big\|\tilde{M}_n^i(t)\big\|^2\Big)^{\frac{p}{2}} \Big(\tilde{\mathbb{E}}\big\|\tilde{W}^j(t)\big\|^{\frac{2p}{2-p}}\Big)^{\frac{2-p}{2}},$$

where the first parenthesis has an estimate independent of n according to the computation (4.5) applied to $\tilde{M}_n^i$. The expression I_2 is estimated in the standard way

$$I_2 \le t^{p-1}\tilde{\mathbb{E}}\int_0^t \big\|\sigma_n^{ij}\big(r\tilde{X}_n(r)\big)\big\|^p \mathrm{d}r \le 2^{p-1}t^{p-1}(K^*)^p\int_0^t \big(2^p+\tilde{\mathbb{E}}\big\|\tilde{X}_n(r)\big\|^p\big)\mathrm{d}r \le$$
$$\le 2^{p-1}t^p(K^*)^p\big(2^p+C_*\big(1+\tilde{\mathbb{E}}\|\psi\|^p\big)\big).$$

Apparently, the same method works for the second system. The process

$$M_0^iW_0^j-\int_0^{\cdot}\sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}r$$

in time v is a Borel function of X_0, W_0 up to time v, which leads to its adaptability to the filtration $(\mathscr{F}_v)$. The integrability and the martingale property are conclusions of (4.18) and (4.19).

□

From the previous propositions we deduce for $i=1,\dots,m$, $j=1,\dots,k$

$$\langle M_0^i\rangle=\int_0^{\cdot}\big\|\sigma_0^i\big(r,X_0(r)\big)\big\|^2\mathrm{d}r,\quad \langle M_0^i,W_0^j\rangle=\int_0^{\cdot}\sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}r.$$

So

$$\Big\langle M_0^i-\sum_{j=1}^k\int_0^{\cdot}\sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}W_0^j(r)\Big\rangle=\langle M_0^i\rangle+\sum_{j=1}^k\Big\langle\int_0^{\cdot}\sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}W_0^j(r)\Big\rangle-$$
$$-2\sum_{j=1}^k\Big\langle M_0^i,\int_0^{\cdot}\sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}W_0^j(r)\Big\rangle+$$
$$+2\sum_{\substack{j,l=1\\ j\neq l}}^k\Big\langle\int_0^{\cdot}\sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}W_0^j(r),\int_0^{\cdot}\sigma_0^{il}\big(r,X_0(r)\big)\mathrm{d}W_0^l(r)\Big\rangle=0.$$

In other words, quadratic variation of the process

$$M_0^i-\sum_{j=1}^k\int_0^{\cdot}\sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}W_0(r)$$

is zero function therefore almost all its paths are zero functions and

$$X_0^i(v)=X_0^i(0)+\int_0^v b_0^i\big(r,X_0(r)\big)\mathrm{d}r+\sum_{j=1}^k\int_0^v\sigma_0^{ij}\big(r,X_0(r)\big)\mathrm{d}W_0^j(r),\quad v\ge 0,\quad \mathbb{P}\text{-a.s.}$$

which means that we have found a weak solution of (4.1). Its initial distribution is clearly μ, since $X_n(0)\to X_0(0)$ $\mathbb{P}$-a.s. implies convergence in distribution and we deal with a constant sequence of laws.

□

Theorem 4.2.8. *Let $b : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{M}_{m\times k}$ be Borel functions which are continuous in x having linear growth in x:*

$$\exists K^* > 0 \quad \forall x \in \mathbb{R}^m \quad \forall t \geq 0 \quad \|b(t,x)\| \vee \|\sigma(t,x)\| \leq K^*(1 + \|x\|).$$

Suppose μ is a Borel probability measure on $\mathbb{R}^m$. Then there exists a weak solution of the equation (4.1) *with the initial distribution μ.*

Proof. The proof consists in a similar construction as was established in 4.2.2 but it is necessary to control the initial condition, which generally need not be integrable.

We shall use the following assertion, which is easy to prove: Let Y_1, Y_2 be two solutions of (4.1) on some stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, set $\Sigma = \{\omega \in \Omega;\ Y_1(0,\omega) = Y_2(0,\omega)\}$. Then

$$\mathbb{I}_\Sigma Y_1 = \mathbb{I}_\Sigma Y_2 \quad \mathbb{P}\text{-a.s.}, \tag{4.20}$$

that is

$$\mathbb{P}\big\{\omega \in \Sigma;\ \sup_{t\geq 0} \|Y_1(t,\omega) - Y_2(t,\omega)\| > 0\big\} = 0.$$

As in the proof of Theorem 4.2.2, there exist functions $b_n : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma_n : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{M}_{m\times k}$ which are globally Lipschitz continuous in x uniformly in t, have the same linear growth in x and for all $t \geq 0$ the function $b_n(t,\,.\,)$ and $\sigma_n(t,\,.\,)$ converge locally uniformly to $b(t,\,.\,)$, $\sigma(t,\,.\,)$ respectively.

Fixing a stochastic basis $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}})$ which satisfies the usual conditions and carries a k-dimensional $(\tilde{\mathscr{F}}_t)$-Brownian motion $\tilde{W}$ as well as a $\tilde{\mathscr{F}}_0$-measurable random variable ψ, $\tilde{\mathbb{P}}\circ\psi^{-1} = \mu$, and applying 4.1.5 we obtain processes $\tilde{X}_n$, $n \in \mathbb{N}$ which are strong solutions of (4.2).

It remains true that $\{\tilde{\mathbb{P}} \circ \tilde{X}_n^{-1};\ n \in \mathbb{N}\}$ is a tight set of measures on $\mathscr{C}_m$. Indeed, fix $\varepsilon > 0$ and find $R > 0$ so that $\tilde{\mathbb{P}}(\|\psi\| > R) < \frac{\varepsilon}{2}$. Denote by $\tilde{Y}_n$ the solution of (4.2) with initial condition $\tilde{Y}_n(0) = \mathbb{I}_{[\|\psi\|\leq R]}\,\psi$. By (4.20), we have $\mathbb{I}_{[\|\psi\|\leq R]}\tilde{X}_n = \mathbb{I}_{[\|\psi\|\leq R]}\tilde{Y}_n$ $\tilde{\mathbb{P}}$-almost surely. Since this initial condition is bounded, Proposition 4.2.3 applies and there exists a compact set $K \subset \mathscr{C}_m$ such that $\tilde{\mathbb{P}}(\tilde{Y}_n \notin K) < \frac{\varepsilon}{2}$. Hence

$$\begin{aligned}\big(\tilde{\mathbb{P}} \circ \tilde{X}_n^{-1}\big)(\mathscr{C}_m \setminus K) = \tilde{\mathbb{P}}\big(\tilde{X}_n \notin K\big) &= \\ &= \tilde{\mathbb{P}}\big(\tilde{X}_n \notin K, \tilde{X}_n = \tilde{Y}_n\big) + \tilde{\mathbb{P}}\big(\tilde{X}_n \notin K, \tilde{X}_n \neq \tilde{Y}_n\big) \leq \\ &\leq \tilde{\mathbb{P}}\big(\tilde{Y}_n \notin K\big) + \tilde{\mathbb{P}}\big(\tilde{X}_n \neq \tilde{Y}_n\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Consequently, $\{\tilde{\mathbb{P}} \circ (\tilde{X}_n, \tilde{W})^{-1};\ n \in \mathbb{N}\}$ is a tight set of measures on $\mathscr{C}_m \times \mathscr{C}_k$ and by similar approach as in 4.2.2, there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and random variables (X_n, W_n), $n \in \mathbb{N}_0$, satisfying

$$(X_n, W_n) \overset{d}{\sim} (\tilde{X}_n, \tilde{W}), \quad n \in \mathbb{N},$$

$$(X_n, W_n) \longrightarrow (X_0, W_0), \quad n \to \infty, \quad \mathbb{P}\text{-a.s..}$$

The set $D = \{R \in \mathbb{R}_+;\ \mathbb{P}(\|X_0(0)\| = R) > 0\}$ is at most countable. To obtain a contradiction, suppose that D is uncountable and denote $D_k = \{R \in \mathbb{R}_+;\ \mathbb{P}(\|X_0(0)\| = R) > \frac{1}{k}\},\ k \in \mathbb{N}$. Then $D = \cup_{k\in\mathbb{N}} D_k$ is a countable union of sets so there exists $k \in \mathbb{N}$ that D_k is uncountable. It follows

$$\mathbb{P}\big(\|X_0(0)\| \in D\big) \geq \mathbb{P}\big(\|X_0(0)\| \in D_k\big) = \sum_{R\in D_k} \mathbb{P}\big(\|X_0(0)\| = R\big) > \sum_{R\in D_k} \frac{1}{k} > 1.$$

Thus we may choose a sequence $R_k \notin D$, i.e. $\mathbb{P}(\|X_0(0)\| = R_k) = 0$, so that $R_k \nearrow \infty$.

Fix $k \in \mathbb{N}$. Let $\tilde{Z}_n$ be a solution to (4.2) with an initial condition $\tilde{Z}_n(0) = \mathbb{I}_{[\|\psi\|\leq R_k]}\psi$. Since it is bounded, we know that

$$\tilde{Z}_n - \tilde{Z}_n(0) - \int_0^{\cdot} b_n\big(r, \tilde{Z}_n(r)\big)\mathrm{d}r$$

is an $\mathscr{L}^p$-martingale for some (fixed) $p > 2$ and

$$\langle\!\langle \tilde{Z}_n \rangle\!\rangle = \int_0^{\cdot} (\sigma_n\sigma_n^*)\big(r, \tilde{Z}_n(r)\big)\mathrm{d}r.$$

Owing to $\tilde{\mathscr{F}}_0$-measurability of $\mathbb{I}_{[\|\psi\|\leq R_k]}$ we have that

$$\mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{Z}_n - \mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{Z}_n(0) - \int_0^{\cdot} \mathbb{I}_{[\|\psi\|\leq R_k]} b_n\big(r, \tilde{Z}_n(r)\big)\mathrm{d}r$$

is an $\mathscr{L}^p$-martingale and

$$\langle\!\langle \mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{Z}_n \rangle\!\rangle = \int_0^{\cdot} \mathbb{I}_{[\|\psi\|\leq R_k]}(\sigma_n\sigma_n^*)\big(r, \tilde{Z}_n(r)\big)\mathrm{d}r.$$

To proceed with our proof we would like to take advantage of the technique used in the proof of 4.2.2. For that purpose, define similarly

$$\tilde{M}_n(t) \;\;=\;\; \tilde{X}_n(t) - \psi - \int_0^t b_n\big(r, \tilde{X}_n(r)\big)\mathrm{d}r, \quad n \in \mathbb{N}.$$

By (4.20), it stands $\mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{X}_n = \mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{Z}_n$ so

$$\begin{aligned}\mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{M}_n(t) &= \mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{X}_n(t) - \mathbb{I}_{[\|\psi\|\leq R_k]}\psi - \int_0^t \mathbb{I}_{[\|\psi\|\leq R_k]} b_n\big(r, \tilde{X}_n(r)\big)\mathrm{d}r = \\ &= \mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{Z}_n(t) - \mathbb{I}_{[\|\psi\|\leq R_k]}\psi - \int_0^t \mathbb{I}_{[\|\psi\|\leq R_k]} b_n\big(r, \tilde{Z}_n(r)\big)\mathrm{d}r\end{aligned}$$

and $\mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{M}_n$ is an $\mathscr{L}^p$-martingale with

$$\langle\!\langle \mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{M}_n \rangle\!\rangle = \int_0^{\cdot} \mathbb{I}_{[\|\psi\|\leq R_k]}(\sigma_n\sigma_n^*)\big(r, \tilde{X}_n(r)\big)\mathrm{d}r.$$

Now, define $Z_n = \mathbb{I}_{[\|X_n(0)\|\leq R_k]} X_n,\ n \in \mathbb{N}_0$. From $\tilde{X}_n \overset{d}{\sim} X_n$ it follows that $\|\tilde{X}_n(0)\| = \|\psi\| \overset{d}{\sim} \|X_n(0)\|$ since it is a continuous function of identically distributed random variables. Consequently, $\mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{Z}_n \overset{d}{\sim} Z_n$. Next, it is needed to show that

$$\mathbb{I}_{[\|X_n(0)\|\leq R_k]} \longrightarrow \mathbb{I}_{[\|X_0(0)\|\leq R_k]}, \quad n \to \infty, \quad \mathbb{P}\text{-a.s..}$$

Towards this end, let us recall that $X_n \to X_0$ $\mathbb{P}$-a.s. in $\mathscr{C}_m$ and deduce that $X_n(0) \to X_0(0)$ $\mathbb{P}$-a.s., therefore $\|X_n(0)\| \to \|X_0(0)\|$ $\mathbb{P}$-a.s.. Denote by Ω^* the corresponding set where the convergence holds, $\mathbb{P}(\Omega^*) = 1$. Apparently, for those $\omega \in \Omega^*$ where $\|X_0(0)\| \neq R_k$, the claim is true. Since $\mathbb{P}(\|X_0(0)\| = R_k) = 0$ we need only to exclude a set of zero probability so it is true $\mathbb{P}$-almost surely. Hence

$$Z_n = \mathbb{I}_{[\|X_n(0)\|\leq R_k]} X_n \longrightarrow \mathbb{I}_{[\|X_0(0)\|\leq R_k]} X_0 = Z_0, \quad \mathbb{P}\text{-a.s..}$$

Analogously, we define

$$\begin{aligned} \mathscr{F}_t &= \sigma\big(\sigma(X_0(s), W_0(s); 0 \leq s \leq t) \cup \{N; \mathbb{P}(N) = 0\}\big), \quad t \geq 0, \\ M_n(t) &= X_n(t) - X_n(0) - \int_0^t b_n\big(r, X_n(r)\big)\mathrm{d}r, \quad n \in \mathbb{N}_0, \end{aligned}$$

and again we have for all $n \in \mathbb{N}_0$

$$\mathbb{I}_{[\|X_n(0)\|\leq R_k]} M_n(t) = Z_n(t) - Z_n(0) - \int_0^t \mathbb{I}_{[\|X_n(0)\|\leq R_k]} b_n\big(r, Z_n(r)\big)\mathrm{d}r.$$

Furthermore, the process Z_0 is clearly $(\mathscr{F}_t)$-adapted. Following the proof of Theorem 4.2.2 and applying Propositions 4.2.4, 4.2.6, 4.2.7 to the processes

$$\mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{M}_n,\ \mathbb{I}_{[\|\psi\|\leq R_k]}\tilde{Z}_n, \quad n \in \mathbb{N}, \qquad \mathbb{I}_{[\|X_n(0)\|\leq R_k]} M_n,\ Z_n, \quad n \in \mathbb{N}_0$$

we conclude that

$$\mathbb{I}_{[\|X_0(0)\|\leq R_k]} M_0 = \mathbb{I}_{[\|X_0(0)\|\leq R_k]} \int_0^\cdot \sigma_0\big(r, X_0(r)\big)\mathrm{d}W_0(r), \quad \mathbb{P}\text{-a.s.},$$

and by Proposition 4.2.5, W_0 is an $(\mathscr{F}_t)$-Brownian motion. This is true for all $k \in \mathbb{N}$ which means that X_0 is the desired solution.

□

Appendix A

Important theorems

The theorems presented in appendix were used throughout the work.

Theorem A.1 (Kolmogorov-Chentsov). *Fix $T > 0$. Suppose that a process $(X(t); 0 \leq t \leq T)$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ satisfies the condition*

$$\mathbb{E}\|X(t) - X(s)\|^a \leq c|t - s|^{1+b}, \quad 0 \leq s, t \leq T$$

for some constants $a, b, c > 0$. Then there exists a continuous modification $\tilde{X}$ of X, which is locally Hölder continuous with exponent $\beta \in [0, \frac{\beta}{\alpha})$.

Proof. see [KS88, Theorem 2.2.8] □

Theorem A.2 (Prochorov). *Suppose that S is a Polish space and let $\mathcal{M}$ be a collection of probability measures on S. Then $\mathcal{M}$ is tight if and only if it is relatively weakly compact.*

Proof. see [Bi99, Theorem 5.1, 5.2] □

Theorem A.3 (Skorokhod). *Suppose that S is a separable metric space and P_n, $n \in \mathbb{N}_0$ are probability measures on S such that P_n converge weakly to P_0. Then on some probability space there exist random variables X_n, $n \in \mathbb{N}_0$ with values in S, such that $X_n \overset{d}{\sim} P_n$, $n \in \mathbb{N}_0$ and $X_n \to X_0$ almost surely.*

Proof. see [Du04, Theorem 11.7.2] □

Theorem A.4 (Arzelà-Ascoli). *Let K be a compact metric space and let $\mathcal{F} \subset \mathscr{C}_m(K)$ be a set of functions. Then the following are equivalent:*

(i) $\mathcal{F}$ is relatively compact,

(ii) $\mathcal{F}$ is equicontinuous and bounded

Proof. see [Ru76] □

Notation

A^C	complement of a set A
$B(x, r)$	closed ball of radius r centered at x
T^*	adjoint operator to an operator T
$\mathrm{cl}\, A$	closure of a set A
$\mathbb{I}_A$	indicator of a set A
$\langle X, Y\rangle$	cross variation of processes X, Y
$\langle X\rangle$	quadratic variation of a local martingale X
$\langle\langle X\rangle\rangle$	tensor variation of a local martingale X
$\lceil b\rceil$	ceiling function of a number b
$\mathscr{D}(\mathbb{R}^m)$	the test function space on $\mathbb{R}^m$
$\mathscr{B}(X)$	the Borel σ-algebra on X
$\mathscr{C}^n$	the space of n-times continuously differentiable functions
$\mathscr{C}_m$	the space of continuous functions from $\mathbb{R}_+$ in $\mathbb{R}^m$
$\mathbb{M}_{L\times K}$	the space of $L \times K$ matrices
$\mathbb{N}$	the set of natural numbers, i.e. $\{1, 2, \dots\}$
$\mathbb{N}_0$	the set of nonnegative integers, i.e. $\{0, 1, 2, \dots\}$
$\mathcal{U}(x, r)$	open ball of radius r centered at x
$\omega_T(f, \delta)$	modulus of continuity of a function f
$\mathrm{supt}\, f$	support of a function f
$\mathrm{D}_h f(x)$	directional derivative of a function f in a direction h at x
$\mathrm{D}f(x)(h)$	Fréchet derivative of a function f at x apllied to h
$f * g$	convolution of functions f, g

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