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DOCTORAL THESIS

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On fluids with pressure-dependent viscosity flowing through a porous medium

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Tekutiny s viskozitou závislou na tlaku proudící porézním prostředím

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Abstrakt: Experimentální údaje jasně dokazují, že viskozita tekutin podléhá značným změnám v závislosti na tlaku. Toto pozorování vede k zobecnění známých modelů, např. Darcyho zákona, Stokesova zákona, Navierových-Stokesových rovnic aj. Tato práce se zabývá třemi takovými modely, které posloužily za podklad k trojici publikovaných článků. Jejich sjednocujícím tématem je vývoj existenční teorie a nalezení slabého řešení systémů parciálních diferenciálních rovnic pocházejících z uvažovaných modelů.

Klíčová slova: Nestlačitelná tekutina, viskozita závislá na tlaku, existenční teorie, proudění porézním prostředím.

Title: On fluids with pressure-dependent viscosity flowing through a porous medium

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Abstract: Experimental data convincingly show that viscosity of a fluid may change significantly with pressure. This observation leads to various generalizations of well-known models, like Darcy's law, Stokes' law or the Navier-Stokes equations, among others. This thesis investigates three such models in a series of three published papers. Their unifying topic is development of existence theory and finding a weak solution to systems of partial differential equations stemming from the considered models.

Keywords: Incompressible fluid, pressure-dependent viscosity, existence theory, flow through porous media.

Contents

Introduction

The topic of flows through porous media is encountered in many areas of engineering and science, e.g. ground water hydrology, reservoir engineering or chemical engineering. It is a subject of intense interest and over time it has emerged as a distinct field of study. But what is a porous medium, actually?

When talking about a porous medium, we mean a material consisting of a solid matrix with interconnected void – the pores. It is the interconnectedness of the pores that allows one or more fluids to flow through the material. Depending on how many fluids saturate the void, we can classify these flows as single-phase (the simplest situation), two-phase (e.g. a liquid and a gas), multi-phase, etc.

Examples of porous media are manifold and can be found at virtually every step — just consider materials like sand, wood, concrete, pastry or even human lungs — and for this sake, their analysis is naturally very attractive. However, from their very definition and variability of examples, one becomes immediately wary. Since the individual pores are expected to be highly irregular, so will be on the pore scale (the microscopic scale) the flow quantities such as velocity or pressure. But these quantities are typically measured over areas spanning many pores and thus space-averaged (macroscopic) quantities then tend to change regularly in time and space, and they are therefore conformable to theoretical treatment.

The balance equations governing macroscopic variables, e.g. the celebrated Darcy's law, are derived similarly – one begins with standard laws abided by the fluid and by means of averaging over volumes containing many pores, the macroscopic equations are acquired. There is a couple of ways how to proceed with the averaging, such as spatial, statistical or, in case of materials with periodic structure, homogenization; see [13] for an elaborate and richly referenced exposition.

A key feature of porous media is their porosity – the fraction of the total volume of the medium that is filled with the void space. We tacitly assume that all the void space is connected (otherwise a different kind of porosity, the so-called effective porosity, is to be defined [13]). As a point of interest, porosity of natural media does not normally exceed 0.6, while in artificial materials it can approach the value of 1, as seen in reticulated metal foams. The porosity is an important consideration in a rock or sedimentary layer in cases when one tries to evaluate the potential volume of water (or hydrocarbons) it may contain.

From a different point of view, time and again it has been evidenced that viscosity of many liquids depends on the pressure. Recognition of this feature dates back to G. G. Stokes [12] according to whom viscosity could be assumed to be constant only for a certain class of flows, e.g. for pipe flows with moderate pressures and pressure gradients. However, there are technologically important processes based on flows through porous media, such as enhanced oil recovery or geological carbon dioxide sequestration, which do not belong into this class of flows. The reason is that one has to deal with a high pressure range [12], in which viscosity of a fluid in question may change dramatically. According to Barus' experiments [2], dependence of viscosity on the pressure may be even exponential, which hints at suitability of including the pressure-dependence into our consideration when analysing pertinent models.

Dissertation summary

This thesis consists of three original articles completed during my third and fourth year of doctoral studies in 2013–2015. Due to the fact that all three have already been published or are to be published soon, with full bibliographic details readily available, they are presented exactly as in their accepted versions, changing the text format only for the sake of unity.

Each of the papers investigates one specific model for fluids with a pressure-dependent viscosity in the context of existence of weak solutions. Interestingly enough, the proofs might seem very similar to each other, at least principally. More concretely, in the background one always finds an approximation scheme based on Galerkin's method and the *quasi-compressible* approximation, replacing the incompressibility condition with an elliptic problem for the pressure (for more information see any of the papers further on). After that, certain uniform bounds on the approximate solutions are derived and, by standard compactness arguments, presence of (usually) weakly convergent subsequences is justified; the weak limit being a candidate for the final solution. However, as the studied problems are highly nonlinear, passing to limit in the equations and justification that the weak limit really is a solution tends to represent an arduous task. It is the identification of weak limits of nonlinear terms that makes each of our papers stand out rather uniquely in its own light, in ways that cannot be carried over between the individual works.

Specifically, this thesis incorporates the following articles:

1. Miroslav Bulíček, Josef Málek and Josef Žabenský. A generalization of the Darcy-Forchheimer equation involving an implicit, pressure-dependent relation between the drag force and the velocity, *J. Math. Anal. Appl.* IF[1](1.120), 424: 785–801, 2015.

   This work marks the deepest excursion into the implicit constitutive theory from the paper triplet. We investigate a generalized Darcy's law, where the *drag term* depends on the velocity as well as on the pressure in a non-explicit, implicit way. Quite curiously, it wasn't the existential proof that was the most challenging part, but rather finding optimal, the most general while still manageable conditions for the relation between the drag, the velocity and the pressure so that the proof, afterwards not so thought-provoking yet quite technical, would work.

   Potency of the existence result is significantly amplified by the minimum and maximum principle for the pressure, also formulated and proved in the article. As a particular, interesting and surprising result, we show that under certain conditions on data, there is a solution to the generalized Darcy equation with the drag depending on the pressure exponentially.

2. Miroslav Bulíček, Josef Málek and Josef Žabenský. On generalized Stokes' and Brinkman's equations with a pressure- and shear-dependent viscosity and drag coefficient, *Nonlinear Anal. Real World Appl.* IF(2.519), 26: 109–132, 2015.

   Looking from the perspective of the previous article, in the second paper we abandoned the requirement for an implicit relation between the drag coefficient, the velocity and the pressure, leaving the drag a function of the latter two quantities (and the shear rate, however). On the other hand, we also added a nonlinear diffusion term, hence the name of the work.

   We established a large-data existence theory for this generalized Brinkman problem with the viscosity and drag coefficients depending on the pressure and the shear rate. To the best of our knowledge, a PDE analysis for similar problems with a pressure- and shear-rate dependent drag had not been investigated before. Within

[1]Impact Factor as of 2014.

the setting considered, even for a generalized Stokes problem (i.e. with zero drag) we established new results when the model parameter r (analogous to the power exponent in the well-known power-law fluids) equals 2, thus improving [4, 7] along the way.

As a footnote, earlier studies concerning PDE analyses of a generalized Stokes problem with the pressure and shear-rate dependent viscosity in general bounded domains suffered a serious drawback. A certain bounding parameter used to be restricted by a constant depending on the geometry of the flow domain. This severe constraint was removed here. The theory presented in this work thus holds under the same restrictions as the theory developed for an (idealized) spatially periodic problem in [10].

3. Miroslav Bulíček and Josef Žabenský. Large data existence theory for unsteady flows of fluids with pressure- and shear-dependent viscosities, *Nonlinear Anal.* IF(1.327), 127: 94–127, 2015.

   Still interested in the model studied in the preceding paper, in the third work we dropped out the drag term completely. Thus we actually cheatingly left the territory of flows through porous media purely for flows of non-Newtonian fluids. From the analytical point of view, the drag did not add much complexity into the problem and its retention would only make proofs unnecessarily messier; the analysis still would have worked though. On the other hand, we added effects of convection and stepped to the unsteady case, thus treating virtually the evolutionary version of [4]. Compared to the previous work, visually practically steady-state version of the problem studied here, we were able to contain the interesting situation $r = 2$ again. However, where the proofs for the steady problem were rather complicated, the time-dependent generalization of mathematical tools and techniques led to a hefty piece of analysis.

   Although the set objective was eventually met with complete success, as opposed to the previous paper we were unable to lift the constraint stemming from the domain geometry discussed above. To spare the reader fumbling in the dark later, let us juxtapose the corresponding situations in the two papers, to demonstrate at least intuitively what goes amiss. In both articles, we used sequences of certain auxiliary functions (see the papers for notation if needed),

$$\begin{aligned} &\{u^n\}_{n\in\mathbb{N}} \subset W^{2,p}(\Omega), && u^n \longrightarrow 0 \quad \text{weakly in } W^{2,p}(\Omega), \\ &\{\varphi^n\}_{n\in\mathbb{N}} \subset L^p(0,T;W^{2,p}(\Omega)), && \varphi^n \longrightarrow 0 \quad \text{weakly in } L^p(0,T;W^{2,p}(\Omega)), \end{aligned}$$

   respectively, for some $p > 1$. Assuming that Ω is Lipschitz, the Rellich-Kondrachov theorem lets us infer strong convergence of $\{u^n\}$ but in no way are we able to justify an analogy thereof for $\{\varphi^n\}$:

$$\begin{aligned} u^n &\longrightarrow 0 \quad \text{strongly in } W^{1,p}(\Omega), \\ \varphi^n &\not\longrightarrow 0 \quad \text{strongly in } L^p(0,T;W^{1,p}(\Omega)). \end{aligned}$$

   This loss of compactness in the Bochner setting, attributable to lack of information about the time derivative of the pressure, spawns additional polluting terms in our computations. In the end we are then made to bound a certain parameter depending on Ω.

Conclusions

Let us recall the famous Hadamard's definition of a well-posed problem [8], according to which a mathematical model describing physical phenomena is well-posed, if each of the following conditions holds:

1. There exists at least one solution to the problem.
2. The solution is unique.
3. The solution is stable with respect to data.

Well-posedness of a problem is a highly desirable quality in general and with the emergence of computer simulations, its significance has increased significantly. Although the three conditions must hold simultaneously, the first one, concerning very existence, apparently protrudes as somewhat superior. In particular, simulations of a mathematical model without a guaranteed solution is a very risky endeavor to say the least and results should be handled with utmost circumspection.

This thesis deals exclusively with the first condition of Hadamard's definition. Not that we deemed the other two unworthy of our attention; we were simply unable to answer them. We would like to note, nevertheless, that the individual papers should not be narrow-mindedly shrunken into mere statements of the principal existence theorems. Regardless of benefit that these theorems deliver themselves in helping answer the question of well-posedness, there is also another thing to take into account—namely the proofs.

More concretely, dissected into individual fragments, the proofs advance by means of relatively well-known tools, such as the Biting lemma [3], the Div-Curl lemma [11, 14] or various versions of the Lipschitz truncation of Sobolev functions [1, 5, 6]. However, their combinations and timing of deployment may often look surprising and maybe even inspiring for posterity; see the second paper for the best illustration thereof.

Without acknowledging it in the articles, on plentiful occasions in the proofs, we are not aware of any alternative method of how to reach our goals, maybe quite as one would expect. However, there are situations where it would be possible to tackle issues differently. As a case in point serves the first article dealing with an implicit dependence between three quantities. In order to construct approximate solutions, we turn to mollification of the graph given by the implicit relation, thus stepping into the purview of ordinary functions. Another path could be to rotate the graph instead, making it a graph of a (completely different) function as well. Perhaps the most interesting alternative approach would be based on Kakutani's fixed-point theorem [9] – a generalization of Brouwer's fixed-point theorem accommodated for set-valued functions. This tool, quite surprising to invoke in this context, would let us retain the implicit nature of the problem throughout.

Let us conclude with reiterating what is the key feature of this thesis. We were able to mathematically prove existence of (weak) solutions to three very general models, which results are completely new. Each of the studied models pertains to reality and observed phenomena (e.g. Barus' law or Bingham fluids in the first paper), thus not serving purely our own self-indulgence. Morevover, it is quite foolhardy or even senseless to numerically compute something not proved to exist in the first place. It is therefore author's hope that our findings might find practical use in the foreseeable future.

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A generalization of the Darcy-Forchheimer equation involving an implicit, pressure-dependent relation between the drag force and the velocity

Miroslav Bulíček, Josef Málek and Josef Žabenský

Journal of Mathematical Analysis and Applications, 424: 785–801, 2015.

Abstract

We study mathematical properties of steady flows described by the system of equations generalizing the classical porous media models of Darcy's and Forchheimer's. The considered generalizations are outlined by implicit relations between the drag force and the velocity, that are in addition parametrized by the pressure. We analyze such drag force–velocity relations which are described through a maximal monotone graph varying continuously with the pressure. Large-data existence of a solution to this system is established, whereupon we show that under certain assumptions on data, the pressure satisfies a maximum or minimum principle, even if the drag coefficient depends on the pressure exponentially.

Keywords

Darcy-Forchheimer equation, pressure dependent material coefficient, implicit constitutive theory, maximal monotone graph, existence theory, maximum/minimum principle

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1.1 Introduction

1.1.1 Setting

Our aim is to develop a mathematical theory for steady, isochoric flows through a saturated porous medium described as the problem of finding a triplet

$$(\boldsymbol{m}, \boldsymbol{v}, p) : \Omega \to \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$$

solving

$$\left.\begin{aligned} \nabla p + \boldsymbol{m} &= \boldsymbol{f} && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} &= 0 && \text{in } \Omega, \\ \boldsymbol{h}(\boldsymbol{m}, \boldsymbol{v}, p) &= \boldsymbol{0} && \text{in } \Omega, \\ (\boldsymbol{v} - \boldsymbol{v}_0) \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_1, \\ p - p_0 &= 0 && \text{on } \Gamma_2. \end{aligned}\right\} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^d$ is supposed to be a Lipschitz domain with an outer normal $\boldsymbol{n}$ and $\Gamma_{1,2} \subset \partial\Omega$ are relatively open parts of the boundary such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma_1 \cup \Gamma_2} = \partial\Omega$. The reader may know Γ_1 as the *exterior boundary* and Γ_2 as the *accessible boundary*, see [2]. A velocity field $\boldsymbol{v}_0 : \Omega \to \mathbb{R}^d$ is given to dictate the normal component of $\boldsymbol{v}$ on Γ_1, as well as $p_0 : \Omega \to \mathbb{R}$, prescribing the boundary pressure on Γ_2. Known external body forces are contained in $\boldsymbol{f} : \Omega \to \mathbb{R}^d$. Throughout the paper there will often appear a real number r, always satisfying $1 < r < \infty$, and we define $r' := r/(r-1)$. Traces and normal traces are not denoted differently from the original functions, i.e. we write, for example, $p_0 \in W^{1,r'}(\Omega)$ as well as $p_0 \in L^\infty(\Gamma_2)$.

The quantity $\boldsymbol{h} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ in $(1.1)_3$ is a given continuous function and we will make the following identification:

$$\boldsymbol{h}(\boldsymbol{m}, \boldsymbol{v}, p) = \boldsymbol{0} \quad \iff \quad (\boldsymbol{m}, \boldsymbol{v}, p) \in \boldsymbol{A},$$

where $\boldsymbol{A}$ denotes a *maximal monotone r-graph* with respect to $\boldsymbol{m}$ and $\boldsymbol{v}$ that is in addition parametrized by p. This means $\boldsymbol{A} \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ satisfies each of the conditions listed below:

(A1) *inclusion of the origin*

$$\forall p \in \mathbb{R} : (\boldsymbol{0}, \boldsymbol{0}, p) \in \boldsymbol{A},$$

(A2) *monotonicity*

$$\forall (\boldsymbol{m}_1, \boldsymbol{v}_1, p), (\boldsymbol{m}_2, \boldsymbol{v}_2, p) \in \boldsymbol{A} : (\boldsymbol{m}_1 - \boldsymbol{m}_2) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) \geq 0,$$

(A3) *maximality*

$$(\boldsymbol{m}', \boldsymbol{v}', p) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R},$$
$$\forall (\boldsymbol{m}, \boldsymbol{v}, p) \in \boldsymbol{A} : (\boldsymbol{m}' - \boldsymbol{m}) \cdot (\boldsymbol{v}' - \boldsymbol{v}) \geq 0 \Rightarrow (\boldsymbol{m}', \boldsymbol{v}', p) \in \boldsymbol{A},$$

(A4) (r, r')*-coercivity for* $\boldsymbol{v}$ *and* $\boldsymbol{m}$

$$\exists\, c_1 > 0, c_2 \geq 0\ \forall (\boldsymbol{m}, \boldsymbol{v}, p) \in \boldsymbol{A} : \boldsymbol{m} \cdot \boldsymbol{v} \geq c_1(|\boldsymbol{v}|^r + |\boldsymbol{m}|^{r'}) - c_2,$$

(A5) *existence of a Carathéodory selection*, i.e. $\boldsymbol{m}^* : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ such that

(i) $\boldsymbol{m}^*(\cdot, p) : \mathbb{R}^d \to \mathbb{R}^d$ is measurable for every $p \in \mathbb{R}$,

(ii) $\boldsymbol{m}^*(\boldsymbol{v}, \cdot) : \mathbb{R} \to \mathbb{R}^d$ is continuous for a.e. $\boldsymbol{v} \in \mathbb{R}^d$,

(iii) $\forall (\boldsymbol{v}, p) \in \mathbb{R}^d \times \mathbb{R} : (\boldsymbol{m}^*(\boldsymbol{v}, p), \boldsymbol{v}, p) \in \boldsymbol{A}$,

(iv) $\exists\, c > 0\, \forall (\boldsymbol{v}, p) \in \mathbb{R}^d \times \mathbb{R} : |\boldsymbol{m}^*(\boldsymbol{v}, p)| \leq c(1 + |\boldsymbol{v}|^{r-1})$.

1.1.2 Motivation and examples

The problem (1.1) describes steady (slow) flows of fluids through porous media (see for example Nield and Bejan [25]). It can be also viewed as a special case in the hierarchical development of the theory of interacting continua (as presented in Rajagopal [28]), where we ignore the viscous effects within the fluid but take into account only the drag due to the flow which is a consequence of the friction at the solid pores as the fluid flows. This leads to the relation between $\boldsymbol{m}$, representing the interaction force (linear momentum) between a fluid and a rigid solid, and the velocity of the fluid $\boldsymbol{v}$. Since $\boldsymbol{v}$ is also the relative velocity between the solid and the liquid, it is frame-indifferent. Taking the simplest case $\boldsymbol{m} = \alpha \boldsymbol{v}$ for certain $\alpha > 0$, one obtains a well known *Darcy's law* for an isotropic medium. Its linearity in the seeping velocity $\boldsymbol{v}$ does not relate well to reality for other than *sufficiently small* velocities [25, 31] and one is driven to a non-linear extension of the form $\boldsymbol{m} = \alpha(|\boldsymbol{v}|)\boldsymbol{v}$, known as *(Darcy-)Forchheimer's equation* if α is an affine function. Moving on to $\boldsymbol{m} = \alpha(p, |\boldsymbol{v}|)\boldsymbol{v}$ as a means of capturing a pressure-related viscosity [18, 32] yields a generalized *Darcy-Forchheimer's model.* As Rajagopal [27] argued, it turns out that not even such setting is always satisfactory in mathematical modelling and one is driven to relate $\boldsymbol{m}$, $\boldsymbol{v}$ and p implicitly, hence $(1.1)_3$.

Apart from Darcy's or Darcy-Forchheimer's models, which are somewhat uninteresting in regard to our setting emphasizing p-dependent interactions, a prime example satisfying **(A1)**–**(A5)** that the reader might have in mind is $\boldsymbol{A}$ with $\boldsymbol{m}$ given as e.g.

$$\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{v}, p) = \alpha(p)|\boldsymbol{v}|^{r-2}\boldsymbol{v}, \tag{1.2}$$

with $r > 1$ and $\alpha \in \mathcal{C}(\mathbb{R})$, satisfying also $0 < \inf_{\mathbb{R}} \alpha \leq \sup_{\mathbb{R}} \alpha < \infty$. Another simple example falling within this category is

$$|\boldsymbol{m}| \leq \sigma(p) \Leftrightarrow \boldsymbol{v} = \boldsymbol{0} \quad \text{and} \quad |\boldsymbol{m}| > \sigma(p) \Leftrightarrow \boldsymbol{m} = \sigma(p)\frac{\boldsymbol{v}}{|\boldsymbol{v}|} + \gamma(p)|\boldsymbol{v}|^{r-2}\boldsymbol{v}, \tag{1.3}$$

with $\sigma(p)$ and $\gamma(p)$ having the same properties as $\alpha(p)$ above. This situation resembles Herschel-Bulkley responses between the Cauchy stress and the velocity gradient in the constitutive theory of non-Newtonian fluids, or Bingham responses in the special case $r = 2$. Note that the relation (1.3) can be rewritten equivalently as

$$\big(\gamma(p)\big)^{\frac{1}{r-1}}\boldsymbol{v} = \big((\boldsymbol{m} - \sigma(p))_+\big)^{\frac{1}{r-1}}\frac{\boldsymbol{m}}{|\boldsymbol{m}|},$$

which corresponds to $\boldsymbol{h}(\boldsymbol{m}, \boldsymbol{v}, p) = \boldsymbol{0}$ with

$$\boldsymbol{h}(\boldsymbol{m}, \boldsymbol{v}, p) = \big(\gamma(p)\big)^{\frac{1}{r-1}}\boldsymbol{v} - \big((\boldsymbol{m} - \sigma(p))_+\big)^{\frac{1}{r-1}}\frac{\boldsymbol{m}}{|\boldsymbol{m}|}.$$

Here, for $z \in \mathbb{R}$ we use $z_+ := \max\{z, 0\}$ to denote its positive part. See Bulíček et al. [27] for an analogon thereof in the case of Bingham fluids.

The two given examples, with α, σ and γ bounded from above, pale into insignificance in the face of interactions of the form

$$\boldsymbol{m}(\boldsymbol{v}, p) = \alpha_1 \exp(\alpha_2 p)\boldsymbol{v}, \qquad \alpha_{1,2} > 0, \tag{1.4}$$

that actually lie at the centre of our attention here. Let us recall that even for simple incompressible fluids, it is known that the viscosity changes significantly at high pressures. In fact, Barus' experimental study (see [3]) led him to the conclusion that the viscosity changes with the pressure exponentially (similarly as the coefficient relating $\boldsymbol{m}$ and $\boldsymbol{v}$ in (1.4)). For flows of fluid through rigid media, the internal fluid friction is frequently neglected as the friction between the fluid and solid is dominant. If such flows take place at high pressures, then one needs to involve the (exponential) dependence of the coefficient relating $\boldsymbol{m}$ and $\boldsymbol{v}$ on the pressure; see Nakshatrala and Rajagopal [24] for more details. Even if the coefficient α_2 in (1.4) is very small ($\alpha_2 \sim 10^{-5}$; see [3]), it is evident that a choice like (1.4) is beyond the purview of **(A4)** and **(A5)**$_{\text{(iv)}}$. Luckily enough, this case and those akin can also be included under certain circumstances into the existence theory developed in this paper; see Sect. 1.5.

We may also take a perturbation of (1.4) in a form

$$\boldsymbol{m}(\boldsymbol{v}, p) = \max\{\alpha_1, \alpha_1 \exp(\alpha_2 p)\}\boldsymbol{v}, \tag{1.5}$$

for existence theory of which we will be able to slightly slacken our hypotheses, see Remark 1.5.2. The reason is that inserting this choice into (1.2) with $r = 2$, the condition $\inf_{\mathbb{R}} \alpha > 0$ is met trivially.

1.1.3 Results

Within the setting of **(A1)**–**(A5)** we are able to establish the existence of a solution to the problem (1.1) fulfilling the first three equations pointwise (almost everywhere) in Ω; see Theorem 1.3.1 below. Although this theorem does not include the models of our main interest such as (1.4), it provides a tool how (1.4) can be analyzed, together with a maximum/minimum principle that is well-known for Darcy's model but is newly discovered for cases like (1.4) in this paper. The maximum/minimum principle is presented in Theorem 1.4.1 and its combination with Theorem 1.3.1 then culminates in Theorem 1.5, where the existence of a solution to situations such as (1.4) or, under less stringent hypotheses (1.5), is established.

It is worth pointing out a remarkable difference between the results presented here and the results concerning those generalizations of incompressible Stokes and Navier-Stokes equations, stationary and evolutionary, in which the viscosity grows more than linearly with the pressure. While here for (1.1) with (1.4) we develop, under certain assumptions, large data existence theory, no such mathematical theory is available for the systems such as

$$\nabla p - \operatorname{div}[2\nu(p,\cdot)\boldsymbol{D}\boldsymbol{v}] + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) = \boldsymbol{f}, \qquad \boldsymbol{D} := \frac{1}{2}(\nabla + \nabla^T), \tag{1.6}$$

if ν depends on p exponentially. With exception of studies concerning flows in special geometries (see [16], [17], [26], [29], [33], [34], [36]), we are aware of merely a few, rather preliminary studies concerning flows in general domains (see [14, 15] and [30]). We remark that in [21] and subsequent studies [12], [7], [8], [6] (that also includes a detailed summary of the available theory), the authors have been able to identify the class of the viscosities depending on the pressure and $|\boldsymbol{D}\boldsymbol{v}|^2$ and to develop large data mathematical theory for relevant boundary and initial boundary value problems. This subclass, however, does not allow to include (1.4). Remarkably enough, there is no maximum principle to eq. (1.6), not even if the equation were stripped of the inertial term $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})$.

There is abundance of available literature on qualitative analysis of Darcy-Forchheimer's equations, or their generalizations like Brinkman-Forchheimer's equations when a diffusive term is added. With the exception of investigating regularity, authors address the evolutionary case right away, see e.g. [2] for the compressible case and [37] for the incompressible one, and papers cited therein. In [35] existence of an attractor for these equations is studied. Regularity of the (unique) solution to Darcy-Forchheimer's equations is examined in [10].

In defiance of a cornucopia of sources, they are all confined to the case where $\boldsymbol{h}$ in $(1.1)_1$ does not depend on the pressure. The p-dependent and implicitly related situation of Darcy-Forchheimer equations analyzed within the current paper seems to have remained, at least to the best of authors' knowledge, almost a terra incognita so far.

1.1.4 Further comments

Behold even at this early phase that **(A4)** hints at setting the stage for working in Lebesgue spaces. It is therefore natural to ask why not plunge ourselves directly into general Orlicz spaces in the vein of Bulíček et al. [4, 5] instead. Even though such an extension should not require much additional effort, we chose the Lebesgue setting for the sake of simplicity, as it allows us to accentuate the ideas concerning p-dependence of the graph $\boldsymbol{A}$ and the maximum and minimum principles.

As far as **(A5)** is concerned, a general question of existence of a measurable selection for the case of $\boldsymbol{h}$ being independent of p is confirmed e.g. in Chiado' Piat et al. [9, Theorem 1.4]. In our setting, we want in addition the selection being continuous with respect to p and also bounded in that variable in the sense of $\mathbf{(A5)}_{\rm (iv)}$. Note that similarly tame behavior is expected in **(A4)** by requiring uniformity in p.

It is not particularly difficult to show that a maximal monotone graph (independent of p) can be rotated so as to form a graph of a 1-Lipschitz function (see [1, 11, 23]). This observation is likely to lead to another feasible way of approaching the existence theory for (1.1), devoid of any need for selections. The path is not followed in our paper save this remark.

Drawing this introduction to its end, in the following brief Sect. 1.2 we deal with a couple of useful mathematical properties to be invoked later on. We then devote an entire Sect. 1.3 to formulate and prove an existence theorem of solutions to the problem (1.1) provided **(A1)**–**(A5)** are all satisfied. The penultimate Sect. 1.4 is somewhat autonomous and serves to state and justify a maximum and a minimum principle for the pressure in (1.1). It will prove invaluable in the last Sect. 1.5, where it authorizes us to somewhat weaken **(A4)** and $\mathbf{(A5)}_{\rm (iv)}$, wherein effect it shows existence for situations like (1.4), supposing certain other hypotheses are satisfied indeed.

1.2 Preliminaries

For $\delta > 0$ denote

$$\omega_\delta(\boldsymbol{x}, t) := \delta^{-(d+1)}\omega\left(\frac{\boldsymbol{x}}{\delta}, \frac{t}{\delta}\right),$$

where ω is the usual mollification kernel on $\mathbb{R}^{d+1}$. With its help we define the *regularized selection*

$$\boldsymbol{m}_\delta(\boldsymbol{x}, t) := \int_{\mathbb{R}^d\times\mathbb{R}} \boldsymbol{m}^*(\boldsymbol{x}-\boldsymbol{y}, t-s)\,\omega_\delta(\boldsymbol{y}, s)\,d\boldsymbol{y}\,ds.$$

Lemma 1.2.1 *The selection $\boldsymbol{m}^*$ and its regularization $\boldsymbol{m}_\delta$ enjoy the following properties, which will be made use of later:*

*(i) (r, r')-coercivity **(A4)** holds for $\boldsymbol{m}_\delta$. The constants may be different but independent of $0 < \delta < 1$.*

*(ii) The property **(A3)** is actually tantamount to apparently a weaker one*

$$(\boldsymbol{m}' - \boldsymbol{m}^*(\boldsymbol{v}, p))\cdot(\boldsymbol{v}' - \boldsymbol{v}) \geq 0 \text{ for a.e. } \boldsymbol{v}\in\mathbb{R}^d \Rightarrow (\boldsymbol{m}', \boldsymbol{v}', p) \in \boldsymbol{A}.$$

Proof. For a proof of (i), we see that $\boldsymbol{m}^*$ is evidently (r, r')-coercive and hence we compute:

$$
\begin{aligned}
\boldsymbol{m}_\delta(\boldsymbol{x},t)\cdot\boldsymbol{x} &= \int_{\mathbb{R}^d\times\mathbb{R}} \boldsymbol{m}^*(\boldsymbol{x}-\boldsymbol{y},t-s)\cdot(\boldsymbol{x}-\boldsymbol{y})\,\omega_\delta(\boldsymbol{y},s)\,d\boldsymbol{y}\,ds \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}} \boldsymbol{m}^*(\boldsymbol{x}-\boldsymbol{y},t-s)\cdot\boldsymbol{y}\,\omega_\delta(\boldsymbol{y},s)\,d\boldsymbol{y}\,ds \\
&\geq \int_{\mathbb{R}^d\times\mathbb{R}} \left[c_1\left(|\boldsymbol{x}-\boldsymbol{y}|^r + |\boldsymbol{m}^*(\boldsymbol{x}-\boldsymbol{y},t-s)|^{r'}\right) - c_2\right]\omega_\delta(\boldsymbol{y},s)\,d\boldsymbol{y}\,ds \\
&\quad - \int_{\mathbb{R}^d\times\mathbb{R}} \left(\frac{c_1}{2}|\boldsymbol{m}^*(\boldsymbol{x}-\boldsymbol{y},t-s)|^{r'} + c_3|\boldsymbol{y}|^r\right)\omega_\delta(\boldsymbol{y},s)\,d\boldsymbol{y}\,ds \\
&\geq c_4(|\boldsymbol{x}|^r + |\boldsymbol{m}_\delta(\boldsymbol{x},t)|^{r'}) - c_5.
\end{aligned}
$$

First we employed Young's inequality and then Jensen's inequality was invoked. Note that neither c_4 nor c_5 depend on $\delta > 0$ as long as δ is bounded.

Towards showing (ii), let

$$\boldsymbol{A}_p = \{(\boldsymbol{m},\boldsymbol{v}) \in \mathbb{R}^d \times \mathbb{R}^d \mid (\boldsymbol{m},\boldsymbol{v},p) \in \boldsymbol{A}\}$$

and $(\boldsymbol{m}',\boldsymbol{v}',p) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ such that we have

$$(\boldsymbol{m}' - \boldsymbol{m}^*(\boldsymbol{v},p))\cdot(\boldsymbol{v}'-\boldsymbol{v}) \geq 0 \quad \text{for a.e. } \boldsymbol{v} \in \mathbb{R}^d. \tag{1.7}$$

The aim is to attest $(\boldsymbol{m}'-\boldsymbol{m})\cdot(\boldsymbol{v}'-\boldsymbol{v}) \geq 0$ for every $(\boldsymbol{m},\boldsymbol{v}) \in \boldsymbol{A}_p$: Let $(\boldsymbol{m},\boldsymbol{v}) \in \boldsymbol{A}_p$ be arbitrary. It is trivial to show that the set

$$M_{\boldsymbol{v}} = \{\widehat{\boldsymbol{m}} \in \mathbb{R}^d \mid (\widehat{\boldsymbol{m}},\boldsymbol{v}) \in \boldsymbol{A}_p\}$$

is convex and closed. Note that $M_{\boldsymbol{v}}$ is also non-empty and bounded, for else one could find $\boldsymbol{u} \in \mathbb{R}^d$ such that $\boldsymbol{m}^*(\boldsymbol{u},p) = \infty$, contradicting **(A5)**$_{\text{(iv)}}$ (but see Remark 1.2.2). Therefore we may express $\boldsymbol{m} = \lambda\boldsymbol{m}_1 + (1-\lambda)\boldsymbol{m}_2$ for some $0 \leq \lambda \leq 1$ and $\boldsymbol{m}_1,\boldsymbol{m}_2 \in \partial M_{\boldsymbol{v}}$.

Now, $\boldsymbol{A}_p$ can be seen as a d-dimensional Lipschitz manifold in $\mathbb{R}^d \times \mathbb{R}^d$ without a boundary [1], whence if $\widetilde{\boldsymbol{m}} \in \partial M_{\boldsymbol{v}}$, there exists $\{(\boldsymbol{m}_n,\boldsymbol{v}_n)\} \subset \boldsymbol{A}_p$, $\boldsymbol{v}_n \neq \boldsymbol{v}$, such that $(\boldsymbol{m}_n,\boldsymbol{v}_n) \to (\widetilde{\boldsymbol{m}},\boldsymbol{v})$, as $n \to \infty$. Of course, otherwise the point $(\widetilde{\boldsymbol{m}},\boldsymbol{v})$ would be a boundary point of $\boldsymbol{A}_p$. Finally, let $\boldsymbol{v}_n$ be chosen so that the set $\{\widehat{\boldsymbol{m}} \in \mathbb{R}^d \mid (\widehat{\boldsymbol{m}},\boldsymbol{v}_n) \in \boldsymbol{A}_p\}$ is a singleton for every n, i.e. $\boldsymbol{m}^*(\boldsymbol{v}_n,p) = \boldsymbol{m}_n$, and (1.7) holds for all $\boldsymbol{v}_n$. It is achievable, since the set of all $\widehat{\boldsymbol{v}} \in \mathbb{R}^d$ such that $M_{\widehat{\boldsymbol{v}}}$ contains more than one element has Hausdorff dimension equal to $d-1$ [1, Remark 2.3].

Thus we find $\{\boldsymbol{v}_1^n\},\{\boldsymbol{v}_2^n\} \subset \mathbb{R}^d$, for which $\boldsymbol{v}_i^n \to \boldsymbol{v}$ and $\boldsymbol{m}^*(\boldsymbol{v}_i^n,p) \to \boldsymbol{m}_i$ as $n \to \infty$, for $i = 1,2$. Given that both $\{\boldsymbol{v}_1^n\}$ and $\{\boldsymbol{v}_2^n\}$ satisfy (1.7), the goal $(\boldsymbol{m}'-\boldsymbol{m})\cdot(\boldsymbol{v}'-\boldsymbol{v}) \geq 0$ follows from passing to limit $n \to \infty$, multiplying by λ and $1-\lambda$, respectively, and finally summing up. □

Remark 1.2.2 A third useful property of $\boldsymbol{m}^*$ is its local boundedness in the sense that $|\boldsymbol{m}^*(\cdot,p)|$ is bounded on bounded domains for every $p \in \mathbb{R}$. This is trivial due to **(A5)**$_{\text{(iv)}}$, yet it would hold even without this requirement. See e.g. [19, Theorem 2], which can be applied to address the question.

1.3 Principal existence theorem

Before formulation of the main result, notation for several function spaces that will often be used shall be introduced. First, for Lebesgue and Sobolev spaces we use the standard notation. To handle the Dirichlet data for the pressure, we define, for $q \in (1, \infty)$,

$$W^{1,q}_{\Gamma_2}(\Omega) := \left\{ u \in W^{1,q}(\Omega) \mid u = 0 \text{ on } \Gamma_2 \right\}.$$

In case of $\Gamma_2 = \emptyset$, we make a natural modification

$$W^{1,q}_{\Gamma_2}(\Omega) := \left\{ u \in W^{1,q}(\Omega) \mid \int_\Omega u = 0 \right\}.$$

Note that in either instance, $W^{1,q}_{\Gamma_2}(\Omega)$ is a closed subspace of $W^{1,q}(\Omega)$. Next, since we will deal with solenoidal functions with a prescribed normal trace on a part of the boundary, we denote

$$L^q_{\mathrm{div}}(\Omega) := \left\{ \boldsymbol{\varphi} \in L^q(\Omega)^d \mid \operatorname{div} \boldsymbol{\varphi} = 0 \right\}.$$

The condition on zero divergence is meant in the sense of distributions. As the zero distribution is regular, we can legally say in particular $\operatorname{div} \boldsymbol{\varphi} = 0$ a.e. in Ω for any $\boldsymbol{\varphi} \in L^q_{\mathrm{div}}(\Omega)$. It is well known (see [13, chapter III.2]) that one can talk about normal traces (remember Ω is Lipschitz) of elements of $L^q_{\mathrm{div}}(\Omega)$, seeing them as elements of $\left(W^{\frac{1}{q},q'}(\partial\Omega)\right)^*$. Understanding $\boldsymbol{\varphi} \cdot \boldsymbol{n}$ on Γ_1 in this generalized sense, we can also introduce

$$L^q_{\mathrm{div},\Gamma_1}(\Omega) := \left\{ \boldsymbol{\varphi} \in L^q_{\mathrm{div}}(\Omega) \mid \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_1 \right\}.$$

To conclude, for $K > 0$ we define a cutoff function $T_K : \mathbb{R} \to \mathbb{R}$ as

$$T_K(x) := \begin{cases} -K & \text{for } x \le -K, \\ x & \text{for } -K < x < K, \\ K & \text{for } x \ge K. \end{cases} \tag{1.8}$$

Here and there we will silently use trivial $|T_K(x)| \le |x|$ for every $x \in \mathbb{R}$. When applying the truncator T_K to vectors, we consider the component-wise truncation, i.e. for $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we set $T_K(\boldsymbol{x}) := (T_K(x_1), \ldots, T_K(x_d))$.

Having finalized indispensable preparations, the promised existence theorem can be formulated:

Theorem 1.3.1 *Let Ω be a Lipschitz domain and $r \in (1, \infty)$ be given. Assume $\boldsymbol{f} \in L^{r'}(\Omega)^d$, $\boldsymbol{v}_0 \in L^r_{\mathrm{div}}(\Omega)$ and $p_0 \in W^{1,r'}(\Omega)$. Moreover, assume that $\boldsymbol{A}$ is a maximal monotone r-graph in the sense of* **(A1)–(A5)**. *Then there exists a triplet*

$$(\boldsymbol{m}, \boldsymbol{v}, p) \in L^{r'}(\Omega)^d \times L^r_{\mathrm{div}}(\Omega) \times W^{1,r'}(\Omega)$$

solving (1.1), *i.e.* $(1.1)_1$–$(1.1)_3$ *are satisfied a.e. in Ω and*

$$\begin{aligned} \boldsymbol{v} - \boldsymbol{v}_0 &\in L^r_{\mathrm{div},\Gamma_1}(\Omega), \\ p - p_0 &\in W^{1,r'}_{\Gamma_2}(\Omega). \end{aligned}$$

Proof. The proof of the theorem makes up the remainder of this section. Let $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}} \subset L^r(\Omega)^d \cap L^\infty(\Omega)^d$ and $\{q_i\}_{i\in\mathbb{N}} \subset W^{1,r'}_{\Gamma_2}(\Omega)$ be linearly independent, with linear spans dense in $L^r(\Omega)^d$ and $W^{1,r'}_{\Gamma_2}(\Omega)$, respectively.

To begin with, we deduce existence of solutions to an approximate problem, i.e. for $n \in \mathbb{N}$ and $\varepsilon, \delta > 0$ to find

$$\boldsymbol{v}_n^{\varepsilon,\delta}(x) = T_n(\boldsymbol{v}_0)(x) + \sum_{i=1}^n a_n^{\varepsilon,\delta,i}\boldsymbol{w}_i(x), \tag{1.9}$$

$$p_n^{\varepsilon,\delta}(x) = p_0(x) + \sum_{i=1}^n b_n^{\varepsilon,\delta,i} q_i(x), \tag{1.10}$$

satisfying

$$\int_\Omega \nabla p_n^{\varepsilon,\delta} \cdot \boldsymbol{w}_i + \int_\Omega \boldsymbol{m}_\delta(\boldsymbol{v}_n^{\varepsilon,\delta}, p_n^{\varepsilon,\delta}) \cdot \boldsymbol{w}_i = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}_i, \ i = 1, \dots, n, \tag{1.11}$$

$$\varepsilon \int_\Omega |\nabla(p_n^{\varepsilon,\delta} - p_0)|^{r'-2}\nabla(p_n^{\varepsilon,\delta} - p_0) \cdot \nabla q_i = \int_\Omega (\boldsymbol{v}_n^{\varepsilon,\delta} - T_n(\boldsymbol{v}_0)) \cdot \nabla q_i, \ i = 1, \dots, n. \tag{1.12}$$

Replacing solenoidality of the velocity field with the eq. (1.12) is a so-called *quasi-compressible* approximation (see [12] and [22, p. 416]), which facilitates construction of the pressure. Note that, at this point at least informally, the limit $\varepsilon \to 0_+$ should produce a divergence-free velocity.

The aim of δ-regularization is to obtain a solution to (1.11) and (1.12). This is actually the first approximation parameter to be dropped due to a limiting process. Since it will require boundedness of $\{\boldsymbol{v}_n^{\varepsilon,\delta}\}_\delta$ in $L^\infty(\Omega)^d$, we need to truncate $\boldsymbol{v}_0$ as seen in (1.9).

Towards showing existence of $\{a_n^{\varepsilon,\delta,i}\}_{i=1}^n$ and $\{b_n^{\varepsilon,\delta,i}\}_{i=1}^n$, we employ the following standard corollary of Brouwer's fixed point theorem, whose justification follows from lines to come and will not be discussed in detail.

Lemma 1.3.2 [20, Lemme 4.3] *Let $\boldsymbol{F} : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function satisfying $\boldsymbol{F}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq 0$ if $|\boldsymbol{\xi}| = \varrho$ for certain $\varrho > 0$. Then there exists $\boldsymbol{\xi}_0 \in \mathbb{R}^d$, $|\boldsymbol{\xi}_0| \leq \varrho$, for which $\boldsymbol{F}(\boldsymbol{\xi}_0) = \boldsymbol{0}$.*

Multiplying eq. $(1.11)_i$ by $a_n^{\varepsilon,\delta,i}$ and eq. $(1.12)_i$ by $b_n^{\varepsilon,\delta,i}$ and summing the resultant $2n$ equalities, we obtain

$$\varepsilon\|\nabla(p_n^{\varepsilon,\delta} - p_0)\|_{r'}^{r'} + \int_\Omega \boldsymbol{m}_\delta(\boldsymbol{v}_n^{\varepsilon,\delta}, p_n^{\varepsilon,\delta}) \cdot (\boldsymbol{v}_n^{\varepsilon,\delta} - \boldsymbol{v}_0) = \int_\Omega (\boldsymbol{f} - \nabla p_0) \cdot (\boldsymbol{v}_n^{\varepsilon,\delta} - T_n(\boldsymbol{v}_0)). \tag{1.13}$$

As we may assume $\delta < 1$, recalling Lemma 1.2.1 for (r, r')-coercivity of $\boldsymbol{m}_\delta$, Hölder's and Young's inequalities, eq. (1.13) is processed into

$$\varepsilon\|\nabla(p_n^{\varepsilon,\delta} - p_0)\|_{r'}^{r'} + \|\boldsymbol{m}_\delta(\boldsymbol{v}_n^{\varepsilon,\delta}, p_n^{\varepsilon,\delta})\|_{r'}^{r'} + \|\boldsymbol{v}_n^{\varepsilon,\delta}\|_r^r \leq C(\|\boldsymbol{f} - \nabla p_0\|_{r'}, \|\boldsymbol{v}_0\|_r). \tag{1.14}$$

In particular, the constant C is independent of δ, n or ε. The *energy inequality* (1.14) will serve us as the starting point for taking the limits $\delta \to 0_+$, $n \to \infty$ and $\varepsilon \to 0_+$, in this order.

1.3.1 δ-limit

In order to accomplish the first limit passage, we start with observation that (1.14) entails

$$\sup\left\{|a_n^{\varepsilon,\delta,i}|, |b_n^{\varepsilon,\delta,i}| \;\middle|\; 0<\delta<1,\, i=1,\dots,n\right\} < C(n,\varepsilon).$$

We may hence assume

$$\begin{aligned} a_n^{\varepsilon,\delta,i} &\to a_n^{\varepsilon,i},\\ b_n^{\varepsilon,\delta,i} &\to b_n^{\varepsilon,i}, \end{aligned} \tag{1.15}$$

as $\delta \to 0_+$, for each $i=1,\dots,n$. This result allows us to observe also the following convergences:

$$\begin{aligned} \boldsymbol{v}_n^{\varepsilon,\delta} &\to \boldsymbol{v}_n^{\varepsilon} && \text{in } L^\infty(\Omega)^d,\\ p_n^{\varepsilon,\delta} - p_0 &\to p_n^{\varepsilon} - p_0 && \text{in } W^{1,r'}_{\Gamma_2}(\Omega),\\ p_n^{\varepsilon,\delta} &\to p_n^{\varepsilon} && \text{a.e. in } \Omega,\\ |\nabla(p_n^{\varepsilon,\delta}-p_0)|^{r'-2}\nabla(p_n^{\varepsilon,\delta}-p_0) &\to |\nabla(p_n^{\varepsilon}-p_0)|^{r'-2}\nabla(p_n^{\varepsilon}-p_0) && \text{in } L^r(\Omega)^d,\\ \boldsymbol{m}_\delta(\boldsymbol{v}_n^{\varepsilon,\delta}, p_n^{\varepsilon,\delta}) &\rightharpoonup \boldsymbol{m}_n^{\varepsilon} && \text{in } L^{r'}(\Omega)^d. \end{aligned} \tag{1.16}$$

For $(1.16)_1$ and $(1.16)_2$ we used (1.9), (1.10) and (1.15); the limits $(1.16)_3$ and $(1.16)_4$ are justified by $(1.16)_2$, and the last passage utilized ineq. (1.14) and reflexivity of $L^{r'}(\Omega)^d$. The subscript in $\boldsymbol{m}_n^{\varepsilon}$ does not correspond to mollification any longer, it is used merely to follow the same notation as p_n^{ε} and $\boldsymbol{v}_n^{\varepsilon}$.

As for what equations the limit quantities satisfy, (1.16) makes passing to limit $\delta \to 0_+$ in equations (1.11) and (1.12) easy and we obtain

$$\begin{aligned} \int_\Omega \nabla p_n^{\varepsilon}\cdot \boldsymbol{w}_i + \int_\Omega \boldsymbol{m}_n^{\varepsilon}\cdot \boldsymbol{w}_i &= \int_\Omega \boldsymbol{f}\cdot \boldsymbol{w}_i,\; i=1,\dots,n,\\ \varepsilon\int_\Omega |\nabla(p_n^{\varepsilon}-p_0)|^{r'-2}\,\nabla(p_n^{\varepsilon}-p_0)\cdot\nabla q_i &= \int_\Omega (\boldsymbol{v}_n^{\varepsilon} - T_n(\boldsymbol{v}_0))\cdot \nabla q_i,\; i=1,\dots,n. \end{aligned} \tag{1.17}$$

Before proceeding to the second passage, we will yet show the limit functions now lie in the graph, i.e. $(\boldsymbol{m}_n^{\varepsilon}, \boldsymbol{v}_n^{\varepsilon}, p_n^{\varepsilon}) \in \boldsymbol{A}$ a.e. in Ω. This objective can be achieved by means of the maximality property **(A3)**, specifically by its version from Lemma 1.2.1. In the given situation, we have to verify

$$(\boldsymbol{m}_n^{\varepsilon} - \boldsymbol{m}^*(\boldsymbol{u}, p_n^{\varepsilon}))\cdot(\boldsymbol{v}_n^{\varepsilon} - \boldsymbol{u}) \ge 0 \quad \text{a.e. in } \Omega \text{ for a.e. } \boldsymbol{u}\in\mathbb{R}^d, \tag{1.18}$$

in order of which it suffices to check

$$\liminf_{\delta\to 0_+}\Big(\boldsymbol{m}_\delta(\boldsymbol{v}_n^{\varepsilon,\delta}, p_n^{\varepsilon,\delta}) - \boldsymbol{m}^*(\boldsymbol{u}, p_n^{\varepsilon})\Big)\cdot(\boldsymbol{v}_n^{\varepsilon,\delta} - \boldsymbol{u}) \ge 0 \quad \text{a.e. in } \Omega \text{ for a.e. } \boldsymbol{u}\in\mathbb{R}^d. \tag{1.19}$$

Indeed it does: Let us consider only $\boldsymbol{u}\in\mathbb{R}^d$ at which $\boldsymbol{m}^*(\boldsymbol{u},\cdot)$ is continuous. For an arbitrary measurable $E\subset\mathbb{R}^d$ of non-zero Lebesgue measure, (1.19) implies

$$\liminf_{\delta\to 0_+}\int_E \Big(\boldsymbol{m}_\delta(\boldsymbol{v}_n^{\varepsilon,\delta}, p_n^{\varepsilon,\delta}) - \boldsymbol{m}^*(\boldsymbol{u}, p_n^{\varepsilon,\delta})\Big)\cdot(\boldsymbol{v}_n^{\varepsilon,\delta} - \boldsymbol{u}) \ge 0.$$

Due to convergences (1.16) and properties of $\boldsymbol{m}^*$, we pass to the limit

$$\int_E (\boldsymbol{m}_n^{\varepsilon} - \boldsymbol{m}^*(\boldsymbol{u}, p_n^{\varepsilon}))\cdot(\boldsymbol{v}_n^{\varepsilon} - \boldsymbol{u}) \ge 0.$$

The arbitrary nature of E yields (1.18). Moving on to the proof of (1.19), monotonicity implies

$$\int_{\mathbb{R}^d\times\mathbb{R}} (\boldsymbol{m}^*(\hat{\boldsymbol{u}},t) - \boldsymbol{m}^*(\boldsymbol{u},t)) \cdot (\hat{\boldsymbol{u}} - \boldsymbol{u})\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt \geq 0,$$

which holds a.e. in Ω. We reshuffle the relation into

$$\begin{aligned} &\int_{\mathbb{R}^d\times\mathbb{R}} (\boldsymbol{m}^*(\hat{\boldsymbol{u}},t) - \boldsymbol{m}^*(\boldsymbol{u},t)) \cdot (\boldsymbol{v}_n^{\varepsilon,\delta} - \boldsymbol{u})\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt \\ &\qquad \geq \int_{\mathbb{R}^d\times\mathbb{R}} (\boldsymbol{m}^*(\hat{\boldsymbol{u}},t) - \boldsymbol{m}^*(\boldsymbol{u},t)) \cdot (\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}})\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt. \end{aligned} \tag{1.20}$$

Limit passage in (1.20) is manageable, for firstly we have

$$\lim_{\delta\to 0_+} \int_{\mathbb{R}^d\times\mathbb{R}} \boldsymbol{m}^*(\boldsymbol{u},t)\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt = \boldsymbol{m}^*(\boldsymbol{u}, p_n^\varepsilon), \tag{1.21}$$

due to continuity and boundedness of $\boldsymbol{m}^*(\boldsymbol{u},\cdot)$ and pointwise convergence of $\{p_n^{\varepsilon,\delta}\}_\delta$. Secondly, $\{\boldsymbol{v}_n^{\varepsilon,\delta}\}_\delta$ is bounded in $L^\infty(\Omega)^d$, and in conjunction with **(A5)**$_{\text{(iv)}}$ we observe

$$\begin{aligned} &\Big|\int_{\mathbb{R}^d\times\mathbb{R}} (\boldsymbol{m}^*(\hat{\boldsymbol{u}},t) - \boldsymbol{m}^*(\boldsymbol{u},t)) \cdot (\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}})\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt\Big| \\ &\qquad \leq C(\boldsymbol{u}, \|\boldsymbol{v}_n^{\varepsilon,\delta}\|_\infty) \underbrace{\Big(\int_{\mathbb{R}^d\times\mathbb{R}} |\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}|^r\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt\Big)^{1/r}}_{\to 0 \text{ a.e. in } \Omega \text{ as } \delta\to 0_+.} \end{aligned} \tag{1.22}$$

Applying (1.21) and (1.22) on (1.20), we obtain (1.19), i.e. $(\boldsymbol{m}_n^\varepsilon, \boldsymbol{v}_n^\varepsilon, p_n^\varepsilon) \in \boldsymbol{A}$ a.e. in Ω.

1.3.2 n-limit

The weak lower semicontinuity of norms applied on (1.14) produces a second level of that energy inequality, meaning

$$\varepsilon \left\|\nabla(p_n^\varepsilon - p_0)\right\|_{r'}^{r'} + \|\boldsymbol{m}_n^\varepsilon\|_{r'}^{r'} + \|\boldsymbol{v}_n^\varepsilon\|_r^r \leq C(\|\boldsymbol{f} - \nabla p_0\|_{r'}, \|\boldsymbol{v}_0\|_r), \tag{1.23}$$

whence we may pass to the limit $n\to\infty$, assuming

$$\begin{aligned} \boldsymbol{v}_n^\varepsilon &\rightharpoonup \boldsymbol{v}^\varepsilon && \text{in } L^r(\Omega)^d, \\ p_n^\varepsilon - p_0 &\rightharpoonup p^\varepsilon - p_0 && \text{in } W^{1,r'}_{\Gamma_2}(\Omega), \\ |\nabla(p_n^\varepsilon - p_0)|^{r'-2}\,\nabla(p_n^\varepsilon - p_0) &\rightharpoonup \boldsymbol{\chi} && \text{in } L^r(\Omega)^d, \\ \boldsymbol{m}_n^\varepsilon &\rightharpoonup \boldsymbol{m}^\varepsilon && \text{in } L^{r'}(\Omega)^d, \\ T_n(\boldsymbol{v}_0) &\to \boldsymbol{v}_0 && \text{in } L^r(\Omega)^d. \end{aligned} \tag{1.24}$$

The last result is an easy consequence of Chebyshev's inequality. Convergences (1.24) let us pass to the limit in eq. (1.17). Using the density property of $\{\boldsymbol{w}_i\}$ in $L^r(\Omega)^d$ and $\{q_i\}$ in $W^{1,r'}_{\Gamma_2}(\Omega)$, we obtain furthermore

$$\begin{aligned} \int_\Omega \nabla p^\varepsilon \cdot \boldsymbol{w} + \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{w} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}, \quad \forall \boldsymbol{w}\in L^r(\Omega)^d, \\ \varepsilon\int_\Omega \boldsymbol{\chi}\cdot\nabla q &= \int_\Omega (\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0)\cdot\nabla q, \quad \forall q\in W^{1,r'}_{\Gamma_2}(\Omega). \end{aligned} \tag{1.25}$$

Like previously, we have to check $(\boldsymbol{m}^\varepsilon, \boldsymbol{v}^\varepsilon, p^\varepsilon) \in \boldsymbol{A}$ a.e. in Ω. Also, weak convergence prevented us from inferring identity of the weak limit $\boldsymbol{\chi}$ and it is necessary yet to verify $\boldsymbol{\chi} = |\nabla(p^\varepsilon - p_0)|^{r'-2} \nabla(p^\varepsilon - p_0)$. We will use the standard monotone operator theory, namely the Minty's method.

From (1.25) we deduce

$$\varepsilon \int_\Omega \boldsymbol{\chi} \cdot \nabla(p^\varepsilon - p_0) + \int_\Omega \boldsymbol{m}^\varepsilon \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0) = \int_\Omega (\boldsymbol{f} - \nabla p_0) \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0), \tag{1.26}$$

while (1.17) implies similarly

$$\varepsilon \int_\Omega |\nabla(p_n^\varepsilon - p_0)|^{r'} + \int_\Omega \boldsymbol{m}_n^\varepsilon \cdot (\boldsymbol{v}_n^\varepsilon - T_n(\boldsymbol{v}_0)) = \int_\Omega (\boldsymbol{f} - \nabla p_0) \cdot (\boldsymbol{v}_n^\varepsilon - T_n(\boldsymbol{v}_0)), \tag{1.27}$$

for every $n \in \mathbb{N}$. Using (1.24), comparing (1.26) with (1.27) yields

$$\lim_{n\to\infty} \varepsilon \int_\Omega |\nabla(p_n^\varepsilon - p_0)|^{r'} + \int_\Omega \boldsymbol{m}_n^\varepsilon \cdot \boldsymbol{v}_n^\varepsilon = \varepsilon \int_\Omega \boldsymbol{\chi} \cdot \nabla(p^\varepsilon - p_0) + \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon. \tag{1.28}$$

This will be our first foothold. Next, recall monotonicity of the p-Laplace operator and $(\boldsymbol{m}_n^\varepsilon, \boldsymbol{v}_n^\varepsilon, p_n^\varepsilon) \in \boldsymbol{A}$ a.e. in Ω. Hence we know that for all $q \in W^{1,r'}_{\Gamma_2}(\Omega)$,

$$\begin{aligned} 0 \le \varepsilon \int_\Omega \big(|\nabla(p_n^\varepsilon - p_0)|^{r'-2} \nabla(p_n^\varepsilon - p_0) - |\nabla q|^{r'-2} \nabla q\big) \cdot \nabla(p_n^\varepsilon - p_0 - q) \\ + \int_\Omega \big(\boldsymbol{m}_n^\varepsilon - \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p_n^\varepsilon)\big) \cdot (\boldsymbol{v}_n^\varepsilon - \boldsymbol{v}^\varepsilon). \end{aligned} \tag{1.29}$$

As we are allowed to assume $p_n^\varepsilon \to p^\varepsilon$ from $(1.24)_2$, properties **(A5)** yield

$$\boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p_n^\varepsilon) \to \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p^\varepsilon) \quad \text{in } L^{r'}(\Omega) \text{ as } \boldsymbol{n} \to \infty,$$

which we on top of that mingle with $(1.24)_1$ and observe

$$\boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p_n^\varepsilon) \cdot (\boldsymbol{v}_n^\varepsilon - \boldsymbol{v}^\varepsilon) \to 0 \quad \text{in } L^1(\Omega) \text{ as } \boldsymbol{n} \to \infty. \tag{1.30}$$

Combining (1.24), (1.28), (1.30) and taking the limit $n \to \infty$ in the monotonicity relation (1.29) gives rise to

$$0 \le \int_\Omega \big(\boldsymbol{\chi} - |\nabla q|^{r'-2} \nabla q\big) \cdot \nabla(p^\varepsilon - p_0 - q), \quad \forall q \in W^{1,r'}_{\Gamma_2}(\Omega). \tag{1.31}$$

Setting $q = p^\varepsilon - p_0 \pm t\varphi$ with $t > 0$ and $\varphi \in W^{1,r'}_{\Gamma_2}(\Omega)$, we divide (1.31) by t and then perform $t \to 0_+$. Arbitrary nature of φ yields

$$\boldsymbol{\chi} = |\nabla(p^\varepsilon - p_0)|^{r'-2} \nabla(p^\varepsilon - p_0). \tag{1.32}$$

Identity (1.28) can now be rewritten

$$\lim_{n\to\infty} \varepsilon \int_\Omega |\nabla(p_n^\varepsilon - p_0)|^{r'} + \int_\Omega \boldsymbol{m}_n^\varepsilon \cdot \boldsymbol{v}_n^\varepsilon = \varepsilon \int_\Omega |\nabla(p^\varepsilon - p_0)|^{r'} + \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon. \tag{1.33}$$

By weak lower semicontinutity of a norm, (1.33) indicates

$$\|\nabla(p^\varepsilon - p_0)\|_{r'} \leq \liminf_{n\to\infty} \|\nabla(p^\varepsilon_n - p_0)\|_{r'} \Rightarrow \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon \geq \limsup_{n\to\infty} \int_\Omega \boldsymbol{m}^\varepsilon_n \cdot \boldsymbol{v}^\varepsilon_n. \tag{1.34}$$

This is actually sufficient for $(\boldsymbol{m}^\varepsilon, \boldsymbol{v}^\varepsilon, p^\varepsilon) \in \boldsymbol{A}$ a.e. in Ω. We will derive it again from the maximality property reformulated in Lemma 1.2.1.

On the one hand, we have $\big(\boldsymbol{m}^\varepsilon_n - \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p^\varepsilon_n)\big) \cdot (\boldsymbol{v}^\varepsilon_n - \boldsymbol{v}^\varepsilon) \geq 0$ a.e. in Ω. However, (1.34) leads us to

$$\begin{aligned} 0 \leq \limsup_{n\to\infty} \int_\Omega \big(\boldsymbol{m}^\varepsilon_n - \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p^\varepsilon_n)\big) \cdot (\boldsymbol{v}^\varepsilon_n - \boldsymbol{v}^\varepsilon) \\ = \limsup_{n\to\infty} \int_\Omega \boldsymbol{m}^\varepsilon_n \cdot \boldsymbol{v}^\varepsilon_n - \boldsymbol{m}^\varepsilon_n \cdot \boldsymbol{v}^\varepsilon - \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p^\varepsilon_n) \cdot (\boldsymbol{v}^\varepsilon_n - \boldsymbol{v}^\varepsilon) \leq 0, \end{aligned}$$

due to (1.30) and (1.34). Therefore $\big(\boldsymbol{m}^\varepsilon_n - \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p^\varepsilon_n)\big) \cdot (\boldsymbol{v}^\varepsilon_n - \boldsymbol{v}^\varepsilon) \to 0$ in $L^1(\Omega)$ for $n \to \infty$. Since a strong convergence implies the weak one, for all $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$ a.e. in Ω, we have

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega \boldsymbol{m}^\varepsilon_n \cdot \boldsymbol{v}^\varepsilon_n \,\varphi\, dx &= \lim_{n\to\infty} \int_\Omega \boldsymbol{m}^\varepsilon_n \cdot \boldsymbol{v}^\varepsilon \,\varphi + \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p^\varepsilon_n) \cdot (\boldsymbol{v}^\varepsilon_n - \boldsymbol{v}^\varepsilon)\,\varphi\, dx \\ &= \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon \,\varphi\, dx. \end{aligned} \tag{1.35}$$

The last equality made again use of (1.30). Now, we take an arbitrary $\boldsymbol{u} \in \mathbb{R}^d$ and use monotonicity to write

$$\int_\Omega \big(\boldsymbol{m}^\varepsilon_n - \boldsymbol{m}^*(\boldsymbol{u}, p^\varepsilon_n)\big) \cdot (\boldsymbol{v}^\varepsilon_n - \boldsymbol{u})\,\varphi\, dx \geq 0, \quad \forall n \in \mathbb{N}.$$

Owing to **(A5)**, (1.24) and (1.35), it is possible to take the limit $n \to \infty$ and infer

$$\int_\Omega \big(\boldsymbol{m}^\varepsilon - \boldsymbol{m}^*(\boldsymbol{u}, p^\varepsilon)\big) \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{u})\,\varphi\, dx \geq 0,$$

yielding $(\boldsymbol{m}^\varepsilon - \boldsymbol{m}^*(\boldsymbol{u}, p^\varepsilon)) \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{u}) \geq 0$ a.e. in Ω, which further begets $(\boldsymbol{m}^\varepsilon, \boldsymbol{v}^\varepsilon, p^\varepsilon) \in \boldsymbol{A}$ a.e. in Ω by Lemma 1.2.1.

1.3.3 ε-limit

In the spirit of the previous limit, (1.24) and the weak lower semicontinuity of a norm applied on (1.23) produce the third energy inequality

$$\varepsilon \|\nabla(p^\varepsilon - p_0)\|^{r'}_{r'} + \|\boldsymbol{m}^\varepsilon\|^{r'}_{r'} + \|\boldsymbol{v}^\varepsilon\|^r_r \leq C(\|\boldsymbol{f} - \nabla p_0\|_{r'}, \|\boldsymbol{v}_0\|_r). \tag{1.36}$$

The first term actually does not pose much of a problem, for eq. $(1.25)_1$ implies a pointwise identity

$$\nabla p^\varepsilon = \boldsymbol{f} - \boldsymbol{m}^\varepsilon \quad \text{a.e. in } \Omega,$$

whence there follows optimization of (1.36), namely

$$\|\nabla(p^\varepsilon - p_0)\|_{r'} + \|\boldsymbol{m}^\varepsilon\|_{r'} + \|\boldsymbol{v}^\varepsilon\|_r \leq C(\|\boldsymbol{f} - \nabla p_0\|_{r'}, \|\boldsymbol{v}_0\|_r).$$

Like twice before already, we can find subsequences

$$\begin{aligned} \boldsymbol{v}^\varepsilon &\rightharpoonup \boldsymbol{v} && \text{in } L^r(\Omega)^d, \\ p^\varepsilon - p_0 &\rightharpoonup p - p_0 && \text{in } W^{1,r'}_{\Gamma_2}(\Omega), \\ \varepsilon \left|\nabla(p_n^\varepsilon - p_0)\right|^{r'-2} \nabla(p_n^\varepsilon - p_0) &\rightharpoonup \mathbf{0} && \text{in } L^r(\Omega)^d, \\ \boldsymbol{m}^\varepsilon &\rightharpoonup \boldsymbol{m} && \text{in } L^{r'}(\Omega)^d, \end{aligned} \tag{1.37}$$

for $\varepsilon \to 0_+$. The limit quantities satisfy

$$\begin{aligned} \int_\Omega \nabla p \cdot \boldsymbol{w} + \int_\Omega \boldsymbol{m} \cdot \boldsymbol{w} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}, \quad \forall \boldsymbol{w} \in L^r(\Omega)^d, \\ 0 &= \int_\Omega (\boldsymbol{v} - \boldsymbol{v}_0) \cdot \nabla q, \quad \forall q \in W^{1,r'}_{\Gamma_2}(\Omega), \end{aligned} \tag{1.38}$$

that is

$$\begin{aligned} \nabla p + \boldsymbol{m} = \boldsymbol{f} \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 \quad & \text{in } \Omega, \\ (\boldsymbol{v} - \boldsymbol{v}_0) \cdot \boldsymbol{n} = 0 \quad & \text{on } \Gamma_1, \\ p - p_0 = 0 \quad & \text{on } \Gamma_2. \end{aligned}$$

In order to reach (1.1), the sole remaining step is showing $(\boldsymbol{m}, \boldsymbol{v}, p) \in \boldsymbol{A}$ a.e. in Ω. Let us first take (1.26) with $\boldsymbol{\chi}$ already identified from (1.32), recall $|\nabla(p^\varepsilon - p_0)|^{r'}$ is bounded in $L^1(\Omega)$ and pass to the limit $\varepsilon \to 0_+$:

$$\lim_{\varepsilon \to 0_+} \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon - \int_\Omega \boldsymbol{m} \cdot \boldsymbol{v}_0 = \int_\Omega (\boldsymbol{f} - \nabla p_0) \cdot (\boldsymbol{v} - \boldsymbol{v}_0).$$

The limit equation (1.38) yields, on the other hand

$$\int_\Omega \boldsymbol{m} \cdot (\boldsymbol{v} - \boldsymbol{v}_0) = \int_\Omega (\boldsymbol{f} - \nabla p_0) \cdot (\boldsymbol{v} - \boldsymbol{v}_0),$$

whereby we infer

$$\lim_{\varepsilon \to 0_+} \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon = \int_\Omega \boldsymbol{m} \cdot \boldsymbol{v}.$$

The rest would follow along the same lines as what came after (1.34). Of course, by (1.37) we may again tacitly assume $p^\varepsilon \to p$ a.e. in Ω. Thus justification of $(\boldsymbol{m}, \boldsymbol{v}, p) \in \boldsymbol{A}$ a.e. in Ω is complete and with it, the proof of Theorem 1.3.1. □

1.4 Maximum and minimum principle

What ensues is an observation that in the case of conservative forces and pure inflow, or pure outflow over Γ_1, one obtains a minimum or a maximum principle, respectively, for the pressure. Note that this result can be relatively easily obtained for the primordial Darcy's model, i.e. $\boldsymbol{m} = \alpha \boldsymbol{v}$, for some $\alpha > 0$ where, after formal application of the divergence operator, one ends up with an elliptic problem $\Delta p = \operatorname{div} f$. The property of maximum and minimum principle thus endured extensions at least up to ours.

We start with introducing an additional assumption on the graph, namely

(A6) *strict monotonicity at the origin*

$$\forall(\boldsymbol{m}, \boldsymbol{v}, p) \in \boldsymbol{A} : \boldsymbol{m} \cdot \boldsymbol{v} = 0 \Rightarrow \boldsymbol{m} = \boldsymbol{0}.$$

Note that this condition follows trivially from **(A4)** provided $c_2 = 0$.

Theorem 1.4.1 *Let assumptions of Theorem 1.3.1 be in force and Ω be additionally connected. Let **(A6)** hold and $\boldsymbol{f} = \nabla g$ for some $g \in W^{1,r'}(\Omega)$. Then*

(i) $\boldsymbol{v}_0 \cdot \boldsymbol{n} \geq 0$ on Γ_1 implies $p - g \leq \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g)$ a.e. in Ω.

(ii) $\boldsymbol{v}_0 \cdot \boldsymbol{n} \leq 0$ on Γ_1 implies $p - g \geq \operatorname*{ess\,inf}_{\Gamma_2}(p_0 - g)$ a.e. in Ω.

In particular, if $\boldsymbol{v}_0 \cdot \boldsymbol{n} = 0$ on Γ_1, Γ_2 is non-trivial in the sense $|\Gamma_2|_{d-1} > 0$, $p_0 \in L^\infty(\Gamma_2)$ and $g \in L^\infty(\Omega) \cap W^{1,r'}(\Omega)$, then $p \in L^\infty(\Omega)$.

Proof. We will concentrate on the maximum principle only, its minimum counterpart would be verified completely analogously.

Without loss of generality assume $\operatorname{ess\,sup}_{\Gamma_2}(p_0 - g) < \infty$. The proof hinges on a proper choice of a test function in the weak formulation (1.38) of the problem (1.1). Define a truncation operator

$$T(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x & \text{for } 0 < x \leq 1, \\ 1 & \text{for } x > 1, \end{cases}$$

and a test function

$$\boldsymbol{w} = T(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g))\boldsymbol{v}.$$

Abbreviating $T := T(p - g - \operatorname{ess\,sup}_{\Gamma_2}(p_0 - g))$ when necessary, we arrive at

$$\int_\Omega \boldsymbol{m} \cdot \boldsymbol{v}\, T\, dx = -\int_\Omega \nabla(p - g) \cdot \boldsymbol{v}\, T\, dx. \tag{1.39}$$

On the one hand, the right-hand side of (1.39) can be rewritten as

$$\begin{aligned} -\int_\Omega \nabla(p - g) \cdot \boldsymbol{v}\, T\, dx &= -\int_\Omega \nabla(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g)) \cdot \boldsymbol{v}\, T\, dx \\ &= -\int_\Omega \nabla H(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g)) \cdot \boldsymbol{v}\, dx, \end{aligned}$$

where $H(x) = \int_0^x T(s)\, ds \geq 0$. Then the integration by parts and $\boldsymbol{v}_0 \cdot \boldsymbol{n} \geq 0$ on Γ_1 yield

$$\begin{aligned} -\int_\Omega \nabla H(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g)) \cdot \boldsymbol{v}\, dx &= -\int_{\Gamma_1 \cup \Gamma_2} H(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g))\boldsymbol{v} \cdot \boldsymbol{n}\, dS \\ &\leq 0. \end{aligned}$$

Eq. (1.39) hence gives $\int_\Omega \boldsymbol{m} \cdot \boldsymbol{v}\, T\, dx \leq 0$. On the other hand, **(A1)** and **(A2)** imply $\boldsymbol{m} \cdot \boldsymbol{v} \geq 0$ a.e. in Ω and therefore

$$\boldsymbol{m} \cdot \boldsymbol{v}\, T(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g)) = 0 \quad \text{a.e. in } \Omega.$$

Denoting $V = \{x \in \Omega \mid (p-g)(x) > \operatorname{ess\,sup}_{\Gamma_2}(p_0+g)\}$, **(A6)** entails $\boldsymbol{m} = 0$ a.e. in V. Fom $(1.1)_1$ we deduce $\nabla(p-g) = 0$ a.e. in V, so that

$$\nabla\Big[\big(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0+g)\big)_+\Big] = \mathbf{0} \quad \text{a.e. in } \Omega.$$

Therefore $\big(p - g - \operatorname{ess\,sup}_{\Gamma_2}(p_0+g)\big)_+ \equiv C$ for some constant C due to connectedness of Ω. However, this constant must be zero, since $p-g$ is a Sobolev function. Therefore $p - g \le \operatorname{ess\,sup}_{\Gamma_2}(p_0 - g)$ a.e. in Ω. □

1.5 Extended existence theorem

The primal benefit of Theorem 1.4.1 is that, at certain price, we can significantly slacken the draconian restrictions imposed by $\mathbf{(A5)}_{(\mathrm{iv})}$, as well as **(A4)**, by allowing the constants c and c_1 to be actually functions of the pressure. Thus we can vastly extend the class of admissible interactions $\boldsymbol{m}$ and cover some physically relevant cases. More precisely, let us consider there exist $\alpha, \beta \in \mathcal{C}(R)$ strictly positive everywhere on $\mathbb{R}$, such that

(A4*) $\exists c_2 \ge 0\ \forall(\boldsymbol{m},\boldsymbol{v},p) \in \boldsymbol{A} : \boldsymbol{m}\cdot\boldsymbol{v} \ge \alpha(p)(|\boldsymbol{v}|^r + |\boldsymbol{m}|^{r'}) - c_2,$

$\mathbf{(A5)}_{(\mathrm{iv}^*)}$ $\forall(\boldsymbol{v},p) \in \mathbb{R}^d \times \mathbb{R} : |\boldsymbol{m}^*(\boldsymbol{v},p)| \le \beta(p)(1 + |\boldsymbol{v}|^{r-1}).$

Theorem 1.5.1 *Let Ω be a connected Lipschitz domain and $r \in (1,\infty)$. Assume $\boldsymbol{f} = \nabla g$ for some $g \in L^\infty(\Omega) \cap W^{1,r'}(\Omega)$, $\boldsymbol{v}_0 \equiv \mathbf{0}$, $|\Gamma_2|_{d-1} > 0$ and $p_0 \in W^{1,r'}(\Omega) \cap L^\infty(\Gamma_2)$. Moreover, assume that $\boldsymbol{A}$ is a maximal monotone r-graph in the sense of* **(A1)–(A6)**, *with **(A4)** and $\mathbf{(A5)}_{(iv)}$ replaced by **(A4*)** and $\mathbf{(A5)}_{(iv^*)}$, respectively. Then the existence result of Theorem 1.3.1 still holds.*

Proof. Take $K := \|g\|_{\infty,\Omega} + \|p_0 - g\|_{\infty,\Gamma_2}$ and recall (1.8) for the definition of T_K. The truncated problem

$$\begin{aligned} \nabla p + \boldsymbol{m} &= \nabla g && \text{in } \Omega,\\ \operatorname{div}\boldsymbol{v} &= 0 && \text{in } \Omega,\\ \boldsymbol{h}(\boldsymbol{m},\boldsymbol{v},T_K(p)) &= \mathbf{0} && \text{in } \Omega,\\ (\boldsymbol{v}-\boldsymbol{v}_0)\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma_1,\\ p - p_0 &= 0 && \text{on } \Gamma_2, \end{aligned} \tag{1.40}$$

is amenable to Theorem 1.3.1. Indeed, setting $c_1 := \min_{[-K,K]}\alpha$ and $c := \max_{[-K,K]}\beta$, we have $c_1 > 0$ and $0 < c < \infty$. Taking $\widehat{\boldsymbol{m}}^*(\boldsymbol{v},p) := \boldsymbol{m}^*(\boldsymbol{v},T_K(p))$ as a selection to be used, invoking the above mentioned theorem is just. Now Theorem 1.4.1 yields $\|p\|_\infty \le K$, which implies $p = T_K(p)$ a.e. in Ω and we are done, as problems (1.1) and (1.40) coincide. □

Remark 1.5.2 We conclude this paper with an easy observation stemming from the foregoing proof. Namely, if $\inf_{\mathbb{R}_+}\alpha > 0$ and $\sup_{\mathbb{R}_+}\beta < \infty$, there is no need for the maximum principle anymore and instead of $\boldsymbol{v}_0\cdot\boldsymbol{n} = 0$ on Γ_1, mere $\boldsymbol{v}_0\cdot\boldsymbol{n} \le 0$ on Γ_1 would suffice to ensure validity of the still indispensable minimum principle. Indeed, in $(1.40)_3$ we could just as well take

$$\boldsymbol{h}(\boldsymbol{m},\boldsymbol{v},\max\{T_K(p),p\}) = 0$$

and $\widehat{\boldsymbol{m}}^*(\boldsymbol{v}, p) := \boldsymbol{m}^*(\boldsymbol{v}, \max\{T_K(p), p\})$ for the selection. Vice versa, we need only the maximum principle, i.e. $\boldsymbol{v}_0 \cdot \boldsymbol{n} \geq 0$ on Γ_1, provided $\inf_{\mathbb{R}_-} \alpha > 0$ and $\sup_{\mathbb{R}_-} \beta < \infty$. The drag coefficient (1.5) is a prime example of such a situation.

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Chapter 2

On generalized Stokes' and Brinkman's equations with a pressure- and shear-dependent viscosity and drag coefficient

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Abstract

We study generalizations of the Darcy, Forchheimer, Brinkman and Stokes problem in which the viscosity and the drag coefficient depend on the shear rate and the pressure. We focus on existence of weak solutions to the problem, with the chief aim to capture as wide a group of viscosities and drag coefficients as mathematically feasible and to provide a theory that holds under minimal, not very restrictive conditions. Even in the case of generalized Stokes system, the established result answers a question on existence of weak solutions that has been open so far.

Keywords

Existence theory, incompressible fluid, pressure-dependent viscosity, shear-dependent viscosity, Lipschitz truncation of Sobolev functions, flow through porous media.

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2.1 Introduction

In this work we study a boundary value problem associated with a system of nonlinear partial differential equations (PDEs) that generalize the classical fluid flow models of Stokes, Darcy, Forchheimer and Brinkman. The problem considered takes the form

$$\left.\begin{aligned} -\operatorname{div}[2\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}] + \beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v} + \nabla p = \boldsymbol{f} &\quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{v} = 0 &\quad \text{in } \Omega,\\ \boldsymbol{v} = \boldsymbol{0} &\quad \text{on } \partial\Omega,\\ \frac{1}{|\Omega|}\int_\Omega p\,dx = p_0. & \end{aligned}\right\} \tag{2.1}$$

We focus on existence of its (generalized) solutions, pursuing the goal to cover as large a class of functions ν and β as possible and to provide a theory that holds under minimal, not very restrictive conditions. In the PDE problem (2.1) the set $\Omega \subset \mathbb{R}^d$ is an open, bounded, connected domain with a Lipschitz boundary and the sought-after quantities $\boldsymbol{v} : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$ correspond to the velocity and the pressure fields, respectively. The symbol $\boldsymbol{D} = \frac{1}{2}(\nabla + \nabla^T)$ stands for the symmetric part of the gradient. The external body forces $\boldsymbol{f}$ are for the sake of convenience supposed to be of the form

$$\boldsymbol{f} = -\operatorname{div}\boldsymbol{F},$$

where $\boldsymbol{F}$ is a given tensor-valued function. A prescribed value of the integral average of the pressure is given by $p_0 \in \mathbb{R}$. The PDE system (2.1) arises in the field of flows through porous media and non-Newtonian fluid mechanics. We provide more information below.

Linear examples Consider first one of the primitive cases $\beta \equiv 0$ and ν being a positive constant called *viscosity*. Then the problem (2.1) reduces to the classical Stokes equation, describing a steady (slow) flow of an incompressible fluid adhering to the boundary (by the no-slip boundary condition $(2.1)_3$) and where the pressure p is determined up to a constant specified by $(2.1)_4$. Conversely, if $\nu \equiv 0$ and β is a positive constant, the PDE system (2.1) simplifies to the standard Darcy's equation [17]. This is virtually the simplest PDE system capable of describing the flow of a single fluid through a rigid porous solid due to the pressure gradient. The number β is then the *drag coefficient*. Thirdly, if both ν and β are positive constants, (2.1) simplifies to Brinkman's equation [10, 11], representing another popular model capable of describing certain flows through porous media.

Note that each of the three aforementioned PDE systems is linear. Still, there is also an ample supply of nonlinear models belonging to the setting of (2.1), that are technologically important as they exhibit experimentally confirmed features, not captured by the said linear models. For instance, taking pressure-dependent viscosity and drag coefficient in (2.1) leads to a *ceiling flux* (a saturation phenomenon; see [43]), while approaches based on classical Darcy's and Brinkman's models result in a flux that is linearly increasing with the pressure.

Our principal interest in the present study is to analyse flows in which the material moduli—the generalized drag coefficient and viscosity—depend on the pressure and the shear rate, where dependence on the latter quantity is usually confined to $|\boldsymbol{Dv}|^2 = \boldsymbol{Dv} \cdot \boldsymbol{Dv} = \mathrm{Tr}(\boldsymbol{Dv})^2$.

Dependence on the shear rate and the pressure It has been convincingly documented in multiple studies (see e.g. [2, 9, 26, 34, 40, 45]) that the viscosity of a fluid can vary by several orders of magnitude with the pressure. Since the friction due to fluid–(rigid) solid interaction usually dominates the friction between layers of the fluid itself, the relation between the drag coefficient and the pressure is even more substantial. Likewise, the viscosity of many fluids varies with the shear rate. See for example [6] and [31] for illustrative lists of areas where incompressible fluids with shear (rate)-dependent viscosity are extensively used, ranging from geophysics, chemical engineering and bio-material science up to the food industry. Both phenomena can also play an important role in understanding the problems of enhanced oil recovery, carbon dioxide sequestration or extraction of unconventional oil deposits.

Compatibility with the second law of thermodynamics A thermodynamic basis for the derivation of the Darcy, Forchheimer and Brinkman models and their generalizations falling within the class given by $(2.1)_{1,2}$ was developed in a recent work by Srinivasan and Rajagopal [44]. The authors of that study set out from the theory of interacting continua as developed in [36, 38, 47]. Following a systematic derivation based on clearly articulated simplifications (as presented earlier in [35]), they arrive at a general reduced thermodynamical system describing steady (slow) flows of a single liquid through a rigid porous solid that takes form[1]

$$\left.\begin{aligned} -\operatorname{div}\boldsymbol{S}+\boldsymbol{m} &= -\nabla p+\rho\boldsymbol{f}, \\ \operatorname{div}\boldsymbol{v} &= 0, \\ \xi &= \boldsymbol{S}\cdot\boldsymbol{D}\boldsymbol{v}+\boldsymbol{m}\cdot\boldsymbol{v}. \end{aligned}\right\} \tag{2.2}$$

Here $\boldsymbol{S}$ stands for the *deviatoric part* of the Cauchy stress $\boldsymbol{T}$ and p for the *mean normal stress*, i.e. the pressure. In other words

$$\boldsymbol{S}=\boldsymbol{T}-\frac{1}{d}\operatorname{Tr}(\boldsymbol{T})\boldsymbol{I} \qquad \text{and} \qquad p=-\frac{1}{d}\operatorname{Tr}(\boldsymbol{T}),$$

so that $\boldsymbol{T}=\boldsymbol{S}-p\boldsymbol{I}$ with $\boldsymbol{I}$ being the identity tensor. The symbol $\boldsymbol{m}$ signifies the force acting on the fluid due to its interaction with the rigid solid and ξ stands for the rate of dissipation, which should be non-negative by the second law of thermodynamics. Note that the choice

$$\boldsymbol{S}=2\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v} \quad \text{with } \nu\geq 0, \tag{2.3}$$

$$\boldsymbol{m}=\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v} \quad \text{with } \beta\geq 0 \tag{2.4}$$

entails $\xi\geq 0$. Consequently, the model considered in our study is thermodynamically compatible. Srinivasan and Rajagopal were actually interested in more delicate issues, namely how to derive (2.3) and (2.4) purely from the knowledge of appropriately chosen constitutive equations for ξ. Towards this objective they apply the *criterion of maximal rate of entropy production*; see [44] for details.

More involved examples The constitutive equations (2.3) and (2.4) (and subsequently also the PDE problem (2.1)) include the following nonlinear models as particular cases[2]:

(i) $\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\equiv 0$ and $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)=\beta_0+\beta_1|\boldsymbol{v}|$ begets the so called Darcy-Forchheimer model [21]. Variants can be obtained by considering $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)=\beta_0+\beta_1|\boldsymbol{v}|^q$ for $q>0$.

(ii) $\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)=\nu_0\exp(\nu_1 p)$ and $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\equiv 0$ leads to the Barus model [5].

(iii) $\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)=\nu_0(\varepsilon+|\boldsymbol{D}\boldsymbol{v}|^2)^{\frac{r-2}{2}}$ and $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\equiv 0$ with $\varepsilon\geq 0$ produces the power-law fluid models (see e.g. [41, 42] and many further references listed in [14]).

(iv) $\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)=\nu_0(\varepsilon+(1+\exp(\nu_1 p))^{-q}+|\boldsymbol{D}\boldsymbol{v}|^2)^{\frac{r-2}{2}}$ and $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\equiv 0$ with $r\in(1,2)$, $\varepsilon>0$ and $q\in(0,\frac{r-1}{2\nu_1(2-r)}\varepsilon^{\frac{2-r}{2}})$ exemplifies a model for which the global-in-time existence of weak solutions was established in [29].

(v) $\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)=\dfrac{2\nu_0 p}{|\boldsymbol{D}\boldsymbol{v}|}$ and $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\equiv 0$ leads to the Schaeffer model [39], proposed to describe flowing granular materials.

[1] The velocity $\boldsymbol{v}$ in the product $\boldsymbol{m}\cdot\boldsymbol{v}$ in $(2.2)_3$ should be understood as the difference of the velocity of the fluid (which is $\boldsymbol{v}$) and the velocity of the rigid solid (which is zero).

[2] The constants ν_0, ν_1, β_0, β_1, β_2 are always assumed to be greater than zero.

(vi) $\nu(p, |\boldsymbol{Dv}|^2) \equiv 0$ and $\beta(p, |\boldsymbol{v}|, |\boldsymbol{Dv}|^2) = \beta(p, |\boldsymbol{v}|)$ brings a generalized Darcy-Forchheimer model that in the special case $\beta(p, |\boldsymbol{v}|) = \beta_0 \exp(\beta_1 p)(1 + \beta_2 |\boldsymbol{v}|^q)\boldsymbol{v}$ has recently been successfully analysed by the authors of this work. The original references concerning the physical context, solutions of semi-inverse problems and some computational results may be found in [33].

The list is meant for illustrative purposes only with no aim to be exhaustive.

Structure of the paper In Section 2.2, having got acquainted with the employed notation, we formulate assumptions specifying the admissible structure of the functions $\nu(p, |\boldsymbol{Dv}|^2)$ and $\beta(p, |\boldsymbol{v}|, |\boldsymbol{Dv}|^2)$. Then in Section 2.3 we state the main result, set it within earlier works and highlight the novel features. Section 2.4 surveys auxiliary mathematical tools used in the proof of the main result. The complete proof is then to be found in Section 2.5.

2.2 Preliminaries

Notation We utilize the standard symbolism with a few perhaps non-obvious exceptions: If $X(\Omega)$ is a Lebesgue or Sobolev space, we denote

$$\mathring{X}(\Omega) := \Big\{ f \in X(\Omega) \;\Big|\; \int_\Omega f(x)\, dx = 0 \Big\}.$$

No explicit distinction between spaces of scalar- and vector-valued functions will be made. Confusion should never come to pass as we employ small boldfaced letters to denote vectors and bolded capitals for tensors. Accordingly, for $r > 1$ we set

$$\begin{aligned} W^{1,r}_{0,\mathrm{div}}(\Omega) &:= \big\{ \boldsymbol{f} \in W^{1,r}_0(\Omega) \;\big|\; \operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega \big\}, \\ W^{-1,r'}(\Omega) &:= \big(W^{1,r}_0(\Omega) \big)^*, \\ \mathcal{C}^\infty_c(\Omega) &:= \big\{ f \in \mathcal{C}^\infty(\Omega) \;\big|\; f \text{ is compactly supported in } \Omega \big\}. \end{aligned}$$

For $f \in L^1(\Omega)$ we denote

$$f_\Omega := \frac{1}{|\Omega|} \int_\Omega f(x)\, dx.$$

For any $K > 0$ we introduce the cut-off function $T_K : \mathbb{R} \to \mathbb{R}$ as

$$T_K(x) := \begin{cases} x & \text{for } |x| < K, \\ K \dfrac{x}{|x|} & \text{for } |x| \geq K. \end{cases}$$

Completely analogously we define the cut-off function $\boldsymbol{T}_K : \mathbb{R}^d \to \mathbb{R}^d$. If $U, V \subset \mathbb{R}^d$, we say V is compactly contained in U, symbolically $V \Subset U$, if V is bounded and $\overline{V} \subset U$. The symbol $\cdot$ stands for the dot product and $\otimes$ signifies the tensor product. When an integral norm misses the set over which the integral is being taken, always Ω is implicitly considered. For $r \in (1, \infty)$ we denote $r' = r/(r-1)$ and $r^* = dr/(d-r)$, provided further $r < d$. If $r = d$, let r^* be an arbitrary number from $[1, \infty)$. The generic constants are denoted simply by C.

Assumptions on nonlinearities For the purpose of brevity, we introduce

$$\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) := 2\nu(p, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}, \tag{2.5}$$

which will be used widely throughout the paper. Let $r \in (1, 2]$ be a fixed number and $d \geq 2$. Inspired by [30], below we reproduce assumptions on the smooth nonlinearity $\nu(p, |\boldsymbol{D}\boldsymbol{v}|^2)$.

Assumption 2.2.1 *Let there be positive constants C_1 and C_2 such that for all $\boldsymbol{B}, \boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and all $p \in \mathbb{R}$*

$$C_1(1 + |\boldsymbol{D}|^2)^{(r-2)/2}|\boldsymbol{B}|^2 \leq \frac{\partial \boldsymbol{S}(p, \boldsymbol{D})}{\partial \boldsymbol{D}} \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \leq C_2(1 + |\boldsymbol{D}|^2)^{(r-2)/2}|\boldsymbol{B}|^2.$$

Assumption 2.2.2 *Let for all $\boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $p \in \mathbb{R}$*

$$\left|\frac{\partial \boldsymbol{S}(p, \boldsymbol{D})}{\partial p}\right| \leq \gamma_0(1 + |\boldsymbol{D}|^2)^{(r-2)/4}, \quad \text{with } 0 < \gamma_0 < \frac{C_1}{C_1 + C_2}.$$

As for the drag term $\beta(p, |\boldsymbol{v}|, |\boldsymbol{D}\boldsymbol{v}|^2)$, not considered in [29], we will assume it meets the following requirements:

Assumption 2.2.3 *Let $\beta : \mathbb{R} \times [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ be a continuous function for which there exist $c > 0$, $q_0 \in [1, d')$, $q_1 \in [1, r^*)$ and $q_2 \in [1, r)$ such that for all $(p, \boldsymbol{v}, \boldsymbol{D}) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$*

$$0 \leq \beta(p, |\boldsymbol{v}|, |\boldsymbol{D}|^2) \leq c(1 + |p|^{q_0} + |\boldsymbol{v}|^{q_1} + |\boldsymbol{D}|^{q_2}).$$

2.3 Main result

Without loss of generality we will suppose that $p_0 = 0$ in $(2.1)_4$, thus getting rid of an expendable symbol and making the presentation neater overall. Our paper is devoted to the justification of the following assertion:

Theorem 2.3.1 *Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be an open, bounded, connected set with a Lipschitz boundary. Consider $\boldsymbol{F} \in L^{r'}(\Omega)$, $r \in (1, 2]$ and suppose that Assumptions 2.2.1–2.2.3 hold. Then there exists a weak solution to the equation (2.1), i.e. a pair*

$$(\boldsymbol{v}, p) \in W^{1,r}_{0,\mathrm{div}}(\Omega) \times \mathring{L}^{d'}(\Omega)$$

satisfying $\beta(p, \boldsymbol{v}, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v} \in L^1(\Omega)$ and

$$\int_\Omega \left[\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) \cdot \boldsymbol{D}\boldsymbol{\varphi} + \beta(p, |\boldsymbol{v}|, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v} \cdot \boldsymbol{\varphi} - p \operatorname{div} \boldsymbol{\varphi}\right] dx = \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi}\, dx$$

for every $\boldsymbol{\varphi} \in W^{1,\infty}_0(\Omega)$.

Importance of and comparison to past results The present paper may be deemed a spiritual descendant of Bulíček and Fišerová [13] who practically further developed the work of Franta et al. [22]. These researchers investigated the model of ours, but without the drag β. On the other hand, their model contains an additional convective term $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})$, which imposes a restriction on the exponent r to be strictly greater than $2d/(d+2)$ at best. In [22], the case $r > 3d/(d+2)$ was investigated and the proof hinged on the fact that the solution velocity field $\boldsymbol{v}$ was an admissible test function. When $r > 2d/(d+2)$, as improved in [13], this is no longer the case and one has to resort to certain additional measures, namely the *Lipschitz approximation of Sobolev functions*. This powerful tool has since its inception in the paper of Acerbi and Fusco [1] been built upon and applied in numerous works (see its evolution e.g. [18], [19] and [8]).

Here we abstain purposefully from incorporating the convective term (but see Theorem 2.6.1). The point is that handling it requires a slightly stronger tool than the drag β alone, namely the Lipschitz approximation lemma from [18] instead of the primordial [1]. Thus we would have obfuscated the procedure needed for β alone. Bear in mind that only due to dropping the convective term are we able to take the generous $r > 1$, otherwise $r > 2d/(d+2)$ would have been necessary. Had we kept the convective term, for $r \in (2d/(d+2), 2)$ it would have been sufficient to copy the proof from [13], yet again at the cost of obscuring issues related to the β-term. Another reason for avoiding the convective term is usability of the PDE system (2.1), as it stands, to real world applications.

Under our assumptions, the solution $\boldsymbol{v}$ is still generally an inadmissible test function. However, as gradients of the test functions do not need to possess a very high integrability (unlike the case with the convective term present), the Lipschitz truncation method might be for $r < 2$ replaced with the *L^∞-truncation* (see [7], [23] or [37]), which may be regarded technically simpler than the Lipschitz truncation. The approach based on the L^∞-truncation method turns out insufficient when trying to cover the case $r = 2$. Interestingly enough, such a situation has been uniformly avoided in the past works ever since the inspiring [29] and [30]. In [15] the case $r = 2$ was treated only due to additional assumptions on the viscosity ν.

In this paper, we are able to contain the case $r = 2$ as a bonus, using a combination of the primeval version of the Lipschitz truncation from [1] with the well known Chacon's biting lemma [12] and the Div-Curl lemma [32, 46]. These tools are summarized in Lemmas 2.4.5–2.4.7. It is worth noting that when $\nu(p, |\boldsymbol{D}\boldsymbol{v}|^2) \equiv \nu_0$ for some $\nu_0 > 0$, the proof of Theorem 2.3.1 could be simplified considerably, although even there we would need certain nontrivial bits, specifically local regularity results (2.11) from Lemma 2.4.2. As an illustration of what specific model the case $r = 2$ covers, consider for example

$$\nu(p, |\boldsymbol{D}\boldsymbol{v}|^2) = \nu_0 + \frac{\alpha(p)}{1 + |\boldsymbol{D}\boldsymbol{v}|},$$

where $\nu_0 > 0$ is a constant and $\alpha(\cdot)$ is a smooth function satisfying

$$0 \le \alpha(\cdot) \le \alpha_0 \quad \text{for some } \alpha_0 > 0 \text{ and } \quad |\alpha'(\cdot)| \le \frac{\nu_0}{2\nu_0 + \alpha_0}.$$

It is not difficult to observe that such a situation, similar to Schaeffer's model [39] mentioned in the introductory part, falls within the framework of Theorem 2.3.1.

A natural question arises and that is whether the case $r = 2$ would admit the reintroduction of the convective term back into the equation. Without going much into details, the answer is positive. One would only have to combine our approach (based on the Biting and Div-Curl lemmas) with the procedure from [13]. We state the corresponding assertion in Theorem 2.6.1 at the end of the paper even though we do not delve into its proof.

As intimated a few lines above, the principal aim of this paper is the inclusion of the drag term β into the PDE analysis, the first such an attempt as far as we can tell.

This term allows for a super-linear growth in the pressure, while add to that, possesses *almost critical growth.* More precisely, under Assumption 2.2.3 with $r \in (1,2]$, we have $\beta(p, \boldsymbol{v}, |\boldsymbol{D}\boldsymbol{v}|^2) \in L^{1+\delta}(\Omega)$ for some $\delta > 0$, provided $(\boldsymbol{v}, p) \in W^{1,r}(\Omega) \times L^{d'}(\Omega)$. Note then $\beta(p, \boldsymbol{v}, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v}$ a priori need not even be integrable, making our investigation of particular interest.

Incidentally, we might replace the requirement on β to be non-negative with

$$\beta(p, |\boldsymbol{v}|, |\boldsymbol{D}|^2)\boldsymbol{v} \cdot \boldsymbol{v} \geq \beta_0 |\boldsymbol{v}|^2 + \beta_1 |\boldsymbol{v}|^q \quad \text{in } \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d \times d}_{\mathrm{sym}} \tag{2.6}$$

for certain $q > 2$, $\beta_0 \in \mathbb{R}$ and $\beta_1 > 0$. This would be quite useful to embrace drag coefficients of the form

$$\beta(p, |\boldsymbol{v}|, |\boldsymbol{D}\boldsymbol{v}|^2) = \beta(|\boldsymbol{v}|) = \beta_0 + \beta_1 |\boldsymbol{v}|^{q-2}.$$

The number β_0 is then called the Darcy coefficient and β_1 the Forchheimer coefficient (see [25]). We will not investigate such a digression for the difference from Assumption 2.2.3 is minimal, at least in terms of the existence theory analysis. The point is that coercivity (2.6) guarantees $\boldsymbol{v}$ to belong in $L^q(\Omega)$, which in turn may allow to slacken the growth conditions on the drag coefficient in Assumption 2.2.3.

Unlike, for instance, the classical Navier-Stokes equations, some kind of a pressure anchorage in (2.1) is necessary, hence $(2.1)_4$. The reason is that in our model, not only the pressure gradient is present but there is dependence also on the pressure itself. From the practical viewpoint it would make more sense to prescribe values of the pressure pointwise, for example along a part of the boundary (the so called *accessible boundary* [3]). Unfortunately, the pressure constructed here is only an integrable function so we cannot refer to its point values. In this case one could take for instance the integral average over a (possibly small) set $\Omega_0 \subset \Omega$, thus approximating the pointwise prescription (see [16]). In this paper we chose fixing p_Ω in the spirit of [22], as the generalization p_{Ω_0} could easily be made but it is not the gist of this paper. A reader requiring more information on this topic should address for example [13] and the references given there. A similar argument applies to our choice of the boundary condition. We are aware of the fact that for problems connected with flows through porous media, the boundary condition $(2.1)_3$ is rather crude as one usually prescribes e.g. the inflow/outflow velocity along parts of the boundary. The no-slip condition could well be generalized but we picked this one as it makes the analysis most translucent. For more information concerning alternative boundary conditions for the velocity and the pressure alike consult e.g. [27].

Lastly, it is worth remarking that the upper bound on the value of γ_0 in Assumption 2.2.2 has since [13, 22] been improved. In other words, our viscosity ν allows a faster growth rate in the pressure variable, albeit still a sublinear one. Aside from C_1 and C_2, the bound γ_0 also used to detrimentally depend on geometry of the set Ω through the Bogovskiĭ operator on Ω (for more information about the constant see [24, Lemma III.3.1]). The idea behind the enhancement in our work is to replace the Bogovskiĭ operator with the Newtonian potential at some point. We recall the key properties of the Newtonian potential in Lemma 2.4.4.

Highlights We want to conclude this part listing the principal contributions of this paper:

1. We establish large-data existence theory for a generalized Brinkman problem with the viscosity and drag coefficients depending on the pressure and the shear rate;

see Theorem 2.3.1. To the best of our knowledge, a PDE analysis for similar problems with a pressure- and shear-dependent drag satisfying Assumption 2.2.3 has not been carried out yet.

2. Within the setting considered, even for a generalized Stokes problem (i.e. $\beta = 0$) we establish new results when $r = 2$, thus improving the works [13] and [22]; see Theorem 2.6.1.

3. The earlier studies concerning the PDE analysis of a generalized Stokes' problem with $\nu(p, |\boldsymbol{Dv}|^2)$ in general bounded domains suffered a serious drawback. The parameter γ_0 appearing in Assumption 2.2.2 used to be restricted by a constant depending on the geometry of the set Ω. This severe constraint has been removed here. The theory presented in this work thus holds under the same restrictions as the theory developed for an (idealized) spatially periodic problem in [29].

2.4 Auxiliary tools

In this section we survey a couple of results exploited in the proof of Theorem 2.3.1. First off, we state what one might call a *compensated monotonicity* of the nonlinearity $\boldsymbol{S}$, as well as coercivity and boundedness thereof.

Lemma 2.4.1 ([22], Lemmas 3.3, 3.4) *Let Assumptions 2.2.1 and 2.2.2 hold. For arbitrary $\boldsymbol{D}^1, \boldsymbol{D}^2 \in \mathbb{R}^{d\times d}_{\text{sym}}$ and $p^1, p^2 \in \mathbb{R}$ we set*

$$I^{1,2} := \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}^1 - \boldsymbol{D}^2|^2 \, ds,$$

with $\overline{\boldsymbol{D}}(s) = \boldsymbol{D}^2 + s(\boldsymbol{D}^1 - \boldsymbol{D}^2)$. Then

$$\frac{1}{2} C_1 I^{1,2} \leq (\boldsymbol{S}(p^1, \boldsymbol{D}^1) - \boldsymbol{S}(p^2, \boldsymbol{D}^2)) \cdot (\boldsymbol{D}^1 - \boldsymbol{D}^2) + \frac{\gamma_0^2}{2C_1} |p^1 - p^2|^2. \tag{2.7}$$

Furthermore

$$|(\boldsymbol{S}(p^1, \boldsymbol{D}^1) - \boldsymbol{S}(p^2, \boldsymbol{D}^2))| \leq \gamma_0 |p^1 - p^2| + C_2 \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}^1 - \boldsymbol{D}^2| \, ds. \tag{2.8}$$

Finally, for all $p \in \mathbb{R}$, $r \in (1, 2]$ and $\boldsymbol{D} \in \mathbb{R}^{d\times d}_{\text{sym}}$

$$\boldsymbol{S}(p, \boldsymbol{D}) \cdot \boldsymbol{D} \geq \frac{C_1}{2r} (|\boldsymbol{D}|^r - 1) \tag{2.9}$$

and

$$|\boldsymbol{S}(p, \boldsymbol{D})| \leq \frac{C_2}{r-1} (1 + |\boldsymbol{D}|)^{r-1}. \tag{2.10}$$

The corresponding statement in [22] does not include (2.8). However, it is only an easy observation stemming from

$$\boldsymbol{S}(p^1, \boldsymbol{D}^1) - \boldsymbol{S}(p^2, \boldsymbol{D}^2) = \int_0^1 \frac{d}{ds} \boldsymbol{S}(p^2 + s(p^1 - p^2), \boldsymbol{D}^2 + s(\boldsymbol{D}^1 - \boldsymbol{D}^2)) \, ds$$

and Assumptions 2.2.1 and 2.2.2.

On occasion, we will use the theory for the Stokes problem. All necessary ingredients are compiled in the lemma below. Beware of our extracting only what is to be needed for purposes of this paper, as we deem stating these theorems in their full form rather distracting.

Lemma 2.4.2 ([24], Theorems IV.1.1, IV.4.1, IV.4.4) *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $d \geq 2$. There exists a continuous linear operator*

$$\boldsymbol{H} : W^{-1,2}(\Omega) \longrightarrow W^{1,2}_{0,\mathrm{div}}(\Omega) \times \mathring{L}^2(\Omega)$$

assigning to any $\boldsymbol{f} \in W^{-1,2}(\Omega)$ the unique weak solution $(\boldsymbol{v}, p)$ of the Stokes problem

$$\begin{aligned} -\Delta \boldsymbol{v} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{v} &= \boldsymbol{0} \quad \text{on } \partial\Omega, \\ p_\Omega &= 0. \end{aligned}$$

Moreover, if $\boldsymbol{f} \in W^{-1,2}(\Omega) \cap W^{k,q}_{loc}(\Omega)$ for certain $1 < q < \infty$ and $k \geq -1$, then $\boldsymbol{H}(\boldsymbol{f}) \in W^{k+2,q}_{loc}(\Omega) \times W^{k+1,q}_{loc}(\Omega)$ and one has the estimate

$$\left\|\nabla^{k+2}\boldsymbol{v}\right\|_{q;\Omega''} + \left\|\nabla^{k+1}p\right\|_{q;\Omega''} \leq c\big(\|\boldsymbol{f}\|_{k,q;\Omega'} + \|\boldsymbol{v}\|_{k+1,q;\Omega'} + \|p\|_{k,q;\Omega'}\big). \tag{2.11}$$

for any $\Omega'' \Subset \Omega' \Subset \Omega$, where $c = c(d, q, k, \Omega', \Omega'')$.

Aside from the Stokes problem, we will have to be capable of dealing effectively with the divergence equation. The following statement about the Bogovskiĭ operator provides us with a necessary tool.

Lemma 2.4.3 ([24], Theorem III.3.3) *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $d \geq 2$ and $1 < q < \infty$. There is a continuous linear operator*

$$\mathcal{B} : \mathring{L}^q(\Omega) \longrightarrow W^{1,q}_0(\Omega)$$

assigning to any $f \in \mathring{L}^q(\Omega)$ a weak solution $\boldsymbol{v}$ of the divergence equation

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= f \quad \text{in } \Omega, \\ \boldsymbol{v} &= \boldsymbol{0} \quad \text{on } \partial\Omega. \end{aligned}$$

The Bogovskiĭ operator will at times be replaced with the Newtonian potential. Then the following result will be used:

Lemma 2.4.4 *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and $f \in L^q(\Omega)$, $q \in (1, \infty)$. Denote $\widetilde{f}$ the zero extension of f on the whole space $\mathbb{R}^d$ and Γ the Newtonian kernel in R^d, i.e.*

$$\Gamma(x) = \begin{cases} -\dfrac{1}{2\pi}\log|x| & \text{for } d = 2, \\ \dfrac{1}{d(d-2)\omega_d}|x|^{2-d} & \text{for } d > 2, \end{cases}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. Define

$$\mathcal{N}(f) := (\widetilde{f} * \Gamma)\big|_{\Omega}.$$

Then $\mathcal{N}$ is continuous from $L^q(\Omega)$ into $W^{2,q}(\Omega)$ and for $q = 2$ one has

$$\left\|\nabla^2 \mathcal{N}(f)\right\|_2 \le \|f\|_2 .$$

Proof. We only sketch out the proof as the result is standard. Continuity from $L^q(\Omega)$ into $W^{1,q}(\Omega)$ follows from Young's inequality for convolutions and boundedness of Ω. In order to bound the second gradients, employ the Calderón-Zygmund theory for singular operators; see [20, Theorem 10.10].

As for the last inequality, we have $-\Delta(\widetilde{f} * \Gamma) = \widetilde{f}$ a.e. in $\mathbb{R}^d$ and $\|\nabla^2 g\|_{2;\mathbb{R}^d} = \|\Delta g\|_{2;\mathbb{R}^d}$ holding for any $g \in W^{2,2}(\mathbb{R}^d)$. Hence

$$\left\|\nabla^2 \mathcal{N}(f)\right\|_2 \le \left\|\nabla^2(\widetilde{f} * \Gamma)\right\|_{2;\mathbb{R}^d} = \left\|\Delta(\widetilde{f} * \Gamma)\right\|_{2;\mathbb{R}^d} = \left\|\widetilde{f}\right\|_{2;\mathbb{R}^d} = \left\|f\right\|_2.$$

□

For the sake of completeness, we explicitly formulate yet three classical results here, namely Chacon's biting lemma [12], Murat's and Tartar's Div-Curl lemma [32, 46] and Acerbi's and Fusco's Lipschitz approximation of Sobolev functions [1]:

Lemma 2.4.5 (Biting lemma, [4]) *Let $\Omega \subset \mathbb{R}^d$ have a finite Lebesgue measure and $\{f^k\}$ be a bounded sequence in $L^1(\Omega)$. Then there exist a function $f \in L^1(\Omega)$, a subsequence $\{f^j\}$ of $\{f^k\}$ and a nonincreasing sequence of measurable sets $E_n \subset \Omega$ with $\lim_{n\to\infty} |E^n| = 0$, such that $f^j \rightharpoonup f$ in $L^1(\Omega \setminus E_n)$ for every fixed n.*

Lemma 2.4.6 (Div-Curl lemma, [20], Theorem 10.21) *Let $\Omega \subset \mathbb{R}^d$ be open. Assume $\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}$ in $L^p(\Omega)$ and $\boldsymbol{v}^n \rightharpoonup \boldsymbol{v}$ in $L^q(\Omega)$, where $1/p + 1/q = 1/r < 1$. In addition, let $\{\operatorname{div} \boldsymbol{u}^n\}$ be relatively compact in $W^{-1,s}(\Omega)$ and $\{\operatorname{curl} \boldsymbol{v}^n\}$[3] be relatively compact in $W^{-1,s}(\Omega)$ for a certain $s > 1$. Then $\boldsymbol{u}^n \cdot \boldsymbol{v}^n \rightharpoonup \boldsymbol{u} \cdot \boldsymbol{v}$ in $L^r(\Omega)$.*

Lemma 2.4.7 (Lipschitz approximation of Sobolev functions, [1]) *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz open set and $p \ge 1$. There exists a constant c such that, for every $u \in W^{1,p}(\Omega)$ and every $\lambda > 0$ there exists $u_\lambda \in W^{1,\infty}(\Omega)$ satisfying*

$$\|u_\lambda\|_{1,\infty} \le \lambda, \tag{2.12}$$

$$|\{u \ne u_\lambda\}| \le c\frac{\|u\|_{1,p}^p}{\lambda^p}, \tag{2.13}$$

$$\|u_\lambda\|_{1,p} \le c\,\|u\|_{1,p} . \tag{2.14}$$

Strictly speaking, the bound (2.14) does not appear in [1]. It is, however, a trivial consequence of (2.12) and (2.13). Secondly, the original result [1] mentions only a *regular* set Ω. Since this regularity is required for a $W^{1,p}$-continuous extension operator, Lipschitz sets are perfectly acceptable.

[3] $\operatorname{curl} = \frac{1}{2}(\nabla - \nabla^T)$

2.5 Proof of the existence theorem

Solutions asserted by Theorem 2.3.1 will be found as a weak limit of a twofold approximation scheme: One is the so called *quasicompressible approximation* (see [22]), which serves to construct at least some kind of a pressure as a solution to an elliptic problem featuring divergence of the velocity field. The term *quasicompressible* is motivated by the fact that the resultant velocity is only *almost* solenoidal (see below). In our exposition we identify this modification with the parameter ε and the goal is to perform $\varepsilon \to 0_+$. The second level is an L^∞*-truncation* of the β-term, the necessity of which is attributable to quite draconian growth conditions in Assumption 2.2.3. This level is associated with the parameter K and our plan is to justify $K \to \infty$. Nonetheless, we have to show any such an approximation exists for each ε and K in the first place.

Lemma 2.5.1 *Under the assumptions of Theorem 2.3.1, for every $\varepsilon, K > 0$ there exist $(\boldsymbol{v}^{\varepsilon,K}, p^{\varepsilon,K}) \in W_0^{1,r}(\Omega) \times \big(\mathring{W}^{1,2}(\Omega) \cap \mathring{L}^{r'}(\Omega)\big)$ satisfying*

$$\varepsilon \int_\Omega \nabla p^{\varepsilon,K} \cdot \nabla \varphi \, dx + \int_\Omega \varphi \operatorname{div} \boldsymbol{v}^{\varepsilon,K} \, dx = 0 \quad \textit{for all } \varphi \in W^{1,2}(\Omega) \cap L^{r'}(\Omega) \tag{2.15}$$

and

$$\int_\Omega \big[S(p^{\varepsilon,K}, \boldsymbol{D}\boldsymbol{v}^{\varepsilon,K}) \cdot \boldsymbol{D}\boldsymbol{\varphi} + T_K \beta(p^{\varepsilon,K}, |\boldsymbol{v}^{\varepsilon,K}|, |\boldsymbol{D}\boldsymbol{v}^{\varepsilon,K}|^2) \boldsymbol{T}_K \boldsymbol{v}^{\varepsilon} \cdot \boldsymbol{\varphi} - p^{\varepsilon,K} \operatorname{div} \boldsymbol{\varphi}\big] dx$$
$$= \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi} \, dx \quad \textit{for all } \boldsymbol{\varphi} \in W_0^{1,r}(\Omega). \tag{2.16}$$

Proof. We will drop the ε, K-indices for the sake of a neater notation. Let $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}} \subset W_0^{1,2}(\Omega)$ and $\{z_i\}_{i\in\mathbb{N}} \subset \mathring{W}^{1,2}(\Omega) \cap \mathring{L}^{r'}(\Omega)$ be linearly independent, with linear spans dense in the respective spaces. To begin with, we will deduce existence of solutions to an approximate problem, i.e. for $n \in \mathbb{N}$ we seek

$$\boldsymbol{v}^n(x) = \sum_{i=1}^n a_i^n \boldsymbol{w}_i(x),$$
$$p^n(x) = \sum_{i=1}^n b_i^n z_i(x),$$

satisfying

$$\varepsilon \int_\Omega \nabla p^n \cdot \nabla z_i \, dx + \int_\Omega z_i \operatorname{div} \boldsymbol{v}^n \, dx = 0, \tag{2.17}$$

$$\int_\Omega \boldsymbol{S}(p^n, \boldsymbol{D}\boldsymbol{v}^n) \cdot \boldsymbol{D}\boldsymbol{w}_i \, dx + \int_\Omega \beta^n \boldsymbol{T}_K \boldsymbol{v}^n \cdot \boldsymbol{w}_i \, dx - \int_\Omega p^n \operatorname{div} \boldsymbol{w}_i \, dx = \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{w}_i \, dx \tag{2.18}$$

L^∞-truncation for $i = 1, \dots, n$, recalling (2.5) and setting $\beta^n := T_K \beta(p^n, |\boldsymbol{v}^n|, |\boldsymbol{D}\boldsymbol{v}^n|^2)$.

Towards showing the existence of $\{a_i^n\}_{i=1}^n$ and $\{b_i^n\}_{i=1}^n$, we employ the standard corollary of Brouwer's fixed point theorem [28, Lemme 4.3]. Its applicability follows from the oncoming lines and will not be discussed in detail. Our undivided attention is zoomed in on the limit passage $n \to \infty$.

Multiplying eq. (2.17) by b_i^n and eq. (2.18) by a_i^n and summing the resultant $2n$ equalities, we obtain

$$\varepsilon \|\nabla p^n\|_2^2 + \int_\Omega \boldsymbol{S}(p^n, \boldsymbol{D}\boldsymbol{v}^n) \cdot \boldsymbol{D}\boldsymbol{v}^n \, dx + \int_\Omega \beta^n \boldsymbol{T}_K \boldsymbol{v}^n \cdot \boldsymbol{v}^n \, dx = \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{v}^n \, dx.$$

Now we recall the coercivity condition (2.9), Korn's, Young's and Hölder's inequalities, non-negativity of β^n and the fact that $\boldsymbol{T}_K \boldsymbol{v}^n \cdot \boldsymbol{v}^n \geq 0$, deducing

$$\sup_n \left(\varepsilon \|\nabla p^n\|_2^2 + \|\boldsymbol{D}\boldsymbol{v}^n\|_r^r \right) < \infty.$$

By Korn's and Poincaré's inequalities and the bound (2.10), we may select a subsequence (labelled again $(p^n, \boldsymbol{v}^n)$) such that for $n \to \infty$[4]

$$\begin{aligned}
\boldsymbol{v}^n &\rightharpoonup \boldsymbol{v} && \text{weakly in } W_0^{1,r}(\Omega),\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{a.e. in } \Omega,\\
p^n &\rightharpoonup p && \text{weakly in } \mathring{W}^{1,2}(\Omega),\\
p^n &\to p && \text{strongly in } L^2(\Omega),\\
p^n &\to p && \text{a.e. in } \Omega,\\
\boldsymbol{S}(p^n, \boldsymbol{D}\boldsymbol{v}^n) &\rightharpoonup \overline{\boldsymbol{S}} && \text{weakly in } L^{r'}(\Omega),\\
\beta^n \boldsymbol{T}_K \boldsymbol{v}^n &\rightharpoonup \overline{\beta \boldsymbol{v}} && \text{weakly in } L^q(\Omega) \text{ for any } q \in [1, \infty).
\end{aligned} \tag{2.19}$$

Letting $n \to \infty$ in the approximate eq. (2.17) and the density of z_i in $\mathring{W}^{1,2}(\Omega)$ guarantee (2.15). Similarly, letting $n \to \infty$ in the approximation (2.18) implies

$$\int_\Omega \overline{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{\varphi} \, dx + \int_\Omega \overline{\beta \boldsymbol{v}} \cdot \boldsymbol{\varphi} \, dx - \int_\Omega p \operatorname{div} \boldsymbol{\varphi} \, dx = \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi} \, dx \quad \text{for all } \boldsymbol{\varphi} \in W_0^{1,2}(\Omega). \tag{2.20}$$

Now we need to show $p \in L^{r'}(\Omega)$. To this end, let $L > 0$ and define ξ_L as the indicator function of $\{|p| < L\}$. Recalling Lemma 2.4.3 on the Bogovskiĭ operator, we set

$$\boldsymbol{\varphi} := \mathcal{B}\big(|p|^{r'-2} p \xi_L - (|p|^{r'-2} p \xi_L)_\Omega\big).$$

In particular, such a $\boldsymbol{\varphi}$ can be used in (2.20) and by the continuity of $\mathcal{B}$

$$\|\boldsymbol{\varphi}\|_{1,r} \leq C \big\| |p|^{r'-1} \xi_L \big\|_r = C \|p \xi_L\|_{r'}^{r'-1}.$$

As $p_\Omega = 0$, plugging $\boldsymbol{\varphi}$ into eq. (2.20) and recalling (2.19) leads to

$$\|p\xi_L\|_{r'}^{r'} = \int_\Omega \overline{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{\varphi} \, dx + \int_\Omega \overline{\beta \boldsymbol{v}} \cdot \boldsymbol{\varphi} \, dx - \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi} \, dx \leq C \|\boldsymbol{\varphi}\|_{1,r} \leq C \|p\xi_L\|_{r'}^{r'-1}.$$

Since C is independent of L, we obtain $p \in L^{r'}(\Omega)$. Thus Eq. (2.20) holds for any $\boldsymbol{\varphi} \in W_0^{1,r}(\Omega)$.

What remains is the identification of the nonlinear terms $\overline{\boldsymbol{S}}$ and $\overline{\beta \boldsymbol{v}}$. Considering the continuity of ν and β and the convergences $(2.19)_2$ and $(2.19)_5$, it is sufficient to verify the pointwise convergence of $\boldsymbol{D}\boldsymbol{v}^n$ a.e. in Ω. Then $\overline{\boldsymbol{S}} = \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$ and $\overline{\beta \boldsymbol{v}} = T_K \beta\big(p, |\boldsymbol{v}|, |\boldsymbol{D}\boldsymbol{v}|^2\big) \boldsymbol{T}_K \boldsymbol{v}$ by Vitali's theorem. We will, however, take these identities for granted and skip the derivation of the pointwise convergence of $\boldsymbol{D}\boldsymbol{v}^n$, as it will once again be reiterated in the following section under more inimical conditions, that time in detail. □

[4]We employ bars for unidentified weak limits.

2.5.1 Vanishing artificial compressibility ($\varepsilon \to 0_+$)

Now we justify the limit passage $\varepsilon \to 0_+$ for solutions yielded by Lemma 2.5.1. Let us again drop the index K and denote the solutions at hand simply $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$.

Uniform estimates Taking $\varphi = p^\varepsilon$ in (2.15), $\boldsymbol{\varphi} = \boldsymbol{v}^\varepsilon$ in (2.16) and summing up the resultant identities, we obtain

$$\varepsilon\big\|\nabla p^\varepsilon\big\|_2^2 + \int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)\cdot \boldsymbol{D}\boldsymbol{v}^\varepsilon\,dx + \int_\Omega \beta^\varepsilon \boldsymbol{T}_K\boldsymbol{v}^\varepsilon\cdot\boldsymbol{v}^\varepsilon\,dx = \int_\Omega \boldsymbol{F}\cdot\nabla\boldsymbol{v}^\varepsilon\,dx,$$

where $\beta^\varepsilon := T_K\beta(p^\varepsilon, |\boldsymbol{v}^\varepsilon|, |\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2)$.

Using $\beta^\varepsilon\boldsymbol{T}_K\boldsymbol{v}^\varepsilon\cdot\boldsymbol{v}^\varepsilon \geq 0$, the property (2.9), Poincaré's, Young's and Korn's inequalities, we observe

$$\sup_\varepsilon \sqrt{\varepsilon}\big\|\nabla p^\varepsilon\big\|_2 < \infty, \tag{2.21}$$

$$\sup_\varepsilon \|\boldsymbol{v}^\varepsilon\|_{1,r} < \infty, \tag{2.22}$$

the latter of which we further combine with (2.10), deducing

$$\sup_\varepsilon \big\|\boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)\big\|_{r'} < \infty. \tag{2.23}$$

As for bounds on the pressure, we can copy the procedure from the previous proof. Setting

$$\boldsymbol{\varphi} := \mathcal{B}\big(|p^\varepsilon|^{r'-2}p^\varepsilon - (|p^\varepsilon|^{r'-2}p^\varepsilon)_\Omega\big),$$

we observe that $\boldsymbol{\varphi} \in W_0^{1,r}(\Omega)$ and furthermore $\big\|\boldsymbol{\varphi}\big\|_{1,r} \leq C\big\||p^\varepsilon|^{r'-1}\big\|_r = C\big\|p^\varepsilon\big\|_{r'}^{r'-1}$ due to continuity of $\mathcal{B}$, with C independent of ε. Recalling that $(p^\varepsilon)_\Omega = 0$, the insertion of $\boldsymbol{\varphi}$ into (2.16) hence produces

$$\begin{aligned}\|p^\varepsilon\|_{r'}^{r'} &= \int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)\cdot\boldsymbol{D}\boldsymbol{\varphi}\,dx + \int_\Omega \beta^\varepsilon\boldsymbol{T}_K\boldsymbol{v}^\varepsilon\cdot\boldsymbol{\varphi}\,dx - \int_\Omega \boldsymbol{F}\cdot\nabla\boldsymbol{\varphi}\,dx \leq C\,\|\boldsymbol{\varphi}\|_{1,r}\\ &\leq C\,\|p^\varepsilon\|_{r'}^{r'-1}\end{aligned}$$

and thus we infer

$$\sup_\varepsilon \|p^\varepsilon\|_{r'} < \infty. \tag{2.24}$$

The bounds (2.21)–(2.24) imply that we may assume the following convergences as $\varepsilon \to 0_+$:

$$\begin{aligned}
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} && \text{weakly in } W_0^{1,r}(\Omega),\\
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} && \text{a.e. in } \Omega,\\
p^\varepsilon &\to p && \text{weakly in } L^{r'}(\Omega),\\
\varepsilon\nabla p^\varepsilon &\to \boldsymbol{0} && \text{strongly in } L^2(\Omega),\\
\boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) &\to \overline{\boldsymbol{S}} && \text{weakly in } L^{r'}(\Omega),\\
\beta^\varepsilon\boldsymbol{T}_K\boldsymbol{v}^\varepsilon &\to \overline{\beta\boldsymbol{v}} && \text{weakly in } L^q(\Omega) \text{ for any } q \in [1,\infty).
\end{aligned} \tag{2.25}$$

The limit $\varepsilon \to 0_+$ applied to eq. (2.15) then guarantees $\operatorname{div} \boldsymbol{v} = 0$ a.e. in Ω and eq. (2.16) yields

$$\int_\Omega \overline{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{\varphi}\, dx + \int_\Omega \overline{\beta \boldsymbol{v}} \cdot \boldsymbol{\varphi}\, dx - \int_\Omega p \operatorname{div} \boldsymbol{\varphi}\, dx = \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi}\, dx \quad \text{for all } \boldsymbol{\varphi} \in W_0^{1,r}(\Omega).$$

Furthermore, since $\mathring{L}^{r'}(\Omega)$ is a weakly closed subset of $L^{r'}(\Omega)$, the property $p_\Omega = 0$ has retained.

We have yet to identify the nonlinear terms $\overline{\boldsymbol{S}}$ and $\overline{\beta \boldsymbol{v}}$. The objective is to verify the pointwise convergence of p^ε and $\boldsymbol{D}\boldsymbol{v}^\varepsilon$. Then $\overline{\boldsymbol{S}} = \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$ and $\overline{\beta \boldsymbol{v}} = T_K \beta\big(p, |\boldsymbol{v}|, |\boldsymbol{D}\boldsymbol{v}|^2\big) \boldsymbol{T}_K \boldsymbol{v}$ by $(2.25)_5$, $(2.25)_6$ and Vitali's theorem. It suffices to prove these pointwise convergences in an arbitrary compactly contained subdomain $\Omega' \Subset \Omega$.

Convergence of p^ε Let $\eta \in \mathcal{C}_c^\infty(\Omega)$ be such that $0 \le \eta \le 1$ and $\eta \equiv 1$ in Ω'. Recall the operator $\mathcal{N}$ from Lemma 2.4.4 and set $u^\varepsilon = \mathcal{N}((p^\varepsilon - p)\eta)$. Note

$$u^\varepsilon \to 0 \quad \text{weakly in } W^{2,2}(\Omega), \tag{2.26}$$

$$u^\varepsilon \to 0 \quad \text{strongly in } W^{1,2}(\Omega) \tag{2.27}$$

by the continuity of $\mathcal{N}$ and a compact embedding, respectively. Now

$$\big\|(p^\varepsilon - p)\eta\big\|_2^2 = -\int_\Omega p^\varepsilon \eta \Delta u^\varepsilon\, dx + \int_\Omega p \eta \Delta u^\varepsilon\, dx, \tag{2.28}$$

and the second integral tends to zero as $\varepsilon \to 0_+$ by (2.26). We develop the first term:

$$-\int_\Omega p^\varepsilon \eta \Delta u^\varepsilon\, dx = -\int_\Omega p^\varepsilon \operatorname{div}(\eta \nabla u^\varepsilon)\, dx + \int_\Omega p^\varepsilon \nabla \eta \cdot \nabla u^\varepsilon\, dx.$$

As $\varepsilon \to 0_+$ the second term again approaches zero by (2.27). At this moment the reason for adding η is becoming apparent, namely to ensure the zero trace of $\eta \nabla u^\varepsilon$. Onwards, by (2.16) we have

$$\begin{aligned} -\int_\Omega p^\varepsilon \operatorname{div}(\eta \nabla u^\varepsilon)\, dx = -\int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \nabla(\eta \nabla u^\varepsilon)\, dx - \int_\Omega \beta^\varepsilon \boldsymbol{T}_K \boldsymbol{v}^\varepsilon \cdot \eta \nabla u^\varepsilon\, dx \\ + \int_\Omega \boldsymbol{F} \cdot \nabla(\eta \nabla u^\varepsilon)\, dx. \end{aligned}$$

The latter two terms tend to zero by (2.26) and (2.27), as $\{\beta^\varepsilon \boldsymbol{T}_K \boldsymbol{v}^\varepsilon\}$ is still bounded in $L^\infty(\Omega)$. Further

$$\begin{aligned} &-\int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \nabla(\eta \nabla u^\varepsilon)\, dx \\ &\qquad = -\int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \eta \nabla^2 u^\varepsilon\, dx - \int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot (\nabla u^\varepsilon \otimes \nabla \eta)\, dx \end{aligned}$$

and the last term converges to zero by (2.27). Lastly

$$\begin{aligned} &-\int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \eta \nabla^2 u^\varepsilon\, dx \\ &\qquad = -\int_\Omega \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) \cdot \eta \nabla^2 u^\varepsilon\, dx + \int_\Omega (\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)) \cdot \eta \nabla^2 u^\varepsilon\, dx. \end{aligned}$$

The first integral on the right-hand side vanishes for $\varepsilon \to 0_+$ by (2.26). The second one will be handled by means of the pointwise estimate (2.8) as

$$\int_\Omega (\boldsymbol{S}(p,\boldsymbol{Dv}) - \boldsymbol{S}(p^\varepsilon,\boldsymbol{Dv}^\varepsilon)) \cdot \eta\nabla^2 u^\varepsilon \,dx$$
$$\leq \gamma_0 \int_\Omega |(p-p^\varepsilon)\eta||\nabla^2 u^\varepsilon|\,dx + C_2 \int_\Omega \int_0^1 (1+|\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)||\nabla^2 u^\varepsilon|\eta \,ds\,dx, \tag{2.29}$$

with $\overline{\boldsymbol{D}}(s) = \boldsymbol{Dv}^\varepsilon + s(\boldsymbol{Dv} - \boldsymbol{Dv}^\varepsilon)$. Denote

$$I^\varepsilon = \int_0^1 (1+|\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)|^2 ds.$$

Since $(1+|\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} \leq (1+|\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/4}$ and $\eta \leq \sqrt{\eta}$, Hölder's inequality and Lemma 2.4.4 applied on (2.29) yield

$$\begin{aligned}\int_\Omega (\boldsymbol{S}(p,\boldsymbol{Dv}) - \boldsymbol{S}(p^\varepsilon,\boldsymbol{Dv}^\varepsilon)) \cdot \eta\nabla^2 u^\varepsilon \,dx &\leq \gamma_0 \big\|(p^\varepsilon - p)\eta\big\|_2^2 \\ &\quad + C_2 \Big(\int_\Omega I^\varepsilon \eta\,dx\Big)^{1/2} \big\|(p^\varepsilon - p)\eta\big\|_2 \\ &\leq \frac{1+\gamma_0}{2}\big\|(p^\varepsilon - p)\eta\big\|_2^2 + \frac{C_2^2}{2(1-\gamma_0)}\left\|I^\varepsilon \eta\right\|_1.\end{aligned} \tag{2.30}$$

It remains to estimate $\left\|I^\varepsilon\eta\right\|_1$. Using (2.7), we have

$$\frac{C_1}{2}\left\|I^\varepsilon\eta\right\|_1 \leq \int_\Omega (\boldsymbol{S}(p,\boldsymbol{Dv}) - \boldsymbol{S}(p^\varepsilon,\boldsymbol{Dv}^\varepsilon)) \cdot \boldsymbol{D}(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)\eta\,dx + \frac{\gamma_0^2}{2C_1}\big\|(p^\varepsilon - p)\eta\big\|_2^2. \tag{2.31}$$

The property (2.10) and the convergence $(2.25)_1$ yield

$$\lim_{\varepsilon\to 0_+} \int_\Omega \boldsymbol{S}(p,\boldsymbol{Dv}) \cdot \boldsymbol{D}(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)\eta\,dx = 0.$$

Towards handling the other integral in (2.31), we set $\boldsymbol{\varphi}_\varepsilon = (\boldsymbol{v}-\boldsymbol{v}^\varepsilon)\eta$ and write

$$\begin{aligned}&\int_\Omega \boldsymbol{S}(p^\varepsilon,\boldsymbol{Dv}^\varepsilon) \cdot \boldsymbol{D}(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)\eta\,dx \\ &\qquad = \int_\Omega \boldsymbol{S}(p^\varepsilon,\boldsymbol{Dv}^\varepsilon) \cdot \boldsymbol{D}\boldsymbol{\varphi}_\varepsilon\,dx - \int_\Omega \boldsymbol{S}(p^\varepsilon,\boldsymbol{Dv}^\varepsilon) \cdot (\nabla\eta \otimes (\boldsymbol{v}-\boldsymbol{v}^\varepsilon))\,dx.\end{aligned}$$

The latter integral vanishes for $\varepsilon \to 0_+$ by (2.25). As for the former, we employ the weak formulation (2.16) tested with $\boldsymbol{\varphi}_\varepsilon = (\boldsymbol{v}-\boldsymbol{v}^\varepsilon)\eta$:

$$\int_\Omega \boldsymbol{S}(p^\varepsilon,\boldsymbol{Dv}^\varepsilon) \cdot \boldsymbol{D}\boldsymbol{\varphi}_\varepsilon\,dx = \int_\Omega p^\varepsilon \operatorname{div}\boldsymbol{\varphi}_\varepsilon\,dx - \int_\Omega \beta^\varepsilon \boldsymbol{T}_K \boldsymbol{v}^\varepsilon \cdot \boldsymbol{\varphi}_\varepsilon\,dx + \int_\Omega \boldsymbol{F}\cdot\nabla\boldsymbol{\varphi}_\varepsilon\,dx. \tag{2.32}$$

The last two terms vanish for $\varepsilon \to 0_+$ by (2.25). As for the first one, we recall eq. (2.15) and write

$$\begin{aligned}\int_\Omega p^\varepsilon \operatorname{div}\boldsymbol{\varphi}_\varepsilon\,dx &= -\int_\Omega p^\varepsilon\eta \operatorname{div}\boldsymbol{v}^\varepsilon\,dx + \int_\Omega p^\varepsilon(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)\cdot\nabla\eta\,dx \\ &= \varepsilon\int_\Omega \nabla(p^\varepsilon\eta)\cdot\nabla p^\varepsilon\,dx + \int_\Omega p^\varepsilon(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)\cdot\nabla\eta\,dx \\ &= \varepsilon\int_\Omega |\nabla p^\varepsilon|^2\eta\,dx + \varepsilon\int_\Omega p^\varepsilon\nabla p^\varepsilon\cdot\nabla\eta\,dx + \int_\Omega p^\varepsilon(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)\cdot\nabla\eta\,dx.\end{aligned}$$

From the convergences (2.25), we hence elicit

$$\liminf_{\varepsilon\to 0_+} \int_\Omega p^\varepsilon \operatorname{div} \boldsymbol{\varphi}_\varepsilon \, dx \geq 0.$$

Plugging this result into (2.32), the entire first integral on the right in (2.31) therefore satisfies

$$\limsup_{\varepsilon\to 0_+} \int_\Omega (\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)) \cdot \boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)\eta \, dx \leq 0. \tag{2.33}$$

Inserting this information back into (2.30) and recalling the steps starting from (2.28), we conclude

$$\limsup_{\varepsilon\to 0_+} \big\|(p^\varepsilon - p)\eta\big\|_2^2 \leq \left(\frac{1+\gamma_0}{2} + \frac{C_2^2\gamma_0^2}{2C_1^2(1-\gamma_0)}\right) \limsup_{\varepsilon\to 0_+} \big\|(p^\varepsilon - p)\eta\big\|_2^2.$$

Hence

$$\lim_{\varepsilon\to 0_+} \big\|(p^\varepsilon - p)\eta\big\|_2 = 0 \tag{2.34}$$

as long as

$$\frac{1+\gamma_0}{2} + \frac{C_2^2\gamma_0^2}{2C_1^2(1-\gamma_0)} < 1,$$

which corresponds to the condition $\gamma_0 < C_1/(C_1 + C_2)$; see Assumption 2.2.2.

Convergence of $\boldsymbol{D}\boldsymbol{v}^\varepsilon$ What remains is to prove the strong convergence of $\boldsymbol{D}\boldsymbol{v}^\varepsilon$ (for a subsequence at least). For $r \in (1, 2]$ we may invoke Hölder's inequality and calculate

$$\begin{aligned}
\big\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\eta\big\|_r^r &= \int_\Omega \Big(\int_0^1 (1 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon + s\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|^2)^{(r-2)/2} \\
&\qquad \times (1 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon + s\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|^2)^{(2-r)/2} |\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})|^2 \, ds \Big)^{r/2} \eta^r \, dx \\
&\leq \int_\Omega \Big(\int_0^1 (1 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon + s\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})|^2 \, ds \Big)^{r/2} \eta^{r/2} \\
&\qquad \times (1 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{r(2-r)/4} \, dx \\
&\leq \Big(\int_\Omega I^\varepsilon \eta \, dx \Big)^{r/2} \Big(\int_\Omega (1 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{r/2} dx \Big)^{(2-r)/2}.
\end{aligned} \tag{2.35}$$

Recalling (2.7) and (2.22), we have thus deduced a useful (though standard) estimate

$$C\big\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\eta\big\|_r^2 \leq \int_\Omega (\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)) \cdot \boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)\eta \, dx + \frac{\gamma_0^2}{2C_1} \big\|(p^\varepsilon - p)\eta\big\|_2^2,$$

which, together with (2.33) and (2.34), implies the required convergence

$$\lim_{\varepsilon\to 0_+} \big\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\eta\big\|_r = 0.$$

Hence we have obtained $(\boldsymbol{v}, p) = (\boldsymbol{v}^K, p^K) \in W^{1,r}_{0,\mathrm{div}}(\Omega) \times \mathring{L}^{r'}(\Omega)$, satisfying

$$\begin{aligned}
\int_\Omega \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) \cdot \boldsymbol{D}\boldsymbol{\varphi} \, dx + \int_\Omega T_K \beta(p^K, |\boldsymbol{v}^K|, |\boldsymbol{D}\boldsymbol{v}^K|^2) \boldsymbol{T}_K \boldsymbol{v}^K \cdot \boldsymbol{\varphi} \, dx - \int_\Omega p^K \operatorname{div} \boldsymbol{\varphi} \, dx \\
= \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi} \, dx \quad \text{for every } \boldsymbol{\varphi} \in W^{1,r}_0(\Omega).
\end{aligned} \tag{2.36}$$

2.5.2 Truncation removal ($K \to \infty$)

The final and key part concerns the limit $K \to \infty$. The essential procedures at this phase will lie in a decomposition of the pressures p^K, followed by an interesting application of the Div-Curl lemma.

Uniform estimates Let us pick $\boldsymbol{\varphi} = \boldsymbol{v}^K$ in the relation (2.36), as in the previous step. Exactly like in (2.22) and (2.23), we obtain bounds

$$\sup_K \left(\|\boldsymbol{v}^K\|_{1,r} + \|\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K)\|_{r'}\right) < \infty \tag{2.37}$$

and, denoting $\beta^K := T_K\beta\big(p^K, |\boldsymbol{v}^K|, |\boldsymbol{D}\boldsymbol{v}^K|^2\big)$, now by non-negativity of β also

$$\sup_K \big\|\beta^K|\boldsymbol{T}_K\boldsymbol{v}^K|^2\big\|_1 \le \sup_K \big\|\beta^K\boldsymbol{T}_K\boldsymbol{v}^K \cdot \boldsymbol{v}^K\big\|_1 < \infty. \tag{2.38}$$

Recall that for each $K \in \mathbb{N}$, $p^K \in L^{r'}(\Omega) \hookrightarrow L^{d'}(\Omega)$. The pressure will be uniformly estimated in the latter space, once again by dint of the Bogovskiĭ operator. This is where we finally give reason for the growth conditions in Assumption 2.2.3. Set

$$\boldsymbol{\varphi} = \mathcal{B}\Big(|p^K|^{d'-2}p^K - \big(|p^K|^{d'-2}p^K\big)_\Omega\Big).$$

Note that $\boldsymbol{\varphi} \in W_0^{1,d}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1,\infty)$ and $\|\boldsymbol{\varphi}\|_{1,d} \le C\|p^K\|_{d'}^{d'-1}$ due to the continuity of $\mathcal{B}$. Using $\boldsymbol{\varphi}$ as a test function in (2.36) yields

$$\|p^K\|_{d'}^{d'} = \int_\Omega \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) : \boldsymbol{D}\boldsymbol{\varphi}\,dx + \int_\Omega \beta^K\boldsymbol{T}_K\boldsymbol{v}^K \cdot \boldsymbol{\varphi}\,dx - \int_\Omega \boldsymbol{F} \cdot \nabla\boldsymbol{\varphi}\,dx. \tag{2.39}$$

Next,

$$\int_\Omega |\beta^K\boldsymbol{T}_K\boldsymbol{v}^K \cdot \boldsymbol{\varphi}|\,dx \le \Big(\int_\Omega \beta^K|\boldsymbol{T}_K\boldsymbol{v}^K|^2 dx\Big)^{1/2}\Big(\int_\Omega \beta^K|\boldsymbol{\varphi}|^2 dx\Big)^{1/2} \tag{2.40}$$

and owing to (2.38) the first term is bounded. Hence we can focus purely on the last integral in (2.40). Recalling Assumption 2.2.3 (with $q_0 < d'$, $q_1 < r^*$ and $q_2 < r$) and the classical Sobolev embedding, we estimate it as follows:

$$\begin{aligned}\int_\Omega \beta^K|\boldsymbol{\varphi}|^2 dx &\le c\int_\Omega |\boldsymbol{\varphi}|^2(1 + |p^K|^{q_0} + |\boldsymbol{v}^K|^{q_1} + |\boldsymbol{D}\boldsymbol{v}^{\boldsymbol{K}}|^{q_2})\,dx \\ &\le C\big(1 + \|p^K\|_{d'}^{q_0} + \|\boldsymbol{v}^K\|_{r^*}^{q_1} + \|\boldsymbol{D}\boldsymbol{v}^K\|_r^{q_2}\big)\,\|\boldsymbol{\varphi}\|_{1,d}^2\,.\end{aligned} \tag{2.41}$$

Combining with (2.37), the above computation amounts to

$$\int_\Omega |\beta^K\boldsymbol{T}_K\boldsymbol{v}^K \cdot \boldsymbol{\varphi}|\,dx \le C\|\boldsymbol{\varphi}\|_{1,d}\big(1 + \|p^K\|_{d'}^{q_0/2}\big),$$

with C independent of K. In light of $W_0^{1,d}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ and (2.37), eq. (2.39) gives rise to

$$\|p^K\|_{d'}^{d'} \le C\|\boldsymbol{\varphi}\|_{1,d}\big(1 + \|p^K\|_{d'}^{q_0/2}\big) \le C\|p^K\|_{d'}^{d'-1}\big(1 + \|p^K\|_{d'}^{q_0/2}\big).$$

Since $q_0 < d' \le 2$, we have arrived at

$$\sup_K \|p^K\|_{d'} < \infty. \tag{2.42}$$

We also observe that (2.40) and (2.41) would just as well work with $\boldsymbol{\varphi} \in L^{1+1/\delta}(\Omega)$ for some small $\delta > 0$, whence

$$\sup_K \left\| \beta^K \boldsymbol{T}_K \boldsymbol{v}^K \right\|_{1+\delta} < \infty. \tag{2.43}$$

Notice that the right-open intervals for q_0, q_1 and q_2 from Assumption 2.2.3 are indispensable for such a claim. It is hence possible by the estimates (2.37), (2.42) and (2.43), to let $K \to \infty$ and presuppose (after a pertinent relabelling of the sequence) that

$$\begin{aligned} \boldsymbol{v}^K &\to \boldsymbol{v} && \text{weakly in } W^{1,r}_{0,\mathrm{div}}(\Omega), \\ \boldsymbol{v}^K &\to \boldsymbol{v} && \text{a.e. in } \Omega, \\ p^K &\to p && \text{weakly in } \mathring{L}^{d'}(\Omega), \\ \boldsymbol{S}(p^K, \boldsymbol{D}(\boldsymbol{v}^K)) &\to \overline{\boldsymbol{S}} && \text{weakly in } L^{r'}(\Omega), \\ \beta^K \boldsymbol{T}_K \boldsymbol{v}^K &\to \overline{\beta \boldsymbol{v}} && \text{weakly in } L^{1+\delta}(\Omega). \end{aligned} \tag{2.44}$$

From (2.36) we have moved on to

$$\int_\Omega \overline{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{\varphi} \, dx + \int_\Omega \overline{\beta \boldsymbol{v}} \cdot \boldsymbol{\varphi} \, dx - \int_\Omega p \operatorname{div} \boldsymbol{\varphi} \, dx = \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi} \, dx$$

for any $\boldsymbol{\varphi} \in W^{1,r}_0(\Omega) \cap L^\infty(\Omega)$ such that $\operatorname{div} \boldsymbol{\varphi} \in L^d(\Omega)$. Not unlike the limit $\varepsilon \to 0_+$, the identification of the weak limits $\overline{\boldsymbol{S}}$ and $\overline{\beta \boldsymbol{v}}$ can and will be performed via the pointwise convergence of p^K and $\boldsymbol{D}\boldsymbol{v}^K$.

Decomposition of p^K Beginning with the pressure, for which we would like to utilize the monotonicity relation (2.7), we run into trouble as $\{p^K\}$ need not be bounded in $L^2(\Omega)$. This is why we decompose p^K into two parts: one being pointwise convergent and the other still converging only weakly, though now in $L^{r'}(\Omega)$, whence the monotonicity property may be used. It is again sufficient to prove the convergence in an arbitrary compactly contained subdomain $\Omega' \Subset \Omega$.

Referring back to Lemma 2.4.2 and noticing that both $\operatorname{div} \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \operatorname{div} \boldsymbol{F}$ and $\beta^K \boldsymbol{T}_K \boldsymbol{v}^K$ belong to $W^{-1,2}(\Omega)$, we may define

$$\begin{aligned} (\boldsymbol{v}_1^K, p_1^K) &:= \boldsymbol{H}(\operatorname{div} \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \operatorname{div} \boldsymbol{F}), \\ (\boldsymbol{v}_2^K, p_2^K) &:= \boldsymbol{H}(-\beta^K \boldsymbol{T}_K \boldsymbol{v}^K). \end{aligned} \tag{2.45}$$

The uniqueness of solutions to the Stokes problem and (2.36) imply

$$\begin{aligned} \boldsymbol{v}_1^K + \boldsymbol{v}_2^K &= 0, \\ p_1^K + p_2^K &= p^K. \end{aligned} \tag{2.46}$$

From (2.37) and the continuity of $\boldsymbol{H}$ we observe

$$\sup_K \left(\left\| \boldsymbol{v}_1^K \right\|_{1,2} + \left\| p_1^K \right\|_2 \right) < \infty. \tag{2.47}$$

Further, tacitly assuming $\delta \le 1/(d-1)$, we may apply (2.11) to $(2.45)_2$ with $k = 0$ and deduce

$$\begin{aligned} \left\| \nabla^2 \boldsymbol{v}_2^K \right\|_{1+\delta;\Omega'} + \left\| \nabla p_2^K \right\|_{1+\delta;\Omega'} &\le c \left(\left\| \beta^K \boldsymbol{T}_K \boldsymbol{v}^K \right\|_{1+\delta} + \left\| \boldsymbol{v}_2^K \right\|_{1,1+\delta} + \left\| p_2^K \right\|_{1+\delta} \right) \\ &\le c \left(\left\| \beta^K \boldsymbol{T}_K \boldsymbol{v}^K \right\|_{1+\delta} + \left\| \boldsymbol{v}_1^K \right\|_{1,2} + \left\| p^K - p_1^K \right\|_{d'} \right) \\ &\le C, \end{aligned}$$

where C is independent of K. Now consider $r < 2$ and Ω'', Ω''' satisfying $\Omega' \Subset \Omega'' \Subset \Omega''' \Subset \Omega$. Since $r' > 2$ we elicit the existence of a $\sigma > 0$ such that (2.11) may be employed again, this time with $k = -1$, leading to

$$\begin{aligned}
&\|\boldsymbol{v}_1^K\|_{1,2+\sigma;\Omega''} + \|p_1^K\|_{2+\sigma;\Omega''} \\
&\qquad \le c\Big(\|\operatorname{div} \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \operatorname{div} \boldsymbol{F}\|_{-1,2+\sigma} + \|\boldsymbol{v}_1^K\|_{2+\sigma;\Omega'''} + \|p_1^K\|_{-1,2+\sigma;\Omega'''}\Big) \\
&\qquad \le c\Big(\|\operatorname{div} \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K)\|_{-1,r'} + \|\boldsymbol{F}\|_{r'} + \|\boldsymbol{v}_1^K\|_{1,2;\Omega'''} + \|p_1^K\|_{2;\Omega'''}\Big) \\
&\qquad \le C;
\end{aligned}$$

the last estimate is due to (2.47). Utilizing the bootstrap argument, the above estimate yields

$$\begin{aligned}
\|\boldsymbol{v}_1^K\|_{1,r';\Omega'} + \|p_1^K\|_{r';\Omega'} &\le c\Big(\|\operatorname{div} \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K)\|_{-1,r'} + \|\boldsymbol{F}\|_{r'} + \|\boldsymbol{v}_1^K\|_{1,2} + \|p_1^K\|_2\Big) \\
&\le C.
\end{aligned}$$

In other words, using (2.46) we observe,

$$\sup_K \Big(\|\boldsymbol{v}_1^K\|_{1,r';\Omega'} + \|\boldsymbol{v}_1^K\|_{2,1;\Omega'} + \|p_1^K\|_{r';\Omega'} + \|p_2^K\|_{1,1;\Omega'}\Big) < \infty, \tag{2.48}$$

for any $r \in (1, 2]$, as the case $r = 2$ is covered directly by (2.47). Hence we may assume

$$\begin{aligned}
p_1^K &\to p_1 \quad \text{weakly in } \mathring{L}^{r'}(\Omega'), \\
p_2^K &\to p_2 \quad \text{a.e. in } \Omega'.
\end{aligned} \tag{2.49}$$

Note that (2.46) yields trivially $p_1 + p_2 = p$. What we are left with is thus to show the pointwise convergence of p_1^K.

Convergence of p_1^K We first notice that (2.45) and (2.48) imply

$$\|\operatorname{div}\big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I} - \boldsymbol{F}\big)\|_{1;\Omega'} \le C.$$

As $L^1(\Omega') \hookrightarrow\hookrightarrow W^{-1,q'}(\Omega')$ for $q > d$, this estimate together with (2.44) and (2.49) allows us to use Div-Curl lemma 2.4.6. Indeed, let $s > r$ and

$$\boldsymbol{\varphi}^K \to \boldsymbol{\varphi} \quad \text{weakly in } W^{1,s}(\Omega'). \tag{2.50}$$

Then $1/r' + 1/s < 1$, $\operatorname{curl} \nabla \boldsymbol{\varphi}^K = 0$ and Div-Curl lemma 2.4.6 implies

$$\big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla \boldsymbol{\varphi}^K \to \big(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\big) \cdot \nabla \boldsymbol{\varphi} \quad \text{weakly in } L^1(\Omega'). \tag{2.51}$$

Note we also tacitly used $\boldsymbol{F} \cdot \nabla \boldsymbol{\varphi}^K \to \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi}$ weakly in $L^1(\Omega')$.

Let $L > 0$. We shall first consider (2.51) with $\boldsymbol{\varphi}^K = \nabla \psi_L^K$, where (see Lemma 2.4.4 for notation)

$$\psi_L^K = \mathcal{N}(T_L(p_1^K - p_1)). \tag{2.52}$$

Note that due to the truncation, we have (for a subsequence if need be)

$$T_L(p_1^K - p_1) \to \overline{T}_L \quad \text{weakly in } L^q(\Omega) \text{ for all } q \in [1, \infty),$$

and hence by the continuity of $\mathcal{N}$ (see Lemma 2.4.4) also

$$\psi_L^K \to \psi_L = \mathcal{N}(\overline{T}_L) \quad \text{weakly in } W^{2,q}(\Omega) \text{ for all } q \in [1,\infty). \tag{2.53}$$

Therefore (2.51) yields

$$\big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla^2 \psi_L^K \to \big(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\big) \cdot \nabla^2 \psi_L \quad \text{weakly in } L^1(\Omega'),$$

which, after a simple rearrangement and using the pointwise convergence of p_2^K from (2.49), leads to

$$\begin{aligned}\big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \boldsymbol{S}(p_1 + p_2^K, \boldsymbol{D}\boldsymbol{v}) - (p_1^K - p_1)\boldsymbol{I}\big) \cdot \nabla^2 \psi_L^K \\ \to \big(\overline{\boldsymbol{S}} - \boldsymbol{S}(p_1 + p_2, \boldsymbol{D}\boldsymbol{v})\big) \cdot \nabla^2 \psi_L \quad \text{weakly in } L^1(\Omega').\end{aligned}$$

As a result, recalling also the definition of ψ_L^K, we find that for an arbitrary measurable $\Omega'' \subset \Omega'$

$$\begin{aligned}\limsup_{K\to\infty} \int_{\Omega''} |p_1^K - p_1||T_L(p_1^K - p_1)|\, dx &= \limsup_{K\to\infty} \int_{\Omega''} (p_1^K - p_1)\, \boldsymbol{I} \cdot \nabla^2 \psi_L^K \, dx \\ &\le \limsup_{K\to\infty} \int_{\Omega''} |\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \boldsymbol{S}(p_1 + p_2^K, \boldsymbol{D}\boldsymbol{v})||\nabla^2 \psi_L^K|\, dx \\ &\quad + \Big| \int_{\Omega''} \big(\overline{\boldsymbol{S}} - \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})\big) \cdot \nabla^2 \psi_L \, dx \Big|. \end{aligned} \tag{2.54}$$

Towards estimating the first term on the right-hand side, the relation (2.8) implies that

$$\begin{aligned}&\int_{\Omega''} |\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \boldsymbol{S}(p_1 + p_2^K, \boldsymbol{D}\boldsymbol{v})||\nabla^2 \psi_L^K|\, dx \\ &\le \gamma_0 \int_{\Omega''} |p_1^K - p_1||\nabla^2 \psi_L^K|\, dx + C_2 \int_{\Omega''} \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}^K - \boldsymbol{v})||\nabla^2 \psi_L^K|\, ds\, dx,\end{aligned} \tag{2.55}$$

where $\overline{\boldsymbol{D}}(s) = \boldsymbol{D}\boldsymbol{v}^K + s(\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}^K)$. Denoting

$$I^K = \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}^K - \boldsymbol{v})|^2 ds,$$

and using Hölder's inequality and $(1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} \le (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/4}$, we turn (2.55) into

$$\begin{aligned}&\int_{\Omega''} |\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \boldsymbol{S}(p_1 + p_2^K, \boldsymbol{D}\boldsymbol{v})||\nabla^2 \psi_L^K|\, dx \\ &\qquad \le \gamma_0 \left\| p_1^K - p_1 \right\|_{2;\Omega''} \left\| \nabla^2 \psi_L^K \right\|_{2;\Omega''} + C_2 \left\| I^K \right\|_{1;\Omega''}^{1/2} \left\| \nabla^2 \psi_L^K \right\|_{2;\Omega''}.\end{aligned}$$

Hence we are able to develop (2.54) as

$$\begin{aligned}&\limsup_{K\to\infty} \int_{\Omega''} |p_1^K - p_1||T_L(p_1^K - p_1)|\, dx \\ &\quad \le \limsup_{K\to\infty} \Big(\gamma_0 \left\| p_1^K - p_1 \right\|_{2;\Omega''} \left\| \nabla^2 \psi_L^K \right\|_{2;\Omega''} + C_2 \left\| I^K \right\|_{1;\Omega''}^{1/2} \left\| \nabla^2 \psi_L^K \right\|_{2;\Omega''} \Big) \\ &\qquad + \Big| \int_{\Omega''} \big(\overline{\boldsymbol{S}} - \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})\big) \cdot \nabla^2 \psi_L \, dx \Big|.\end{aligned} \tag{2.56}$$

Next, due to (2.53) we may assume without loss of generality that both $|\nabla^2\psi_L^K|^2$ and $|\Delta\psi_L^K|^2$ converge weakly in $L^q(\Omega')$ for any $q \in [1,\infty)$ as $k \to \infty$. To compare these weak limits, it suffices to investigate

$$\lim_{K\to\infty}\int_{\Omega'}\big(|\nabla^2\psi_L^K|^2 - |\Delta\psi_L^K|^2\big)\,\varphi\,dx$$

for arbitrary $\varphi \in \mathcal{C}_c^\infty(\Omega')$. Using the integration by parts, we find that

$$\begin{aligned}
\lim_{K\to\infty}&\int_{\Omega'}\big(|\nabla^2\psi_L^K|^2 - |\Delta\psi_L^K|^2\big)\varphi\,dx\\
&= \lim_{K\to\infty}\int_{\Omega'}\big(\nabla^2\psi_L^K\cdot\nabla^2\psi_L^K\varphi - |\Delta\psi_L^K|^2\varphi\big)\,dx\\
&= \lim_{K\to\infty}\int_{\Omega'}\big(-\nabla\psi_L^K\cdot\nabla\Delta\psi_L^K\varphi - (\nabla\psi_L^K\otimes\nabla\varphi)\cdot\nabla^2\psi_L^K - |\Delta\psi_L^K|^2\varphi\big)\,dx\\
&= \lim_{K\to\infty}\int_{\Omega'}\big(\nabla\psi_L^K\cdot\nabla\varphi\Delta\psi_L^K - (\nabla\psi_L^K\otimes\nabla\varphi)\cdot\nabla^2\psi_L^K\big)\,dx\\
&= \int_{\Omega'}\big(\nabla\psi_L\cdot\nabla\varphi\Delta\psi_L - (\nabla\psi_L\otimes\nabla\varphi)\cdot\nabla^2\psi_L\big)\,dx\\
&= \int_{\Omega'}\big(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\big)\varphi\,dx.
\end{aligned}$$

By the density argument, we therefore get for all measurable $\Omega'' \subset \Omega'$

$$\lim_{K\to\infty}\int_{\Omega''}\big(|\nabla^2\psi_L^K|^2 - |\Delta\psi_L^K|^2\big)\,dx = \int_{\Omega''}\big(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\big)\,dx,$$

in particular then

$$\limsup_{K\to\infty}\int_{\Omega''}|\nabla^2\psi_L^K|^2\,dx \le \limsup_{K\to\infty}\int_{\Omega''}|\Delta\psi_L^K|^2\,dx + \int_{\Omega''}\big(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\big)\,dx.$$

Hence, substituting this relation into (2.56), using the pointwise estimate

$$|\Delta\psi_L^K|^2 = |T_L(p_1^K - p_1)|^2 \le |p_1^K - p_1|^2$$

and the a priori estimates (2.44) and (2.47), we find out

$$\begin{aligned}
\limsup_{K\to\infty}&\int_{\Omega''}|p_1^K - p_1||T_L(p_1^K - p_1)|\,dx\\
&\le \limsup_{K\to\infty}\Big(\gamma_0\left\|p_1^K - p_1\right\|_{2;\Omega''} + C_2\left\|I^K\right\|_{1;\Omega''}^{1/2}\Big)\\
&\quad\times\Big(\left\|p_1^K - p_1\right\|_{2;\Omega''}^2 + \int_{\Omega''}\big(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\big)\,dx\Big)^{1/2}\\
&\quad + \Big|\int_{\Omega''}\big(\overline{\boldsymbol{S}} - \boldsymbol{S}(p,\boldsymbol{D}\boldsymbol{v})\big)\cdot\nabla^2\psi_L\,dx\Big|\\
&\le \limsup_{K\to\infty}\Big(\gamma_0\left\|p_1^K - p_1\right\|_{2;\Omega''}^2 + C_2\left\|I^K\right\|_{1;\Omega''}^{1/2}\left\|p_1^K - p_1\right\|_{2;\Omega''}\Big)\\
&\quad + C\Big|\int_{\Omega''}\big(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\big)\,dx\Big|^{1/2} \qquad (2.57)\\
&\quad + \Big|\int_{\Omega''}\big(\overline{\boldsymbol{S}} - \boldsymbol{S}(p,\boldsymbol{D}\boldsymbol{v})\big)\cdot\nabla^2\psi_L\,dx\Big|.
\end{aligned}$$

Finally, we choose Ω'' so that the truncator T_L could be disregarded. For this sake recall the Biting lemma 2.4.5 that we are going to apply to

$$f^K = |p_1^K|^{r'} + |\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K)|^{r'}. \tag{2.58}$$

Note that $\{f^K\}$ form a bounded sequence in $L^1(\Omega')$ by (2.44) and (2.48). Hence, the Biting lemma guarantees the existence of a nonincreasing sequence of measurable sets $E_n \subset \Omega'$ fulfilling $\lim_{n\to\infty} |E_n| = 0$ such that (modulo a subsequence) for each $n \in \mathbb{N}$ the sequence $\{f^K\}$ is uniformly equi-integrable in $\Omega_n := \Omega' \setminus E_n$.

The estimate (2.57) with $\Omega'' = \Omega_n$ entails for each $n \in \mathbb{N}$

$$\begin{aligned}\limsup_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n}^2 \le{}& \limsup_{K\to\infty} \int_{\Omega_n} |p_1^K - p_1||p_1^K - p_1 - T_L(p_1^K - p_1)|\,dx \\ &+ \limsup_{K\to\infty} \left(\gamma_0 \left\|p_1^K - p_1\right\|_{2;\Omega_n}^2 + C_2 \left\|I^K\right\|_{1;\Omega_n}^{1/2} \left\|p_1^K - p_1\right\|_{2;\Omega_n}\right) \\ &+ C\Big|\int_{\Omega_n} \left(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\right)dx\Big|^{1/2} + C\Big|\int_{\Omega_n} \left(\overline{\boldsymbol{S}} - \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})\right)\cdot\nabla^2\psi_L\,dx\Big|.\end{aligned} \tag{2.59}$$

We further let $L \to \infty$ in order to eliminate the terms depending on L. Denoting $\Omega_L^K = \{|p_1^K - p_1| > L\}$, we observe from (2.47) that $|\Omega_L^K| \le C/L^2$ whence, using the uniform equi-integrability of $|p_1^K|^{r'}$ in Ω_n,

$$\begin{aligned}\limsup_{L\to\infty}\limsup_{K\to\infty} \int_{\Omega_n} |p_1^K - p_1||p_1^K - p_1 - T_L(p_1^K - p_1)|\,dx \\ \le \limsup_{L\to\infty}\limsup_{K\to\infty} \int_{\Omega_n\cap\Omega_L^K} |p_1^K - p_1|^2\,dx = 0.\end{aligned}$$

At this point we wish to highlight the importance of the uniform equi-integrability of $|p_1^K|^2$. For $r < 2$ it would be trivial from (2.49), whereas when $r = 2$, the Biting lemma seems to be essential.

The remaining L-dependent terms in (2.59) tend with $L \to \infty$ likewise to zero, towards which it is evidently enough to prove

$$\psi_L \to 0 \quad \text{strongly in } W^{2,2}(\Omega). \tag{2.60}$$

Due to continuity of the Newtonian potential $\mathcal{N}$ (see Lemma 2.4.4), the problem (2.52) implies that (2.60) holds so long as (see (2.53))

$$\overline{T}_L \to 0 \quad \text{strongly in } L^2(\Omega). \tag{2.61}$$

To achieve this, we first draw from (2.47) that

$$T_L(p_1^K - p_1) - (p_1^K - p_1) \to \overline{T}_L \quad \text{weakly in } L^2(\Omega)$$

and therefore from the weak lower semicontinuity of the L^1-norm we find that

$$\left\|\overline{T}_L\right\|_1 \le \liminf_{K\to\infty} \left\|T_L(p_1^K - p_1) - (p_1^K - p_1)\right\|_1 \le 2\limsup_{K\to\infty} \int_{\Omega_L^K} |p_1^K - p_1|\,dx \le C/L,$$

whence

$$\overline{T}_L \to 0 \quad \text{strongly in } L^1(\Omega).$$

To strengthen the strong convergence from $L^1(\Omega)$ into $L^2(\Omega)$, it is enough to find a dominating function belonging to $L^2(\Omega)$. However, denoting $\nu \in L^2(\Omega)$ the weak limit

$$|p_1^K - p_1| \to \nu \quad \text{weakly in } L^2(\Omega),$$

a simple estimate $|T_L(p_1^K - p_1)| \le |p_1^K - p_1|$ implies $|\overline{T}_L| \le \nu$ and Lebesgue's dominated convergence theorem finishes the proof of (2.61). As a consequence, we conclude from (2.59) that

$$\limsup_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n}^2 \le \limsup_{K\to\infty} \Big(\gamma_0 \left\|p_1^K - p_1\right\|_{2;\Omega_n}^2 + C_2 \left\|I^K\right\|_{1;\Omega_n}^{1/2} \left\|p_1^K - p_1\right\|_{2;\Omega_n}\Big),$$

ultimately implying for each n (note that $\gamma_0 < 1$)

$$\limsup_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n} \le \frac{C_2}{1-\gamma_0} \limsup_{K\to\infty} \left\|I^K\right\|_{1;\Omega_n}^{1/2}. \tag{2.62}$$

We want to develop (2.62) into

$$\limsup_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n} \le \alpha \limsup_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n} \quad \text{for some } \alpha \in (0,1).$$

We are again going to utilize the observation based on the Div-Curl lemma (2.51). To this end, take a fixed $\lambda > 0$, recall Lemma 2.4.7 about Lipschitz approximations of Sobolev functions and set $\boldsymbol{\varphi}^K := \boldsymbol{v}_\lambda^K$ in (2.51), where $\boldsymbol{v}_\lambda^K$ denotes the Lipschitz approximation of $\boldsymbol{v}^K$. Note that due to (2.12), fulfilment of the condition (2.50) may be taken for granted. Hence

$$\big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla \boldsymbol{v}_\lambda^K \to \big(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\big) \cdot \nabla \overline{\boldsymbol{v}}_\lambda \quad \text{weakly in } L^1(\Omega'), \tag{2.63}$$

where

$$\boldsymbol{v}_\lambda^K \to \overline{\boldsymbol{v}}_\lambda \quad \text{weakly in } W^{1,q}(\Omega) \text{ for all } q \in [1,\infty).$$

Note that (2.63) directly implies

$$\lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla \boldsymbol{v}_\lambda^K \, dx = \int_{\Omega_n} \big(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\big) \cdot \nabla \overline{\boldsymbol{v}}_\lambda \, dx \tag{2.64}$$

for each n, where Ω_n are still the subsets specified above, when we applied the Biting lemma to the sequence given in (2.58). By (2.64), we have

$$\begin{aligned}
&\lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla \boldsymbol{v}^K \, dx \\
&\quad = \lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla (\boldsymbol{v}^K - \boldsymbol{v}_\lambda^K) \, dx \\
&\qquad + \lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla \boldsymbol{v}_\lambda^K \, dx \\
&\quad = \lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla (\boldsymbol{v}^K - \boldsymbol{v}_\lambda^K) \, dx + \int_{\Omega_n} \big(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\big) \cdot \nabla \overline{\boldsymbol{v}}_\lambda \, dx,
\end{aligned}$$

and consequently, as the left-hand side is independent of λ,

$$\lim_{K\to\infty}\int_{\Omega_n}\big(\boldsymbol{S}(p^K,\boldsymbol{D}\boldsymbol{v}^K)-p_1^K\boldsymbol{I}\big)\cdot\nabla\boldsymbol{v}^K\,dx$$
$$=\lim_{\lambda\to\infty}\lim_{K\to\infty}\int_{\Omega_n}\big(\boldsymbol{S}(p^K,\boldsymbol{D}\boldsymbol{v}^K)-p_1^K\boldsymbol{I}\big)\cdot\nabla(\boldsymbol{v}^K-\boldsymbol{v}_\lambda^K)\,dx+\lim_{\lambda\to\infty}\int_{\Omega_n}\big(\overline{\boldsymbol{S}}-p_1\boldsymbol{I}\big)\cdot\nabla\overline{\boldsymbol{v}}_\lambda\,dx. \tag{2.65}$$

First we notice the first term on the right vanishes. Indeed, thanks to (2.14) we have

$$\int_{\Omega_n}\big|\big(\boldsymbol{S}(p^K,\boldsymbol{D}\boldsymbol{v}^K)-p_1^K\boldsymbol{I}\big)\cdot\nabla(\boldsymbol{v}^K-\boldsymbol{v}_\lambda^K)\big|\,dx$$
$$\le C\left\|\boldsymbol{v}^K\right\|_{1,r}\left\|\boldsymbol{S}(p^K,\boldsymbol{D}\boldsymbol{v}^K)-p_1^K\boldsymbol{I}\right\|_{r';\Omega_n\cap\{\boldsymbol{v}^K\neq\boldsymbol{v}_\lambda^K\}}$$

and the claim follows from the uniform equi-integrability of

$$|p_1^K|^{r'}+|\boldsymbol{S}(p^K,\boldsymbol{D}\boldsymbol{v}^K)|^{r'}$$

in Ω_n (see (2.58)) and (2.13), i.e. $|\{\boldsymbol{v}^K\neq\boldsymbol{v}_\lambda^K\}|\le C/\lambda^r$.

The second term on the right-hand side of (2.65) can easily be identified: The weak lower semicontinuity of a norm and (2.14) bring about

$$\|\overline{\boldsymbol{v}}_\lambda\|_{1,r}\le\liminf_{K\to\infty}\left\|\boldsymbol{v}_\lambda^K\right\|_{1,r}\le C\limsup_{K\to\infty}\left\|\boldsymbol{v}^K\right\|_{1,r}\le C.$$

Accordingly, we may safely assume

$$\overline{\boldsymbol{v}}_\lambda\to\overline{\boldsymbol{v}}\quad\text{weakly in } W^{1,r}(\Omega).$$

On the other hand, it follows from the compact embedding, (2.13) and (2.14) that

$$\|\boldsymbol{v}-\overline{\boldsymbol{v}}_\lambda\|_1=\lim_{K\to\infty}\int_\Omega|\boldsymbol{v}^K-\boldsymbol{v}_\lambda^K|\,dx=\lim_{K\to\infty}\int_{\{\boldsymbol{v}^K\neq\boldsymbol{v}_\lambda^K\}}|\boldsymbol{v}^K-\boldsymbol{v}_\lambda^K|\,dx\le C/\lambda^{r-1},$$

meaning

$$\overline{\boldsymbol{v}}_\lambda\to\boldsymbol{v}\quad\text{strongly in } L^1(\Omega),$$

and finally, due to uniqueness of weak limits

$$\overline{\boldsymbol{v}}_\lambda\to\boldsymbol{v}\quad\text{weakly in } W^{1,r}(\Omega).$$

Thus we are able to pass $\lambda\to\infty$ on the right-hand side of (2.65), obtaining

$$\lim_{K\to\infty}\int_{\Omega_n}\big(\boldsymbol{S}(p^K,\boldsymbol{D}\boldsymbol{v}^K)-p_1^K\boldsymbol{I}\big)\cdot\nabla\boldsymbol{v}^K dx=\int_{\Omega_n}\big(\overline{\boldsymbol{S}}-p_1\boldsymbol{I}\big)\cdot\nabla\boldsymbol{v}\,dx.$$

Since $\boldsymbol{v}^K$ and $\boldsymbol{v}$ are both divergence-free, this is actually tantamount to

$$\lim_{K\to\infty}\int_{\Omega_n}\boldsymbol{S}(p^K,\boldsymbol{D}\boldsymbol{v}^K)\cdot\boldsymbol{D}\boldsymbol{v}^K dx=\int_{\Omega_n}\overline{\boldsymbol{S}}\cdot\boldsymbol{D}\boldsymbol{v}\,dx$$

and thanks to the strong convergence of p_2^K (see (2.49)) also

$$\lim_{K\to\infty}\int_{\Omega_n}(\boldsymbol{S}(p^K,\boldsymbol{D}\boldsymbol{v}^K)-\boldsymbol{S}(p_1+p_2^K,\boldsymbol{D}\boldsymbol{v}))\cdot\boldsymbol{D}(\boldsymbol{v}^K-\boldsymbol{v})\,dx=0. \tag{2.66}$$

At long last, recalling (2.7) we see that (2.66) implies

$$\limsup_{K\to\infty}\left\|I^K\right\|_{1;\Omega_n}\le\frac{\gamma_0^2}{C_1^2}\limsup_{K\to\infty}\left\|p_1^K-p_1\right\|_{2;\Omega_n}^2. \tag{2.67}$$

Substituting (2.67) into (2.62), we have

$$\limsup_{K\to\infty}\left\|p_1^K-p_1\right\|_{2;\Omega_n}\le\frac{C_2\gamma_0}{C_1(1-\gamma_0)}\limsup_{K\to\infty}\left\|p_1^K-p_1\right\|_{2;\Omega_n}.$$

Considering

$$\frac{C_2\gamma_0}{C_1(1-\gamma_0)}<1\iff\gamma_0<\frac{C_1}{C_1+C_2},$$

by Assumption 2.2.2 we have

$$\lim_{K\to\infty}\left\|p_1^K-p_1\right\|_{2;\Omega_n}=0. \tag{2.68}$$

Since $\lim_{n\to\infty}|\Omega'\setminus\Omega_n|=0$, we may suppose $p_1^K\to p_1$ a.e. in Ω'.

Convergence of $\boldsymbol{D}\boldsymbol{v}^K$ We have yet to affirm the strong convergence of $\boldsymbol{D}\boldsymbol{v}^K$. A simple task, in fact, for we can proceed just as in the ε-passage. Exactly like in (2.35), we would use Hölder's inequality to show

$$\left\|\boldsymbol{D}(\boldsymbol{v}^K-\boldsymbol{v})\right\|_{r;\Omega_n}^2\le C\int_{\Omega_n}\boldsymbol{I}^K dx,$$

whence by (2.7) also

$$\begin{aligned}&C\left\|\boldsymbol{D}(\boldsymbol{v}^K-\boldsymbol{v})\right\|_{r;\Omega_n}^2\\&\quad\le\int_{\Omega_n}(\boldsymbol{S}(p^K,\boldsymbol{D}\boldsymbol{v}^K)-\boldsymbol{S}(p_1+p_2^K,\boldsymbol{D}\boldsymbol{v}))\cdot\boldsymbol{D}(\boldsymbol{v}^K-\boldsymbol{v})\,dx+\frac{\gamma_0^2}{2C_1}\left\|p_1^K-p_1\right\|_{2;\Omega_n}^2.\end{aligned}$$

The right-hand side tends to zero as $K\to\infty$ by (2.66) and (2.68). This fact implies we may assume $\boldsymbol{D}\boldsymbol{v}^K\to\boldsymbol{D}\boldsymbol{v}$ a.e. in Ω' and also eventually finishes the proof. □

2.6 Closing remarks

We would like to finish this paper with a theorem directly improving the results of [13] and [22]:

Theorem 2.6.1 *Let $d\ge 2$ and $\Omega\subset\mathbb{R}^d$ be a Lipschitz domain. Consider $\boldsymbol{f}\in W^{-1,r'}(\Omega)$, $p_0\in\mathbb{R}$, $r\in(2d/(d+2),2]$ and let Assumptions 2.2.1 and 2.2.2 hold. Enforcing a slightly strengthened Assumption 2.2.3, namely let $q_0\in\left[1,\min\left\{d',\frac{dr}{2(d-r)}\right\}\right)$, there exists a pair $(\boldsymbol{v},p)\in W_{0,\mathrm{div}}^{1,r}(\Omega)\times L^{\min\left\{d',\frac{dr}{2(d-r)}\right\}}(\Omega)$ satisfying $p_\Omega=p_0$, $\beta(p,\boldsymbol{v},|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v}\in L^1(\Omega)$ and*

$$\int_\Omega\left[2\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}\cdot\boldsymbol{D}\boldsymbol{\varphi}-(\boldsymbol{v}\otimes\boldsymbol{v})\cdot\nabla\boldsymbol{\varphi}+\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v}\cdot\boldsymbol{\varphi}-p\operatorname{div}\boldsymbol{\varphi}\right]dx=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle$$

for every $\boldsymbol{\varphi}\in W_0^{1,\infty}(\Omega)$.

As insinuated in Section 2.2, we are not going to establish this result in detail. The proof would lie in a straightforward combination of the procedure implemented here and steps used in [13] to control the convective term. To be more specific, one would need a stronger version of the Lipschitz approximation lemma than Lemma 2.4.7, namely that from [18]. The second change would be in the decomposition of the pressure (2.45). Informally speaking, we would add one more partial pressure corresponding to the convective term, i.e. $(\boldsymbol{v}_3^K, p_3^K) := \boldsymbol{H}(-\operatorname{div}(\boldsymbol{v}^K \otimes \boldsymbol{v}^K))$. The new pressure would, like p_2^K, also converge pointwise due to estimates based on the regularity theory for the Stokes problem. In reality however, there would have to appear an additional regularizing term in the argument of $\boldsymbol{H}$; see [13] for details.

As far as the possible deterioration of the pressure integrability is concerned, the culprit is again the convective term. Note that $\frac{dr}{2(d-r)} < d'$ for $r < \frac{2d}{d+1}$, so that the exponent of integrability becomes worse for low values of r. If $\frac{dr}{2(d-r)} < d'$ then $p \in L^{\frac{dr}{2(d-r)}}(\Omega)$ and it is necessary to restrict growth of the drag β in the pressure accordingly, as the original $q_0 < d'$ from Assumption 2.2.3 requires being able to bound the pressure in $L^{d'}(\Omega)$. It is an easy exercise to perform variants of the estimates (2.39)–(2.42) again with the convective term present.

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Chapter 3

Large data existence theory for unsteady flows of fluids with pressure- and shear-dependent viscosities

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Abstract

A generalization of Navier-Stokes' model is considered, where the Cauchy stress tensor depends on the pressure as well as on the shear rate in a power-law-like fashion, for values of the power-law index $r \in \left(\frac{2d}{d+2}, 2\right]$. We develop existence of generalized (weak) solutions for the resultant system of partial differential equations, including also the so far uncovered cases $r \in \left(\frac{2d}{d+2}, \frac{2d+2}{d+2}\right]$ and $r = 2$. By considering a maximal sensible range of the power-law index r, the obtained theory is in effect identical to the situation of dependence on the shear rate only.

Keywords

Existence theory, weak solution, incompressible fluid, pressure-dependent viscosity, shear-dependent viscosity, Navier boundary condition, parabolic Lipschitz truncation of Sobolev functions, flow through porous media, Neumann problem.

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3.1 Introduction

Let $T > 0$, $\Omega \in \mathbb{R}^d$ be an open Lipschitz domain and denote $Q = (0,T) \times \Omega$. We would like to study unsteady flows of incompressible homogeneous fluids in Ω. Setting density to be identically one for simplicity, balance of linear momentum and balance of mass for such fluids can be written down as

$$\begin{aligned} \partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{T} &= \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{v} &= 0, \end{aligned} \tag{3.1}$$

both holding in Q, where $\boldsymbol{f}$ represents the external forces acting on the fluid and $\boldsymbol{T}$ is the Cauchy stress tensor. When the fluid is additionally supposed to be Newtonian, the Cauchy stress is of the form

$$\boldsymbol{T} = -p\boldsymbol{I} + \nu \boldsymbol{D}\boldsymbol{v}, \tag{3.2}$$

where p is the pressure (the indeterminate part of the stress),

$$\boldsymbol{D}\boldsymbol{v} = \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v}\right)$$

is the symmetric part of the velocity gradient and $\nu > 0$ is the shear viscosity. When $\boldsymbol{T}$ is of the form (3.2), Eq. (3.1) becomes the notorious Navier-Stokes model. Unfortunately, despite all the rapt attention that this model has drawn in renown mathematicians throughout the last century and beyond, the hitherto obtained results are still far from satisfactory. Worse yet, it is well known that this model is incapable of capturing manifold features manifested by non-Newtonian fluids, such as shear-thinning or -thickening, pressure-dependent viscosity etc.

In this paper we are interested in the situation where the Cauchy stress is of the form

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}(p, \boldsymbol{Dv}) = -p\boldsymbol{I} + \nu(p, |\boldsymbol{Dv}|^2)\boldsymbol{Dv}, \tag{3.3}$$

in which the viscous stress tensor $\boldsymbol{S}$ is supposed to meet certain requirements; see Assumptions 3.2.1 and 3.2.2. This particular model goes back to two papers by Málek et al. [23, 24] and has been dealt with on multiple occasions ever since (see e.g. [5, 14, 17, 22] and the discussion below Theorem 3.3.1).

It has been convincingly documented in experiments that viscosity of a fluid may vary significantly with the pressure (exponentially or even more dramatically; see e.g. [1, 3] or comprehensive references in [27]). Likewise, the already mentioned shear-thinning or shear-thickening behavior can be captured through a non-constant viscosity $\nu = \nu(|\boldsymbol{Dv}|^2)$ like in the mathematically popular model of Ladyzhenskaya's. By means of the constitutive relation (3.3), we can capture both these dependences in a single model. It comes at a price, sadly, for instance we are able to handle only shear-thinning, not shear-thickening, behavior (see the main result, Theorem 3.3.1, and the upper bound for the power exponent r).

The objective we set is to prove existence of weak solutions for the model. Therefore we have to add initial and suitable boundary conditions, for which sake let us denote $\Gamma = (0, T) \times \partial\Omega$. We consider an impermeable boundary, that is

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma,$$

where $\boldsymbol{n}$ is the unit outer normal vector of Ω. We cannot, however, resort to the no-slip boundary condition

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma,$$

for in that case we would be unable to construct the pressure (see the discussion below Theorem 3.3.1). Instead, we choose the Navier slip condition

$$\alpha \boldsymbol{v}_\tau = -(\boldsymbol{Sn})_\tau \quad \text{on } \Gamma$$

for some $\alpha \geq 0$, which is the heart of the matter here due to the dependence of $\boldsymbol{S}$ on p. For $\boldsymbol{u} : \partial\Omega \to \mathbb{R}^d$, a vector field on the boundary, we define its tangential component as

$$\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}.$$

Note that from an instinctive point of view, the Navier slip may be regarded as a bridge between the no-slip condition ($\alpha \to \infty$) and the perfect slip condition ($\alpha = 0$).

On account of the pressure-dependent viscous stress, we have yet to add some kind of pressure anchoring, which we take in the form

$$\frac{1}{|\Omega|} \int_\Omega p(t, x)\, dx = h(t) \quad \text{in } (0, T) \tag{3.4}$$

for a given function h. Ideally one should like to prescribe the pressure locally (at some point) but since our pressure will be merely an integrable function, dictating its pointwise values is out of the question. A possible approximation could lie in the integral average over a given subset $\Omega_0 \subset \Omega$ but in our case, corresponding attempts led to insurmountable technical difficulties, hence (3.4) for simplicity.

All in all, the model to be analyzed reads

$$
\left.
\begin{aligned}
\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\boldsymbol{S}(p,\boldsymbol{D}\boldsymbol{v}) + \nabla p &= \boldsymbol{f} && \text{in } Q,\\
\operatorname{div}\boldsymbol{v} &= 0 && \text{in } Q,\\
\boldsymbol{v}\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma,\\
\alpha\boldsymbol{v}_\tau &= -(\boldsymbol{S}\boldsymbol{n})_\tau && \text{on } \Gamma,\\
\boldsymbol{v}(0) &= \boldsymbol{v}_0 && \text{in } \Omega,\\
\frac{1}{|\Omega|}\int_\Omega p\,dx &= h && \text{in } (0,T).
\end{aligned}
\right\}
\tag{3.5}
$$

As far as the structure of this paper goes, next we are about to introduce our notation and certain assumptions, in particular those on the viscous stress $\boldsymbol{S}$, i.e. Assumptions 3.2.1 and 3.2.2. In the ensuing section, we present the result of this paper, Theorem 3.3.1 on existence of weak solutions to problem (3.5), and devote a few lines to the discussion of its relevance to past works and to the sketch of the fundamental techniques employed in the proof. In Section 3.4, we list various nontrivial results that are exploited in the proof of Theorem 3.3.1, to which the entire Section 3.5 and Appendix are dedicated.

3.2 Preliminaries

For $0 < t < T$ we write $Q_t = (0,t)\times\Omega$ and $\Gamma_t = (0,t)\times\partial\Omega$. For $r \in (1,\infty)$ we denote $r' = r/(r-1)$. For a Lebesgue measurable set Ω we denote $|\Omega|$ its Lebesgue measure. If $X(\Omega)$ is a Lebesgue or Sobolev space, we denote

$$\mathring{X}(\Omega) := \Big\{ f \in X(\Omega) \;\Big|\; \int_\Omega f(x)\,dx = 0 \Big\}.$$

For $f \in L^1(\Omega)$ we denote

$$f_\Omega := \frac{1}{|\Omega|}\int_\Omega f(x)\,dx.$$

Usually, no explicit distinction between spaces of scalar- and vector-valued functions will be made. Confusion should never come to pass as we employ small boldfaced letters to denote vectors and bold capitals for tensors. The same applies also to traces of Sobolev functions, which we denote like the original functions. Only when in need, we use Tr for a trace. Accordingly, for $r > 1$ we set

$$
\begin{aligned}
W^{1,r}_{\boldsymbol{n}}(\Omega) &:= \big\{ \boldsymbol{f} \in W^{1,r}(\Omega) \;\big|\; \operatorname{Tr}\boldsymbol{f}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega \big\},\\
W^{1,r}_{\boldsymbol{n},\mathrm{div}}(\Omega) &:= \big\{ W^{1,r}_{\boldsymbol{n}}(\Omega) \;\big|\; \operatorname{div}\boldsymbol{f} = 0 \text{ in } \Omega \big\},\\
W^{-1,r'}_{\boldsymbol{n}}(\Omega) &:= \big(W^{1,r}_{\boldsymbol{n}}(\Omega) \big)^*,\\
X^r_{\boldsymbol{n}} &:= L^r(0,T; W^{1,r}_{\boldsymbol{n}}(\Omega)) \cap L^2(0,T; L^2(\partial\Omega)),\\
X^r_{\boldsymbol{n},\mathrm{div}} &:= L^r(0,T; W^{1,r}_{\boldsymbol{n},\mathrm{div}}(\Omega)) \cap L^2(0,T; L^2(\partial\Omega)),\\
\mathcal{C}^\infty_c(\Omega) &:= \big\{ f \in \mathcal{C}^\infty(\Omega) \;\big|\; f \text{ is compactly supported in } \Omega \big\}.
\end{aligned}
$$

If $r > 0$ and $x \in \mathbb{R}^d$, let $B_r(x) = \{|y-x| < r\}$. For $f \in L^1_{loc}(\mathbb{R}^{d+1})$ and $(t,x) \in \mathbb{R}^{d+1}$, we define the parabolic maximal operator

$$\mathcal{M}^*(f)(t,x) := \sup_{0<\varrho<\infty} \frac{1}{2\varrho}\int_{t-\varrho}^{t+\varrho} \sup_{0<r<\infty} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(s,y)|\,dy\,ds.$$

When applied to functions not defined on the whole $\mathbb{R}^{d+1}$, we implicitly consider their zero extension. For more details about maximal operators see [29] or, only for the fundamental properties of $\mathcal{M}^*$ needed here, Appendix A of [15].

The symbol $\cdot$ stands for the scalar product and $\otimes$ signifies the tensor product. For open subsets A, B of $\mathbb{R}^d$, we write $A \Subset B$ if $A \subset \overline{A} \subset B$ and $\overline{A}$ is compact. We denote $(\cdot, \cdot)$ the inner product in $L^2(\Omega)$, while $(\cdot,\cdot)_S$ stands for the inner product in $L^2(S)$ for a measurable set S other than Ω. Generic constants are denoted simply by C and, when circumstances require it, we may also include quantities on which the constants depend, e.g. $C(\|\boldsymbol{v}_0\|_2)$.

The external body forces $\boldsymbol{f}$ are for the sake of convenience supposed to be of the form

$$\boldsymbol{f} = -\operatorname{div} \boldsymbol{F},$$

Consider $r \in (1,2]$ a fixed number and $d \geq 2$. Below we reproduce assumptions on the viscous stress, i.e. the smooth nonlinearity $\boldsymbol{S}$:

Assumption 3.2.1 *Let there be positive constants C_1 and C_2 such that for all $\boldsymbol{B}, \boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $p \in \mathbb{R}$*

$$C_1(1+|\boldsymbol{D}|^2)^{(r-2)/2}|\boldsymbol{B}|^2 \leq \frac{\partial \boldsymbol{S}(p,\boldsymbol{D})}{\partial \boldsymbol{D}} \cdot (\boldsymbol{B}\otimes\boldsymbol{B}) \leq C_2(1+|\boldsymbol{D}|^2)^{(r-2)/2}|\boldsymbol{B}|^2.$$

Assumption 3.2.2 *Let for all $\boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $p \in \mathbb{R}$*

$$\left|\frac{\partial \boldsymbol{S}(p,\boldsymbol{D})}{\partial p}\right| \leq \gamma_0 (1+|\boldsymbol{D}|^2)^{(r-2)/4}, \quad \text{with } 0 < \gamma_0 < \frac{C_1}{C_{reg}(C_1+C_2)},$$

where C_{reg} is attributed to the solution operator of Neumann's problem on Ω; see (3.15) *and below.*

Both these requirements date back to [23, 24]. The authors offer several examples of viscosities meeting these criteria, among others

$$\nu_i(p, |\boldsymbol{D}|^2) = \left(A + \mu_i(p) + |\boldsymbol{D}|^2\right)^{\frac{r-2}{2}}, \qquad i = 1,2,3,$$

where $A \in (0,1]$ is a (typically small) number and $\mu_i(p)$ takes one of the following forms $(\alpha, \beta > 0)$:

$$\begin{aligned}
\mu_1(p) &= (1+\alpha^2 p^2)^{-\frac{\beta}{2}},\\
\mu_2(p) &= (1+\exp(\alpha p))^{-\beta},\\
\mu_3(p) &= \begin{cases} \exp(-\alpha p) & \text{if } p > 0,\\ 1 & \text{if } p \leq 0.\end{cases}
\end{aligned}$$

In [23], there can be found also examples of viscosities that do not fulfill the assumptions above, yet they can be approximated (in a suitable manner) with such viscosities.

One should indeed generally suppose the constants C_1, C_2 in Assumption 3.2.1 depend on pressure. Bluntly speaking, the assumptions are chosen in such a way that the employed techniques work and the existence theory can be developed. As for treatment of the more realistic case where viscosity is an unbounded function of the pressure (i.e. $C_i = C_i(p)$, $i = 1,2$), we refer to [11].

3.3 Main result

Theorem 3.3.1 *Let $d \geq 2$, $T > 0$, $\alpha > 0$, $2d/(d+2) < r \leq 2$ and $\Omega \in \mathcal{C}^{1,1}$ be a bounded domain in $\mathbb{R}^d$. Denote*

$$q = \frac{r(d+2)}{2d} > 1 \tag{3.6}$$

and consider $\boldsymbol{F} \in L^{r'}(Q)$, $h \in L^q(0,T)$ and $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}(\Omega)$. Finally suppose that Assumptions 3.2.1 and 3.2.2 hold. Then there exists a weak solution $(\boldsymbol{v}, p)$ to the problem (3.5)*, that is*

$$\boldsymbol{v} \in \mathcal{C}_w([0,T]; L^2(\Omega)) \cap X^r_{\boldsymbol{n},\mathrm{div}}, \quad \partial_t \boldsymbol{v} \in L^q(0,T; W^{-1,q}_{\boldsymbol{n}}(\Omega)),$$

$$p \in L^q(0,T; L^q(\Omega)) \text{ and } \int_\Omega p(t,x)\,dx = h(t) \text{ for a.e. } t \in (0,T)$$

and the weak formulation is satisfied, i.e. for all $\boldsymbol{\varphi} \in W^{1,q'}_{\boldsymbol{n}}(\Omega)$ and a.e. $t \in (0,T)$ we have

$$\langle \partial_t \boldsymbol{v}(t), \boldsymbol{\varphi}\rangle - ((\boldsymbol{v} \otimes \boldsymbol{v})(t), \nabla \boldsymbol{\varphi}) + (\boldsymbol{S}(t), \boldsymbol{D}\boldsymbol{\varphi}) + \alpha(\boldsymbol{v}(t), \boldsymbol{\varphi})_{\partial\Omega} - (p(t), \operatorname{div} \boldsymbol{\varphi}) = (\boldsymbol{F}(t), \nabla \boldsymbol{\varphi}), \tag{3.7}$$

with $\boldsymbol{S}(t) = \boldsymbol{S}(p(t), \boldsymbol{D}\boldsymbol{v}(t))$. The initial condition is attained through

$$\lim_{t \to 0_+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_{L^2(\Omega)} = 0.$$

With this result, we practically conclude the existence theory for the corresponding class of models conceived by Málek et al. in [23, 24]. More precisely, with the condition $r > 2d/(d+2)$ we have reached the same lower bound as in the case of pressure-independent viscosity $\nu = \nu(|\boldsymbol{D}\boldsymbol{v}|^2)$; see Diening et al. [15]. This bound is the best one guaranteeing compactness of the convective term $\boldsymbol{v} \otimes \boldsymbol{v}$ in $L^1(Q)$ and in this regard it may be considered optimal.

Although the range $r \in (2d/(d+2), 2)$ has already been investigated in [5], it was in the steady case and therefore the situation was considerably simpler, although the bedrock of the proof was quite similar. As for the evolutionary system like that of ours, the best result so far comes from [12], where existence for $r \in ((2d+2)/(d+2), 2)$ was proven. In [8], the problem has already been grappled with $\Omega = \mathbb{R}^3$ and $r \in (9/5, 2)$. For local results (small data, short times), see [19, 20, 28]. In [13], the model of ours is investigated, enriched additionally by the temperature dependence, in which case only $r \in (3d/(d+2), 2)$ can be handled, imposing a restriction $d = 2, 3$.

Apart from optimization from below, we have also finally incorporated the value $r = 2$ among amenable values of the exponent r, which has only recently been achieved for the steady-state problem in [14]. The work [12] also covers the value $r = 2$, yet under a slightly different analogue of Assumption 3.2.2. Similarly in [10], where the case $d = 2$ with the periodic boundary conditions is treated. Inclusion of the *critical value* $r = 2$ in our paper not only makes the theory cover the Navier-Stokes model but, more importantly, allows us to consider balance equations $(3.5)_1$ of the form

$$\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \Delta \boldsymbol{v} - \operatorname{div} \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) + \nabla p = \boldsymbol{f},$$

with $\boldsymbol{S}$ fulfilling Assumptions 3.2.1 and 3.2.2 with $r < 2$ if need be.

As for the issue of strong solutions, the authors in [7] investigated planar, steady case of our model with periodic boundary conditions and proved existence of strong solutions. The unsteady case in three dimensions is, however, far from clear, as the question of regularity is unanswered even for the popular *Ladyzhenskaya model* for low values of the power index (i.e. close to $2d/(d+2)$), and yet that system can be considered a pressure-independent simplification of our model. For a nicely organized survey of regularity results concerning these generalized Navier-Stokes' models, see [26].

It is important to notice that we actively avoid the homogeneous Dirichlet boundary condition, corresponding informally to $\alpha = \infty$. The reason is that we need a measurable pressure for the sake of the pressure-dependent viscous stress, which in the case of zero boundary condition remains an insurmountable task. The snag lies in *incompatibility* of the Helmholtz decomposition with the Dirichlet boundary condition or, in other words, the fact that in the Neumann problem for Poisson's equation, the trace of the gradient cannot be required to be zero; only its normal component can (see (3.13)). This obstacle will be experienced in the flesh in (3.36) and below.

Even though $\alpha = \infty$ is out of the question, in Theorem 3.3.1 we could take $\alpha = 0$ without scruples. This situation would correspond to *the perfect-slip condition*, accounting for the fluid slipping along the boundary. From the analytical point of view, the proof would be simplified slightly as we would be completely unflapped by the trace of the velocity field. Navier's condition $(3.5)_4$ can be further generalized; see [9] where the so called *threshold slip* was investigated. This condition is a very natural approximation of the no-slip condition as it models a fluid adhering to the boundary until a certain *threshold stress* is experienced, after which the fluid abides by Navier's condition.

Although, as stated, the result of Theorem 3.3.1 is optimal in terms of the range of r, there are still opportunities for improvement. Firstly, the condition from Assumption 3.2.2,

$$\gamma_0 < \frac{C_1}{C_{reg}(C_1 + C_2)},$$

now depends on the set Ω through the constant C_{reg}. It is highly probable, however, that like in the steady case (see [14]), one may relax the condition to the point

$$\gamma_0 < \frac{C_1}{C_1 + C_2}. \tag{3.8}$$

It would require replacing the solving operator of the Neumann problem $\mathcal{N}$ (see (3.13)) with something *more refined*, i.e. an operator with all the properties we want from $\mathcal{N}$, enjoying additionally $C_{reg} = 1$. In [14], we were able to do so by means of the Newtonian potential. In the time-dependent case, however, this choice is no longer viable due to the loss of certain necessary compactness with respect to the time derivative.

Secondly, in $(3.5)_6$ it would seem more appropriate to prescribe $p_{\Omega_0}(t)$ over some (possibly small) measurable $\Omega_0 \subset \Omega$, thus to approximately fix the pressure at some point. Unluckily, not only does such a generalization lead to severe technical difficulties in the proof but, perhaps even more importantly, Assumption 3.2.2 was then altered to

$$\gamma_0 < \sqrt{\frac{|\Omega_0|}{|\Omega|}} \frac{C_1}{C_{reg}(C_1 + C_2)},$$

see [13]. This condition is sufficiently deterring in itself as $|\Omega_0| \to 0$ implies $\gamma_0 \to 0$. Bear in mind that this is again not the case for the steady problem, where (3.8) would suffice.

As far as the proof of Theorem 3.3.1 is concerned, we employ a two-level approximation scheme (see (3.128)). The inner level (limit parameter k) consists in truncation of the convective and boundary terms so that up to that point we have a sufficiently regular pressure and the velocity field is a legal test function. Getting rid of this approximation level lies virtually at the heart of this paper and the entire Section 3.5 is devoted to it. It is based on a pressure decomposition (see p. 68) into a lowly integrable but compact part and a highly integrable part that is at first sight only weakly convergent. Besides this decomposition, we resort to the Lipschitz truncation of functions lying in Bochner spaces (see Lemmas 3.4.5 and 3.4.6) to deal with the issue of insufficient regularity of the velocity field to make it an admissible test function in (3.7).

The primary objective of the outer level is to introduce the pressure. Unlike the traditional Navier-Stokes model, we cannot invoke De Rham's theorem in our situation, for the viscous stress tensor itself is pressure-dependent – the resultant pressure would be a distribution in time. Also, there would then appear two possibly distinct pressures (one in $\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$ and the other generated by De Rham's theorem) and we might have to resort to some fixed-point argument to equate them. Here we construct the pressure by means of an auxiliary elliptic problem, the so called *quasicompressible approximation* (see [17]), replacing the condition on solenoidality $(3.5)_2$ by

$$\varepsilon p = \mathcal{N}(\operatorname{div} \boldsymbol{v}),$$

(see (3.13) for the definition of $\mathcal{N}$), intuitively making the velocity field only *almost* divergence-free. Since this level of approximation is comparatively simpler to lift than the truncation, we leave it for Appendix.

Moving on to the following section, we survey several nontrivial results exploited in the proof of Theorem 3.3.1.

3.4 Auxiliary tools

To begin with, we list a couple of crucial properties exhibited by the nonlinear viscous stress tensor $\boldsymbol{S}$.

Lemma 3.4.1 ([17], Lemmas 3.3, 3.4) *Let Assumptions 3.2.1 and 3.2.2 hold. For arbitrary $\boldsymbol{D}^1, \boldsymbol{D}^2 \in \mathbb{R}^{d\times d}_{\text{sym}}$ and $p^1, p^2 \in \mathbb{R}$ we set*

$$I^{1,2} := \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}^1 - \boldsymbol{D}^2|^2 \, ds,$$

with $\overline{\boldsymbol{D}}(s) = \boldsymbol{D}^2 + s(\boldsymbol{D}^1 - \boldsymbol{D}^2)$. Then

$$\frac{1}{2} C_1 I^{1,2} \le (\boldsymbol{S}(p^1, \boldsymbol{D}^1) - \boldsymbol{S}(p^2, \boldsymbol{D}^2)) \cdot (\boldsymbol{D}^1 - \boldsymbol{D}^2) + \frac{\gamma_0^2}{2C_1} |p^1 - p^2|^2. \tag{3.9}$$

Furthermore

$$|(\boldsymbol{S}(p^1, \boldsymbol{D}^1) - \boldsymbol{S}(p^2, \boldsymbol{D}^2))| \le \gamma_0 |p^1 - p^2| + C_2 \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}^1 - \boldsymbol{D}^2| \, ds. \tag{3.10}$$

Finally, for all $p \in \mathbb{R}$, $r \in (1,2]$ *and* $\boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$\boldsymbol{S}(p,\boldsymbol{D}) \cdot \boldsymbol{D} \geq \frac{C_1}{2r}(|\boldsymbol{D}|^r - 1) \tag{3.11}$$

and

$$|\boldsymbol{S}(p,\boldsymbol{D})| \leq \frac{C_2}{r-1}(1+|\boldsymbol{D}|)^{r-1}. \tag{3.12}$$

The corresponding statement in [17] does not include (3.10). However, it is only an easy observation stemming from

$$\boldsymbol{S}(p^1,\boldsymbol{D}^1) - \boldsymbol{S}(p^2,\boldsymbol{D}^2) = \int_0^1 \frac{d}{ds}\boldsymbol{S}(p^2 + s(p^1-p^2), \boldsymbol{D}^2 + s(\boldsymbol{D}^1 - \boldsymbol{D}^2))\, ds$$

and Assumptions 3.2.1 and 3.2.2.

We also recall the Helmholtz decomposition and the L^q-regularity theory of the Neumann problem for Poisson's equation: If $q \in (1,\infty)$ and $\Omega \in \mathcal{C}^{1,1}$, let

$$\mathcal{N} : \mathring{L}^q(\Omega) \to \mathring{W}^{2,q}(\Omega)$$

ascribe to $z \in \mathring{L}^q(\Omega)$ the unique solution v of

$$\Delta v = z \text{ in } \Omega, \qquad \nabla v \cdot \boldsymbol{n} = 0 \text{ at } \partial\Omega, \qquad v_\Omega = 0. \tag{3.13}$$

The Helmholtz decomposition of the space $W^{1,q}_{\boldsymbol{n}}(\Omega)^d$ lets us resolve any $\boldsymbol{u} \in W^{1,q}_{\boldsymbol{n}}(\Omega)^d$ as a sum

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{div}} + \nabla \boldsymbol{g}_{\boldsymbol{u}}, \tag{3.14}$$

where $\boldsymbol{g}_{\boldsymbol{u}} = \mathcal{N}(\operatorname{div}\boldsymbol{u})$ and $\boldsymbol{u}_{\mathrm{div}} = \boldsymbol{u} - \nabla \boldsymbol{g}_{\boldsymbol{u}}$. The L^q-continuity of $\boldsymbol{u} \mapsto \boldsymbol{u}_{\mathrm{div}}$ [18, Remark III.1.1] and the L^q-regularity for $\mathcal{N}$ with $\Omega \in \mathcal{C}^{1,1}$ [21, Proposition 2.5.2.3] imply

$$\begin{aligned} \|\mathcal{N}(z)\|_{W^{2,q}(\Omega)} &\leq C_{reg,q}\,\|z\|_{L^q(\Omega)}\,, \qquad \|\boldsymbol{u}_{\mathrm{div}}\|_{W^{1,q}(\Omega)} \leq (C_{reg,q}+1)\,\|\boldsymbol{u}\|_{W^{1,q}(\Omega)}\,, \\ \|\boldsymbol{g}_{\boldsymbol{u}}\|_{W^{1,q}(\Omega)} &\leq C(\Omega,q)\,\|\boldsymbol{u}\|_{L^q(\Omega)}\,, \qquad \|\boldsymbol{u}_{\mathrm{div}}\|_{L^q(\Omega)} \leq C(\Omega,q)\,\|\boldsymbol{u}\|_{L^q(\Omega)}\,, \end{aligned} \tag{3.15}$$

for any $z \in \mathring{L}^q(\Omega)$ and $\boldsymbol{u} \in W^{1,q}_{\boldsymbol{n}}(\Omega)^d$. Later on we will need especially $C_{reg} = C_{reg,2}$ which is why we utilize different notation for these constants.

Lemma 3.4.2 (Korn's inequality, [16], Theorem 10.15) *Let* $\Omega \in \mathcal{C}^{0,1}$ *and* $r \in (1,\infty)$. *Then there exists a positive constant* $C = C(\Omega, r)$ *such that for all* $\boldsymbol{u} \in W^{1,r}(\Omega)$ *it holds that*

$$\|\boldsymbol{u}\|_{W^{1,r}(\Omega)} \leq C\big(\|\boldsymbol{D}\boldsymbol{u}\|_{L^r(\Omega)} + \|\boldsymbol{u}\|_{L^1(\Omega)}\big). \tag{3.16}$$

Lemma 3.4.3 (Compactness of traces) *Let* r *and* q *retain their meaning from Theorem 3.3.1 and suppose that* $\{\boldsymbol{v}^i\}_{i=1}^\infty$ *is bounded in*

$$L^r(0,T;W^{1,r}_{\boldsymbol{n}}(\Omega)) \cap W^{1,q}(0,T;W^{-1,q}_{\boldsymbol{n}}).$$

Then $\{\operatorname{Tr}\boldsymbol{v}^i\}_{i=1}^\infty$ *is precompact in* $L^r(0,T;L^r(\partial\Omega))$.

Proof. The Aubin-Lions lemma implies precompactness of $\{\boldsymbol{v}^i\}_{i=1}^\infty$ in $L^r(0,T;L^r(\Omega))$. Interpolation (see e.g. [25, Lemma 2.18]) then yields precompactness of $\{\boldsymbol{v}^i\}_{i=1}^\infty$ in $L^r(0,T;W^{1-\varepsilon,r}(\Omega))$ for an arbitrarily small $\varepsilon > 0$. There is also a continuous trace operator from $W^{p_1,p_2}(\Omega)$ into $W^{p_1-1/p_2,p_2}(\partial\Omega)$ for any $p_1 \in R_+$ and $p_2 \geq 1$ such that $p_1 p_2 > 1$ (see [30] and the remark in [4, Lemma B.3]). Taking $\varepsilon > 0$ so small that $(1-\varepsilon)r > 1$, we have $L^r(0,T;W^{1-\varepsilon-\frac{1}{r},r}(\partial\Omega)) \hookrightarrow L^r(0,T;L^r(\partial\Omega))$ and thus also the claim. □

Lemma 3.4.4 (Biting lemma, [2]) *Let $S \subset \mathbb{R}^d$ have a finite Lebesgue measure and $\{f^k\}$ be a bounded sequence in $L^1(S)$. Then there exist a function $f \in L^1(S)$, a subsequence $\{f^j\}$ of $\{f^k\}$ and a nonincreasing sequence of measurable sets $D^m \subset S$ with $\lim_{m\to\infty} |D^m| = 0$, such that $f^j \to f$ weakly in $L^1(S \setminus D^m)$ for every fixed m.*

Lemma 3.4.5 (Parabolic Lipschitz approximation I, [15], Lemma 3.11, Theorem 3.21) *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set, $\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega)^d) \cap L^q(0,T;W^{1,q}(\Omega)^d)$ $(1 < q < \infty)$ and $\boldsymbol{H} \in L^\sigma(0,T;L^\sigma(\Omega)^{d\times d})$ $(1 < \sigma < \infty)$ be such that*

$$-\int_Q \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi}\, dx\, dt = \int_Q \boldsymbol{H} \cdot \nabla \boldsymbol{\varphi}\, dx\, dt \tag{3.17}$$

for all $\boldsymbol{\varphi} \in \mathcal{C}_c^\infty(Q)$. For $\Lambda > 0$ we define

$$\mathcal{O}_\Lambda = \{\mathcal{M}^*(|\nabla \boldsymbol{u}^k|) + \mathcal{M}^*(|\boldsymbol{H}|) > \Lambda\}.$$

Let $E \subset \mathbb{R}^{d+1}$ be an open set such that $Q \cap \mathcal{O}_\Lambda \subset E \subset Q$.

Then there exists $\mathcal{L}_E \boldsymbol{u} \in L^\infty_{loc}(0,T;W^{1,\infty}_{loc}(\Omega)^d)$ such that $\mathcal{L}_E\boldsymbol{u} = \boldsymbol{u}$ in $Q \setminus E$ and[1]

$$\|\mathcal{L}_E \boldsymbol{u}\|_{L^p(Q)} \leq C \|\boldsymbol{u}\|_{L^p(Q)} \qquad \textit{for any } 1 \leq p \leq \infty. \tag{3.18}$$

Let $K \subset Q$ be a compact set. There is a constant $C_K > 0$ depending on K such that

$$\|\nabla \mathcal{L}_E \boldsymbol{u}\|_{L^\infty(K)} \leq C\big(\Lambda + C_K \|\boldsymbol{u}\|_{L^1(E)}\big). \tag{3.19}$$

Furthermore, the function $(\partial_t \mathcal{L}_E \boldsymbol{u}) \cdot (\mathcal{L}_E \boldsymbol{u} - \boldsymbol{u})$ belongs to $L^1(K \cap E)$ and we have

$$\|(\partial_t \mathcal{L}_E \boldsymbol{u}) \cdot (\mathcal{L}_E \boldsymbol{u} - \boldsymbol{u})\|_{L^1(K\cap E)} \leq C|E|\big(\Lambda + C_K \|\boldsymbol{u}\|_{L^1(E)}\big)^2. \tag{3.20}$$

Finally, for all $g \in \mathcal{C}_c^\infty(Q)$ holds the identity

$$\begin{aligned}\int_0^T &\big\langle \partial_t \boldsymbol{u}(t), (\mathcal{L}_E \boldsymbol{u}(t)) g(t)\big\rangle\, dt \\ &= \frac{1}{2}\int_Q \big(|\mathcal{L}_E \boldsymbol{u}|^2 - 2\boldsymbol{u} \cdot \mathcal{L}_E \boldsymbol{u}\big)\partial_t g\, dx\, dt + \int_E (\partial_t \mathcal{L}_E \boldsymbol{u}) \cdot (\mathcal{L}_E \boldsymbol{u} - \boldsymbol{u}) g\, dx\, dt.\end{aligned} \tag{3.21}$$

The original version of the stated lemma contains also a certain scaling parameter[2] $\alpha > 0$. For our purposes we need the case $\alpha = 1$ only and we have adapted the statement of the lemma accordingly.

[1]The generic constants C below depend only on the dimension d.
[2]This scaling parameter α is completely unrelated to that in the boundary condition $(3.5)_4$.

Lemma 3.4.6 (Parabolic Lipschitz approximation II, [6], Lemma 2.5) *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set, $T > 0$ and $r \in (1, \infty)$. For any functions $\boldsymbol{H}$, $\overline{\boldsymbol{H}}$ and arbitrary sequences $\{\boldsymbol{u}^k\}$ and $\{\boldsymbol{H}^k\}$ we set*

$$a^k = |\boldsymbol{H}^k| + |\boldsymbol{H}| + |\overline{\boldsymbol{H}}| \quad \textit{and} \quad b^k = |\boldsymbol{D}\boldsymbol{u}^k|$$

and suppose that for certain $C^* > 1$ and all k we have

$$\|a^k\|_{L^r(Q)} + \|b^k\|_{L^{r'}(Q)} + \sup_{t\in(0,T)} \|\boldsymbol{u}^k(t)\|_{L^2(\Omega)} \le C^*,$$
$$\boldsymbol{u}^k \to \boldsymbol{0} \quad \textit{a.e. in } Q.$$

In addition, let $\{\boldsymbol{G}^k\}$ consist of symmetric $\boldsymbol{G}^k$ such that

$$\boldsymbol{G}^k \to \boldsymbol{0} \quad \textit{strongly in } L^1(Q)^{d\times d} \tag{3.22}$$

and let us have the distributional identity

$$\partial_t \boldsymbol{u}^k + \operatorname{div}(\boldsymbol{H}^k - \boldsymbol{H} + \boldsymbol{G}^k) = \boldsymbol{0}.$$

Then there is $\beta > 0$ such that for arbitrary $\widehat{Q} \Subset Q$, $\lambda^* \in (r^{\frac{1}{(r-1)}}, \infty)$ and $n \in \mathbb{N}$ there exist sequences $\{\lambda^{k,n}\}_k \subset \mathbb{R}$, $\{B^{k,n}\}_k$ of open sets $B^{k,n} \subset Q$ and $\{\boldsymbol{u}^{k,n}\}_k$ bounded in $L^\infty_{loc}(0, T; W^{1,\infty}_{loc}(\Omega)^d)$ such that

$$\{\lambda^{k,n}\}_k \subset [\lambda^*, r^{\frac{1-r^n}{r-1}} (\lambda^*)^{r^n}], \tag{3.23}$$
$$\limsup_{k\to\infty} |\widehat{Q} \cap B^{k,n}| \le \frac{C(\widehat{Q})}{(\lambda^*)^r}, \tag{3.24}$$
$$\boldsymbol{u}^{k,n} \to \boldsymbol{0} \quad \textit{strongly in } L^s(\widehat{Q})^d \textit{ as } k \to \infty \textit{ for any } 1 \le s < \infty, \tag{3.25}$$
$$\boldsymbol{u}^{k,n} = \boldsymbol{u}^k \quad \textit{in } \widehat{Q} \setminus B^{k,n}, \tag{3.26}$$
$$\|\boldsymbol{D}\boldsymbol{u}^{k,n}\|_{L^\infty(\widehat{Q})} \le C(\widehat{Q})\lambda^{k,n}. \tag{3.27}$$

Moreover, for all $\tau \in \mathcal{C}^\infty_c(\widehat{Q}; [0, 1])$ the following estimates hold:

$$\limsup_{k\to\infty} \int_{\widehat{Q}\cap B^{k,n}} (|\boldsymbol{H}^k| + |\boldsymbol{H}| + |\overline{\boldsymbol{H}}|)|\boldsymbol{D}\boldsymbol{u}^{k,n}| \, dx\, dt \le C(\widehat{Q})(r(\lambda^*)^{1-r} + n^{-\beta}), \tag{3.28}$$
$$-\liminf_{k\to\infty} \int_0^T \langle \partial_t \boldsymbol{u}^k, \boldsymbol{u}^{k,n}\tau\rangle \, dt \le C(\widehat{Q})(r(\lambda^*)^{1-r} + n^{-1})^\beta. \tag{3.29}$$

Strictly speaking, the above lemma as we state it is not a precise reproduction of [6]. To avoid unnecessary generality of Orlicz spaces, our theorem pertains to a special choice of the N-function $\psi(x) = x^r/r$, to which we adapted all parameters of the original theorem. Dependences of constants on fixed parameters, e.g. Ω or r, were also omitted. In (3.29), the estimate should also hang on $\|\tau\|_{L^\infty(\Omega)}$ but since we restrict ourselves to $\|\tau\|_{L^\infty(\Omega)} \le 1$, we may assume that the bound is really independent of the truncating function τ and that each $C(\widehat{Q})$ in (3.24)–(3.29) are the same.

3.5 Proof of the existence theorem

Without loss of generality we will assume $h \equiv 0$, that is to say

$$\int_\Omega p(t,x)\,dx = 0$$

for almost every time. We can think so since in the general case we would first investigate the equation for $\overline{p} = p - h$. Due to $h \in L^q(0,T)$, if $\overline{p} \in L^q(Q)$ then of course also $p \in L^q(Q)$.

There is a couple of strategies how to deal with the convective term, be it the addition of a penalty term, its mollification, or truncation (see e.g. [6, 13, 15], respectively). Here, we choose the truncation and for this purpose, let $\Phi \in \mathcal{C}^1([0,\infty))$ be a non-increasing function such that $\Phi(x) = 1$ if $x \le 1$, $\Phi(x) = 0$ if $x \ge 2$ and $\Phi(x) \in (0,1)$ otherwise, with $|\Phi'(x)| \le 2$. For $k > 0$ then define

$$\Phi_k(x) = \Phi(k^{-1}x).$$

With fixed $k > 0$, the original system (3.5) will be approximated by

$$\left.\begin{aligned}
\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}\Phi_k(|\boldsymbol{v}|)) - \operatorname{div}\boldsymbol{S} + \nabla p = -\operatorname{div}\boldsymbol{F} &\quad \text{in } Q,\\
\operatorname{div}\boldsymbol{v} = 0 &\quad \text{in } Q,\\
\boldsymbol{v}\cdot\boldsymbol{n} = 0 &\quad \text{on } \Gamma,\\
\alpha\boldsymbol{v}_\tau\Phi_k(|\boldsymbol{v}_\tau|) = -(\boldsymbol{S}\boldsymbol{n})_\tau &\quad \text{on } \Gamma,\\
\boldsymbol{v}(0) = \boldsymbol{v}_0 &\quad \text{in } \Omega,\\
p_\Omega = 0 &\quad \text{in } (0,T).
\end{aligned}\right\} \tag{3.30}$$

The boundary conditions imply $\boldsymbol{v}_\tau = \boldsymbol{v}$ on Γ and therefore we will not distinguish between these two entities (see the weak formulation (3.31)).

Existence of weak solutions for thus truncated system can be shown by standard means (see e.g. [12, 13]) and we postpone it for Appendix. To be more precise, we suppose momentarily that the following lemma holds:

Lemma 3.5.1 *Under the assumptions of Theorem 3.3.1, for every $k > 0$ there exists a weak solution to the truncated problem* (3.30)*, i.e. a couple $(\boldsymbol{v}^k, p^k)$ such that*

$$\boldsymbol{v}^k \in L^r(0,T;W^{1,r}_{\boldsymbol{n},\operatorname{div}}(\Omega)), \quad \partial_t\boldsymbol{v}^k \in L^{r'}(0,T;W^{-1,r'}_{\boldsymbol{n}}(\Omega)), \quad p^k \in L^{r'}(0,T;\mathring{L}^{r'}(\Omega)),$$

satisfying[3] $\lim_{t\to 0_+} \|\boldsymbol{v}^k(t) - \boldsymbol{v}_0\|_{L^2(\Omega)} = 0$ *and*

$$\begin{aligned}
\langle\partial_t\boldsymbol{v}^k(t),\boldsymbol{\varphi}\rangle - (\boldsymbol{v}^k\otimes\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|)(t),\nabla\boldsymbol{\varphi}) + (\boldsymbol{S}^k(t),\boldsymbol{D}\boldsymbol{\varphi})&\\
+\alpha(\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|),\boldsymbol{\varphi})_{\partial\Omega} - (p^k(t),\operatorname{div}\boldsymbol{\varphi}) = (\boldsymbol{F}(t),\nabla\boldsymbol{\varphi})&
\end{aligned} \tag{3.31}$$

with $\boldsymbol{S}^k(t) = \boldsymbol{S}(p^k(t),\boldsymbol{D}\boldsymbol{v}^k(t))$, for every $\boldsymbol{\varphi} \in W^{1,r}_{\boldsymbol{n}}(\Omega)$ and a.e. $t \in (0,T)$.

3.5.1 Truncation removal ($k \to \infty$)

The reinstatement of the full-fledged convective term is the key limit process. Our first steps will be devoted to finding bounds independent of $k > 0$ in suitable function spaces.

[3]Note that $\boldsymbol{v}^k \in \mathcal{C}([0,T];L^2(\Omega))$.

Uniform estimates Taking $\boldsymbol{\varphi} = \boldsymbol{v}^k(t)$ in (3.31) and exploiting integration by parts and solenoidality of $\boldsymbol{v}^k$, we note that

$$(\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|), \nabla \boldsymbol{v}^k)_Q = \Big(\boldsymbol{v}^k, \nabla \int_0^{|\boldsymbol{v}^k|} s\,\Phi_k(s)\,ds\Big)_Q = 0,$$

owing to which (ensuing relations hold for a.e. $t \in (0,T)$)

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^k(t)\|_{L^2(\Omega)}^2 + (\boldsymbol{S}^k(t), \boldsymbol{D}\boldsymbol{v}^k(t)) + \alpha\|\Phi_k^{1/2}(|\boldsymbol{v}^k|)\boldsymbol{v}^k(t)\|_{L^2(\partial\Omega)}^2 = (\boldsymbol{F}(t), \nabla \boldsymbol{v}^k(t)).$$

Due to coercivity of the stress tensor (3.11), the fact that $\Phi_k \leq \Phi_k^{1/2}$ and Hölder's inequality,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^k(t)\|_{L^2(\Omega)}^2 + \frac{C_1}{2r}\|\boldsymbol{D}\boldsymbol{v}^k(t)\|_{L^r(\Omega)}^r + \alpha\|\Phi_k(|\boldsymbol{v}^k|)\boldsymbol{v}^k(t)\|_{L^2(\partial\Omega)}^2 \\ \leq \|\boldsymbol{F}(t)\|_{L^{r'}(\Omega)}\|\nabla \boldsymbol{v}^k(t)\|_{L^r(\Omega)} + \frac{C_1|\Omega|}{2r}. \quad (3.32)$$

By means of Hölder's, Young's and Korn's inequality (3.16), we then obtain

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^k(t)\|_{L^2(\Omega)}^2 + \|\boldsymbol{v}^k\|_{L^r(0,T;W^{1,r}(\Omega))}^r + \|\Phi_k(|\boldsymbol{v}^k|)\boldsymbol{v}^k\|_{L^2(\Gamma)}^2 \\ \leq C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \quad (3.33)$$

Using boundedness of the stress tensor (3.12), we get in addition

$$\|\boldsymbol{S}^k\|_{L^{r'}(Q)}^{r'} \leq C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \quad (3.34)$$

Combined with $L^\infty(0,T;L^2(\Omega)) \cap L^r(0,T;W^{1,r}(\Omega)) \hookrightarrow L^{2q}(Q)$ with $q > 1$ (defined in (3.6)), we have also

$$\|\boldsymbol{v}^n\|_{L^{2q}(Q)} \leq C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \quad (3.35)$$

As for a bound on the pressure p^k, due to the convective term we have to relax our requirements from the current integrability $p^k \in L^{r'}(Q)$ – we will estimate it in $L^q(Q)$. Let us consider the equation (3.31) with the test function

$$\boldsymbol{\varphi}^k = \nabla\mathcal{N}\big(|p^k|^{q-2}p^k - (|p^k|^{q-2}p^k)_\Omega\big), \quad (3.36)$$

which due to (3.15) satisfies

$$\|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \leq C\||p^k|^{q-1}\|_{L^{q'}(Q)} = C\|p^k\|_{L^q(Q)}^{q-1}, \\ \operatorname{div}\boldsymbol{\varphi}^k = |p^k|^{q-2}p^k - (|p^k|^{q-2}p^k)_\Omega \quad \text{a.e. in } Q.$$

Here we want to point out that had we chosen Dirichlet's boundary conditions instead of Navier's, we would now have run into serious trouble. The culprit is $\operatorname{Tr}\boldsymbol{\varphi}^k$ – in the Dirichlet setting we would be unable to justify it is actually zero, making the choice of (3.36) illegal for the weak formulation corresponding to Dirichlet's problem. indeed, we could choose $\boldsymbol{\varphi}^k$ differently so that $\operatorname{Tr}\boldsymbol{\varphi}^k = \mathbf{0}$ (e.g. by means of the Bogovskiĭ operator) but then we would face new problems stemming from the time derivative $\partial_t \boldsymbol{v}^k$ (see I_5 below and how it vanishes with our choice of $\boldsymbol{\varphi}^k$).

From (3.31) it holds that

$$\|p^k\|^q_{L^q(Q)} = (p^k, \operatorname{div} \boldsymbol{\varphi}^k)_Q = \sum_{i=1}^{5} I_i,$$

where due to Hölder's inequality and estimates (3.33), (3.34) and (3.35) (note $q' \geq 2$),

$$I_1 = -(\boldsymbol{F}, \nabla\boldsymbol{\varphi}^k)_Q \leq \|\boldsymbol{F}\|_{L^{r'}(Q)}\|\nabla\boldsymbol{\varphi}^k\|_{L^r(Q)} \leq C\|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))},$$

$$I_2 = (\boldsymbol{S}^k, \boldsymbol{D}\boldsymbol{\varphi}^k)_Q \leq \|\boldsymbol{S}^k\|_{L^{r'}(Q)}\|\nabla\boldsymbol{\varphi}^k\|_{L^r(Q)} \leq C\|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))},$$

$$I_3 = \alpha(\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|), \boldsymbol{\varphi}^k)_\Gamma \leq \alpha\|\Phi_k(|\boldsymbol{v}^k|)\boldsymbol{v}^k\|^2_{L^2(\Gamma)}\|\boldsymbol{\varphi}^k\|_{L^2(\Gamma)} \leq C\|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))},$$

$$I_4 = -(\boldsymbol{v}^k \otimes \boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|), \nabla\boldsymbol{\varphi}^k)_Q \leq \|\boldsymbol{v}^k\|^2_{L^{2q}(Q)}\|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}$$
$$\leq C\|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))},$$

$$I_5 = \int_0^T \langle \partial_t\boldsymbol{v}^k, \boldsymbol{\varphi}^k\rangle\, dt = -\big(\partial_t \operatorname{div}\boldsymbol{v}^k, \mathcal{N}\big(|p^k|^{q-2}p^k - (|p^k|^{q-2}p^k)_\Omega\big)\big)_Q = 0.$$

Thus we have the desired estimate

$$\|p^k\|_{L^q(Q)} \leq C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \tag{3.37}$$

Next, estimates (3.33), (3.34), (3.35) and (3.37) divulge that functionals Ψ^k defined on $L^\infty(0,T;W^{1,\infty}_{\boldsymbol{n}})$ as

$$\Psi^k(\boldsymbol{\varphi}) = (\boldsymbol{v}^k \otimes \boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|), \nabla\boldsymbol{\varphi})_Q - (\boldsymbol{S}^k, \boldsymbol{D}\boldsymbol{\varphi})_Q - \alpha(\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|), \boldsymbol{\varphi})_\Gamma + (p^k, \operatorname{div}\boldsymbol{\varphi})_Q$$
$$+ (\boldsymbol{F}, \nabla\boldsymbol{\varphi})_Q,$$

satisfy

$$\big|\Psi^k(\boldsymbol{\varphi})\big| \leq C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big)\|\boldsymbol{\varphi}\|_{L^{q'}(0,T;W^{1,q'}_{\boldsymbol{n}}(\Omega))}$$

uniformly in k. In other words, from eq. (3.31) it follows that

$$\|\partial_t\boldsymbol{v}^k\|_{L^q(0,T;W^{-1,q}_{\boldsymbol{n}}(\Omega))} \leq C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \tag{3.38}$$

Limit $k \to \infty$ By the uniform bounds (3.33)–(3.38) and the compactness lemma 3.4.3, we may select a subsequence $(\boldsymbol{v}^k, p^k)$ satisfying[4]

$$\boldsymbol{v}^k \to \boldsymbol{v} \quad \text{weakly in } L^r(0,T;W^{1,r}_{\boldsymbol{n},\operatorname{div}}(\Omega)), \tag{3.39}$$
$$\boldsymbol{v}^k \to \boldsymbol{v} \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{3.40}$$
$$\partial_t\boldsymbol{v}^k \to \partial_t\boldsymbol{v} \quad \text{weakly in } L^q(0,T;W^{-1,q}_{\boldsymbol{n}}(\Omega)), \tag{3.41}$$
$$\boldsymbol{v}^k \to \boldsymbol{v} \quad \text{strongly in } L^s(Q) \text{ for all } s \in [1,2q), \tag{3.42}$$
$$\boldsymbol{v}^k \to \boldsymbol{v} \quad \text{strongly in } L^r(\Gamma), \tag{3.43}$$
$$\Phi_k(|\boldsymbol{v}^k|)\boldsymbol{v}^k \to \boldsymbol{v} \quad \text{weakly in } L^2(\Gamma), \tag{3.44}$$
$$\Phi_k(|\boldsymbol{v}^k|)\boldsymbol{v}^k \to \boldsymbol{v} \quad \text{strongly in } L^s(\Gamma) \text{ for all } s \in [1,2), \tag{3.45}$$
$$\boldsymbol{v}^k \to \boldsymbol{v} \quad \text{a.e. in } Q, \tag{3.46}$$
$$p^k \to p \quad \text{weakly in } L^q(0,T;\mathring{L}^q(\Omega)), \tag{3.47}$$
$$\boldsymbol{S}^k \to \overline{\boldsymbol{S}} \quad \text{weakly in } L^{r'}(Q). \tag{3.48}$$

[4] We employ bars for unidentified weak limits.

We also have $\boldsymbol{v} \in \mathcal{C}_w([0,T];L^2(\Omega))$ by (3.40) and (3.41). These convergences, when applied to equation (3.31), produce

$$\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi} \rangle \, dt - (\boldsymbol{v} \otimes \boldsymbol{v}, \nabla \boldsymbol{\varphi})_Q + (\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{\varphi})_Q + \alpha(\boldsymbol{v}, \boldsymbol{\varphi})_\Gamma - (p, \operatorname{div} \boldsymbol{\varphi})_Q = (\boldsymbol{F}, \nabla \boldsymbol{\varphi})_Q \tag{3.49}$$

for every $\boldsymbol{\varphi} \in L^{q'}(0,T;W^{1,q'}_{\boldsymbol{n}}(\Omega))$.

The next step, basically the core of this paper, consists in showing $\overline{\boldsymbol{S}} = \boldsymbol{S}$ (i.e. $\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$) and this will be achieved through Vitali's theorem, since $\boldsymbol{S}(\cdot,\cdot)$ is continuous. To this end we have to show the pointwise convergence of $\boldsymbol{D}\boldsymbol{v}^k$ and p^k a.e. in Q.

Decomposition of p^k We will overcome the problem of low[5] pressure integrability by decomposing the pressure into two parts – one keeping the low q-integrability, yet converging pointwise, and the other r'-integrable, for which we then prove the pointwise convergence.

According to (3.31), for any $\varphi \in W^{2,q'}(\Omega)$ such that $\nabla\varphi \cdot \boldsymbol{n} = 0$ at $\partial\Omega$ and a.e. $t \in (0,T)$, we have

$$\begin{aligned}(p^k(t), \Delta\varphi) = &-(\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|)(t), \nabla^2\varphi) + (\boldsymbol{S}^k(t), \nabla^2\varphi) \\ &+ \alpha(\boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|)(t), \nabla\varphi)_{\partial\Omega} - (\boldsymbol{F}(t), \nabla^2\varphi).\end{aligned} \tag{3.50}$$

We will decompose the pressure as $p^k = p_1^k + p_2^k$, where $p_2^k \in L^{r'}(0,T;\mathring{L}^{r'}(\Omega))$ is the unique solution to

$$\begin{aligned}(p_2^k(t), \Delta\varphi) &= (\boldsymbol{S}^k(t), \nabla^2\varphi) - (\boldsymbol{F}(t), \nabla^2\varphi), \\ (p_2^k(t))_\Omega &= 0\end{aligned} \tag{3.51}$$

for any $\varphi \in W^{2,r}(\Omega)$ such that $\nabla\varphi \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ and a.e. $t \in (0,T)$. For details about solvability of this equation, formally corresponding to

$$\Delta p_2^k(t) = \operatorname{div}\operatorname{div}\big(\boldsymbol{S}^k(t) - \boldsymbol{F}(t)\big),$$

see [4, (3.51)], where a procedure based on an approximation of what is here $\boldsymbol{S}^k(t) - \boldsymbol{F}(t)$ by compactly supported smooth functions is explained in more depth. Let us show $\{p_2^k\}$ is bounded in $L^{r'}(Q)$. To this end take

$$\varphi(t) = \mathcal{N}\big(|p_2^k(t)|^{r'-2}p_2^k(t) - (|p_2^k(t)|^{r'-2}p_2^k(t))_\Omega\big)$$

and recall that for $\mathcal{N}$ we have L^q-regularity (3.15), implying

$$\|\varphi(t)\|_{W^{2,r}(\Omega)} \le C(\Omega,r)\||p_2^k(t)|^{r'-1}\|_{L^r(\Omega)} = C(\Omega,r)\|p_2^k(t)\|^{r'-1}_{L^{r'}(\Omega)}. \tag{3.52}$$

Next we insert such φ into (3.51), obtaining

$$\begin{aligned}\|p_2^k\|^{r'}_{L^{r'}(Q)} &= (p_2^k, \Delta\varphi)_Q = (\boldsymbol{S}^k - \boldsymbol{F}, \nabla^2\varphi)_Q \\ &\le \big(\|\boldsymbol{S}^k\|_{L^{r'}(Q)} + \|\boldsymbol{F}\|_{L^{r'}(Q)}\big)\|\varphi\|_{L^r(0,T;W^{2,r}(\Omega))} \\ &\le C\|p_2^k\|^{r'-1}_{L^{r'}(Q)}\end{aligned}$$

[5]Relative to the ε-limit, cf. Subsection 3.6.1.

by means of Hölder's inequality and the estimates (3.34) and (3.52). Therefore we may assume there exists $p_2 \in L^{r'}(0,T;\mathring{L}^{r'}(\Omega))$ such that

$$p_2^k \to p_2 \quad \text{weakly in } L^{r'}(Q). \tag{3.53}$$

By (3.50) and (3.54), the other partial pressure $p_1^k = p^k - p_2^k$ must satisfy

$$(p_1^k(t), \Delta\varphi) = -(\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|)(t), \nabla^2\varphi) + \alpha(\boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|)(t), \nabla\varphi)_{\partial\Omega}, \tag{3.54}$$

for any $\varphi \in W^{2,q'}(\Omega)$ such that $\nabla\varphi \cdot \boldsymbol{n} = 0$ at $\partial\Omega$ and $(p_1^k(t))_\Omega = 0$ for a.e. $t \in (0,T)$. It follows from (3.47) and (3.53) that $\{p_1^k\}$ is bounded in $L^q(0,T;\mathring{L}^q(\Omega))$. We will show it also converges strongly in $L^1(0,T;L^1(\Omega))$. Let $k, l \in \mathbb{N}$ and $1 < s < q$ be arbitrary. Take

$$\varphi(t) = \mathcal{N}\big(|p_1^k - p_1^l|^{s-2}(p_1^k - p_1^l)(t) - (|p_1^k - p_1^l|^{s-2}(p_1^k - p_1^l)(t))_\Omega\big)$$

and like in (3.52), observe that due to L^q-regularity (3.15),

$$\|\varphi(t)\|_{W^{2,s'}(\Omega)} \le C(\Omega,s)\||p_1^k - p_1^l|^{s-1}(t)\|_{L^{s'}(\Omega)} = C(\Omega,s)\|(p_1^k - p_1^l)(t)\|^{s-1}_{L^s(\Omega)}. \tag{3.55}$$

Plugging φ into (3.54) yields

$$\|p_1^k - p_1^l\|^s_{L^s(Q)} = (p_1^k - p_1^l, \Delta\varphi)_Q = I_1 + I_2,$$

where, using (3.55),

$$\begin{aligned}
I_1 &= (\boldsymbol{v}^l \otimes \boldsymbol{v}^l \Phi_l(|\boldsymbol{v}^l|)(t) - \boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|)(t), \nabla^2\varphi)_Q,\\
&\le \|\boldsymbol{v}^l \otimes \boldsymbol{v}^l \Phi_l(|\boldsymbol{v}^l|) - \boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|)\|_{L^s(Q)} \|\varphi\|_{L^{s'}(0,T;W^{2,s'}(\Omega))}\\
&\le C(\Omega,s)\|\boldsymbol{v}^l \otimes \boldsymbol{v}^l \Phi_l(|\boldsymbol{v}^l|) - \boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|)\|_{L^s(Q)} \|p_1^k - p_1^l\|^{s-1}_{L^s(Q)}
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \alpha(\boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|)(t) - \boldsymbol{v}^l \Phi_l(|\boldsymbol{v}^l|)(t), \nabla\varphi)_\Gamma\\
&\le \alpha\|\boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}^l \Phi_l(|\boldsymbol{v}^l|)\|_{L^s(\Gamma)} \|\varphi\|_{L^{s'}(\Gamma)}\\
&\le C(\Omega,s)\|\boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}^l \Phi_l(|\boldsymbol{v}^l|)\|_{L^s(\Gamma)} \|\varphi\|_{L^{s'}(0,T;W^{2,s'}(\Omega))}\\
&\le C(\Omega,s)\|\boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}^l \Phi_l(|\boldsymbol{v}^l|)\|_{L^s(\Gamma)} \|p_1^k - p_1^l\|^{s-1}_{L^s(Q)}.
\end{aligned}$$

The above computations imply that $\{p_1^k\}$ is a Cauchy sequence in $L^s(Q)$ since by the estimate (3.35) and the strong convergence (3.42), we observe

$$\begin{aligned}
\lim_{k,l\to\infty} &\|\boldsymbol{v}^l \otimes \boldsymbol{v}^l \Phi_l(|\boldsymbol{v}^l|) - \boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|)\|_{L^s(Q)}\\
&\le 2 \lim_{k\to\infty} \|\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v} \otimes \boldsymbol{v}\|_{L^s(Q)}\\
&\le 2 \lim_{k\to\infty} \|\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}^k \otimes \boldsymbol{v}^k\|_{L^s(Q)}\\
&\le 4 \lim_{k\to\infty} \|\boldsymbol{v}^k\|^2_{L^{2s}(Q\cap\{|\boldsymbol{v}^k|>k\})}\\
&\le C \lim_{k\to\infty} \big|Q \cap \{|\boldsymbol{v}^k| > k\}\big|^{\frac{q-s}{qs}}\\
&= 0
\end{aligned}$$

and similarly, using and the strong convergence (3.45)

$$\lim_{k,l\to\infty} \|\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}^l\Phi_l(|\boldsymbol{v}^l|)\|_{L^s(\Gamma)} = 0.$$

Hence there exists $p_1 \in L^q(0,T;\mathring{L}^q(\Omega))$ such that

$$\begin{aligned} p_1^k &\to p_1 \quad \text{weakly in } L^q(Q), \\ p_1^k &\to p_1 \quad \text{strongly in } L^1(Q). \end{aligned} \tag{3.56}$$

The first convergence was trivial by the already shown weak convergences (3.47) and (3.53). In particular, we may assume

$$p_1^k \to p_1 \quad \text{a.e. in } Q.$$

From (3.56) we also infer by the dominated convergence theorem and (3.12) that

$$\boldsymbol{S}(p_1^k + p_2, \boldsymbol{Dv}) \to \boldsymbol{S} \quad \text{strongly in } L^{r'}(Q). \tag{3.57}$$

Showing the pointwise convergence of p_2^k is all that remains. We will treat the cases $r < 2$ and $r = 2$ separately. The procedure necessitated by the former case may be accommodated to deal also with the latter (and vice versa, actually). Nonetheless, it would require to prove an improved version of Lemma 3.4.6, which we do not find necessary. Even though it may not be the most elegant way of tackling the issue, we have taken the path of least resistance and resolved to cover the case $r = 2$ rather by the spiritual ancestor of the aforementioned Lemma 3.4.6, i.e. by Lemma 3.4.5. This result could be in turn utilized to handle also the case $r < 2$ but it would be considerably messier than with Lemma 3.4.6.

3.5.2 Convergence for $r < 2$

Let $N \in \mathbb{N}$ be fixed. Take $Q_N \Subset \widehat{Q}_N \Subset Q$ such that

$$|Q \setminus Q_N| \le \frac{1}{N}. \tag{3.58}$$

Now we invoke the parabolic Lipschitz truncation lemma 3.4.6, set up as follows:

$$\begin{aligned} \boldsymbol{H} &= p_2\boldsymbol{I} - \overline{\boldsymbol{S}}, \\ \boldsymbol{H}^k &= p_2^k\boldsymbol{I} - \boldsymbol{S}^k, \\ \overline{\boldsymbol{H}} &= |\boldsymbol{S}| + |\overline{\boldsymbol{S}}|, \\ \boldsymbol{u}^k &= \boldsymbol{v}^k - \boldsymbol{v}, \\ \boldsymbol{G}^k &= \boldsymbol{v}^k \otimes \boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}\otimes\boldsymbol{v} + (p_1^k - p_1)\boldsymbol{I}. \end{aligned}$$

Next we take numbers $\lambda^* = \lambda^*(N)$ and $n = n(N)$ large enough so that the constant $C(\widehat{Q}_N)$ from (3.28) and (3.29) satisfies

$$C(\widehat{Q}_N)(r(\lambda^*)^{1-r} + n^{-\beta}) \le \frac{1}{N}, \tag{3.59}$$

$$C(\widehat{Q}_N)(r(\lambda^*)^{1-r} + n^{-1})^{\beta} \le \frac{1}{N}, \tag{3.60}$$

where the number $\beta > 0$ is produced by the said Lemma 3.4.6. Note that (3.59) also implies

$$\frac{C(\widehat{Q}_N)}{(\lambda^*)^r} \leq \frac{1}{N}. \tag{3.61}$$

To finish the setup of Lemma 3.4.6, we take

$$\widehat{Q} = \widehat{Q}_N.$$

As a result, there exist sequences $\{\lambda^{k,n}\}_k \subset \mathbb{R}$, $\{B^{k,n}\}_k$ of open sets $B^{k,n} \subset Q$ and $\{\boldsymbol{u}^{k,n}\}_k$ bounded in $L^\infty_{loc}(0,T;W^{1,\infty}_{loc}(\Omega)^d)$ such that (3.23)–(3.29) hold.

Furthermore, we take $\tau^N \in \mathcal{C}^\infty_c(\widehat{Q}_N;[0,1])$ such that

$$\tau^N \equiv 1 \quad \text{in } Q_N \tag{3.62}$$

and

$$C(\widehat{Q}_N)\lambda^{k,n}\big|\{0 < \tau^N < 1\}\big|^{1/r} \leq \frac{1}{N} \quad \text{for every } k, \tag{3.63}$$

which is possible by (3.23).

Finally, we define bad sets $E^{k,n}$ and good sets $G^{k,n}$ as

$$E^{k,n} = B^{k,n} \cup \{\tau^N < 1\}, \tag{3.64}$$

$$G^{k,n} = Q \setminus E^{k,n}. \tag{3.65}$$

Informally speaking, the bad set consists of points near the boundary or those where the Lipschitz approximation does not match the original function; see (3.26). From the estimate (3.24), bounds (3.58) and (3.61) and the property (3.62), it follows that

$$\limsup_{k\to\infty} |E^{k,n}| \leq \frac{2}{N}. \tag{3.66}$$

Convergence of p_2^k Denote $\pi^k = p_2^k - p_2$. We are going to show

$$\lim_{k\to\infty} \|\pi^k\|_{L^2(Q)} = 0. \tag{3.67}$$

Towards this goal, we set

$$\boldsymbol{\varphi}^k = \mathcal{N}(\pi^k) \tag{3.68}$$

and observe that by (3.15) and (3.53), $\boldsymbol{\varphi}^k$ satisfies

$$\|\boldsymbol{\varphi}^k\|_{L^2(0,T;W^{2,2}(\Omega))} \leq C_{reg}\|\pi^k\|_{L^2(Q)}, \tag{3.69}$$

$$\boldsymbol{\varphi}^k \to 0 \quad \text{weakly in } L^{r'}(0,T;W^{2,r'}(\Omega)). \tag{3.70}$$

Let $O(k^{-1})$ signify a quantity satisfying $\limsup_{k\to\infty} O(k^{-1}) \leq 0$. For quantities A^k, B^k we write $A^k \overset{k}{\lesssim} B^k$ if $A^k \leq B^k + O(k^{-1})$. With this notation[6] we develop

$$\|\pi^k\|^2_{L^2(Q)} = (\pi^k, \Delta\boldsymbol{\varphi}^k)_Q \overset{k}{\lesssim} (p_2^k, \Delta\boldsymbol{\varphi}^k)_Q \overset{k}{\lesssim} (\boldsymbol{S}^k, \nabla^2\boldsymbol{\varphi}^k)_Q$$

[6] We exploit it analogously also for other limit quantities, so for instance $O(N^{-1})$ or, later on, $O(\varepsilon)$.

by (3.51) and the weak convergence (3.70). Since

$$(\boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \nabla^2 \boldsymbol{\varphi}^k)_Q \overset{k}{\sim} 0$$

by (3.57) and (3.70) (note $r' \geq 2$), we may write

$$\begin{aligned}
&\|\pi^k\|_{L^2(Q)}^2 \overset{k}{\sim} (\boldsymbol{S}^k, \nabla^2 \boldsymbol{\varphi}^k)_Q \overset{k}{\sim} (\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \nabla^2 \boldsymbol{\varphi}^k)_Q \\
&\leq \gamma_0 \int_Q |\pi^k||\nabla^2 \boldsymbol{\varphi}^k| \, dx\, dt + C_2 \int_Q \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}\boldsymbol{u}^k||\nabla^2 \boldsymbol{\varphi}^k| \, ds\, dx\, dt, \quad (3.71)
\end{aligned}$$

by (3.10) with $\overline{\boldsymbol{D}}(s) = \boldsymbol{D}\boldsymbol{v} + s(\boldsymbol{D}\boldsymbol{v}^k - \boldsymbol{D}\boldsymbol{v})$. Denote

$$I^k = \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}\boldsymbol{u}^k|^2 ds. \quad (3.72)$$

As $(1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} \leq (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/4}$, Hölder's inequality and bound (3.69) applied to (3.71) yield

$$\|\pi^k\|_{L^2(Q)}^2 \overset{k}{\sim} \gamma_0 C_{reg} \|\pi^k\|_{L^2(Q)}^2 + C_2 C_{reg} \Big(\int_{G^{k,n}} I^k \, dx\, dt \Big)^{1/2} \|\pi^k\|_{L^2(Q)} + O(N^{-1}),$$

where we got rid of the bad set $E^{k,n}$ (see its definition (3.64)) by means of the bound on its measure (3.66), boundedness stemming from (3.39) and (3.53) and $r < 2$ as follows:

$$\begin{aligned}
&C_2 \int_{E^{k,n}} \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}\boldsymbol{u}^k||\nabla^2 \boldsymbol{\varphi}^k| \, ds\, dx\, dt \\
&\qquad \leq C \|\boldsymbol{D}\boldsymbol{u}^k\|_{L^r(Q)} \|\pi^k\|_{L^2(Q)} |E^{k,n}|^{\frac{2-r}{2r}} \overset{k}{\sim} O(N^{-1}). \quad (3.73)
\end{aligned}$$

Consequently

$$\|\pi^k\|_{L^2(Q)}^2 \overset{k}{\sim} \left(\frac{C_2 C_{reg}}{1 - \gamma_0 C_{reg}} \right)^2 \int_{G^{k,n}} I^k \, dx\, dt + O(N^{-1}). \quad (3.74)$$

The integral on the right can be estimated by means of (3.9):

$$\begin{aligned}
\int_{G^{k,n}} I^k \, dx\, dt &\leq \frac{\gamma_0^2}{C_1^2} \|\pi^k\|_{L^2(Q)}^2 + \frac{2}{C_1} \big(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k \big)_{G^{k,n}} \\
&\overset{k}{\sim} \frac{\gamma_0^2}{C_1^2} \|\pi^k\|_{L^2(Q)}^2 + O(N^{-1}), \quad (3.75)
\end{aligned}$$

provided

$$I_1 = (\boldsymbol{S}^k, \boldsymbol{D}\boldsymbol{u}^k)_{G^{k,n}} \overset{k}{\sim} O(N^{-1}), \quad (3.76)$$

$$I_2 = -(\boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k)_{G^{k,n}} \overset{k}{\sim} O(N^{-1}). \quad (3.77)$$

The limit inequalities (3.74) and (3.75) would then yield

$$\|\pi^k\|_{L^2(Q)}^2 \overset{k}{\sim} \left(\frac{\gamma_0 C_2 C_{reg}}{C_1 (1 - \gamma_0 C_{reg})} \right)^2 \|\pi^k\|_{L^2(Q)}^2 + O(N^{-1}),$$

implying the desired convergence (3.67), as long as

$$\frac{\gamma_0 C_2 C_{reg}}{C_1(1-\gamma_0 C_{reg})} < 1,$$

which does hold, however, due to Assumption 3.2.2, namely

$$\gamma_0 < \frac{C_1}{C_{reg}(C_1 + C_2)}.$$

We must therefore justify (3.76) and (3.77). Recall that in $G^{k,n}$ we have $\boldsymbol{u}^k = \boldsymbol{u}^{k,n}$; see the definition (3.65). Also note that by (3.23), (3.25) and (3.27), for any fixed N (hence also for $n = n(N)$ and $\lambda^* = \lambda^*(N)$), we may assume

$$\nabla \boldsymbol{u}^{k,n} \to 0 \quad \text{weakly in } L^r(\widehat{Q}_N) \text{ as } k \to \infty. \tag{3.78}$$

We rewrite I_1 as

$$I_1 = (\boldsymbol{S}^k, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_{G^{k,n}} = (\boldsymbol{S}^k, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_Q - (\boldsymbol{S}^k, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_{\{\tau^N>0\}\setminus G^{k,n}}. \tag{3.79}$$

Since $\nabla \tau^N = 0$ a.e. in $G^{k,n}$ and

$$\{\tau^N > 0\} \setminus G^{k,n} = \big(\{\tau^N > 0\} \cap B^{k,n}\big) \cup \big(\{0 < \tau^N < 1\} \setminus B^{k,n}\big),$$

we recast (3.79) as

$$\begin{aligned} I_1 = (\boldsymbol{S}^k, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_Q &- (\boldsymbol{S}^k, \boldsymbol{u}^{k,n} \otimes \nabla \tau^N)_{\{\tau^N>0\}\setminus G^{k,n}} \\ &- (\boldsymbol{S}^k, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} - (\boldsymbol{S}^k, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{0<\tau^N<1\}\setminus B^{k,n}}. \end{aligned}$$

According to the strong convergence (3.25), it holds that

$$\lim_{k\to\infty} (\boldsymbol{S}^k, \boldsymbol{u}^{k,n} \otimes \nabla \tau^N)_{\{\tau^N>0\}\setminus G^{k,n}} = 0. \tag{3.80}$$

Additionally, by the Lipschitz bound (3.27), the weak convergence of $\boldsymbol{S}^k$ from (3.48) and then by (3.63),

$$\big|(\boldsymbol{S}^k, \tau^n \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{0<\tau^n<1\}\setminus B^{k,n}}\big| \le C(\widehat{Q}_N)\lambda^{k,n}\big|\{0 < \tau^N < 1\}\big|^{1/r} \|\boldsymbol{S}^k\|_{L^{r'}(Q)} \le \frac{C}{N}. \tag{3.81}$$

As a result of (3.80) and (3.81),

$$\begin{aligned} I_1 &\overset{k}{\sim} (\boldsymbol{S}^k, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_Q - (\boldsymbol{S}^k, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} + O(N^{-1}) \\ &\overset{k}{\sim} (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_Q - (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} + O(N^{-1}) \end{aligned}$$

by (3.25) and (3.78) in the first term and (3.28) and (3.59) in the second one. Now we recall the weak formulations (3.31) and (3.49) and notice that $\tau^N \boldsymbol{u}^{k,n}$ is a legal test function in either of them (which mere $\boldsymbol{u}^k$ fails to meet). Substituting the term $(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_Q$ accordingly, we obtain

$$I_1 \overset{k}{\sim} J_1 + J_2 + J_3 + O(N^{-1}),$$

where

$$J_1 = -\int_0^T \langle \partial_t \boldsymbol{u}^k, \tau^N \boldsymbol{u}^{k,n} \rangle \, dt \overset{k}{\sim} C(\widehat{Q}_N)(r(\lambda^*)^{1-r} + n^{-1})^\beta \leq \frac{1}{N}$$

by (3.29) and (3.60). Next,

$$J_2 = (\boldsymbol{G}^k, \operatorname{div}(\tau^N \boldsymbol{u}^{k,n}))_Q \overset{k}{\sim} 0$$

by (3.22), (3.23) and (3.27). Finally,

$$\begin{aligned} J_3 &= (\pi^k, \operatorname{div}(\tau^N \boldsymbol{u}^{k,n}))_Q - (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} \\ &\overset{k}{\sim} (\pi^k, \tau^N \operatorname{div} \boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} - (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} \\ &= (\boldsymbol{H}^k - \boldsymbol{H}, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} \leq \frac{1}{N}. \end{aligned}$$

by dint of (3.25), (3.28) and (3.59) since evidently

$$\{\tau^N > 0\} \cap B^{k,n} \subset \widehat{Q}_N.$$

Thus (3.76) has been shown.

As far as I_2 in (3.77) is concerned, we recall that $\boldsymbol{u}^k = \boldsymbol{u}^{k,n}$ in $G^{k,n}$ and notice

$$G^{k,n} = \{\tau^N \equiv 1\} \setminus \big(\{\tau^N \equiv 1\} \cap B^{k,n}\big).$$

Since $\{\tau^N \equiv 1\} \subset \widehat{Q}_N$, we recall the strong convergence (3.57) and the weak convergence (3.78) to deduce

$$\begin{aligned} I_2 &\overset{k}{\sim} -(\boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n})_{G^{k,n}} = (\boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N\equiv 1\}\cap B^{k,n}} - (\boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N\equiv 1\}} \\ &\overset{k}{\sim} (\boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N\equiv 1\}\cap B^{k,n}} \overset{k}{\sim} C(\widehat{Q}_N)(r(\lambda^*)^{1-r} + n^{-\beta}) \leq \frac{1}{N}, \end{aligned}$$

by (3.28) and (3.59), thus showing (3.77) and ultimately proving also (3.67) for $r < 2$.

Convergence of $\boldsymbol{D}\boldsymbol{u}^k$ Recalling the definition (3.72), we can infer by Hölder's inequality that

$$\begin{aligned} \|\boldsymbol{D}\boldsymbol{u}^k\|_{L^r(S)}^r &\leq \int_S \Big(\int_0^1 (1+|\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}\boldsymbol{u}^k|^2 (1+|\boldsymbol{D}\boldsymbol{v}^k|^2+|\boldsymbol{D}\boldsymbol{v}|^2)^{(2-r)/2} \, ds \Big)^{r/2} dx\, dt \\ &\leq \Big(\int_S I^k \Big)^{r/2} \Big(\int_Q (1+|\boldsymbol{D}\boldsymbol{v}^k|^2+|\boldsymbol{D}\boldsymbol{v}|^2)^{r/2} \Big)^{(2-r)/2}, \end{aligned}$$

for any measurable $S \subset Q$, implying with help of the uniform estimate (3.33) in the end

$$C\|\boldsymbol{D}\boldsymbol{u}^k\|_{L^r(S)}^2 \leq \int_S I^k \quad \text{for any measurable } S \subset Q. \tag{3.82}$$

Applying Biting lemma 3.4.4 to

$$f^k(t,x) = |\boldsymbol{D}\boldsymbol{u}^k(t,x)|^r, \quad (t,x) \in Q,$$

there is a nonincreasing sequence of measurable sets $D^m \subset Q$ with $\lim_{m\to\infty} |D^m| = 0$, such that (without loss of generality) f^k converge weakly in $L^1(Q \setminus D^m)$ for every m. Our aim is to prove

$$\|\boldsymbol{D}\boldsymbol{u}^k\|_{L^r(Q\setminus D^m)} \overset{k}{\sim} 0 \tag{3.83}$$

for any $m \in \mathbb{N}$. Since $\lim_{m\to\infty} |D^m| = 0$, the pointwise convergence (for a subsequence) follows.

Let $D \in \{D^m\}$. Since f^k converge weakly in $L^1(Q \setminus D)$, they are uniformly equi-integrable in $Q \setminus D$. Let us take an arbitrary $\sigma > 0$ and $N > \sigma^{-1}$, such that

$$S \subset Q \setminus D,\ |S| < \frac{2}{N} \Rightarrow \|f^k\|_{L^1(S)} < \sigma \quad \text{for every } k. \tag{3.84}$$

We use this N as the starting parameter for the Lipschitz approximation scheme started at the beginning of Subsection 3.5.2. We may assume $|E^{k,n}| < 2N^{-1}$ for all k by (3.66), which combined with (3.82) and (3.84) yields

$$C\|\boldsymbol{D}\boldsymbol{u}^k\|^2_{L^r(Q\setminus D)} \le \int_{G^{k,n}\setminus D} I^k + C\|\boldsymbol{D}\boldsymbol{u}^k\|^2_{L^r(E^{k,n}\setminus D)} \le \int_{G^{k,n}} I^k + O(\sigma), \tag{3.85}$$

given that $I^k \ge 0$. Relations (3.75) and (3.67) then imply

$$\int_{G^{k,n}} I^k \, dx \, dt \overset{k}{\sim} \frac{\gamma_0^2}{C_1^2}\|\pi^k\|^2_{L^2(Q)} + O(N^{-1}) \overset{k}{\sim} O(N^{-1}) = O(\sigma).$$

If we plug this observation back into (3.85), we obtain the desired (3.83). Together with the compactness of the partial pressures (3.56) and (3.67), we may assume both p^k and $\boldsymbol{D}\boldsymbol{v}^k$ converge pointwise a.e. in Q, which yields ultimately $\overline{\boldsymbol{S}} = \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$ for $r < 2$ by Vitali's theorem.

3.5.3 Convergence for $r = 2$

The above procedure, followed step by step, is rendered useless when $r = 2$ for we cannot get rid of the polluting term in (3.73). On the other hand, the strong convergence in $L^2(Q)$ is not essential for the pointwise convergence of a subsequence. Now we show only the strong convergence in $L^1(Q)$, arriving at the same conclusion. Although we could have skipped the case $r < 2$ entirely, given that the method applied to $r = 2$, resting on Lemma 3.4.5, may be presented in such a way that it conquers also the former case, we treat this situation apart for two reasons: Firstly, it is much more convenient to use Lemma 3.4.6 when applicable (see [15] for usage of Lemma 3.4.5 for a wider range of exponents). Secondly, dealing with the case $r = 2$ individually lets us balance out its slightly increased technicality with simplification of certain terms; consider e.g. I^k in (3.72).

Several definitions We set

$$g^k = \mathcal{M}^*\big(|\nabla \boldsymbol{u}^k|\big) + \mathcal{M}^*\big(|\boldsymbol{S}^k - \overline{\boldsymbol{S}}|\big) + \mathcal{M}^*\big(|\pi^k|\big). \tag{3.86}$$

By the properties of $\mathcal{M}^*$ and boundedness of the individual arguments in $L^2(Q)$ (see (3.39), (3.48) and (3.53)), the sequence $\{g^k\}$ is also bounded in $L^2(Q)$. Therefore[7]

$$\sum_{i=0}^{n} \int_{\left\{2^{2^{n+i}} < g^k \le 2^{2^{n+i+1}}\right\}} \big(g^k\big)^2 \, dx \, dt \le C \quad \text{for any } n \in \mathbb{N},$$

[7]Notice that $2^{2^{n+i+1}} = \big(2^{2^{n+i}}\big)^2$, which is the reason for our choice of such numbers.

independently of k and n, which guarantees there are

$$2^{2^n} \leq \lambda^{k,n} \leq 2^{2^{2n}} \tag{3.87}$$

such that

$$\int_{\left\{\lambda^{k,n} < g^k \leq (\lambda^{k,n})^2\right\}} \left(g^k\right)^2 dx\, dt \leq \frac{C}{n} \quad \text{for any } k, n \in \mathbb{N}. \tag{3.88}$$

Let us define level sets related to g^k:

$$\begin{aligned} A_1^{k,n} &= \left\{g^k \leq \lambda^{k,n}\right\}, \\ A_2^{k,n} &= \left\{\lambda^{k,n} < g^k \leq \left(\lambda^{k,n}\right)^2\right\}, \\ A_3^{k,n} &= \left\{\left(\lambda^{k,n}\right)^2 < g^k\right\}. \end{aligned} \tag{3.89}$$

By (3.88), we can bound the measure of $A_2^{k,n}$ as

$$\left|A_2^{k,n}\right| = \int_{\left\{\lambda^{k,n} < g^k \leq (\lambda^{k,n})^2\right\}} 1\, dx\, dt \leq \int_{\left\{\lambda^{k,n} < g^k \leq (\lambda^{k,n})^2\right\}} \frac{\left(g^k\right)^2}{\left(\lambda^{k,n}\right)^2} dx\, dt \leq \frac{C}{n\left(\lambda^{k,n}\right)^2}. \tag{3.90}$$

Chebyshev's inequality also implies

$$(\lambda^{k,n})^4 \left|A_3^{k,n}\right| \leq C. \tag{3.91}$$

Furthermore, we define

$$F^k = \{\mathcal{M}^*(|\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v} \otimes \boldsymbol{v} + (p_1^k - p_1)\boldsymbol{I}|) > 1\}.$$

By means of the strong-type estimate for $\mathcal{M}^*$ and (3.35), (3.42) and (3.56), we obtain

$$\lim_{k\to\infty} \left|F^k\right| \leq C \lim_{k\to\infty} \left\|\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v} \otimes \boldsymbol{v} + (p_1^k - p_1)\boldsymbol{I}\right\|_{L^\sigma(Q)}^\sigma = 0 \quad \text{for some } \sigma > 1. \tag{3.92}$$

For fixed $n \in \mathbb{N}$ we also find $\tau^n \in \mathcal{C}_c^\infty(Q; [0,1])$ such that

$$\left|\{\tau^n < 1\}\right| \leq \frac{1}{2^{2^{2n+1}} n}. \tag{3.93}$$

Finally we include all the adverse sets into one so that we define

$$E^{k,n} = \left(A_2^{k,n} \cup A_3^{k,n} \cup F^k \cup \{\tau^n < 1\}\right) \cap Q, \tag{3.94}$$

$$G^{k,n} = Q \setminus E^{k,n}. \tag{3.95}$$

It follows easily from the definition of $E^{k,n}$, (3.87), (3.92) and (3.93) that

$$\left(\lambda^{k,n}\right)^2 \left|E^{k,n} \cap A_1^{k,n}\right| \overset{k}{\sim} O(n^{-1}). \tag{3.96}$$

We would like to engage Lemma 3.4.5 with $E^{k,n}$ playing the role of E. Setting

$$\boldsymbol{H}^k = \boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{S}^k + \overline{\boldsymbol{S}} + (p^k - p)\boldsymbol{I},$$

Eq. (3.17) evidently holds with $\boldsymbol{u}^k$ and $\boldsymbol{H}^k$. The sets $E^{k,n}$ are open due to the lower semicontinuity of $\mathcal{M}^*$. Finally, subadditivity of $\mathcal{M}^*$ yields

$$\begin{aligned}\{g^k > \lambda^{k,n}\} \cup F^k &\supset \{\mathcal{M}^*(|\nabla \boldsymbol{u}^k|) > \lambda^{k,n}\} \cup \{\mathcal{M}^*(|\boldsymbol{S}^k - \overline{\boldsymbol{S}} - \pi^k \boldsymbol{I}|) > \lambda^{k,n}\} \cup F^k \\ &\supset \{\mathcal{M}^*(|\nabla \boldsymbol{u}^k|) > \lambda^{k,n}\} \cup \{\mathcal{M}^*(|\boldsymbol{H}^k|) > \lambda^{k,n} + 1\} \\ &\supset \{\mathcal{M}^*(|\nabla \boldsymbol{u}^k|) + \mathcal{M}^*(|\boldsymbol{H}^k|) > 3\lambda^{k,n}\},\end{aligned}$$

implying the required property

$$\{\mathcal{M}^*(|\nabla \boldsymbol{u}^k|) + \mathcal{M}^*(|\boldsymbol{H}^k|) > 3\lambda^{k,n}\} \cap Q \subset E^{k,n}.$$

Therefore we may invoke Lemma 3.4.5 with $\Lambda = 3\lambda^{k,n}$. Let us denote

$$\boldsymbol{u}^{k,n} = \mathcal{L}_{E^{k,n}} \boldsymbol{u}^k.$$

Note that due to the L^p-estimate (3.18) and the strong convergence (3.42) it holds that

$$\boldsymbol{u}^{k,n} \to 0 \quad \text{strongly in } L^2(Q) \text{ as } k \to \infty \text{ for any } n \in \mathbb{N}. \tag{3.97}$$

Accessory calculation In this part we show a result that will be useful in a while, namely

$$\big(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} \overset{k}{\sim} O(n^{-1}). \tag{3.98}$$

The individual steps to be taken will be

$$\begin{aligned}\big(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} &\overset{k}{\sim} \big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} + O(n^{-1}) && (3.99)\\ &\overset{k}{\sim} \big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}(\tau^n \boldsymbol{u}^{k,n})\big)_Q + O(n^{-1}) && (3.100)\\ &\overset{k}{\sim} O(n^{-1}). && (3.101)\end{aligned}$$

As for the first relation (3.99), in view of the strong convergence (3.57), it boils down to showing

$$\big(\overline{\boldsymbol{S}} - \boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} \overset{k}{\sim} O(n^{-1}). \tag{3.102}$$

By Lemma 3.4.5 we have $\boldsymbol{u}^k = \boldsymbol{u}^{k,n}$ in $G^{k,n}$, which set we rewrite by (3.94) as

$$G^{k,n} = \{\tau^n = 1\} \setminus \big((F^k \cap A_1^{k,n} \cap \{\tau^n = 1\}) \cup ((A_2^{k,n} \cup A_3^{k,n}) \cap \{\tau^n = 1\})\big). \tag{3.103}$$

The Lipschitz bound (3.19) combined with the strong convergence (3.97) allows us to assume that for any $n \in \mathbb{N}$,

$$\nabla \boldsymbol{u}^{k,n} \to 0 \quad \text{weakly in } L^2(\{\tau^n = 1\}) \text{ as } k \to \infty. \tag{3.104}$$

By (3.19) again and the shrinkage of F^k expressed in (3.92), it follows that

$$\big|\big(\overline{\boldsymbol{S}} - \boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n}\big)_{F^k \cap A_1^{k,n} \cap \{\tau^n = 1\}}\big| \le C|F^k|^{1/2}\big(\lambda^{k,n} + C_{\{\tau^n=1\}}\|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} 0. \tag{3.105}$$

Lastly, we easily deduce by (3.19), bounds on $|A_2^{k,n} \cup A_3^{k,n}|$ given in (3.90) and (3.91) that

$$\begin{aligned}\left|\left(\overline{\boldsymbol{S}} - \boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n}\right)_{(A_2^{k,n} \cup A_3^{k,n}) \cap \{\tau^n = 1\}}\right| &\le \|\overline{\boldsymbol{S}} - \boldsymbol{S}\|_{L^2(A_2^{k,n} \cup A_3^{k,n})} \|\nabla \boldsymbol{u}^{k,n}\|_{L^\infty(\{\tau^n=1\})} \\ &\quad \times |A_2^{k,n} \cup A_3^{k,n}|^{1/2} \\ &\overset{k}{\sim} O(n^{-1}).\end{aligned} \tag{3.106}$$

Combining (3.103)–(3.106), we obtain (3.102) and hence also the first step of (3.99).

Towards showing the second step (3.100), we start noticing that

$$\left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\right)_{E^{k,n} \cap \{\tau^n = 1\}} \overset{k}{\sim} O(n^{-1}). \tag{3.107}$$

Indeed, treating the level sets (3.89) individually, we estimate

$$\left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\right)_{E^{k,n} \cap A_1^{k,n} \cap \{\tau^n = 1\}} = \left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\right)_{F^k \cap A_1^{k,n} \cap \{\tau^n = 1\}} \overset{k}{\sim} 0$$

as in (3.105) due to boundedness of $\boldsymbol{S}^k$ in $L^2(Q)$ (see (3.48)). Then

$$\left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\right)_{A_2^{k,n} \cap \{\tau^n = 1\}} \le C n^{-1/2} (\lambda^{k,n})^{-1} \left(\lambda^{k,n} + C_{\{\tau^n = 1\}} \|\boldsymbol{u}^k\|_{L^1(Q)}\right) \overset{k}{\sim} O(n^{-1})$$

by the bounds (3.19) and (3.90). Very similarly, using the bounds (3.19) and (3.91),

$$\left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\right)_{A_3^{k,n} \cap \{\tau^n = 1\}} \le C (\lambda^{k,n})^{-2} \left(\lambda^{k,n} + C_{\{\tau^n = 1\}} \|\boldsymbol{u}^k\|_{L^1(Q)}\right) \overset{k}{\sim} O(n^{-1}).$$

Hence (3.107) holds and therefore also

$$\left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^k\right)_{G^{k,n}} = \left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\right)_{G^{k,n}} \overset{k}{\sim} \left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\right)_{\{\tau^n = 1\}} + O(n^{-1}). \tag{3.108}$$

Next, we would like to add another negligible term, namely the Lipschitz bound (3.19) and properties (3.87) and (3.93) imply

$$\left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \tau^n \boldsymbol{D}\boldsymbol{u}^{k,n}\right)_{\{0 < \tau^n < 1\}} \overset{k}{\sim} C \left|\{\tau^n < 1\}\right|^{1/2} \left(\lambda^{k,n} + C_{\operatorname{spt} \tau^n} \|\boldsymbol{u}^k\|_{L^1(Q)}\right) \overset{k}{\sim} O(n^{-1}).$$

As a result, we may improve (3.108) into

$$\begin{aligned}\left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^k\right)_{G^{k,n}} &\overset{k}{\sim} \left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \tau^n \boldsymbol{D}\boldsymbol{u}^{k,n}\right)_Q + O(n^{-1}) \\ &\overset{k}{\sim} \left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}(\tau^n \boldsymbol{u}^{k,n})\right)_Q + O(n^{-1}),\end{aligned}$$

recalling also the strong convergence of the Lipschitz approximations (3.97). The last inequality justifies the second step (3.100) and we may jubilate, for $\tau^n \boldsymbol{u}^{k,n}$ is a legal test function in both the weak formulations (3.31) and (3.49). We exploit this fact to rewrite

$$\begin{aligned}\left(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}(\tau^n \boldsymbol{u}^{k,n})\right)_Q &= \left(p^k - p, \operatorname{div}(\tau^n \boldsymbol{u}^{k,n})\right)_Q \\ &\quad + \left(\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v} \otimes \boldsymbol{v}, \nabla(\tau^n \boldsymbol{u}^{k,n})\right)_Q \\ &\quad - \int_0^T \langle \partial_t \boldsymbol{u}^k, \tau^n \boldsymbol{u}^{k,n} \rangle \, dt = I_1 + I_2 + I_3.\end{aligned} \tag{3.109}$$

We will demonstrate $I_i \overset{k}{\sim} O(n^{-1})$ for each $i = 1, 2, 3$. Beginning with I_1, the strong convergence (3.56) and the bound (3.19) yield

$$\begin{aligned} I_1 &\overset{k}{\sim} \big(p_2^k - p_2, \operatorname{div}(\tau^n \boldsymbol{u}^{k,n})\big)_Q = \big(\pi^k, \boldsymbol{u}^{k,n} \cdot \nabla \tau^n\big)_Q + \big(\pi^k, \tau^n \operatorname{div} \boldsymbol{u}^{k,n}\big)_Q \\ &\overset{k}{\sim} \big(\pi^k, \tau^n \operatorname{div} \boldsymbol{u}^{k,n}\big)_Q = \big(\pi^k, \tau^n \operatorname{div} \boldsymbol{u}^{k,n}\big)_{E^{k,n}}. \end{aligned} \tag{3.110}$$

We could ignore the term $(\pi^k, \boldsymbol{u}^{k,n} \cdot \nabla \tau^n)_Q$ due to the strong convergence (3.97) and boundedness coming from (3.53). Classical properties of Sobolev functions also guarantee $\operatorname{div} \boldsymbol{u}^{k,n} = \operatorname{div} \boldsymbol{u}^k = 0$ a.e. in $G^{k,n}$, which we exploited in the last equality. The rest follows the track of (3.107). More precisely,

$$\big(\pi^k, \tau^n \operatorname{div} \boldsymbol{u}^{k,n}\big)_{E^{k,n} \cap A_1^{k,n}} \le C \big|E^{k,n} \cap A_1^{k,n}\big|^{1/2} \big(\lambda^{k,n} + C_{\operatorname{spt} \tau^n} \|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} O(n^{-1})$$

by the observation (3.96). Then

$$\big(\pi^k, \tau^n \operatorname{div} \boldsymbol{u}^{k,n}\big)_{E^{k,n} \cap A_2^{k,n}} \le C n^{-1/2} (\lambda^{k,n})^{-1} \big(\lambda^{k,n} + C_{\operatorname{spt} \tau^n} \|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} O(n^{-1})$$

by estimates stemming from (3.19), (3.53) and (3.90). And similarly, only switching to (3.91) in order to bound $|A_3^{k,n}|$,

$$\big(\pi^k, \tau^n \operatorname{div} \boldsymbol{u}^{k,n}\big)_{E^{k,n} \cap A_3^{k,n}} \le C (\lambda^{k,n})^{-2} \big(\lambda^{k,n} + C_{\operatorname{spt} \tau^n} \|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} O(n^{-1}).$$

Thus we have shown that (3.110) can be concluded as

$$I_1 \overset{k}{\sim} O(n^{-1}). \tag{3.111}$$

The term I_2 is quite effortless to tackle. Due to the strong convergences (3.42) and (3.97), we have

$$\begin{aligned} I_2 &= \big(\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v} \otimes \boldsymbol{v}, \nabla(\tau^n \boldsymbol{u}^{k,n})\big)_Q \overset{k}{\sim} \big(\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v} \otimes \boldsymbol{v}, \tau^n \nabla \boldsymbol{u}^{k,n}\big)_Q \\ &\le C \|\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v} \otimes \boldsymbol{v}\|_{L^1(Q)} \big(\lambda^{k,n} + C_{\operatorname{spt} \tau^n} \|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} 0. \end{aligned} \tag{3.112}$$

To process the last term I_3, corresponding to the time derivative, we recall the *integration by parts* formula (3.21), according to which we can rewrite I_3 as

$$\begin{aligned} I_3 &= \frac{1}{2} \int_Q \big(2\boldsymbol{u} \cdot \boldsymbol{u}^{k,n} - |\boldsymbol{u}^{k,n}|^2\big) \partial_t \tau^n \, dx \, dt + \int_{E^{k,n}} (\partial_t \boldsymbol{u}^{k,n}) \cdot (\boldsymbol{u} - \boldsymbol{u}^{k,n}) \tau^n \, dx \, dt \\ &\overset{k}{\sim} \int_{E^{k,n}} (\partial_t \boldsymbol{u}^{k,n}) \cdot (\boldsymbol{u} - \boldsymbol{u}^{k,n}) \tau^n \, dx \, dt \le C \big|E^{k,n}\big| \big(\lambda^{k,n} + C_{\operatorname{spt} \tau^n} \|\boldsymbol{u}^k\|_{L^1(Q)}\big)^2, \end{aligned} \tag{3.113}$$

first by the strong convergence (3.97) and then by the estimate (3.20). However, the sets $E^{k,n}$ by their very definition (3.94) satisfy trivially

$$\big|E^{k,n}\big| \le \big|A_2^{k,n}\big| + \big|A_3^{k,n}\big| + \big|F^k\big| + \big|\{\tau^n < 1\}\big|.$$

Estimates for the individual summands are contained in (3.90)–(3.93) and we plug them into (3.113) to infer

$$I_3 \overset{k}{\sim} C \big|E^{k,n}\big| \big(\lambda^{k,n} + C_{\operatorname{spt} \tau^n} \|\boldsymbol{u}^k\|_{L^1(Q)}\big)^2 \overset{k}{\sim} O(n^{-1}).$$

We insert this last result into (3.109) together with (3.111) and (3.112), procuring the third and final relation (3.101). The longed-for (3.98) has been hereby justified.

Pressure test function For $K > 0$ we consider the usual truncation operator $T_K : \mathbb{R} \to \mathbb{R}$

$$T_K(x) = \begin{cases} x & \text{for } |x| \le K, \\ K \operatorname{sgn} x & \text{for } |x| > K. \end{cases}$$

In contrast to the case $r < 2$ (cf. (3.68)), now we take

$$\boldsymbol{\varphi}^{k,n} = \mathcal{N}\big(T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega\big).$$

For all $p < \infty$ we may assume due to the convergence (3.53) and the boundedness of $\lambda^{k,n}$ (3.87) that

$$T_{\lambda^{k,n}}\pi^k \to \overline{T}^n \quad \text{weakly in } L^p(Q) \text{ as } k \to \infty, \tag{3.114}$$
$$\overline{T}^n \to \overline{T} \quad \text{weakly in } L^2(Q) \text{ as } n \to \infty. \tag{3.115}$$

By the weak convergences (3.53) and (3.114) evidently

$$T_{\lambda^{k,n}}\pi^k - \pi^k \to \overline{T}^n \quad \text{weakly in } L^2(Q) \text{ as } k \to \infty.$$

Due to (3.53) and the bound (3.87), we may estimate

$$\int_Q |T_{\lambda^{k,n}}\pi^k - \pi^k| \le 2\int_{\{|\pi^k|>\lambda^{k,n}\}} |\pi^k| \le 2\int_{\{|\pi^k|>\lambda^{k,n}\}} \frac{|\pi^k|^2}{\lambda^{k,n}} = O(n^{-1}),$$

specifying the weak convergence (3.115) more closely as

$$\overline{T}^n \to 0 \quad \text{weakly in } L^2(Q) \text{ as } n \to \infty.$$

By the same token (up to a subsequence)

$$T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega \to \overline{T}_0^n \quad \text{weakly in } L^p(Q) \text{ as } k \to \infty,$$
$$\overline{T}_0^n \to 0 \quad \text{weakly in } L^2(Q) \text{ as } n \to \infty. \tag{3.116}$$

Back to $\boldsymbol{\varphi}^{k,n}$, the property (3.15) entails for any $1 < p < \infty$ that

$$\|\boldsymbol{\varphi}^{k,n}\|_{L^p(0,T;W^{2,p}(\Omega))} \le C_{reg,p}\|T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega\|_{L^p(Q)} \tag{3.117}$$
$$\le C_{reg,p}\,\lambda^{k,n}. \tag{3.118}$$

As a result, and also owing to (3.116), we may assume that for all $p < \infty$

$$\boldsymbol{\varphi}^{k,n} \to \overline{\boldsymbol{\varphi}}^n \quad \text{weakly in } L^p(0,T;W^{2,p}(\Omega)) \text{ as } k \to \infty, \tag{3.119}$$
$$\overline{\boldsymbol{\varphi}}^n \to 0 \quad \text{weakly in } L^2(0,T;W^{2,2}(\Omega)) \text{ as } n \to \infty. \tag{3.120}$$

Convergence of p_2^k Let $n \in \mathbb{N}$. We are going to show

$$(\pi^k, T_{\lambda^{k,n}}\pi^k)_Q \overset{k}{\sim} O(n^{-1}), \tag{3.121}$$

implying $\pi^k \to 0$ strongly in $L^1(Q)$, hence $\pi^k \to 0$ a.e. in Q for a subsequence. We write

$$(\pi^k, T_{\lambda^{k,n}}\pi^k)_Q = (\pi^k, T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega)_Q = (\pi^k, \Delta\boldsymbol{\varphi}^{k,n})_Q \overset{k}{\sim} (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \nabla^2\boldsymbol{\varphi}^{k,n})_Q$$

by the weak formulation for p_2^k (3.51), strong convergence (3.43) and weak convergences (3.53) and (3.119). We carry on by means of the strong convergence (3.57):

$$\begin{aligned}(\pi^k, T_{\lambda^{k,n}}\pi^k)_Q &\overset{k}{\sim} (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \nabla^2\boldsymbol{\varphi}^{k,n})_Q \\ &\overset{k}{\sim} (\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{Dv}), \nabla^2\boldsymbol{\varphi}^{k,n})_Q - (\boldsymbol{S} - \overline{\boldsymbol{S}}, \nabla^2\overline{\boldsymbol{\varphi}}^n)_Q \\ &\leq \gamma_0 \int_Q |\pi^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt + C_2\int_Q |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt + O(n^{-1}),\end{aligned} \tag{3.122}$$

by (3.10) and (3.120). We will concentrate on the second integral, decomposing Q into four subdomains (see (3.89), (3.94) and (3.95) for definitions):

$$Q = (E^{k,n} \cap A_1^{k,n}) \cup (E^{k,n} \cap A_2^{k,n}) \cup (E^{k,n} \cap A_3^{k,n}) \cup G^{k,n}.$$

Accordingly

$$\int_Q |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}I_1 &= \int_{E^{k,n}\cap A_1^{k,n}} |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt \leq \|\boldsymbol{Du}^k\|_{L^\infty(A_1^{k,n})}\|\nabla^2\boldsymbol{\varphi}^{k,n}\|_{L^2(Q)}|E^{k,n}\cap A_1^{k,n}|^{1/2} \\ &\leq C\lambda^{k,n}|E^{k,n}\cap A_1^{k,n}|^{1/2} \overset{k}{\sim} O(n^{-1}),\end{aligned}$$

by the observation (3.96), $|\boldsymbol{Du}^k| \leq \lambda^{k,n}$ a.e. in $A_1^{k,n}$ and the estimate of $\boldsymbol{\varphi}^{k,n}$ (3.117). Next

$$\begin{aligned}I_2 &= \int_{E^{k,n}\cap A_2^{k,n}} |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt \leq \|\boldsymbol{Du}^k\|_{L^2(A_2^{k,n})}\|\nabla^2\boldsymbol{\varphi}^{k,n}\|_{L^2(Q)} \leq C\|g^k\|_{L^2(A_2^{k,n})} \\ &= O(n^{-1}),\end{aligned}$$

by the key property of $A_2^{k,n}$ (3.88) and the estimate (3.117), and

$$\begin{aligned}I_3 &= \int_{E^{k,n}\cap A_3^{k,n}} |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt \leq \|\boldsymbol{Du}^k\|_{L^2(Q)}\|\nabla^2\boldsymbol{\varphi}^{k,n}\|_{L^p(Q)}|A_3^{k,n}|^{\frac{p-2}{2p}} \\ &\leq C(\lambda^{k,n})^{\frac{4-p}{p}} \overset{k}{\sim} O(n^{-1})\end{aligned}$$

for any $p > 4$ by the bound on $|A_3^{k,n}|$ (3.91) and (3.118) for a fixed $p > 4$. Finally,

$$\begin{aligned}I_4 &= \int_{G^{k,n}} |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt \\ &\leq \Big(\frac{\gamma_0^2}{C_1^2}\|\pi^k\|^2_{L^2(A_1^{k,n})} + \frac{2}{C_1}\big(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{Dv}), \boldsymbol{Du}^k\big)_{G^{k,n}}\Big)^{1/2}\|\nabla^2\boldsymbol{\varphi}^{k,n}\|_{L^2(Q)} \\ &\overset{k}{\sim} \frac{\gamma_0}{C_1}\|\pi^k\|_{L^2(A_1^{k,n})}\|\nabla^2\boldsymbol{\varphi}^{k,n}\|_{L^2(Q)} + O(n^{-1}),\end{aligned}$$

by (3.9), $G^{k,n} \subset A_1^{k,n}$, the accessory calculation (3.98) and the estimate (3.117). We see that only the last term I_4 adds a palpable contribution to (3.122), which hence simplifies into

$$
\begin{aligned}
(\pi^k, T_{\lambda^{k,n}}\pi^k)_Q &\overset{k}{\sim} \gamma_0 \left(1 + \frac{C_2}{C_1}\right) \|\pi^k\|_{L^2(A_1^{k,n})} \|\nabla^2 \boldsymbol{\varphi}^{k,n}\|_{L^2(Q)} + O(n^{-1}) \\
&\leq \gamma_0 C_{reg} \left(\frac{C_1 + C_2}{C_1}\right) \|\pi^k\|_{L^2(A_1^{k,n})} \|T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega\|_{L^2(Q)} + O(n^{-1}) \\
&\leq \gamma_0 C_{reg} \left(\frac{C_1 + C_2}{C_1}\right) \|\pi^k\|_{L^2(A_1^{k,n})} \|T_{\lambda^{k,n}}\pi^k\|_{L^2(Q)} + O(n^{-1}),
\end{aligned}
\tag{3.123}
$$

by (3.117) and an elementary manipulation

$$
\|T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega\|^2_{L^2(Q)} = \|T_{\lambda^{k,n}}\pi^k\|^2_{L^2(Q)} - |\Omega|(T_{\lambda^{k,n}}\pi^k)^2_\Omega \leq \|T_{\lambda^{k,n}}\pi^k\|^2_{L^2(Q)}.
$$

What remains is to relate $(\pi^k, T_{\lambda^{k,n}}\pi^k)_Q$ to the right-hand side in a better way: Recalling the definition of g^k (3.86), we have trivially

$$
|T_{\lambda^{k,n}}\pi^k| \leq |\pi^k| \leq g^k \quad \text{a.e. in } Q.
$$

Therefore, and by the estimates (3.88) and (3.91), we observe

$$
\|T_{\lambda^{k,n}}\pi^k\|^2_{L^2(Q)} \leq \|\pi^k\|^2_{L^2(A_1^{k,n})} + \|g^k\|^2_{L^2(A_2^{k,n})} + \|\lambda^{k,n}\|^2_{L^2(A_3^{k,n})} \leq \|\pi^k\|^2_{L^2(A_1^{k,n})} + O(n^{-1}).
$$

Then we add an obvious inequality

$$
\|\pi^k\|^2_{L^2(A_1^{k,n})} \leq (\pi^k, T_{\lambda^{k,n}}\pi^k)_Q
$$

and (3.123) combined with $0 < \gamma_0 < \dfrac{C_1}{C_{reg}(C_1 + C_2)}$ from Assumption 3.2.2 becomes the desired (3.121) and we may hence assume (bearing in mind the already proved result for p_1^k (3.56))

$$
p^k \to p \quad \text{a.e. in } Q. \tag{3.124}
$$

Convergence of $\boldsymbol{Du}^k$ This time the Biting lemma will be engaged on

$$
f^k(t, x) = |\pi^k(t, x)|^2 + |\boldsymbol{Du}^k(t, x)|^2, \quad (t, x) \in Q,
$$

with our sight set on

$$
\|\boldsymbol{Du}^k\|_{L^r(Q \setminus D^m)} \overset{k}{\sim} 0
$$

for any $m \in \mathbb{N}$, where D^m are the sets provided by the Biting lemma, like in (3.83). Assuming without loss of generality that f^k are themselves weakly convergent in $L^1(Q \setminus D_m)$, in particular they are equi-integrable in $Q \setminus D^m$, for any $m \in \mathbb{N}$, Vitali's theorem and the pointwise convergence (3.124) imply

$$
\pi^k \to 0 \quad \text{strongly in } L^2(Q \setminus D^m) \text{ for every } m \in \mathbb{N}. \tag{3.125}
$$

Let $m_0 \in \mathbb{N}$ be fixed. Equi-integrability of f^k and the definition of $E^{k,n}$ (3.94) imply

$$
\|\boldsymbol{Du}^k\|^2_{L^2(Q \setminus D^{m_0})} \overset{k}{\sim} \|\boldsymbol{Du}^k\|^2_{L^2(G^{k,n} \setminus D^{m_0})} + O(n^{-1}) \leq \|\boldsymbol{Du}^k\|^2_{L^2(G^{k,n} \setminus D^m)} + O(n^{-1})
$$

for any $m \geq m_0$. Take $m(n) \geq m_0$ fulfilling

$$\left|D^{m(n)}\right| \leq \frac{1}{2^{2^{2n+1}} n}. \tag{3.126}$$

Applying the estimate (3.9) and convergence (3.125), we obtain

$$\begin{aligned} C\|\boldsymbol{D}\boldsymbol{u}^k\|^2_{L^2(G^{k,n}\setminus D^{m(n)})} &\overset{k}{\sim} (\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k)_{G^{k,n}\setminus D^{m(n)}} \\ &\overset{k}{\sim} -(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^{k,n})_{G^{k,n}\cap D^{m(n)}} + O(n^{-1}), \end{aligned}$$

where we recalled the accessory calculation (3.98), i.e.

$$\left(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k\right)_{G^{k,n}} \overset{k}{\sim} O(n^{-1}),$$

for the second relation. The rest is assured by the Lipschitz bound (3.19) and (3.126):

$$\begin{aligned} (\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^{k,n})_{G^{k,n}\cap D^{m(n)}} &\leq C|D^{m(n)}|^{1/2}\left(\lambda^{k,n} + C_{\{\tau^n=1\}}\|\boldsymbol{u}^k\|_{L^1(Q)}\right) \\ &\overset{k}{\sim} O(n^{-1}), \end{aligned}$$

yielding

$$\|\boldsymbol{D}\boldsymbol{u}^k\|^2_{L^2(Q\setminus D^{m_0})} \overset{k}{\sim} O(n^{-1}),$$

hence also the pointwise convergence (for a subsequence) of $\boldsymbol{D}\boldsymbol{u}^k$. Together with the compactness of the pressure (3.124) we obtain also $\overline{\boldsymbol{S}} = \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$ for $r = 2$.

3.5.4 Initial condition

Proceeding exactly like in the Galerkin approximation (see Appendix), we could justify

$$(\boldsymbol{v}_0 - \boldsymbol{v}(0), \boldsymbol{w}) = 0 \quad \text{for all } \boldsymbol{w} \in W^{1,q'}_{\boldsymbol{n}}(\Omega),$$

i.e. $\boldsymbol{v}(0) = \boldsymbol{v}_0$. Next we will show

$$\boldsymbol{v}^k(t) \to \boldsymbol{v}(t) \quad \text{weakly in } L^2(\Omega) \text{ for all } t \in (0, T). \tag{3.127}$$

Let $t \in (0, T)$, then $\{\boldsymbol{v}^k(t)\}_k$ is bounded in $L^2(\Omega)$ and we may assume that for a subsequence

$$\boldsymbol{v}^{k_m}(t) \to \overline{\boldsymbol{v}} \quad \text{weakly in } L^2(\Omega).$$

Recall (3.31) and take $\boldsymbol{\varphi} = \boldsymbol{w}\chi_{(0,t)}$ for an arbitrary $\boldsymbol{w} \in W^{1,q'}_{\boldsymbol{n}}(\Omega)$. Then

$$\begin{aligned} (\boldsymbol{v}^{k_m}(t), \boldsymbol{w}) - (\boldsymbol{v}_0, \boldsymbol{w}) &= (\boldsymbol{v}^{k_m} \otimes \boldsymbol{v}^{k_m}\Phi_{k_m}(|\boldsymbol{v}^{k_m}|), \nabla\boldsymbol{w})_{Q_t} - (\boldsymbol{S}^{k_m}, \boldsymbol{D}\boldsymbol{w})_{Q_t} \\ &\quad - \alpha(\boldsymbol{v}^{k_m}\Phi_{k_m}(|\boldsymbol{v}^{k_m}|), \boldsymbol{w})_{\Gamma_t} + (p^{k_m}, \operatorname{div}\boldsymbol{w})_{Q_t} + (\boldsymbol{F}, \nabla\boldsymbol{w})_{Q_t}, \end{aligned}$$

which tends for $m \to \infty$ to

$$\begin{aligned} (\overline{\boldsymbol{v}}, \boldsymbol{w}) - (\boldsymbol{v}_0, \boldsymbol{w}) &= (\boldsymbol{v} \otimes \boldsymbol{v}, \nabla\boldsymbol{w})_{Q_t} - (\boldsymbol{S}, \boldsymbol{D}\boldsymbol{w})_{Q_t} - \alpha(\boldsymbol{v}, \boldsymbol{w})_{\Gamma_t} + (p, \operatorname{div}\boldsymbol{w})_{Q_t} + (\boldsymbol{F}, \nabla\boldsymbol{w})_{Q_t} \\ &= (\boldsymbol{v}(t), \boldsymbol{w}) - (\boldsymbol{v}_0, \boldsymbol{w}), \end{aligned}$$

by the already proved weak formulation (3.7). Therefore $\overline{\boldsymbol{v}} = \boldsymbol{v}(t)$ and we may extend the result beyond a mere subsequence, in other words (3.127) holds.

Regarding the strong convergence to the initial value in $L^2(\Omega)$, in the weak formulation (3.31) we can take $\boldsymbol{\varphi} = \boldsymbol{v}^k \chi_{(0,t)}$ for any $t \in (0,T)$, obtaining

$$\begin{aligned}\|\boldsymbol{v}^k(t)\|^2_{L^2(\Omega)} - \|\boldsymbol{v}_0\|^2_{L^2(\Omega)} &= (\boldsymbol{F}, \nabla \boldsymbol{v}^k)_{Q_t} - (\boldsymbol{S}^k, \boldsymbol{D}\boldsymbol{v}^k)_{Q_t} - \alpha(\boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|), \boldsymbol{v}^k)_{\Gamma_t} \\ &\leq (\boldsymbol{F}, \nabla \boldsymbol{v}^k)_{Q_t} + Ct,\end{aligned}$$

by means of the property (3.11) and non-negativity of the boundary term.

Adding (3.127) and the lower semicontinuity of the norm then yields

$$\begin{aligned}\lim_{t\to 0_+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|^2_{L^2(\Omega)} &= \lim_{t\to 0_+} \|\boldsymbol{v}(t)\|^2_{L^2(\Omega)} - \|\boldsymbol{v}_0\|^2_{L^2(\Omega)} \\ &\leq \lim_{t\to 0_+} \liminf_{k\to\infty} \|\boldsymbol{v}^k(t)\|^2_{L^2(\Omega)} - \|\boldsymbol{v}_0\|^2_{L^2(\Omega)} \\ &\leq \lim_{t\to 0_+} \big((\boldsymbol{F}, \nabla \boldsymbol{v})_{Q_t} + Ct\big) = 0.\end{aligned}$$

With this last fragment we have established the claim of Theorem 3.3.1. □

3.6 Appendix

In this ancillary part we prove Lemma 3.5.1. Towards that aim, with fixed $\varepsilon, k > 0$, the original problem (3.5) will be further approximated by the following *quasicompressible* system:

$$\left.\begin{aligned}\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}\Phi_k(|\boldsymbol{v}|)) - \operatorname{div} \boldsymbol{S} + \nabla p &= -\operatorname{div} \boldsymbol{F} && \text{in } Q, \\ \operatorname{div} \boldsymbol{v} &= \varepsilon \Delta p && \text{in } Q, \\ \nabla p \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma, \\ \boldsymbol{v} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma, \\ \alpha \boldsymbol{v}_\tau \Phi_k(|\boldsymbol{v}_\tau|) &= -(\boldsymbol{S}\boldsymbol{n})_\tau && \text{on } \Gamma, \\ \boldsymbol{v}(0) &= \boldsymbol{v}_0 && \text{in } \Omega, \\ p_\Omega &= 0 && \text{in } (0,T).\end{aligned}\right\} \tag{3.128}$$

Like in the case of the system with only the convective term truncated, we are interested in existence of weak solutions. In the following lemma we both particularize this concept and affirm the existential question.

Lemma 3.6.1 *Under the assumptions of Theorem 3.3.1, for every $\varepsilon, k > 0$ there exists a weak solution to the approximate problem* (3.128)*, i.e. a couple $(\boldsymbol{v}^{\varepsilon,k}, p^{\varepsilon,k})$ satisfying*

$$\begin{aligned}\boldsymbol{v}^{\varepsilon,k} &\in L^r(0,T; W^{1,r}_{\boldsymbol{n}}(\Omega)), \\ \partial_t \boldsymbol{v}^{\varepsilon,k} &\in L^{r'}(0,T; W^{-1,r'}_{\boldsymbol{n}}(\Omega)), \\ p^{\varepsilon,k} &\in L^2(0,T; \mathring{W}^{1,2}(\Omega)) \cap L^{r'}(Q)\end{aligned}$$

and for all $\boldsymbol{\varphi} \in W^{1,r}_{\boldsymbol{n}}(\Omega)$ and a.e. $t \in (0,T)$, it holds that

$$\begin{aligned}\langle \partial_t \boldsymbol{v}^{\varepsilon,k}(t), \boldsymbol{\varphi}\rangle - (\boldsymbol{v}^{\varepsilon,k} \otimes \boldsymbol{v}^{\varepsilon,k}\Phi_k(|\boldsymbol{v}^{\varepsilon,k}|)(t), \nabla\boldsymbol{\varphi}) + (\boldsymbol{S}(p^{\varepsilon,k}(t), \boldsymbol{D}\boldsymbol{v}^{\varepsilon,k}(t)), \boldsymbol{D}\boldsymbol{\varphi}) \\ + \alpha(\boldsymbol{v}^{\varepsilon,k}\Phi_k(|\boldsymbol{v}^{\varepsilon,k}|)(t), \boldsymbol{\varphi})_{\partial\Omega} - (p^{\varepsilon,k}(t), \operatorname{div}\boldsymbol{\varphi}) = (\boldsymbol{F}(t), \nabla\boldsymbol{\varphi}),\end{aligned} \tag{3.129}$$

as well as for every $\psi \in W^{1,2}(\Omega)$ *and a.e.* $t \in (0,T)$ *the identity*

$$\varepsilon\,(\nabla p^{\varepsilon,k}(t), \nabla\psi) = -(\operatorname{div} \boldsymbol{v}^{\varepsilon,k}(t), \psi). \tag{3.130}$$

The initial condition is being attained in the form $\lim_{t\to 0_+} \|\boldsymbol{v}^{\varepsilon,k}(t) - \boldsymbol{v}_0\|_{L^2(\Omega)} = 0$.

Proof. Let $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}} \subset W^{1,2}_{\boldsymbol{n}}(\Omega)$ be an orthogonal basis in $W^{1,2}_{\boldsymbol{n}}(\Omega)$ and an orthonormal basis in $L^2(\Omega)$. We also standardly require of the basis that L^2-projections

$$P^n \boldsymbol{u} = \sum_{i=1}^{n} (\boldsymbol{u}, \boldsymbol{w}_i)\boldsymbol{w}_i, \qquad \boldsymbol{u} \in L^2(\Omega),\ n \in \mathbb{N},$$

be orthogonal in $W^{1,2}_{\boldsymbol{n}}(\Omega)$. Note that $P^n\boldsymbol{v}_0$ converges to $\boldsymbol{v}_0$ in $L^2(\Omega)$ for $n \to \infty$.

Galerkin approximation Dropping the ε, k-indices (both parameters stay fixed), for $n \in \mathbb{N}$ we construct Faedo-Galerkin approximations

$$\begin{aligned} \boldsymbol{v}^n(t,x) &= \sum_{i=1}^{n} c_i^n(t)\boldsymbol{w}_i(x), \\ p^n(t,x) &= \mathcal{N}\left(\frac{\operatorname{div} \boldsymbol{v}^n}{\varepsilon}\right)(t,x) = \frac{1}{\varepsilon}\sum_{i=1}^{n} c_i^n(t)\mathcal{N}(\operatorname{div} \boldsymbol{w}_i)(x). \end{aligned} \tag{3.131}$$

Recall (3.13) for the definition of $\mathcal{N}$. What is to be found are absolutely continuous functions $\{c_i^n\}_{i=1}^n$, extensible to the whole $[0,T]$ and satisfying

$$\begin{aligned} (\partial_t \boldsymbol{v}^n(t), \boldsymbol{w}_i) - (\boldsymbol{v}^n \otimes \boldsymbol{v}^n \Phi_k(|\boldsymbol{v}^n|)(t), \nabla \boldsymbol{w}_i) + (\boldsymbol{S}^n(t), \boldsymbol{D}\boldsymbol{w}_i) + \alpha(\boldsymbol{v}^n\Phi_k(|\boldsymbol{v}^n|)(t), \boldsymbol{w}_i)_{\partial\Omega} \\ - (p^n(t), \operatorname{div} \boldsymbol{w}_i) = (\boldsymbol{F}(t), \nabla \boldsymbol{w}_i) \qquad \text{for all } i = 1, \dots, n, \end{aligned} \tag{3.132}$$

where $\boldsymbol{S}^n(t) = \boldsymbol{S}(p^n(t), \boldsymbol{D}\boldsymbol{v}^n(t))$. We also set $\boldsymbol{v}^n(0) = P^n\boldsymbol{v}_0$.

The functions $\{c_i^n\}_{i=1}^n$ would be found standardly with help of the Carathéodory theory, at least for a short time interval. The extensibility onto the whole of $[0,T]$ will follow from the uniform estimates derived presently.

Uniform estimates Multiplying eq. (3.132) by $c_i^n(t)$ and summing the n equalities yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^n(t)\|^2_{L^2(\Omega)} - (\boldsymbol{v}^n \otimes \boldsymbol{v}^n\Phi_k(|\boldsymbol{v}^n|)(t), \nabla\boldsymbol{v}^n(t)) + (\boldsymbol{S}^n(t), \boldsymbol{D}\boldsymbol{v}^n(t)) \\ + \alpha\|\Phi_k^{1/2}(|\boldsymbol{v}^n|)\boldsymbol{v}^n(t)\|^2_{L^2(\partial\Omega)} - (p^n(t), \operatorname{div} \boldsymbol{v}^n(t)) = (\boldsymbol{F}(t), \nabla\boldsymbol{v}^n(t)). \end{aligned}$$

Due to eq. (3.131), boundedness of the truncated convective term and (3.11),

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^n(t)\|^2_{L^2(\Omega)} + \frac{C_1}{2r}\|\boldsymbol{D}\boldsymbol{v}^n(t)\|^r_{L^r(\Omega)} + \varepsilon\,\|\nabla p^n(t)\|^2_{L^2(\Omega)} \\ \le \left(\|\boldsymbol{F}(t)\|_{L^{r'}(\Omega)} + C(k)\right)\|\nabla\boldsymbol{v}^n(t)\|_{L^r(\Omega)} + \frac{C_1|\Omega|}{2r}. \end{aligned} \tag{3.133}$$

Hölder's inequality now implies

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|^2_{L^2(\Omega)} \le 2\big(\|\boldsymbol{F}\|_{L^{r'}(Q)} + C(k)\big)\, \|\nabla \boldsymbol{v}^n\|_{L^r(Q)} + \|\boldsymbol{v}_0\|^2_{L^2(\Omega)} + \frac{TC_1|\Omega|}{r},$$

which we apply in (3.133), getting

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|^2_{L^2(\Omega)} + \frac{C_1}{r}\, \|\boldsymbol{D}\boldsymbol{v}^n\|^r_{L^r(Q)} + 2\varepsilon\, \|\nabla p^n\|^2_{L^2(Q)} \\ \le 2\big(\|\boldsymbol{F}\|_{L^{r'}(Q)} + C(k)\big)\, \|\nabla \boldsymbol{v}^n\|_{L^r(Q)} + \|\boldsymbol{v}_0\|^2_{L^2(\Omega)} + \frac{TC_1|\Omega|}{r}.$$

Now we recall Korn's inequality (3.16) and then utilize Young's inequality to deduce

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}^n\|^r_{L^r(0,T;W^{1,r}(\Omega))} + \varepsilon\, \|\nabla p^n\|^2_{L^2(Q)} \le C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big),$$

finally implying, using (3.12) for the stress tensor $\boldsymbol{S}$ and Poincaré's inequality for the pressure,

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}^n\|^r_{L^r(0,T;W^{1,r}(\Omega))} + \|\boldsymbol{S}^n\|^{r'}_{L^{r'}(Q)} + \varepsilon\, \|p^n\|^2_{L^2(0,T;W^{1,2}(\Omega))} \\ \le C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \quad (3.134)$$

The time derivative $\partial_t \boldsymbol{v}^n$ will be momentarily estimated in $L^2(0,T;W^{-1,2}_{\boldsymbol{n}}(\Omega))$. Noting that $W^{1,2}_{\boldsymbol{n}}(\Omega)$ is densely and continuously embedded in $L^2(\Omega)$, for $\boldsymbol{\varphi} \in W^{1,2}_{\boldsymbol{n}}(\Omega)$ we may write

$$\begin{aligned} \langle \partial_t \boldsymbol{v}^n(t), \boldsymbol{\varphi}\rangle &= (\partial_t \boldsymbol{v}^n(t), P^n\boldsymbol{\varphi}) \\ &\le 4k^2 \big\|\nabla P^n\boldsymbol{\varphi}\big\|_{L^1(\Omega)} + \big\|\boldsymbol{S}^n(t)\big\|_{L^{r'}(\Omega)} \big\|\boldsymbol{D}P^n\boldsymbol{\varphi}\big\|_{L^r(\Omega)} + 2\alpha k \big\|P^n\boldsymbol{\varphi}\big\|_{L^2(\partial\Omega)} \\ &\quad + \big\|p^n(t)\big\|_{L^2(\Omega)} \big\|\nabla P^n\boldsymbol{\varphi}\big\|_{L^2(\Omega)} + \big\|\boldsymbol{F}(t)\big\|_{L^{r'}(\Omega)} \big\|\nabla P^n\boldsymbol{\varphi}\big\|_{L^r(\Omega)} \\ &\le C\, \|\nabla\boldsymbol{\varphi}\|_{L^2(\Omega)} \big(4k^2 + \|\boldsymbol{S}^n(t)\|_{L^{r'}(\Omega)} + 2\alpha k + \|p^n(t)\|_{L^2(\Omega)} + \|\boldsymbol{F}(t)\|_{L^{r'}(\Omega)}\big). \end{aligned}$$

The first inequality follows from Eq. (3.132), while the latter step made use of orthogonality of P^n on $W^{1,2}_{\boldsymbol{n}}(\Omega)$, as well as Hölder's inequality ($r \le 2$) and the trace theorem for Sobolev functions. Combining the last inequality with (3.134) yields the desired

$$\int_0^T \|\partial_t \boldsymbol{v}^n(t)\|^2_{W^{-1,2}_{\boldsymbol{n}}(\Omega)}\, dt \le C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \quad (3.135)$$

Limit $n \to \infty$ With bounds (3.134)–(3.135), we may invoke the traditional compactness arguments like reflexivity, the Banach-Alaoglu theorem, the Aubin-Lions lemma with $W^{1,r}_{\boldsymbol{n}}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,2}_{\boldsymbol{n}}(\Omega)$ and Lemma 3.4.3, to select a subsequence (labeled again $(p^n, \boldsymbol{v}^n)$) such that for $n \to \infty$

$$
\begin{aligned}
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{weakly in } L^r(0,T;W^{1,r}_{\boldsymbol{n}}(\Omega)), && (3.136)\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)), && (3.137)\\
\partial_t \boldsymbol{v}^n &\to \partial_t \boldsymbol{v} && \text{weakly in } L^2(0,T;W^{-1,2}_{\boldsymbol{n}}(\Omega)), && (3.138)\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{strongly in } L^2(Q), && (3.139)\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{strongly in } L^r(\Gamma), && (3.140)\\
\|\boldsymbol{v}^n(t)\|_2 &\to \|\boldsymbol{v}(t)\|_2 && \text{a.e. in } (0,T), && (3.141)\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{a.e. in } Q, && (3.142)\\
p^n &\to p && \text{strongly in } L^2(0,T;\mathring{W}^{1,2}(\Omega)), && (3.143)\\
p^n &\to p && \text{a.e. in } Q, && (3.144)\\
\boldsymbol{S}^n &\to \overline{\boldsymbol{S}} && \text{weakly in } L^{r'}(Q). && (3.145)
\end{aligned}
$$

We were able to deduce the strong convergence of p^n from (3.15) and (3.139).

Considering the continuity of $\mathcal{N}$ and properties of $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}}$, we apply the convergence results (3.136)–(3.145) to the equations (3.131) (3.132) to acquire

$$\varepsilon p = \mathcal{N}(\operatorname{div}\boldsymbol{v}) \tag{3.146}$$

and

$$
\begin{aligned}
\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi}\rangle\, dt = (\boldsymbol{v}\otimes\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \nabla\boldsymbol{\varphi})_Q - (\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{\varphi})_Q - \alpha(\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \boldsymbol{\varphi})_\Gamma + (p, \operatorname{div}\boldsymbol{\varphi})_Q\\
+ (\boldsymbol{F}, \nabla\boldsymbol{\varphi})_Q
\end{aligned} \tag{3.147}
$$

for every $\boldsymbol{\varphi} \in L^2(0,T;W^{1,2}_{\boldsymbol{n}}(\Omega))$.

Improved pressure integrability The bound (3.134) is insufficient to infer $p \in L^{r'}(Q)$ but we are able to deduce it all the same, even uniformly in ε. The first thing we notice is that

$$p \in L^2(0,T;L^{r'}(\Omega))$$

since $r' < 2d/(d-2)$. This observation carries over to Eq. (3.147), where it allows us to infer $\partial_t\boldsymbol{v} \in L^2(0,T;W^{-1,r'}_{\boldsymbol{n}}(\Omega))$ and we may take $\boldsymbol{\varphi} \in L^2(0,T;W^{1,r}_{\boldsymbol{n}}(\Omega))$.

For $L > 0$ denote χ_L the indicator function of the set $\{\|p(t)\|_{L^{r'}(\Omega)} < L\}$. We will consider

$$\boldsymbol{\varphi} = \chi_L \nabla\mathcal{N}\big(|p|^{r'-2}p - (|p|^{r'-2}p)_\Omega\big).$$

Notice from (3.15) that

$$
\begin{aligned}
\|\boldsymbol{\varphi}(t)\|_{W^{1,r}(\Omega)} &\le C(\Omega,r)\chi_L(t)\||p(t)|^{r'-1}\|_{L^r(\Omega)} = C(\Omega,r)\|\chi_L(t)p(t)\|^{r'-1}_{L^{r'}(\Omega)},\\
\|\boldsymbol{\varphi}\|_{L^r(0,T;W^{1,r}(\Omega))} &\le C(\Omega,r)\|\chi_L p\|^{r'-1}_{L^{r'}(Q)},\\
\|\boldsymbol{\varphi}\|_{L^\infty(0,T;W^{1,r}(\Omega))} &\le C(\Omega,r)L^{r'-1},\\
\operatorname{div}\boldsymbol{\varphi} &= \big(|p|^{r'-2}p - (|p|^{r'-2}p)_\Omega\big)\chi_L \quad \text{a.e. in } Q.
\end{aligned} \tag{3.148}
$$

In particular, we can make use of $\boldsymbol{\varphi}$ in the equation (3.147), implying

$$\|p\chi_L\|_{L^{r'}(Q)}^{r'} = (p, \operatorname{div}\boldsymbol{\varphi})_Q = \sum_{i=1}^{5} I_i, \tag{3.149}$$

where, by (3.147) and Hölder's inequality,

$$\begin{aligned}
I_1 &= -(\boldsymbol{F}, \nabla\boldsymbol{\varphi})_Q \le \|\boldsymbol{F}\|_{L^{r'}(Q)}\|\nabla\boldsymbol{\varphi}\|_{L^r(Q)} \le C\,\|\boldsymbol{\varphi}\|_{L^r(0,T;W^{1,r}(\Omega))}\,,\\
I_2 &= (\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{\varphi})_Q \le \|\overline{\boldsymbol{S}}\|_{L^{r'}(Q)}\|\nabla\boldsymbol{\varphi}\|_{L^r(Q)} \le C\,\|\boldsymbol{\varphi}\|_{L^r(0,T;W^{1,r}(\Omega))}\,,\\
I_3 &= -(\boldsymbol{v}\otimes\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \nabla\boldsymbol{\varphi})_Q \le C(k)\,\|\boldsymbol{\varphi}\|_{L^r(0,T;W^{1,r}(\Omega))}\,,\\
I_4 &= \alpha(\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \boldsymbol{\varphi})_\Gamma \le C(k)\|\boldsymbol{\varphi}\|_{L^r(\Gamma)} \le C(k)\,\|\boldsymbol{\varphi}\|_{L^r(0,T;W^{1,r}(\Omega))}\,,\\
I_5 &= \int_0^T \langle\partial_t\boldsymbol{v}, \boldsymbol{\varphi}\rangle\,dt = \int_0^T \langle\partial_t\nabla\mathcal{N}(\operatorname{div}\boldsymbol{v}), \boldsymbol{\varphi}\rangle\,dt = \varepsilon\int_0^T \langle\partial_t\nabla p, \boldsymbol{\varphi}\rangle\,dt,
\end{aligned} \tag{3.150}$$

by the Helmholtz decomposition (3.14) and the relation (3.146). If p were smooth, then

$$I_5 = -\varepsilon\int_0^T (\partial_t p, |p|^{r'-2}p)\chi_L\,dt = -\frac{\varepsilon}{r'}\,\|p(T)\|_{L^{r'}(\Omega)}^{r'}\,\chi_L(T) \le 0.$$

In the general case we could use an approximation by smooth functions to conclude $I_5 \le 0$. All in all, from (3.148), (3.149) and the estimates on I_1–I_5 we have

$$\|p\chi_L\|_{L^{r'}(Q)}^{r'} \le C(k).$$

independently of $L > 0$, which entails $L^{r'}$-integrability of the pressure

$$\|p\|_{L^{r'}(Q)} \le C(k). \tag{3.151}$$

Therefore the right-hand side of Eq. (3.147) is well-defined for any $\boldsymbol{\varphi} \in L^r(0,T;W^{1,r}_{\boldsymbol{n}}(\Omega))$ and we conclude $\partial_t\boldsymbol{v} \in L^{r'}(0,T;W^{-1,r'}_{\boldsymbol{n}}(\Omega))$.

Initial condition Attainment of the initial condition is almost trivial: Let $\zeta \in C^1_c([0,T))$, such that $\zeta(0) = -1$. Multiply Eq. (3.132) with ζ, integrate over $(0,T)$ and perform the limit $n \to \infty$. Then

$$\begin{aligned}
(\boldsymbol{v}_0, \boldsymbol{w}_i) = \lim_{n\to\infty}(\boldsymbol{v}^n(0), \boldsymbol{w}_i) &= (\boldsymbol{v}, \zeta'\boldsymbol{w}_i)_Q + (\boldsymbol{v}\otimes\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \nabla(\zeta\boldsymbol{w}_i))_Q - (\overline{\boldsymbol{S}}, \boldsymbol{D}(\zeta\boldsymbol{w}_i))_Q\\
&- \alpha(\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), (\zeta\boldsymbol{w}_i))_\Gamma + (p, \operatorname{div}(\zeta\boldsymbol{w}_i))_Q + (\boldsymbol{F}, \nabla(\zeta\boldsymbol{w}_i))_Q \qquad \text{for all } i \in \mathbb{N}.
\end{aligned} \tag{3.152}$$

If we in (3.147) take $\boldsymbol{\varphi} = \zeta\boldsymbol{w}_i$ and compare the equation with (3.152), we obtain

$$(\boldsymbol{v}_0 - \boldsymbol{v}(0), \boldsymbol{w}_i) = 0 \quad \text{for all } i \in \mathbb{N},$$

so that $\boldsymbol{v}(0) = \boldsymbol{v}_0$. Since $\boldsymbol{v} \in \mathcal{C}([0,T];L^2(\Omega))$, we are finished.

Identification of $\overline{\boldsymbol{S}}$ What remains is to show $\overline{\boldsymbol{S}} = \boldsymbol{S}$ (i.e. $\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$). Since $\boldsymbol{S}(\cdot,\cdot)$ is continuous and we already have (3.144), it suffices to verify the pointwise convergence of $\boldsymbol{D}\boldsymbol{v}^n$ a.e. in Q. Then $\overline{\boldsymbol{S}} = \boldsymbol{S}$ by Vitali's theorem.

Observe that we may without loss of generality assume in (3.141) that

$$\|\boldsymbol{v}^n(T)\|_{L^2(\Omega)} \to \|\boldsymbol{v}(T)\|_{L^2(\Omega)} \quad \text{for } n \to \infty.$$

Indeed so; if it were otherwise, we would solve our equation from the beginning on a larger time interval, say $(0, T+1)$. Then we could assume there is $T \le \tau \le T+1$ such that $\|\boldsymbol{v}^n(\tau)\|_{L^2(\Omega)} \to \|\boldsymbol{v}(\tau)\|_{L^2(\Omega)}$ for $n \to \infty$, and we would prove all convergences on $(0, \tau)$, only to restrict ourselves to $(0, T)$ in the end.

Define

$$I^n = \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}\boldsymbol{v}^n - \boldsymbol{D}\boldsymbol{v}|^2 \, ds, \qquad \overline{\boldsymbol{D}}(s) = \boldsymbol{D}\boldsymbol{v} + s(\boldsymbol{D}\boldsymbol{v}^n - \boldsymbol{D}\boldsymbol{v}).$$

With the strong convergence (3.143), the relation (3.9) implies

$$\begin{aligned} 0 \le C \limsup_{n\to\infty} \int_Q I^n &\le \limsup_{n\to\infty} (\boldsymbol{S}^n - \boldsymbol{S}, \boldsymbol{D}(\boldsymbol{v}^n - \boldsymbol{v}))_Q \\ &= \limsup_{n\to\infty} (\boldsymbol{S}^n, \boldsymbol{D}\boldsymbol{v}^n)_Q - (\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{v})_Q \le \sum_{i=1}^5 \limsup_{n\to\infty} I_i, \end{aligned} \tag{3.153}$$

where, by (3.132) and (3.147), the terms I_i are are handled by convergences (3.136)–(3.143) as follows:[8]

$$\begin{aligned} I_1 &= (\boldsymbol{F}, \nabla(\boldsymbol{v}^n - \boldsymbol{v}))_Q \overset{n}{\sim} 0 \\ I_2 &= (p^n, \operatorname{div} \boldsymbol{v}^n)_Q - (p, \operatorname{div} \boldsymbol{v})_Q = \varepsilon \|\nabla p\|^2_{L^2(Q)} - \varepsilon \|\nabla p^n\|^2_{L^2(Q)} \overset{n}{\sim} 0, \\ I_3 &= (\boldsymbol{v}^n \otimes \boldsymbol{v}^n \Phi_k(|\boldsymbol{v}^n|), \nabla \boldsymbol{v}^n)_Q - (\boldsymbol{v} \otimes \boldsymbol{v} \Phi_k(|\boldsymbol{v}|), \nabla \boldsymbol{v})_Q \overset{n}{\sim} 0, \\ I_4 &= \frac{1}{2} \int_0^T \frac{d}{dt} \big(\|\boldsymbol{v}\|^2_{L^2(\Omega)} - \|\boldsymbol{v}^n\|^2_{L^2(\Omega)} \big) \, dt \\ &= \frac{1}{2} \big(\|\boldsymbol{v}(T)\|^2_{L^2(\Omega)} - \|\boldsymbol{v}^n(T)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}^n(0)\|^2_{L^2(\Omega)} - \|\boldsymbol{v}(0)\|^2_{L^2(\Omega)} \big) \overset{n}{\sim} 0, \\ I_5 &= \alpha(\boldsymbol{v} \Phi_k(|\boldsymbol{v}|), \boldsymbol{v})_\Gamma - \alpha(\boldsymbol{v}^n \Phi_k(|\boldsymbol{v}^n|), \boldsymbol{v}^n)_\Gamma \overset{n}{\sim} 0. \end{aligned}$$

Therefore (3.153) entails

$$\lim_{n\to\infty} \int_Q I^n = 0 \tag{3.154}$$

and now we are practically finished, for Hölder's inequality yields

$$\begin{aligned} \|\boldsymbol{D}(\boldsymbol{v}^n - \boldsymbol{v})\|^r_{L^r(Q)} &\le \int_Q \Big(\int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}^n - \boldsymbol{v})|^2 \, ds \Big)^{r/2} \\ &\qquad \times (1 + |\boldsymbol{D}\boldsymbol{v}^n|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{r(2-r)/4} \, dx \, dt \\ &\le \Big(\int_Q I^n \Big)^{r/2} \Big(\int_Q (1 + |\boldsymbol{D}\boldsymbol{v}^n|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{r/2} \Big)^{(2-r)/2}, \end{aligned} \tag{3.155}$$

which tends to zero with $n \to \infty$ by (3.154). □

[8] The symbol $\overset{n}{\sim}$ has an analogical meaning to $\overset{k}{\sim}$ introduced under (3.166).

3.6.1 Vanishing artificial compressibility ($\varepsilon \to 0_+$)

Now we justify the limit $\varepsilon \to 0_+$ for solutions yielded by Lemma 3.6.1, proving thus Lemma 3.5.1. Let us again drop the index k and denote the solutions at hand simply $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$.

Uniform estimates Taking $\varphi = p^\varepsilon$ in (3.130), $\boldsymbol{\varphi} = \boldsymbol{v}^\varepsilon$ in (3.129) and summing up the resultant identities, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^\varepsilon(t)\|^2_{L^2(\Omega)} - (\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon \Phi_k(|\boldsymbol{v}^\varepsilon|)(t), \nabla \boldsymbol{v}^\varepsilon(t)) + (\boldsymbol{S}^\varepsilon(t), \boldsymbol{D}\boldsymbol{v}^\varepsilon(t)) \\ + \alpha \|\Phi_k^{1/2}(|\boldsymbol{v}^\varepsilon|)\boldsymbol{v}^\varepsilon(t)\|^2_{L^2(\partial\Omega)} + \varepsilon\|\nabla p^\varepsilon(t)\|^2_{L^2(\Omega)} = (\boldsymbol{F}(t), \nabla \boldsymbol{v}^\varepsilon(t)),$$

where $\boldsymbol{S}^\varepsilon(t) = \boldsymbol{S}(p^\varepsilon(t), \boldsymbol{D}\boldsymbol{v}^\varepsilon(t))$. Following the same steps as in the proof of Lemma 3.6.1, we could show

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^\varepsilon(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}^\varepsilon\|^r_{L^r(0,T;W^{1,r}(\Omega))} + \|\boldsymbol{S}^\varepsilon\|^{r'}_{L^{r'}(\Omega)} + \varepsilon\|p^\varepsilon\|^2_{L^2(0,T;W^{1,2}(\Omega))} \\ \le C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big), \quad (3.156)$$

which can be combined with the weak formulation for the pressure (3.130) to obtain

$$\int_0^T \|\operatorname{div} \boldsymbol{v}^\varepsilon\|^2_{W^{-1,2}_{\boldsymbol{n}}(\Omega)} \le \sqrt{\varepsilon}\, C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big).$$

As far as an ε-uniform estimate of p^ε is concerned, we still have (3.151). Combining (3.156) with (3.151) and the starting equation (3.129) also yields the last estimate

$$\|\partial_t \boldsymbol{v}^\varepsilon\|_{L^{r'}(0,T;W^{-1,r'}_{\boldsymbol{n}}(\Omega))} \le C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big).$$

Limit $\varepsilon \to 0_+$ The uniform bounds hitherto deduced allow us to pick a subsequence $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ satisfying

$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \quad \text{weakly in } L^r(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}}(\Omega)), \quad (3.157)$$

$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)), \quad (3.158)$$

$$\partial_t \boldsymbol{v}^\varepsilon \to \partial_t \boldsymbol{v} \quad \text{weakly in } L^{r'}(0,T;W^{-1,r'}_{\boldsymbol{n}}(\Omega)), \quad (3.159)$$

$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \quad \text{strongly in } L^2(Q), \quad (3.160)$$

$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \quad \text{strongly in } L^r(\Gamma), \quad (3.161)$$

$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \quad \text{a.e. in } Q, \quad (3.162)$$

$$p^\varepsilon \to p \quad \text{weakly in } L^{r'}(0,T;\mathring{L}^{r'}(\Omega)), \quad (3.163)$$

$$\boldsymbol{S}^\varepsilon \to \overline{\boldsymbol{S}} \quad \text{weakly in } L^{r'}(Q). \quad (3.164)$$

Applying (3.157)–(3.164) to eq. (3.129), we get

$$\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi}\rangle\, dt - (\boldsymbol{v}\otimes\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \nabla\boldsymbol{\varphi})_Q + (\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{\varphi})_Q + \alpha(\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \boldsymbol{\varphi})_\Gamma \\ - (p, \operatorname{div}\boldsymbol{\varphi})_Q = (\boldsymbol{F}, \nabla\boldsymbol{\varphi})_Q$$

for every $\boldsymbol{\varphi} \in L^r(0,T;W^{1,r}_{\boldsymbol{n}}(\Omega))$. As far as attainment of the initial condition is concerned, we could proceed identically like in the Galerkin approximation (notice $\boldsymbol{v}^\varepsilon(0) = \boldsymbol{v}_0$ for all $\varepsilon > 0$) and hence we skip it.

Identification of the weak limit $\overline{\boldsymbol{S}}$ is thus the only remaining issue of the ε-limit. Yearning to invoke Vitali's theorem again, we are in a slightly more problematic situation at this moment as we have lost compactness of the pressure. The equality $\overline{\boldsymbol{S}} = \boldsymbol{S}$ (i.e. $\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$) now therefore demands showing not only the pointwise convergence of $\boldsymbol{D}\boldsymbol{v}^\varepsilon$ but also of p^ε a.e. in Q.

Convergence of p^ε We will deduce

$$p^\varepsilon \to p \quad \text{strongly in } L^2(Q).$$

Define $\boldsymbol{\varphi}^\varepsilon = \mathcal{N}(p^\varepsilon - p)$ and observe that by (3.15) and (3.163)

$$\|\boldsymbol{\varphi}^\varepsilon\|_{L^2(0,T;W^{2,2}(\Omega))} \le C_{reg} \|p^\varepsilon - p\|_{L^2(Q)}\,, \tag{3.165}$$

$$\boldsymbol{\varphi}^\varepsilon \to 0 \quad \text{weakly in } L^{r'}(0,T;W^{2,r'}(\Omega)). \tag{3.166}$$

Let $O(\varepsilon)$ signify a quantity satisfying $\limsup_{\varepsilon\to 0_+} O(\varepsilon) \le 0$. For quantities A^ε, B^ε we write $A^\varepsilon \overset{\varepsilon}{\lesssim} B^\varepsilon$ if $A^\varepsilon \le B^\varepsilon + O(\varepsilon)$. Then

$$\|p^\varepsilon - p\|^2_{L^2(Q)} = (p^\varepsilon - p, \Delta\boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\lesssim} (p^\varepsilon, \Delta\boldsymbol{\varphi}^\varepsilon)_Q = (\boldsymbol{S}^\varepsilon, \nabla^2\boldsymbol{\varphi}^\varepsilon)_Q + \sum_{i=1}^5 I_i, \tag{3.167}$$

where by Eq. (3.129), convergences (3.157)–(3.162) and (3.166), the individual summands are dealt with as

$$I_1 = -(\boldsymbol{F}, \nabla^2\boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\lesssim} 0,$$

$$I_2 = \alpha(\boldsymbol{v}^\varepsilon \Phi_k(|\boldsymbol{v}^\varepsilon|), \nabla\boldsymbol{\varphi}^\varepsilon)_\Gamma \overset{\varepsilon}{\lesssim} 0,$$

$$I_3 = -(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon \Phi_k(|\boldsymbol{v}^\varepsilon|), \nabla^2\boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\lesssim} 0,$$

$$I_4 = -\int_0^T \langle \partial_t \boldsymbol{v}^\varepsilon, \nabla\mathcal{N}(p)\rangle \, dt \overset{\varepsilon}{\lesssim} -\int_0^T \langle \partial_t \boldsymbol{v}, \nabla\mathcal{N}(p)\rangle \, dt = 0,$$

$$I_5 = \int_0^T \langle \partial_t \boldsymbol{v}^\varepsilon, \nabla\mathcal{N}(p^\varepsilon)\rangle \, dt \overset{\varepsilon}{\lesssim} 0,$$

being a clone of I_5 in (3.150) with r' changed to 2. Hence the sum in (3.167) can be ignored and

$$\begin{aligned} &\|p^\varepsilon - p\|^2_{L^2(Q)} \overset{\varepsilon}{\lesssim} (\boldsymbol{S}^\varepsilon, \nabla^2\boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\lesssim} (\boldsymbol{S}^\varepsilon - \boldsymbol{S}, \nabla^2\boldsymbol{\varphi}^\varepsilon)_Q \\ &\le \gamma_0 \int_Q |p^\varepsilon - p||\nabla^2\boldsymbol{\varphi}^\varepsilon| \, dx\, dt + C_2 \int_Q \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})||\nabla^2\boldsymbol{\varphi}^\varepsilon| \, ds\, dx\, dt, \end{aligned} \tag{3.168}$$

by the property (3.10) with $\overline{\boldsymbol{D}}(s) = \boldsymbol{D}\boldsymbol{v} + s(\boldsymbol{D}\boldsymbol{v}^\varepsilon - \boldsymbol{D}\boldsymbol{v})$. Denote

$$I^\varepsilon = \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})|^2 ds.$$

Since $(1+|\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} \le (1+|\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/4}$, Hölder's inequality and bound (3.165) applied to (3.168) yield

$$\|p^\varepsilon - p\|^2_{L^2(Q)} \overset{\varepsilon}{\lesssim} \gamma_0 C_{reg} \|p^\varepsilon - p\|^2_{L^2(Q)} + C_2 C_{reg} \Big(\int_Q I^\varepsilon \, dx \, dt \Big)^{1/2} \|p^\varepsilon - p\|_{L^2(Q)}$$

entailing (note $1 - \gamma_0 C_{reg} > 0$ by Assumption 3.2.2)

$$\|p^\varepsilon - p\|^2_{L^2(Q)} \overset{\varepsilon}{\lesssim} \left(\frac{C_2 C_{reg}}{1 - \gamma_0 C_{reg}} \right)^2 \int_Q I^\varepsilon \, dx \, dt. \tag{3.169}$$

Using (3.9), we can estimate the integral on the right as

$$\begin{aligned} \int_Q I^\varepsilon \, dx \, dt &\le \frac{2}{C_1} \big(\boldsymbol{S}^\varepsilon - \boldsymbol{S}, \boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \big)_Q + \frac{\gamma_0^2}{C_1^2} \|p^\varepsilon - p\|^2_{L^2(Q)} \\ &\overset{\varepsilon}{\lesssim} \frac{2}{C_1} \big(\boldsymbol{S}^\varepsilon, \boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \big)_Q + \frac{\gamma_0^2}{C_1^2} \|p^\varepsilon - p\|^2_{L^2(Q)} \overset{\varepsilon}{\lesssim} \frac{\gamma_0^2}{C_1^2} \|p^\varepsilon - p\|^2_{L^2(Q)}, \end{aligned} \tag{3.170}$$

as long as

$$(\boldsymbol{S}^\varepsilon, \boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}))_Q \overset{\varepsilon}{\lesssim} 0. \tag{3.171}$$

Notice that (3.169) and (3.170) would then imply

$$\lim_{\varepsilon \to 0_+} \|p^\varepsilon - p\|_{L^2(Q)} = 0 \tag{3.172}$$

provided also

$$\frac{\gamma_0 C_2 C_{reg}}{C_1 (1 - \gamma_0 C_{reg})} < 1,$$

which does hold, however, due to Assumption 3.2.2, namely

$$\gamma_0 < \frac{C_1}{C_{reg}(C_1 + C_2)}.$$

We must therefore justify (3.171). Set $\boldsymbol{\varphi}^\varepsilon = \boldsymbol{v}^\varepsilon - \boldsymbol{v}$ in the weak formulation (3.129), whence

$$\big(\boldsymbol{S}^\varepsilon, \boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \big)_Q = \sum_{i=1}^5 I_i,$$

where, exploiting convergences (3.157)–(3.162),

$$\begin{aligned} I_1 &= (\boldsymbol{F}, \nabla \boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\lesssim} 0, \\ I_2 &= -\alpha (\boldsymbol{v}^\varepsilon \Phi_k(|\boldsymbol{v}^\varepsilon|), \boldsymbol{\varphi}^\varepsilon)_\Gamma \overset{\varepsilon}{\lesssim} 0, \\ I_3 &= (\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon \Phi_k(|\boldsymbol{v}^\varepsilon|), \nabla \boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\lesssim} 0, \\ I_4 &= (p^\varepsilon, \operatorname{div} \boldsymbol{\varphi}^\varepsilon)_Q = -\varepsilon (\nabla p^\varepsilon, \nabla p^\varepsilon)_Q \overset{\varepsilon}{\lesssim} 0, \\ I_5 &= -\int_0^T \langle \partial_t \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}^\varepsilon \rangle \, dt = -\frac{1}{2} \int_0^T \frac{d}{dt} \|\boldsymbol{v}^\varepsilon - \boldsymbol{v}\|^2_{L^2(\Omega)} \, dt - \int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi}^\varepsilon \rangle \, dt \overset{\varepsilon}{\lesssim} 0, \end{aligned}$$

thus proving (3.171) and justifying (3.172).

Convergence of $\boldsymbol{D}\boldsymbol{v}^\varepsilon$ The inequality (3.155) in the current situation takes form

$$\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\|^r_{L^r(Q)} \leq \Big(\int_Q I^\varepsilon\Big)^{r/2}\Big(\int_Q (1+|\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{r/2}\Big)^{(2-r)/2} \overset{\varepsilon}{\sim} 0,$$

by (3.170) and (3.172). Consequently, we may assume the pointwise convergence of both p^ε and $\boldsymbol{D}\boldsymbol{v}^\varepsilon$ a.e. in Q, which proves $\overline{\boldsymbol{S}} = \boldsymbol{S}$ and thus concludes the entire ε-limit. □

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