Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

Marek Sterzik

Pokrývání sečen konvexní oblasti
—
Covering All Lines Intersecting a Convex Domain

Katedra aplikované matematiky

Vedoucí práce: Doc. RNDr. Pavel Valtr, Dr.
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Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 4.8.2008 Marek Sterzik

Contents

Title: Covering All Lines Intersecting a Convex Domain
Author: Marek Sterzik
Department: Department of Applied Mathematics
Supervisor: Doc. RNDr. Pavel Valtr Dr.
Supervisor's e-mail address: valtr@kam.mff.cuni.cz

Abstract: For a given covnex body we try to find the shortest possible set (optionally admitting some prescribed properties) meeting all lines meeting the given body. The size of the covering set is measured by the Hausdorff 1-dimensional measure λ_1. In the first chapter there is given an introduction to the problem. In the second chapter we discuss the upper bound for the minimal covering set. In the third chapter we discuss the existence and properties of the minimal covering. In the fourth chapter we show some lower bounds for the size of a covering. In the fifth chapter we study some related topics and a generalization of the problem.
Keywords: lines, convex domain, Hausdorff measure, Steiner tree, opaque covering

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Autor: Marek Sterzik
Katedra (ústav): katedra aplikované matematiky
Vedoucí bakalářské práce: Doc. RNDr. Pavel Valtr Dr.
e-mail vedoucího: valtr@kam.mff.cuni.cz

Abstrakt: Pro danou konvexní oblast v rovině se snažíme nalézt co možná nejkratší množinu (navíc volitelně splňující předepsané vlastnosti), která protíná všechny přímky, které protínají danou oblast. Velikost pokrývacích množin měříme Hausdorffovou 1-dimenzionální mírou λ_1. V první kapitole je podán úvod do problému. Druhá kapitola se zabývá problémem horního odhadu velikosti minimální pokrývací množiny. Třetí kapitola se zabývá existencí a vlastnostmi nejmenšího pokrytí. Ve čtvrté kapitole je rozebírán problém dolního odhadu pro velikost pokrytí. V páté kapitole jsou studovány další souvislosti a zobecnění problému.
Klíčová slova: sečna, konvexní oblast, Hausdorffova míra, Steinerův strom

Chapter 1

Introduction

1.1 Motivation

A phone company needs to repair a buried cable, which was damaged when doing earthwork on a given convex area. The cable is about four feet under the ground and straight. But because of missing documentation and the construction mess, nothing more is known. Not even the place where the cable was damaged. The task is to find the cable (any point of it) with minimal effort for the diggers.

More formally, for a given convex domain $D \subseteq \mathbb{R}^2$ we ask for a set $T \subset \mathbb{R}^2$, such that T meets all lines which meets D, and T is minimal in some sence. It is possible to take only such sets T, which have a special form. For example only (finite) unions of line segments can be allowed (when the diggers are able only to dig straight trenches). In this case, the size of T, which we try to minimize is the sum of lengths of the line segment. In all cases, the size of T will be measured by the Hausdorff 1-dimensional measure, which is an natural extension of measuring lengths of arcs.

1.2 Overview of the problem

In [FMP84] Faber, Mycielski and Pedersen showed, that the minimal curve meeting all lines meeting a unit circle has lenght $\pi+2$.

The shortest curve is shown in figure 1.1.

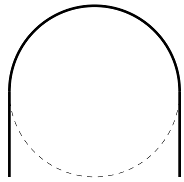


Figure 1.1: The shortest curve, which meets all lines meeting a circle

Later, in [FM86] Faber and Mycielski introduced the general problem studied in this paper. They also showed, that there exist a minimal covering of any compact set in the plane among these with at most n connected components. For this theorem (theorem 5) there is given the same proof as in [FM86], but it is explained in more details. This theorem can be easy modified to the statement, that for a compact set in the plane there exists a minimal covering among all being unions of at most n line segments. This is stated in theorem 6.

In [FM86] Faber and Mycielski also introduced the shortest known opaque coverings for a triangle or for a square. And also for other convex compact sets. They showed, that the minimal connected opaque covering of a polygon is the Steiner tree of the vertices of the polygon. They also showed that the minimal covering of a polygon may be disconnected. The following figure shows the minimal connected covering and the minimal known (not necessarily connected) covering of a square. But the ques-

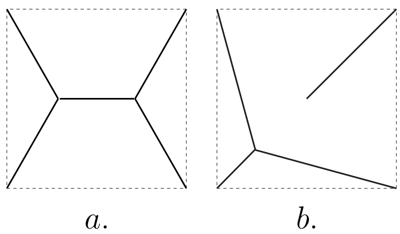


Figure 1.2: (a.) The minimal connected covering of a square. (b.) The minimal known covering of a square.

tion, if there exists a covering of a triangle, which is shorter than the shortest connected covering is still open. The answer to this question (as shown in the second section of chapter 3) will also give some results for the structure of the minimal opaque covering of an arbitrary polygon.

Techniques for the lower bound were introduced in [FMP84]. In this paper they are introduced in a different way, in terms of a measure on all lines in the plane. But both concepts are equivalent. More about measures on lines can be found in [San04].

In this paper are also introduced some new lower bounds, which idea cames from the basic lower bound. The basic lower bound cannot be used for general λ_1-measurable coverings, [1] but only for countable unions of arcs. It is possible to make the lower bound more general, allowing all λ_1-measurable coverings, but then only a weaker bound can be proven. The basic lower bound says, that every opaque covering of a convex body has lenght at least half of the perimeter of the convex body. when allowing all λ_1-measurable coverings, the constant $1/2$ has to be replaced by $1/\pi$. If the constant $1/\pi$ is the best possible is not known.

When trying to make a stronger lower bound, one can try to replace the canonical measure μ on all lines in the basic lower bound by an another measure ν. We will show, that (except some special cases) this generalization will never be better than the basic bound.

Also a generalization for all λ_1-measurable coverings can be made with an arbitrary measure ν on the lines.

Chapter 5 introduces two new views on the problem. It seems to be dificult to find the minimal covering in general. Therefore in the first section we handle, what can be said when the number of connected components is bounded. A theorem, which says that in a minimal covering with unions of line segments with two connected components the two components can be separated by a line. This also cannot be generalized for more than two components.

In the second section of chapter 5 the notion of opaque covering is generalized requiring, that every line meets the covering set,

[1] meaning all sets in the plane, which are measurable with the Hausdorff 1-dimensional measure

not only once, but at least n times. We show, that the problem is trivial for n even and is never trivial for n odd.

There are many results in this area and it is not easy to find the authors of some of them. So, at least, we want so say, what definitions or results are introduced in this paper. In chapter 3 the definition of the opaque hull $O(T)$ is introduced. In chapter 4 the generalizations of the basic lower bound (namely theorems 18, 21 and 24) are also introduced in this paper. All definitions and results from chapter 5 are introduced in this paper as well.

1.3 Formal definitions

Let $\mathcal{L}$ denote the set of all lines in the plane. If $D \subseteq \mathbb{R}^2$, then the set of all lines intersecting D let be denoted by $\mathcal{L}(D)$.

Definition 1. *A set $D \subseteq \mathbb{R}^2$ is (opaque) covered by $T \subseteq \mathbb{R}^2$ if $\mathcal{L}(D) \subseteq \mathcal{L}(T)$. In other words, every line l, such that $l \cap D \neq \emptyset$ has an non-empty intersection with T also.*

We are asking for in some sense minimal T, such that D is covered by T. The measured size of T will be in all cases the Hausdorff 1-dimensional measure λ_1 of T, which is defined as

$$\lambda_1(T) := \lim_{\delta \to 0} \left(\inf \left\{ \sum_{i=0}^{\infty} \text{diam } B_i \;\middle|\; T \subseteq \bigcup_{i=0}^{\infty} B_i, \text{ diam} B_i \leq \delta \text{ for all } i \right\} \right),$$

where all B_i are balls. This also means, that $\lambda_1(T)$ we obtain taking all coverings of T with a set of balls. Every set of balls has the size of the sum of diameters of all balls. The size of T is then obtained from the infimum over all coverings with all balls smaller than δ taking $\delta \to 0$.

The measure λ_1 is an extension of measuring lengths of arcs. If T is an arc, then $\lambda_1(T)$ is the length of T.

We do not allow every possible $T \subseteq \mathbb{R}^2$, but only these, which are from a given fixed set system $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^2)$. In all cases we require T to be λ_1-measurable. We may also require additional restrictions. For example we may require that every possible T has at most n connected components. An another important examble is to take T as a countable, finite, or bounded union of line segments. It is also possible to combine this requirements. For examble $\mathcal{S}$ can be the system of all unions of 18 line segments with at most 3 connected components. Most important for us are systems of unions (finite or infinite) of line segments.

In the whole paper we are trying to hold the notation, that D is the set, which we want to cover, T is the covering set and $\mathcal{S}$ is the system of allowed coverings.

Definition 2. *Let $\mathcal{S}$ be a system of some λ_1-measurable subsets of the plane. For a given set $D \subseteq \mathbb{R}^2$ we define the* covering number $C_{\mathcal{S}}(D)$ *as*

$$C_{\mathcal{S}}(D) := \inf \left\{ \lambda_1(T) \;\middle|\; T \in \mathcal{S} \text{ and } T \text{ covers } D \right\}$$

We may also require, that $T \subseteq D$. If this holds, then T is an *inner covering* of D. As in the definition above, we can define the *inner covering number*

$$I_{\mathcal{S}}(D) := \inf \left\{ \lambda_1(T) \;\middle|\; T \in \mathcal{S}, T \subseteq D \text{ and } T \text{ covers } D \right\}$$

.

We are mainly interested in the case D being a ,,simple" set. At least, D (or every connected component of D) should be convex. The main point of interest will be only sets D being polygons. However, other sets will be studied as well.

This definitions can be also generalized to higher dimensions. But this is not a subject of this paper. The reader can find more about this case in [Bra92].

Chapter 2

The upper bound

In this chapter we will study the upper bound for $C_{\mathcal{S}}(D)$ or $I_{\mathcal{S}}(D)$. We will show, that the perimeter of a convex body D is an upper bound for a general convex set D. Then we show, that if D is a convex polygon and $\mathcal{S}$ is the system of countable unions of line segments forming a connected set, then $I_{\mathcal{S}}(D)$ is equal to the length of the Steiner tree of the vertices of D. We also show, that the Steiner tree is only an upper bound for $C_{\mathcal{S}}(D)$ or when omiting the condition of connectedness in the definition of $\mathcal{S}$.

For a general polygon there is one upper bound. It is obvious that the boundary of a polygon opaque covers it. More, we can omit one edge of the boundary and the set remains an opaque covering. This gives the result:

Observation 3. *For $\mathcal{S}$ being the set of unions of line segments and $D \subseteq \mathbb{R}^2$ being convex it holds $C_{\mathcal{S}}(D) < P(D)$ and also $I_{\mathcal{S}}(D) < P(D)$.* [1]

Some upper bound for a given fixed polygon D can be found by finding an opaque covering of D.

If we require, that the covering sets should be connected (i.e. $\mathcal{S}$ is the set of unions of line segments with just one connected component), we can find the minimal inner covering. How to find it says the next observation.

[1] $P(D)$ denotes the perimeter of D

Observation 4. *Let D be a convex polygon and let T be a connected set such that all vertices of D are in T. Then T opaque covers D.*

Proof. For a contradiction let l be a line meeting D and not meeting T. Let l^+ and l^- be the open half-planes given by l. Let V be the set of all vertices of D. Since $V \subseteq T$, it is $l \cap V = \emptyset$. It is also $l^+ \cap V \neq \emptyset$ and $l^- \cap V \neq \emptyset$, since it would be $l \cap D = \emptyset$ othervise. Since l does not meet T, it is $T = (l^+ \cap T) \cup (l^- \cap T)$. But both, $l^+ \cap T$ and also $l^- \cap T$ are (in T) open non-empty sets. So T is not connected, which is a contradiction. □

This observation tells us, that instead of finding some opaque covering of D we can search for some set connecting all vertices of D.

For inner coverings holds, that every inner covering of D must contain all vertices of D. Therefore the following holds: T is an inner connected covering of D if and only if $T \subseteq D$ and T connects all vertices of D.

If we search for coverings being (at most) countable unions of line segments, then finding an inner covering of D means exactly finding the Steiner tree of the vertices of D. The *Steiner tree* of a finite number of vertices is a set T being a finite union of linesegments connecting this vertices and having minimal length.

The Steiner tree exists for all finite set of vertices and has finite number of line segments. Therefore it is minimal in the set of finite and also countable unions of line segments.

The problem of finding the Steiner tree is NP-hard. It is also more known about Steiner trees. The reader can find more in the book [HRW92]. The only thing important for us is, that the Steiner tree exists for every finite set of vertices and finding the Steiner tree is algorithmically solvable.

Figure 2.1 shows the Steiner tree for triangles and for the square.

Note that the Steiner tree is only equal to the minimal inner connected opaque covering. When we omit the requirement of inner coverings, there are polygons (especially regular polygons with at least 6 vertices) where there exist a connected opaque covering, which is shorter than the Steiner tree of the vertices

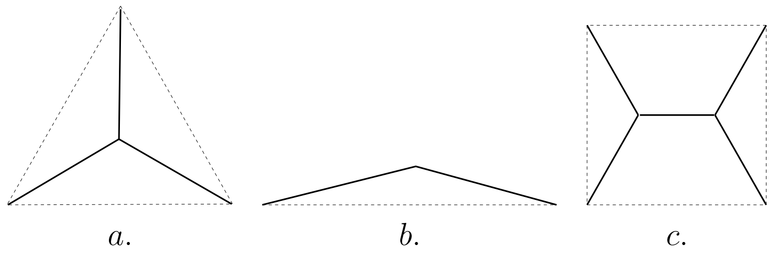


Figure 2.1: The Steiner tree for (a.) a triangle with all angles less than $2\pi/3$, (b.) a triangle with one angle at least $2\pi/3$, (c.) a square.

of the polygon. This is because the result of Du, Hwang and Chao (the reader can find it in [HRW92]). The result tells, that for $n \geq 6$ the Steiner tree of the vertices of the regular polygon with n vertices is its boundary without one edge. But obviously, there is one connected opaque covering, which is shorter than the Steiner tree. It is shown on figure 2.2.

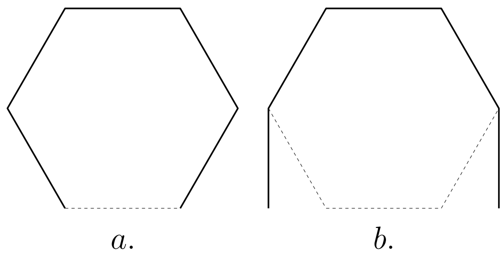


Figure 2.2: (a.) The Steiner tree of the regular hexagon. (b.) The shortest known connected opaque covering of the regular hexagon.

Also when omiting the requirement being a connected covering, the Steiner tree becomes an upper bound only. There exist coverings of convex polygons which are shorter than the size of the Steiner tree of the vertices of the polygon. In all cases, the minimal covering which does not have to be connected is not known. A covering with lower size then the Steiner tree for the square is shown on figure 2.3.

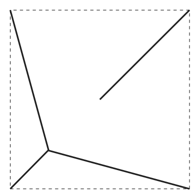


Figure 2.3: The best known opaque covering for the square

Chapter 3

The minimal covering

In this chapter we will study the properties of a minimal covering. In the first section we will study the existence of the minimal covering and in the second section we will study some cases how does a minimal covering look like (if it exists).

3.1 Existence of the minimal covering

In general, it is not clear, if there exists a minimal opaque covering of a given convex domain. It may depend on the choice of $\mathcal{S}$. In this section we will mention some theorems, which show the existence for some choices of $\mathcal{S}$.

Theorem 5 (Faber, Mycielski [FM86])**.** *Let $\mathcal{S}$ be the system of all λ_1-measurable sets in the plane with at most n connected components. Then for every compact set $D \subseteq \mathbb{R}^2$ there exists a set $T \in \mathcal{S}$, such that $\lambda_1(T) = C_{\mathcal{S}}(D)$.*

Proof. Let $\gamma := C_{\mathcal{S}}(D)$. By the presumption of the theorem there exists a sequence $\{T_i\}_{i=0}^{\infty}$ of sets from $\mathcal{S}$, such that

$$\lim_{i\to\infty} \lambda_1(T_i) = \gamma.$$

We have to find some set $T \in \mathcal{S}$, such that T covers D and $\lambda_1(T) = \gamma$. For each $i \in \mathbb{N}$ order the connected components of T_i in some way (which is not important, but will be choosen later just for comfort). Then the j-th component (in the given order) of T_i let be denoted by T_{ij}. Without lost of generality we may assume, that T_i has exactly n connected components.

By choosing a suitable subsequence of $\{T_i\}_{i=0}^{\infty}$, we can assume, that for each $0 \leq j < n$ $\text{dist}(T_{ij}, D)$ is bounded or

$$\lim_{i \to \infty} \text{dist}(T_{ij}, D) = \infty.$$

By choosing a suitable order of the connected components, we can find some $m \leq n$, such that $\text{dist}(T_{ij}, D)$ is bounded for $j < m$ and is unbounded (has an infinite limit) for $m \leq j < n$.

By another choosing an appropriate subsequence we can assume, that for every j such that $m \leq j < n$ the angles of all lines intersecting D and T_{ij} converges to some angle α_j. This means, that for every $\epsilon > 0$ there exists some i_0, such that for each $i \geq i_0$ the angle of any line intersecting D and T_{ij} is in the interval $(\alpha_j - \epsilon, \alpha_j + \epsilon)$.

Finally we can assume by choosing an appropriate subsequence, that for $j < m$ the sequence $\{T_{ij}\}_{i=0}^{\infty}$ converges to some closed set T_j^*. We assume the convergence in the Hausdorff distance of sets defined as

$$d_H(X, Y) := \max\{\sup_{x \in X} \text{dist}(x, Y), \sup_{y \in Y} \text{dist}(X, y)\}.$$

Then $\lim_{i \to \infty} \lambda_1(T_{ij}) = \lambda_1(T_j^*)$. Taking $T := T_0^* \cup T_1^* \cup ... \cup T_{m-1}^*$ we have $\lambda_1(T) \leq \gamma$.

We claim, that T covers D. For a contradiction, let l be a line, which intersects D, but does not intersect T. The angle of l must be equal to some angle α_j for $m \leq j < n$ (otherwise l intersects infinitely many sets T_{ij} for some $j < m$, but then l must intersect T_j^* too, since T_j^* is the limit of T_{ij} and closed).

Since T is compact (it is bounded and closed), the distance between T and l is non-zero. Fix some $x \in l \cap D$ and take the function $f(\alpha) := \text{dist}(T, l_{x\alpha})$ where $l_{x\alpha}$ is the line meeting x and having angle α. Then f is continuous, since T is compact. But that means if $\text{dist}(T, l)$ is non-zero, then $\text{dist}(T, l_{x\alpha})$ is non-zero for all $\alpha \in (\alpha_j - \epsilon, \alpha_j + \epsilon)$ for a suitable $\epsilon > 0$, so there infinitely many such angles α. This is a contradiction with the presumtion having only finitely many such angles. □

Note, that generalizing the theorem to countably many connected components cannot be done in an easy way. The point, where this proof fails is at the end when supposing T to be compact. T as a union of infinitely many closed sets does not have necessarily be closed.

Since the λ_1-size of $\overline{T}$ [1] is equal to the λ_1-size of T, one could take $\overline{T}$ as the covering set. $\overline{T}$ really covers D, but the number of connected components of $\overline{T}$ can be uncountable.

An example of a set T with countably many connected components such that $\overline{T}$ has uncountably many connected components can be constructed as follows. For each $i \in \mathbb{N}$ let l_i be defined inductively. Let l_0 be the closed interval $[0, 1]$. Define l_i as

$$l_i = l_{i-1} \cap \bigcup_{j=0}^{(3^i-1)/2} [\frac{2j}{3^i}, \frac{2j+1}{3^i}].$$

This corresponds to the sets converging to the Cantor's set. That means, that the intersection of all sets l_i is the Cantor's set. Now we can embed the sets l_i in the plane such that the y-coordinate of l_i will be $1/(i + 1)$. The set is shown in figure 3.1.

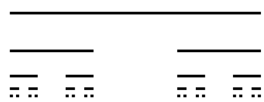


Figure 3.1: The two-dimensional Cantor's set

So T has countably many connected components. We can write $\overline{T} = T \cup \mathcal{C}$, where $\mathcal{C}$ is the standard Cantor's set with the y-coordinate 0. $\mathcal{C}$ is uncountable and totally disconnected. Moreover no point of $\mathcal{C}$ is in the same connected component as any point of T. So $\overline{T}$ has uncountably many connected components.

Theorem 6. *Let $\mathcal{S}$ be the system of all unions of at most n line segments. Then for every compact set $D \subseteq \mathbb{R}^2$ there exists a set $T \in \mathcal{S}$, tuch that $\lambda_1(T) = C_{\mathcal{S}}(D)$.*

Proof. We can proceed as in the proof of theorem 5, because every line segment is a connected component. We only need to

[1] $\overline{T}$ denotes the topological closure of T

show, that we can choose the subsequence in such way, that line segments converge to line segments in the Hausdorff distance. But this is obvious. □

It is easy to see, that theorem 5 and also theorem 6 holds for inner coverings as well.

3.2 Properties of minimal coverings

Once, the minimal covering exists, we can study how does it look like.

Definition 7. *For a given set $T \subseteq \mathbb{R}^2$ we can define the* opaque hull *of T, $O(T)$, as the maximal set D, which is opaque covered by T:*

$$O(T) := \bigcup \left\{ \ D \subseteq \mathbb{R}^2 \ \middle| \ \mathcal{L}(D) \subseteq \mathcal{L}(T) \ \right\}$$

Since the union of sets opaque covered by T is opaque covered by T as well, the definition is correct.

Here are some simple properties of $O(T)$:

1. $T \subseteq O(T)$
2. $O(O(T)) = O(T)$
3. Every connected component of $O(T)$ is convex.

Properties 1 and 2 are obvious. To see, that property 3 holds, we need to realize, that

$$O(T) = \mathbb{R}^2 \setminus \bigcup_{l \in \mathcal{L} \setminus \mathcal{L}(T)} l \quad = \bigcap_{l \in \mathcal{L} \setminus \mathcal{L}(T)} (\mathbb{R}^2 \setminus l).$$

But every connected component of $O(T)$ lies in exactly one of two open half-planes forming $\mathbb{R}^2 \setminus l$. But this is convex. So every connected component is an intersection of convex sets and therefore convex.

Observation 8. *Let $T \subseteq \mathbb{R}^2$ be a minimal opaque covering of some $D \subseteq \mathbb{R}^2$. Then every $S \subseteq T$ is a minimal opaque covering of $O(S)$.*

Proof. For a contradiction let S' be a covering of $O(S)$, such that $\lambda_1(S') < \lambda_1(S)$. Then taking $T' := (T \setminus S) \cup S'$ we have

$$\lambda_1(T') < \lambda_1(T).$$

We claim, that T' also opaque covers D, which will be a contradiction.

Let $l \in \mathcal{L}$ be a line meeting D. Then l meets also T. If l meets $T \setminus S$, we are done. So let l meets S. But then l also meets S', since S' opaque covers S. But $S' \subseteq T'$. So l meets T' and D is opaque covered by T'. □

In the remaining part of the section we will study only the case when $\mathcal{S}$ is the system of unions of line segments and D is a polygon.

Some properties of the minimal covering of any polygon depends (using observation 8) on the properties of the minimal covering of a triangle. Recall, that the minimal known opaque covering of the triangle is shown in figure 2.1.

For the triangles and their minimal coverings the following holds:

Theorem 9.
1. *If the minimal covering of a triangle with all angles less than $2\pi/3$ is its Steiner tree, then the minimal covering of any triangle with angles less or equal $2\pi/3$ is its Steiner tree.*
2. *If the minimal covering of a triangle with one angle greater or equal $2\pi/3$ is its Steiner tree, then the minimal covering of any triangle with the same biggest angle is its Steiner tree.*

Proof. It follows imediately from observation 8 and the fact, that the opaque covering is preserved by the similarity. □

The following theorem gives the relationship between the minimal opaque covering of the triangle and any polygon:

Theorem 10. *Let $\mathcal{S}$ be the system of countable unions of line segments. Let D be a convex polygon and let $T \in \mathcal{S}$ be the minimal opaque covering of D. Then the following holds:*

1. *If in T there exist 3 line segments meeting at one point,* [2] *then the minimal covering of any triangle with all angles less or equal $2\pi/3$ is its Steiner tree.*

2. *If in T there exist 2 line segments meeting at one point,* [3] *then the minimal covering of any triangle with an angle equal to the angle given by the two line segments is its Steiner tree.*

Proof. As in the preceding theorem, it follows imediately from observation 8 and the fact, that the opaque covering is preserved by the similarity. □

[2] then they have to meet in such way, that the angles of adjacent line segments is exactly $2\pi/3$

[3] then they have to meet in such way, that the angle they contain must be at least $2\pi/3$

Chapter 4

The lower bound

This chapter will show some lower bounds for $C_{\mathcal{S}}(D)$ or $I_{\mathcal{S}}(D)$. The first section gives a simple lower bound, which holds for $\mathcal{S}$ being all λ_1-measurable sets. A less general, but stronger lower bound can be done using a measure (called μ) on the set of lines $\mathcal{L}$. Using this, we will show, that $C_{\mathcal{S}}(D) \geq P(D)/2$ if $\mathcal{S}$ is the system of countable unions of line segments. We also show that $C_{\mathcal{S}}(D) \geq P(D)/\pi$ for general λ_1-measurable sets. This is the subject of the second section. The last, third, section shows, how can be the lower bound from section 2 generalized replacing the measure μ by an another (arbitrary) measure ν.

4.1 A simple lower bound

Definition 11. *Let D be a convex set in the plane. Let α be a direction; $\alpha \in [0, \pi)$. The* diameter of D in direction α *(denoted by* $\mathrm{diam}_\alpha(D)$*) is defined as*

$$\mathrm{diam}_\alpha(D) = \sup \left\{ \ |(x-y)u_\alpha| \ \middle| \ x, y \in D \ \right\},$$

where u_α is a unit vector of direction α.

In other words, $\mathrm{diam}_\alpha(D)$ is the size of the orthogonal projection of D to any line with direction α. If D is convex, then the projection is an interval.

The equation

$$\mathrm{diam}(D) = \sup_{\alpha \in [0,\pi)} (diam_\alpha(D))$$

holds for any convex set D.

Lemma 12. *Let D be a convex set in the plane. then for each $\alpha \in [0, \pi)$ there is $C_{\mathcal{S}}(D) \geq \operatorname{diam}_\alpha(D)$ for any possible system $\mathcal{S}$.*

Proof. It is sufficient to show the lemma for the system $\mathcal{S}$ of all λ_1-measurable subsets of $\mathbb{R}^2$. Let $X \in \mathcal{S}$ be a set covering D. We show, that $\lambda_1(X) \geq diam_\alpha(D)$. Let π be an orthogonal projection from the plane to a line with direction α. Then $\pi(D) \subseteq \pi(X)$ and $\pi(D)$ is an interval on the line. Let $\mathcal{B}$ be a system of balls, which cover [1] X (from the definition of $\lambda_1(X)$). Then the union of $\pi(\mathcal{B}) = \{\pi(B) | B \in \mathcal{B}\}$ contains $\pi(X)$. Since $\operatorname{diam}(B) = \operatorname{diam}(\pi(B))$ for each $B \in \mathcal{B}$, so $\lambda_1(X) \geq \lambda_1(\pi(X))$. But $\lambda_1(\pi(X)) \geq \lambda_1(\pi(D)) = \operatorname{diam}_\alpha(D)$. □

4.2 A better lower bound

For a better lower bound we need some definitions. The set of all lines $\mathcal{L}$ can be divided into sets $\mathcal{L}_\alpha$ for $\alpha \in [0, \pi)$, where $\mathcal{L}_\alpha$ is the set of all lines having direction α. Thus

$$\mathcal{L} = \bigcup_{\alpha \in [0,\pi)} \mathcal{L}_\alpha.$$

For each $\alpha \in [0, \pi)$ we can define a measure μ_α on $\mathcal{L}_\alpha$ [2] as

$$\mu_\alpha(L) = \lambda_1((\bigcup L) \cap l_{\alpha+\pi/2}),$$

where l_α is some fixed line of direction α. That means, that $l_{\alpha+\pi/2}$ is some fixed line orthogonal to the direction α. In other words, $\mu_\alpha(L)$ is obtained as the λ_1 measure of the projection of L to a line orthogonal to α. Each line projects to exactly one point in $l_{\alpha+\pi/2}$.

μ_α can be also viewed as a measure on $\mathcal{L}$ meaning that for $L \subseteq \mathcal{L}$ it is

$$\mu_\alpha(L) = \mu_\alpha(L \cap \mathcal{L}_\alpha)$$

[1] not in sence of opaque covering

[2] in fact on some subset of $\mathcal{P}(\mathcal{L}_\alpha)$ called the measurable sets on $\mathcal{L}_\alpha$, which is not important for us since all ,,interesting" sets will be measurable

Observation 13. *μ_α is a measure on $\mathcal{L}_\alpha$.*

Now we can construct a measure μ on $\mathcal{L}$ as

$$\mu(L) = \int_0^\pi \mu_\alpha(L \cap \mathcal{L}_\alpha)\ \mathrm{d}\alpha.$$

Then μ is a translation-invariant and rotation-invariant measure on $\mathcal{L}$. By a *translation-invariant* (*rotation-invariant*) measure μ on $\mathcal{L}$ we mean that for $L \subseteq \mathcal{L}$ it holds

$$\mu(L) = \mu(\tau(L)),$$

where τ is some translation (rotation) on $\mathbb{R}^2$ (and also on $\mathcal{L}$).

Lemma 14. *Let s be a line segment. Then $\mu(\mathcal{L}(s)) = 2\lambda_1(s)$.*

Proof. Let $l := \lambda_1(s)$. Since μ is rotation-invariant, it is sufficient to take a line segment parallel to the x-axis. Then $\mu_\alpha(\mathcal{L}(s) \cap \mathcal{L}_\alpha) = l \sin\alpha$. Thus

$$\mu(\mathcal{L}(s)) = \int_0^\pi l \sin\alpha\ \mathrm{d}\alpha = 2l = 2\lambda_1(s)$$

□

Lemma 15. *Let D be a convex polygon. Then $\mu(\mathcal{L}(D)) = P(D)$.*

Proof. Let $s_0, s_1, ..., s_{n-1}$ be the line segments forming the boundary of D. Let $l_0, l_1, ..., l_{n-1}$ be the lenghts of the line segments.

Then almost every line from $\mathcal{L}(D)$ [3] crosses exactly two distinct segments s_i, s_j. Therefore in the sum

$$\sum_{i=0}^{n-1} \mu(\mathcal{L}(s_i))$$

almost every line from $\mathcal{L}(D)$ is calculated two times. Therefore

$$\sum_{i=0}^{n-1} \mu(\mathcal{L}(s_i)) = 2\mu(\mathcal{L}(D)).$$

[3]that means every line except a set of measure zero

But by lemma 14 we have

$$\sum_{i=0}^{n-1} \mu(\mathcal{L}(s_i)) = \sum_{i=0}^{n-1} 2l_i = 2P(D)$$

which gives the result. □

Lemma 15 also holds for general bounded convex sets. The proof can be found in [San04].

Theorem 16. *Let $\mathcal{S}$ be the system of countable unions of line segments and let D be a convex polygon. Then $C_{\mathcal{S}}(D) \geq \frac{1}{2}P(D)$.*

Proof. Let $T \in \mathcal{S}$ be a set covering D. We show, that $\lambda_1(T) \geq \frac{1}{2}P(D)$. Since T opaque covers D, it holds $\mathcal{L}(D) \subseteq \mathcal{L}(T)$. Therefore, using lemma 15, it is $\mu(\mathcal{L}(T)) \geq P(D)$. Since $T = \bigcup_{i=0}^{\infty} s_i$, where all s_i are line segments, it is $\mathcal{L}(T) = \bigcup_{i=0}^{\infty} \mathcal{L}(s_i)$. Therefore using the basic properties of a measure (which can be found for example in [LM05]) it holds

$$\sum_{i=0}^{\infty} \mu(\mathcal{L}(s_i)) \geq \mu(\mathcal{L}(T)) \geq P(D).$$

By lemma 14 it is $\mu(\mathcal{L}(s_i)) = 2\lambda_1(s_i)$ and therefore

$$\lambda_1(T) = \sum_{i=0}^{\infty} \lambda_1(s_i) = \frac{1}{2}\sum_{i=0}^{\infty} \mu(\mathcal{L}(s_i)) \geq \frac{1}{2}P(D).$$

□

Note, that if D is not a line segment, then in the proof of theorem 16 the equality will never hold. Therefore, $\lambda_1(S) > \frac{1}{2}P(D)$. The equality would be possible only in the case, that almost all lines crosses exactly one line segment s_i. But if D is not a line segment, there exist $i \neq j$ such that $\mu(\mathcal{L}(s_i) \cap \mathcal{L}(s_j)) > 0$.

It is also possible to make a lower bound when $\mathcal{S}$ is the system of all λ_1-measurable sets. The bound is given by the following lemma:

Lemma 17. *Let $T \subseteq \mathbb{R}^2$ be a λ_1-measurable set. Then $\mu(\mathcal{L}(T)) \leq \pi\lambda_1(T)$.*

Proof. Let $\mathcal{B} = \{ B_i \mid i = 0, ..., n-1 \}$ be a system of balls, which covers T (as in the definition of $\lambda_1(T)$). Then also $\bigcup \mathcal{B}$ opaque covers T. Therefore it is

$$\mathcal{L}(T) \subseteq \bigcup_{i=0}^{n-1} \mathcal{L}(B_i).$$

Therefore

$$\mu(\mathcal{L}(T)) \leq \sum_{i=0}^{n-1} \mu(\mathcal{L}(B_i)).$$

Since it is $\mu(\mathcal{L}(B_i)) = \pi \cdot \mathrm{diam}(B_i)$, it holds

$$\mu(\mathcal{L}(T)) \leq \pi \sum_{i=0}^{n-1} \mathrm{diam}(B_i).$$

Since $\lambda_1(T)$ is at least the infimum over all coverings of balls, it holds $\mu(\mathcal{L}(T)) \leq \pi\lambda_1(T)$. □

The following theorem is an easy consequence:

Theorem 18. *Let $\mathcal{S}$ be the system of all λ_1-measurable sets in the plane. Let D be a covex polygon.* [4] *Then $C_{\mathcal{S}}(D) \geq \frac{1}{\pi} P(D)$.*

Proof. Let T be an opaque covering of D. Then it is $\mu(\mathcal{L}(T)) \geq \mu(\mathcal{L}(D))$. By lemma 17 it is $\pi\lambda_1(T) \geq \mu(\mathcal{L}(D))$. Therefore it is

$$\lambda_1(T) \geq \frac{1}{\pi} \mu(\mathcal{L}(D)) = \frac{1}{\pi} P(D).$$

The statement theorem immediately follows. □

Note, that it is unclear, if in lemma 17 the constant π is the best possible. For T being a (at most) countable union of line segments, the constant can be replaced by 2. Also for many other curves the lemma holds with constant 2. But if the lemma holds with constant 2 for all λ_1-measurable sets is not clear.

[4] As stated in the note below the proof of lemma 15, D can be also an arbitrary convex bounded set.

4.3 Generalizing the lower bound

A generalized lower bound for $C_{\mathcal{S}}(D)$ can be done proceeding the same way as in theorem 16, but using an another measure ν on $\mathcal{L}$. In the first approach we will target only on translation-invariant bounded measures.

By a *bounded measure* on $\mathcal{L}$ we mean a measure ν, such that $\nu(\mathcal{L}(D)) < \infty$ whenever $D \subseteq \mathbb{R}^2$ is a bounded set.

Two properties of translation-invariant bounded measures are given by the following two lemmas. The proof of both lemmas is not hard, but quite technical.

Lemma 19. *Let ν be a translation-invariant bounded measure on $\mathcal{L}$. Then $\nu(\mathcal{L}(\{x\})) = 0$ for each $x \in \mathbb{R}^2$.*

Proof. First we prove, that there is no line l, such that $\nu(\{l\}) > 0$. For a contradiction, let l be such a line. Any line parallel with l have a non-zero measure, since ν is translation-invariant. Therefore, if s is a line segment not parallel with l, it is $\nu(\mathcal{L}(s)) = \infty$, since $\mathcal{L}(s)$ contains infinitely many lines parallel with l. Therefore ν is not bounded, which is a contradiction.

Now, we can prove the lemma. For a contradiction, let $\nu(\mathcal{L}(\{x\})) > 0$. for some $x \in \mathbb{R}^2$. Then, since ν is translation-invariant, it is $\nu(\mathcal{L}(\{y\})) = \nu(\mathcal{L}(\{x\})) > 0$ for each $y \in \mathbb{R}^2$. Let l be a line and let s be a line segment liyng on l. Since $\nu(\{l\}) = 0$, it is $\nu(\mathcal{L}(\{x\}) \setminus \{l\}) = \nu(\mathcal{L}\{x\}) > 0$. But it is

$$\mathcal{L}(s) = \left(\bigcup_{x \in s} (\mathcal{L}(\{x\}) \setminus \{l\}) \right) \cup \{l\}.$$

So $\mathcal{L}(s)$ is a union of infinitely many disjoint sets with the same non-zero measure and therefore $\nu(\mathcal{L}(s)) = \infty$. Therefore ν is not bounded, which is a contradiction. Therefore the lemma holds. □

This lemma says, that a translation-invariant bounded measure has no singular points. By a *singular point* we mean some $x \in \mathbb{R}^2$ such that $\nu(\mathcal{L}(\{x\})) > 0$.

Lemma 20. *Let ν be a translation-invariant bounded measure on $\mathcal{L}$. Let s, t be line segments of the same direction, such that $\lambda_1(t) = 1$. Then $\nu(\mathcal{L}(s)) = \lambda_1(s) \cdot \nu(\mathcal{L}(t))$*

Proof. Let $l := \lambda_1(s)$. First, we prove the lemma for l rational. Let $l = \frac{p}{q}$. Let r be a line segment of the same direction as s and t with length p. Then we can write

$$r = \bigcup_{i=0}^{q-1} s_i = \bigcup_{i=0}^{p-1} t_i,$$

where each s_i is a copy of s and each t_i is a copy of t. In both cases, the copies are obtained by translation. For $i \neq j$ there is $|s_i \cap s_j| \leq 1$ and $|t_i \cap t_j| \leq 1$.

By lemma 19, ν has no singular point and therefore

$$\sum_{i=0}^{q-1} \nu(\mathcal{L}(s_i)) = \nu(\mathcal{L}(r)) = \sum_{i=0}^{p-1} \nu(\mathcal{L}(t_i)). \tag{4.1}$$

But ν is translation-invariant, which means, that for each possible i it is

$$\nu(\mathcal{L}(s_i)) = \nu(\mathcal{L}(s)) \text{ and } \nu(\mathcal{L}(t_i)) = \nu(\mathcal{L}(t)).$$

Therefore, using (4.1) it is

$$q \cdot \nu(\mathcal{L}(s)) = \nu(\mathcal{L}(r)) = p \cdot \nu(\mathcal{L}(t)),$$

which gives the result.

Now, let l be irrational. There exist an increasing sequence $\{l_i\}_{i=0}^{\infty}$ of rationals, converging to l. For each i we can find a line segment s_i of length l_i, such that $s_i \subseteq s$. Therefore it is $\nu(\mathcal{L}(s_i)) \leq \nu(\mathcal{L}(s))$.

Since l_i is rational, it is $\nu(\mathcal{L}(s_i)) = l_i \cdot \nu(\mathcal{L}(t))$. That means, that for each $i \in \mathbb{N}$ there is $l_i \cdot \nu(\mathcal{L}(t)) \leq \nu(\mathcal{L}(s))$ and therefore

$$l \cdot \nu(\mathcal{L}(t)) \leq \nu(\mathcal{L}(s)).$$

The opposite inequality can be done similary using a decreasing sequence of rationals and line segments containing s.

□

This lemma says, that the measure of all lines meeting a given line segment depends lineary on the length of the line segment. The linear coeficient depends only on the direction of the line segment.

Let s_α be a unit line segment of direction α. Define a function $f : [0, \pi] \to \mathbb{R}$ by the formula

$$f(\alpha) := \nu(\mathcal{L}(s_\alpha)).$$

Since ν is bounded and has no singular point, f is bounded and continuous. Therefore (and since $\mathrm{dom}(f)$ is compact) f has a maximal and minimal value. Therefore we can define

$$\begin{aligned} M_\nu &:= \max \ \left\{ \ \nu(\mathcal{L}(s_\alpha)) \ \middle| \ \alpha \in [0, \pi] \ \right\} \\ m_\nu &:= \min \ \left\{ \ \nu(\mathcal{L}(s_\alpha)) \ \middle| \ \alpha \in [0, \pi] \ \right\} \quad . \end{aligned}$$

Theorem 21. *Let $\mathcal{S}$ be the system of countable unions of line segments. Let ν be a translation-invariant bounded measure on $\mathcal{L}$. Let D be a convex polygon and let $s_0, ..., s_{n-1}$ be line segments forming the boundary of D. Then*

$$C_{\mathcal{S}}(D) \geq \frac{1}{2M_\nu} \sum_{i=0}^{n-1} \nu(\mathcal{L}(s_i))$$

Proof. Let $T = \bigcup_{i=0}^{m-1} t_i$ be a covering of D. Then $\mathcal{L}(D) \subseteq \mathcal{L}(T)$. It is

$$\nu(\mathcal{L}(D)) = \frac{1}{2} \sum_{i=0}^{n-1} \nu(\mathcal{L}(s_i)),$$

and

$$\nu(\mathcal{L}(T)) \leq \sum_{i=0}^{m-1} \nu(\mathcal{L}(t_i)) \leq M_\nu \sum_{i=0}^{m-1} \lambda_1(t_i) = M_\nu \cdot \lambda_1(T).$$

Therefore it is

$$M_\nu \cdot \lambda_1(T) \geq \frac{1}{2} \sum_{i=0}^{n-1} \nu(\mathcal{L}(s_i)),$$

which gives the result.

□

Corollary 22. *Let $\mathcal{S}$, ν and D be the same asi in theorem 21. Then*

$$C_{\mathcal{S}}(D) \geq \frac{m_\nu}{2M_\nu} P(D).$$

Proof. It follows imediately from the fact that $\nu(\mathcal{L}(s_i)) \geq m_\nu \lambda_1(s_i)$. □

Note, that theorem 16 and also lemma 12 (at least a special case where not all systems $\mathcal{S}$ are allowed) are consequences of theorem 21. Theorem 16 we get by choosing $\nu := \mu$ and lemma 12 we get by choosing $\nu := \mu_\alpha$.

Corollary 22 is not useful every time. This is since for $\nu := \mu_\alpha$ it is $m_\nu = 0$ and therefore it says only that $C_{\mathcal{S}}(D) \geq 0$, which is not a very interesting result.

Choosing $\nu := \mu$ in theorem 16 is the best possible choice. This is since

$$\sum_{i=0}^{n-1} \nu(\mathcal{L}(s_i)) \leq M_\nu \cdot \sum_{i=0}^{n-1} \lambda_1(s_i),$$

but for $\nu = \mu$ there holds an equality.

This does not mean, that all translation-invariant bounded measures except μ are completely useless. In some cases it is not possible to use the measure μ. For example when we are trying to bound the total length of line segments contained in some $D_0 \subseteq D$ (for the inner covering). Then we need, that the measure of all lines meeting D_0 and also $D \setminus D_0$ is zero since such lines can be covered by some line segment in $D \setminus D_0$. In this cases choosing μ_α for an appropriate α would be a good choice many times.

This lower bound can be used in some cases, but it has some essential limitations. For example, the covering set must be composed from line segments and the measure has to be translation-invariant. The translation-invariancy of the measure can be limiting. The following lower bound will be based on more general measures and will be usable for general λ_1-measurable coverings, but will be not as strong as the first one.

For an arbitrary measure ν on $\mathcal{L}$ we can define a parameter K_ν as

$$K_\nu := \lim_{\delta \to 0} \sup_{B \in \mathcal{B}_\delta} \frac{\nu(\mathcal{L}(B))}{\operatorname{diam}(B)},$$

where $\mathcal{B}_\delta$ is the set of all balls of diameter at most δ. We will be interested only in measures, where K_ν is finite. Such measures obviously does not have singular points.

Note, that for the measure μ it holds $K_\mu = \pi$.

Since for $\delta_1 < \delta_2$ it is $\mathcal{B}_{\delta_1} \subseteq \mathcal{B}_{\delta_2}$, K_ν is the limit of an decreasing sequence and therefore it exist every time.

The following lemma is the main part of the bound:

Lemma 23. *Let T be a λ_1-measurable set and let ν be a measure on $\mathcal{L}$ such that $K_\nu < \infty$. Then $\nu(\mathcal{L}(T)) \leq K_\nu \lambda_1(T)$.*

Proof. If $\lambda_1(T)$ is infinite, then the lemma is trivial. So let $\lambda_1(T)$ be finite.

We show, that for each $\epsilon > 0$ it holds

$$\nu(\mathcal{L}(T)) \leq (K_\nu + \epsilon)\lambda_1(T).$$

So let ϵ be fixed. From the definition of K_ν there exist δ such that

$$\sup_{B \in \mathcal{B}_\delta} \frac{\nu(\mathcal{L}(B))}{\operatorname{diam}(B)} \cdot \lambda_1(T) \leq K_\nu \lambda_1(T) + \epsilon.$$

Therefore it is sufficient to show, that for every $\delta > 0$ and every $\kappa > 0$ it holds

$$\nu(\mathcal{L}(T)) \leq \sup_{B \in \mathcal{B}_\delta} \frac{\nu(\mathcal{L}(B))}{\operatorname{diam}(B)} \cdot (\lambda_1(T) + \kappa).$$

This gives the result, since if the inequality holds for each $\kappa > 0$, it holds also for $\kappa = 0$.

Let $\mathcal{B} = \left\{ B_i \;\middle|\; i = 0, ..., n-1 \right\}$ be a system of balls, such that $\mathcal{B} \subseteq \mathcal{B}_\delta$ and $\mathcal{B}$ covers T. We can choose $\mathcal{B}$ in such way, that

$$\sum_{i=0}^{n-1} \operatorname{diam}(B_i) \leq \lambda_1(T) + \kappa.$$

The theorem also reduces to show, that the following holds:

$$\nu(\mathcal{L}(T)) \leq \sup_{B \in \mathcal{B}_\delta} \frac{\nu(\mathcal{L}(B))}{\operatorname{diam}(B)} \cdot \sum_{i=0}^{n-1} \operatorname{diam}(B_i).$$

Since $B_i \in \mathcal{B}_\delta$, it holds

$$\sup_{B \in \mathcal{B}_\delta} \frac{\nu(\mathcal{L}(B))}{\operatorname{diam}(B)} \cdot \operatorname{diam}(B_i) \geq \frac{\nu(\mathcal{L}(B_i))}{\operatorname{diam}(B_i)} \cdot \operatorname{diam}(B_i) = \nu(\mathcal{L}(B_i)).$$

Therefore it is

$$\sup_{B \in \mathcal{B}_\delta} \frac{\nu(\mathcal{L}(B))}{\operatorname{diam}(B)} \cdot \sum_{i=0}^{n-1} \operatorname{diam}(B_i) \geq \sum_{i=0}^{n-1} \nu(\mathcal{L}(B_i)) \geq$$

$$\geq \nu\left(\mathcal{L}\left(\bigcup \mathcal{B}\right)\right) \geq \nu(\mathcal{L}(T)).$$

□

From this lemma imediately follows the lower bound stated in the next theorem.

Theorem 24. *Let $\mathcal{S}$ be the system of all λ_1-measurable sets and let ν be a measure on $\mathcal{L}$, such that $K_\nu < \infty$. Let $D \subseteq \mathbb{R}^2$. Then $C_{\mathcal{S}}(D) \geq \frac{\nu(\mathcal{L}(D))}{K_\nu}$.*

Proof. If T is an opaque covering of D, then it is $\nu(\mathcal{L}(T)) \geq \nu(\mathcal{L}(D))$. By lemma 23 it is $\nu(\mathcal{L}(T)) \leq K_\nu \lambda_1(T)$. The statement of the theorem imediately follows. □

This theorem can be easy made stronger for inner coverings. For a measure ν on $\mathcal{L}$ and for a set $D \subseteq \mathbb{R}^2$ we can define

$$K_\nu^D := \lim_{\delta \to 0} \sup_{B \in \mathcal{B}_\delta^D} \frac{\nu(\mathcal{L}(B))}{\operatorname{diam}(B)},$$

where $\mathcal{B}_\delta^D$ is the set of all balls with diameter at most δ and intersecting the set D.

Then the following theorem holds:

Theorem 25. *Let $\mathcal{S}$ be the system of all λ_1-measurable sets. Let $D \subseteq \mathbb{R}^2$. Let ν be a measure on $\mathcal{L}$, such that $K_\nu^D < \infty$. Then $I_{\mathcal{S}}(D) \geq \frac{\nu(\mathcal{L}(D))}{K_\nu^D}$.*

The proof of this theorem can be easy done as a modification of the proof of theorem 24. It is kept to the reader as an easy exercise.

Chapter 5

Some related topics

In this chapter we will introduce two related problems. In the first section we will show the problem, how does look like the minimal covering when only bounded number of connected components is allowed for the opaque covering. We will show, that when there are exactly two connected components in a minimal covering of a set, then the components are separated by a line. In the second section we will introduce a generalization of the opaque-covering problem, where we require, that a line is not covered only by one point (which is the ordinary case), but at least by n points, where n is a given natural number. We will show, that the case if n is even is quite simple to solve while the case when n is odd is difficult and still open.

5.1 Covering with bounded number of connected components

The question of finding a minimal opaque covering of a given convex polygon seems to be difficult. But requiring connectivity of the covering will simplify the problem, as shown in chapter 2. At least for inner coverings. One can therefore ask, what will happen if we allow more, but a bounded nuber, of connected components.

We will not answer the question, what is the minimal covering of a convex body, but we will show a property of the minimal covering in the case of two connected components. The property is given by the following theorem.

Theorem 26. *Let $\mathcal{S}$ be the system of finite unions of line segments. Let $T \in \mathcal{S}$ be a minimal opaque covering of some bounded set $D \subseteq \mathbb{R}^2$. Let T have exactly two connected components T_0 and T_1 where none of them is a point. Then there exist a line $l \in \mathcal{L}$, such that $T_0 \subseteq \bar{l^+}$ and $T_1 \subseteq \bar{l^-}$, where $\bar{l^+}$ and $\bar{l^-}$ denote the closed half-planes given by l.*

Proof. Let H_0 be the convex hull of T_0 and let H_1 be the convex hull of T_1. Then both, H_0 and also H_1 are polygons. We show, that the interiors of H_0 and H_1 does not intersect. Then there exist a line l, which separates $\text{int}(H_0)$ and $\text{int}(H_1)$, since both sets are convex and bounded. But such a line satisfy the requirements from the statement of the theorem.

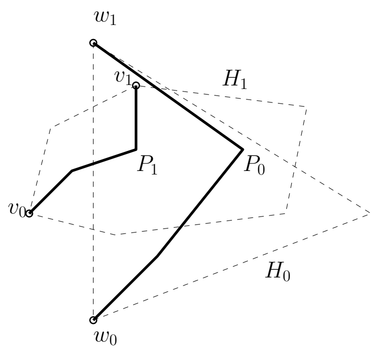
It remains to show, that $\text{int}(H_0)$ and $\text{int}(H_1)$ are disjoint. For a contradiction, let $x \in \text{int}(H_0) \cap \text{int}(H_1)$. If $H_0 = H_1$, then we are done, since T_0 opaque covers D as well, but is shorter than $T_0 \cup T_1$.

If $H_0 \neq H_1$, fix a direction and go from x in this direction until you reach an edge of one of the polygons H_0 and H_1. It is also possible to choose the direction in such a way, that we reach only one edge (without lost of generality the edge of H_0). Otherwise it has to be $H_0 = H_1$. We found an edge e of H_0, which intersect the interior of H_1.

Now, there are two possible cases: One endpoint of e lies in the interior of H_1 or $H_1 \setminus e$ has two components.

If one endpoint of e (which is a vertex of H_0) lies in the interior of H_1, there exist a line segment s of T_0 meeting this endpoint. Removing a part of s, which is short enough we does not change the opaque hull of T and therefore T still opaque covers D. But T will become shorter by this operation. Therefore T was not minimal and we are done.

If $H_1 \setminus e$ has two components, then in each of this component lies a vertex of H_1. Let this two vertices of H_1 be denoted by v_0 and v_1. The two endpoints of e let be denoted by w_0 and w_1. It holds: $\{v_0, v_1\} \subseteq T_1$, $\{w_0, w_1\} \subseteq T_0$. Since T_0 and T_1 are connected, there exist a path P_0 connecting w_0 and w_1 and there exist a path P_1 connecting v_0 and v_1. The paths cannot intersect, since otherwise T consists of only one connected component. The situation is shown on the following figure.



But P_1 has to intersect the edge e. Therefore v_1 lies in the interior of the polygon given by the path P_0 and the edge e. But this polygon is all contained in H_0. Therefore v_1 is a vertex of H_1, which lies in the interior of H_0 and we can proceed as in the first case. □

It also seems to be, that this theorem holds as well for $\mathcal{S}$ being the system of all λ_1-measurable sets, but the proof of such a theorem would be much more complicated in technical details.

Note, that generalizing this theorem for more than two connected components in an easy way is not possible. An example is given in figure 5.1. When a line separates two connected components in the ,,flower", there is one other component of the set, which does not lie in any closed half-plane given by the line. If this situation can happen for minimal coverings is unclear.

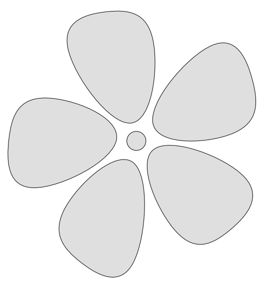


Figure 5.1: the flower

5.2 Covering every line more than once

The definiton of opaque covering can be generalized in that way, that we require, that every line is covered n times instead of covering it only once. An exact definition of this can be:

Definition 27. *A set $D \subseteq \mathbb{R}^2$ is* opaque n-covered *by $T \subseteq \mathbb{R}^2$ (for $n \in \mathbb{N}$) if for every $l \in \mathcal{L}(D)$ it is $|l \cap T| \geq n$.*

It seems to be useful to allow T to be a multiset. That means that every point can be in T more than one time and $|l \cap T|$ should be understood as the sum of all occuurrences of all points of l in T.

If we deal with multisets, we need to modify the definition of the λ_1 measure accordingly. In the definition of $\lambda_1(T)$ we took a limit of sizes of some coverings of T with balls. We only need to modify the definition of a covering with balls. A multiset T is covered with a system of balls $\mathcal{B}$ if for every $t \in T$ it is $|\left\{ B \in \mathcal{B} \mid t \in B \right\}| \geq n$ if n is the occurence of t in T. $\mathcal{B}$ may be represented as a multiset as well, but this is not necessary. [1]

The covering number and the inner covering number can be then generalized as follows:

Definition 28. *Let $\mathcal{S}$ be a system of multisets in the plane and let $D \subseteq \mathbb{R}^2$. The* n-th covering number *of D, $C^n_{\mathcal{S}}(D)$ is defined as*

$$C^n_{\mathcal{S}}(D) := \inf \left\{ \lambda_1(T) \mid T \in \mathcal{S} \text{ and } T\ n\text{−covers } D \right\}$$

The n-th inner covering number *of D, $I^n_{\mathcal{S}}(D)$ is defined as*

$$I^n_{\mathcal{S}}(D) := \inf \left\{ \lambda_1(T) \mid T \in \mathcal{S} \text{ and } T \subseteq D \text{ and } T\ n\text{−covers } D \right\}$$

It is obvious that $C^1_{\mathcal{S}}(D) = C_{\mathcal{S}}(D)$ and $I^1_{\mathcal{S}}(D) = I_{\mathcal{S}}(D)$.

Theorems from chapter 4 can be generalized to n-coverings. Mainly theorem 16:

Theorem 29. *Let $\mathcal{S}$ be the system of countable unions of line segments and let D be a convex polygon. Then $C^n_{\mathcal{S}}(D) \geq \frac{n}{2} P(D)$.*

[1] the results will be same

Proof. By lemma 15 it is $\mu(\mathcal{L}(D)) = P(D)$. We need to show, that for each multiset $T \in \mathcal{S}$ which n-covers D we have

$$\mu(\mathcal{L}^n(T)) \leq \frac{2}{n}\lambda_1(T). \tag{5.1}$$

By $\mathcal{L}^n(T)$ we mean the set of all lines $l \in \mathcal{L}$ such that $|l \cap T| \geq n$. That means the set of all by T n-covered lines.

The statement of the theorem imediately follows, since $\mathcal{L}(D) \subseteq \mathcal{L}^n(T)$. So the remaining part is to prove (5.1).

Let $\mathcal{T}$ be the multiset of line segments forming T. Thus, $\bigcup \mathcal{T} = T$.

We can divide $\mathcal{L}$ to

$$\mathcal{L} = \bigcup_{i=0}^{\infty} L_i \cup L_\infty,$$

where L_i is the set of lines intersecting T exactly n times. L_∞ is the set of all lines intersecting T infinitely many times. It is easy to see, that all sets L_i (and also L_∞) are measurable.

First we prove, that $\mu(L_\infty) = 0$. This holds by the following argument: Every line in L_∞ has with T an intersection of uncountable size or countable size. If $l \in L_\infty$ has with T an intersection of uncountable size, then there exist a line segment $s \in \mathcal{T}$, such that $s \subseteq l$. So there are only countably many such lines and therefore such lines have measure zero. If $l \in L_\infty$ has with T an intersection with countable size, then l has to intersect infinitely many line segments from $\mathcal{T}$. In other words, if we order $\mathcal{T}$ arbitrarily, $\mathcal{T} = \{s_0, s_1, ...\}$, then

$$l \in \bigcup_{i=N}^{\infty} \mathcal{L}(s_i)$$

for N arbitrarily large. But it holds:

$$\mu\left(\bigcup_{i=N}^{\infty} \mathcal{L}(s_i)\right) \leq \sum_{i=N}^{\infty} \mu(\mathcal{L}(s_i)) = \sum_{i=N}^{\infty} 2\lambda_1(s_i)$$

But

$$\lim_{N\to\infty} \sum_{i=N}^{\infty} 2\lambda_1(s_i) = 0$$

since $\lambda_1(T) < \infty$. Therefore the set of all lines having with T a countable intersecition has measure zero as well. Therefore also L_∞ has measure zero, since it is a union of two sets of measure zero.

Now, we can write

$$\mu(\mathcal{L}^n(T)) = \mu\left(\bigcup_{i=n}^{\infty} L_i\right) = \sum_{i=n}^{\infty} \mu(L_i) = \frac{1}{n}\sum_{i=n}^{\infty} n\mu(L_i) \leq$$

$$\leq \frac{1}{n}\sum_{i=0}^{\infty} i\mu(L_i) = \frac{1}{n}\sum_{s\in\mathcal{T}} \mu(\mathcal{L}(s)) = \frac{2}{n}\sum_{s\in\mathcal{T}} \lambda_1(s) = \frac{2}{n}\lambda_1(T).$$

□

For n even an observation can be made:

Observation 30. *Let $\mathcal{S}$ be the system of countable unions of line segments and let D be a convex polygon. Let n be even. Then $C^n_{\mathcal{S}}(D) = \frac{n}{2}P(D)$.*

Proof. Since theorem 29 holds, it is sufficient to show the upper bound. But this follows imediately from the fact, that taking $n/2$ times the boundary of D n-covers D. In fact we need to take every line segment forming the boundary $n/2$ times. Then the vertices of D are n-covered as well. □

Note, that observation 30 holds for $\mathcal{S}$ being the system of *finite* unions of line segments as well.

For n odd the situation becomes more complicated. We cannot take only one half of a point. [2] One upper bound can be done as taking $(n-1)/2$ times the boundary of D and one more ordinary opaque covering. As shown in chapter 4, the size of this opaque n-covering is never equal to the bound given before. Also the size of any opaque n-covering is never equal to the bound given before. This is the subject of the following theorem:

Theorem 31. *Let $\mathcal{S}$ be the system of finite unions of line segments and let D be a convex polygon which is not a line segment.*

[2] Taking one half of point could be possible in some weighted version of the problem. But then the the problem becomes trivial.

Let n be odd. Then there is no multiset $T \in \mathcal{S}$, which opaque n-covers D such that

$$\lambda_1(T) = \frac{n}{2}P(D).$$

Proof. Let T be an n-covering of D. In theorem 29 we showed that $\lambda_1(T) \geq \frac{n}{2}P(D)$. But the equalitiy is possible only when in (5.1) there holds the equality. But this is possible only in the case when in the inequality in the last part of the proof of theorem 29 there holds an equality. The remaining part is also to show, that with the conditions of this theorem it holds

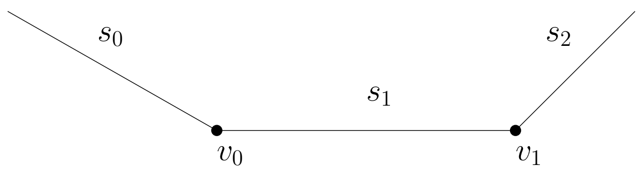
$$\sum_{i=n}^{\infty} n\mu(L_i) < \sum_{i=0}^{\infty} i\mu(L_i)$$

But this inequality holds if and only if there exist some $i \neq n$, such that $\mu(L_i) \neq 0$. Recall, that L_i is the set of all lines meeting T exactly i times. It remains to show, that there is some $i \neq n$ and a set L of nonzero measure meeting exactly i line segments forming T.

Let C denotes the convex hull of T. Then C is a polygon containing D and therefore not a line segment. Moreover, every vertex of C is adjacent to some line segment forming T.

If there is some vertex v of C, which is adjacent with more or less line segments of T than n, then we are done, since we can take L as the set of lines meeting some ϵ-neighbourhood of v but not meeting the rest of C. This is obviously a set of lines with nonzero measure.

So let all vertices of C be adjacent with exactly n line segments. Fix two adjacent vertices v_0, v_1. Let s_0, s_1, s_2 be the line segments of the boundary of C adjacent with v_0 and v_1 as shown in the following figure:



Wihtout lost of generality we can choose v_0 and v_1 in such way, that there are less than $n/2$ line segments of T adjacent with v_0 and going along the edge s_1. Therefore we can choose a line segment t adjacent with v_0 and going along the edge s_0 such that all lines meeting t and an ϵ-neighbourhood of v_1 [3] meet more than $n/2$ line segments of T adjacent with v_0.

Now, there are two possibilities. If there are more than $n/2$ line segments adjacent with v_1 and going along the edge s_1, choose t' as the shortest such line segment. Otherwise choose t' as the a (short enough) line segment adjacent with v_1 and going along the edge s_2. In both cases all lines meeting t and also t' intersect more than $n/2$ line segments of T adjacent with v_0 and also more than $n/2$ line segments of T adjacent with v_1.

That means, that all lines meeting t and t' intersect at least (but without lost of generality exactly) $n+1$ line segments of T, since n is odd. But the set of these lines has a non-zero measure. Therefore taking L as the set of lines meeting t and t' gives the result. □

[3] in fact meeting only the intersection of the ϵ-neighbourhood and C

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