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DIPLOMOVÁ PRÁCE

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Tilting Modules over Gorenstein Rings

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Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 10.4.2007 David Pospíšil

Contents

Název práce: Vychylující moduly nad Gorensteinovými okruhy
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Abstrakt: Nechť R je komutativní 1-Gorensteinův okruh. Hlavním výsledkem této práce je charakterizace všech vychylujících a kovychylujících modulů nad R, až na ekvivalenci, jsou charakterizovány podmnožinami množiny všech prvoideálů výšky jedna. Přesněji, každý vychylující (kovychylující) R-modul je ekvivalentní nějakému Bassovu vychylujícímu (kovychylujícímu) modulu. Tato charakterizace byla známa ve speciálním případě Dedekindových oborů integrity, v kapitole 4 je uveden nový a jednodušší důkaz tohoto faktu. Důkaz hlavního výsledku je proveden v kapitole 5 a kapitola 6 zahrnuje kovychylující případ. V kapitole 4 je ještě uveden důkaz nepříliš známého faktu, že konečně generované vychylující moduly nad komutativními okruhy jsou projektivní.
Klíčová slova: komutativní algebra, Gorensteinovy okruhy, vychylující moduly

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Abstract: Let R be a commutative 1-Gorenstein ring. Our main result characterizes all tilting and cotilting R-modules: up to equivalence: they are parametrized by subsets of the set of all prime ideals of height one. More precisely, every tilting (cotilting) R-module is equivalent to some Bass tilting (cotilting) module. This characterization was known in the particular case of Dedekind domains: Chapter 4 contains a new and simpler proof of this fact. Our main result is proved in Chapter 5, while Chapter 6 deals with the cotilting case. In Chapter 4, there is also a proof of the less well-known fact that all finitely generated tilting modules over commutative rings are projective.
Keywords: commutative algebra, Gorenstein rings, tilting modules

1 List of symbols

Mod-R	the class of all right R-modules
R-Mod	the class of all left R-modules
$\mathcal{P}$	the class of all modules of finite projective dimension
$\mathcal{P}_n$	the class of all modules of projective dimension $\leq n$
$\mathcal{I}$	the class of all modules of finite injective dimension
$\mathcal{I}_n$	the class of all modules of injective dimension $\leq n$
$\mathcal{F}$	the class of all modules of finite flat dimension
$\mathcal{F}_n$	the class of all modules of flat dimension $\leq n$
mod-R	the class of all modules possesing a projective resolution consisting of finitely generated modules
$\mathcal{C}^{<\kappa}$	the subclass of $\mathcal{C}$ formed by all the modules possessing a projective resolution consisting of $< \kappa$-generated projective modules
$\mathcal{C}^{<\omega}$	$= \mathcal{C} \cap \text{mod-}R$
$\mathcal{CM}$	the class of all cyclic modules
Add(T)	the class of all direct summands of arbitrary direct sums of copies of a module T
Prod(C)	the class of all direct summands of arbitrary direct products of copies of a module C
$\mathcal{C}^{\perp}$	$= \text{Ker Ext}^1_R(\mathcal{C}, -)$ $(= \{N \in \text{Mod-}R \mid \text{Ext}^1_R(C, N) = 0 \text{ for all } C \in \mathcal{C}\})$
$\mathcal{C}^{\perp_i}$	$= \text{Ker Ext}^i_R(\mathcal{C}, -)$
$\mathcal{C}^{\perp_\infty}$	$= \bigcap_{1 \leq i < \omega} \mathcal{C}^{\perp_i}$
${}^{\perp}\mathcal{C}$	$= \text{Ker Ext}^1_R(-, \mathcal{C})$ $(= \{N \in \text{Mod-}R \mid \text{Ext}^1_R(N, C) = 0 \text{ for all } C \in \mathcal{C}\})$
${}^{\perp_i}\mathcal{C}$	$= \text{Ker Ext}^i_R(-, \mathcal{C})$
${}^{\perp_\infty}\mathcal{C}$	$= \bigcap_{1 \leq i < \omega} {}^{\perp_i}\mathcal{C}$
$\mathcal{C}^{\intercal}$	$= \text{Ker Tor}^1_R(\mathcal{C}, -)$ $(= \{N \in \text{Mod-}R \mid \text{Tor}^1_R(C, N) = 0 \text{ for all } C \in \mathcal{C}\})$
$\mathcal{C}^{\intercal_i}$	$= \text{Ker Tor}^i_R(\mathcal{C}, -)$
$\mathcal{C}^{\intercal_\infty}$	$= \bigcap_{1 \leq i < \omega} \mathcal{C}^{\intercal_i}$
$\Omega^i(M)$	the class of all the i-th syzygies occurring in all projective resolutions of a module M
$\Omega^{-i}(M)$	the class of all the i-th cosyzygies occurring in all injective coresolutions of a module M
mSpec R	the set of all maximal ideals of a ring R
Spec R	the set of all prime ideals of a commutative ring R
dim R	the Krull dimension of a commutative ring R

In the following, ring will allways mean an associative ring with a unit.

2 Basics

2.1 General case

In this subsection we will prove some basic facts from the theory of modules over generally non-commutative rings.

Definition 2.1. Let $\mathcal{C}$ be a class, for each pair $A, B \in \mathcal{C}$, let $\text{mor}_C(A, B)$ be a set. Write the elements of $\text{mor}_C(A, B)$ as 'arrows' $f : A \to B$ for which A is called the *domain* and B the *codomain*. Finally, suppose that for each triple $A, B, C \in \mathcal{C}$ there is a mapping

$$\circ \colon \text{mor}_C(B, C) \times \text{mor}_C(A, B) \to \text{mor}_C(A, C).$$

We denote the arrow assigned to a pair

$$g \colon B \to C \qquad f \colon A \to B$$

by the arrow $gf : A \to C$. The system $\mathbf{C} = (\mathcal{C}, \text{mor}_\text{C}, \circ)$ consisting of the class $\mathcal{C}$, the mapping $\text{mor}_C \colon (A, B) \mapsto \text{mor}_C(A, B)$, and the partial mapping $\circ$ is a *category* in case

(i) for every triple $h \colon C \to D$, $g \colon B \to C$, $f : A \to B$,

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

(ii) for each $A \in \mathcal{C}$, there is a unique $\text{id}_A \in \text{mor}_C(A, A)$ such that if $f : A \to B$ and $g \colon C \to A$, then

$$f \circ \text{id}_A = f \quad \textit{and} \quad \text{id}_A \circ g = g.$$

If $\mathbf{C}$ is a category, then the elements of the class $\mathcal{C}$ are called the *objects* of the category, the 'arrows' $f : A \to B$ are called the *morphisms*, the partial mapping $\circ$ is called the *composition*, and the morphisms id_A are called the *identities* of the category.

Example 2.2.
1. Let $\mathcal{R}$ be the class of all rings, let $\text{mor}_R(R, S)$ be the set of all ring homomorphisms from R to S and $\circ$ be the usual composition of mappings. Then $\mathbf{R} = (\mathcal{R}, \text{mor}_\text{R}, \circ)$ is the *category of rings.*

2. Let R be a ring, let $\mathcal{M}_R$ be the class of all right R-modules, let $\text{mor}_{M_R}(M, N)$ be the set of all right R-module homomorphisms from M to N and $\circ$ be the usual composition of mappings. Then Mod-$R = (\mathcal{M}_R, \text{mor}_{M_R}, \circ)$ is the *category of right R-modules.*

3. Let R be a ring, let ${}_R\mathcal{M}$ be the class of all left R-modules, let $\mathrm{mor}_{{}_RM}(M,N)$ be the set of all left R-module homomorphisms from M to N and $\circ$ be the usual composition of mappings. Then R-Mod $= ({}_R\mathcal{M}, \mathrm{mor}_{{}_RM}, \circ)$ is the *category of left R-modules.*

Definition 2.3. A category $\mathbf{D} = (\mathcal{D}, \mathrm{mor}_{\mathrm{D}}, \circ)$ is a *subcategory* of $\mathbf{C} = (\mathcal{C}, \mathrm{mor}_{\mathrm{C}}, \circ)$ provided $\mathcal{D} \subseteq \mathcal{C}$, $\mathrm{mor}_D(A,B) \subseteq \mathrm{mor}_C(A,B)$ for each pair $A, B \in \mathcal{D}$, $\circ$ in $\mathbf{D}$ is the restriction of $\circ$ in $\mathbf{C}$. If in addition $\mathrm{mor}_D(A,B) = \mathrm{mor}_C(A,B)$ for each $A, B \in \mathcal{D}$, then $\mathbf{D}$ is a *full* subcategory of $\mathbf{C}$.

Definition 2.4. Let $\mathbf{C} = (\mathcal{C}, \mathrm{mor}_{\mathrm{C}}, \circ)$ and $\mathbf{D} = (\mathcal{D}, \mathrm{mor}_{\mathrm{D}}, \circ)$ be two categories. A pair of mapping (F', F'') is a *covariant functor* from $\mathbf{C}$ to $\mathbf{D}$ in case F' is a mapping from $\mathcal{C}$ to $\mathcal{D}$, F'' is a mapping from the morphisms of $\mathbf{C}$ to those of $\mathbf{D}$ such that for all $A, B, C \in \mathcal{C}$ and all $f\colon A \to B$ and $g\colon B \to C$ in $\mathbf{C}$,

(F1) $F''(f)\colon F'(A) \to F'(B)$ in $\mathbf{D}$,

(F2) $F''(g \circ f) = F''(g) \circ F''(f)$,

(F3) $F''(\mathrm{id}_A) = \mathrm{id}_{F'(A)}$.

A *contravariant functor* is a pair $F = (F', F'')$ satysfying instead of (F1) and (F2) their duals

(F1)* $F''(f)\colon F'(B) \to F'(A)$ in $\mathbf{D}$,

(F2)* $F''(g \circ f) = F''(f) \circ F''(g)$,

(F3) $F''(\mathrm{id}_A) = \mathrm{id}_{F'(A)}$.

Remark 2.5. Given a functor $F = (F', F'')$, we will write $F(A)$ and $F(f)$ instead of $F'(A)$ and $F''(f)$.

Definition 2.6. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Let F and G be functors from $\mathbf{C}$ to $\mathbf{D}$, both covariant (the 'contravariant version' is at the end of this definition). Let $\eta = (\eta_A \mid A \in \mathcal{C})$ be a family of morphisms in $\mathbf{D}$ such that for each $A \in \mathcal{C}$,

$$\eta_A \in \mathrm{mor}_D(F(A), G(A)).$$

Then η is a *natural transformation* from F to G, denoted $\eta\colon F \to G$, in case for each pair, $A, B \in \mathcal{C}$, and each $f \in \mathrm{mor}_C(A,B)$ the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

commutes, that is $\eta_B \circ F(f) = G(f) \circ \eta_A$. (If both F and G were contravariant, the only change would be to reverse the arrows $F(f)$ and $G(f)$.)

Remark 2.7. Let R, S be rings. Let $F, G\colon \text{Mod-}R \to \text{Mod-}S$ be additive functors, both covariant or contravariant. Let $h_M : F(M) \to G(M)$, $M \in \text{Mod-}R$ be a homomorphism such that $h = (h_M \mid M \in \text{Mod-}R)$ is a natural transformation from F to G. Then we say that h_M is *natural* and we often write h instead of h_M when it is clear which h_M is ment.

Definition 2.8. Let R, S be rings. Let $\mathbf{C}$ be a full subcategory of the category of right (left) R-modules and $\mathbf{D}$ be a full subcategory of the category of right (left) S-modules. Then a functor F (covariant or contravariant) from $\mathbf{C}$ to $\mathbf{D}$ is *additive* in case for each M, N, modules in $\mathbf{C}$, and each pair $f, g\colon M \to N$ in $\mathbf{C}$,

$$F(f+g) = F(f) + F(g).$$

In particular, if F is additive and covariant, then the restriction

$$F\colon \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_S(F(M), F(N))$$

is an abelian group homomorphism, whereas if F is additive and contravariant, then the restriction

$$F\colon \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_S(F(N), F(M)$$

is an abelian group homomorphism.

Definition 2.9. Let R be a ring. A non-zero element $a \in R$ is called *left zero-divisor* if there is a non-zero element $b \in R$ such that $ab = 0$. A non-zero element $a \in R$ is called *right zero-divisor* if there is a non-zero element $b \in R$ such that $ba = 0$. A non-zero element $a \in R$ is called *zero-divisor* if it is both a left and a right zero-divisor. Note that if R is commutative then a non-zero element $a \in R$ is a left zero-divisor iff it is a right zero-divisor iff it is a zero-divisor.

A non-zero element $a \in R$ is *left regular* if it is not a left zero-divisor. A non-zero element $a \in R$ is *right regular* if it is not a right zero-divisor. A non-zero element $a \in R$ is *regular* if it is both left and right regular.

Note that if R is commutative then a non-zero element $a \in R$ is left regular iff it is right regular iff it is regular.

Definition 2.10. Let R be a ring. A right (left) ideal m of R is *maximal* if the following two conditions hold

(i) $m \neq R$,

(ii) there is no right (left) ideal I of R satisfying $m \subsetneq I \subsetneq R$.

The set of all maximal right (left) ideals of R is denoted by $\operatorname{mSpec} R$.

Definition 2.11. Let R be a ring and M be a right (left) R-module. Then a submodule A of M is *maximal* if

1. $A \neq M$,
2. there is no other right (left) R-submodule A' of M satysfying $A \subsetneq A' \subsetneq M$.

And a submodule B of M is *minimal* if

1. $B \neq 0$,
2. there is no other right (left) R-submodule B' of M satysfying $0 \subsetneq B' \subsetneq B$.

Remark 2.12. Let R be a ring, M be a right (left) R-module and N be a submodule of M. If $N \neq M$ then we say that N is a *proper* submodule of M.

Definition 2.13. Let R be a ring and let M be a right (left) R-module. Then we define a cardinal *gen(M)* in the following way

$$\operatorname{gen}(M) = \min \{|X| \mid X \text{ is a generating subset of } M\}.$$

If $\operatorname{gen}(M) < \kappa$, where κ is an infinite cardinal, we say that M is $<\kappa$*-generated*, if M is $<\aleph_1$-generated we say that M is *countably generated*, if M is $<\aleph_0$-generated we say that M is *finitely generated* and if $\operatorname{gen}(M) = 1$, we say that M is *cyclic*. The class of all cyclic right (left) R-modules will be denoted $\mathcal{CM}$.

Theorem 2.14. *Let R be a ring and M be a finitely generated right (left) R-module. Then every proper submodule of M is contained in a maximal submodule. In particular, M has a maximal submodule.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Let K be a proper submodule of M. Then there is a finite sequence $x_1, x_2, \ldots, x_n \in M$ such that

$$M = K + x_1R + x_2R + \cdots + x_nR.$$

So certainly among all such sequences there is one of minimal length (presumably there are several such sequences), and so we may assume that $x_1, x_2, \ldots, x_n$ has minimal lenght. Then

$$L = K + x_2R + x_3R + \cdots + x_nR$$

is a proper submodule of M (otherwise the too short sequence $x_2, x_3, \ldots, x_n$ would do for $x_1, x_2, \ldots, x_n$). Let P be the set of all proper submodules of M that contain L. By The Zorn's Lemma, P has a maximal element, say N. Because N is maximal in P any strictly larger submodule of M is not in P, and so contains x_1. But then any such submodule must contain $N + x_1R \supseteq L + x_1R = M$. Thus N is a maximal submodule of M. For the final statement of the Theorem let $K = 0$. □

Definition 2.15. Let R be a ring, M be a right (left) R-module and $\{M_\alpha \mid \alpha \in A\}$ be a set of submodules of M. Then we say that the set $\{M_\alpha \mid \alpha \in A\}$ is *independent* if $M_\alpha \cap (\sum_{\beta \neq \alpha} M_\beta) = 0$ for all $\alpha \in A$.

Remark 2.16. Let R be a ring, M be a right (left) R-module and $\{M_\alpha \mid \alpha \in A\}$ be an independent set of submodules of M. Then $\sum_{\alpha \in A} M_\alpha = \bigoplus_{\alpha \in A} M_\alpha$.

Definition 2.17. Let R be a ring and S be a non-zero right (left) R-module. Then S is called *simple* if S has no non-zero proper submodules.

Lemma 2.18. *Let R be a ring and S be a right (left) R-module. Then S is simple iff $S \simeq R/m$, where m is a maximal right (left) ideal of R.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. For every $m \in \operatorname{mSpec} R$, R/m is clearly a simple right R-module.

For the implication to the right, define an R-module homomorphism $\varphi\colon R \to S$ by $\varphi(r) = mr$, where m is an arbitrary non-zero element of S. By The First Isomorphism Theorem, $S \simeq R/\operatorname{Ann}(m)$ and by the simplicity of S, $\operatorname{Ann}(m)$ is a maximal right ideal of R. ☐

Definition 2.19. Let R be a ring and M be a right (left) R-module. Then the *socle* of M, denoted $\operatorname{Soc}(M)$, is defined by

$$\operatorname{Soc}(M) = \sum \{S \mid S \text{ is a simple submodule of } M\},$$

if M has no simple submodules we set $\operatorname{Soc}(M) = 0$.

Lemma 2.20. *Let R be a ring, M be a right (left) R-module and let $\{S_\alpha \mid \alpha \in A\}$ be a set of all simple submodules of M. Then for each submodule K of $Soc(M)$, there is a subset $B \subseteq A$ such that the set $\{S_\beta \mid \beta \in B\}$ is independent and $Soc(M) = K \oplus (\bigoplus_{\beta \in B} S_\beta)$.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. By Definition 2.19, we have $\operatorname{Soc}(M) = \sum_{\alpha \in A} S_\alpha$. Let K be an arbitrary submodule of $\operatorname{Soc}(M)$. By the Zorn's Lemma, there is a subset $B \subseteq A$ maximal with respect to the conditions that $\{S_\beta \mid \beta \in B\}$ is independent and $K \cap (\sum_{\beta \in B} S_\beta) = 0$. Then the sum

$$N = K + (\sum_{\beta \in B} S_\beta) = K \oplus (\bigoplus_{\beta \in B} S_\beta)$$

is direct. We claim that $N = \operatorname{Soc}(M)$. For let $\alpha \in A$. Since S_α is simple, either $S_\alpha \cap N = S_\alpha$ or $S_\alpha \cap N = 0$. But $S_\alpha \cap N = 0$ would contradict the maximality of B. Thus $S_\alpha \subseteq N$ for all $\alpha \in A$, so $N = \operatorname{Soc}(M)$. So the claim is true. ☐

Corollary 2.21. *Let R be a ring and M be right (left) R-module. Then*

1. *there is a set* $\{S_\alpha \mid \alpha \in A\}$ *of simple submodules of* M *such that* $Soc(M) = \bigoplus_{\alpha \in A} S_\alpha$,
2. *every submodule of* $Soc(M)$ *is a direct summand in* $Soc(M)$.

Proof. (1) follows from Lemma 2.20 by setting $K = 0$.
(2) follows directly from Lemma 2.20. □

Lemma 2.22. *Let* R *be a ring and* M, N *be right (left)* R*-modules and* $f \colon M \to N$ *be an* R*-module homomorphism. Then* $f(Soc(M)) \subseteq Soc(N)$.

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Since $f(\mathrm{Soc}(M))$ is generated by it's submodules of the form $f(S)$, where S is a simple submodule of M, it is enough to prove that $f(S) \subseteq \mathrm{Soc}(N)$ for all simple submodules S of M. But since S is simple we have either $\mathrm{Ker}\, f_{\restriction S} = 0$ or $\mathrm{Ker}\, f_{\restriction S} = S$, so either $f(S) \simeq S \subseteq \mathrm{Soc}(N)$ or $f(S) \simeq 0 \subseteq \mathrm{Soc}(N)$. So the claim is true. □

Lemma 2.23. *Let* R *be a ring and* m *be a maximal right ideal of* R*. Then*

$$R/m \simeq Hom_R(R/m, E(R/m))$$

as abelian groups.

Proof. Clearly $\mathrm{Soc}(R/m) = R/m$ and $\mathrm{Soc}(E(R/m)) \supseteq R/m$. Since R/m is essential in $E(R/m)$, Corollary 2.21 implies that $\mathrm{Soc}(E(R/m)) = R/m$. By Lemma 2.22 we have

$$\mathrm{Hom}_R(R/m, E(R/m)) \simeq \mathrm{Hom}_R(R/m, R/m) \simeq R/m.$$

So the claim is true. □

Lemma 2.24. *Let* R *be a ring,* M, M' *be right (left)* R*-modules,* N *be a submodule of* M *and let* $\delta \in Hom_R(M, M')$ *be an arbitrary* R*-module homomorphism such that* $N \subseteq Ker\, \delta$*. Then there exists a unique* R*-module homomorphism* $\delta' \in Hom_R(M/N, M')$ *such that* $\delta'\pi = \delta$*, where* π *is the canonical projection.*

Proof. Define $\delta'(m + N) = \delta(m)$. □

Definition 2.25. Let R be a ring and M be a right R-module. Then the *right annihilator* of an element $m \in M$, denoted $\mathrm{Ann}(m)$, is defined by $\mathrm{Ann}(m) = \{r \in R \mid mr = 0\}$. The *right annihilator* of M, denoted $\mathrm{Ann}(M)$, is defined by $\mathrm{Ann}(M) = \{r \in R \mid mr = 0 \text{ for all } m \in M\} = \bigcap_{m \in M} \mathrm{Ann}(m)$.

Analogously, we can define the *left annihilator* of an element of a left R-module and the *left annihilator* of a left R-module. If the ring R is commutative we call the left (= right) annihilator just an *annihilator*.

If $r \in \mathrm{Ann}(m)$ then we say that r *annihilates* m and if $r \in \mathrm{Ann}(M)$ then we say that r *annihilates* M.

Lemma 2.26. *Let R be a ring and M be a right (left) R-module. Then*

1. *right (left) annihilator of any element of M is a right (left) ideal of R,*
2. *right (left) annihilator of M is a two-sided ideal of R.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. The part (1) is clear.

Ann(M) is clearly a right ideal of R. But since $m(sr) = (ms)r = 0$ for each $r \in \text{Ann}(M)$, $s \in R$ and $m \in M$, Ann(M) is also a left ideal of R. So the claim is true. □

Remark 2.27. Let R be a ring and M be a right (left) R-module. If I is a right (left) ideal of R such that $I \subseteq \text{Ann}(M)$, then M is a right (left) R/I-module via scalar multiplication $m(r + I) = mr$ ($(r + I)m = rm$). This is well-defined for if $r + I = s + I$, then $r - s \in I \subseteq \text{Ann}(M)$ and so $m(r - s) = 0$ ($(r - s)m = 0$). In particular, we have that M is always a right (left) $(R/\text{Ann}(M))$-module.

Definition 2.28. Let R be a ring. Then the *Jacobson radical*, denoted $J(R)$, of the ring R is defined as the intersection of all maximal right ideals of R (in the following we will prove that $J(R)$ is also the intersection of all maximal left ideals of R).

Lemma 2.29. *Let R be a ring. Then $J(R)$ is the intersection of all right annihilators of simple right R-modules.*

Proof. Assume $r \in J(R)$. If M is a simple right R-module, choose any non-zero element $m \in M$. Analogously as in the proof of Lemma 2.18, $M \simeq R/\text{Ann}(m)$ and Ann(m) is a maximal right ideal of R. Thus $r \in \text{Ann}(m)$ for each element $m \in M$, and so by Definition 2.25, $r \in \text{Ann}(M)$.

If r annihilates each simple right R-module then by Lemma 2.18, r annihilates each right R-module R/m, where m is a maximal right ideal of R. Thus in particular $(1+m)r = 0$ for each maximal right ideal m of R and it is iff $r \in m$ for each maximal right ideal m of R. So $r \in J(R)$. □

Corollary 2.30. *Let R be a ring. Then $J(R)$ is a two-sided ideal.*

Proof. This follows from Lemmas 2.29 and 2.26. □

Definition 2.31. Let R be a ring. Then an element $r \in R$ is *right quasi-regular*, (rqr) if $1 - r$ has a right inverse, *left quasi-regular*, (lqr) if $1 - r$ has a left inverse, and *quasi-regular*, (qr) if $1 - r$ is invertible.

Lemma 2.32. *Let R be a ring and $r \in R$. The the following are equivalent*

1. *r is rqr and lqr,*

2. r is qr.

Proof. This follows from the fact that if $(1-r)s = t(1-r) = 1$, then $t = t1 = t(1-r)s = 1s = s$. □

Lemma 2.33. *Let R be a ring and I be a right ideal of R. If every element of I is rgr, then every element of I is qr.*

Proof. If $r \in I$, then we have $(1-r)s = 1$ for some $s \in R$. Let $t = 1-s$, so that $(1-r)(1-t) = 1-r-t+rt = 1$. Thus $t = rt - r = r(t-1) \in I$. By hypotesis, t is rqr, so $(1-t)$ has a right inverse. But we know that $(1-t)$ has a left inverse $(1-r)$, so t is also lqr. By Lemma 2.32, t is qr and $(1-t)$ is the two-sided inverse of $(1-r)$. So the claim is true. □

Lemma 2.34. *Let R be a ring. Then the Jacobson radical $J(R)$ is the largest two-sided ideal consisting entirely of quasi-regular elements.*

Proof. First, $J(R)$ is a two-sided ideal by Corollary 2.30.

We show that each $r \in J(R)$ is rqr, so by Lemma 2.33, each $r \in J(R)$ is qr. If $(1-r)$ has no right inverse, then $(1-r)R$ is a proper right ideal of R, which is contained in a maximal right ideal I by Theorem 2.14. But then $r \in I$ and $(1-r) \in I$, and therefore $1 \in I$, a contradiction.

Now we show that every right ideal (hence every two-sided ideal) I consistinq entirely of quasi-regular elements is contained in $J(R)$. If $r \in I$ but $r \notin J(R)$, then for some maximal right ideal K we have $r \notin K$. By maximality of K, we have $R = I + K$, so $1 = i + k$ for some $i \in I$, $k \in K$. But then i is quasi-regular, so $k = 1 - i$ has an inverse, and consequently $1 \in K$, a contradiction. □

Corollary 2.35. *Let R be a ring. Then the Jacobson radical $J(R)$ is the intersection of all maximal left ideals of R.*

Proof. We can reproduce the entire discussion beginning with Definition 2.28 with right and left ideals interchanged, and reach exactly the same conclusion, namely that the 'right' Jacobson radical is the largest two-sided ideal consisting entirely of quasi-regular elements. It follows that the 'right' and 'left' Jacobson radicals are identical. □

Definition 2.36. Let R be a ring. Then R is called *local* if R has a unique maximal right ideal.

Lemma 2.37. *Let R be a local ring. Then R has a unique maximal left ideal.*

Proof. Since R is local it has a unique maximal right ideal m, it follows that $m = J(R)$.

Let $r \in R \backslash J(R)$, then $rR = R$, otherwise rR is contained in the unique maximal ideal $J(R)$, but it is not possible since $r \notin J(R)$. So r has a right inverse.

Suppose now that r has a right inverse and that $r \in J(R)$. Then there is an $s \in R$ such that $rs = 1$ and since $J(R)$ is a right ideal of R, we have that $1 \in J(R)$, a contradiction. So $r \in R \setminus J(R)$ iff r has a right inverse.

Suppose that $r \in R$ has a left inverse, i.e. there is an $s \in R$ such that $sr = 1$. Then $r \notin J(R)$, otherwise $sr = 1 \in J(R)$ since $J(R)$ is a left ideal of R, so by the previous, r has a right inverse.

Suppose now that $r \in R$ has a right inverse, i.e. there is an $s \in R$ such that $rs = 1$. So $srs = s$ and thus $(sr - 1)s = 0$. Denote $I = \{t \in R \mid (sr - 1)t = 0\}$. It is easy to see that I is a right ideal of R. We have $I = R$, otherwise I is contained in the unique maximal right ideal $J(R)$, but it is not possible since $s \notin J(R)$ (s has a left inverse, so s has a right inverse and thus $s \notin J(R)$). So $(sr - 1)1 = 0$ which implies $sr = 1$ and thus r has a left inverse.

So $r \notin J(R)$ iff R has a right inverse and by the previous, it is iff r is invertible. Thus every proper left ideal of R is contained in $J(R)$, so by Lemma 2.34, R has a unique maximal left ideal $J(R)$. $\square$

Lemma 2.38 (Nakayama)**.** *Let R be a ring, M be a right (left) R-module and I be a subgroup of the additive group of R such that either*

1. *I is nilpotent (that is, $I^n = 0$ for some $n \geq 1$),*

   or

2. *$I \subseteq J(R)$ and M is finitely generated.*

Then $MI = M$ ($IM = M$) implies $M = 0$.

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. (1) is trivial for $M = MI = MI^2 = \cdots = 0$. For (2) suppose $MI = M$ and $M \neq 0$. Then let $\{x_1, x_2, \ldots, x_n\}$ be a minimal set of generators of M. So $x_1 = \sum_{i=1}^{n} x_i r_i$ for some $r_i \in I$ since $M = MI$. But by Lemma 2.34, $(1 - r_1)$ is invertible. Thus $x_1 \in x_2R + x_3R + \cdots + x_nR$ which contradicts the minimality of $\{x_1, x_2, \ldots, x_n\}$. So the claim is true. $\square$

Definition 2.39. Let R be a ring and

$$\mathcal{E} : 0 \longrightarrow A \xrightarrow{i} B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of right (left) R-modules. We say that $\mathcal{E}$ is *split exact* if $i(A)$ is a direct summand of B. In this case clearly $B \simeq A \oplus C$ as right (left) R-modules.

Lemma 2.40. *Let R be a ring and $\mathcal{E}\colon 0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$ be a short exact sequence of right (left) R-modules. Then the following conditions are equivalent*

1. *$\mathcal{E}$ is split exact,*
2. *there is a homomorphism $f\colon B \to A$ such that $fi = id_A$,*
3. *there is a homomorphism $g\colon C \to B$ such that $\pi g = id_C$,*
4. *there are homomorphisms $f\colon B \to A$ and $g\colon C \to B$ such that $\pi i = fg = 0$, $fi = id_A$, $\pi g = id_C$ and $if + g\pi = id_B$.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Assume first, that the sequence $\mathcal{E}$ is split exact, i.e. that $B = i(A) \oplus D$ for some submodule D of B. Denoting $f\colon B \to A$ and $g\colon C \to B$ the homomorphism given by $f(i(a) + d) = a$ and $g(c) = d$ whenever $\pi(b) = c$, it is an easy excercise to verify, that f is a homomorphism satisfying $fi = \mathrm{id}_A$. Concerning g we first note that for each $c \in C$ there is $b \in B$ with $\pi(b) = c$. The element b can be uniquely expressed in the form $b = i(a) + d$ for some $a \in A$ and $d \in D$. If $\overline{b}$ is another element with $\pi(\overline{b}) = c$ and $\overline{b} = i(\overline{a}) + \overline{d}$, then $b - \overline{b} \in \mathrm{Ker}\,\pi = \mathrm{Im}\,i$ yields that $b - \overline{b} = i(a')$ for some $a' \in A$ and consequently $b - \overline{b} = i(a) - i(\overline{a}) + d - \overline{d} = i(a')$ yields that $d = \overline{d}$ and the mapping g is well-defined. Moreover, it is obvious, that g is a homomorphism and that $\pi g = \mathrm{id}_C$. Finaly, $\pi i = 0$ by the exactness of $\mathcal{E}$, $fg(c) = \pi(d) = 0$ by the definition of π and $(if + g\pi)(i(a) + d) = i(a) + d$ showing that (1) implies (2), (3) and (4).

Assuming (2) we denote $D = \mathrm{Ker}\, f$. For $u \in D \cap i(A)$ we have $u = i(a)$ for some $a \in A$ and so $0 = f(u) = fi(a) = a$. Hence $u = i(a) = 0$ and $D \cap i(a) = 0$. Moreover, for an arbitrary $b \in B$ we have $b = if(b) + (b - if(b))$, where $f(b - if(b)) = 0$ showing that $B = i(A) \oplus D$ and so (2) implies (1).

Similarly, assuming (3), we are going to verify that $B = i(A) \oplus g(C)$. So if $i(a) = g(c) \in i(A) \cap g(C)$ is arbitrary, then $0 = \pi i(a) = \pi g(c) = c$ yields $i(A) \cap g(C) = 0$. Further, if $b \in B$ is arbitrary, then $b - g\pi(b) = i(a)$ for some $a \in A$ in view of the fact that $\pi(b - g\pi(b)) = 0$ and $\mathrm{Ker}\,\pi = \mathrm{Im}\,i$. Thus $b \in i(A) + g(C)$ and (3) implies (1).

The implication (4) $\Rightarrow$ (2) is obvious and the proof is complete. $\square$

Remark 2.41. If the condition (1) ((2)) from Lemma 2.40 is satisfyied for the short exact sequence $\mathcal{E}$, we say that $\mathcal{E}$ is *left (right) split exact.* It is now clear that $\mathcal{E}$ is left (right) split exact iff $\mathcal{E}$ is split exact.

Lemma 2.42. *Let R, S be rings. Let $\mathbf{C}$ be a full subcategory of the category of all right (left) R-modules and let $\mathbf{D}$ be a full subcategory of the category of all right (left) S-modules. Let F and G be additive functors (both covariant or both contravariant) from $\mathbf{C}$ to $\mathbf{D}$ and let $\eta\colon F \to G$ be a natural transformation. If*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is split exact in $\mathbf{C}$, then η_M is injective (surjective) iff both $\eta_{M'}$ and $\eta_{M''}$ are injective (surjective).

Proof. We will prove the 'right and covariant version', the proof of the 'rest versions' is analogical. By Lemma 7.1, we have the following two commutative diagrams with split exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(M') & \longrightarrow & F(M) & \longrightarrow & F(M'') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_{M'}} & & \downarrow{\scriptstyle \eta_{M}} & & \downarrow{\scriptstyle \eta_{M''}} & & \\ 0 & \longrightarrow & G(M') & \longrightarrow & G(M) & \longrightarrow & G(M'') & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(M'') & \longrightarrow & F(M) & \longrightarrow & F(M') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_{M''}} & & \downarrow{\scriptstyle \eta_{M}} & & \downarrow{\scriptstyle \eta_{M'}} & & \\ 0 & \longrightarrow & G(M'') & \longrightarrow & G(M) & \longrightarrow & G(M') & \longrightarrow & 0 \end{array}$$

obtained from

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and

$$0 \longrightarrow M'' \longrightarrow M \longrightarrow M' \longrightarrow 0.$$

Now, it is an easy excercise to verify that the claim is true. □

Lemma 2.43. *Let R be a ring, P be a right (left) R-module and κ be an infinite cardinal. Then P is $< \kappa$-generated and projective iff P is a direct summand in $< \kappa$-generated free right (left) R-module.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. The implication $\Leftarrow$ is clear.

A right R-module P is $< \kappa$-generated iff there is a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$$

with F free and $< \kappa$-generated. But since P is projective, this exact sequence is split exact and hence by Definition 2.39, P is a direct summand in F. So the claim is true. □

Remark 2.44. Lemma 2.42 implies that if $\eta_{M_1}, \eta_{M_2}, \ldots, \eta_{M_n}$ are isomorphisms, then so is $\eta_{M_1 \oplus M_2 \oplus \cdots \oplus M_n}$. Therefore by Lemma 2.43, if η_R is an isomorphism, then so is η_P for every finitely generated projective right R-module P.

Lemma 2.45. *Let R be a ring and M be a right (left) R-module. Then $Hom_R(R, M) \simeq M$ as right (left) R-modules.*

Proof. It is an easy excercise to verify that the mapping

$$\begin{aligned} \varphi\colon \operatorname{Hom}_R(R, M) &\to M \\ \varphi &\mapsto \varphi(1) \end{aligned}$$

has demanded features. □

Lemma 2.46. *Let R, S be rings, A be a right S-module, B be a (S, R)-bimodule and C be a right R-module. Then*

$$Hom_S(A, Hom_R(B, C)) \simeq Hom_R(A \otimes_S B, C)$$

as abelian groups.

Proof. It is an easy excercise to verify that the mapping

$$\varphi\colon \operatorname{Hom}_S(A, \operatorname{Hom}_R(B, C)) \to \operatorname{Hom}_R(A \otimes_S B, C)$$

defined by $\varphi(f)(a \otimes b) = (f(a))(b)$ where $f \in \operatorname{Hom}_S(A, \operatorname{Hom}_R(B, C))$, $a \in A$ and $b \in B$, has demanded features. □

Lemma 2.47. *Let R be a ring, M be a right R-module and N be a left R-module. Then $M \otimes_R R \simeq M$ as right R-modules and $R \otimes_R N \simeq N$ as left R-modules.*

Proof. It is an easy excercise to verify that the mappings

$$\begin{aligned} \varphi\colon M \otimes_R R &\to M \\ m \otimes r &\mapsto mr \end{aligned}$$

and

$$\begin{aligned} \varphi'\colon R \otimes_R N &\to N \\ r \otimes n &\mapsto rn \end{aligned}$$

have demanded features. □

Lemma 2.48. *Let R, S be rings and U be an (R,S)-bimodule, N be a right S-module and P be a left R-module. Then there is an abelian group homomorphism*

$$\nu\colon Hom_S(U,N)\otimes_R P \to Hom_S(Hom_R(P,U),N)$$

defined via

$$\nu(\varphi\otimes p)\colon \psi\mapsto \varphi(\psi(p))$$

which is natural in U, N and P. Moreover, if P is finitely generated and projective, then ν_{UNP} is an isomorphism for each (R,S)-bimodule U and each right S-module N.

Proof. It is tedious but no difficult to check that ν is an abelian group homomorphism that is natural in all three variables. Now for each (R,S)-bimodule U and each right S-module N we have by Lemmas 2.47 and 2.45 that

$$\mathrm{Hom}_S(U,N)\otimes_R R\simeq \mathrm{Hom}_S(U,N)\simeq \mathrm{Hom}_S(\mathrm{Hom}_R(R,U),N)$$

as abelian groups via

$$\varphi\otimes r\mapsto \varphi r\mapsto \delta\colon \psi\mapsto (\varphi r)(\psi(1))=\varphi(r\psi(1))=\varphi(\psi(r))$$

where $\varphi\in\mathrm{Hom}_S(U,N)$, $r\in R$, $\delta\in\mathrm{Hom}_S(\mathrm{Hom}_R(R,U),N)$ and $\psi\in\mathrm{Hom}_R(R,U)$. Thus ν_{UNR} is the composition of previous isomorphisms, and so is itself an isomorphism for each (R,S)-bimodule U and each right S-module N. So by Remark 2.44, the 'moreover' part is also true. □

Definition 2.49. Let R be a ring, M be a right (left) R-module and κ be an infinite cardinal. Then M is $<\kappa$*-presented* if

(i) M is $<\kappa$-generated,

(ii) in every short exact sequence of right (left) R-modules

$$0\longrightarrow K\longrightarrow F\longrightarrow M\longrightarrow 0$$

with F free and $<\kappa$-generated, the module K is also $<\kappa$-generated.

If M is $<\aleph_1$-presented we say that M is *countably presented* and if M is $<\aleph_0$-presented we say that M is finitely presented.

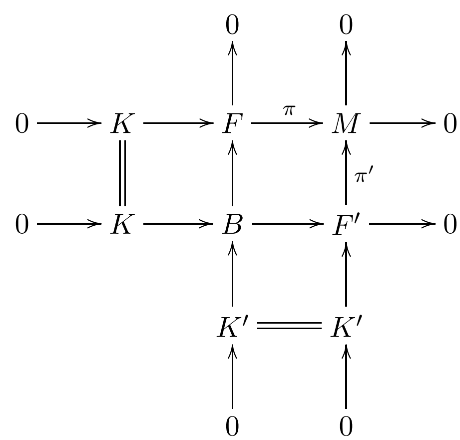
Lemma 2.50. *Let R be a ring, M be a right (left) R-module and κ be an infinite cardinal. Then M is $<\kappa$-presented iff there exists a short exact sequence of right (left) R-modules*

$$0\longrightarrow K\longrightarrow F\longrightarrow M\longrightarrow 0$$

with F free and $<\kappa$-generated and K $<\kappa$-generated.

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. The implication to the right is clear.

Let $0 \longrightarrow K \longrightarrow F \stackrel{\pi}{\longrightarrow} M \longrightarrow 0$ be a short exact sequence of right R-modules with F free and $< \kappa$-generated and $K < \kappa$-generated. M is clearly $< \kappa$-generated. In order to prove that M is $< \kappa$-presented we must show that in every short exact sequence of right R-modules $0 \longrightarrow K' \longrightarrow F' \stackrel{\pi'}{\longrightarrow} M \longrightarrow 0$ with F' free and $< \kappa$-generated, the module K' is also $< \kappa$-generated. So let $0 \longrightarrow K' \longrightarrow F' \stackrel{\pi'}{\longrightarrow} M \longrightarrow 0$ be an arbitrary short exact sequence of right R-modules with F' free and $< \kappa$-generated. Denote B the pullback of π and π'. We have the following diagram



with exact rows and collums (the exactness is an easy excercise). The modules F and F' are projective, so by Lemma 2.40, the short exact sequences $0 \longrightarrow K \longrightarrow B \longrightarrow F' \longrightarrow 0$ and $0 \longrightarrow K' \longrightarrow B \longrightarrow F \longrightarrow 0$ are split exact and thus by Definition 2.39, we have

$$K \oplus F' \simeq B \simeq K' \oplus F.$$

Since K and F' are $< \kappa$-generated, so is B. And since F is $< \kappa$-generated, so is K'. And we are done. □

Lemma 2.51. *Let R, S be rings and A be a finitely presented left R-module, B be an (R, S)-bimodule and C be an injective right S-module. Then*

$$Hom_S(B, C) \otimes_R A \simeq Hom_S(Hom_R(A, B), C)$$

as abelian groups. Where the isomorphism is given by

$$\nu(f \otimes a)(g) = f(g(a)).$$

Proof. Since A is finitely presented there is a short exact sequence of left R-modules $0 \longrightarrow K \longrightarrow F_0 \longrightarrow A \longrightarrow 0$ with F_0 free and finitely generated and K finitely generated. So we can consider the exact sequence $F_1 \xrightarrow{\varphi} F_0 \longrightarrow A \longrightarrow 0$ of left R-modules with F_0, F_1 finitely generated and free (φ is the composite mapping of $F_1 = R^{(X)} \longrightarrow K \longrightarrow 0$ and $0 \longrightarrow K \longrightarrow F_0$, where X is the finite generating subset of K). Then by Lemma 2.48, we have the following commutative diagram of abelian groups

$$\begin{array}{ccccc}
\mathrm{Hom}_S(B,C)\otimes_R F_1 & \longrightarrow & \mathrm{Hom}_S(B,C)\otimes_R F_0 & \twoheadrightarrow & \mathrm{Hom}_S(B,C)\otimes_R A \\
\downarrow & & \downarrow & & \downarrow \nu \\
\mathrm{Hom}_S(\mathrm{Hom}_R(F_1,B),C) & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(F_0,B),C) & \twoheadrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(A,B),C)
\end{array}$$

with exact rows (C is injective). But by Lemma 2.48, the first two vertical mappings are isomorphisms. So ν is also an isomorphism. So the claim is true. ☐

Lemma 2.52. *Let R be a ring, $(M_i \mid i \in I)$ be a family of (S,R)-bimodules and N be an (R,T)-bimodule. Then*

$$(\bigoplus_{i\in I} M_i) \otimes_R N \simeq \bigoplus_{i\in I}(M_i \otimes_R N)$$

as (S,T)-bimodules.

Proof. The map $(\bigoplus_{i\in I} M_i) \times N \to \bigoplus_{i\in I}(M_i \otimes_R N)$ given by $((x_i)_I, y) \mapsto (x_i \otimes y)_I$ is R-balanced and so we have a unique homomorphism of abelian groups $h\colon (\bigoplus_{i\in I} M_i) \otimes_R N \to \bigoplus_{i\in I}(M_i \otimes_R N)$ such that $h((x_i)_I \otimes y) = (x_i \otimes y)_I$. Similarly one gets a unique homomorphism of abelian groups $h'\colon \bigoplus_{i\in I}(M_i \otimes_R N) \to (\bigoplus_{i\in I} M_i) \otimes_R N$ given by $h'((x_i \otimes y_i)_I) = \sum_{i\in I}(e_i(x_i) \otimes y_i)$, where $e_i\colon M_i \to \bigoplus_{i\in I} M_i$ is a natural embedding. It is easy to see that h, h' are (S,T)-bimodule homomorphisms and that $h' = h^{-1}$. ☐

Lemma 2.53. *Let R be a ring, M be an (S,R)-bimodule and $(M_i \mid i \in I)$ be a family of (R,T)-bimodules. Then*

$$M \otimes_R (\bigoplus_{i\in I} N_i) \simeq \bigoplus_{i\in I}(M \otimes_R N_i)$$

as (S,T)-bimodules.

Proof. It is analogical to the proof of Lemma 2.52. ☐

Lemma 2.54. *Let R be a ring, M be a right R-module and I be a left ideal of R. Then*

$$M \otimes_R (R/I) \simeq M/MI$$

as abelian groups.

Let R be a ring and M be a left R-module and I be a right ideal of R. Then

$$(R/I) \otimes_R M \simeq M/IM$$

as abelian groups.

Proof. We will prove the 'first' version, the proof of the 'second' version is analogical. We consider the short exact sequence of left R-modules $0 \longrightarrow I \xrightarrow{\mu} R \longrightarrow R/I \longrightarrow 0$. Since the covariant functor $M \otimes_R -$ is right exact and using Lemma 2.47, we have the following exact sequence of abelian groups

$$M \otimes_R I \overset{\varphi\circ(\mathrm{id}_M \otimes \mu)}{\longrightarrow} M \longrightarrow M \otimes_R (R/I) \longrightarrow 0,$$

where φ is the isomorphism $M \otimes_R R \overset{\varphi}{\simeq} M$ from the Lemma 2.47. But $\mathrm{Im}\,(\varphi \circ (\mathrm{id}_M \otimes \mu)) = \{\sum_i m_i r_i \mid m_i \in M, r_i \in I\} = MI$. Hence the result follows. □

Definition 2.55. Let R be a ring. A left (right) R-module F is said to be *flat* if given any exact sequence $0 \longrightarrow A \longrightarrow B$ of right (left) R-modules, the tensored sequence of abelian groups $0 \longrightarrow F \otimes_R A \longrightarrow F \otimes_R B$ is exact.

Lemma 2.56. *Let R be a ring. Then the direct sum $\bigoplus_{i \in I} F_i$ of left (right) R-modules is flat if and only if each F_i is a flat left (right) R-module.*

Proof. This follows from Lemma 2.53. □

Corollary 2.57. *Let R be a ring. Then every projective left (right) R-module is flat.*

Proof. We will prove the 'left' version, the proof of the 'right' version is analogical. Let P be a projective left R-module. Then P is a summand of a free left R-module. But by Lemma 2.47, R is a flat left R-module and so every free left R-module is flat by Lemma 2.56 above. Thus P is a direct summand of a flat left R-module and hence is flat again by Lemma 2.56. □

Lemma 2.58. *Let R be a ring, F be a flat left R-module and I be a right ideal of R. Then $I \otimes_R F \simeq IF$ as abelian groups.*

Let R be a ring, F be a flat right R-module and I be a left ideal of R. Then $F \otimes_R I \simeq FI$ as abelian groups.

Proof. We will prove the 'first' version, the proof of the 'second' version is analogical. We consider the exact sequence $0 \longrightarrow I \longrightarrow R$ of right R-modules. Then $0 \longrightarrow I \otimes_R F \longrightarrow F$ is an exact sequence of abelian groups. But the image of $I \otimes_R F$ in F under this embedding is IF. So we are done. □

Remark 2.59. Let $R \xrightarrow{\varphi} S$ be a ring homomorphism and M be a right (left) S-module. Then M is a right (left) R-module via $mr = m\varphi(r)$ $(rm = \varphi(r)m)$.

Lemma 2.60. *Let $R \xrightarrow{\varphi} S$ be a ring homomorphism and E be an injective right (left) R-module. Then $Hom_R(S, E)$ is an injective right (left) S-module.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Note that by Remark 2.59, S is an (S, R)-bimodule. Let $N \subseteq M$ be a submodule of the right S-module M. Then by Lemmas 2.46, 2.47 and Remark 2.59,

$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, E)) \simeq \mathrm{Hom}_R(N \otimes_S S, E) \simeq \mathrm{Hom}_R(N, E)$$

and likewise for $\mathrm{Hom}_S(M, \mathrm{Hom}_R(S, E))$. So we have that

$$\mathrm{Hom}_S(M, \mathrm{Hom}_R(S, E)) \longrightarrow \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, E)) \longrightarrow 0$$

is exact since by injectivity of E

$$\mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(N, E) \longrightarrow 0$$

is exact. Hence $\mathrm{Hom}_R(S, E)$ is an injective right S-module. $\square$

Remark 2.61. We note that it follows from the above that $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ is an injective right and left R-module for any ring R when G is a divisible (= injective) abelian group.

Definition 2.62. Let R be a ring and E be an injective right (left) R-module. Then E is said to be an *injective cogenerator* for right (left) R-modules, if for each non-zero right (left) R-module M and each non-zero element $m \in M$, there is $\varphi \in \mathrm{Hom}_R(M, E)$ such that $\varphi(m) \neq 0$.

This is equivalent to the condition that $\mathrm{Hom}_R(M, E) \neq 0$ for any right (left) R-module $M \neq 0$. For if $m \in M$, $m \neq 0$, any $\varphi' \in \mathrm{Hom}_R(mR, E)$ with $\varphi' \neq 0$ has $\varphi'(m) \neq 0$. And since E is injective, such φ' has an extension $\varphi \in \mathrm{Hom}_R(M, E)$.

It is well-known fact that the group $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator for abelian groups. Hence if M is a non-zero right (left) R-module, then the *character module* M^+ of M, defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, is a non-zero left (right) R-module.

Remark 2.63. Let R be a ring and M be a right (left) R-module. Then by Lemma 2.46, $\mathrm{Hom}_R(M, R^+) \simeq M^+$ as abelian groups. Hence $R^+ = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is an injective cogenerator for right (left) R-modules since R^+ is an injective right (left) R-module by Remark 2.61. Thus there exists an injective cogenerator for right (left) R-modules for any ring R.

Lemma 2.64. *Let R be a ring and E be an injective right (left) R-module. The the following are equivalent*

1. *E is an injective cogenerator for right (left) R-modules,*

2. *$Hom_R(T, E) \neq 0$ for all simple right (left) R-modules T.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. The implication (1) $\Rightarrow$ (2) is clear from Definiton 2.62.

Assume that E satisfies (2). Let M be a right R-module and $0 \neq m \in M$. Since mR is cyclic, by Theorem 2.14, it contains a maximal submodule N, so by (2) there is a non-zero homomorphism $\varphi = h \circ \pi \colon mR \to E$, where π is the canonical projection $mR \xrightarrow{\pi} (mR)/N$. But E is injective, so φ can be extended to a homomorphism $\overline{\varphi} \colon M \to E$ with $\overline{\varphi}(x) = \varphi(x) \neq 0$. Thus E is an injective cogenerator for right R-modules by Definiton 2.62. □

Lemma 2.65. *Let R, S be rings and E be an injective cogenerator for right (left) R-modules. Then a sequence*

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

of $(S, R)((R, S))$-bimodules is exact iff the sequence

$$0 \longrightarrow Hom_R(C, E) \xrightarrow{\psi^*} Hom_R(B, E) \xrightarrow{\varphi^*} Hom_R(A, E) \longrightarrow 0$$

or right (left) S-modules is exact.

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. The implication to the right is clear since E is an injective right R-module.

For the implication to the left, first we will prove that $\text{Im}\ \varphi = \text{Ker}\ \psi$. Suppose that $\text{Im}\ \varphi \nsubseteq \text{Ker}\ \psi$. Then choose $b \in \text{Im}\ \varphi \setminus \text{Ker}\ \psi$. So $\psi(b) \neq 0$. But $\psi(b) \in C$. So there is an $f \in \text{Hom}_R(C, E)$ such that $f(\psi(b)) \neq 0$ since E is an injective cogenerator for right R-modules. But $b = \varphi(a)$ for some $a \in A$. Thus $f \circ \psi \circ \varphi \neq 0$. But then $(\varphi^* \circ \psi^*)(f) \neq 0$, a contradiction. So $\text{Im}\ \varphi \subseteq \text{Ker}\ \psi$.

Now suppose $\text{Im}\ \varphi \nsupseteq \text{Ker}\ \psi$. Then let $b \in \text{Ker}\ \psi \setminus \text{Im}\ \varphi$. So $b + \text{Im}\ \varphi$ is non-zero in $B/\text{Im}\ \varphi$. Thus there is an $f \in \text{Hom}_R(B/\text{Im}\ f, E)$ such that $f(b + \text{Im}\ f) \neq 0$. Hence the composite mapping $g : B \xrightarrow{\pi} B/\text{Im}\ \varphi \xrightarrow{f} E$, where π is the canonical projection, is such that $g(b) \neq 0$. But $\varphi^*(g) = g \circ \varphi = 0$ since $g(\text{Im}\ \varphi) = 0$. So $g \in \text{Ker}\ \varphi^* = \text{Im}\ \psi^*$. That is $g = \psi^*(h) = h \circ \psi$ for some $h \in \text{Hom}_R(C, E)$. But $b \in \text{Ker}\ \psi$. So $g(b) = h(\psi(b)) = 0$, a contradiction since $g(b) \neq 0$. So $\text{Im}\ \varphi = \text{Ker}\ \psi$ and thus the claim is true. □

Lemma 2.66. *Let R be a ring and F be a left (right) R-module. If F is finitely presented and flat then F is projective.*

Proof. We will prove the 'left' version, the proof of the 'right' version is analogical. Let F be a finitely presented flat left R-module and $B \longrightarrow C \longrightarrow 0$ be an exact sequence of left R-modules. We want to show that the sequence of abelian groups $\mathrm{Hom}_R(F, B) \longrightarrow \mathrm{Hom}_R(F, C) \longrightarrow 0$ is exact, or equivalently by Lemma 2.65, $0 \longrightarrow \mathrm{Hom}_R(F, C)^+ \longrightarrow \mathrm{Hom}_R(F, B)^+$ is an exact sequence of abelian groups. But by Lemma 2.48, we have the following commutative diagram of abelian groups

$$\begin{array}{ccccc} 0 & \longrightarrow & C^+ \otimes_R F & \longrightarrow & B^+ \otimes_R F \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_R(F, C)^+ & \longrightarrow & \mathrm{Hom}_R(F, B)^+ \end{array}$$

where the first row is exact since F is flat. But the vertical mappings are isomorphisms by Lemma 2.51 since F is finitely presented, hence the second row is also exact and thus we are done. □

Definition 2.67. Let R be a ring. A right (left) R-module M is called *noetherian* if every right (left) R-submodule of M is finitely generated. This implies in particular that M itself is finitely generated.

The ring R is *right (left) noetherian* if it is itself noetherian as a right (left) R-module, that is, every right (left) ideal of R is finitely generated.

Note that a ring may be right noetherian but not left noetherian. The term *noetherian* ring will mean a ring which is both left and right noetherian. It is clear that, when R is commutative, R is left noetherian precisely when it is right noetherian.

Remark 2.68. It is a well-know fact (see [1]) that a right (left) R-module M is noetherian iff every ascending chain of right (left) R-submodules of M terminates and it is iff every non-empty set of right (left) R-submodules of M has an inclusion-maximal element.

Lemma 2.69. *Let R be a ring and let*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence of right (left) R-modules. Then M is noetherian iff both M' and M'' are noetherian.

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Suppose that M is noetherian. A submodule of M' is isomorphic to a submodule of M, and so is finitely generated. A submodule N of M'' is the homomorphic image of its inverse image

$$\beta^{-1}(N) = \{m \in M \mid \beta(m) \in N\}$$

in M. Since $\beta^{-1}(N)$ is finitely generated, so is N. Thus M' and M'' are noetherian.

Conversely, consider a submodule N of M. Let $N' = N \cap \alpha(M)$ and let N'' be the β-image of N in M'', so that there is a short exact sequence of right R-modules

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0.$$

Since both N' and N'' are finitely generated, so also is N. □

Corollary 2.70. *Let R be a ring and let $\{M_1, \dots, M_k\}$ be a finite set of noetherian right (left) R-modules. Then the direct sum $M_1 \oplus \cdots \oplus M_k$ is a noetherian right R-module.*

In particular, every free right (left) module of finite rank over a right (left) noetherian ring is noetherian.

Proof. This follows from Lemma 2.69. □

Theorem 2.71. *Let R be a right (left) noetherian ring and M be a finitely generated right (left) R-module. Then M is noetherian.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. We have the following short exact sequence of right R-modules

$$0 \longrightarrow K \longrightarrow R^{(X)} \xrightarrow{\pi} M \longrightarrow 0,$$

where X is the finite set of generators of M and K is the kernel of π. The module $R^{(X)}$ is noetherian by Corollary 2.70 and thus (using Lemma 2.69) M is noetherian. □

Lemma 2.72. *Let R be right (left) noetherian ring and M be a right (left) R-module. Then M is finitely generated iff M is finitely presented.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Let M be a finitely presented right R-module, then M is finitely generated by Definition 2.49.

Let M be a finitely generated right R-module. If we have a short exact sequence of right R-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F free and finitely generated, then F is noetherian by Theorem 2.71 and thus K is finitely generated since K is isomorphic to some submodule of F. So M is finitely presented. □

Definition 2.73. Let R be a ring and M be a right (left) R-module. Then a *projective resolution* of M is an (finite or infinite) exact sequence of right (left) R-modules

$$\mathcal{E}_P\colon \ldots \longrightarrow P_2 \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

with every P_i projective. For $i \geq 0$, the image of π_i in the previous exact sequence is called the *i-th syzygy* of M in $\mathcal{E}_P$. We denote $\Omega^i(M)$ the class of all the i-th syzygies occurring in all projective resolutions of M.

An *injective coresolution* (sometimes called an *injective resolution*) of M is an (finite or infinite) exact sequence of right (left) R-modules

$$\mathcal{E}_I\colon 0 \longrightarrow M \xrightarrow{\iota_0} I_0 \xrightarrow{\iota_1} I_1 \xrightarrow{\iota_2} I_2 \longrightarrow \ldots$$

with every I_i injective. For $i \geq 0$, the image of ι_i in the previous exact sequence is called the *i-th cosyzygy* of M in $\mathcal{E}_I$. We denote $\Omega^{-i}(M)$ the class of all the i-th cosyzygies occurring in all injective coresolutions of M. If every I_i in $\mathcal{E}_I$ is an injective hull of the i-th cosyzygy of M in $\mathcal{E}_I$, then $\mathcal{E}_I$ is called the *minimal injective coresolution* of M (or the *minimal injective resolution* of M).

A *flat resolution* of M is an (finite or infinite) exact sequence right (left) R-modules

$$\mathcal{E}_F\colon \ldots \longrightarrow F_2 \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

with every F_i projective. For $i \geq 0$, the image of π_i in the previous exact sequence is called the *i-th flat-syzygy* of M in $\mathcal{E}_F$.

Lemma 2.74. *Let R be a ring and M be a right (left) R-module. Then M has a projective (therefore flat) resolution.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Clearly, M is a homomorphic image of a free (hence projective) right R-module P_0. Let K_0 be the kernel of the homomorphism P_0 onto M. In turn, there is a homomorphism with kernel K_1 from a free right R-module P_1 onto K_0, and we have the following sequence of right R-modules

$$0 \longrightarrow K_1 \longrightarrow P_1 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Composing the homomorphisms $P_1 \longrightarrow K_0$ and $K_0 \longrightarrow P_0$, we get

$$0 \longrightarrow K_1 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

which is an exact sequenceof right R-modules. But now we can find a free right R-module P_2 and a homomorphism with kernel K_2 mapping P_2 onto K_1. The above process can be iterated to produce the desired projective resolution of M. □

Lemma 2.75. *Let R be a ring and M be a right (left) R-module. Then M has an injective coresolution.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. By the classic result, M can be embedded in an injective right R-module I_0. Let C_0 be the cokernel of $M \longrightarrow I_0$, and map canonically I_0 onto C_0. Embed C_0 in an injective right R-module I_1, and let C_1 be the cokernel of the embedding map. We have the following sequence of right R-modules

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow C_0 \longrightarrow I_1 \longrightarrow C_1 \longrightarrow 0.$$

Composing the homomorphisms $I_0 \longrightarrow C_0$ and $C_0 \longrightarrow I_1$, we get

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow C_1 \longrightarrow 0$$

which is an exact sequence of right R-modules. Iterate to produce the desired injective coresolution of M. □

Definition 2.76. Let R be a ring and M be a right (left) R-module. Then M is said to have *projective dimension* at most n, denoted $\operatorname{proj\,dim} M \leq n$, if there is a projective resolution of the form $0 \longrightarrow P_n \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$. If n is the least, then we set $\operatorname{proj\,dim} M = n$ and if there is no such n, we set $\operatorname{proj\,dim} M = \infty$. The class of all right (left) R-modules of projective dimension at most n will be denoted $\mathcal{P}_n$, the class of all right (left) R-modules of finite projective dimension will be denoted $\mathcal{P}$.

Dually, M is said to have *injective dimension* at most n, denoted $\operatorname{inj\,dim} M \leq n$, if there is an injective coresolution of the form $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \dots \longrightarrow I_n \longrightarrow 0$. If n is the least, then we set $\operatorname{inj\,dim} M = n$ and if there is no such n, we set $\operatorname{inj\,dim} M = \infty$. The class of all right (left) R-modules of injective dimension at most n will be denoted $\mathcal{I}_n$, the class of all right (left) R-modules of finite injective dimension will be denoted $\mathcal{I}$.

Lemma 2.77. *Let R be a ring, M be a right (left) R-module and $0 \leq n < \omega$. Then the following are equivalent*

(i) $M \in \mathcal{P}_n$.

(ii) $Ext_R^{n+k}(M, N) = 0$ *for all right (left) R-modules N and every $k \geq 1$,*

(iii) $Ext_R^{n+1}(M, N) = 0$ *for all right (left) R-modules N,*

(iv) every n-th syzygy of M is projective.

Proof. This is a well-known fact which can be found in [10]. □

Lemma 2.78. *Let R be a ring, N be a right (left) R-module and $0 \leq n < \omega$. Then the following are equivalent*

(i) $N \in \mathcal{I}_n$.

(ii) $Ext_R^{n+k}(M, N) = 0$ *for all right (left) R-modules M and every $k \geq 1$,*

(iii) $Ext_R^{n+1}(M, N) = 0$ *for all right (left) R-modules M,*

(iv) every n-th cosyzygy of N is injective,

(v) $Ext_k^{n+1}(R/I, N) = 0$ *for all right (left) ideals I of R.*

Proof. This is a well-known fact which can be found in [10]. □

Lemma 2.79. *Let R be a ring, N be a left R-module and $0 \leq n < \omega$. Then the following are equivalent*

(i) $N \in \mathcal{F}_n$.

(ii) $Tor_R^{n+k}(M, N) = 0$ *for all right R-modules M and every $k \geq 1$,*

(iii) $Tor_R^{n+1}(M, N) = 0$ *for all right R-modules M,*

(iv) every n-th flat-syzygy of N is flat.

(v) $Tor_R^{n+1}(R/I, N)$ *for all right ideals I of R.*

And let M be a right R-module and $0 \leq n < \omega$. Then the following are equivalent

(i) $M \in \mathcal{F}_n$.

(ii) $Tor_R^{n+k}(M, N) = 0$ *for all left R-modules N and every $k \geq 1$,*

(iii) $Tor_R^{n+1}(M, N) = 0$ *for all left R-modules N,*

(iv) every n-th flat-syzygy of M is flat.

(v) $Tor_R^{n+1}(M, R/I)$ *for all left ideals I of R.*

Proof. This is a well-known fact which can be found in [10]. □

Lemma 2.80. *Let R be a ring, M, N be right (left) R-modules, $S_i \in \Omega^i(M)$ be an i-th syzygy of M in some projective resolution of M and $C_{-i} \in \Omega^{-i}(N)$ be an i-th cosyzygy of N in some injective coresolution of N. Then*

$$Ext_R^1(S_{i-1}, N) \simeq Ext_R^i(M, N) \simeq Ext_R^1(M, C_{-i+1})$$

as abelian groups for all $i \geq 1$.

Proof. This is a well-known fact called *dimension shifting*, which can be found in [11]. ☐

Definition 2.81. Let R be a ring and M be a right (left) R-module. Then M is called *strongly finitely presented* if M posses a projective resolution (finite or infinite) consisting of finitely generated right (left) R-modules. That is, there exists a long exact sequence (finite or infinite) of right (left) R-modules

$$\ldots \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with P_i projective and finitely generated for all $i \geq 0$.

The class of all strongly finitely presented right (left) R-modules is denoted by mod-R.

Lemma 2.82. *Let R be a ring, M be a right (left) R-module and κ be an infinite cardinal. If M posses a projective resolution consisting of $< \kappa$-generated projective right (left) R-modules, then M is $< \kappa$-presented.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Let

$$\ldots \longrightarrow P_1 \longrightarrow P_0 \stackrel{\pi}{\longrightarrow} M \longrightarrow 0$$

be a projective resolution (finite or infinite) of M with each $P_i < \kappa$-generated and projective. Denote by K the first syzygy of M in the previous projective resolution of M. Then by Lemma 2.43, there exists a right R-module M_0 such that $P_0 \oplus M_0$ is free and $< \kappa$-generated right R-module. It is easy to see that the following sequence of right R-modules

$$0 \longrightarrow K \oplus M_0 \stackrel{i \oplus \mathrm{id}_{M_0}}{\longrightarrow} P_0 \oplus M_0 \stackrel{\pi \oplus 0_{M_0}}{\longrightarrow} M \longrightarrow 0,$$

where i is an inclusion $0 \longrightarrow K \stackrel{i}{\longrightarrow} P_0$, is exact. So by Lemma 2.50, M is $< \kappa$-presented. ☐

Lemma 2.83. *Let R be a right (left) noetherian ring. Then mod-R is equal to the class of all finitely generated right (left) R-modules.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Let M be a strongly finitely presented right R-module. By Definition 2.81, we have the following long exact sequence of right R-modules

$$\ldots \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with P_i projective and finitely generated for all $i \geq 0$. Since P_0 is finitely generated, M is finitely generated.

Let M be a finitely generated right R-module. Let X be a finite generating subset of M. By Lemma 2.72, M is finitely presented, so in the following short exact sequence of right R-modules

$$0 \longrightarrow K \xrightarrow{\mu} R^{(X)} \longrightarrow M \longrightarrow 0 \tag{1}$$

K is finitely generated. If K is projective we are done and if K is not projective we can use previous arguments again but now for the finitely generated right R-module K and we get the following short exact sequence of right R-modules

$$0 \longrightarrow L \longrightarrow R^{(Y)} \xrightarrow{\pi} K \longrightarrow 0, \tag{2}$$

where Y is the finite generating subset of K and L is a finitely generated.

Composing (1) and (2) together we get the following long exact sequence of right R-modules

$$0 \longrightarrow L \longrightarrow R^{(Y)} \xrightarrow{\mu\circ\pi} R^{(X)} \longrightarrow M \longrightarrow 0.$$

If L is projective we are done and if L is not projective we can continue analogously and we get the projective resolution (finite or infinite) of M consisting of finitely generated projective right R-modules thus M is strongly finitely presented. $\square$

2.2 Commutative case

In this subsection we will prove some basic facts from the theory of modules over commutative rings.

Definition 2.84. Let R be a commutative ring. An ideal p of R is *prime* if the following two conditions hold

(i) $p \neq R$,

(ii) for all x,$y \in R$, if $xy \in p$ then $x \in p$ or $y \in p$.

The set of all prime ideals is denoted by $\operatorname{Spec} R$.

Definition 2.85. A commutative ring R is called an *integral domain* (or simply a *domain*) if $ab = 0$ implies $a = 0$ or $b = 0$.

An integral domain F is called a *field* if every non-zero element of F has an inverse under multiplication.

Lemma 2.86. *Let R be a commutative ring and p be an ideal of R. Then*

1. *p is prime iff the factor ring R/p is a domain,*

2. *p is maximal iff the factor ring R/m is a field.*

In particular, in any commutative ring, maximal ideals are prime.

Proof. This is a well-known fact which can be found in [6]. □

Definition 2.87. Let R be a commutative ring. The *height* (ht) of a prime ideal p of R is the supremum of the lenghts s of strictly decreasing chains $p = p_0 \supsetneq p_1 \supsetneq \cdots \supsetneq p_{s-1} \supsetneq p_s$ of prime ideals of R.

The *Krull dimension* of R, denoted dim R, is defined by

$$\dim R = \sup \{\operatorname{ht} p \mid p \in \operatorname{Spec} R\}.$$

It follows from the definition above that $\operatorname{ht} p + \dim R/p \leq \dim R$ and $\operatorname{ht} p = \dim R_p$.

If $\dim R = 0$, then every prime ideal of R is minimal, and if R is a principal ideal domain which is not a field, then $\dim R = 1$.

Definition 2.88. Let R be a domain. We construct a field F in which every non-zero element r of R has an inverse $1/r$, and further any element of f can be written in the form r/s for $r, s \in R$. The field F is called the *field of fractions* of R. The technique is exactly the same as that used to manufacture the rational numbers $\mathbb{Q}$ from the ring of integers $\mathbb{Z}$.

Let $\Sigma = R \setminus \{0\}$ be the set of non-zero elements in R. We introduce a relation $\sim$ on the set of pairs $(r, s) \in R \times \Sigma$ by stipulating that $(r, s) \sim (r', s')$ if and only if $rs' = r's$. It is easy to verify that this relation is an equivalence relation.

The fraction r/s is defined to be the equivalence class (r, s) under this relation and F is the set of equivalence classes; thus $r/s = r'/s'$ if and only if $rs' = r's$.

We define addition by

$$r/s + r'/s' = (rs' + r's)/ss',$$

and multiplication by

$$(r/s)(r'/s') = rr'/ss'.$$

Another routine check shows that these rules are well-defined and make F into a ring with zero element $0/1$ and identity $1/1$.

Furthermore, $r/1 = 0$ only if $r = 0$, so that we can identify R as the subring of F consisting of all elements of the form $r/1$.

Then the identity $r/r = 1/1$ holds for all non-zero r in R, which confirms that r has an inverse in F, and it is easy to see that F is a field.

Definition 2.89. Let R be a commutative ring. The subset S of R is called *multiplicative* in case

(i) $0 \notin S$,

(ii) S is closed under multiplication.

Definition 2.90. Let R be a commutative ring and S be a multiplicative subset of R. Then the *localization* of R with respect to S, denoted $S^{-1}R$, is the set of all equivalence classes (r, s) with $r \in R$, $s \in S$ under equivalence relation $(r, s) \sim (r', s')$ if there is an $t \in S$ such that $(rs' - r's)t = 0$. It is easy to check that this relation is indeed an equivalence relation. The equivalence class (r, s) is denoted by r/s.

We now define addition and multiplication on $S^{-1}R$ by

$$\begin{aligned} r/s + r'/s' &= (rs' + r's)/ss' \\ (r/s)(r'/s') &= rr'/ss'. \end{aligned}$$

These operations are well-defined and $S^{-1}R$ is then a commutative ring with identity $1/1$.

Remark 2.91. The mapping $\varphi\colon R \to S^{-1}R$ defined by $\varphi(r) = r/1$ is a ring hommomorphism with $\operatorname{Ker} \varphi = \{r \in R \mid rs = 0 \text{ for some } s \in S\}$. As a consequence, φ is injective iff S is without zero-divisors. Moreover, if R is a domain and S is the set of all non-zero elements of R, then $S^{-1}R$ is the field of fractions of R.

Definition 2.92. Let R be a commutative ring and T be ring. Then T is said to be an R-*algebra* if there is a ring homomorphism $\varphi\colon R \to T$. It is easy to see that T is an R-module via $tr = t\varphi(r)$ for all $r \in R$ and $t \in T$. For example every ring is a $\mathbb{Z}$-algebra and we have just seen that if R is a commutative ring, then $S^{-1}R$ is an R-algebra for every multiplicative subset S of R.

Remark 2.93. Let R be a commutative ring, S be a multiplicative subset of R and J be an ideal of $S^{-1}R$. Define a set $J \cap R$ as an inverse image of J under the mapping φ from 2.91. Then $J \cap R$ is an ideal of R, moreover if J is prime then $J \cap R$ is also such.

Definition 2.94. Let R be a commutative ring, S be a multiplicative subset of R and M be an R-module. Then the *localization* of M with respect to S, denoted $S^{-1}M$, is defined as for $S^{-1}R$. $S^{-1}M$ is an abelian group under addition and is an $S^{-1}R$-module via $(r/s) \cdot (m/s') = rm/ss'$.

Remark 2.95. Let R be a commutative ring and S be a multiplicative subset of R. We note that an $S^{-1}R$-module N is also an R-module via $r \cdot n = (r/1) \cdot n$. In the following, the R-module structure on some $S^{-1}R$-module will always mean this R-module structure.

Lemma 2.96. *Let R be a commutative ring, S be a multiplicative subset of R and M,N be $S^{-1}R$-modules. Then $\varphi\colon M \to N$ is an $S^{-1}R$-module homomorphism iff φ is an R-module homomorphism.*

Proof. Every $S^{-1}R$-module homomorphism is clearly an R-module homomorphism.

Let $\varphi : M \to N$ be an R-module homomorphism. We need to prove that $\varphi((r/s)m) = (r/s)\varphi(m)$, for every $r \in R$, $s \in S$. But $\varphi((r/s)m) = \varphi(r(1/s)m) = r\varphi((1/s)m) = r(s/s)\varphi((1/s)m) = (r/s)s\varphi((1/s)m) = (r/s)\varphi(m)$. So the claim is true. $\square$

Lemma 2.97. *Let R be a commutative ring and S be a multiplicative subset of R. Then*

1. *If $f : M \to N$ is an R-module homomorphism, then $S^{-1}f : S^{-1}M \to S^{-1}N$ defined by $(S^{-1}f)(m/s) = f(m)/s$ is an $S^{-1}R$-module homomorphism.*

2. *If $M' \longrightarrow M \longrightarrow M''$ is a sequence of R-modules which is exact at M, then $S^{-1}M' \longrightarrow S^{-1}M \longrightarrow S^{-1}M''$ is a sequence of $S^{-1}R$-modules which is exact at $S^{-1}M$.*

3. *If $N \subseteq M$ are R-modules, then $S^{-1}(M/N) \simeq S^{-1}(M)/S^{-1}(N)$.*

4. *If M is an R-module, then $S^{-1}R \otimes_R M \simeq S^{-1}M$ as $S^{-1}R$-modules.*

5. *$S^{-1}R$ is a flat R-module.*

Proof. (1) and (2) are easy.

(3) follows from (2) by considering the short exact sequence of R-modules $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$.

For (4) define a map $\varphi : S^{-1}R \otimes_R M \to S^{-1}M$ by $\varphi(r/s \otimes m) = (rm)/s$. Then φ is well-defined $S^{-1}R$-homomorphism and φ is clearly onto. Now suppose $(rm)/s = 0$. Then there is an $s' \in S$ such that $rs'm = 0$. So $(r/s) \otimes m = (rs'/ss') \otimes m = (1/ss') \otimes rs'm = 0$. Thus φ is one-to-one.

(5) follows from parts (2) and (4). $\square$

Lemma 2.98. *Let R be a commutative ring, S_1 be a multiplicative subset of R and M,N be $S_1^{-1}R$-modules. Then*

1. $$M \otimes_{S_1^{-1}R} N \simeq M \otimes_R N$$
   as $S_1^{-1}R$-modules,

2. *if moreover, $S_2 \subseteq S_1$ is a multiplicative subset of R, then*

   (a) M is an $S_2^{-1}R$-module via restriction of the scalar multiplication on $S_2^{-1}R$,

   (b) $S_2^{-1}M \simeq M$ as $S_2^{-1}R$-modules.

Proof. (1). This follows from the fact that in $M \otimes_R N$ we have

$$((r/s)m) \otimes n = (rm/s) \otimes (sn/s) = (sm/s) \otimes (rn/s) = m \otimes ((r/s)n)$$

for any $m \in M$, $n \in N$, $r \in R$ and $s \in S_1$.

(2)(a). This is easy.

(2)(b). By (1) and (2)(a), we have

$$S_2^{-1}M \simeq M \otimes_R S_2^{-1}R \simeq M \otimes_{S_2^{-1}R} S_2^{-1}R \simeq M$$

as $S_2^{-1}R$-modules. □

Lemma 2.99. *Let R be a commutative ring, S be a multiplicative subset of R and M, N be R-modules. Then*

$$S^{-1}(M \otimes_R N) \simeq S^{-1}M \otimes_{S^{-1}R} S^{-1}N$$

as $S^{-1}R$-modules.

Proof. By Lemmas 2.97, 2.47 and 2.98 we have

$$\begin{aligned} S^{-1}(M \otimes_R N) &\simeq S^{-1}R \otimes_R (M \otimes_R N) \simeq (S^{-1}R \otimes_R M) \otimes_R N \simeq S^{-1}M \otimes_R N \simeq \\ &\simeq (S^{-1}M \otimes_{S^{-1}R} S^{-1}R) \otimes_R N \simeq (S^{-1}M \otimes_R S^{-1}R) \otimes_R N \simeq \\ &\simeq S^{-1}M \otimes_R (S^{-1}R \otimes_R N) \simeq S^{-1}M \otimes_R S^{-1}N \simeq \\ &\simeq S^{-1}M \otimes_{S^{-1}R} S^{-1}N. \end{aligned}$$

So the claim is true. □

Lemma 2.100. *Let R be a commutative ring, S be a multiplicative subset of R and M be an R-module. Then*

1. *if M is finitely generated, then $S^{-1}M$ is a finitely generated $S^{-1}R$-module,*
2. *if M is free, then $S^{-1}M$ is a free $S^{-1}R$-module,*
3. *if M is projective, then $S^{-1}M$ is a projective $S^{-1}R$-module.*

Proof. (1) is easy.

Let X be a free basis of M. Then $\overline{X} = \{x/1 \mid x \in X\}$ is clearly a generating subset of an $S^{-1}R$-module $S^{-1}M$. Let N be an arbitrary $S^{-1}R$-module and let $f\colon \overline{X} \to N$ be an arbitrary mapping. Define a mapping $g\colon X \to N$ by $g(x) = f(x/1)$. Since M is a free R-module, there is an R-module homomorphism $\varphi : M \to N$ which extends g. By Lemmas 2.97 and 2.98, $S^{-1}\varphi\colon S^{-1}M \to N$ is an $S^{-1}R$-module homomorphism which extends f. So $S^{-1}M$ is a free $S^{-1}R$-module.

(3) follows from Lemma 2.97 and (2) using the fact that M is a projective R-module iff M is a direct summand of a free R-module. □

Lemma 2.101. *Let R be a commutative ring, S be a multiplicative subset of R and M, N be R-modules. Then*

$$S^{-1}(Tor_R^1(M,N)) \simeq Tor_{S^{-1}R}^1(S^{-1}M, S^{-1}N)$$

as $S^{-1}R$-modules.

Proof. Consider a short exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 0$$

where F is a free (hence projective) R-module and K is the kernel of φ. Applying $-\otimes_R N$ and using Lemmas 2.57 and 2.79, we get the following exact sequence of R-modules

$$0 \longrightarrow \mathrm{Tor}_R^1(M,N) \longrightarrow K \otimes_R N \longrightarrow F \otimes_R N \longrightarrow M \otimes_R N \longrightarrow 0.$$

Applying $-\otimes_R S^{-1}R$ and using Lemma 2.97, we get the following exact sequence of R-modules

$$0 \longrightarrow S^{-1}(\mathrm{Tor}_R^1(M,N)) \longrightarrow K \otimes_R S^{-1}N \longrightarrow F \otimes_R S^{-1}N \longrightarrow M \otimes_R S^{-1}N \longrightarrow 0.$$

Applying $S^{-1}R \otimes_R -$ and using Lemmas 2.98, 2.47 and 2.96, we get the following exact sequence of $S^{-1}R$-modules

$$\begin{aligned} 0 \quad &\longrightarrow \quad S^{-1}(\mathrm{Tor}_R^1(M,N)) \longrightarrow S^{-1}K \otimes_{S^{-1}R} S^{-1}N \longrightarrow S^{-1}F \otimes_{S^{-1}R} S^{-1}N \longrightarrow \\ &\longrightarrow \quad S^{-1}M \otimes_{S^{-1}R} S^{-1}N \longrightarrow 0. \end{aligned}$$

Using Lemma 2.100, it is now easy to see that we have $S^{-1}(\mathrm{Tor}_R^1(M,N)) \simeq \mathrm{Tor}_{S^{-1}R}^1(S^{-1}M, S^{-1}N)$ as $S^{-1}R$-modules. □

Lemma 2.102. *Let R be a commutative ring and S be a multiplicative subset of R. If J is an ideal of $S^{-1}R$, then $J = IS^{-1}R \simeq S^{-1}I$, where $I = J \cap R$ is an ideal of R and the previous isomorphism is an isomorphism of $S^{-1}R$-modules.*

Proof. I is an ideal of R by Remark 2.93. Clearly $IS^{-1}R \subseteq J$. Now let $a = r/s \in J$. Then $a = (r/1)(1/s)$. So it suffices to show that $r \in I$. For then $a \in IS^{-1}R$. But $r/1 = a(1/s) \in J$ and so $r \in J \cap R = I$. Thus $J = IS^{-1}R$. But from Lemma 2.58 it easily follows that $IS^{-1}R \simeq S^{-1}R \otimes_R I$ as R-modules since by Lemma 2.97, $S^{-1}R$ is a flat R-module. By Lemma 2.96, $IS^{-1}R \simeq S^{-1}R \otimes_R I$ as $S^{-1}R$-modules. And hence, by Lemma 2.97, $IS^{-1}R \simeq S^{-1}I$ as $S^{-1}R$-modules. So the claim is true. □

Definition 2.103. Let R be a commutative ring and let $p \in \operatorname{Spec} R$. Then $S = R \backslash p$ is a multiplicative subset of R. In this case $S^{-1}R$, $S^{-1}M$ and $S^{-1}f$ are denoted by $R_{(p)}$, $M_{(p)}$, $f_{(p)}$ respectively, where M is an R-module and f is an R-module homomorphism. We say that $M_{(p)}$ is the *localization* of M at p.

Lemma 2.104. *Let R be a commutative ring and S be a multiplicative subset of R. Then there is ono-to-one inclusion-order preserving correspondence between the prime ideals of $S^{-1}R$ and the prime ideals of R disjoint from S given by $S^{-1}p \leftrightarrow p$.*

Proof. Let J be a prime ideal of $S^{-1}R$, and let $p = J \cap R$. Then p is a prime ideal of R by Remark 2.93. But then $J = S^{-1}p$ by Lemma 2.102. If $p \cap S \neq \emptyset$, then $1/1 \in S^{-1}p = J$, a contradiction. Hence $p \cap S = \emptyset$.

Now suppose p is a prime ideal of R disjoint from S. We claim that $S^{-1}p$ is a prime ideal of $S^{-1}R$. But $1 \notin S^{-1}p$ since $p \cap S = \emptyset$. Moreover if $(a/s) \cdot (b/t) \in S^{-1}p$ with $s, t \in S$, then $(a/s) \cdot (b/t) = c/r$ for some $c \in p$, $r \in S$. So there is an $s' \in S$ such that $(abr - stc)s' = 0$. But $stcs' \in p$. So $abrs' \in p$ where $rs' \in S$. But then $ab \in p$ and so $a \in p$ or $b \in p$. That is, $a/s \in S^{-1}p$ or $b/s \in S^{-1}p$. Hence $S^{-1}p$ is a prime ideal of $S^{-1}R$. The rest is easy. □

Theorem 2.105. *Let R be a commutative ring and let $p \in \operatorname{Spec} R$. Then there is ono-to-one inclusion-order preserving correspondence between the prime ideals of $R_{(p)}$ and the prime ideals of R contained in p.*

Proof. This follows from Lemma 2.104. □

Remark 2.106. Let R be a commutative ring and let $p \in \operatorname{Spec} R$. Then pR_p is a prime ideal of $R_{(p)}$ from the above. But if J is an ideal of $R_{(p)}$, then $J = IR_p$ where I is an ideal of R such that $I \cap (R \setminus p) = \emptyset$. So $I \subseteq p$ and hence $J = IR_{(p)} \subseteq pR_{(p)}$. Thus $pR_{(p)}$ is the maximal ideal of $R_{(p)}$, moreover it is the only one of $R_{(p)}$. So the localization of a commutative ring R at a prime ideal p is a local commutative ring with maximal ideal pR_p. The field $R_{(p)}/pR_{(p)}$ is called the *residue field* of $R_{(p)}$ and it is denoted by $k(p)$.

Definition 2.107. Let R be a commutative ring and M be an R-module. Then a prime ideal p of R is said to be an *associated prime ideal* of M if $p = \operatorname{Ann}(m)$ for some $m \in M$. It is easy to see that this is equivalent to M containing a cyclic submodule isomorphic to R/p. The set of all asociated prime ideals of M is denoted by $\operatorname{Ass}(M)$.

Lemma 2.108. *Let R be a noetherian commutative ring and M be an R-module. Then $M = 0$ iff $\mathit{Ass}(M) = \emptyset$.*

Proof. If $M = 0$ then clearly $\operatorname{Ass}(M) = \emptyset$.

Let $M \neq 0$ and $0 \neq m \in M$. If $\operatorname{Ann}(m)$ is a prime ideal of R, we are through. If not, let $rs \in \operatorname{Ann}(m)$ with $r, s \notin \operatorname{Ann}(m)$. Then $rm \neq 0$ and $s \in \operatorname{Ann}(rm)$. So $\operatorname{Ann}(m) \subsetneq \operatorname{Ann}(rm)$. If $\operatorname{Ann}(rm)$ is not a prime ideal of R then we can repeat the procedure. If the procedure did not stop we would contradict the fact that R is noetherian. Hence the procedure stops and we see that $\operatorname{Ass}(M) \neq \emptyset$. □

Lemma 2.109. *Let R be a noetherian commutative ring and M be a non-zero finitely generated R-module. Then there exists a chain $0 = M_0 \subsetneq M_1 \subsetneq \ldots \cdots \subsetneq M_{n-1} \subsetneq M_n = M$ of submodules of M such that for each $1 \leq i \leq n$, $M_i/M_{i-1} \simeq R/p_i$ for some $p_i \in \mathit{Spec}\, R$.*

Proof. Let $p_1 \in \mathrm{Ass}(M)$ (see Lemma 2.108). Then there is a submodule M_1 of M such that $M_1 \simeq R/p_1$. If $M_1 = M$, then we are done. Otherwise let $p_2 \in \mathrm{Ass}(M/M_1)$. Then there is a submodule M_2 of M containing M_1 such that $M_2/M_1 \simeq R/p_2$. One then repeats this procedure to get the required submodules noting that the process stops since M is noetherian. □

Lemma 2.110. *Let R be a noetherian commutative ring, M be an R-module and p be a prime ideal of R. Then $p \in \mathit{Ass}(M)$ iff $pR_{(p)} \in \mathit{Ass}_{R_{(p)}}(M_{(p)})$.*

Proof. If $p \in \mathrm{Ass}(M)$, then $R/p \simeq Rm$ for some $m \in M$, $m \neq 0$. So R/p is isomorphic to a submodule of M. Thus $R_{(p)}/pR_{(p)}$ is isomorphic to a submodule of $M_{(p)}$. Hence $pR_{(p)} \in \mathrm{Ass}_{R_{(p)}}(M_{(p)})$.

If $pR_{(p)} \in \mathrm{Ass}_{R_{(p)}}(M_{(p)})$, then $pR_{(p)} = \mathrm{Ann}_{R_{(p)}}(m/s)$ where $m/s \in M_{(p)}$ for some $m \in M$ and $s \in R \setminus p$. Since p is finitely generated, let $p =< a_1, a_2, \ldots, a_n >$. Then $(a_i/1)(m/s) = 0$ for each i. So there is an $r_i \in R \setminus p$ such that $r_i a_i m = 0$ for each i. Now set $r = r_1 r_2 \ldots r_n$. Then $ram = 0$ for all $a \in p$. Thus $p \subseteq \mathrm{Ann}_R(rm)$. If $t \in \mathrm{Ann}_R(rm)$, then $trm = 0$ and so $(t/1)(m/s) = 0$. But then $t/1 \in pR_{(p)}$. Consequently $t \in p$. Thus $\mathrm{Ann}_R(rm) \subseteq p$. Hence $p = \mathrm{Ann}_R(rm)$ and so $p \in \mathrm{Ass}_R(M)$. □

Definition 2.111. Let R be a commutative ring. The *support* of an R-module M, denoted $\mathrm{Supp}(M)$, is the set of all prime ideals of R such that $M_{(p)} \neq 0$.

Lemma 2.112. *Let R be a commutative ring and M be an R-module. Then $M = 0$ iff $\mathit{Supp}(M) = 0$ (moreover, $M = 0$ iff $\mathit{Supp}(M) \cap \mathit{mSpec}\, R = 0$).*

Proof. If $M = 0$ then obviously $\mathrm{Supp}(M) = 0$.

If $M \neq 0$, let $m \in M$, $m \neq 0$, then $\mathrm{Ann}(m) \subseteq p$ for p maximal ideal of R. Obviously p is also a prime ideal of R. But $m/1 \neq 0$ in $M_{(p)}$ and so $p \in \mathrm{Supp}(M)$. Thus $\mathrm{Supp}(M) \neq 0$. The 'moreover' part follows from the previous part of the proof. □

Lemma 2.113. *Let R be a noetherian commutative ring and M be an R-module. Then*

1. *$\mathit{Ass}(M) \subseteq \mathit{Supp}(M)$,*
2. *if p is an inclusion-minimal element in $\mathit{Supp}(M)$, then $p \in \mathit{Ass}(M)$.*

Proof. (1). If $p \in \text{Ass}(M)$, then $pR_{(p)} \in \text{Ass}_{R_{(p)}}(M_{(p)})$ by Lemma 2.110. So $R_{(p)}/pR_{(p)}$ is isomorphic to a submodule of $M_{(p)}$. Hence $M_{(p)} \neq 0$ and so $p \in \text{Supp}(M)$. Thus $\text{Ass}(M) \subseteq \text{Supp}(M)$.

(2). Let p be a minimal element in $\text{Supp}(M)$. By Lemma 2.110, it suffices to prove the result for a local noetherian commutative ring R with maximal ideal p and a non-zero R-module M (note that a localization of a noetherian ring is clearly a noetherian ring). Since p is minimal, we further assume that $M_{(q)} = 0$ for all prime ideals q contained in p. So $\text{Supp}(M) = \{p\}$. But $\text{Ass}(M) \subseteq \text{Supp}(M)$ by (1). So $p \in \text{Ass}(M)$ since $\text{Ass}(M) \neq \emptyset$. □

Lemma 2.114. *Let R be a commutative ring and M be a finitely generated R-module. Then $Supp(M) = \{p \in Spec\, R \mid Ann(M) \subseteq p\}$.*

Proof. If $M = m_1R + m_2R + \cdots + m_nR$ for some $m_1, m_2, \ldots, m_n \in M$, then $p \in \text{Supp}(M)$ iff there is an i such that $m_i/1 \neq 0$ in $M_{(p)}$. But this means that there is an i such that $\text{Ann}(m_i) \subseteq p$. But this helds iff $\text{Ann}(M) = \bigcap_{i=1}^n \text{Ann}(m_i) \subseteq p$. □

Definition 2.115. Let R be a commutative ring and I be an ideal of R. Then the *radical* of I, denoted $\sqrt{I}$, is defined by $\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n > 0\}$. We note that $I \subseteq \sqrt{I}$. If $I = 0$, then $\sqrt{I}$ is called the *nilradical*. It is easy to see that the nilradical is the set of all nilpotent elements of R.

Lemma 2.116. *Let R be a commutative ring and I be an ideal of R. Then $\sqrt{I}$ is the intersection of all prime ideals p of R containing I, i.e. $\sqrt{I} = \bigcap_{I \subseteq p} p$.*

Proof. Let p be a prime ideal containing I. If $r \in \sqrt{I}$, then $r^n \in I \subseteq p$ and so $r \in p$. Hence $\sqrt{I} \subseteq \bigcap_{I \subseteq p} p$.

Now let $r \notin \sqrt{I}$. Then $r^n \notin I$ for each $n \geq 0$. So $S = \{1, r, r^2, \ldots\}$ is a multiplicative subset of R disjoint from I. Then the set of all ideals J such that $I \subseteq J$ and $J \cap S = \emptyset$ has a maximal element q by the Zorn's Lemma. We claim that q is a prime ideal. We first note that if $s \notin q$, then $(q+sR) \cap S \neq \emptyset$ for otherwise $q+sR$ would contradict the maximality of q. So $s \in q$ iff $(q+sR) \cap S = \emptyset$. Thus $s_1 \notin q$, $s_2 \notin q$ implies that $(q+s_iR) \cap S \neq \emptyset$. Since S is multiplicative, $((q+s_1R)(q+s_2R)) \cap S \neq \emptyset$. But $(q + s_1R)(q + s_2R) \subseteq (q + s_1s_2R)$. So $(q + s_1s_2R) \cap S \neq \emptyset$ and thus $s_1s_2 \notin q$. So q is a prime ideal of R. Hence $r \notin \bigcap_{I \subseteq p} p$. Thus $\sqrt{I} = \bigcap_{I \subseteq p} p$. □

Corollary 2.117. *Let R be a commutative ring. Then the set of all nilpotent elements of R is the intersection of all prime ideals of R.*

Proof. This follows from Definition 2.115 and Lemma 2.116. □

Lemma 2.118. *Let R be a noetherian commutative ring and I be an ideal of R. Then $(\sqrt{I})^n \subseteq I$ for some $n > 0$.*

Proof. Since R is noetherian, let $\sqrt{I} =< r_1, r_2, \ldots, r_s >$. Then $r_i^{n_i} \in I$ for some $n_i > 0$. Let $n = (n_1 - 1) + (n_2 - 1) + \cdots + (n_s - 1) + 1$. Then $(\sqrt{I})^n$ is generated by monomials $r_1^{m_1} r_2^{m_2} \ldots r_s^{m_s}$ where $\sum_{i=1}^{s} m_i = n$ and $m_i \geq n_i$ for some i. Thus $r_1^{m_1} r_2^{m_2} \ldots r_s^{m_s} \in I$ and so $(\sqrt{I})^n \subseteq I$. □

Definition 2.119. Let R be a ring and M be a right (left) R-module. Then M is *indecomposable* if there are no non-zero submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$.

Lemma 2.120. *Let R be a commutative ring and M be an injective R-module. Then M is indecomposable iff it is the injective envelope of each of its non-zero submodules.*

Proof. Let N be a non-zero submodule of M. Then $M \simeq E(N) \oplus N'$ for some R-module N'. Thus $N' = 0$ since M is indecomposable. Conversely, suppose $M = M_1 \oplus M_2$. If $M_1 \neq 0$, then $M_1 \subseteq M$ is an essential extension by assumption. But $M_1 \cap M_2 = 0$. So $M_2 = 0$ and we are done. □

Lemma 2.121. *Let R be a commutative ring and $p, q \in Spec\, R$. Then*

1. *$E(R/p)$ is indecomposable R-module,*
2. *if $s \in R \backslash p$, then the mapping multiplication by s is an R-module automorphism on $E(R/p)$,*
3. *$E(R/p) \simeq E(R/q)$ iff $p = q$,*
4. *$Ass(E(R/p)) = \{p\}$,*
5. *$E(R/p)$ is an $R_{(p)}$-module and it is an injective hull of $(R/p)_{(p)} = R_{(p)}/(pR_{(p)})$, that is*

$$E_R(R/p) = E_{R_{(p)}}(R_{(p)}/(pR_{(p)})).$$

Proof. (1). Suppose there are non-zero submodules E_1 and E_2 of $E(R/p)$ such that $E(R/p) = E_1 \oplus E_2$. Then $E_i \cap R/p \neq 0$ for $i = 1, 2$ since $R/p \subseteq E(R/p)$ is an essential extension. So let $x_i \in E_i \cap R/p$ be a non-zero elements. $(E_i \cap R/p)$, $i = 1, 2$ are ideals of R/p, thus $x_1 x_2 \in (E_1 \cap R/p) \cap (E_2 \cap R/p)$. But $(E_1 \cap R/p) \cap (E_2 \cap R/p) = 0$. So $x_1 x_2$ are non-zero elements in R/p such that $x_1 x_2 = 0$. This contradicts the fact that R/p is a domain (see Lemma 2.86). Hence $E(R/p)$ is indecomposable.

(2). Let $\varphi \colon E(R/p) \to E(R/p)$ be the mapping multiplication by s. Since p is a prime ideal, φ is injective on (R/p). So Ker $\varphi \cap (R/p) = 0$. But $(R/p) \subseteq E(R/p)$ is an essential extension. So Ker $\varphi = 0$ and thus φ is injective. But then $\varphi(E(R/p))$ is an injective submodule of $E(R/p)$, thus $\varphi(E(R/p))$ is a direct summand of $E(R/p)$. So φ is an automorphism since $E(R/p)$ is indecomposable by (1).

(3). Suppose $p \neq q$. Let $p \nsubseteq q$. Then the mapping multiplication by $s \in p \setminus q$ is an automorphism on $E(R/q)$ but clearly not on $E(R/p)$. So $E(R/p) \not\simeq E(R/q)$.

(4). First, $R/p \subseteq E(R/p)$, thus $p \in \mathrm{Ass}(E(R/p))$. Let $q \in \mathrm{Ass}(E(R/p))$, then R/q is isomorphic to a submodule of $E(R/p)$ and since $E(R/p)$ is indecomposable by (1), we have that $E(R/q) \simeq E(R/p)$. Hence $p = q$ by (3).

(5). For each $s \in R \setminus p$ denote $\varphi_s \colon E(R/p) \to E(R/p)$ mapping multiplication by s. Then by (2), $E(R/p)$ is an $R_{(p)}$-module via $m(r/s) = \varphi_s^{-1}(mr)$, where $r \in R$, $s \in R \setminus p$. Using Lemma 2.98, it is now easy to see that $E(R/p) \supseteq (R/p)_{(p)}$. Since $E(R/p)$ is an essential extension of (R/p) and $E(R/p) \supseteq (R/p)_{(p)} \supseteq (R/p)$, $E(R/p)$ is also an essential extension of an $R_{(p)}$-module $(R/p)_{(p)}$. And since $E(R/p)$ is injective as R-module, it is also injective as $R_{(p)}$-module by Lemma 2.96. Thus $E(R/p)$ is an injective hull of an $R_{(p)}$-module $(R/p)_{(p)}$. □

Lemma 2.122. *Let R be a noetherian commutative ring and $p, q \in \mathrm{Spec}\, R$. Then*

1. *if $m \in E(R/p)$, then there exists an $n > 0$ such that $mp^n = 0$,*
2. *$\mathrm{Hom}_R(E(R/p), E(R/q)) \neq 0$ iff $p \subseteq q$,*
3. *if S is a multiplicative subset of R, then*
   (a) *if $S \cap p = \emptyset$ then $E(R/p)$ is an $S^{-1}R$-module,*
   (b)
$$S^{-1}E(R/p) \simeq \begin{cases} E(R/p), & \text{if } S \cap p = \emptyset \\ 0, & \text{if } S \cap p \neq \emptyset \end{cases}$$
   as $S^{-1}R$-modules.

Proof. (1). Let $m \in E(R/p)$, $m \neq 0$. Then $mR \simeq R/\mathrm{Ann}(m)$. But $\mathrm{Ass}(E(R/p)) = \{p\}$ by 2.121. So $\mathrm{Ass}(mR) = \{p\}$ since $\mathrm{Ass}(mR) \neq \emptyset$. But then by Lemma 2.113, p is the unique minimal element in $\mathrm{Supp}(mR)$. But $\mathrm{Supp}(mR) = \{p \in \mathrm{Spec}\, R \mid \mathrm{Ann}(mR) \subseteq p\}$ by Lemma 2.114. Hence p is the radical of $\mathrm{Ann}(m)$ (note that every ideal of R is finitely generated). By Lemma 2.118, we have $p^n = \left(\sqrt{\mathrm{Ann}(m)}\right)^n \subseteq \mathrm{Ann}(m)$. So (1) is true.

(2). If $p \subseteq q$, then we have a homomorphism $R/p \xrightarrow{\varphi} R/q$ induced by the inclusion $p \subseteq q$. Now embed R/q into $E(R/q)$. Then the composition of φ and the inclusion $R/q \subseteq E(R/q)$ can be extended to a non-zero homomorphism in $\mathrm{Hom}_R(E(R/p), E(R/q))$ since $E(R/q)$ is injective.

Now let $\varphi \in \mathrm{Hom}_R(E(R/p), E(R/q))$ be non-zero. Then let $m \in E(R/p)$ be such that $\varphi(m) \neq 0$. If $r \in p$, then $r^n m = 0$ for some $n > 0$ by (1) above. So $r^n \in \mathrm{Ann}(m)$. But by 2.121, $\mathrm{Ass}(\varphi(m)R) = \{q\}$ thus there is an $s \in R$ such that $\mathrm{Ann}(\varphi(m)s) = q$, it implies that $\mathrm{Ann}(\varphi(m)) \subseteq \mathrm{Ann}(\varphi(m)s) = q$. Therefore $\mathrm{Ann}(m) \subseteq \mathrm{Ann}(\varphi(m)) \subseteq q$. So $r^n \in q$ and thus $r \in q$. Hence $p \subseteq q$.

(3)(a). If $S \cap p = \emptyset$, then for each $s \in R \setminus p$ denote $\varphi_s : E(R/p) \to E(R/p)$ mapping multiplication by s. Then by 2.121 (2), $E(R/p)$ is an $S^{-1}R$-module via $m(r/s) = \varphi_s^{-1}(mr)$ where $r \in R$, $s \in S$.

(3)(b). If $S \cap p = \emptyset$, then using Lemma 2.98, we have that $S^{-1}E(R/p) \simeq E(R/p)$ as $S^{-1}R$-modules.

If $S \cap p \neq \emptyset$, then let $s \in S \cap p$, by (1) above we have that for each $m \in E(R/p)$ there is an $n > 0$ such that $ms^n = 0$. Thus for each $m/s' \in S^{-1}E(R/p)$ we have that $m/s' = (m/s')(s^n/s^n) = (ms^n)/(s's^n) = 0$. So (3) is true. □

Theorem 2.123. *Let R be a commutative noetherian ring. Then*

1. *if E is an indecomposable injective R-module, then $E \simeq E(R/p)$ for some $p \in \mathrm{Spec}\, R$,*

2. *every injective R-module E is a direct summand of indecomposables R-modules. This decomposition is unique in the sense that for each $p \in \mathrm{Spec}\, R$, the number of summands isomorphic to $E(R/p)$ depends only on p and E.*

Proof. (1). Let $p \in \mathrm{Ass}(E)$ (see Lemma 2.108). Then R/p is isomorphic to a submodule of E. Thus $E \simeq E(R/p)$ by Lemma 2.120.

(2). This is part of the Theorem 3.3.10. from [10]. □

Lemma 2.124. *Let R be a commutative ring and F be an R-module. Then F is flat iff F_p is flat as $R_{(p)}$-module for all $p \in \mathrm{Spec}\, R$ (moreover, F is flat iff F_p is flat as $R_{(p)}$-module for all $p \in \mathrm{mSpec}\, R$).*

Proof. Let F be flat and let $p \in \mathrm{Spec}\, R$. Let $A \longrightarrow B$ be an injective $R_{(p)}$-module homomorphism, by Lemma 2.96, it is also an injective R-module homomorphism. Since F is a flat R-module, the induced R-module homomorphism $A \otimes_R F \longrightarrow B \otimes_R F$ is injective and since by Lemma 2.97, $R_{(p)}$ is a flat R-module, the induced R-module homomorphism $A \otimes_R F \otimes_R R_{(p)} \longrightarrow B \otimes_R F \otimes_R R_{(p)}$ is also injective. Now using the fact that $A \otimes_R F \otimes_R R_{(p)} \simeq A \otimes_R F_{(p)} \simeq A \otimes_{R_{(p)}} F_{(p)}$ as $R_{(p)}$-modules and analogously $B \otimes_R F \otimes_R R_{(p)} \simeq B \otimes_{R_{(p)}} F_{(p)}$ as $R_{(p)}$-modules (see Lemmas 2.97 and 2.98), it is easy to see that the induced $R_{(p)}$-module homomorphism $A \otimes_{R_{(p)}} F_{(p)} \longrightarrow B \otimes_{R_{(p)}} F_{(p)}$ is injective, so $F_{(p)}$ is a flat $R_{(p)}$-module.

Let $F_{(p)}$ be a flat $R_{(p)}$-module for all $p \in \mathrm{mSpec}\, R$. Let $A \longrightarrow B$ be an injective R-module homomorphism. Denote K the kernel of the induced R-module homomorphism $A \otimes_R F \longrightarrow B \otimes_R F$. So the sequence $0 \longrightarrow K \longrightarrow A \otimes_R F \longrightarrow B \otimes_R F$

is the exact sequence of R-modules. By Lemma 2.97, the following sequence of $R_{(p)}$-modules

$$0 \longrightarrow K_{(p)} \longrightarrow (A \otimes_R F)_{(p)} \longrightarrow (B \otimes_R F)_{(p)}$$

is exact. Since by Lemma 2.99, $(A \otimes_R F)_{(p)} \simeq A_{(p)} \otimes_{R_{(p)}} F_{(p)}$ as $R_{(p)}$-modules and $(B \otimes_R F)_{(p)} \simeq B_{(p)} \otimes_{R_{(p)}} F_{(p)}$ as $R_{(p)}$-modules, it follows that $K_{(p)} = 0$. Thus $K_{(p)} = 0$ for all $p \in \operatorname{mSpec} R$, so $K = 0$ by Lemma 2.112. So the claim is true. $\square$

Definition 2.125. A domain R is called a *valuation domain* if the set of all ideals of R form a chain under inclusion.

Lemma 2.126. *Let R be a valuation domain and I be a finitely generated ideal of R. Then I is a principal ideal.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be the generating subset of I. Since the set of all ideals of R form a chain under inclusion, there exists $k \in \{1, 2, \ldots, n\}$ such that $x_k R \supseteq \bigcup_{i=1}^{i=n} x_i R$. But then we have $I = x_n R$, thus I is principal. $\square$

3 Tilting modules

Definition 3.1. Let R be a ring and $\mathcal{C}$ be a class of right R-modules. We define a *right orthogonal class* of $\mathcal{C}$, denoted $\mathcal{C}^{\perp_1}$, as

$$\mathcal{C}^{\perp_1} = \{M \in \text{Mod-}R \mid \text{Ext}^1_R(C, M) \text{ for all } C \in \mathcal{C}\},$$

and a *left orthogonal class* of $\mathcal{C}$, denoted ${}^{\perp_1}\mathcal{C}$, as

$${}^{\perp_1}\mathcal{C} = \{M \in \text{Mod-}R \mid \text{Ext}^1_R(M, C) \text{ for all } C \in \mathcal{C}\}.$$

Let $i \geq 1$, the class $\mathcal{C}^{\perp_i}$ is defined by

$$\mathcal{C}^{\perp_i} = \{M \in \text{Mod-}R \mid \text{Ext}^i_R(C, M) \text{ for all } C \in \mathcal{C}\},$$

the class $\mathcal{C}^{\perp_\infty}$ is defined by

$$\mathcal{C}^{\perp_\infty} = \bigcap_{1 \leq j < \omega} \mathcal{C}^{\perp_j},$$

the classes ${}^{\perp_i}\mathcal{C}$ and ${}^{\perp_\infty}\mathcal{C}$ are defined analogicaly.

Remark 3.2. Let R be a ring and $\mathcal{C}$ be a class of right R-modules. Then $\mathcal{C} \subseteq {}^{\perp_1}(\mathcal{C}^{\perp_1})$ and $\mathcal{C} \subseteq ({}^{\perp_1}\mathcal{C})^{\perp_1}$. Also $\mathcal{C}_1 \subseteq \mathcal{C}_2$ implies ${}^{\perp_1}\mathcal{C}_2 \subseteq {}^{\perp_1}\mathcal{C}_1$ and $\mathcal{C}_2^{\perp_1} \subseteq \mathcal{C}_1^{\perp_1}$. From this, it follows that $({}^{\perp_1}(\mathcal{C}^{\perp_1}))^{\perp_1} = \mathcal{C}^{\perp_1}$ and ${}^{\perp_1}(({}^{\perp_1}\mathcal{C})^{\perp_1}) = {}^{\perp_1}\mathcal{C}$.

We also note that each right orthogonal class is closed under extensions, direct summands and arbitrary direct products and contains all the injective modules and each left orthogonal class is closed under extensions, direct summands and arbitrary direct sums and contains all the projective modules.

Definition 3.3. Let R be a ring and $\mathcal{A}, \mathcal{B}$ be two classes of right R-modules. Then the ordered pair $(\mathcal{A}, \mathcal{B})$ is called a *cotorsion pair* (or *cotorsion theory*) if $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$.

From Remark 3.2, it follows that $({}^{\perp_1}(\mathcal{C}^{\perp_1}), \mathcal{C}^{\perp_1})$ and $({}^{\perp_1}\mathcal{C}, ({}^{\perp_1}\mathcal{C})^{\perp_1})$ are cotorsion pairs, they are called cotorsion pairs *generated* and *cogenerated*, respectively, by the class $\mathcal{C}$.

In case when $\mathcal{C}$ consists of a single right R-module C, we simply write ${}^{\perp_1}C$ and $C^{\perp_1}$ in place of ${}^{\perp_1}\{C\}$ and $\{C\}^{\perp_1}$.

Remark 3.4. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair, then by Remark 3.2, we have that

1. $\mathcal{P}_0 \subseteq \mathcal{A}$ and $\mathcal{A}$ is closed under extensions, direct summands and arbitrary direct sums,

2. $\mathcal{I}_0 \subseteq \mathcal{B}$ and $\mathcal{B}$ is closed under extensions, direct summands and arbitrary direct products.

We also note that for any ring R, the cotorsion pairs of right R-modules are partialy ordered by inclusion of their first component. The largest element under this order is $(\text{Mod-}R, \mathcal{I}_o)$, the least is $(\mathcal{P}_0, \text{Mod-}R)$, these are called the *trivial cotorsion pairs* (or *trivial cotorsion theories*).

Definition 3.5. Let R be a ring, $\mathcal{C}$ be a class of right R-modules and M be a right R-module. A homomorphism $f : M \to C$ with $C \in \mathcal{C}$ is a *$\mathcal{C}$-preenvlope* of M if for each homomorphism $f' : M \to C'$ with $C' \in \mathcal{C}$ there is a homomorphism $g : C \to C'$ such that $f' = gf$. The $\mathcal{C}$-preenvlope f of M is a *$\mathcal{C}$-envlope* of M if for each $g : C \to C$ the equation $f = gf$ implies that g is an automorphism of C. The $\mathcal{C}$-preenvlope f of M is called *special* if f is injective and $\text{Coker } f \in {}^{\perp_1}\mathcal{C}$

A homomorphism $f : C \to M$ with $C \in \mathcal{C}$ is a *$\mathcal{C}$-precover* of M if for each homomorphism $f' : C' \to M$ with $C' \in \mathcal{C}$ there is a homomorphism $g : C' \to C$ such that $f' = fg$. The $\mathcal{C}$-precover f of M is a *$\mathcal{C}$-cover* of M if for each $g : C \to C$ the equation $f = fg$ implies that g is an automorphism of C. The $\mathcal{C}$-precover f of M is called *special* if f is surjective and $\text{Ker } f \in \mathcal{C}^{\perp_1}$

If $\mathcal{C}$ is a class of right R-modules such that each right R-module has a special preenlope (special precover) then $\mathcal{C}$ is called *special preenvloping* (*special precovering*).

Note that both the $\mathcal{C}$-preenvlope of M and the $\mathcal{C}$-precover of M need not to be unique.

Definition 3.6. Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair of right R-modules. Then $(\mathcal{A}, \mathcal{B})$ is called *complete* if each right R-module has a special $\mathcal{A}$-precover and each right R-module has a special $\mathcal{B}$-preenvlope.

Definition 3.7. Let R be a ring and $\mathcal{C}$ be a class of right R-modules. Then

(i) $\mathcal{C}$ is called *resolving* if $\mathcal{C}$ is closed under extensions, $\mathcal{P}_0 \subseteq \mathcal{C}$ and $A \in \mathcal{C}$, whenever $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence such that $B, C \in \mathcal{C}$,

(ii) $\mathcal{C}$ is called *coresolving* if $\mathcal{C}$ is closed under extensions, $\mathcal{I}_0 \subseteq \mathcal{C}$ and $C \in \mathcal{C}$, whenever $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence such that $A, B \in \mathcal{C}$.

Definition 3.8. Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair of right R-modules. Then $(\mathcal{A}, \mathcal{B})$ is called *hereditary* if $\mathcal{A}$ is resolving and $\mathcal{B}$ is coresolving.

Definition 3.9. Let R be a ring and $\mathcal{C}$ be a class of right R-modules. Then $\mathcal{C}$ is of *finite type* if there exist $n < \omega$ and a class (equivalently a set) $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{C} = \mathcal{S}^{\perp_\infty}$.

Let T be a right R-module. Then T is of *finite type* if the class $T^{\perp_\infty}$ is of finite type.

Lemma 3.10. *Let R be a ring and T a right R-module of projective dimension n. Let $0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_0 \longrightarrow T \longrightarrow 0$ be a projective resolution of T with syzygies $T = S_0, S_1, \ldots, S_{n-1}, S_n = P_n, S_{n+1} = 0, S_{n+2} = 0, \ldots$ and let $S = \bigoplus_{0\le i\le n} S_i$. Then*

1. *$({}^{\perp_1}(T^{\perp_\infty}), T^{\perp_\infty})$ is the cotorsion pair generated by S,*

2. *${}^{\perp_1}(T^{\perp_\infty}) \subseteq \mathcal{P}_n$.*

Proof. (1) by Lemma 2.80 we have

$$\begin{aligned}
T^{\perp_\infty} &= \bigcap_{1\le i<\omega} \{M \in \text{Mod-}R \mid \text{Ext}^i_R(T, M) = 0\} = \\
&= \bigcap_{1\le i<\omega} \{M \in \text{Mod-}R \mid \text{Ext}^1_R(S_{i-1}, M) = 0\} = \\
&= \bigcap_{0\le i<n} \{M \in \text{Mod-}R \mid \text{Ext}^1_R(S_i, M) = 0\} = \\
&= \{M \in \text{Mod-}R \mid \prod_{0\le i\le n} \text{Ext}^1_R(S_i, M) = 0\} = \\
&= \{M \in \text{Mod-}R \mid \text{Ext}^1_R(\bigoplus_{0\le i\le n} S_i, M) = 0\} = (\bigoplus_{0\le i\le n} S_i)^{\perp_1} = S^{\perp_1}.
\end{aligned}$$

So the (1) is true.

(2) by assumption, $S \in \mathcal{P}_n$, so $S^{\perp_1} \supseteq {\mathcal{P}_n}^{\perp_1}$. By (1), Remark 3.2 and Theorem 7.10, ${}^{\perp_1}(T^{\perp_\infty}) = {}^{\perp_1}(S^{\perp_1}) \subseteq {}^{\perp_1}({\mathcal{P}_n}^{\perp_1}) = \mathcal{P}_n$. □

Definition 3.11. Let R be a ring. A right R-module T is *tilting* provided that

(T1) T has finite projective dimension (that is, $T \in \mathcal{P}$),

(T2) $\text{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $1 \le i < \omega$ and all cardinals κ,

(T3) there are $r \ge 0$ and a long exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$, where $T_i \in \text{Add}(T)$ for all $i \le r$.

The class $T^{\perp_\infty}$ is called *tilting class* induced by T and the cotorsion pair $({}^{\perp_1}(T^{\perp_\infty}), T^{\perp_\infty})$ is called *tilting cotorsion pair* induced by T.

If $n < \omega$ and T is tilting of projective dimension $\le n$, then T is *n-tilting*, the class $T^{\perp_\infty}$ is called *n-tilting class* induced by T and the cotorsion pair $({}^{\perp_1}(T^{\perp_\infty}), T^{\perp_\infty})$ is called *n-tilting cotorsion pair* induced by T.

If T and T' are tilting right R-modules, then T is said to be *equivalent* to T' if the induced tilting classes coincide, that is, $T^{\perp_\infty} = (T')^{\perp_\infty}$.

Definition 3.12. Let R be a ring and let μ be an ordinal. The sequence $\mathcal{A} = (A_\alpha \mid \alpha \leq \mu)$ of right (left) R-modules is called a *continuous chain of R-modules* in case following three conditions hold

(i) $A_0 = 0$,

(ii) $A_\alpha \subseteq A_{\alpha+1}$ for all $\alpha < \mu$,

(iii) $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for all limit ordinals $\alpha \leq \mu$.

If μ is finite, the previous sequence is called a *finite chain of R-modules.*

Definition 3.13. Let R be a ring, M be a right (left) R-module, and $\mathcal{C}$ be a class of right (left) R-modules. Then M is *$\mathcal{C}$-filtered*, provided that there are an ordinal μ and a continuous chain of right (left) R-modules $(M_\alpha \mid \alpha \leq \mu)$, consisting of submodules of M such that $M = M_\mu$, and each of the right (left) R-module $M_{\alpha+1}/M_\alpha$ $(\alpha < \mu)$ is isomorphic to an element of $\mathcal{C}$. The chain $(M_\alpha \mid \alpha \leq \mu)$ is called a *$\mathcal{C}$-filtration* of M. If μ is finite, then M is said to be *finitely $\mathcal{C}$-filtered* and the coresponding finite chain of R-modules is called a *finite $\mathcal{C}$-filtration* of M.

Now, we will prove that each tilting module over an arbitrary ring is strongly finitely presented. We will need this result in order to prove that finitely generated tilting modules over commutative rings are projective.

Lemma 3.14. *Let R be a ring, $(\mathcal{A}, \mathcal{B})$ be a tilting cotorsion pair. Then each countably generated right R-module M from $\mathcal{A}$ is countably presented.*

Proof. By Theorem 7.14, there is a $\mathcal{A}^{<\aleph_1}$-filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ of M. Thus each right R-module $M_{\alpha+1}/M_\alpha$ $(\alpha < \sigma)$ posses a projective resolution consisting of $< \aleph_1$-generated projective right R-modules. Using Lemma 2.82, we see that each $M_{\alpha+1}/M_\alpha$ $(\alpha < \sigma)$ is $< \aleph_1$-presented. By Theorem 7.11 (in setting $\kappa = \aleph_1$, $N = 0$ and X be a generating subset of M of cardinality $< \kappa$), we have that M is countably presented. □

Lemma 3.15. *Let R be a ring and T be a finitely generated tilting right R-module. Then T is strongly finitely presented.*

Proof. Denote $(\mathcal{A}, \mathcal{B})$ the cotorsion pair induced by T. By Lemma 3.10, $T^{\perp_\infty} = S^{\perp_1}$, so $S \in {}^{\perp_1}(T^{\perp_\infty})$ and since T is a direct summand in S, we have that $T \in {}^{\perp_1}(T^{\perp_\infty})$. Using Lemma 3.14 and the fact that T is finitely generated, we have the following short exact sequence of right R-modules

$$0 \longrightarrow K \longrightarrow R^{(m)} \longrightarrow T \longrightarrow 0$$

where $m < \omega$ and K is countably generated. Write $K = \bigcup_{0 \leq i < \omega} K_i$ as the union of the strongly increasing continuous chain of finitely generated submodules K_i of K. Let E_i denote the injective hull of K/K_i. Define $f : K \to \prod_{0 \leq i < \omega} E_i$ by $f(k) = (k + K_i)_{0 \leq i < \omega}$. For every $k \in K$, there is an $i_k < \omega$ such that $k \in K_{i_k}$, so the image of f is contained in $\bigoplus_{0 \leq i < \omega} E_i$. Using Remark 3.4 and Theorem 7.13, we have that $\bigoplus_{0 \leq i < \omega} E_i \in \mathcal{B}$ and since $T \in \mathcal{A}$, there is $g \in \mathrm{Hom}_R(R^{(m)}, \bigoplus_{0 \leq i < \omega} E_i)$ such that $g \restriction_K = f$. But, the image of g is finitely generated, so there exists $i < \omega$ such that $\mathrm{Im}\, f \subseteq \bigoplus_{0 \leq j < i} E_j$ and hence $K_i = K$ proving that K is finitely generated. If K is projective we are done, otherwise repeat the previous procedure again but now for the following short exact sequence of right R-modules

$$0 \longrightarrow L \longrightarrow R^{(n)} \longrightarrow K \longrightarrow 0$$

where L is countably generated (see Lemma 3.10 and use the fact that K is the first syzygy of T). We get that L is finitely generated and if L is projective we are done, otherwise we can repeat the previous procedure again, etc. So T is strongly finitely presented. □

The following Lemma is crucial in proving that finitely generated tilting modules over commutative rings are projective. The technique of the proof is taken from Proposition 2.2. from [9] and its modification is due to S. Bazzoni.

Lemma 3.16. *Let R be a commutative ring and M be a strongly finitely presented R-module such that proj dim$_R\, M \leq n$ and Ext$^i_R(M, M) = 0$ for all $1 \leq i \leq n$. Then M is projective.*

Proof. Suppose that $\mathrm{proj\,dim}\, M = k$, $0 < k \leq n$. Let $0 \longrightarrow P_k \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be the projective resolution of M consisting of finitely generated projective R-modules. Denote by S the $(k-1)$th syzygy of this resolution of M. We will prove that S is projective, it will be the contradiction proving that M is projective.

Since S is strongly finitely presented, by Lemmas 2.82, 2.66 and 2.124, it is enough to prove that for every maximal ideal I of R, $S_{(I)}$ is a projective $R_{(I)}$-module.

Let I be a maximal ideal of R. We can assume without loss of generality, that $M_{(I)} \neq 0$, because $S_{(I)}$ is the $(k-1)$th syzygy of the following projective reolution of $M_{(I)}$ (see Lemma 2.100)

$$0 \longrightarrow (P_k)_{(I)} \longrightarrow \ldots \longrightarrow (P_0)_{(I)} \longrightarrow M_{(I)} \longrightarrow 0.$$

By Lemma 2.100, $M_{(I)}$ is a finitely generated $R_{(I)}$-module. By Remark 2.106, $R_{(I)}$ is a local ring with a maximal ideal $IR_{(I)}$. So by Nakayma's Lemma 2.38, we obtain that $M_{(I)} \neq M_{(I)}I = (MI)_{(I)}$ and hence $M \neq MI$. Therefore by Remark 2.27, $M/(MI)$ is a non-zero (R/I)-vector space. So that we have an (R/I)-module epimorphism $M/(MI) \xrightarrow{\varphi} (R/I) \longrightarrow 0$ (it is a projection to some of it's one-dimensional subspace). φ is clearly also an R-module epimorphism and if we define $\psi = \varphi \circ \pi$ as a composite mapping of a canonical projection $M \xrightarrow{\pi} M/(MI)$ and φ, we have the following short exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\psi} R/I \longrightarrow 0$$

of R-modules (K is the kernel of ψ). Applying $\mathrm{Hom}_R(M, -)$ to the previous short exact sequence we get part of the induced long exact sequence

$$\mathrm{Ext}_R^k(M, M) \longrightarrow \mathrm{Ext}_R^k(M, R/I) \longrightarrow \mathrm{Ext}_R^{k+1}(M, K).$$

Since $\mathrm{Ext}_R^k(M, M) = \mathrm{Ext}_R^{k+1}(M, K) = 0$ ($\operatorname{proj\,dim} M = k$), using Lemma 2.80 we obtain that $\mathrm{Ext}_R^k(M, R/I) = \mathrm{Ext}_R^1(S, R/I) = 0$.

Now using Lemmas 2.23 and 7.2 we get that

$$\begin{aligned} 0 &= \mathrm{Ext}_R^1(S, R/I) \simeq \mathrm{Ext}_R^1(S, \mathrm{Hom}_R(R/I, E(R/I))) \simeq \\ &\simeq \mathrm{Hom}_R(\mathrm{Tor}_R^1(S, R/I), E(R/I)). \end{aligned}$$

Since by Lemma 2.121, $E_R(R/I) = E_{R_{(I)}}(R_{(I)}/(IR_{(I)}))$ as $R_{(I)}$-modules and therefore as R-modules, we obtain by Lemmas 2.45, 2.96 and 2.46 that

$$\begin{aligned} 0 &= \mathrm{Hom}_R(\mathrm{Tor}_R^1(S, R/I), E_{R_{(I)}}(R_{(I)}/IR_{(I)})) \simeq \\ &\simeq \mathrm{Hom}_R(\mathrm{Tor}_R^1(S, R/I), \mathrm{Hom}_{R_{(I)}}(R_{(I)}, E_{R_{(I)}}(R_{(I)}/IR_{(I)}))) \simeq \\ &\simeq \mathrm{Hom}_R(\mathrm{Tor}_R^1(S, R/I), \mathrm{Hom}_R(R_{(I)}, E_{R_{(I)}}(R_{(I)}/IR_{(I)}))) \simeq \\ &\simeq \mathrm{Hom}_R(\mathrm{Tor}_R^1(S, R/I) \otimes_R R_{(I)}, E_{R_{(I)}}(R_{(I)}/IR_{(I)})) \simeq \\ &\simeq \mathrm{Hom}_{R_{(I)}}(\mathrm{Tor}_R^1(S, R/I) \otimes_R R_{(I)}, E_{R_{(I)}}(R_{(I)}/IR_{(I)})). \end{aligned}$$

Remark 2.106 and Lemma 2.64 imply that $E_{R_{(I)}}(R_{(I)}/(IR_{(I)}))$ is an injective cogenerator for $R_{(I)}$-modules, thus

$$\mathrm{Tor}_R^1(S, R/I) \otimes_R R_{(I)} = 0.$$

Hence by Lemma 2.101,

$$\mathrm{Tor}^1_{R_{(I)}}(S_{(I)}, R_{(I)}/(IR_{(I)})) = 0.$$

Therefore in view of Theorem 7.5, $S_{(I)}$ is a projective $R_{(I)}$-module and we are done. □

Corollary 3.17. *Let R be a commutative ring and T be a finitely generated tilting R-module. Then T is projective.*

Proof. This follows from Lemmas 3.15 and 3.16. □

Now we will define Gorenstein rings and Bass tilting modules and we will prove that Bass tilting modules are 1-tilting.

Definition 3.18. A ring R is called *Iwanaga-Gorenstein* (or simply *Gorenstein*) if R is both left and right noetherian and if R has finite self-injective dimension on both the left and the right. A Gorenstein ring with $\operatorname{inj}\dim {}_R R \leq n$ (or equivalently with $\operatorname{inj}\dim R_R \leq n$) is called *n-Iwana-Gorenstein* (or simply *n-Gorenstein* ring).

Lemma 3.19. *Let R be a commutative noetherian ring. Then the following are equivalent*

1. *R is n-Gorenstein,*
2. *Krull dimension of R is at most n, i.e. $\dim R \leq n$,*
3. *$\mathcal{P} = \mathcal{I} = \mathcal{F} = \mathcal{P}_n = \mathcal{I}_n = \mathcal{F}_n$,*
4. *the minimal injective coresolution of R is of the form*

$$0 \longrightarrow R \longrightarrow \bigoplus_{ht\, p=0} E(R/p) \longrightarrow \bigoplus_{ht\, p=1} E(R/p) \longrightarrow \dots \longrightarrow \bigoplus_{ht\, p=n} E(R/p) \longrightarrow 0.$$

Proof. These are the classical results on Gorenstein rings and can be found in [12, §18]. □

Definition 3.20. Let R be a commutative 1-Gorenstein ring. Let P_0 and P_1 denote the sets of all prime idelas of height 0 and 1, respectively. By Lemma 3.19, the minimal injective coresolution of R has the form

$$0 \longrightarrow R \longrightarrow \bigoplus_{q \in P_0} E(R/q) \xrightarrow{\pi} \bigoplus_{p \in P_1} E(R/p) \longrightarrow 0.$$

Consider a subset $P \subseteq P_1$. Put $R_P = \pi^{-1}(\bigoplus_{p \in P} E(R/p))$ and $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$. We define the *Bass tilting module* (with respect to $P \subseteq P_1$) as T_P.

The following can also be found in [3] as Example 4.1.

Lemma 3.21. *Let R be a commutative 1-Gorenstein ring. Then the Bass tilting module T_P is a 1-tilting module for any $P \subseteq P_1$.*

Proof. Let $P \subseteq P_1$ and consider the T_P.

(T1). First note that the R-modules $\bigoplus_{p\in P_1\setminus P} E(R/p)$ and $\bigoplus_{p\in P} E(R/p)$ are injective because R is noetherian. By Definition 3.20, we have the following short exact sequence

$$0 \longrightarrow R_P \longrightarrow E(R) \longrightarrow \bigoplus_{p\in P_1\setminus P} E(R/p) \longrightarrow 0.$$

We see that R_P has an injective dimension ≤ 1. Since both R_P and $\bigoplus_{p\in P} E(R/p)$ have injective dimension ≤ 1, so does T_P. By Lemma 3.19 we have that T_P has also projective dimension ≤ 1, so $T_P \in \mathcal{P}_1$ and (T1) is satisfied.

(T2). First we will prove that $\operatorname{Ext}^1_R(E(R/p), R_P^{(\kappa)}) = 0$ for any $p \in P$ and any cardinal κ. Consider the short exact sequence

$$0 \longrightarrow R_P^{(\kappa)} \longrightarrow E(R)^{(\kappa)} \longrightarrow \bigoplus_{p\in P_1\setminus P} E(R/p)^{(\kappa)} \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(E(R/p), -)$, we get part of the induced long exact sequence

$$\operatorname{Hom}_R(E(R/p), \bigoplus_{p\in P_1\setminus P} E(R/p)^{(\kappa)}) \longrightarrow \operatorname{Ext}^1_R(E(R/p), R_P^{(\kappa)}) \longrightarrow \operatorname{Ext}^1_R(E(R/p), E(R)^{(\kappa)}).$$

But by Lemma 2.122, $\operatorname{Hom}_R(E(R/p), \bigoplus_{p\in P_1\setminus P} E(R/p)^{(\kappa)}) = 0$ and since $E(R)^{(\kappa)}$ is an injective R-module, we also have $\operatorname{Ext}^1_R(E(R/p), E(R)^{(\kappa)}) = 0$. So we have just proved that $\operatorname{Ext}^1_R(E(R/p), R_P^{(\kappa)}) = 0$ for any $p \in P$ and any cardinal κ.

By Definition 3.20, we have the following short exact sequence

$$0 \longrightarrow R \longrightarrow R_P \longrightarrow \bigoplus_{p\in P} E(R/p) \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(-, R_P^{(\kappa)})$, we get part of the induced long exact sequence

$$\operatorname{Ext}^1_R(\bigoplus_{p\in P} E(R/p), R_p^{(\kappa)}) \longrightarrow \operatorname{Ext}^1_R(R_P, R_p^{(\kappa)}) \longrightarrow \operatorname{Ext}^1_R(R, R_P^{(\kappa)}).$$

We already know that $\operatorname{Ext}^1_R(\bigoplus_{p\in P} E(R/p), R_p^{(\kappa)}) \simeq \prod_{p\in P} \operatorname{Ext}^1_R(E(R/p), R_p^{(\kappa)}) = 0$ and we also have $\operatorname{Ext}^1_R(R, R_p^{(\kappa)}) = 0$ because R is a projective R-module, so we have just proved that $\mathrm{Ext}^1_R(R_P, R_p^{(\kappa)}) = 0$ for any κ. Now we have

$$\begin{aligned}\mathrm{Ext}^1_R(T_P, T_P^{(\kappa)}) &\simeq \mathrm{Ext}^1_R(R_P \oplus \bigoplus_{p\in P} E(R/p), T_P^{(\kappa)}) \simeq \\ &\simeq \mathrm{Ext}^1_R(R_p, T_P^{(\kappa)}) \oplus \prod_{p\in P} \mathrm{Ext}^1_R(E(R/p), T_P^{(\kappa)}) \simeq \\ &\simeq \mathrm{Ext}^1_R(R_P, R_P^{(\kappa)}) \oplus \mathrm{Ext}^1_R(R_P, \bigoplus_{p\in P} E(R/p)^{(\kappa)}) \oplus \\ &\oplus \prod_{p\in P} \mathrm{Ext}^1_R(E(R/p), R_P^{(\kappa)}) \oplus \mathrm{Ext}^1_R(E(R/p), \bigoplus_{p\in P} E(R/p)^{(\kappa)}).\end{aligned}$$

Using $\mathrm{Ext}^1_R(E(R/p), R_P^{(\kappa)}) = 0$ for any $p \in P$ and any cardinal κ, $\mathrm{Ext}^1_R(R_P, R_p^{(\kappa)}) = 0$ for any cardinal κ and $\mathrm{Ext}^1_R(M, I) = 0$ for any R-module M and any injective R-module I, we have just proved that $\mathrm{Ext}^1_R(T_P, T_P^{(\kappa)}) = 0$ for any cardinal κ. By the previous part, T_P has projective dimension ≤ 1, so (using Lemma 2.77) $\mathrm{Ext}^i_R(T_P, T_P^{(\kappa)}) = 0$ for all $i \geq 1$ and all cardinals κ, thus the condition (T2) is satisfied for T_P.

(T3). The short exact sequence $0 \to R \to R_P \to \bigoplus_{p\in P} E(R/p) \to 0$ yields that the condition (T3) is satisfied for T_P. □

Remark 3.22. Consider the short exact sequence

$$0 \longrightarrow R \longrightarrow R_P \longrightarrow \bigoplus_{p\in P} E(R/p) \longrightarrow 0.$$

Applying $\mathrm{Hom}_R(-, M)$ (M is an arbitrary R-module), we get part of the induced long exact sequence

$$\mathrm{Ext}^1_R(\bigoplus_{p\in P} E(R/p), M) \longrightarrow \mathrm{Ext}^1_R(R_P, M) \longrightarrow \mathrm{Ext}^1_R(R, M).$$

We have $\mathrm{Ext}^1_R(R, M) = 0$ because R is a projective R-module and since

$$\mathrm{Ext}^1_R(\bigoplus_{p\in P} E(R/p), M) \simeq \prod_{p\in P} \mathrm{Ext}^1_R(E(R/p), M),$$

we have that if $\mathrm{Ext}^1_R(E(R/p), M) = 0$ for all $p \in P$ then $\mathrm{Ext}^1_R(R_P, M) = 0$.

By Definition 3.11 and Lemma 2.77 (T_P is 1-tilting R-module), the 1-tilting class induced by T_P is $\{M \in \text{Mod-}R \mid \mathrm{Ext}^1_R(T_P, M) = 0\}$. But we have

$$\begin{aligned}\mathrm{Ext}^1_R(T_P, M) &\simeq \mathrm{Ext}^1_R(R_P \oplus \bigoplus_{p\in P} E(R/p), M) \simeq \\ &\simeq \mathrm{Ext}^1_R(R_P, M) \oplus \prod_{p\in P} \mathrm{Ext}^1_R(E(R/p), M).\end{aligned}$$

So by the previous part we get that $T_P^{\perp_\infty} = \{M \in \text{Mod-}R \mid \text{Ext}^1_R(E(R/p), M) = 0$ for all $p \in P\} = \bigcap_{p \in P} (E(R/p))^{\perp_1}$.

Lemma 3.23. *Let R be a ring and C a left R-module of injective dimension n. Let $0 \longrightarrow C \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \dots \longrightarrow I_n \longrightarrow 0$ be an injective coresolution of C with cosyzygies $C = S_0, S_1, \dots, S_{n-1}, S_n = I_n, S_{n+1} = 0, S_{n+2} = 0, \dots$ and let $S = \prod_{0 \le i \le n} S_i$. Then $({}^{\perp_\infty}C, ({}^{\perp_\infty}C)^{\perp_1})$ is the cotorsion pair cogenerated by S.*

Proof. By Lemma 2.80 we have

$$\begin{aligned}
{}^{\perp_\infty}C &= \bigcap_{1 \le i < \omega} \{M \in \text{Mod-}R \mid \text{Ext}^i_R(M, C) = 0\} = \\
&= \bigcap_{1 \le i < \omega} \{M \in \text{Mod-}R \mid \text{Ext}^1_R(M, S_{i-1}) = 0\} = \\
&= \bigcap_{0 \le i < n} \{M \in \text{Mod-}R \mid \text{Ext}^1_R(M, S_i) = 0\} = \\
&= \{M \in \text{Mod-}R \mid \prod_{0 \le i \le n} \text{Ext}^1_R(M, S_i) = 0\} = \\
&= \{M \in \text{Mod-}R \mid \text{Ext}^1_R(M, \prod_{0 \le i \le n} S_i) = 0\} = {}^{\perp_1}(\prod_{0 \le i \le n} S_i) = {}^{\perp_1}S.
\end{aligned}$$

So the claim is true. ☐

Definition 3.24. Let R be a ring. A left R-module C is *cotilting* provided that

(C1) C has finite injective dimension (that is, $C \in \mathcal{I}$),

(C2) $\text{Ext}^i_R(C^\kappa, C) = 0$ for all $1 \le i < \omega$ and all cardinals κ,

(C3) there are $r \ge 0$ and a long exact sequence $0 \to C_r \to \cdots \to C_1 \to C_0 \to W \to 0$, where $C_i \in \text{Prod}(C)$ for all $i \le r$ and W is an injective cogenerator for R-Mod.

The class ${}^{\perp_\infty}C$ is called *cotilting class* induced by C and the cotorsion pair $({}^{\perp_\infty}C, ({}^{\perp_\infty}C)^{\perp_1})$ is called *cotilting cotorsion pair* induced by C.

If $n < \omega$ and C is cotilting of injective dimension $\le n$, then C is *n-cotilting*, the class ${}^{\perp_\infty}C$ is called *n-cotilting class* induced by C and the cotorsion pair $({}^{\perp_\infty}C, ({}^{\perp_\infty}C)^{\perp_1})$ is called *n-cotilting cotorsion pair* induced by C.

If C and C' are cotilting left R-modules, then C is said to be *equivalent* to C' if the induced cotilting classes coincide, that is, ${}^{\perp_\infty}C = {}^{\perp_\infty}C'$.

4 Tilting modules over Dedekind domains

In this chapter, we will prove that every tilting module over a Dedekind domain is equivalent to some Bass tilting module.

Definition 4.1. A ring R is *right (left) hereditary* in case every right (left) ideal of R is a projective right (left) R-module.

Remark 4.2. Note that a ring may be right hereditary but not left hereditary. The term *hereditary* ring will mean a ring which is both left and right hereditary. It is clear that, when R is commutative, R is left hereditary precisely when it is right hereditary.

Lemma 4.3. *Let R be a ring. Then the following are equivalent*

1. *R is right (left) hereditary,*
2. *if M is an injective right (left) R-module, then M/M' is injective for every submodule $M' \subseteq M$,*
3. *if M is a projective right (left) R-module, then M' is projective for every submodule $M' \subseteq M$,*
4. *$\mathrm{Ext}^1_R(M, N) = 0$ implies $\mathrm{Ext}^1_R(M, N/N') = 0$ for all right (left) R-modules M, $N' \subseteq N$,*
5. *$\mathrm{Ext}^1_R(M, N) = 0$ implies $\mathrm{Ext}^1_R(M', N) = 0$ for all right (left) R-modules $M' \subseteq M$, N,*
6. *$\mathrm{Ext}^i_R(M, N) = 0$ for all $i \geq 2$ and for all right (left) R-modules M, N,*
7. *$M^{\perp_\infty} = M^{\perp_1}$ for all right (left) R-modules M.*

Proof. This is a well-known fact which can be found in [8]. □

Definition 4.4. A hereditary integral domain is called a *Dedekind domain.*

Lemma 4.5. *Let R be a Dedekind domain. Then*

1. *R is noetherian and $\mathrm{inj\,dim}\, R \leq 1$, in particular R is a hereditary 1-Gorenstein domain,*
2. *every non-zero prime ideal p of R is maximal, i.e. $\mathrm{ht}\, p = 1$ iff $p \in \mathrm{mSpec}\, R$,*
3. *if $p \in \mathrm{Spec}\, R$, then $R_{(p)}$ is a valuation domain.*

Proof. (1). It is a well-known fact that every Dedekind domain is noetherian (see [12] or [6]) and by Lemma 4.3, the following short exact sequence

$$0 \longrightarrow R \longrightarrow E(R) \longrightarrow E(R)/R \longrightarrow 0$$

is an injective coresolution of R.

(2) and (3) are well-known facts and can be found in [12]. □

Lemma 4.6 (Eklof Lemma)**.** *Let R be a ring, N be a right (left) R-module, and M be a ${}^{\perp_1}N$-filtered right (left) R-module. Then $M \in {}^{\perp_1}N$. (Or equivalently: Let R be a ring and M, N be right (left) R-modules. If there is a continuous chain $(M_\alpha \mid \alpha \leq \mu)$ of submodules of M such that $M = M_\mu$ and $Ext^1_R(M_{\alpha+1}/M_\alpha, N) = 0$ for all ordinals $\alpha < \mu$. Then $Ext^1_R(M, N) = 0$.)*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Let $(M_\alpha \mid \alpha \leq \mu)$ be a ${}^{\perp_1}N$-filtration of M. So by Definition 3.12, $\mathrm{Ext}^1_R(M_0, N) = 0$ and by Definition 3.13, $\mathrm{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) = 0$ for each $\alpha < \mu$. We will prove that $\mathrm{Ext}^1_R(M, N) = 0$.

By induction on $\alpha < \mu$ we will prove that $\mathrm{Ext}^1_R(M_\alpha, N) = 0$. This is clear for $\alpha = 0$. Applying $\mathrm{Hom}_R(-, N)$ to the following short exact sequence

$$0 \longrightarrow M_\alpha \longrightarrow M_{\alpha+1} \overset{\pi_{\alpha+1}}{\longrightarrow} M_{\alpha+1}/M_\alpha \longrightarrow 0$$

we get a part of the induced long exact sequence

$$0 = \mathrm{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) \longrightarrow \mathrm{Ext}^1_R(M_{\alpha+1}, N) \longrightarrow \mathrm{Ext}^1_R(M_\alpha, N) = 0$$

which proves the induction step for all non-limit ordinals $\alpha + 1 \leq \mu$. Assume $\alpha \leq \mu$ is a limit ordinal and let I denote the injective hull of N. We have the following short exact sequence $0 \longrightarrow N \longrightarrow I \overset{\pi}{\longrightarrow} I/N \longrightarrow 0$. In order to prove that $\mathrm{Ext}^1_R(M_\alpha, N) = 0$, we show that the abelian group homomorphism $\mathrm{Hom}_R(M_\alpha, \pi)\colon \mathrm{Hom}_R(M_\alpha, I) \to \mathrm{Hom}_R(M_\alpha, I/N)$ is surjective.

Let $\varphi \in \mathrm{Hom}_R(M_\alpha, I/N)$. By induction we define homomorphisms $\psi_\beta \in \mathrm{Hom}_R(M_\beta, N)$, $\beta < \alpha$, so that $\varphi \restriction M_\beta = \pi\psi_\beta$ and $\psi_\beta \restriction M_\gamma = \psi_\gamma$ for all $\gamma < \beta < \alpha$. First define $M_{-1} = 0$ and $\psi_{-1} = 0$. If ψ_β is already defined, the injectivity of I yields the existence of $\eta \in \mathrm{Hom}_R(M_{\beta+1}, I)$ such that $\eta \restriction M_\beta = \psi_\beta$. Put $\delta = \varphi \restriction M_{\beta+1} - \pi\eta \in \mathrm{Hom}_R(M_{\beta+1}, I/N)$. Then $\delta \restriction M_\beta = 0$. By Lemma 2.24, there exists a unique homomorphism $\delta' \in \mathrm{Hom}_R(M_{\beta+1}/M_\beta, I/N)$ such that $\delta'\pi_{\beta+1} = \delta$. Since $\mathrm{Ext}^1_R(M_{\beta+1}/M_\beta, N) = 0$, there is an $\epsilon' \in \mathrm{Hom}_R(M_{\beta+1}/M_\beta, I)$ such that $\pi\epsilon' = \delta'$. Now we define a homomorphism $\epsilon \in \mathrm{Hom}_R(M_{\beta+1}, I)$ in the following way

$$\epsilon(m) = \epsilon'(m + M_\beta)$$

for all $m \in M_{\beta+1}$, thus we have $\epsilon \restriction M_\beta = 0$ and $\pi\epsilon = \delta$. Put $\psi_{\beta+1} = \eta + \epsilon$. Then $\psi_{\beta+1} \restriction M_\beta = \psi_\beta$ and $\pi\psi_{\beta+1} = \pi\eta + \delta = \varphi \restriction M_{\beta+1}$. For a limit ordinal $\beta < \alpha$, put $\psi_\beta = \bigcup_{\gamma<\beta} \psi_\gamma$. Finally, put $\psi_\alpha = \bigcup_{\beta<\alpha} \psi_\beta$. By the construction, $\pi\psi_\alpha = \varphi$.

The claim is just the case of $\alpha = \mu$. □

Lemma 4.7. *Let R be a ring and let $(X_i \mid i < \omega)$ be a chain of right (left) R-modules such that for every $i < \omega$, the module (X_{i+1}/X_i) is $\mathcal{C}$-filtered. Then the right (left) R-module $\bigcup_{i<\omega} X_i$ is $\mathcal{C}$-filtered.*

Proof. This is really easy, but very difficult to write it down in some well-arranged way, so we only show the idea of the proof. Assume for simplicity that $X_1 = X_1/X_0$ and X_2/X_1 are finitely $\mathcal{C}$-filtered. Let $(M_i \mid i < k)$, $(N_j \mid j < l)$ be a finite $\mathcal{C}$-filtration of X_1, X_2/X_1 respectively. Then the chain $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = X_1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_l = X_2$ is a $\mathcal{C}$-filtration of X_2. □

Lemma 4.8. *Let R be a commutative noetherian ring and let $p \in mSpecR$. Then the R-module $E(R/p)$ is $\{R/p\}$-filtered.*

Proof. Define a chain of submodules of $E(R/p)$ in the following way

$$\begin{aligned} X_0 &= 0, \\ X_n &= \{x \in E(R/p) \mid xp^n = 0\}, \;\; 1 \le n < \omega. \end{aligned}$$

By Lemma 2.122, we have $0 = X_0 \subseteq X_1 \subseteq X_2 \ldots$ and $\bigcup_{i<\omega} X_n = E(R/p)$ and $p(X_{n+1}/X_n) = 0$ for every $n < \omega$.

Let $n < \omega$. From the previous fact that $p(X_{n+1}/X_n) = 0$, we have that $p \subseteq \mathrm{Ann}(X_{n+1}/X_n)$, so X_{n+1}/X_n is an R/p-module (see Definition 2.27). Since p is a maximal ideal, by Lemma 2.86, R/p is a field, thus X_{n+1}/X_n is an R/p-vector space. Let $\lambda = \dim_{R/p}(X_{n+1}/X_n)$. Thus we have the following isomorphism of R/p-vector spaces

$$X_{n+1}/X_n \overset{\varphi}{\simeq} \bigoplus_{i<\lambda} R/p$$

We would like to prove that the φ is also an R-module isomorphism. For this it is enough to prove that $\varphi(xr) = \varphi(x)r$ for all $r \in R$ and all $x \in X_{n+1}/X_n$. From the definition of multiplication in the factor ring R/p, we know that $\varphi(x)(r+p) = \varphi(x)r$ for every $r \in R$ and every $x \in X_{n+1}/X_n$. So we have $\varphi(xr) = \varphi(x(r+p)) = \varphi(x)(r+p) = \varphi(x)r$ for every $r \in R$ and every $x \in X_{n+1}/X_n$. Thus φ is also an R-module isomorphism.

Now, define a continuous chain of submodules of (X_{n+1}/X_n) in the following way

$$\begin{aligned} Y_0 &= 0 \\ Y_j &= \bigoplus_{i<j} R/p, \;\; 1 \le j \le \lambda. \end{aligned}$$

The continuous chain $(Y_j \mid j \leq \lambda)$ is obviously an $\{R/p\}$-filtration of (X_{n+1}/X_n). So (X_{n+1}/X_n) is $\{R/p\}$-filtered for all $n < \omega$.

By Lemma 4.7, the R-module $\bigcup_{i<\omega} X_n = E(R/p)$ is $\{R/p\}$-filtered. □

Lemma 4.9. *Let R be a commutative noetherian ring and let $p \in mSpec\, R$. Then the R-module R/p^k is $\{R/p\}$-filtered for all $k \geq 1$.*

Proof. Define a finite chain of submodules of R/p^k in the following way

$$\begin{aligned} X_0 &= 0, \\ X_n &= \{x \in R/p^k \mid xp^n = 0\}, \ \ 1 \leq n \leq k. \end{aligned}$$

We have $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k = R/p^k$ and $p(X_{n+1}/X_n) = 0$ for all $n < k$.

Analogously as in the proof of Lemma 4.8, we prove that the module (X_{n+1}/X_n) is $\{R/p\}$-filtered for all $n < k$.

By Lemma 4.7 (set $X_j = R/p^k$ for all $k < j < \omega$), the R-module $X_k = R/p^k$ is $\{R/p\}$-filtered. □

Lemma 4.10. *Let R be a noetherian hereditary commutative ring and $p \in mSpec\, R$. Then $Ext^1_R(E(R/p), M) = 0$ iff $Ext^1_R(R/p, M) = 0$.*

Proof. Suppose that $\mathrm{Ext}^1_R(E(R/p), M) = 0$. Since R is hereditary, by Lemma 4.3, we have that $\mathrm{Ext}^1_R(R/p, M) = 0$.

Suppose that $\mathrm{Ext}^1_R(E(R/p), M) = 0$. By Lemma 4.8, the R-module $E(R/p)$ is $\{R/p\}$-filtered and thus, using Eklof Lemma 4.6, we get that $\mathrm{Ext}^1_R(E(R/p), M) = 0$. □

Corollary 4.11. *Let R be a commutative hereditary 1-Gorenstein ring (in particular a Dedekind domain (see Remark 4.5)) . Then the 1-tilting class $T_P^{\perp_\infty}$ induced by the Bass tilting module T_P is equal to the class $\{M \in Mod\text{-}R \mid Ext^1_R(R/p, M) = 0$ for all $p \in P\} = \bigcap_{p \in P} (R/p)^{\perp_1}$.*

Proof. Just combine Remark 3.22 and Lemma 4.10. □

Lemma 4.12. *Let R be a noetherian hereditary commutative ring and $p \in mSpec\, R$. Then for every $M \in Mod\text{-}R$ and every $k \geq 1$, we have $Ext^1_R(R/p^k, M) = 0$ iff $Ext^1_R(R/p, M) = 0$.*

Proof. Assume $\mathrm{Ext}^1_R(R/p^k, M) = 0$. Since R is hereditary and $R/p \subseteq R/p^k$, by Lemma 4.3, we have that $\mathrm{Ext}^1_R(R/p, M) = 0$.

Assume $\mathrm{Ext}^1_R(R/p, M) = 0$. Since by Lemma 4.9 the module R/p^k is $\{R/p\}$-filtered, we can use Eklof Lemma 4.6 and we get that $\mathrm{Ext}^1_R(R/p^k, M) = 0$. □

The following Theorem can also be found in [5] as Theorem 5.3., but since we know that every tilting module is of finite type (see Theorem 7.15), we can prove it in much simpler way.

Theorem 4.13. *Let R be a Dedekind domain and T be a tilting R-module. Then there is a set $P \subseteq mSpec\, R$ such that T is equivalent to T_P.*

Proof. By Theorem 7.15, T is of finite type, thus there exists a set $\mathcal{S}$ of finitely generated R-modules such that $T^{\perp_\infty} = \mathcal{S}^{\perp_\infty}$. By Theorem 7.4, an R-module M is finitely generated iff M is of the form

$$M \simeq P \oplus \bigoplus_{p \in \mathrm{mSpec}\, R} M_p, \tag{3}$$

where P is a finitely generated projective R-module and each R-module M_p which is non-zero is of the form

$$M_p \simeq R/p^{\delta(p,1)} \oplus R/p^{\delta(p,2)} \oplus \cdots \oplus R/p^{\delta(p,l(p))}, \tag{4}$$

where $0 < \delta(p,1) \leq \delta(p,2) \leq \cdots \leq \delta(p,l(p))$ are positive integers, moreover, this decomposition is uniquely determined by M. By Lemma 4.3, we have

$$\begin{aligned} M^{\perp_\infty} &= M^{\perp_1} = \{N \in \text{Mod-}R \mid \mathrm{Ext}^1_R(M,N) = 0\} = \\ &= \{N \in \text{Mod-}R \mid \mathrm{Ext}^1_R(P \oplus \bigoplus_{p \in \mathrm{mSpec}\, R} M_p, N) = 0\} = \\ &= \{N \in \text{Mod-}R \mid \mathrm{Ext}^1_R(P,N) \oplus \prod_{p \in \mathrm{mSpec}\, R} \mathrm{Ext}^1_R(M_p,N) = 0\} = \\ &= \{N \in \text{Mod-}R \mid \prod_{p \in \mathrm{mSpec}\, R} \mathrm{Ext}^1_R(M_p,N) = 0\} = \bigcap_{p \in \mathrm{mSpec}\, R} M_p^{\perp_1}. \end{aligned}$$

Now using (4) and Lemma 4.12, we have the following for every non-zero R-module M_p

$$\begin{aligned} M_p^{\perp_1} &= \{N \in \text{Mod-}R \mid \mathrm{Ext}^1_R(M_p,N) = 0\} = \\ &= \{N \in \text{Mod-}R \mid \mathrm{Ext}^1_R(R/p^{\delta(p,1)} \oplus R/p^{\delta(p,2)} \oplus \cdots \oplus R/p^{\delta(p,l(p))}, N) = 0\} = \\ &= \{N \in \text{Mod-}R \mid \prod_{i=1}^{i=l(p)} \mathrm{Ext}^1_R(R/p^{\delta(p,i)}, N) = 0\} = \\ &= \{N \in \text{Mod-}R \mid \mathrm{Ext}^1_R(R/p,N) = 0) = (R/p)^{\perp_1}. \end{aligned}$$

Thus $M^{\perp_\infty} = \bigcap_{p \in \mathrm{mSpec}\, R} M_p^{\perp_1} = \bigcap_{\substack{p \in \mathrm{mSpec}\, R \\ M_p \neq 0}} M_p^{\perp_1} = \bigcap_{\substack{p \in \mathrm{mSpec}\, R \\ M_p \neq 0}} (R/p)^{\perp_1}$.

And finally if we define $P = \{p \in \mathrm{mSpec}\, R \mid \exists\, M \in \mathcal{S}$ such that $M_p \neq 0$ in the decomposition (3) of $M\}$, we have $\mathcal{S}^{\perp_\infty} = \bigcap_{M \in \mathcal{S}} M^{\perp_\infty} = \bigcap_{p \in P} (R/p)^{\perp_1} =$

$\{M \in \text{Mod-}R \mid \text{Ext}^1_R(R/p, M) = 0 \text{ for all } p \in P\}$, but by Remark 3.22, this is exactly the $T_P^{\perp_\infty}$.

Thus we have $T^{\perp_\infty} = \mathcal{S}^{\perp_\infty} = T_P^{\perp_\infty}$ and we have just proved that T is equivalent to T_P. □

Now we will show how the induced classes of Bass tilting modules T_P look like. They are the classes of all modules which are p-divisible for all $p \in P$.

Definition 4.14. Let R be a ring, I be a right (left) ideal of R and M be right (left) R-module. Then M is *I-divisible* if $\text{Ext}^1_R(R/I, M) = 0$.

Lemma 4.15. *Let R be a Dedekind domain, I be a non-zero ideal of R and M be an R-module. Then M is I-divisible iff $MI = M$.*

Proof. First denote $E = \text{Ext}^1_R(R/I, M)$. By Lemma 2.112, the R-module $E = 0$ iff $E_{(p)} = 0$ for all $p \in \text{Spec}\, R$. Let $p \in \text{Spec}\, R$. By Theorem 7.3, we have

$$E_{(p)} \simeq \text{Ext}^1_R((R/I)_{(p)}, M_{(p)}).$$

as $R_{(p)}$-modules. Moreover (using Lemma 2.97), $(R/I)_{(p)} \simeq R_{(p)}/I_{(p)}$ as $R_{(p)}$-modules. Since I is finitely generated (R is noetherian), so is $I_{(p)}$ and since $R_{(p)}$ is a valuation domain (see Lemma 4.5), by Lemma 2.126, the ideal $I_{(p)}$ of $R_{(p)}$ is principal.

We have $E_{(p)} = 0$ iff a natural abelian group homomorphism

$$\text{Hom}_{R_{(p)}}(R_{(p)}, M_{(p)}) \overset{\text{Hom}_{R_{(p)}}(\mu, M_{(p)})}{\longrightarrow} \text{Hom}_{R_{(p)}}(I_{(p)}, M_{(p)})$$

(induced by an inclusion $I_{(p)} \overset{\mu}{\longrightarrow} R_{(p)}$) is surjective and it is iff $M_{(p)} I_{(p)} = M_{(p)}$. The latter says (using Lemma 2.54) that

$$M_{(p)} \otimes_{R_{(p)}} R_{(p)}/I_{(p)} = 0.$$

Now using previous facts and Lemmas 2.97 and 2.99 we have $E_{(p)} = 0$ iff

$$0 = M_{(p)} \otimes_{R_{(p)}} R_{(p)}/I_{(p)} \simeq M_{(p)} \otimes_{R_{(p)}} (R/I)_{(p)} \simeq (M \otimes_R (R/I))_{(p)}.$$

Altogether we have $E = 0$ iff $E_{(p)} = 0$ for all $p \in \text{Spec}\, R$, iff $(M \otimes_R (R/I))_{(p)} = 0$ for all $p \in \text{Spec}\, R$, iff $M \otimes_R (R/I) = 0$, iff $MI = M$. □

Corollary 4.16. *Let R be a Dedekind domain. Then the 1-tilting class $T_P^{\perp_\infty}$ induced by the Bass tilting module T_P is equal to the class $\{M \in \text{Mod-}R \mid Mp = M$ for all $p \in P\}$.*

Proof. Just combine Corolarry 4.11 and previous Lemma 4.15. □

Theorem 4.17. *Let R be a Dedekind domain and T be a tilting R-module. Then there is a set $P \subseteq mSpec\, R$ such that the tilting class induced by T is equal to the class $\{M \in Mod\text{-}R \mid Mp = M \text{ for all } p \in P\}$.*

Proof. Just combine Theorem 4.13 and previous Corollary 4.16. □

5 Tilting modules over 1-Gorenstein commutative rings

Lemma 5.1. *Let R be a 1-Gorenstein commutative ring with Krull dimension 0 (or equivalently: let R be a 0-Gorenstein commutative ring (see Lemma 3.19)). Then each tilting R-module is projective and thus each tilting class is equal to the Mod-R and thus each tilting R-module is equivalent to the Bass tilting R-module $T_\emptyset$.*

Proof. By Definition 3.11, every tilting R-module T is of finite projective dimension thus by Lemma 3.19, T is projective. The rest is clear. □

5.1 Generalization of the Dedekind case

Now we will generalize Theorems 4.13 and 4.17 for finite direct products of Dedekind domains.

Definition 5.2. Let $R_1, R_2, \ldots, R_n$ be rings. Define a ring R as a direct product of rings R_1, $R_2, \ldots, R_n$ in the category of all rings, i.e.

$$R = R_1 \times R_2 \times \cdots \times R_n$$

Remark 5.3. Now, we will describe a structure of the ring R from Definition 5.2 more precisely. From the definition of a direct product in the category of all rings, it is easy to see that R is a set

$$\{(r_1, r_2, \ldots, r_n) \mid r_i \in R_i\}$$

with following operations

$$\begin{aligned} 0 &= (0, 0, \ldots, 0) \\ 1 &= (1, 1, \ldots, 1) \\ (r_1, r_2, \ldots, r_n) + (s_1, s_2, \ldots, s_n) &= (r_1 + s_1, r_2 + s_2, \ldots, r_n + s_n) \\ (r_1, r_2, \ldots, r_n) \cdot (s_1, s_2, \ldots, s_n) &= (r_1 \cdot s_1, r_2 \cdot s_2, \ldots, r_n \cdot s_n). \end{aligned}$$

Remark 5.4. In the following in this subsection.

1. Sometimes, for better understanding, we will write subscripts to the elements of R_i, for example $(0_1, 0_2, \ldots, 0_n) = (0, 0, \ldots, 0)$.

2. The order of the rings R_i is fixed, this means, that even if R_i and R_j are the same rings and $i \neq j$, then we make a difference between them.

3. R will always mean the ring from Definition 5.2.

Lemma 5.5. *Let R be a ring from Definition 5.2 and M be a right R-module. Then there are modules $M_1, M_2, \dots, M_n$ such that each M_i is a right R_i-module and if we define a right R-module structure on each M_i in the following way*

$$m(r_1, \dots, r_i, \dots, r_n) = mr_i \quad m \in M_i$$

then $M \simeq M_1 \oplus M_2 \oplus \cdots \oplus M_n$ as right R-modules.

Proof. For each $1 \leq i \leq n$ define a set

$$M_i = \{m(0_1, 0_2, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n) \mid m \in M\}$$

and define the following operations on M_i

$$0 = 0(0_1, 0_2, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n), \quad 0 \in M,$$

$$\begin{aligned} &m(0_1, 0_2, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n) + m'(0_1, 0_2, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n) = \\ &= (m + m')(0_1, 0_2, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n), \quad m, m' \in M, \end{aligned}$$

$$\begin{aligned} &m(0_1, 0_2, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n) \cdot r_i = \\ &= (m(0_1, 0_2, \dots, 0_{i-1}, r_i, 0_{i+1}, \dots, 0_n))(0_1, 0_2, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n), \\ &m \in M, r_i \in R_i. \end{aligned}$$

It is easy to see that M_i with these operations is a right R_i-module and it is easy to see that each right R_i-module is a right R-module via the definition from assumption.

Now define a mapping φ in the following way

$$\begin{aligned} \varphi \colon M &\rightarrow M_1 \oplus M_2 \oplus \cdots \oplus M_n \\ m &\mapsto (m(1, 0, 0, \dots, 0), m(0, 1, 0, 0, \dots, 0), \dots, m(0, 0, \dots, 0, 1)). \end{aligned}$$

It is easy to see that φ is a right R-module isomorphism. □

Remark 5.6. In the following in this subsection, the right R-module structure on some right R_i-module will mean the right R-module structure which was defined in Lemma 5.5.

Lemma 5.7. *Let R be a ring from Definition 5.2, A, B be right R_i-modules and C be a right R_j-module ($i \neq j$). Then $Hom_R(A,B) = Hom_{R_i}(A,B)$ and $Hom_R(A,C) = 0$.*

Proof. Let $\varphi\colon A \to B$ be a right R-module homomorphism. Then

$$\begin{aligned}\varphi(mr_i) &= \varphi(m(1_1, 1_2, \dots, 1_{i-1,} r_i, 1_{i+1}, \dots, 1_n)) = \\ &= \varphi(m)(1_1, 1_2, \dots, 1_{i-1}, r_i, 1_{i+1}, \dots, 1_n) = \\ &= \varphi(m)r_i, \quad r_i \in R_i.\end{aligned}$$

So φ is a right R_i-module homomorphism.

Let $\varphi\colon A \to B$ be a right R_i-module homomorphism. Then

$$\varphi(mr) = \varphi(m(r_1, r_2, \dots, r_n)) = \varphi(mr_i) = \varphi(m)r_i = \varphi(m)r, \quad r \in R.$$

So φ is a right R-module homomorphism.

Let $\varphi\colon A \to C$ be a right R-module homomorphism. Then

$$\begin{aligned}\varphi(m) &= \varphi(m(0_1, 0_2, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n)) = \\ &= \varphi(m)(0_1, 0_2, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n) = 0.\end{aligned}$$

So $\mathrm{Hom}_R(A,C) = 0$. □

Remark 5.8. By Lemma 5.5, for every R-module M, there are R_i-modules M_i, $1 \leq i \leq n$ such that $M \simeq M_1 \oplus M_2 \oplus \dots \oplus M_n$ as R-modules. It is now easy to see that M_i are uniquely (up to R_i-isomorphism) determined by M. For if $M \simeq M_1 \oplus M_2 \oplus \dots \oplus M_n \overset{\varphi}{\simeq} M_1' \oplus M_2' \oplus \dots \oplus M_n'$ as R-modules, then by Lemma 5.7, $\varphi\restriction_{M_i}$ is an R_i-module isomorphism of M_i and M_i'.

Corollary 5.9. *Let R be a ring from Definition 5.2 and let $A, B \in$ Mod-R. Then $A \subseteq B$ iff for all $1 \leq i \leq n$, $A_i \subseteq B_i$ as right R_i-modules. Moreover, if $A \subseteq B$, then $A_i \simeq B_i \cap A$ as R_i-modules for all $1 \leq i \leq n$.*

Proof. This follows from Lemma 5.5 and Remark 5.8. □

Corollary 5.10. *Let R be a ring from Definition 5.2 and $M, N \in$ Mod-R. Then $N \in Add(M)$ iff for all $1 \leq i \leq n$, $N_i \in Add(M_i)$ as right R_i-modules.*

Proof. This follows from Corollary 5.9. □

Corollary 5.11. *Let R be a ring from Definition 5.2. Then I is a right ideal of R iff*

$$I = J_1 \oplus J_2 \oplus \dots \oplus J_n,$$

where J_i is a right ideal of R_i for each $1 \leq i \leq n$. Moreover, if I is a right ideal of R, then $J_i = I \cap R_i$ for each $1 \leq i \leq n$.

Proof. This follows from Corollary 5.9 . ☐

Corollary 5.12. *Let R be a ring from Definition 5.2. Then R is a right noetherian iff each R_i is a right noetherian ring.*

Proof. Let R be right noetherian. If J_i is a right ideal of R_i, then $I = R_1 \oplus R_2 \oplus \cdots \oplus R_{i-1} \oplus J_i \oplus R_{i+1} \oplus \cdots \oplus R_n$ is a right ideal of R, thus I is finitely generated as a right R-module. It follows that J_i is finitely generated as a right R_i-module.

Let $R_1, R_2, \ldots, R_n$ be right noetherian rings. If I is a right ideal of R, then by Corollary 5.11 $I = J_1 \oplus J_2 \oplus \cdots \oplus J_n$, where each J_i is a right ideal of R_i. Thus each J_i is finitely generated as a right R_i-module. Let $X_i = \{x_i^1, x_i^2, \ldots, x_i^{m(i)}\}$ be a finite generating subset of J_i. Then the set $X = \bigcup_{i=1}^n \overline{X_i}$, where $\overline{X_i} = \{(0_1, 0_2, \ldots, 0_{i-1}, x_i^j, 0_{i+1}, \ldots, 0_n) \mid 1 \leq j \leq m(i)\}$, is a generating subset of I as a right R-module. ☐

Lemma 5.13. *Let R be a ring from Definition 5.2 and M_i be a right R_i-module. Then M_i is injective (projective) as a right R_i-module iff M_i is injective (projective) as a right R-module.*

Proof. We will prove the injective version, the proof of the projective version is analogical.

The implication to the left is easy (see Lemma 5.7).

Suppose that M_i is injective as a right R_i-module. Let

$$0 \longrightarrow A \longrightarrow B$$

be an exact sequence of right R-modules and suppose that there is a right R-module homomorphism $\varphi\colon A \to M_i$. By Lemma 5.5, we have that $A \simeq A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and $B \simeq B_1 \oplus B_2 \oplus \cdots \oplus B_n$. In order to prove that M_i is injective as a right R-module, it is enough prove that $\varphi \restriction_{A_j} = 0$ for all $j \neq i$. But the last follows from Lemma 5.7. So M_i is an injective right R-module and thus the claim is true. ☐

Corollary 5.14. *Let R be a ring from Definition 5.2, A, B be right R_i-modules and C be a right R_j-module ($i \neq j$). Then $Ext_R^k(A, B) = Ext_{R_i}^k(A, B)$ and $Ext_R^k(A, C) = 0$ for all $0 \leq k < \omega$.*

Proof. This follows from the definition of an Ext, Lemma 5.13 and Lemma 5.7. ☐

Corollary 5.15. *Let R be a ring from Definition 5.2 and M be a right R-module. Then M is injective (projective) iff each M_i is injective (projective) as a right R_i-module.*

Proof. This follows from Lemma 5.13 and from the fact that that the class of all injective modules over an arbitrary ring is closed under direct summands and under finite direct sums. □

Corollary 5.16. *Let R be a ring from Definition 5.2. Then R is a right hereditary ring iff each R_i is a right hereditary ring.*

Proof. Let R be right hereditary. If J_i is a right ideal of R_i, then $I = R_1 \oplus R_2 \oplus \cdots \oplus R_{i-1} \oplus J_i \oplus R_{i+1} \oplus \cdots \oplus R_n$ is a right ideal of R, thus I is projective as a right R-module. It follows from Corollary 5.15 that J_i is projective as a right R_i-module.

Let $R_1, R_2, \dots, R_n$ be right hereditary rings. If I is a right ideal of R, then by Corollary 5.11 $I = J_1 \oplus J_2 \oplus \cdots \oplus J_n$, where J_i is a right ideal of R_i. Thus each J_i is projective as a right R_i-module. It follows from Corollary 5.15 that I is projective as a right R-module. □

Lemma 5.17. *Let R be a ring from Definition 5.2. Then*

$$\mathit{inj\,dim}_R\, M = \mathit{max}\,\{\mathit{inj\,dim}_{R_i}\, M_i \mid 1 \leq i \leq n\},$$

where $\mathit{inj\,dim}_R\, M$ denotes the injective dimension of M as a right R-module.

Proof. If $\max\,\{\operatorname{inj\,dim}_{R_i} M_i \mid 1 \leq i \leq n\} = \infty$, then clearly $\operatorname{inj\,dim}_R M \leq \max\,\{\operatorname{inj\,dim}_{R_i} M_i \mid 1 \leq i \leq n\}$, so suppose that $\max\,\{\operatorname{inj\,dim}_{R_i} M_i \mid 1 \leq i \leq n\}$ is finte, let $m = \max\,\{\operatorname{inj\,dim}_{R_i} M_i \mid 1 \leq i \leq n\}$ and let

$$0 \longrightarrow M_i \xrightarrow{\varphi_i^1} I_i^1 \xrightarrow{\varphi_i^2} I_i^2 \longrightarrow \dots \xrightarrow{\varphi_i^m} I_i^m \longrightarrow 0$$

be an injective coresolution of each M_i as R_i-module. Then by Corollary 5.15

$$0 \longrightarrow M \xrightarrow{\bigoplus_{j=1}^n \varphi_j^1} \bigoplus_{j=1}^n I_j^1 \xrightarrow{\bigoplus_{j=1}^n \varphi_j^2} \bigoplus_{j=1}^n I_j^2 \longrightarrow \dots \xrightarrow{\bigoplus_{j=1}^n \varphi_j^m} \bigoplus_{j=1}^n I_j^m \longrightarrow 0$$

is an injective coresolution of M. So $\operatorname{inj\,dim}_R M \leq \max\,\{\operatorname{inj\,dim}_{R_i} M_i \mid 1 \leq i \leq n\}$.

Now suppose, that $\operatorname{inj\,dim}_R M < \max\,\{\operatorname{inj\,dim}_{R_i} M_i \mid 1 \leq i \leq n\}$. Let $k = \operatorname{inj\,dim}_R M$ and let

$$0 \longrightarrow M \xrightarrow{\varphi^1} I^1 \xrightarrow{\varphi^2} I^2 \longrightarrow \dots \xrightarrow{\varphi^k} I^k \longrightarrow 0$$

be an injective coresolution of M. Then by Corollary 5.15

$$0 \longrightarrow M_i \xrightarrow{\varphi^1\restriction_{M_i}} I_i^1 \xrightarrow{\varphi^2\restriction_{I_i^1}} I_i^2 \longrightarrow \dots \xrightarrow{\varphi^k\restriction_{I_i^{k-1}}} I_i^k \longrightarrow 0$$

is an injective resolution of each M_i as a right R_i-module. Thus $\max\,\{\operatorname{inj\,dim}_{R_i} M_i \mid 1 \leq i \leq n\} \leq k$, the contradiction. So the claim is true. □

Lemma 5.18. *Let R be a ring from Definition 5.2. Then*

$$\mathit{proj\,dim}_R\, M = \mathit{max}\,\{\mathit{proj\,dim}_{R_i}\, M_i \mid 1 \leq i \leq n\},$$

where $\mathit{proj\,dim}_R\, M$ denotes the projective dimension of M as a right R-module.

Proof. Analogously as in the proof of Lemma 5.17. □

Remark 5.19. Lemma 5.17, 5.18 follows also from Lemmas 5.14 and 2.78, 2.77 respectively.

Lemma 5.20. *Let $2 \leq n < \omega$ and let $R_1, R_2, \ldots, R_n$ be Dedekind domains. Define a ring R as in 5.2, i.e.*

$$R = R_1 \times R_2 \times \cdots \times R_n.$$

Then R is a commutative hereditary 1-Gorenstein ring which is not a domain.

Proof. R is obviously commutative, it is hereditary by Corollary 5.16 and it is noetherian by Corollary 5.12. Since by Lemma 4.5, every Dedekind domain has a self-injective dimension ≤ 1, so has R by Lemma 5.17. Thus R is commutative hereditary 1-Gorenstein ring. In order to prove that R is not a domain, consider two following elements of R

$$\begin{aligned} r_1 &= (1, 0, 0, \ldots, 0) \\ r_2 &= (0, 1, 0, \ldots, 0). \end{aligned}$$

These elements are non-zero, but $r_1 r_2$ is a zero element of R, thus R is not a domain. □

Lemma 5.21. *Let R be a ring from Definition 5.2 and T be a right R-module. Then T is tilting iff each T_i is a tilting right R_i-module.*

Proof. (T1) (see Definition 3.11). By Lemma 5.18, T has a finite projective dimension as a right R-module iff each T_i has a finite projective dimension as a right R_i-module.

(T2). By Corollary 5.14, we have

$$\begin{aligned} \mathrm{Ext}^i_R(T, T^{(\kappa)}) &\simeq \mathrm{Ext}^i_R\Big(\bigoplus_{j=1}^{n} T_j, \bigoplus_{j'=1}^{n} T_{j'}^{(\kappa)}\Big) \simeq \prod_{j=1}^{n}\prod_{j'=1}^{n} \mathrm{Ext}^i_R(T_j, T_{j'}^{(\kappa)}) \simeq \\ &\simeq \prod_{j=1}^{n} \mathrm{Ext}^i_R(T_j, T_j^{(\kappa)}) \simeq \prod_{j=1}^{n} \mathrm{Ext}^i_{R_j}(T_j, T_j^{(\kappa)}) \end{aligned}$$

where κ is an arbitrary cardinal and $1 \leq i < \omega$. So $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all cardinals κ and all $1 \leq i < \omega$ iff $\mathrm{Ext}^i_{R_j}(T_j, T_j^{(\kappa)}) = 0$ for all cardinals κ, all $1 \leq i < \omega$ and all $1 \leq j \leq n$.

(T3). Let the condition (T3) be satisfied for T. Then there exist $r \geq 0$ and a long exact sequence

$$0 \longrightarrow R \xrightarrow{\varphi^0} T^0 \xrightarrow{\varphi^1} T^1 \longrightarrow \dots \xrightarrow{\varphi^r} T^r \longrightarrow 0,$$

where $T^j \in \mathrm{Add}(T)$ for all $0 \leq j \leq r$. Corollary 5.10 and the long exact sequence

$$0 \longrightarrow R_i \xrightarrow{\varphi^0 \restriction_{R_i}} T_i^0 \xrightarrow{\varphi^1 \restriction_{T_i^0}} T_i^1 \longrightarrow \dots \xrightarrow{\varphi^r \restriction_{T_i^{r-1}}} T_i^r \longrightarrow 0$$

prove the condition (T3) for each T_i as a right R_i-module.

Let the condition (T3) be satisfied for each T_i as a right R_i-module. Then for each $1 \leq i \leq n$ there exist $r_i \geq 0$ and a long exact sequence

$$0 \longrightarrow R_i \xrightarrow{\varphi_i^0} T_i^0 \xrightarrow{\varphi_i^1} T_i^1 \longrightarrow \dots \xrightarrow{\varphi_i^{r_i}} T_i^{r_i} \longrightarrow 0,$$

where $T_i^j \in \mathrm{Add}(T_i)$ for all $0 < j \leq r_i$. Let $r = \max\{r_i \mid 1 \leq i \leq n\}$ and set $\varphi_i^j = 0$, $T_i^j = 0$ if $r_i < j \leq r$. Then Corollary 5.10 and the long exact sequence

$$0 \longrightarrow R \xrightarrow{\bigoplus_{i=1}^n \varphi_i^0} \bigoplus_{i=1}^n T_i^0 \xrightarrow{\bigoplus_{i=1}^n \varphi_i^1} \bigoplus_{i=1}^n T_i^1 \longrightarrow \dots \xrightarrow{\bigoplus_{i=1}^n \varphi_i^r} \bigoplus_{i=1}^n T_i^r \longrightarrow 0$$

prove the condition (T3) for T. So the claim is true. □

Lemma 5.22. *Let $R_1, R_2, \dots, R_n$ be commutative rings, define a ring R as in 5.2, i.e.*

$$R = R_1 \times R_2 \times \dots \times R_n.$$

Then p is a prime ideal of R iff there exist $1 \leq i \leq n$ and

$$p = R_1 \oplus R_2 \oplus \dots \oplus R_{i-1} \oplus p_i \oplus R_{i+1} \oplus \dots \oplus R_n,$$

where p_i is a prime ideal of R_i.

Proof. Implication to the left is easy.

Suppose that p is a prime ideal of R. By Corollary 5.11, $p = I_1 \oplus I_2 \oplus \dots \oplus I_n$ where I_i is an ideal of R_i. Suppose that there are $1 \leq i, j \leq n$ such that $i \neq j$, $I_i \neq R_i$ and $I_j \neq R_j$. Then $\overline{r_i} = (0_1, 0_2, \dots, 0_{i-1}, r_i, 0_{i+1}, \dots, 0_n)$, where $r_i \in R_i \setminus I_i$ and $\overline{r_j} = (0_1, 0_2, \dots, 0_{j-1}, r_j, 0_{j+1}, \dots, 0_n)$, where $r_j \in R_j \setminus I_j$ are two elements of R which are not in p, but $\overline{r_i r_j} \in p$, the contradiction. Thus there exists $1 \leq i \leq n$ such that $p = R_1 \oplus R_2 \oplus \dots \oplus R_{i-1} \oplus p_i \oplus R_{i+1} \oplus \dots \oplus R_n$, where p_i is an ideal of R_i and $p_i \neq R_i$ (see Definition 2.84). Using Remark 5.3, it is easy to prove that p_i is a prime ideal of R_i. □

Corollary 5.23. *Let $R_1, R_2, \dots, R_n$ be commutative rings, define a ring R as in 5.2, i.e.*

$$R = R_1 \times R_2 \times \dots \times R_n.$$

Then p is a prime ideal of R of height 1 iff there exists $1 \leq i \leq n$ and

$$p = R_1 \oplus R_2 \oplus \dots \oplus R_{i-1} \oplus p_i \oplus R_{i+1} \oplus \dots \oplus R_n,$$

where p_i is a prime ideal of R_i of height 1.

Proof. This follows from Lemma 5.22. □

Theorem 5.24. *Let $2 \leq n < \omega$ and let $R_1, R_2, \dots, R_n$ be Dedekind domains. Define a ring R in the following way*

$$R = R_1 \times R_2 \times \dots \times R_n.$$

Then R is a commutative hereditary 1-Gorenstein ring which is not a domain. Moreover, let T be a tilting R-module. Then there exists a subset P of the set of all prime ideals of R of height 1 such that T is equivalent to the Bass tilting module T_P.

Proof. The first part of the assertion follows from Lemma 5.20.

We will prove the 'moreover' part. By Corollary 5.14, we have

$$\begin{aligned} \operatorname{Ext}^j_R(T, M) &\simeq \prod_{i=1}^{n} \prod_{i'=1}^{n} \operatorname{Ext}^j_R(T_i, M_{i'}) \simeq \\ &\simeq \prod_{i=1}^{n} \operatorname{Ext}^j_R(T_i, M_i) \simeq \prod_{i=1}^{n} \operatorname{Ext}^j_{R_i}(T_i, M_i) \end{aligned}$$

for all $1 \leq j < \omega$. Thus $M \in T^{\perp\infty}$ iff $M_i \in T_i^{\perp\infty}$ for each $1 \leq i \leq n$ as R_i-module. By Lemma 5.21 and Theorem 4.13, we have that T_i is a tilting R_i-module and there exists a set $P_i \subset \operatorname{mSpec} R_i$ such that T_i is equivalent to the Bass tilting module T_{i,P_i}. So by Corollary 4.11, $M_i \in T_i^{\perp\infty}$ iff $\operatorname{Ext}^1_{R_i}(R_i/p_i, M_i) = 0$ for all $p_i \in P_i$ and it is iff $\operatorname{Ext}^1_R(R/\overline{p_i}, M) = 0$ for all $\overline{p_i} \in \overline{P_i}$, where $\overline{P_i} = \{R_1 \oplus R_2 \oplus \dots R_{i-1} \oplus p_i \oplus R_{i+1} \oplus \dots \oplus R_n \mid p_i \in P_i\}$. So $M \in T^{\perp\infty}$ iff $\operatorname{Ext}^1_R(R/p, M) = 0$ for all $p \in P$, where $P = \bigcup_{i=1}^n \overline{P_i}$. Thus by Lemma 4.5, Corollary 5.23 and Corollary 4.11, T is equivalent to the Bass tilting module T_p. So the claim is true. □

Lemma 5.25. *Let R be a ring from Theorem 5.24, p be a prime ideal of R and M be an R-module. Then M is p-divisible iff $Mp = M$.*

Proof. Let $p \in \operatorname{Spec} R$. By Lemma 5.22, there is a $1 \leq j \leq n$ such that $p = R_1 \oplus R_2 \oplus \cdots \oplus R_{j-1} \oplus p_j \oplus R_{j+1} \oplus \cdots \oplus R_n$, where p_j is a prime ideal of R_j. By Corollary 5.14, we have

$$\operatorname{Ext}^1_R(R/p, M) \simeq \prod_{i=1}^{n} \operatorname{Ext}^1_{R_i}((R/p)_i, M_i) \simeq \operatorname{Ext}^1_{R_j}(R_j/p_j, M_j).$$

So $\operatorname{Ext}^1_R(R/p, M) = 0$ iff $\operatorname{Ext}^1_{R_j}(R_j/p_j, M_j) = 0$ and by Lemma 4.15, it is iff $M_j p_j = M_j$ and by Remark 5.3, it is iff $Mp = M$. So the claim is true. □

Corollary 5.26. *Let R be a ring from Theorem 5.24 and let P be some subset of a set of all prime ideals of R of height 1. Then the 1-tilting class $T_P^{\perp_\infty}$ induced by the Bass tilting module T_P is equal to the class $\{M \in \text{Mod-}R \mid Mp = M \text{ for all } p \in P\}$.*

Proof. This follows from Corollary 4.11 and from Lemma 5.25. □

Theorem 5.27. *Let $2 \leq n < \omega$ and let $R_1, R_2, \ldots, R_n$ be Dedekind domains. Define a ring R in the following way*

$$R = R_1 \times R_2 \times \cdots \times R_n.$$

Let T be a tilting R-module. Then there exists a subset P of the set of all prime ideal of height 1 of R such that the tilting class induced by T is equal to the class $\{M \in \text{Mod-}R \mid Mp = M \text{ for all } p \in P\}$.

Proof. This follows from Theorem 5.24 and from Corollary 5.26. □

5.2 An important difference from the Dedekind case

In proving that every tilting module over a Dedekind domain is equivalent to some Bass tilting module, we used Corollary 4.11, namely that $(E(R/p))^{\perp_1} = (R/p)^{\perp_1}$. Now we will show that there exist a 1-Gorenstein rings in which the previous is not true.

Lemma 5.28. *Let R be a ring and M be a right (left) R-module. Then M is $\mathcal{CM}$-filtered.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Let $\operatorname{gen}(M) = \kappa$ and let $\{x_\mu \mid \mu < \kappa\}$ be a generating subset of M. Define a sequence $(M_\alpha \mid \alpha \leq \kappa)$ of submodules of M in the following way

$$\begin{aligned} M_0 &= 0 \\ M_\alpha &= \sum_{\mu < \alpha} x_\mu R \quad \alpha \leq \kappa. \end{aligned}$$

Since $M_0 = 0$, $M_\alpha \subseteq M_{\alpha+1}$ ($\alpha < \kappa$), and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for α a limit ordinal, the sequence $(M_\alpha \mid \alpha \leq \kappa)$ is a continuous chain of submodules of M. In order to prove that $(M_\alpha \mid \alpha \leq \kappa)$ is a $\mathcal{CM}$-filtration of M, it remains to prove that $M_\kappa = M$ and that $M_{\alpha+1}/M_\alpha \in \mathcal{CM}$. But $M_\kappa = \sum_{\mu<\kappa} x_\mu R = M$. And for every $\alpha < \kappa$ we have

$$\begin{aligned} M_{\alpha+1}/M_\alpha &= (\sum_{\mu<\alpha+1} x_\mu R)/(\sum_{\mu<\alpha} x_\mu R) = \\ &= \{\sum_{\mu<\alpha+1} x_\mu r_\mu + \sum_{\mu<\alpha} x_\mu R \mid r_\mu \in R \text{ and } r_\mu = 0 \text{ for almost all } \mu < \alpha+1\} = \\ &= \{x_{\alpha+1} r_{\alpha+1} + \sum_{\mu<\alpha} x_\mu R \mid r_{\alpha+1} \in R\}, \end{aligned}$$

so the module $M_{\alpha+1}/M_\alpha$ is cyclic. □

Lemma 5.29 (Auslander Lemma)**.** *Let R be a ring, $n < \omega$ and M be a right (left) R-module. Assume that M is $\mathcal{P}_n$-filtered. Then $M \in \mathcal{P}_n$.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Denote $\mathcal{C}_{-n} = \{\Omega^{-n}(N) \mid N \in \text{Mod-}R\}$. First note that $\mathcal{P}_n = {}^{\perp_1}\mathcal{C}_{-n}$, for this by Lemmas 2.77 and 2.80, $M \in \mathcal{P}_n$ iff $\text{Ext}_R^{n+1}(M, N) = 0$ for all $N \in \text{Mod-}R$ iff $\text{Ext}_R^1(M, \Omega^{-n}(N)) = 0$ for all $N \in \text{Mod-}R$ iff $M \in {}^{\perp_1}\mathcal{C}_{-n}$. Thus $M \in \mathcal{P}_n$ iff $\text{Ext}_R^1(M, C) = 0$ for all $C \in \mathcal{C}_{-n}$.

Let $C \in \mathcal{C}_{-n}$. Since M is ($\mathcal{P}_n = {}^{\perp_1}\mathcal{C}_{-n}$)-filtered there is a continuous chain $(M_\alpha \mid \alpha \leq \mu)$ of submodules of M such that $M_\mu = M$ and $\text{Ext}_R^1(M_{\alpha+1}/M_\alpha, C') = 0$ for all $C' \in \mathcal{C}_{-n}$ and all cardinals $\alpha < \mu$, specially $\text{Ext}_R^1(M_{\alpha+1}/M_\alpha, C) = 0$ for all cardinals $\alpha < \mu$. Using Eklof Lema 4.6, we have that $\text{Ext}_R^1(M, C) = 0$. So $\text{Ext}_R^1(M, C) = 0$ for all $C \in \mathcal{C}_{-n}$ and thus $M \in {}^{\perp_1}\mathcal{C}_{-n} = \mathcal{P}_n$. So the claim is true. □

Lemma 5.30. *Let R be a 1-Gorenstein domain of Krull dimension 1 which is not hereditary. Then there exists $p \in \operatorname{Spec} R$ such that $\operatorname{ht} p = 1$ and $\operatorname{proj\,dim}(R/p) = \infty$.*

Proof. First note that since R is a domain and $\dim R = 1$ we have the following for every prime ideal p of R

$$\operatorname{ht} p = 1 \Leftrightarrow p \in \operatorname{mSpec} R \Leftrightarrow p \in \operatorname{Spec} R \setminus \{0\}.$$

Since R is not a Dedekind domain, R is not hereditary thus there exists an R-module M such that $\operatorname{proj\,dim} M > 1$ (see Lemma 4.3), it follows that $\operatorname{proj\,dim} M = \infty$. By Lemma 5.28 and Auslander Lemma 5.29, we have that there exists a finitely generated (cyclic) R-module N such that $\operatorname{proj\,dim} N = \infty$. Since $R/0 = R \in \mathcal{P}_0$, by Lemma 2.109 and Auslander Lemma 5.29, we have that there exists a prime ideal p of R such that $\operatorname{ht} p = 1$ and $\operatorname{proj\,dim}(R/p) = \infty$. □

Definition 5.31. Let R be a Gorenstein ring and M be a right or left R-module. Then M is *Gorenstein projective* (*Gorenstein injective*), if $M \in {}^{\perp_1}\mathcal{P} = {}^{\perp_1}\mathcal{I}$ ($M \in \mathcal{P}^{\perp_1}$). Denote by $\mathcal{GP}$ ($\mathcal{GI}$) the class of all Gorenstein projective (injective) modules. By Lemma 3.19, Theorems 7.9 and 7.10, the pairs $(\mathcal{GP}, \mathcal{P}) = (\mathcal{GP}, \mathcal{I})$ and $(\mathcal{P}, \mathcal{GI})$ are complete hereditary cotorsion pairs.

Lemma 5.32. *Let R be a ring and $\mathcal{C}$ be a class of right (left) R-modules such that $\mathcal{C} \subseteq \mathcal{I}_1$. Then the class ${}^{\perp_1}\mathcal{C}$ is closed under submodules.*

Proof. We will prove the 'right' version, the proof of the 'left' version is analogical. Let $M \in {}^{\perp_1}\mathcal{C}$ and let N be an arbitrary submodule of M. In order to prove that $N \in {}^{\perp_1}\mathcal{C}$, we need to prove that $\mathrm{Ext}^1_R(N, C) = 0$ for an arbitrary $C \in \mathcal{C}$. Let $C \in \mathcal{C}$. Applying $\mathrm{Hom}_R(-, C)$ to the following short exact sequence of right R-modules $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$, we get part of the induced long exact sequence of abelian groups

$$\mathrm{Ext}^1_R(M, C) \longrightarrow \mathrm{Ext}^1_R(N, C) \longrightarrow \mathrm{Ext}^2_R(M/N, C).$$

Since $\mathrm{Ext}^1_R(M, C) = 0$ by assumption and $\mathrm{Ext}^2_R(M/N, C) = 0$ by Lemma 2.77, we have that $\mathrm{Ext}^1_R(N, C) = 0$. Thus $N \in {}^{\perp_1}\mathcal{C}$. So the claim is true. □

Lemma 5.33. *Let R be a commutative 1-Gorenstein ring and let $p \in \mathrm{Spec}\, R$. Then*

1. *all modules from $(E(R/p))^{\perp_1} \setminus (R/p)^{\perp_1}$ have an infinite injective (and hence an infinite projective) dimension,*
2. *if $\mathrm{proj\,dim}\,(R/p) = \infty$, then $(E(R/p))^{\perp_1} \supsetneq (R/p)^{\perp_1}$.*

Proof. (1). We will prove that if an R-module I has a finite injective dimension then $\mathrm{Ext}^1_R(E(R/p), I) = 0$ implies $\mathrm{Ext}^1_R(R/p, I) = 0$. Let $I \in \mathcal{I}$. Then by Lemma 3.19, $N \in \mathcal{I}_1$ and by Lemma 5.32, the class ${}^{\perp_1}\mathcal{I}_1$ is closed under submodules, thus $\mathrm{Ext}^1_R(E(R/p), I) = 0$ implies $\mathrm{Ext}^1_R(R/p, I) = 0$. So the claim is true.

(2). By Definitions 3.3 and 5.31, we have two cotorsion pairs $(\text{Mod-}R, \mathcal{I}_0) \supseteq (\mathcal{P}, \mathcal{GI})$. By Lemma 4.8 and Eklof Lemma 4.6, we have that $(E(R/p))^{\perp_1} \supseteq (R/p)^{\perp_1}$. Suppose that $(E(R/p))^{\perp_1} = (R/p)^{\perp_1}$. Since $E(R/p) \in \mathcal{I}_1 = \mathcal{P}$ we have that $(E(R/p))^{\perp_1} \supseteq \mathcal{GI}$. And thus $(R/p)^{\perp_1} = (E(R/p))^{\perp_1} \supseteq \mathcal{GI}$, which implies that $(R/p) \in \mathcal{P}$, the contradiction. Thus $(E(R/p))^{\perp_1} \supsetneq (R/p)^{\perp_1}$.

□

5.3 One positive result

By [2], if R is a 1-Gorenstein commutative ring of Krull dimension 1 and S is a multiplicative subset of R which is without zero-divisors, then $S^{-1}R \oplus S^{-1}R/R$ is a 1-tilting module with induced class equal to the class of all S-divisible modules. Now we are going to test whether each of these tilting modules is equivalent to some Bass tilting R-module.

Definition 5.34. Let R be a ring, S be a subset of R and M be a right (left) R-module. Then M is S-divisible if $Ms = M$ ($sM = M$) for every $s \in S$.

Definition 5.35. Let R be a commutative ring and S be a multiplicative subset of R. Then S is called *saturated* if $ab \in S$ implies $a \in S$ and $b \in S$.

Let R be a commutative ring and S be a multiplicative subset of R. Then the set $S' = \{t \in R \mid \exists\, t' \in R\colon tt' \in S\} \supseteq S$ is called the *saturation* of S.

Lemma 5.36. *Let R be a commutative ring, S be a multiplicative subset of R and S' be a saturation of S. Then*

1. *if S is moreover saturated, then $S' = S$,*
2. *S' is a saturated multiplicative subset of R,*
3. *S is without zero-divisors iff S' is without zero-divisors,*
4. *an R-module M is S-divisible iff it is S'-divisible,*
5. *if S is moreover saturated, then $S = R \setminus \bigcup_{p \in V(S)} p$ where $V(S) = \{p \in Spec\, R \mid p \cap S = \emptyset$.*

Proof. (1) is clear from Definition 5.35.

(2) clearly $0 \notin S'$. Let $a, b \in S'$, then there are $a', b' \in R$ such that $aa' \in S$ and $bb' \in S$, so $ab(a'b') \in S$, it follows that $ab \in S'$. If $ab \in S'$ then there is a $c \in R$ such that $(ab)c \in S$, thus $a(bc) \in S$ and $b(ac) \in S$. So (1) is true.

(3) the implication $\Leftarrow$ is trivial.

Let S be without zero-divisors. Suppose that there is a zero-divisor $0 \neq a \in S'$. We have that there is a non-zero $b \in R$ such that $ab = 0$ and there is a $c \in R$ such that $ac \in S$. But then $(ac)b = (ab)c = 0$, a contradiction with the assumption that S is without zero-divisors.

(4) the implication $\Leftarrow$ is trivial.

Suppose that M is S-divisible. Let $0 \neq m \in M$ and $t \in S'$. We have $tt' \in S$ for some $t' \in R$. Thus $m = n(tt')$ for some $n \in M$. It follows that $m = (nt')t$. So (3) is true.

(5) clearly $S \subseteq R \setminus \bigcup_{p \in V(S)} p$.

Let $x \in R \setminus S$, since S is saturated $xR \cap S = \emptyset$. Analogicaly as in the proof of Lemma 2.116, we show that there is a prime ideal from $V(S)$ containing x. So $S = R \setminus \bigcup_{p \in V(S)} p$. $\square$

Lemma 5.37. *Let R be a commutative ring and S be a multiplicative subset of R which is without zero-divisors. Then as an R-module*

$$Supp(S^{-1}R/R) = V(S)^c = \{p \in Spec\, R \mid p \cap S \neq \emptyset\}.$$

Proof. First recall that since S is without zero-divisors, we have that $R \subseteq S^{-1}R$. Let $p \notin V(S)^c$. Then $S \subseteq R \setminus p$, thus by Lemms 2.97 and 2.98,

$$\begin{aligned}(S^{-1}R/R)_{(p)} &\simeq (S^{-1}R)_{(p)}/R_{(p)} \simeq (S^{-1}R \otimes_R R_{(p)})/R_{(p)} \simeq \\ &\simeq (R_{(p)} \otimes_R S^{-1}R)/R_{(p)} \simeq (R_{(p)} \otimes_{S^{-1}R} S^{-1}R)/R_{(p)} \simeq \\ &\simeq R_{(p)}/R_{(p)} \simeq 0.\end{aligned}$$

Let p be a prime ideal of R such that $p \in V(S)^c$. As above, we have $(S^{-1}R/R)_{(p)} \simeq (R_{(p)} \otimes_R S^{-1}R)/R_{(p)}$ as $R_{(p)}$-modules. Now, view $R_{(p)}$ as an R-module, thus we have $(R_{(p)} \otimes_R S^{-1}R)/R_{(p)} \simeq S^{-1}(R_{(p)})/R_{(p)}$ as R-modules. Altogether $(S^{-1}R/R)_{(p)} \simeq S^{-1}(R_{(p)})/R_{(p)}$ as R-modules. Let $s \in p \cap S$. Then $1/s + R_{(p)}$ is a non-zero element of $S^{-1}(R_{(p)})/R_{(p)}$, thus $(S^{-1}R/R)_{(p)} \neq 0$. So the claim is true. □

Theorem 5.38. *Let R be a 1-Gorenstein commutative ring of Krull dimension 1 and S be a multiplicative subset of R which is without zero-divisors. Then the class $\mathcal{C} = \{M \in Mod\text{-}R \mid Ms = M \text{ for all } s \in S\}$ is a 1-tilting class. Denote $P = \{p \in mSpec\, R \mid p \cap S \neq \emptyset\}$. Then the 1-tilting class induced by the Bass 1-tilting R-module T_P is equal $\mathcal{C}$.*

Proof. By 2.97, $S^{-1}R$ is a flat R-module and thus by Lemma 3.19, $\operatorname{proj\,dim} S^1R \leq 1$. By Theorem 7.18, $T = S^{-1}R \oplus S^{-1}R/R$ is a 1-tilting R-module and the 1-tilting class induced by T is equal $\mathcal{C}$. We will prove that T is isomorphic to the Bass 1-tilting R-module T_P as R-modules.

Denote Σ the set of all regular elements of R. By Lemma 3.19, $\Sigma^{-1}R \simeq \bigoplus_{\text{ht } p=0} E(R/p)$ as R-modules. It is an easy excercise to verify that $S^{-1}R \subseteq \Sigma^{-1}R$ as R-modules. So we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & \Sigma^{-1}R & \xrightarrow{\pi} & \Sigma^{-1}R/R & \longrightarrow & 0 \\ & & \Vert & & \uparrow{\scriptstyle \iota_1} & & \uparrow{\scriptstyle \iota_2} & & \\ 0 & \longrightarrow & R & \longrightarrow & S^{-1}R & \xrightarrow{\pi\restriction_{S^{-1}R}} & S^{-1}R/R & \longrightarrow & 0. \end{array}$$

where ι_1 and ι_2 are inclusions. By Lemma 3.19, we have that $\Sigma^{-1}R \simeq E(R)$ and $\Sigma^{-1}R/R \simeq \bigoplus_{p \in \text{mSpec}\, R} E(R/p)$ as R-modules. By Lemma 7.17, we have that $S^{-1}R/R$ is a direct summand of $\Sigma^{-1}R/R$ and since each $E(R/p)$ is indecomposable, we have that $S^{-1}R/R \simeq \bigoplus_{p\in P'} E(R/p)$ as R-modules for some $P' \subseteq \operatorname{mSpec} R$. It is now easy to see that $T \simeq T'_P$ as R-modules.

Using Lemmas 2.52 and 2.122, we have for every maximal ideal q of R that $(\bigoplus_{p\in P'} E(R/p))_{(q)} \neq 0$ iff $q \in P'$. But for every maximal ideal q of R, $(\bigoplus_{p\in P'} E(R/p))_{(q)} \neq 0$ iff $q \in \operatorname{Supp}(\bigoplus_{p\in P'} E(R/p)) \cap \operatorname{mSpec} R$. So $P' = \operatorname{Supp}(\bigoplus_{p\in P'} E(R/p)) \cap \operatorname{mSpec} R$. Using the fact that $\bigoplus_{p\in P'} E(R/p) \simeq S^{-1}R/R$ as R-modules and Lemma 5.37, we have that $P' = \{p \in \operatorname{mSpec} R \mid p \cap S \neq \emptyset\} = P$. $\square$

5.4 Another positive result, an important one

Definition 5.39. Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair of right (left) R-modules. Then $(\mathcal{A}, \mathcal{B})$ is said to be of *weak-finite type* if there is a class (equivalently a set) $\mathcal{S} \subseteq \text{mod-}R$ of right (left) R-modules such that $\mathcal{S}^{\perp} = \mathcal{B}$. Note that in this case clearly $\mathcal{S} \subseteq \mathcal{A}^{<\omega}$.

Lemma 5.40. *Let R be a Gorenstein ring and $(\mathcal{A}, \mathcal{B})$ be a tilting cotorsion pair. Then the class $\mathcal{B}$ (and therefore $\mathcal{A}$) is uniquely determined by the class $\mathcal{B} \cap \mathcal{P}$, more precisely $\mathcal{B} = \{B \in \text{Mod-}R \mid$ there exists a short exaxt sequence $0 \longrightarrow G \longrightarrow C \longrightarrow B \longrightarrow 0$ with $G \in \mathcal{GI}$ and $C \in \mathcal{B} \cap \mathcal{P}\}$.*

Proof. Denote $\mathcal{B}' = \{B \in \text{Mod-}R \mid$ there exists a short exaxt sequence $0 \longrightarrow G \longrightarrow C \longrightarrow B \longrightarrow 0$ with $G \in \mathcal{GI}$ and $C \in \mathcal{B} \cap \mathcal{P}\}$. Let $B \in \mathcal{B}$. By 5.31, the class $\mathcal{P}$ is special precovering so there is a short exact sequence

$$0 \longrightarrow G \longrightarrow C \longrightarrow B \longrightarrow 0$$

with $G \in \mathcal{GI}$ and $C \in \mathcal{P}$. By Lemma 3.10, we have $\mathcal{GI} = \mathcal{P}^{\perp_1} \subseteq \mathcal{A}^{\perp_1} = \mathcal{B}$, so $G \in \mathcal{B}$ and so $C \in \mathcal{B} \cap \mathcal{P}$, thus $B \in \mathcal{B}'$.

Let $B \in \mathcal{B}'$. Let

$$0 \longrightarrow G \longrightarrow C \longrightarrow B \longrightarrow 0$$

be a short exact sequence with $G \in \mathcal{GI}$ and $C \in \mathcal{B} \cap \mathcal{P}$. By 7.12, the class $\mathcal{B}$ is coresolving and since $G \in \mathcal{GI} \subseteq \mathcal{B}$ and $C \in \mathcal{B} \cap \mathcal{P} \subseteq \mathcal{B}$, we have $B \in \mathcal{B}$. So the claim is true. $\square$

Lemma 5.41. *Let R be a noetherian commutative ring and N be an R-module. Then the following are equivalent*

1. $N \in \mathcal{I}_0$,
2. $\operatorname{Ext}^1_R(R/p, N) = 0$ *for all* $p \in \operatorname{Spec} R$.

Proof. The implication (1) $\Rightarrow$ (2) is trivial.

Let N be an R-module such that $\mathrm{Ext}^1_R(R/p, N) = 0$ for all $p \in \operatorname{Spec} R$. Since R is noetherian, every ideal I of R is finitely generated. So by Lemma 2.109, every ideal I of R is finitely $\{R/p \mid p \in \operatorname{Spec} R\}$-filtered. So by Eklof Lemma 4.6, $\mathrm{Ext}^1_R(I, N) = 0$ for every ideal I of R. So by Lemma 2.78, N is injective. □

Corollary 5.42. *Let R be a noetherian commutative ring and N be an R-module. Then the following are equivalent*

1. $N \in \mathcal{I}_n$,
2. $\mathit{Ext}^{n+1}_R(R/p, N) = 0$ *for all* $p \in \mathit{Spec}\, R$.

Proof. By Lemmas 2.78, 2.80 and 5.41 we have

$$\begin{aligned} N \in \mathcal{I}_n &\Leftrightarrow \mathrm{Ext}^{n+1}_R(M, N) = 0 \text{ for all } M \in \text{Mod-}R \Leftrightarrow \\ &\Leftrightarrow \mathrm{Ext}^1_R(M, \Omega^{-n}(N)) = 0 \text{ for all } M \in \text{Mod-}R \Leftrightarrow \\ &\Leftrightarrow \Omega^{-n}(N) \in \mathcal{I}_0 \Leftrightarrow \mathrm{Ext}^1_R(R/p, \Omega^{-n}(N)) = 0 \text{ for all } p \in \operatorname{Spec} R \Leftrightarrow \\ &\Leftrightarrow \mathrm{Ext}^{n+1}_R(R/p, N) = 0 \text{ for all } p \in \operatorname{Spec} R \end{aligned}$$

So the claim is true. □

Corollary 5.43. *Let R be a noetherian commutative ring and N be an R-module. Then $N \in \mathcal{I}_1$ iff $N \in (\mathit{Spec}\, R)^{\perp_1}$.*

Proof. By Corollary 5.42 and Lemma 2.80, we have

$$\begin{aligned} N \in \mathcal{I}_1 &\Leftrightarrow \mathrm{Ext}^2_R(R/p, N) = 0 \text{ for all } p \in \operatorname{Spec} R \Leftrightarrow \\ &\Leftrightarrow \mathrm{Ext}^1_R(\Omega^1(R/p), N) = 0 \text{ for all } p \in \operatorname{Spec} R \Leftrightarrow \\ &\Leftrightarrow \mathrm{Ext}^1_R(p, N) = 0 \text{ for all } p \in \operatorname{Spec} R \Leftrightarrow \\ &\Leftrightarrow N \in (\operatorname{Spec} R)^{\perp_1}. \end{aligned}$$

The third equivalence follows from the fact that p is the first syzygy of R/p in the projective resolution beginning with

$$\ldots \longrightarrow R \longrightarrow R/p \longrightarrow 0.$$

So the claim is true. □

Lemma 5.44. *Let R be a commutative Gorenstein ring. Then $\mathit{Ass}(R) = \{p \in \mathit{Spec}\, R \mid \mathit{ht}\, p = 0\}$.*

Proof. Suppose that $p \in \mathrm{Ass}(R)$. Then $R/p \subseteq R$, so $R/p \subseteq E(R)$ and thus $E(R/p) \subseteq E(R)$. By Lemmas 3.19 and 2.121, ht $p = 0$.

Suppose that ht $p = 0$. Then $E(R/p) \subseteq E(R)$, which implies that $E(R/p) \cap R \neq 0$. So by Lemmas 2.108 and 2.121, we have that $\mathrm{Ass}(E(R/p) \cap R) \supseteq \{p\}$ and thus $p \in \mathrm{Ass}(R)$. □

Remark 5.45. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair of weak-finite type. Then the pair $(\mathcal{A}, \mathcal{B})$ is uniquely determined by $\mathcal{A}^{<\omega}$. For this, denote $\mathcal{S}$ the set of strongly finitely presented modules such that $\mathcal{S}^{\perp_1} = \mathcal{B}$. By Definition 5.39, we have that $\mathcal{S} \subseteq \mathcal{A}^{<\omega} \subseteq \mathcal{A}$. So $\mathcal{B} = (\mathcal{A}^{<\omega})^{\perp_1}$ and thus $\mathcal{A} = {}^{\perp_1}(\mathcal{A}^{<\omega})^{\perp_1}$.

So if if we have two cotorsion pairs $(\mathcal{A}, \mathcal{B})$, $(\mathcal{C}, \mathcal{D})$ both of weak-finite type such that $\mathcal{A}^{<\omega} = \mathcal{C}^{<\omega}$ then $(\mathcal{A}, \mathcal{B}) = (\mathcal{C}, \mathcal{D})$.

Lemma 5.46. *Let R be a 1-Gorenstein commutative ring of Krull dimension 1 and $(\mathcal{A}, \mathcal{B})$ be a tilting cotorsion pair of R-modules. Denote $\mathcal{B}' = \mathcal{B} \cap \mathcal{P}$ and $\mathcal{A}' = {}^{\perp_1}\mathcal{B}'$. Then the pair $(\mathcal{A}', \mathcal{B}')$ is a cotorsion pair of weak-finite type, the class $\mathcal{A}'$ is closed under submodules and $\mathcal{A}'^{<\omega} \supseteq \mathit{Spec}\, R \cup \{R/p \mid p \in \mathit{Spec}\, R \wedge \mathit{ht}\, p = 0\}$.*

Proof. Since $(\mathcal{A}, \mathcal{B})$ is a tilting cotorsion pair (thus 1-tilting cotosion pair) and since every tilting module is of finite type, we have that there is a set $\mathcal{S} \subseteq \mathcal{P}_1^{<\omega}$ such that $\mathcal{S}^{\perp_1} = \mathcal{S}^{\perp_\infty} = \mathcal{B}$. By Corollary 5.43, we have $\mathcal{P} = \mathcal{I} = \mathcal{I}_1 = (\mathrm{Spec}\, R)^{\perp_1}$. Denote $\mathcal{S}' = \mathcal{S} \cup \mathrm{Spec}\, R$, so $\mathcal{S}'^{\perp_1} = \mathcal{B} \cap \mathcal{P} = \mathcal{B}'$. Using Lemma 2.83 we have that $\mathcal{S}' \subseteq$ mod-R and using lemma 3.3 we have that the pair $(\mathcal{A}', \mathcal{B}')$ is a cotorsion pair of weak-finite type.

Since $\mathcal{B}' = \mathcal{B} \cap \mathcal{P} \subseteq \mathcal{P} = \mathcal{I} = \mathcal{I}_1$, by Lemma 5.32, we have that $\mathcal{A}' = {}^{\perp_1}\mathcal{B}'$ is closed under submodules.

By Definition 5.39, we know that $\mathrm{Spec}\, R \subseteq \mathcal{A}'^{<\omega}$. By Remark 3.4, $R \in \mathcal{A}'$ and by Lemma 5.44, $R/p \subseteq R$ for each $p \in \mathrm{Spec}\, R$ such that ht $p = 0$. So since $\mathcal{A}'$ is closed under submodules, we have that $\mathcal{A}'^{<\omega} \supseteq \mathrm{Spec}\, R \cup \{R/p \mid p \in \mathrm{Spec}\, R, \mathrm{ht}\ p = 0\}$. □

Lemma 5.47. *Let R be a 1-Gorenstein commutative ring of Krull dimension 1 and $(\mathcal{A}, \mathcal{B})$ be a tilting cotorsion pair of R-modules. Denote $\mathcal{A}'$, $\mathcal{B}'$ as in Lemma 5.46 and $P_1 = \{p \in \mathit{Spec}\, R \mid \mathit{ht}\, p = 1 \wedge R/p \in \mathcal{A}'\}$. Then*

$$\mathcal{B}' = \mathcal{P} \cap \bigcap_{p \in P_1} (R/p)^{\perp_1} \Leftrightarrow \mathcal{B} = \bigcap_{p \in P_1} (E(R/p))^{\perp_1}.$$

Proof. First suppose that $\mathcal{B}' = \mathcal{P} \cap \bigcap_{p \in P_1} (R/p)^{\perp_1}$. Let $B \in \mathcal{B}$. By Definition 5.31, $\mathcal{P}$ is a special precovering class so there is a short exact sequence

$$\mathcal{E}\colon\ 0 \longrightarrow G \longrightarrow P \longrightarrow B \longrightarrow 0$$

with $G \in \mathcal{GI} \subseteq \mathcal{B}$ (see the proof of Lemma 5.40) and $P \in \mathcal{P}$. Since $\mathcal{B}$ is closed under extensions, using Lemma 5.33, we get that

$$\begin{aligned} P \in \mathcal{B} \cap \mathcal{P} &= \mathcal{B}' = \mathcal{P} \cap \bigcap_{p \in P_1} (R/p)^{\perp_1} = \\ &= \mathcal{P} \cap \bigcap_{p \in P_1} (E(R/p))^{\perp_1} \subseteq \bigcap_{p \in P_1} (E(R/p))^{\perp_1}. \end{aligned}$$

So we have that $P \in \bigcap_{p \in P_1} (E(R/p))^{\perp_1}$ and it is iff $\mathrm{Ext}^1_R(E(R/p), P) = 0$ for all $p \in P_1$. Let $p \in P_1$. Applying $\mathrm{Hom}_R(E(R/p), -)$ to the short exact sequence $\mathcal{E}$ we get part of the induced long exact sequence

$$\mathrm{Ext}^1_R(E(R/p), P) \longrightarrow \mathrm{Ext}^1_R(E(R/p), B) \longrightarrow \mathrm{Ext}^2_R(E(R/p), G).$$

Since $\mathrm{Ext}^1_R(E(R/p), P) = \mathrm{Ext}^2_R(E(R/p), G) = 0$ we get that $\mathrm{Ext}^1_R(E(R/p), B) = 0$. So $B \in \bigcap_{p \in P_1} (E(R/p))^{\perp_1}$.

Let $B \in \bigcap_{p \in P_1} (E(R/p))^{\perp_1}$. We have the short exact sequence $\mathcal{E}$ with $G \in \mathcal{GI} \subseteq \mathcal{B} \subseteq \bigcap_{p \in P_1} (E(R/p))^{\perp_1}$ (by previous part) and $P \in \mathcal{P}$. It follows that $P \in \mathcal{P} \cap \bigcap_{p \in P_1} (E(R/p))^{\perp_1} = \mathcal{P} \cap \bigcap_{p \in P_1} (R/p)^{\perp_1} = \mathcal{B}' \subseteq \mathcal{B}$. By Theorem 7.12, $\mathcal{B}$ is coresolving class and thus $B \in \mathcal{B}$.

Suppose now that $B = \bigcap_{p \in P_1} (E(R/p))^{\perp_1}$. We have

$$\mathcal{B}' = \mathcal{P} \cap \mathcal{B} = \mathcal{P} \cap \bigcap_{p \in P_1} (E(R/p))^{\perp_1} = \mathcal{P} \cap \bigcap_{p \in P_1} (R/p)^{\perp_1}.$$

So the claim is true. □

Proposition 5.48. *Let R be a 1-Gorenstein commutative ring of Krull dimension 1 and M be an R-module. Then*

1. *if $M \in \mathcal{P}^{<\omega}$, and $E(M) \simeq \bigoplus_{ht\, p=0} E(R/p)^{\alpha_p}$ for some $\alpha_p \geq 0$, then M is projective,*
2. *if R is moreover local with maximal ideal m, then*
   (a) $\mathcal{P} \cap (R/m)^{\perp_1} = \mathcal{I}_0$.
   (b) $({}^{\perp_1}R)^{<\omega} = ({}^{\perp_1}\mathcal{P})^{<\omega} = \mathcal{GP}^{<\omega}$

Proof. (1). Let

$$0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0$$

be a minimal injective resolution of M. By Lemma 7.6, $E(M)$ is a flat R-module. Since $E(M)/M \in \mathcal{I}_0 \subseteq \mathcal{I}_1 = \mathcal{F}_1$, we have by Lemma 2.79, that M is flat and since M is finitely generated, Lemmas 2.83 and 2.66 imply that M is projective.

(2)(a). By Corollary 5.43, we have $\mathcal{P} \cap (R/m)^{\perp_1} = (\operatorname{Spec} R)^{\perp_1} \cap (R/m)^{\perp_1} = (\operatorname{Spec} R \cup \{R/m\})^{\perp_1}$. Denote $\mathcal{C} = {}^{\perp_1}(\mathcal{P} \cap (R/m)^{\perp_1})$. Thus $(\mathcal{C}, \mathcal{P} \cap (R/m)^{\perp_1})$ is a cotorsion pair. Since $\mathcal{P} \cap (R/m)^{\perp_1} \subseteq \mathcal{P} = \mathcal{I}_1$, Lemma 5.32 implies that $\mathcal{C}$ is closed under submodules. Clearly $R \in \mathcal{C}$ and thus by Lemma 5.44, we have that $\{R/p \mid p \in \operatorname{Spec} R \wedge \operatorname{ht} p = 0\} \subseteq \mathcal{C}$. So $\{R/p \mid p \in \operatorname{Spec} R\} \subseteq \mathcal{C}$. But by Lemma 5.41, $\{R/p \mid p \in \operatorname{Spec} R\}^{\perp_1} = \mathcal{I}_0$, so $\mathcal{C}^{\perp_1} = \mathcal{P} \cap (R/m)^{\perp_1} = \mathcal{I}_0$.

(2)(b). Inclusion $({}^{\perp_1}R)^{<\omega} \supseteq ({}^{\perp_1}\mathcal{P})^{<\omega}$ and the second equation are clear. Let $M \in ({}^{\perp_1}R)^{<\omega}$. By Lemma 7.8, $\operatorname{Ext}^1_R(M, R^{(\kappa)}) = 0$ for every cardinal κ. Let $N \in \mathcal{P} = \mathcal{P}_1$. Thus there is a short exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow R^{(\lambda)} \longrightarrow N \longrightarrow 0$$

with K projective. Applying $\operatorname{Hom}_R(M, -)$ to the previous short exact sequence, we get part of the induced long exact sequence of abelian groups

$$\operatorname{Ext}^1_R(M, R^{(\lambda)}) \longrightarrow \operatorname{Ext}^1_R(M, N) \longrightarrow \operatorname{Ext}^2_R(M, K).$$

Since $\operatorname{Ext}^1_R(M, R^{(\lambda)}) = \operatorname{Ext}^2_R(M, K) = 0$ ($K \in \mathcal{I}_1$), we get that $\operatorname{Ext}^1_R(M, N) = 0$. So $M \in ({}^{\perp_1}\mathcal{P})^{<\omega}$. □

Theorem 5.49. *Let R be a 1-Gorenstein commutative local ring of Krull dimension 1 with maximal ideal m and T be a tilting R-module. Then there is a set $P_1 \subseteq \{p \in \operatorname{Spec} R \mid \operatorname{ht} p = 1\}$ such that T is equivalent to the Bass tilting R-module T_{P_1}. Moreover if we denote $(\mathcal{A}, \mathcal{B})$ the tilting cotorsion pair induced by T and $\mathcal{A}'$, as in Lemma 5.46, then we have that*

$$T^{\perp_\infty} = T^{\perp_1} = \begin{cases} \{M \in \text{Mod-}R \mid \operatorname{Ext}^1_R(E(R/m), M) = 0\}, & \text{if } R/m \in \mathcal{A}' \\ \text{Mod-}R, & \text{if } R/m \notin \mathcal{A}'. \end{cases}$$

Proof. Denote $\mathcal{B}'$ and P_1 as in Lemma 5.47. We are going to show that $\mathcal{B} = \{M \in \text{Mod-}R \mid \operatorname{Ext}^1_R(E(R/p), M) = 0 \text{ for all } p \in P_1\}$ (and thus T is equivalent to the Bass tilting R-module T_{P_1}). By Lemma 5.47, it is enough to show that $\mathcal{B}' = \mathcal{P} \cap \bigcap_{p \in P_1} (R/p)^{\perp_1}$. Since R is local we only need to prove following two cases

1. if $R/m \in \mathcal{A}'$, then $\mathcal{B}' = \mathcal{I}_0$ (see Proposition 5.48),
2. if $R/m \notin \mathcal{A}'$, then $\mathcal{B}' = \mathcal{P} = \mathcal{I}$ (or equivalently $\mathcal{A}' = \mathcal{GP}$).

Suppose $R/m \in \mathcal{A}'$. By Lemma 5.46, $\{R/p \mid p \in \operatorname{Spec} R \wedge \operatorname{ht} p = 0\} \subseteq \mathcal{A}'$. So $\{R/p \mid p \in \operatorname{Spec} R\} \subseteq \mathcal{A}'$, thus by Lemma 5.41, $\mathcal{B}' = \mathcal{I}_0$, so the case (1) is clear.

Suppose $R/m \notin \mathcal{A}'$. Let $M \in \mathcal{A}'$. Suppose that $E(M) \simeq (E(R/m))^{\alpha_m} \oplus \bigoplus_{\operatorname{ht} p = 0} (E(R/p))^{\alpha_p}$ for some $\alpha_m \geq 1$, $\alpha_p \geq 0$ (see Theorem 2.123). Then $M \cap E(R/m) \neq 0$, so $M \cap R/m \neq 0$ and since R/m is a simple R-module, we have

$R/m \subseteq M$, thus $R/m \in \mathcal{A}'$, a contradiction. Thus if $M \in \mathcal{A}'$, then $E(M) = \bigoplus_{\text{ht } p=0}(E(R/p))^{\alpha_p}$ for some $\alpha_p \geq 0$.

Now let $F \in \mathcal{A}'^{<\omega}$. By Lemma 7.7 and Proposition 5.48, there is a short exact sequence

$$0 \longrightarrow F \longrightarrow F' \longrightarrow G \longrightarrow 0$$

with $F' \in \mathcal{P}^{<\omega}$ and $G \in ({}^{\perp_1}R)^{<\omega} \subseteq \mathcal{GP}$. Since $\mathcal{B}' \subseteq \mathcal{P} = \mathcal{I}$ we have that $\mathcal{A}' \supseteq \mathcal{GP}$, so $G \in \mathcal{A}'$ and thus $F' \in (\mathcal{A}' \cap \mathcal{P})^{<\omega}$. By the previous part and by Proposition 5.48, $F' \in \mathcal{P}_0$ and thus $F' \in \mathcal{GP}$. Since $\mathcal{GP}$ is a resolving class, we have that $F \in \mathcal{GP}$, so $\mathcal{A}'^{<\omega} \subseteq \mathcal{GP}^{<\omega}$. We have already proved that $\mathcal{A}' \supseteq \mathcal{GP}$, so $\mathcal{A}'^{<\omega} = \mathcal{GP}^{<\omega}$. By Remark 5.45 ($(\mathcal{GP}, \mathcal{I})$ is of weak-finite type by Corollary 5.43), we have that $\mathcal{A}' = \mathcal{GP}$. So the claim is true. □

5.5 Solution of the problem

Definition 5.50. Let R be a commutative ring, S be a multiplicative subset of R and $\mathcal{B}$ be a class of R-modules. Then the class $\mathcal{B}_S$ of $S^{-1}R$-modules is defined by $\mathcal{B}_S = \{N \in \text{Mod-}S^{-1}R \mid N \simeq S^{-1}M \text{ for some } M \in \mathcal{B}\}$. For a prime ideal p of R and $S = R \setminus p$, we also use the notation $\mathcal{B}_{(p)} = \mathcal{B}_S$.

Proposition 5.51. *Let R be a 1-Gorenstein commutative ring with Krull dimension 1 and m, m' be maximal ideals of R and m be of height 1. Denote $T_{\{m\}}$ and $R_{\{m\}}$ as in Definition 3.20. Then*

$$((T_{\{m\}})_{(m')})^{\perp_1} = \begin{cases} (E(R/m))^{\perp_1}, & \text{if } m' = m \\ \text{Mod-}R_{(m')}, & \text{if } m' \neq m, \end{cases}$$

where $E(R/m)$ is taken as an $R_{(m)}$-module.

Proof. First note that $((T_{\{m\}})_{(m')})^{\perp_1} = ((R_{\{m\}})_{(m')} \oplus (E(R/m))_{(m')})^{\perp_1}$, where $E(R/m)$ is taken as an R-module. As in Remark 3.22, we have the following short exact sequence of R-modules

$$0 \longrightarrow R \longrightarrow R_{\{m\}} \longrightarrow E(R/m) \longrightarrow 0.$$

Applying $- \otimes_R R_{(m')}$, we get the following short exact sequence of $R_{(m')}$-modules

$$0 \longrightarrow R_{(m')} \longrightarrow (R_{\{m\}})_{(m')} \longrightarrow (E(R/m))_{(m')} \longrightarrow 0.$$

Applying $\text{Hom}_{R_{(m')}}(-, M)$ where M is an arbitrary $R_{(m')}$-module, we get part of the induced long exact sequence of abelian groups

$$\text{Ext}^1_{R_{(m')}}((E(R/m))_{(m')}, M) \longrightarrow \text{Ext}^1_{R_{(m')}}((R_{\{m\}})_{(m')}, M) \longrightarrow \text{Ext}^1_{R_{(m')}}(R_{(m')}, M).$$

First note that $\operatorname{Ext}^1_{R_{(m')}}(R_{(m')}, M) = 0$ since $R_{(m')}$ is a projective $R_{(m')}$-module. By Lemma 2.122, we have that $E(R/m)$ is an $R_{(m)}$-module and

$$(E(R/m))_{(m')} \simeq \begin{cases} E(R/m), & \text{if } m' = m \\ 0, & \text{if } m' \neq m \end{cases}$$

as $R_{(m')}$-modules. So the claim is true. □

Theorem 5.52. *Let R be a 1-Gorenstein commutative ring and T be a tilting R-module. Then there is a set $P \subseteq \{p \in \operatorname{Spec} R \mid ht\, p = 1\}$ such that T is equivalent to the Bass tilting R-module T_P.*

Proof. If dim $R = 0$, we can use Lemma 5.1. So suppose that dim $R = 1$. Denote $\mathcal{B}$ the 1-tilting class induced by T. First note that Lemma 3.19 implies that $R_{(m)}$ is a 1-Gorenstein commutative local ring for all $m \in \operatorname{mSpec} R$ and also note that the Theorem 7.16 implies that $\mathcal{B}_{(m)}$ is a 1-tilting class in Mod-$R_{(m)}$ for all $m \in \operatorname{mSpec} R$. Denote $\mathcal{A}'_{(m)}$ and $\mathcal{B}'_{(m)}$ as in Lemma 5.46. Let M be an arbitrary R-module. By Theorem 7.16, we have that $M \in \mathcal{B}$ iff $M_{(m)} \in \mathcal{B}_{(m)}$ for all $m \in \operatorname{mSpec} R$. Note that if $m \in \operatorname{mSpec} R$ is such that ht $m = 0$, then $R_{(m)}$ is 0-Gorenstein and so by Lemma 5.1, $M_m \in \mathcal{B}_{(m)}$ every time. By Theorem 5.49, we have for every maximal ideal m of R of height 1 that

$$M_{(m)} \in \mathcal{B}_{(m)} \Leftrightarrow \begin{cases} \operatorname{Ext}^1_{R_{(m)}}(E_{R_{(m)}}(R_{(m)}/mR_{(m)}), M_{(m)}) = 0, & \text{if } R_{(m)}/mR_{(m)} \in \mathcal{A}'_{(m)} \\ \text{every time}, & \text{if } R_{(m)}/mR_{(m)} \notin \mathcal{A}'_{(m)}. \end{cases}$$

Denote $P = \{m \in \operatorname{mSpec} R \mid \text{ht } m = 1 \wedge R_{(m)}/mR_{(m)} \in \mathcal{A}'_{(m)}\}$. So we have that

$$M \in \mathcal{B} \Leftrightarrow \operatorname{Ext}^1_{R_{(m)}}(E_{R_{(m)}}(R_{(m)}/mR_{(m)}), M_{(m)}) = 0 \text{ for all } m \in P.$$

By Lemma 2.120, we have that $E(R/m)$ is an $R_{(m)}$-module and that $E_{R_{(m)}}(R_{(m)}/mR_{(m)}) \simeq E(R/m)$ as $R_{(m)}$-modules. So, to the claim, it is enough to prove that

$$\operatorname{Ext}^1_{R_{(m)}}(E(R/m), M_{(m)}) = 0 \Leftrightarrow \operatorname{Ext}^1_R(E(R/m), M) = 0$$

for all $m \in \{p \in \operatorname{Spec} R \mid \text{ht } p = 1\}$, where $E(R/m)$ on the left hand side is taken as an $R_{(m)}$-module and $E(R/m)$ on the right hand side is taken as an R-module (then we have that T is equivalent to the Bass tilting R-module T_P). The previous statement is equivalent to the following statement

$$M_{(m)} \in (E(R/m))^{\perp_1} \Leftrightarrow M \in (E(R/m))^{\perp_1}$$

for all $m \in \{p \in \operatorname{Spec} R \mid \text{ht } p = 1\}$, where $E(R/m)$ on the left hand side is taken as an $R_{(m)}$-module and $E(R/m)$ on the right hand side is taken as an R-module.

Let $m \in \{p \in \operatorname{Spec} R \mid \operatorname{ht} p = 1\}$. By Lemma 3.21 and Remark 3.22, we have that $T_{\{m\}} = R_{\{m\}} \oplus E(R/m)$ (we use the notation from Definition 3.20) is a 1-tilting R-module with the induced 1-tilting class equal to $(E(R/m))^{\perp_1}$, where $E(R/m)$ is taken as an R-module. So $M \in (E(R/m))^{\perp_1}$ iff $M \in (T_{\{m\}})^{\perp_1}$ and by Theorem 7.16, it is iff $M_{(m')} \in ((T_{\{m\}})_{(m')})^{\perp_1}$ for all $m' \in \operatorname{mSpec} R$. But by Proposition 5.51, it is iff $M_{(m)} \in (E(R/m))^{\perp_1}$, where $E(R/m)$ is taken as an $R_{(m)}$-module. So the claim is true. ☐

6 Cotilting modules over 1-Gorenstein commutative rings

Definition 6.1. Let R be a ring and S be a commutative ring such that R is an S-algebra (see Definition 2.92) and denote φ the ring homomorphism from S to R. Let E be an injective cogenerator for S-Mod, which exists by Remark 2.63. Let M be an arbitrary right R-module. Then M is clearly a left S-module via $sm = m\varphi(s)$. The *dual module* M^d is defined by $M^d = \operatorname{Hom}_S({}_SM_R, {}_SE)$, it is clearly a left R-module.

Theorem 6.2. *Let R be a ring and, $n < \omega$ and T be an n-tilting right R-module. Then the dual module T^d is an n-cotilting left R-module.*

Proof. This is part of the Theorem 8.1.2. from [11]. ☐

Definition 6.3. Let R be a commutative 1-Gorenstein ring and let P be a subset of the set of all prime ideals of R of height 1. By Definition 3.20 and Lemma 3.21, T_P is a 1-tilting R-module. Consider the injective cogenerator $E = \bigoplus_{p \in mSpecR} E(R/p)$ (see Lemma 2.64). By Theorem 6.2, $C_P = (T_P)^d = \operatorname{Hom}_R(T_P, E)$ is a 1-cotilting R-module, called *Bass cotilting R-module.*

Definition 6.4. Let R be a ring and $\mathcal{C}$ be a class of left R-modules. Then $\mathcal{C}$ is of *cofinite type* if there exist $n < \omega$ and a class (equivalently a set) $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{C} = \mathcal{S}^{\intercal_\infty}$.

Let C be a left R-module. Then C is of *cofinite type* if the class ${}^{\perp_\infty}C$ is of cofinite type.

Theorem 6.5. *Let R be a ring and $n < \omega$.*

1. *Let C be an n-cotilting left R-module. Then C is of cofinite type iff there is an n-tilting right R-module T_C such that C is equivalent to $(T_C)^d$.*
2. *If C and C' are n-cotilting left R-modules of cofinite type, then C' is equivalent C iff the n-tilting right R-modules T_C and $T_{C'}$ are equivalent.*

Proof. This is part of the Theorem 8.1.13. from [11]. □

Theorem 6.6. *Let R be a left noetherian ring such that $\mathcal{F}_1 = \mathcal{P}_1$ (in particular, let R be a 1-Gorenstein ring). Then all 1-cotilting classes are of cofinite type.*

Proof. This is part of the Theorem 8.2.8. from [11]. □

Theorem 6.7. *Let R be a 1-Gorenstein commutative ring and C be a cotilting R-module. Then there is a set $P \subseteq \{p \in \mathit{Spec}\, R \mid \mathit{ht}\, p = 1\}$ such that C is equivalent to the Bass cotilting R-module C_P.*

Proof. Firts note, that C is a 1-cotilting R-module. By Theorem 6.6, C is of cofinite type. By Theorem 6.5, there exists a 1-tilting R-module T_C such that $(T_C)^d$ is equivalent to C. By Theorem 5.52, there is a set $P \subseteq \{p \in \operatorname{Spec} R \mid \operatorname{ht}\, p = 1\}$ such that T_C equivalent to the Bass tilting R-module T_P. By Theorem 6.5, C is equivalent to the Bass cotilting R-module C_P. □

7 Appendix

Lemma 7.1. *Let R, S be rings. Let $\mathbf{C}$ be a full subcategory of the category of all right (left) R-modules and $\mathbf{D}$ be a full subcategory of the category of all right (left) S-modules. Let $F\colon \mathbf{C} \to \mathbf{D}$ ($G\colon \mathbf{C} \to \mathbf{D}$) be an additive covariant (contravariant) functor. If*

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is split exact in $\mathbf{C}$, then both

$$0 \longrightarrow F(K) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N) \longrightarrow 0,$$

$$0 \longrightarrow G(N) \xrightarrow{G(g)} G(M) \xrightarrow{G(f)} G(K) \longrightarrow 0$$

are split exact in $\mathbf{D}$. In particular, if $g\colon M \to N$ is an isomorphism, then both $F(g)$ and $G(g)$ are isomrphisms.

Proof. This is the Proposition 16.2. from [1]. □

Lemma 7.2. *Let R, S be rings, A be a left R-module, B be an (S,R)-bimodule and C be an injective left S-module. Then*

$$Ext^i_R(A, Hom_S(B,C)) \simeq Hom_S(Tor^i_R(B,A), C)$$

as abelian groups for all $i \geq 0$.

Proof. This is the Theorem 3.2.1. from [10] □

Theorem 7.3. *Let R, S be commutative rings, S be a flat R-algebra and M, N be R-modules. If R is noetherian and M is finitely generated, then*

$$Ext^i_R(M,N) \otimes_R S \simeq Ext^i_S(M \otimes_R S, N \otimes_R S)$$

as S-modules for all $i \geq 0$.

Specially if R is noetherian and M is finitely generated, then

$$Ext^i_R(M,N)_{(p)} \simeq Ext^i_{R_{(p)}}(M_{(p)}, N_{(p)})$$

as $R_{(p)}$-modules for all $i \geq 0$.

Proof. The first part is the Theorem 3.2.5 from [10], the second part follows from Definition 2.92. □

Theorem 7.4. *Let R be a Dedekind domain and M be a finitely generated R-module. Then*

$$M \simeq P \oplus \bigoplus_{p \in mSpec\, R} M_p,$$

where P is a finitely generated projective R-module and each R-module M_p which is non-zero is of the form

$$M_p \simeq R/p^{\delta(p,1)} \oplus R/p^{\delta(p,2)} \oplus \cdots \oplus R/p^{\delta(p,l(p))},$$

where $0 < \delta(p,1) \leq \delta(p,2) \leq \cdots \leq \delta(p,l(p))$ are positive integers. Moreover, this decomposition is uniquely determined by M.

Proof. This is part of the Theorem 6.3.23. from [6]. □

Theorem 7.5. *Let R be a commutative local ring with maximal ideal m and M be a finitely generated R-module. Then M is projective iff $Tor_R^1(M, R/m) = 0$.*

Proof. This is the Corollary 2 to Proposition 5 in Chapter II, Section 3 from [7]. □

Lemma 7.6. *Let R be a commutative noetherian ring. Then the following are equivalent*

1. *R is Gorenstein*
2. *$\text{flat dim}\, E(R/m) = \text{ht}\, m$ for any maximal ideal m,*
3. *$\text{flat dim}\, E(R/m) < \infty$ for any maximal ideal m,*
4. *$\text{flat dim}\, E(R/p) = \text{ht}\, m$ for any $p \in Spec\, R$,*
5. *$\text{flat dim}\, E(R/p) =< \infty$ for any $p \in Spec\, R$.*

Proof. This is the Proposition 2.1. from [13]. □

Lemma 7.7. *Let R be a Gorenstein ring. Then for each finitely generated R-module M, there exist short exact sequences*

$$0 \longrightarrow A_M \longrightarrow B_M \longrightarrow M \longrightarrow 0$$

with $A_M \in \mathcal{P}^{<\omega}$ and $B_M \in ({}^{\perp_1}R)^{<\omega}$, and

$$0 \longrightarrow M \longrightarrow C_M \longrightarrow D_M \longrightarrow 0$$

with $C_M \in \mathcal{P}^{<\omega}$ and $D_M \in ({}^{\perp_1}R)^{<\omega}$.

Proof. This is part of the Proposition 1.8. from [4]. □

Lemma 7.8. *Let R be ring, M be a strongly finitely presented right R-module and $(N_\alpha \mid \alpha < \kappa)$ be a family of right R-modules. Then for each $0 \leq i < \omega$*

$$Ext^i_R(M, \bigoplus_{\alpha<\kappa} N_\alpha) \simeq \bigoplus_{\alpha<\kappa} Ext^i_R(M, N_\alpha)$$

as abelian groups.

Proof. This is part of the Lemma 3.1.6. from [11]. □

Theorem 7.9. *Let R be a ring and $n < \omega$. Then $({}^{\perp_1}\mathcal{I}_n, \mathcal{I}_n)$ is a complete hereditary cotorsion pair.*

Proof. This is part of the Theorem 4.1.7. from [11]. □

Theorem 7.10. *Let R be a ring and $n < \omega$. Then $(\mathcal{P}_n, {\mathcal{P}_n}^{\perp_1})$ is a complete hereditary cotorsion pair.*

Proof. This is part of the Theorem 4.1.12. from [11]. □

Theorem 7.11. *Let R be a ring, κ be an infinite regular cardinal and $\mathcal{C}$ be a set of $< \kappa$-presented right R-modules. Let M be a right R-module with a $\mathcal{C}$-filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$. Then there is a set $\mathcal{F}$ consisting of submodules of M such that*

1. *$M_\alpha \in \mathcal{F}$ for all $\alpha \leq \sigma$,*
2. *let $N \in \mathcal{F}$ and let X be a subset of M of cardinality $< \kappa$. Then there is a $P \in \mathcal{F}$ such that $N \cup X \subseteq P$ and P/N is $< \kappa$-presented.*

Proof. This is the part of the Theorem 4.2.6. (Hill Lemma) from [11]. □

Theorem 7.12. *Let R be a ring, $n < \omega$ and $\mathcal{C}$ be a class of right R-modules. Then the following are equivalent*

1. *$\mathcal{C}$ is n-tilting,*
2. *$\mathcal{C}$ is coresolving, special preenvloping, closed under direct sums and direct summands and ${}^{\perp_1}\mathcal{C} \subseteq P_n$.*

Proof. This is the Theorem 5.1.14. from [11]. □

Theorem 7.13. *Let R be a ring, $n < \omega$ and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then the following are equivalent*

1. *$(\mathcal{A}, \mathcal{B})$ is n-tilting,*

2. $(\mathcal{A}, \mathcal{B})$ *is hereditary and complete,* $\mathcal{A} \subseteq P_n$ *and* $\mathcal{B}$ *is closed under direct sums.*

Proof. This is the Corollary 5.1.16. from [11]. □

Theorem 7.14. *Let* R *be a ring and* $(\mathcal{A}, \mathcal{B})$ *be a tilting cotorsion pair. Then each right* R*-module* $A \in \mathcal{A}$ *is* $\mathcal{A}^{<\aleph_1}$*-filtered.*

Proof. This is the part of the Theorem 5.2.10. (Deconstruction to countable type) from [11]. □

Theorem 7.15. *Let* R *be a ring and* T *be a tilting right* R*-module. Then* T *is of finite type.*

Proof. This is the part of the Theorem 5.2.20 from [11]. □

Theorem 7.16. *Let* R *be a commutative ring,* $n < \omega$, T *be an* n*-tilting* R*-module and* $\mathcal{B} = T^{\perp_\infty}$ *be the* n*-tilting class induced by* T.

1. *Let* S *be a multiplicative subset of* R*. Then* $S^{-1}T$ *is an* n*-tilting* $S^{-1}R$*-module, the corresponding* n*-tilting class being*

$$\mathcal{B}_S = (S^{-1}T)^{\perp_\infty} = \mathcal{B} \cap \textit{Mod-}S^{-1}R.$$

2. *Let* $M \in$ *Mod-*R*. Then* $M \in \mathcal{B}$*, iff* $M_{(m)} \in \mathcal{B}_{(m)}$ *for all maximal ideals* m *of* R*.*

Proof. This is the Theorem 5.2.24. from [11]. □

Lemma 7.17. *Let* R *be a 1-Gorenstein commutative ring of Krull dimension* 1*,* S *be a multiplicative subset of* R *which is without zero-divisors and* Σ *be a set of all regular elements of* R*. Then*

1. $\Sigma^{-1}R \simeq \bigoplus_{ht\, p=0} E(R/p)$ *as* R*-modules,*

2. $S^{-1}R/R$ *is a direct summand of* $\Sigma^{-1}R/R$ *as* R*-modules.*

Proof. This is the part of the Example 7.13 from [2]. □

Theorem 7.18. *Let* R *be a commutative ring and* S *be a multiplicative subset of* R *which is without zero-divisors. Then the following conditions are equivalent*

1. $\textit{proj dim}\, S^{-1}R \leq 1$,

2. $T = S^{-1}R \oplus S^{-1}R/R$ *is a 1-tilting* R*-module.*

Moreover, if T *is 1-tilting then the 1-tilting class induced by* T *is equal* $\{M \in \textit{Mod-}R \mid Ms = M \textit{ for all } s \in S\}$.

Proof. This is the part of the Theorem 6.3.16 from [11]. □

References

[1] Anderson F. W., Fuller K. R.: *Rings and Categories of Modules*, Second Edition, Springer-Verlag, New York, 1992.

[2] Angeleri-Hügel L., Herbera D., Trlifaj J.: *Divisible modules and localization*, Journal of Algebra 294 (2005), 519-551.

[3] Angeleri-Hügel L., Herbera D., Trlifaj J.: *Tilting modules and Gorenstein rings*, Forum Math. 18 (2006), no. 2, 211-229.

[4] Auslander M., Buchweitz R. O.: *The homological theory of maximal Cohen-Macaulay approximations*, Memories of Soc. Math. de France (Suppl.), Nouvelle Serie 38 (1989), 5-37.

[5] Bazzoni S., Eklof P. C., Trlifaj J.: *Tilting cotorsion pairs*, Bull. London Math. Soc. 37 (2005), no. 5, 683-696.

[6] Berrick A. J., Keating M. E.: *An Introduction to Rings and Modules*, Cambridge University Press, Cambridge, 2000.

[7] Bourbaki N.: *Éléments de mathématique, Algébre Commutative*, Vol. XXVII, Hermann, Paris, 1961.

[8] Cartan H., Eilenberg S.: *Homological algebra*, with an appendix by David A. Buchsbaum, reprint of the 1956 original, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999.

[9] Colpi R., Menini C.: *On the structure of $*$-modules*, Journal of Algebra 158 (1993), no. 2, 400-419.

[10] Enochs E. E., Jenda O. M. G.: *Relative Homological Algebra*, Walter de Gruyter, Berlin, 2000.

[11] Göbel R., Trlifaj J.: *Approximations and Endomorphism Algebras of Modules*, Walter de Gruyter, Berlin, 2006.

[12] Matsumura H.: *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[13] Xu J.: *Minimal Injective and Flat Resolutions of Modules over Gorenstein Rings*, Journal of Algebra 175 (1995), 451-477.