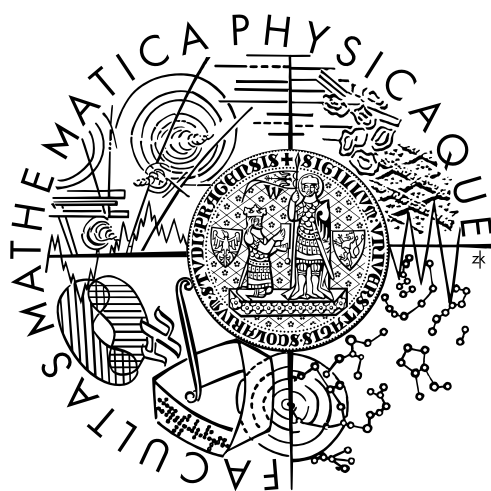


Univerzita Karlova v Praze  
Matematicko-fyzikální fakulta

# DIPLOMOVÁ PRÁCE



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Ramseyova teorie a kombinatorické hry

Katedra Aplikované Matematiky

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Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

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Tomáš Valla

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**Název práce:** Ramseyova teorie a kombinatorické hry

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**Abstrakt:** Ramseyova teorie studuje vnitřní homogenitu matematických struktur (grafů, číselných oborů), jejichž části (podgrafy, podmnožiny) jsou libovolně obarveny. Často platí, že je-li studovaný objekt dostatečně velký, lze v něm najít určitý jednobarevný podobjekt. Kombinatorické hry jsou hry dvou hráčů s plnou informací, kde záleží pouze na jejich inteligenci. Teorie kombinatorických her studuje především otázky existence vyhrávajících či neprohrávajících strategií. Vezmeme-li ramseyovskou větu a necháme-li objekt, který tato věta studuje, střídavě barvit dvěma hráči, jejichž cílem je vytvořit určitý monochromatický podobjekt, dostaneme kombinatorickou hru. Předmětem našeho zájmu je jednak nejmenší velikost objektu, při které platí ramseyovská věta, tzv. *ramseyovské číslo*, a jednak nejmenší velikost téhož objektu, při které má první hráč vyhrávající strategii v příslušné kombinatorické hře, tzv. *herní číslo*. V této práci popisujeme takové ramseyovské věty, u nichž je ramseyovské číslo podstatně větší než číslo herní. To znamená, že podáváme důkazy existence vyhrávajících strategií prvního hráče spolu s horními odhady na ramseyovská a herní čísla a obě čísla porovnáváme.

**Klíčová slova:** Ramseyova teorie, kombinatorické hry, potenciálová metoda, strategie

**Title:** Ramsey theory and combinatorial games

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**Abstract:** Ramsey theory studies the internal homogeneity of mathematical structures (graphs, number sets), parts of which (subgraphs, number subsets) are arbitrarily coloured. Often, the sufficient object size implies the existence of a monochromatic sub-object. Combinatorial games are 2-player games of skill with perfect information. The theory of combinatorial games studies mostly the questions of existence of winning or drawing strategies. Let us consider an object that is studied by a particular Ramsey-type theorem. Assume two players alternately colour parts of this object by two colours and their goal is to create certain monochromatic sub-object. Then this is a combinatorial game. We focus on the minimum object size such that the appropriate Ramsey-type theorem holds, called *Ramsey number*, and on the minimum object size such that the first player has a winning strategy in the corresponding combinatorial game, called *game number*. In this thesis, we describe such Ramsey-type theorems where the Ramsey number is substantially greater than the game number. This means, we show the existence of first player's winning strategies, together with Ramsey and game numbers upper bounds, and we compare both numbers.

**Keywords:** Ramsey theory, combinatorial games, resource counting, strategy

# 1. Introduction

There are many aspects of combinatorics, but only few areas form such a compact body of concepts and results (and thus in turn form a theory in the classical sense) as Ramsey theory. Also, very few areas of combinatorics display such a variety of techniques from various parts of mathematics. Very roughly speaking, Ramsey theory studies the chromatic number of hypergraphs. Many results of Ramsey theory (including Ramsey's theorem itself) have a character of a combinatorial principle which may be viewed as a generalisation of the pigeon-hole principle. Moreover, many mathematicians admit there is an elegance in Ramsey theory statements.

The theory of combinatorial games is an exceptionally attractive field. Combinatorial games are 2-player games of skill (no chance moves) with perfect information (the player cannot hide anything), and there are only three possible outcomes of the game: "win", "draw" and "loss". This class includes Chess, Go, Checkers, Tic-Tac-Toe, Hex, Nim, etc. The goal is to answer questions like "who wins", "how to win" and "how long does it take to win". In other words, the theory of combinatorial games tries to find good *strategies*. Strategies can be also viewed as a special class of *online algorithms*. As a contrary to Ramsey theory, the theory of combinatorial games is still very young and at an early stage of development. Therefore, the underdeveloped state of the theory is a great opportunity to make major discoveries.

What is the thing that connects Ramsey theory and combinatorial games? In one direction, almost every object studied by Ramsey theory can be taken and considered a "playground" of a combinatorial game. Two players keep colouring parts of this object and both wants to win, this means, to colour certain sub-object by their own colour. There can be also many different game rules. In the other direction, Ramsey theory can serve as a powerful tool to at least partially answer questions like "who wins" in a particular game.

Given a certain object, Ramsey theory states that there exists an *internal regularity* inside, some homogeneous sub-object. To the contrary, the goal of many combinatorial games is to *create* such a homogeneous sub-objects. Usually, the validity of Ramsey-type theorems depends only on the size of the object; given object large enough, the theorem holds. We call such minimal sufficient size *Ramsey number*. Similar concept exists in combinatorial games. By *game number* we mean the minimum object size such that certain player (usually the first) wins, provided he uses the best strategy possible. Often, there is large gap between the Ramsey number and the appropriate game number.

The goal of this thesis is to study various Ramsey-type theorems and the corresponding games, establish good upper bounds on both numbers and discover large gaps between them. In many cases, establishing a reasonable Ramsey number upper bound is an enormously complicated task which has been a subject of effort of many great mathematicians. Surprisingly, when considering the corresponding combinatorial game, it is often quite easy to find good upper bound on the game number, usually much lower than the Ramsey number bound. Therefore, we consider this topic exceptionally interesting.

In Chapter 2, we give a survey of selected interesting parts of Ramsey theory. We formulate all theorems we later study in the “game-centric” aspect and prove a majority of them. These include: Ramsey’s theorem, van der Waerden theorem, Hales-Jewett theorem (together with the revolutionary proof by Shelah), and others. We mostly follow the write-ups by Nešetřil [Ne] and Graham [GRS].

In Chapter 3, we build the theory of positional games from scratch. We give the basic definitions and fundamental theorems and we introduce the powerful technique of *resource counting*, which is the main tool in our work. We also present the *degree game* in a comprehensible way as previous write-ups were unsatisfactory. The chapter more or less closely follows (with the exception of the degree game) the great work of József Beck [Be].

Chapter 4 contains the main and original results of this thesis. As we have already mentioned before, we study Ramsey-type theorems from Chapter 2 in the view of combinatorial games and we find reasonable strategies together with appropriate game numbers upper bounds. We compare the Ramsey and game numbers and show the sometimes surprisingly large gap between them.

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## 1.1 Basic notions

For the sake of completeness, we define the basic notation used in this book. However, we restrict ourselves only to bare definitions, for details and explanation, see e.g. Matoušek and Nešetřil [MN].

By the symbol  $\mathbb{N}$ , we shall mean the set  $\{1, 2, \dots\}$  of all positive integers, and by  $\mathbb{R}$  we mean the set of all real numbers. For  $n \in \mathbb{N}$ , we often denote the set  $\{1, 2, \dots, n\}$  by  $[n]$ . For an integer  $c$ , by  $c[n]$  we denote the set  $\{ci; i \in [n]\}$ . To emphasise that a certain element is vector, we use bold symbols  $(\mathbf{x}, \mathbf{y}, \dots)$ . For  $0 \leq k \leq n$ , the symbol  $\binom{n}{k}$  denotes the number of  $k$ -element subsets of an  $n$ -element set. For a set  $X$ , the symbol  $2^X$  means the set of all possible subsets of  $X$ , the symbol  $\binom{X}{k}$  means the set of all  $k$ -element subsets of  $X$ , by  $|X|$  we denote the cardinality of  $X$ , and we define

$$X^k = \underbrace{X \times \dots \times X}_k,$$

where  $\times$  is the Cartesian product of two sets. The difference of two set  $A$  and  $B$  is denoted by  $A \setminus B$ .

Let us define the *asymptotic estimates*.

- (1) We say that a function  $f: \mathbb{N} \rightarrow \mathbb{N}$  is  $\mathcal{O}(g)$  for a function  $g: \mathbb{N} \rightarrow \mathbb{N}$ , if there is a constant  $c > 0$  such that  $f(n) \leq c \cdot g(n)$  for every  $n \in \mathbb{N}$ .

- (2) We say that a function  $f: \mathbb{N} \rightarrow \mathbb{N}$  is  $\Omega(g)$  for a function  $g: \mathbb{N} \rightarrow \mathbb{N}$ , if there is a constant  $c > 0$  such that  $c \cdot g(n) \leq f(n)$  for every  $n \in \mathbb{N}$ .
- (3) We say that a function  $f$  is  $\Theta(g)$  if  $f$  is both  $\mathcal{O}(g)$  and  $\Omega(g)$ .
- (4) To express that  $f$  is  $\mathcal{O}(g)$ , we often write  $f(n) = \mathcal{O}(g(n))$ . Similarly,  $f(n) = \Omega(g(n))$  means that  $f$  is  $\Omega(g)$ .

A *graph*  $G$  is a tuple  $(V, E)$  where  $V$ , called the *vertex set*, is an arbitrary finite set and  $E \subseteq \binom{V}{2}$  is called the *edge set*. Elements of  $E$  are called *edges*. For a graph  $G$ , the vertex set of  $G$  is denoted by  $V(G)$ , and the edge set by  $E(G)$ .

A finite *hypergraph*  $\mathcal{F} = (V, F)$  is a set system where  $V$  is an arbitrary finite set and  $F \subseteq 2^V$ . Similarly,  $V(\mathcal{F})$  is the vertex set and  $E(\mathcal{F})$  is the edge set. Elements of  $F$  are usually called *hyperedges*. A hypergraph  $\mathcal{F}$  is *k-uniform* (or simply a *k-graph*) if  $|S| = k$  for every  $S \in E(\mathcal{F})$ . Thus, graph is a special case of hypergraph, a 2-graph. We always use standard letters to denote graphs, and we use both standard and caligraphic symbols  $(\mathcal{A}, \mathcal{B}, \dots)$  to denote hypergraphs.

We say that hypergraph  $\mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}})$  is a *subgraph* of hypergraph  $\mathcal{F} = (V_{\mathcal{F}}, E_{\mathcal{F}})$  if  $V_{\mathcal{H}} \subseteq V_{\mathcal{F}}$  and  $E_{\mathcal{H}} \subseteq 2^{V_{\mathcal{H}}} \cap E_{\mathcal{F}}$ . We denote this fact by  $\mathcal{H} \subseteq \mathcal{F}$ .

For a hypergraph  $\mathcal{F} = (V, F)$  and a vertex  $v \in V$ , we define the *vertex degree*

$$\deg_{\mathcal{F}}(v) = |\{S \in F; v \in S\}|.$$

We also define a *minimum degree*

$$\delta(\mathcal{F}) = \min_{v \in V} \deg_{\mathcal{F}}(v)$$

and a *maximum degree*

$$\Delta(\mathcal{F}) = \max_{v \in V} \deg_{\mathcal{F}}(v).$$

For two distinct vertices  $u, v \in V$ , we define the *double degree*

$$\deg_{\mathcal{F}}(u, v) = |\{S \in F; u, v \in S\}|$$

and a *maximum double degree*

$$\Delta_2(\mathcal{F}) = \max_{\substack{u, v \in V \\ u \neq v}} \deg_{\mathcal{F}}(u, v).$$

If  $\Delta_2(\mathcal{F}) = 1$ , then  $\mathcal{F}$  is called *almost disjoint*.

For two hypergraphs  $\mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}})$  and  $\mathcal{F} = (V_{\mathcal{F}}, E_{\mathcal{F}})$ , the bijection  $f: V_{\mathcal{H}} \rightarrow V_{\mathcal{F}}$  is called *isomorphism* if the condition

$$\{v_1, v_2, \dots, v_k\} \in E_{\mathcal{H}} \Leftrightarrow \{f(v_1), f(v_2), \dots, f(v_k)\} \in E_{\mathcal{F}}$$

holds for every  $k$  and for every subset  $\{v_1, \dots, v_k\} \subseteq V_{\mathcal{H}}$ . If there exists an isomorphism of two hypergraphs  $\mathcal{H}$  and  $\mathcal{F}$ , we denote this fact by  $\mathcal{H} \simeq \mathcal{F}$  and we say that  $\mathcal{H}$  and  $\mathcal{F}$  are *isomorphic*.



For two hypergraphs  $\mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}})$  and  $\mathcal{F} = (V_{\mathcal{F}}, E_{\mathcal{F}})$ , the function  $h: V_{\mathcal{H}} \rightarrow V_{\mathcal{F}}$  is called *homomorphism* if the condition

$$\{v_1, v_2, \dots, v_k\} \in E_{\mathcal{H}} \Rightarrow \{f(v_1), f(v_2), \dots, f(v_k)\} \in E_{\mathcal{F}}$$

holds for every  $k$  and for every subset  $\{v_1, \dots, v_k\} \subseteq V_{\mathcal{H}}$ .

These are the important graphs used throughout the whole thesis:

- (1) A *complete  $k$ -graph*  $K_n^k$  is a  $k$ -graph  $(V, \binom{V}{k})$ ,  $|V| = n$ . If  $k = 2$ , we use just the symbol  $K_n$ .
- (2) A *cycle*  $C_\ell$  of length  $\ell$  is every hypergraph  $(V_C, E_C)$  where  $V_C = \{v_0, v_1, \dots, v_{\ell-1}\}$  and  $E_C = \{S_0, S_1, \dots, S_{\ell-1}\}$  such that  $\{v_i, v_{(i+1) \bmod \ell}\} \subseteq S_i$  for every  $0 \leq i < \ell$ .
- (3) A *path*  $P$  of length  $p$  is every hypergraph  $(V_P, E_P)$  such that  $V_P = \{v_1, v_2, \dots, v_p\}$  and  $E_P = \{S_1, S_2, \dots, S_{p-1}\}$  such that  $\{v_i, v_{i+1}\} \subseteq S_i$  for every  $0 \leq i < p$ . Often, we talk about a path from one vertex (which is  $v_1$ ) to another vertex (which is  $v_p$ ).

A hypergraph  $\mathcal{F}$  is said to be *connected* if for every two distinct vertices  $u, v \in V(\mathcal{F})$  there exists a path in  $\mathcal{F}$  from  $u$  to  $v$ . A *tree* is a connected hypergraph that contains no cycle. A *star* is a tree where all edges share precisely one common vertex.

A *vertex-colouring* (or just *colouring* to be short) of a hypergraph  $\mathcal{F} = (V, F)$  by  $t$  colours is a mapping  $c_t: V \rightarrow \{1, \dots, t\}$ . We say that a colouring  $c_t$  is *proper* if every  $S \in F$  contains at least two vertices with different colours. The *chromatic number* of  $\mathcal{F}$  is

$$\chi(\mathcal{F}) = \min\{t; \text{there exists a proper colouring } c_t \text{ of } \mathcal{F}\}.$$

## 2. Ramsey Theory

In this chapter we give a survey of the most famous Ramsey-type theorems, which later in Chapter 4 serve as a basis for Ramsey-type combinatorial games. Most of information here comes from Graham, Rothschild and Spencer [GRS], from Nešetřil [Ne], and other resources.

### 2.1 Introduction

**Proposition 2.1.** *Consider a group of six people. In this group, there are 3 people who know each other, or there are 3 people who do not know each other. (We assume that the relation “to know someone” is symmetric.)*

*Proof.* Let us model the society as a graph  $G$  such that its vertices are the people and if they know each other, they are connected by an edge. Choose an arbitrary vertex  $v$  of the graph  $G$ . The vertex  $v$  is adjacent either with at least 3 edges or at least 3 non-edges. Let us consider the first case, and let  $v$  be adjacent with vertices  $x$ ,  $y$  and  $z$ .

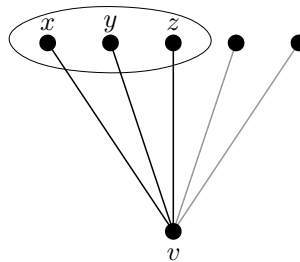


FIG. 2.1. The group of people as a graph.

If there is some edge defined on these three vertices, we have just found a triangle. If there is no edge on  $x, y, z$ , they form an independent set. The second case, i.e. non-edges, is analogous.  $\square$

The previous simple proposition is one of the first nontrivial claims, which we call *Ramsey-type theorems*. These theorems states that in every sufficiently large object there is some homogeneous sub-object. In many cases, the surprising fact occurs that for the existence of an internal regularity, only the assumption of large object size is needed. Speaking in a popular and rather inaccurate manner, “total chaos is impossible” inside large objects. In this chapter, we show some of the most important Ramsey-type theorems.

In other words, Ramsey-type theorems study the chromatic number of certain hypergraphs. For some family of hypergraphs, a typical Ramsey-type theorem states that starting from a certain number of vertices, the chromatic number of every such hypergraph is big.

Section 2.2 is an improved translation of the appropriate chapter by Valla and Matoušek [VM], presenting the original results of Ramsey [Ram]. In Section 2.3, we present the Hales-Jewett theorem with the proof by Shelah, loosely following the write-up by Nešetřil [Ne]. In Section 2.4, we present some famous Ramsey-type results for arithmetic progressions, following e.g. Graham et al. [GRS]. Section 2.5 contains formulation of the result of Rado [Rad], as presented by Graham et al. [GRS]. Sections 2.6 and 2.8 loosely follow the write-up by Nešetřil [Ne].

## 2.2 Graph and hypergraph Ramsey theorems

Let us extend Proposition 2.1 into more complex theorem, which was published in a slightly different form in 1930 by the English mathematician, economist and philosopher Frank Ramsey [Ram].

**Theorem 2.2.** (Ramsey, for graphs) *For every positive integer  $n$ , there exists positive integer  $N$  such that an arbitrary graph on  $N$  vertices contains a complete graph on  $n$  vertices or an independent set on  $n$  vertices.*

*In general: For every positive integers  $n$  and  $r$ , there exists positive integer  $N$  such that if every edge of the graph  $K_N$  is coloured by one of the  $r$  colours, then this  $K_N$  contains a monochromatic  $K_n$  subgraph, that is, a complete subgraph on  $n$  vertices with all edges of the same colour.*

How does the second part imply the first part? Let us replace every edge of the given graph on  $N$  vertices by red edge and every non-edge by blue edge. This gives a  $K_N$  graph with edges coloured by red and blue and we apply the second part of the theorem on this graph.

*Proof.* We first show the theorem holds for two colours ( $r = 2$ ). From this we prove the theorem for any number of colours.

Let us define the number  $\mathcal{R}(k, \ell)$  as follows:

$$\mathcal{R}(k, \ell) := \min \left\{ N; \begin{array}{l} \text{every } K_N \text{ with edges coloured red and blue} \\ \text{contains red } K_k \text{ or blue } K_\ell \end{array} \right\}.$$

The number  $\mathcal{R}(k, \ell)$  is called *Ramsey number* for graphs and two colours.

We only need to show  $\mathcal{R}(n, n) < \infty$ , but we actually prove that also  $\mathcal{R}(k, \ell)$  is finite for every  $k, \ell$ . We do it by induction on  $k + \ell$ .

If  $k = 1$  or  $\ell = 1$ , one can choose an arbitrary vertex. Therefore, we have  $\mathcal{R}(1, \ell) = 1$  and  $\mathcal{R}(k, 1) = 1$ .

Let us assume  $\mathcal{R}(k - 1, \ell)$  is finite and  $\mathcal{R}(k, \ell - 1)$  is finite. We prove that also  $\mathcal{R}(k, \ell)$  is finite. In particular, we verify

$$\mathcal{R}(k, \ell) \leq \mathcal{R}(k - 1, \ell) + \mathcal{R}(k, \ell - 1).$$

Let  $N = \mathcal{R}(k - 1, \ell) + \mathcal{R}(k, \ell - 1)$ , and consider a graph  $K_N$  with edges arbitrarily coloured by two colours, red and blue. We show it contains a  $K_k$ -subgraph with all edges red or a  $K_\ell$ -subgraph with all edges blue.

Consider an arbitrary vertex  $v$  of this  $K_N$ . We divide the remaining vertices into two sets  $A$  and  $B$ : The set  $A$  contains vertices which are adjacent to  $v$  by red edge, and the set  $B$  contains vertices which are adjacent to  $v$  by blue edge.

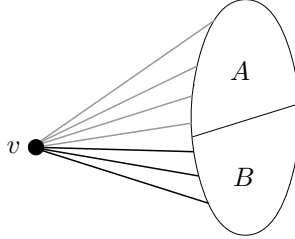


FIG. 2.2. Splitting the vertices into the sets  $A$  and  $B$ .

We have  $|A| + |B| = N - 1 = \mathcal{R}(k - 1, \ell) + \mathcal{R}(k, \ell - 1) - 1$ , this means that  $|A| \geq \mathcal{R}(k - 1, \ell)$  or  $|B| \geq \mathcal{R}(k, \ell - 1)$ .

Let us first assume  $|A| \geq \mathcal{R}(k - 1, \ell)$ . If the subgraph induced by  $A$  contains red  $K_{k-1}$ , we can add the vertex  $v$  to it and get a red  $K_k$ . If there is no red  $K_{k-1}$  in  $A$ , then there must be a red  $K_\ell$ .

For  $|B| \geq \mathcal{R}(k, \ell - 1)$  is the argument similar. If  $B$  contains red  $K_{\ell-1}$ , we can add the vertex  $v$  to it and get a blue  $K_\ell$ . Otherwise,  $B$  contains complete red  $K_k$ .

All cases thus lead to existence of a monochromatic  $K_k$  or  $K_\ell$ , and due to previous reasoning also to finality of  $\mathcal{R}(k, \ell)$ . This finishes the proof of Ramsey theorem for two colours.

Let us consider three colours now. Let  $M = \mathcal{R}(n, n)$  and  $N = \mathcal{R}(n, M)$  where  $\mathcal{R}(k, \ell)$  is the Ramsey number defined above. Given a graph  $K_N$  with edges arbitrarily coloured by red, blue and yellow, we first merge the colours blue and yellow into one, green. Thus, we have coloured the edges of  $K_N$  by red and green, and by definition of  $\mathcal{R}(n, M)$ , there exist a red  $K_n$  or green  $K_M$  in the graph  $K_N$ . In the first case, we are done. In the second case, we have  $K_M$  such that its edges in the original red-blue-yellow colouring are only blue and yellow. Because we have set  $M = \mathcal{R}(n, n)$ , we can find a blue  $K_n$  or a yellow  $K_n$  in the  $K_M$ . This proves the theorem for three colours.

For four colours we reduce the problem in a similar way as before on theorem for three colours, and so on. Therefore, Ramsey theorem holds for any finite number of colours.

One can also observe that for any finite number of colours, the same approach we have used for two colours works. We define the Ramsey number  $\mathcal{R}(k_1, k_2, \dots, k_r)$  for  $r$  colours, and in induction we choose the size of the graph as the sum of  $r$  these numbers.  $\square$

We further generalise Theorem 2.2 for edge-coloured complete  $p$ -graphs.

**Theorem 2.3.** (Ramsey, for hypergraphs) *For every positive integers  $n, r, p$ , there exists a positive integer  $N$  with the following property: Given a set  $X$  of  $N$  elements where every  $p$ -element subset of  $X$  is coloured by one of the  $r$  colours,  $X$  contains a monochromatic subset  $Y \subset X$ , that is, all  $p$ -element subsets of  $Y$  are of the same colour. In other words, every complete  $p$ -graph on  $N$  vertices with edges coloured by  $r$  colours contains a monochromatic complete subgraph on  $n$  vertices.*

*Proof.* We proceed by induction on  $p$ . For  $p = 1$  the theorem reduces to pigeonhole principle and for  $p = 2$  to already proven Ramsey theorem for graphs (the element tuples are the graph edges).

Assume we already know the sufficient size  $N$  of the set  $X$ . We establish  $N$  at the end of the proof. Consider a set  $X$  of size  $N$  with all  $p$ -tuples coloured by  $r = 2$  colours. Let  $X_0 := X$  and we do the following step for  $i = 1, 2, \dots, 2n - 1$ :

- Consider an arbitrary element  $x_i \in X_{i-1}$ , and colour every  $(p - 1)$ -element subset  $S$  of the set  $X_{i-1} \setminus \{x_i\}$  by the colour of the  $p$ -tuple  $S \cup \{x_i\}$ . By induction hypothesis there exists sufficiently large subset  $X_i$  of the set  $X_{i-1} \setminus \{x_i\}$  such that all  $(p - 1)$ -tuples of  $X_i$  have got the same colour  $b_i$ .

One of the colours appears (by pigeonhole principle) among the colours  $b_i$  at least  $n$ -times, without loss of generality the red. Then  $\{x_i; b_i = \text{red}\}$  is the desired  $n$ -element subset  $Y$ . Therefore, the size  $N$  of the set  $X$  can be chosen huge enough (but still finite) such that  $X$  is sufficient for all the steps.

There are more ways to prove the theorem for  $r > 2$  colours. One can perform more steps, for  $i = 1, 2, \dots, rn - 1$ , and again observe it is possible to choose the number  $N$  sufficiently large such that all arguments are valid. Or one may use the “colour merging” method in the similar way as in Theorem 2.2. We merge two colours into one, apply the theorem for the lesser number of colours, which yields a number  $N$ , and we use the theorem once more, this time for  $n := N$ .  $\square$

The Ramsey theorem for  $p$ -tuples is also often denoted by the following abbreviation:

$$N \rightarrow (n)_r^p.$$

We read this notation: “The size  $N$  of an arbitrary set  $X$  where each  $p$ -tuple is coloured by one of  $r$  colours is sufficient for the existence of a homogeneous subset of size  $n$ .” We denote the minimum  $N$  such that  $N \rightarrow (n)_r^p$  by  $\mathcal{R}_p(n, r)$ .

## 2.3 Hales-Jewett theorem

This section contains formulation of the famous theorem by Hales and Jewett [HJ], together with the proof by Shelah [Sh].

### 2.3.1 Definitions and theorem formulation

We begin by the notation. We define  $C_t^n$ , the  $n$ -cube over  $t$  elements, by

$$C_t^n = \{(x_1, \dots, x_n); x_i \in \{0, 1, \dots, t-1\}\}.$$

By a *line* in  $C_t^n$  we mean a set of (suitably ordered) points  $\mathbf{x}_0, \dots, \mathbf{x}_{t-1}$ ,  $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,n})$  so that in each coordinate  $j$ ,  $1 \leq j \leq n$ , either

$$x_{0,j} = x_{1,j} = \dots = x_{t-1,j}$$

or

$$x_{s,j} = s \quad \text{for } 0 \leq s < t,$$

and the latter occurs for at least one  $j$  (otherwise all  $\mathbf{x}_i$  would be constant). For example, with  $t = 4$ ,  $n = 3$ ,  $\{020, 121, 222, 323\}$  forms a line, as does  $\{031, 131, 231, 331\}$ .

Our definition differs from the ordinary geometric definition as, for example the set  $\{02, 11, 20\}$  is not a line in  $C_3^2$ . The reason for this is that the cube is meant to be independent of the underlying set  $\{0, 1, \dots, t-1\}$ . In other words, for any set  $A = \{a_1, \dots, a_t\}$  we may define

$$C_t^n = \{(x_1, \dots, x_n); x_i \in A\}$$

and lines of  $C_t^n$  as those  $\mathbf{x}_0, \dots, \mathbf{x}_{t-1}$  so that in each coordinate  $j$  either the  $x_{i,j}$  are constant or  $x_{i,j} = a_i$ . All such cubes are combinatorially isomorphic.

By  $[t]^n$  we denote the cubes  $C_t^n$  based on the set  $\{1, 2, \dots, t\}$ .

**Theorem 2.4.** (Hales, Jewett) *Let  $C$  be a finite set (alphabet) and let  $t$  be a positive integer. Then there exists positive integer  $N$  with the following property: For every colouring of the elements of the cube  $C^N$  by  $t$  colours, one of the colour classes contains a combinatorial line.*

The original proof by Hales and Jewett [HJ] was quite simple, however, the upper bound on  $N$  was immensely high—not even primitive recursive (see Section 2.8 for the informal definition). We present a relatively new proof. In 1988, Shelah [Sh] found a proof which avoids use of double induction and yields a primitive recursive upper bound. We loosely follow the write-up by Nešetřil [Ne].

By  $\mathcal{H}(n, t)$  denote the minimal number  $N$  for which the statement of the Hales-Jewett theorem (for  $C = [n]$  and  $t$  colours) holds.

### 2.3.2 Shelah's pigeonhole lemma

In the proof we shall use the following technical lemma.

**Lemma 2.5.** (Shelah's pigeonhole) *For all positive integers  $n$  and  $t$ , there exists a positive integer  $N$  with the following property: Consider any choice of  $n$  colourings of the cube  $[N]^{2n-1}$  by  $t$  colours, that is, every  $\alpha_i: [N]^{2n-1} \rightarrow [t]$  for  $i = 1, 2, \dots, n$ . Then for every  $i = 1, \dots, n$  there exist integers  $1 \leq a_i < b_i \leq N$  such that we have*

$$\begin{aligned} & \alpha_i(a_1, b_1, \dots, a_{i-1}, b_{i-1}, a_i, a_{i+1}, b_{i+1}, \dots, a_n, b_n) \\ & = \alpha_i(a_1, b_1, \dots, a_{i-1}, b_{i-1}, b_i, a_{i+1}, b_{i+1}, \dots, a_n, b_n). \end{aligned} \quad (2.1)$$

Denote by  $f(n, t)$  the minimal such number  $N$ .

*Proof.* We proceed by induction on  $n$ . Obviously,  $f(1, t) = t + 1$  by pigeonhole principle. For the inductive step we prove

$$f(n + 1, t) \leq t^{f(n, t)^{2n}} + 1. \quad (2.2)$$

To simplify the notation, put

$$M = f(n, t) \quad \text{and} \quad N = t^{M^{2n}} + 1.$$

(The reason for setting the number  $N$  like this will be clear later.) Let  $\alpha_i: [N]^{2n+1} \rightarrow [t]$ ,  $i = 1, \dots, n + 1$ , be arbitrary colourings. Consider the induced colouring

$$\alpha': [N] \rightarrow [t]^{M^{2n}}$$

of the set  $[N]$  by  $t^{M^{2n}}$  "vector-colours", defined as

$$\alpha'(a) = (\alpha_{n+1}(x_1, \dots, x_{2n}, a); (x_1, \dots, x_{2n}) \in [M]^{2n})$$

for every  $a \in [N]$ . Note that each such "vector-colour" has  $M^{2n}$  coordinates. We have set the number  $N$  large enough, therefore, by the pigeonhole principle, there are two integers  $1 \leq a_{n+1} < b_{n+1} \leq N$  such that

$$\alpha'(a_{n+1}) = \alpha'(b_{n+1}). \quad (2.3)$$

Now, we define colourings of the cube  $[M]^{2n-1}$  by  $t$  colours. Precisely, we define the  $t$ -colourings  $\alpha_i'': [M]^{2n-1} \rightarrow [t]$  for  $i = 1, \dots, n$ , such that

$$\alpha_i''(x_1, \dots, x_{2n-1}) = \alpha_i(x_1, \dots, x_{2n-1}, a_{n+1}, b_{n+1})$$

for every  $(x_1, \dots, x_{2n-1}) \in [M]$ . By the induction hypothesis, there are numbers

$$a_1 < b_1, \quad a_2 < b_2, \quad \dots, \quad a_n < b_n$$

such that for every  $i = 1, \dots, n$  we have

$$\begin{aligned}
& \alpha_i''(a_1, b_2, \dots, a_{i-1}, b_{i-1}, a_i, a_{i+1}, b_{i+1}, \dots, a_n, b_n) \\
&= \alpha_i''(a_1, b_1, \dots, a_{i-1}, b_{i-1}, b_i, a_{i+1}, b_{i+1}, \dots, a_n, b_n) \\
&= \alpha_i(a_1, b_1, \dots, a_{i-1}, b_{i-1}, b_i, a_{i+1}, b_{i+1}, \dots, a_n, b_n) \\
&= \alpha_i(a_1, b_1, \dots, a_{i-1}, b_{i-1}, a_i, a_{i+1}, b_{i+1}, \dots, a_n, b_n).
\end{aligned} \tag{2.4}$$

The equality (2.3) means

$$\alpha_{n+1}(x_1, \dots, x_{2n}, a_{n+1}) = \alpha_{n+1}(x_1, \dots, x_{2n}, b_{n+1})$$

for every  $(x_1, \dots, x_{2n}) \in [M]^{2n}$ , thus also

$$\alpha_{n+1}(a_1, b_1, \dots, a_n, b_n, a_{n+1}) = \alpha_{n+1}(a_1, b_1, \dots, a_n, b_n, b_{n+1}).$$

This together with (2.4) gives (2.1). □

### 2.3.3 Main proof

Let us proceed with the main proof of the Hales-Jewett theorem.

*Proof.* Obviously,  $\mathcal{H}(1, t) = 1$ . By induction on  $n$  we prove

$$\mathcal{H}(n+1, t) \leq \mathcal{H}(n, t) \cdot f\left(\mathcal{H}(n, t), t^{(n+1)^{\mathcal{H}(n, t)}}\right), \tag{2.5}$$

where  $f$  is the function occurring in Shelah's pigeonhole lemma (Lemma 2.5).

To simplify the notation, set

$$N = \mathcal{H}(n, t) \quad \text{and} \quad m = f\left(N, t^{(n+1)^N}\right).$$

Thus, the desired upper bound (2.5) is  $N \cdot m$ . Also set

$$M_k = \{mk + 1, \dots, m(k+1)\}$$

for  $k = 1, \dots, N$ .

Now, let  $\alpha: [n+1]^{N \cdot m} \rightarrow [t]$  be some fixed colouring.

In fact, it suffices to consider a colouring of a subset of  $[n+1]^{N \cdot m}$  formed by all *cascade functions*. Note that each vector  $\mathbf{x} \in X^k$  can be interpreted as a function  $[k] \rightarrow X$  that maps each coordinate index to some element of  $X$ . A cascade function  $f$  is a function  $[N \cdot m] \rightarrow [n+1]$  determined by

- (1) a family  $\langle a_k, b_k; k = 1, \dots, N \rangle$  where  $a_k \leq b_k$  and  $a_k, b_k \in M_k$ ,
- (2) a function  $g: [N] \rightarrow [n+1]$ .



A cascade function  $f$  further satisfies

$$f(i) = \begin{cases} n+1 & \text{for } i < a_k, i \in M_k, \\ g(k) & \text{for } a_k \leq i \leq b_k, \\ n & \text{for } b_k < i \in M_k. \end{cases}$$

The  $(2N)$ -tuple  $\langle a_k, b_k; k = 1, \dots, N \rangle$  is called the *schema*  $S$  of the cascade function  $f$ .

For a fixed schema  $S$ , the mapping  $g \mapsto f$  is a bijection which carries a line from  $[n+1]^N$  into a line in  $[n+1]^{N \cdot m}$ . We put  $H_S(g) = f$ . See Figure 2.3 for the illustrated meaning of the definition.

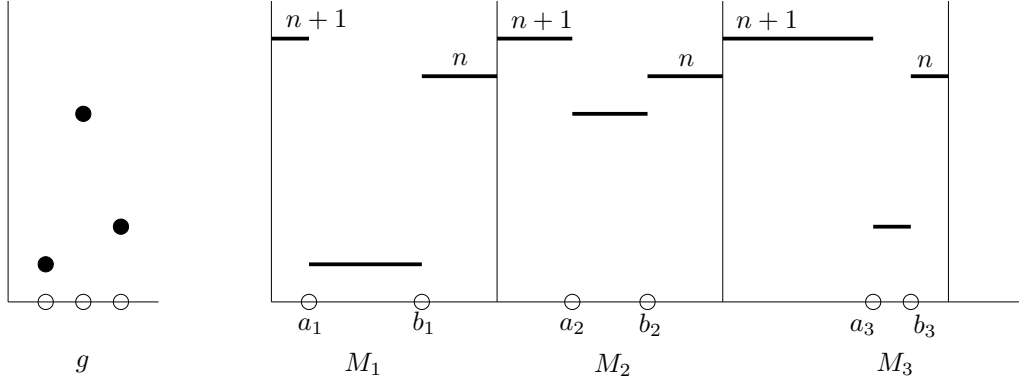


FIG. 2.3. Illustrating the definition of  $H_S(g)$ .

For  $\ell = 1, \dots, N$ , let us define the colourings

$$\beta_\ell: [m]^{2N-1} \rightarrow \{f; f: [n+1]^N \rightarrow [t]\}$$

where each vector  $\mathbf{x} = (a_1, b_1, \dots, a_{\ell-1}, b_{\ell-1}, a_\ell = b_\ell, a_{\ell+1}, b_{\ell+1}, \dots, a_N, b_N) \in [m]$  gets the colour

$$\beta_\ell(\mathbf{x}) = (\alpha(H_{(a_\ell, b_\ell)}(g)); g \in [n+1]^N). \quad (2.6)$$

If  $(a_\ell, b_\ell)$  does not correspond to a schema of a cascade function, then we define  $\beta_\ell$  arbitrarily.

Let us apply Shelah's pigeonhole lemma on the colourings  $\beta_\ell$ : there exists a schema  $S = \langle a_1 < b_1, a_2 < b_2, \dots, a_N < b_N \rangle$  such that (2.1) holds. Explicitly, for every  $\ell = 1, \dots, N$ ,

$$\begin{aligned} & \beta_\ell(a_1, b_1, \dots, a_{\ell-1}, b_{\ell-1}, a_\ell, a_{\ell+1}, b_{\ell+1}, \dots, a_N, b_N) \\ &= \beta_\ell(a_1, b_1, \dots, a_{\ell-1}, b_{\ell-1}, b_\ell, a_{\ell+1}, b_{\ell+1}, \dots, a_N, b_N). \end{aligned} \quad (2.7)$$

Let us consider all cascade functions with schema  $S$ , i.e. all functions  $H_S(g)$ ,  $g \in [n+1]^N$ . By the choice  $N = \mathcal{H}(n, t)$ , there exists a subset  $I \subseteq [N]$  and a vector  $\mathbf{x}^0 = (x_1^0, \dots, x_N^0) \in$

$[n]^N$  such that if we denote by  $L$  the line in  $[n]^N$  determined by  $\mathbf{x}^0$  and  $I$  (the indices of non-constant coordinates), then  $H_S(L)$  is a monochromatic subset of  $[n+1]^{N \cdot m}$ .

Let the points (functions) of the line  $L$  be denoted by  $\mathbf{x}^1, \dots, \mathbf{x}^n$ . Observe that all the cascade functions  $H_S(\mathbf{x}^1), H_S(\mathbf{x}^2), \dots, H_S(\mathbf{x}^n)$  have schema  $S$ . Define  $\mathbf{x}^{n+1} = (x_1^{n+1}, \dots, x_N^{n+1})$  by

$$x_i^{n+1} = \begin{cases} x_i^0 & \text{for } i \notin I, \\ n+1 & \text{for } i \in I. \end{cases}$$

The cascade function  $H_S(\mathbf{x}^{n+1})$  has schema  $S$ . However, both cascade functions  $H_S(\mathbf{x}^n)$  and  $H_S(\mathbf{x}^{n+1})$  may be also thought of as having any of the following schemas  $((a'_\ell, b'_\ell); \ell = 1, \dots, N)$  where  $a'_\ell = a_\ell$  and  $b'_\ell = b_\ell$ , for every  $\ell \notin I$ , and

$$a'_\ell, b'_\ell \in \{a_\ell, b_\ell\}, \quad a'_\ell \leq b'_\ell \quad \text{for } \ell \in I.$$

From this follows (by repeated use of (2.6) and (2.7)) that

$$\alpha(H_S(\mathbf{x}^n)) = \alpha(H_S(\mathbf{x}^{n+1}))$$

and thus  $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^{n+1}$  is a monochromatic line in  $[n+1]^{N \cdot m}$ .  $\square$

## 2.4 Arithmetic progressions

This section contains one of the oldest and most famous Ramsey-type results, studying arithmetic progressions. In 1927, B. L. van der Waerden [Wae] published a proof of the following unexpected result.

**Theorem 2.6.** (Van der Waerden) *For every choice of positive integers  $t$  and  $n$ , there exists an integer  $N$  such that for every colouring of the set  $\{1, \dots, N\}$  by  $t$  colours, one of the colour classes contains an arithmetic progression with  $n$  terms.*

However, the original proof was very complicated and provided enormous upper bound on the number  $N$ . We present a simpler proof by Shelah [Sh], who applies his own proof of Hales-Jewett theorem, that gives primitive recursive (see Section 2.8) upper bound.

*Proof.* Van der Waerden theorem can be easily proved using Hales-Jewett theorem (Theorem 2.4). Assume we know the number  $N$  already. Let us put  $C = \{0, 1, \dots, n-1\}$  and consider the cube  $C^N$ . With every point  $\mathbf{x} = (x_1, \dots, x_N) \in C^N$  we associate an integer

$$w(\mathbf{x}) = \sum_{i=1}^N x_i \cdot n^i.$$

The mapping  $w: C^N \rightarrow \{0, \dots, n^N - 1\}$  is bijective; one can view the number  $w(\mathbf{x})$  as an  $N$ -coordinate number in the scale of  $n$ . Therefore, every combinatorial line is mapped to an arithmetic progression of length  $n$ , and vice versa. By Hales-Jewett theorem, there exists a number  $N$  such that the cube  $C^N$  coloured by  $t$  colours contains a monochromatic line, i.e.  $\{1, \dots, N\}$  contains a monochromatic arithmetic progression.  $\square$

## 2.5 Rado theorem and related theorems

Let us first start with the following simple theorem by Schur [Sch]. We show a proof based on Ramsey theorem, presented e.g. by Graham et al. [GRS].

**Theorem 2.7.** (Schur) *For every positive integer  $t$ , there exists a positive integer  $N$  such that for every colouring of the set  $S = \{1, \dots, N\}$  by  $t$  colours, one of the colour classes contains number  $x$  and  $y$  together with their sum  $x + y$ .*

*Proof.* Schur's theorem easily follows from Ramsey's theorem. Let  $\alpha$  be a given colouring of the elements of  $S$  by  $t$  colours. We define the colouring  $\alpha'$  of pairs by  $\alpha'(\{i, j\}) = \alpha(|i - j|)$  for  $i, j \in S, i \neq j$ . By Ramsey theorem (Theorem 2.2), we may choose  $N$  such that there exists an  $\alpha'$ -monochromatic triangle with vertices  $i < j < k$ . However,  $(j - i) + (k - j) = k - i$ , and we know that  $\alpha(i - j) = \alpha(k - j) = \alpha(k - i)$ . The theorem follows.  $\square$

Both Schur's theorem and Van der Waerden's theorem fit the following more general schema. Let  $A = (a_{i,j})$  be an integer  $m \times n$  matrix. Then for every integer  $t \geq 1$ , there exists an integer  $N = N(A, t)$  which has the following properties: If the set  $\{1, 2, \dots, N\}$  is partitioned into  $t$  classes, then in one class of the partition there is a solution  $x_1, \dots, x_n$  of a system of equations

$$\begin{aligned} a_{1,1}x_1 + \dots + a_{1,n}x_n &= 0 \\ a_{2,1}x_1 + \dots + a_{2,n}x_n &= 0 \\ &\vdots \\ a_{m,1}x_1 + \dots + a_{m,n}x_n &= 0. \end{aligned} \tag{2.8}$$

We can abbreviate (2.8) by writing

$$A\mathbf{x} = \mathbf{0}, \quad \mathbf{x} = (x_1, \dots, x_n)^T.$$

The basic problem is to characterise those integral matrices  $A$  for which a result analogous to Schur's and Van der Waerden's theorems holds. This leads to the following notions:

The set of equations  $A\mathbf{x} = \mathbf{0}$  is said to be *partition regular* if for any finite partition of the set  $[N]$  there is always a solution of the system (2.8) in one of the classes.

Note that obviously not every set of equations is partition regular, e.g. consider  $x = 2y - 1$  and a partition by parity. However, one can characterise all partition regular systems as follows:

An  $m \times n$  matrix  $A = (a_{i,j})$  is said to satisfy the *columns condition* if it is possible to order its column vectors  $\mathbf{a}_1, \dots, \mathbf{a}_n$  so that for some choice of indices  $1 \leq n_1 < n_2 < \dots < n_k = n$ , if we set

$$\mathbf{b}_i = \sum_{j=n_{i-1}+1}^{n_i} \mathbf{a}_j,$$

then

- (1)  $\mathbf{b}_1 = \mathbf{0}$ ,
- (2) for  $1 < i \leq k$ , the vector  $\mathbf{b}_i$  can be expressed as a *rational* linear combination of columns  $\mathbf{a}_j$  for  $1 \leq j \leq n_{i-1}$ .

Now we can formulate the following.

**Theorem 2.8.** (Rado [Rad]) *The system  $A\mathbf{x} = \mathbf{0}$  is partition regular if and only if  $A$  satisfies the columns condition.*

In neither direction this is a trivial result. For the proof see Graham et al. [GRS]. For the Schur theorem, consider the matrix  $A$  consisting of the single row

$$A = (1, 1 - 1).$$

For the Van der Waerden theorem, consider the following matrix  $M$ :

$$M = \begin{pmatrix} 1 & -1 & & & 1 \\ & 1 & -1 & & 1 \\ & & 1 & -1 & 1 \\ & & & \ddots & \vdots \\ & & & & 1 & -1 & 1 \end{pmatrix}$$

The matrix  $M$  actually proves a stronger statement (also conjectured by Schur) and originally proved by his student Brauer [Br].

**Theorem 2.9.** (Brauer, 1928) *For a positive integer  $n$ , there exists a positive integer  $N$  such that in an arbitrary colouring of the set  $[N]$  by  $r$  colours, we can find in one of the colour classes the arithmetic progression  $a_0, a_0 + d, \dots, a_0 + nd$  together with the difference  $d$ .*

*Proof.* The first column  $\mathbf{m}_1$  of the matrix  $M$  corresponds to the variable  $a_0$ , second column  $\mathbf{m}_2$  to  $a_0 + d$ , etc., and the last column  $\mathbf{m}_{n+2}$  corresponds to the difference  $d$ . To apply Rado Theorem, one has to show that  $M$  satisfies the columns condition. To see this, consider the vector  $\mathbf{b}_1 = \sum_{i=1}^{n+1} \mathbf{m}_i$  and vector  $\mathbf{b}_2 = \mathbf{m}_{n+2}$ . It is easy to see that  $\mathbf{b}_1 = \mathbf{0}$ . By considering the rational combination  $\mathbf{b}_2 = \sum_{i=1}^{n+1} (n - i + 1)\mathbf{m}_i$ , the columns condition holds, therefore, due to Rado theorem, Theorem 2.9 follows.  $\square$

We also present another proof of Theorem 2.9, which is using van der Waerden theorem. In spite of the results by Shelah, this gives better upper bound on the number  $N$ . For the original write-up, see Graham et al. [GRS], Chapter 3.

*Proof.* We use induction on  $r$ . In the case  $r = 1$ , we may clearly take  $N(n, 1) = n + 1$ . Let  $W(t, r)$  be the minimal number  $W$  such that if  $[W]$  is  $r$ -coloured, there exists a monochromatic arithmetic progression of length  $t$ . Here, of course, we are using van der Waerden's theorem.

For given  $n$  and  $r$ , we claim that we may take

$$N = N(n, r) = W(n \cdot N(n, r - 1), r). \tag{2.9}$$

We fix an  $r$ -colouring of  $[N]$ . Among the first  $W(n \cdot N(n, r - 1), r)$  integers, we find a monochromatic, say red, set

$$\{a_0 + id'; 0 \leq i \leq n \cdot N(n, r - 1)\}.$$

If, for some  $j$ ,  $1 \leq j \leq N(n, r - 1)$ ,  $d'j$  is red, then the set  $\{a_0, a_0 + d, \dots, a_0 + nd, d\}$  is red with  $d = jd'$ . Otherwise,  $\{d'j; 1 \leq j \leq N(n, r - 1)\}$  is  $(r - 1)$ -coloured. Using the equivalence between colourings of  $[N]$  and  $d'[N]$ , we find that the desired monochromatic set exists.  $\square$

## 2.6 Restricted theorems

We start this section by chronologically first example of restricted Ramsey problem (i.e.  $K_4$ -free Ramsey graphs for the triangle). This problem was fully solved by Nešetřil and Rödl [NR2].

**Theorem 2.10.** *Let  $G$  be a graph not containing a complete graph  $K_k$ . Let  $t$  be a positive integer. Then there exists a graph  $H$  not containing a complete graph  $K_k$  such that for every  $t$ -colouring of edges of  $H$  we get a subgraph isomorphic to  $G$  with all its edges in one of the classes of the partition.*

Results of this type are called *restricted* Ramsey-type theorems. However, it is possible to prove a much more general statement. We shall do so by means of the following definitions.

Given objects  $A$  and  $B$ , denote by  $\binom{B}{A}$  the set of all sub-objects of  $B$  which are isomorphic to  $A$ . We say that object  $C$  is  $(t, A)$ -*Ramsey* for object  $B$  if for every  $t$ -colouring of the set  $\binom{C}{A}$  there exists a sub-object  $B'$  of  $C$  which is isomorphic to  $B$  such that the set  $\binom{B'}{A}$  is monochromatic. We denote this by  $C \rightarrow (B)_t^A$ .

For  $A \in K$  we say that the class  $K$  has the  $A$ -*Ramsey property* if for every object  $B$  of  $K$  and every positive integer  $t$  there exists  $C$  of  $K$  such that  $C \rightarrow (B)_t^A$ . In the extreme case where  $K$  has the  $A$ -Ramsey property for each of its objects  $A$  we say that  $K$  is a *Ramsey class*.

A *type* is a sequence  $(n_\delta; \delta \in \Delta)$  of positive integers. A type will be fixed. A *structure* (set system) of type  $\Delta$  is a pair  $(X, \mathcal{M})$  where:

- (1)  $X$  is linearly ordered set,
- (2)  $\mathcal{M} = (\mathcal{M}_\delta; \delta \in \Delta)$  and  $\mathcal{M}_\delta \subseteq \binom{X}{n_\delta}$  for each  $\delta \in \Delta$ .

Given two structures  $(X, \mathcal{M})$  and  $(Y, \mathcal{N})$ ,  $\mathcal{N} = (\mathcal{N}_\delta; \delta \in \Delta)$ , a mapping  $f: X \rightarrow Y$  is said to be an *embedding* if

- (1)  $f$  is bijection and monotone with respect to standard orderings,
- (2) for every  $\delta \in \Delta$  and each subset  $M \subseteq X$ , we have  $M \in \mathcal{M}_\delta \Leftrightarrow f(M) \in \mathcal{N}_\delta$ .

The tuple  $(X, \mathcal{M})$  is a *substructure* of  $(Y, \mathcal{N})$  if the inclusion  $X \subseteq Y$  is embedding.

A structure  $A = (X, \mathcal{M})$ ,  $\mathcal{M} = (\mathcal{M}_\delta; \delta \in \Delta)$  of type  $\Delta$ , is said to be *irreducible* if for every pair  $x, y \in X$ , there exists  $\delta \in \Delta$  and  $M \in \mathcal{M}_\delta$  such that  $x, y \in M$ . Let  $\mathcal{F}$

be a (possibly infinite) set of structures of type  $\Delta$ . Denote by  $\text{Forb}_\Delta(\mathcal{F})$  the class of all structures  $A$  of type  $\Delta$  which do not contain any member of  $\mathcal{F}$  as a substructure.

Now we can formulate the principal result for set structures due to Nešetřil and Rödl [NR3, NR4]. We omit the proof.

**Theorem 2.11.** (Ramsey classes of structures) *Let  $\Delta$  be a type. Let  $\mathcal{F}$  be a (possibly infinite) set of irreducible structures of type  $\Delta$ . Then  $\text{Forb}_\Delta(\mathcal{F})$  is a Ramsey class.*

As the special case of the immensely general Theorem 2.11, we formulate the following corollary.

**Corollary 2.12.** *Let  $k \geq 2$  and  $t \geq 2$  be integers. Let  $F$  and  $W$  be two  $k$ -graphs such that  $F$  is irreducible and  $F \not\subseteq W$ . Then there exists an  $F$ -free  $k$ -graph  $H$  such that in any edge-colouring of  $H$  by  $t$  colours,  $H$  contains a monochromatic  $F$ -subgraph.*

*Proof.* Follows from Theorem 2.11 by considering the type  $\Delta = (k)$ . The one-element set of forbidden structures contains only  $\mathcal{F} = \{F\}$ . Note that  $W \in \text{Forb}_\Delta(\mathcal{F})$  and the one-edge  $k$ -graph  $S = (T, T) \in \text{Forb}_\Delta(\mathcal{F})$ . Then by Theorem 2.11, the class  $\text{Forb}_\Delta(\mathcal{F})$  contains a  $k$ -graph  $H$  such that  $H \rightarrow (W)_t^S$ .  $\square$

## 2.7 Other Ramsey-type theorems

The following challenging conjecture has been formulated by Martin Loeb1.

**Conjecture 2.13.** (Loeb1) *Let  $T$  be a tree on  $n$  vertices. Then every  $K_{2n}$  whose edges are coloured by two colours contains monochromatic  $T$ .*

Only partial results have been achieved, e.g. Haxell, Łuczak and Tingley [HLT] have proved the conjecture for trees with small maximum degree.

## 2.8 Ramsey numbers upper bounds

Perhaps the first question which one is tempted to consider is the problem of the actual size of a set which guarantees the validity of Ramsey's (and Ramsey-type) theorem. However, it is well known that Ramsey numbers are difficult to determine and even good asymptotic estimates are difficult to find (and improve).

First we should define the family of functions, which we will often use in our estimates. For each positive integer  $n$ , define the functions  $f_n: \mathbb{N} \rightarrow \mathbb{N}$  as follows:

$$\begin{aligned} f_1(i) &= i + 1 \\ f_2(i) &= 2i \\ f_{n+1}(i) &= \underbrace{f_n \circ \cdots \circ f_n}_i(i) \end{aligned}$$

Thus,  $f_3(i) = 2^i$  and  $f_4(i)$  is a stack of 2's of height  $i$  (the *tower function*). This family of functions is called the *Ackermann hierarchy*. The *Ackermann function*  $A$  is the diagonal

function

$$A(n) = f_n(n).$$

Many results could be shown about Ackermann hierarchy. All these function can be also expressed in the terms of *partially recursive functions*. We do not wish to introduce the formal machinery of partially recursive functions. However, we mention that these functions contain instruments like “for-cycles” and “while-cycles” (in the meaning of programming language) and also simple arithmetic operations. For details see e.g. [So].

A function that can be computed without “while-cycles” is called *primitive recursive*. It can be shown that every function  $f_i$  from the Ackermann hierarchy is asymptotically the fastest growing function among all functions using at most  $i$  “for-cycles”. Ackermann function fails to be primitive recursive; it cannot be expressed by a combination of the usual function operations without “while-cycles”.

Back to the Ramsey numbers estimates. For Ramsey numbers the situation is not as dramatic. It follows from Erdős and Rado [ER] and Erdős et al. [EHR] that both upper and lower bound for the Ramsey number  $\mathcal{R}_p(n, t)$  (see Section 2.2 for definition) are of the form

$$T_p(c_{p,t} \cdot n), \tag{2.10}$$

where  $c_{p,t}$  is a constant depending solely on  $p$  and  $t$  and  $T_p$  is the two-variables version of the tower function  $f_4$ :

$$T_p(x) = 2^{2^{\cdot^{2^x}}} \tag{2.11}$$

and there are  $p - 1$  2’s on the stack.

Let us derive an upper bound for the Hales-Jewett theorem.

**Corollary 2.14.** *For every positive integer  $n$  and  $t$*

$$\mathcal{H}(n, t) \leq f_5(c(n + t)),$$

where  $f_5$  is the function introduced in the Ackermann hierarchy and  $c$  is an absolute positive constant.

*Proof.* The function  $f(n, t)$  from Lemma 2.5 has a growth of the tower function, which in our notation is the function  $f_4$ . This follows from the inequality (2.2). The bound on  $\mathcal{H}(n, t)$  (see the inequality (2.5)) then involves the iteration of  $f(n, t)$  as the principal term, which gives the function  $f_5$ .  $\square$

Recall the van der Waerden theorem (Theorem 2.6). By  $\mathcal{A}(n)$ , we shall mean the smallest integer  $N$  such that van der Waerden theorem holds for  $n$  and  $t = 2$  colours. By Corollary 2.14 and the proof of Theorem 2.6, we know that  $\mathcal{A}(n) \leq f_5(c \cdot n)$ . However, Gowers recently proved the following celebrated result [Go].

**Theorem 2.15.** (Gowers, 2001) *Let  $n$  be an integer. Then*

$$\mathcal{A}(n) \leq 2^{2^{2^{2^{n+9}}}}.$$

Let  $\mathcal{S}(t)$  denote the minimum  $n$  such that the theorem of Schur (Theorem 2.7) holds. An examination of proof of Theorem 2.7 yields

$$\mathcal{S}(t) \leq \mathcal{R}_2(3, t).$$

The second proof of Brauer theorem (Theorem 2.9) gives a reasonable upper bound. Let  $\mathcal{A}_d(n)$  be the smallest number  $N$  such that Brauer theorem holds for  $r = 2$  colours. An examination of the second proof of Brauer theorem, particularly (2.9), yields

$$\mathcal{A}_d(n) \leq W(n(n+1), 2) = \mathcal{A}(n(n+1)) \leq 2^{2^{2^{2^{n^2+n+9}}}}, \quad (2.12)$$

using Theorem 2.15 by Gowers.

However, for the Rado theorem (Theorem 2.8), no reasonable upper bounds are known, i.e. they are not even primitive recursive. The same applies for Theorem 2.11 and Corollary 2.12, and these results remain to be improved.



## 3. Combinatorial Games

In this chapter, we build all the necessary theory of positional games from scratch, to be later used in Chapter 4. The write-up more or less closely follows the great work by Beck [Be], with the exception of Section 3.4.

### 3.1 Informal introduction into combinatorial games

Games belong to the oldest experiences of mankind, well before the appearance of any kind of serious mathematics. How the playing of games has long been a natural instinct of all humans, is how the solving of games is a natural instinct of mathematicians. “Recreational mathematics” is a vast collection of all kinds of clever observations about games and puzzles, the perfect empirical background for a mathematical theory. It is well known how games of chance played a crucial role in the early developments of probability theory. Similarly, graph theory grew out of puzzles (i.e. one-player games) like the “Königsberg bridge problem”, solved by Euler, or Hamilton’s “round-trip game” on the graph of dodecahedron. Unlike these two very successful theories, we still do not have a really satisfying quantitative theory of games of pure skill with perfect information, or as they are called nowadays: *combinatorial games*.

In technical terms, combinatorial games are 2-player games, mostly finite, with perfect information and no chance moves, and the payoff function has three values  $\pm 1, 0$  as the first player wins or loses the play, or it ends in a draw. Combinatorial game theory attempts to answer the questions of “who wins”, “how to win”, and “how long does it take to win”. Naturally, “win” means “forced win”, i.e. a winning strategy.

A great effort to bring the enormously rich branch of “recreational mathematics” up to “academic mathematics” is the two-volumed *Winning Ways* by Berlekamp, Conway and Guy [BCG], published in 1982. We must admit, however, that combinatorial game theory is still at an early stage of its developments. We do not feel that this is discouraging. Just to the contrary: we think the underdeveloped state of the theory is a great challenge and a great opportunity to make major discoveries in this exceptionally attractive field.

### 3.1.1 Classification of games

One natural way to classify games is the following:

- (1) games of pure chance,
- (2) games of mixed chance and skill,
- (3) games of pure skill.

Another possible classification is

- (1) games of perfect information (players cannot hide anything),
- (2) games of imperfect information (players can keep some secrets).

Throughout this work, we are interested exclusively in the games of pure skill and perfect information.

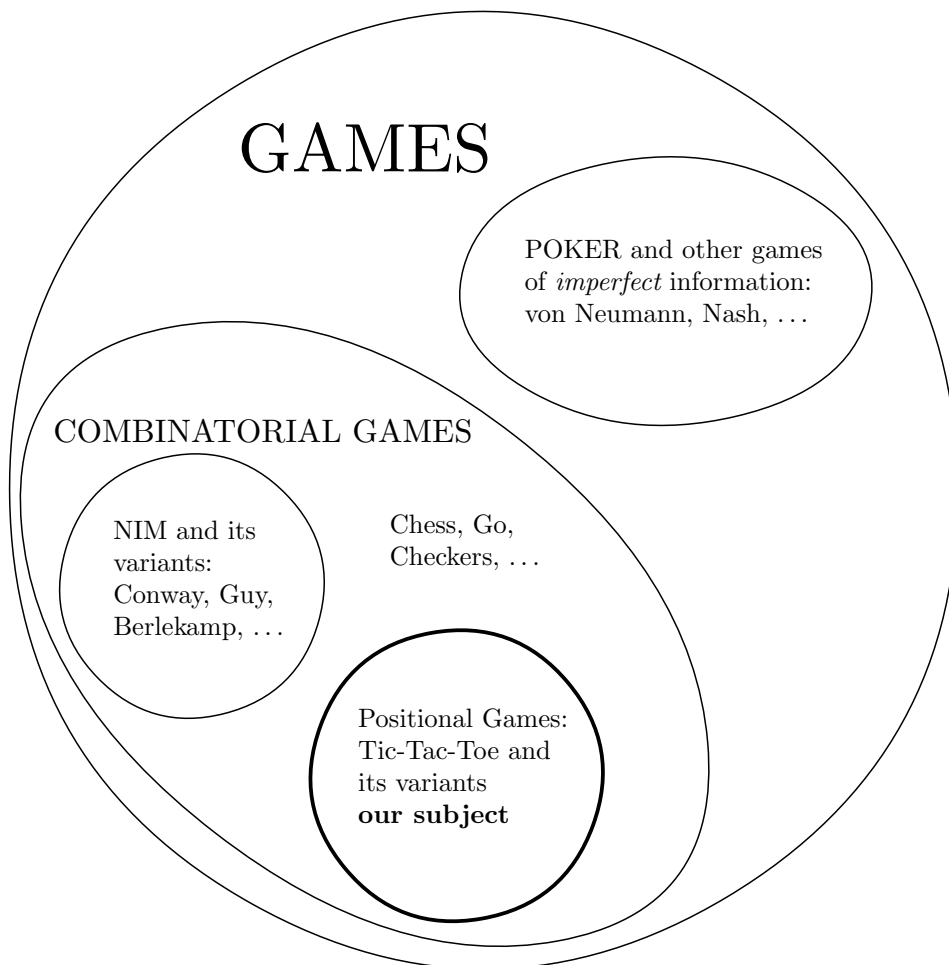


FIG. 3.4. One of the game classifications.

### 3.1.2 Traditional game theory

Traditional game theory, initiated by John von Neumann, deals with an extremely wide concept of games, including both games of perfect and imperfect information, chance and skill, with two or more players, and with arbitrary payoff functions. Each player has a choice, called *strategy*, and his object is to maximise a payoff which depends both on his own choice and on his opponent's.

The crucial *minimax* theorem for 2-player games (i.e. pure conflict situation) is that it is always possible for both players to find *mixed* strategies forming an “equilibrium”, which means the best compromise for both players. Mixed strategy means to randomly choose a strategy from a set of available strategies by some given probability distribution. The interesting consequence of Neumann's minimax theorem is that the best play often involves *random*, unpredictable moves. A typical example of such a game is Poker, which was in fact von Neumann's main motivation.

In the years since 1944, traditional game theory has developed rapidly. It plays a fundamental role in the mathematical fields like linear programming and optimisation. It has an important place in economics. Traditional game theory is certainly very useful in games of imperfect informations. However, for combinatorial games like Chess, Go, Checkers, traditional theory does not give too much insight.

For perfect-information games, including the subclass of combinatorial games, the von Neumann theory provided a more efficient general method to find a pure optimal strategy. It is called the *backward labelling* of the game-tree, or, in other words, the brute force case study. However, backward labelling still needs exponential running time in terms of the size of the board, which means it is impractical.

We can say, therefore, that the basic challenge of the theory of combinatorial games is the *complexity problem*. Even for games with relatively small boards, the backward labelling of the game-tree or any other brute force case study is far beyond the capacity of the fastest computers.

### 3.1.3 Combinatorial games theory

The object of combinatorial game theory is to describe wide classes of games for which there is a substantial “shortcut”. This means to find a fast way of answering the question “who wins”, and also, if possible, to find a tractable way of answering the other question of “how to win”, avoiding the exhaustive search in full depth. A natural way to cut down the alternatives is to be able to judge the value of a position at a level of a few moves depth only. This requires human intelligence, an essential thing that computers lack.

If we do not have time for the exhaustive search through all branches of the game-tree, then we have to develop a new approach. The basic idea is to use score systems which have some natural probabilistic interpretations related to the random walk on the game-tree. The strategy is to keep the “danger” (i.e. the “loss-chance”) at a low level during the entire course of the play.

How can one keep the “danger” at a low level? Well, the trick is the *potential technique*, a popular method in applied mathematics. We find the origin of the potential

technique in physics. Consider for example a pendulum. When a pendulum is at the top of its swing, it has certain potential energy and unless it receives extra energy, it cannot attain more than this speed, nor can it swing higher than its starting point. The same argument is widely used in one-player games. This technique is called *resource counting*.

We apply this approach on the wide class of “Tic-Tac-Toe-like games”, i.e. *positional games*. It means to play according to the rules of the Tic-Tac-Toe on an arbitrary family of winning sets (*strong game*). Or to play the asymmetric “maker-breaker version” on the same family: maker wants to completely occupy a winning set, and breaker simply wants to prevent his opponent from doing so (we call it the *weak game*).

### 3.1.4 Why are games interesting?

Why should the reader be interested in combinatorial games? Well, beside the obvious reason that games are great fun, we are going to present four reasons.

First, positional games are *interesting*, because they include such all-time favourites as Tic-Tac-Toe and its “grown-up” variants like Go-Moku, multidimensional Tic-Tac-Toe, Hex, Bridgit, graph games like Sim, etc.

The second reason why the reader might find our subject interesting is that the theory of positional games forms a natural bridge between the two well-established combinatorial theories: random graphs and Ramsey theory.

Third, games are ideal models for all kinds of research problems. Expressing a mathematical or social problem in terms of games is half of the problem.

Finally, this theory already has some very interesting applications in algorithms and complexity theory.

The “bad news” is that we could not solve such famous open problems like to find an *explicit* first-player’s winning strategy in Hex or similar games (in these cases *strategy stealing* implies the *existence*). One also cannot expect the potential function method to solve delicately balanced “head-to-head” games where single mistake could be fatal. But the theory can recognise and solve large classes of complex “one-sided” games where a few mistakes do not change the outcome.

The “good news” is that this theory is *not* about long, boring case studies. Quite to the contrary, we hardly have any case study. We focus on finding general principles which explain the behaviour of large classes of games.

## 3.2 Positional games and strategies formally

We start building the theory of positional games precisely.

### 3.2.1 Definitions

We define the class of (finite) *strong games*. Let  $\mathcal{F} = (V, F)$  be an arbitrary finite hypergraph. Here  $V$  is a finite set (the “vertex set” or “point set”) that we call the *board* of the game. The set  $F$  is an arbitrary collection of subsets of  $V$  (formally  $F \subseteq 2^V$ ), a collection of hyperedges, that we prefer to call the family of *winning sets*. Two players alternately occupy previously unoccupied points of the board  $V$ . Each player occupies one point per move. That player wins who occupies all points of some winning set  $A \in F$  *first*; otherwise the play ends in a draw.

If a player can block every winning set in a strong game, then he can force a *blocking draw*. The complementary concept is *weak win*: if a player does not have a blocking draw, then the opponent has a weak win. In other words, a player has a weak win if he can completely occupy a winning set (but not necessarily first).

In similar way we define the class of (finite) *weak games*. Let  $\mathcal{F} = (V, F)$  be an arbitrary finite hypergraph, the set  $V$  is a finite set that we call the board of the game and the set  $F \subseteq 2^V$  is a collection of hyperedges that we call the winning sets. Two players alternately occupy previously unoccupied points of the board  $V$ , each player occupies one point per move. The first player wins if he occupies all points of some winning set  $A \in F$ . Otherwise the second player wins. Note that the second player can completely occupy some winning set, but this is not considered as victory, his only goal is to prevent the first player from winning. Also note that draw is impossible in a weak game. In a weak game, the first player is usually called *maker* and the second player is called *breaker*. Sometimes the weak games are thus called *maker-breaker games*.

Note that a drawing strategy can be a winning strategy, and a blocking draw can be a pairing draw.

Consider a strong game on a finite hypergraph  $\mathcal{F} = (V, F)$ . A *strategy* for the first (second) player formally means a function  $S$  such that the domain of  $S$  is a set of even (odd) length subsequences of different elements of the board  $V$ , and the range is  $V$ . If the moves of the first player are denoted by  $x_1, x_2, x_3, \dots$ , and the moves of the second player are  $y_1, y_2, y_3, \dots$ , then the  $i$ -th move  $x_i$  ( $y_i$ ) is determined from the “past” by  $S$  as follows:

$$\begin{aligned} x_i &= S(x_1, y_1, x_2, y_2, \dots, y_{i-1}) \in V \setminus \{x_1, y_1, x_2, y_2, \dots, y_{i-1}\} \\ \left( y_i &= S(x_1, y_1, x_2, y_2, \dots, y_{i-1}, x_i) \in V \setminus \{x_1, y_1, x_2, y_2, \dots, y_{i-1}, x_i\} \right) \end{aligned}$$

defines the  $i$ -th move of the first (second) player. In other words, a *strategy* for the *first* (*second*) player is a function which assigns a legal next move to all partial plays of *even* (*odd*) length.

A *winning* (or *drawing*) *strategy*  $S$  for the first player means that in all possible plays where the first player follows  $S$  to find his next move is a win for him (a win or a draw). Formally, each play

$$x_1 = S(\emptyset), \forall y_1, x_2 = S(x_1, y_1), \forall y_2, x_3 = S(x_1, y_1, x_2, y_2), \forall y_3, \dots, \forall y_{n/2} \quad (3.1)$$

if  $n = |V|$  is *even*, and

$$x_1 = S(\emptyset), \forall y_1, \dots, \forall y_{(n-1)/2}, x_{(n+1)/2} = S(x_1, y_1, x_2, y_2, \dots, y_{(n-1)/2}) \quad (3.2)$$

if  $n$  is *odd*, is a win for the first player (a win or a draw).

Similarly, a winning (drawing) strategy  $S$  for the second player means that in all possible plays where the second player uses  $S$  to find his next move is a win for him (a win or a draw). Formally, each play

$$\forall x_1, y_1 = S(x_1), \dots, \forall x_{n/2}, y_{n/2} = S(x_1, y_1, x_2, y_2, \dots, x_{n/2}) \quad (3.3)$$

if  $n = |V|$  is *even*, and

$$\forall x_1, y_1 = S(x_1), \forall x_2, y_2 = S(x_1, y_1, x_2), \forall x_3, \dots, \forall x_{(n+1)/2} \quad (3.4)$$

if  $n$  is *odd*, is a win for the second player (a win or a draw). In both cases

$$x_i \in V \setminus \{x_1, y_2, x_2, y_2, \dots, y_{i-1}\} \quad \text{and} \quad y_i \in V \setminus \{x_1, y_2, x_2, y_2, \dots, y_{i-1}, x_i\} \quad (3.5)$$

hold for all  $i \geq 1$ .

The ultimate questions of game theory are about strategies, in fact about *optimal* strategies. *Optimal strategies* are the winning strategies, and the drawing strategies when winning strategy does not exist.

### 3.2.2 Possible game outcomes

There are only three possible outcomes of a strong game, as precisely formulated in the following theorem.

**Theorem 3.1.** (Strategy Theorem) *Let  $\mathcal{F} = (V, F)$  be an arbitrary finite hypergraph, and consider the strong game on this hypergraph. Then there are three alternatives: either the first player has a winning strategy, or the second player has a winning strategy, or both of them have a drawing strategy.*

*Proof.* The formal proof is a simple modification of the De Morgan's law. Indeed, we have the following three possibilities:

- (a) *either* the first player (**I**) has a winning strategy;
- (b) *or* the second player (**II**) has a winning strategy;
- (c) *or* the negation of (a)  $\vee$  (b).

First assume that  $n = |V|$  is even. In view of (3.1)–(3.5), case (a) formally means that

$$\exists x_1 \forall y_1 \exists x_2 \forall y_2 \cdots \exists x_{n/2} \forall y_{n/2} \quad (3.6)$$

such that **I** wins (the sequence in (3.6) has to satisfy (3.5)).

Indeed,

$$S(x_1, y_1, x_2, y_2, \dots, y_{i-1}) = x_i \in V \setminus \{x_1, y_1, x_2, y_2, \dots, y_{i-1}\}$$

defines a winning strategy  $S$  for **I**.

By the De Morgan's law,  $\neg(a)$  is equivalent to

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots \forall x_{n/2} \exists y_{n/2} \quad (3.7)$$

such that **I** loses or it is a draw (the sequence in (3.7) has to satisfy (3.5)). Therefore,  $\neg(a)$  means that **I** has a drawing strategy.

Case (c) is equivalent to  $\neg(a) \wedge \neg(b)$ , which means that both players have a drawing strategy. The case “ $n$  is odd” is analogous and we omit it.  $\square$

We should also mention that for the class of weak games, there are only two outcomes: either the first player (maker) has a winning strategy or the second player (breaker) has a winning strategy, which follows directly from the definition.

### 3.2.3 Strategy stealing argument

The following theorem is extremely important and also quite surprising; it states that in the class of strong games the first player has a big advantage—he cannot lose.

**Theorem 3.2.** (Strategy Stealing) *Let  $\mathcal{F} = (V, F)$  be an arbitrary finite hypergraph. Then playing the strong game on  $\mathcal{F}$ , first player can force at least a draw, i.e. a draw or possibly a win.*

*Proof.* The real meaning of Theorem 3.2 is that for the subclass of strong games one of the 3 possible outcomes of a game (see Theorem 3.1) cannot occur. Indeed, we show that second player cannot have a winning strategy.

Assume that second player (**II**) has a winning strategy  $S$ , and we want to obtain a contradiction. The idea is to see what happens if first player (**I**) steals and uses  $S$ . A winning strategy for a player is a list of instructions telling the player that if the opponent does this, then he does that, so if the player follows the instructions, he will always win.

Now **I** can use **II**'s winning strategy  $S$  to win as follows. **I** takes an arbitrary first move, and then *pretends* to be the second player, that is, he ignores his first move. After **II**'s each move, **I**, as a fake second player, reads the instruction in  $S$  to take action. If **I** is told to take a move that is still available, he takes it. If this move was taken by him before as his ignored “arbitrary” first move, he takes another “arbitrary move”. The crucial point here is that an extra move, namely the last “arbitrary move”, only *benefits* **I** in a strong game.

The formal execution of this idea is very simple and goes as follows. We use the notation  $x_1, x_2, x_3, \dots$  for the moves of **I**, and  $y_1, y_2, \dots$  for the moves of **II**. By using **II**'s moves  $y_1, y_2, \dots$  and **II**'s winning strategy  $S$ , we are going to define **I**'s moves  $x_1, x_2, \dots$  (satisfying (3.5)), and also two auxiliary sequences  $z_1, z_2, z_3, \dots$  and  $w_1, w_2, w_3, \dots$ .

Let  $x_1$  be an “arbitrary” first move of **I**. Let  $w_1 = x_1$  and  $z_1 = S(y_1)$ . We distinguish two cases. If  $z_1 \neq w_1$ , then let  $x_2 = z_1$  and  $w_2 = w_1$ . If  $z_1 = w_1$ , then let  $x_2$  be another “arbitrary” move, and let  $w_2 = x_2$ . Next let  $z_2 = S(y_1, z_1, y_2)$ . Again we distinguish two cases. If  $z_2 \neq w_2$ , then let  $x_3 = z_2$  and  $w_3 = w_2$ . If  $z_2 = w_2$ , then let  $x_3$  be another “arbitrary” move, and let  $w_3 = x_3$ , and so on.

In general, let  $z_i = S(y_1, z_1, y_2, z_2, \dots, y_i)$ . We distinguish two cases: if  $z_i \neq w_i$ , then let  $x_{i+1} = z_i$  and  $w_{i+1} = w_i$ ; if  $z_i = w_i$ , then let  $x_{i+1}$  be another “arbitrary” move, and let  $w_{i+1} = x_{i+1}$ .

It follows from the construction that

$$\{x_1, x_2, \dots, x_i, x_{i+1}\} = \{z_1, z_2, \dots, z_i\} \cup \{w_{i+1}\} \quad (3.8)$$

for each  $i \geq 1$ . In view of (3.8) the “virtual play”  $y_1, z_1, y_2, z_2, y_3, z_3, \dots$  is a legitimate one, i.e. it satisfies principle (3.5). We call the two players of this “virtual play” Mr. Y (who starts) and Mr. Z (of course Mr. Y is **II**, and Mr. Z is “almost” **I**). The only minor technical difficulty is to see what happens at the end. We consider two cases according to the parity of the board size. The complete “virtual play” between Mr. Y and Mr. Z is

$$y_1, z_1, y_2, z_2, y_3, z_3, \dots, y_m, z_m, w_{m+1} \quad (3.9)$$

if the board size  $|V| = 2m + 1$  is odd, and

$$y_1, z_1, y_2, z_2, y_3, z_3, \dots, y_{m-1}, z_{m-1}, y_m, w_m \quad (3.10)$$

if the board size  $|V| = 2m$  is even.

We recall that  $z_i = S(y_1, z_1, y_2, z_2, \dots, z_{i-1}, y_i)$  for each  $i \geq 1$ . Since  $S$  is a winning strategy for the second player, it follows that Mr. Z wins the virtual play (3.9) (i.e. when  $|V|$  is odd) even if the last move  $w_{m+1}$  belongs to Mr. Y. In view of (3.8) this implies a **I**’s win in the “real play”

$$x_1, y_1, x_2, y_2, x_3, y_3, \dots, x_m, y_m.$$

We used the fact that in a strong game an extra point cannot possibly harm **I**. This is how **I** “steals” **II**’s winning strategy  $S$ .

The conclusion is that if **II** had a winning strategy, so would **I**. But it is impossible that both players have a winning strategy. Indeed, if each player follows his own winning strategy, then this particular play has two winners, which is a contradiction. So the supposed winning strategy for **II** cannot exist. This implies that **I** can always force at least a draw.  $\square$

**Corollary 3.3.** *Let  $\mathcal{F}$  be an arbitrary finite hypergraph. If the second player has a winning strategy in a weak game on  $\mathcal{F}$ , then the strong game on  $\mathcal{F}$  is draw.*

*Proof.* Let  $S$  be the winning strategy of the second player in the weak game on  $\mathcal{F}$ . Consider a strong game on  $\mathcal{F}$ . If the second player uses the strategy  $S$ , he cannot lose. Due to Strategy Stealing (Theorem 3.2), he also cannot win, therefore the game is draw.  $\square$

However, we should emphasise that the proof of Strategy Stealing is not constructive, i.e. it gives only existential proof of the first player’s drawing strategy.



### 3.3 Resource counting

The resource counting technique will be the main tool in our work. In this section we present the most important theorems based on resource counting.

#### 3.3.1 Solitaire Army puzzle

The best way to illustrate the *resource counting* method is to discuss Conway's solution of a puzzle called *Solitaire Army*.

Solitaire Army is a particular case of the class of *solitaire puzzles*. These puzzles are not “real games” because there is only one player. The common feature of these puzzles is that each one is played with a board and men or pegs, the board contains a number of holes each of which can hold one man. Each move consists of a jump by one man over one or more other men, the man jumped over being removed from the board. Each move therefore reduces the number of men on the board.

The Solitaire Army is played on the infinite plane and the holes are in the lattice points. The permitted move is to jump a man horizontally or vertically but not diagonally. Let us draw a horizontal line across the infinite board and start with all men behind this line. Assume this line is the horizontal axis, so all men are in the lower half-plane. How many men do we need to send one man forward 1, 2, 3, 4 or 5 holes into the upper half-plane?

Obviously, two men are needed to send a man forward one hole, and four men are needed to send a man forward two holes. Eight men are enough to send a man forward three holes. Twenty men are enough to send a man forward four holes, see Figure 3.5.

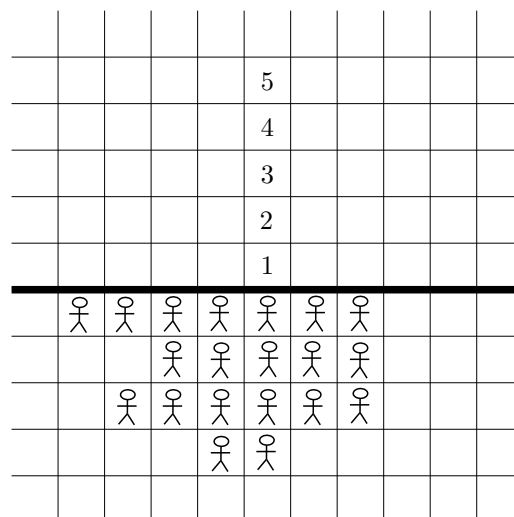


FIG. 3.5. A configuration of Solitaire Army able to jump to distance 4.

But the really surprising result is the case of five holes: it is *impossible* to send a man forward five holes into the upper half-plane. This striking result was discovered by Conway in 1961 (see e.g. [BCG] or [Be]).

The idea behind Conway's resource counting is the following. We assign a weight to each hole subject to the condition that if  $H_1$ ,  $H_2$  and  $H_3$  are *any* three consecutive holes in a row or in a column, and  $w(H_1)$ ,  $w(H_2)$  and  $w(H_3)$  are the corresponding weights, then  $w(H_1) + w(H_2) \geq w(H_3)$ . We can evaluate a position by the sum of the weights of those holes which are occupied by men—this sum is called the *value* of the position.

The meaning of the inequality  $w(H_1) + w(H_2) \geq w(H_3)$  is following. The effect of a move where a man in  $H_1$  jumps over another man in  $H_2$  and arrives at  $H_3$  is that we replace men with weights  $w(H_1)$  and  $w(H_2)$  by a man with weight  $w(H_3)$ . Since  $w(H_1) + w(H_2) \geq w(H_3)$ , this change cannot be an increase in the value of the new position.

Inequality  $w(H_1) + w(H_2) \geq w(H_3)$  guarantees that *no play is possible from an initial position to a target position if the target position has a higher value*.

Let  $w$  be a positive number which satisfies  $w + w^2 = 1$ . The number  $w$  equals the *golden section*  $\frac{\sqrt{5}-1}{2}$ . Now Conway's resource counting goes as follows. Assume that one succeeded to send a man 5 holes forward into the upper half-plane by starting from a configuration of a finite number of men in the lower half-plane. Write 1 where the man stands 5 holes forward into the upper half-plane, and extend it in the following way:

				1					
				$w$					
				$w^2$					
				$w^3$					
				$w^4$					
	$\dots$	$w^8$	$w^7$	$w^6$	$w^5$	$w^6$	$w^7$	$w^8$	$\dots$
		$w^9$	$w^8$	$w^7$	$w^6$	$w^7$	$w^8$	$w^9$	
	$\dots$	$w^{10}$	$w^9$	$w^8$	$w^7$	$w^8$	$w^9$	$w^{10}$	$\dots$
			$\vdots$		$\vdots$		$\vdots$		

FIG. 3.6. Evaluating the board by  $w = \frac{\sqrt{5}-1}{2}$ .

The value of the top line of the lower half-plane is

$$w^5 + 2w^6 + 2w^7 + 2w^8 + \dots = w^5 + 2\frac{w^6}{1-w} = w^5 + 2\frac{w^6}{w^2} = w^5 + 2w^4 = w^3 + w^4 = w^2.$$

So the value of the whole lower half-plane is

$$w^2(1 + w + w^2 + w^3 + \dots) = w^2\frac{1}{1-w} = w^2\frac{1}{w^2} = 1,$$

which is exactly the value of the target position. So no finite number of men in the lower half-plane will suffice to send a man forward five holes into the upper half-plane. If it was

possible, then, using rules which do not increase the total weight, it would be possible to gain the final value 1 from the original value less than 1. This is a contradiction.

One can even show that eight men are in fact needed to send a man forward three holes, and similarly, twenty men are needed to send a man forward four holes (i.e. the  $2^i$  pattern breaks for four holes). The fact that eight men are necessary can be seen from the resource count of Figure 3.6, for the target position has value  $w^2$  and the highest value that can be achieved with only seven men below the line is  $w^5 + 3w^6 + 3w^7$ . One can easily check that  $w^2 > w^5 + 3w^6 + 3w^7$ .

To send a man forward 4 holes requires 20 men (not 16). The proof that 20 men are necessary is more complicated, but the idea is almost the same as for the case of 3 holes, and we skip it.

### 3.3.2 Sufficient condition for blocking draw

An application of resource counting for 2-player games is the following result of Erdős and Selfridge from 1973 [ESe]. This is a sufficient condition for blocking draw.

**Theorem 3.4.** (Erdős, Selfridge) *If  $\mathcal{F} = (V, F)$  is an  $k$ -uniform hypergraph and  $|F| < 2^{k-3}$ , then the second player can force a draw, in fact a blocking draw, in the strong game on  $\mathcal{F}$ .*

*Proof.* Let  $F = \{A_1, A_2, \dots, A_m\}$  where  $m < 2^{k-1}$ . Assume we are at the stage of the game where the first player already occupied  $x_1, x_2, \dots, x_i$ , and the second player occupied  $y_1, y_2, \dots, y_{i-1}$ . The question is how to choose the second player's next point  $y_i$ . Those winning sets which contain at least one  $y_j$ ,  $j \leq i-1$ , are "harmless"—we call them *dead sets*. The winning sets which are not dead are called *survivors*. The survivors have a chance to be completely occupied by the first player at the end of the play, so they each represent some "danger". What is the "total danger" of the whole position? We evaluate the given position by the following expression, called *danger function*:  $D_i = \sum_{s \in S} 2^{-u_s}$ , where  $S$  is the set of survivors indices,  $u_s$  is the number of unoccupied elements of the survivor  $A_s$ ,  $s \in S$  and index  $i$  indicates that we are at the stage of choosing the  $i$ -th point  $y_i$  of the second player.

A natural choice for  $y_i$  is to minimise the danger  $D_{i+1}$  at the next stage. How to do that? Because of the simple linear structure of the danger function  $D_i$ , it is quite easy to answer this question. Let  $y_i$  and  $x_{i+1}$  denote the next two moves. What is their effect on  $D_{i+1}$ ? The point  $y_i$  "kills" all the survivors  $A_s \ni y_i$ , which means we have to subtract the sum

$$\sum_{A_s \ni y_i} 2^{-u_s}$$

from  $D_i$ . On the other hand,  $x_{i+1}$  doubles the danger of each survivor  $A_s \ni x_{i+1}$ , that is, we have to add the sum  $\sum_{A_s \ni x_{i+1}} 2^{-u_s}$  back. If some survivor  $A_s$  contains both  $y_i$  and  $x_{i+1}$ , then we do not have to give the corresponding term  $2^{-u_s}$  back because that  $A_s$  was previously killed by  $y_i$ .

Now the natural choice for  $y_i$  is that unoccupied  $z$  for which  $\sum_{A_s \ni z} 2^{-u_s}$  is maximum. Then clearly

$$\begin{aligned} D_{i+1} &\leq D_i - \sum_{A_s \ni y_i} 2^{-u_s} + \sum_{A_s \ni x_{i+1}} 2^{-u_s} \\ &\leq D_i - \sum_{A_s \ni y_i} 2^{-u_s} + \sum_{A_s \ni y_i} 2^{-u_s} = D_i. \end{aligned}$$

In other words, the second player can force the *decreasing property*  $D_0 \geq D_1 \geq D_2 \geq \dots \geq D_{\text{end}}$ .

Second player's ultimate goal is to prevent first player from completely occupying some  $A_0 \in F$ , that is, to avoid  $u_0 = 0$ . If  $u_0 = 0$ , then  $D_{\text{end}} \geq 2^{-u_0} = 1$ . On the other hand,

$$D_0 = \sum_{A: x_1 \in A \in F} 2^{-k+1} + \sum_{A: x_1 \notin A \in F} 2^{-n} \leq |F| \cdot 2^{-n+1} < 1,$$

so by the decreasing property of the danger function,  $D_{\text{end}} < 1$ . Therefore, no play is possible from an initial position if the target position has a higher value. This completes the proof of the Erdős-Selfridge theorem.  $\square$

Theorem 3.4 is sharp. The full branches of a binary tree  $T$  with  $n$  levels form an  $n$ -uniform family of  $2^{n-1}$  winning sets (see Figure 3.7) such that the first player can occupy a full branch in  $n$  moves.

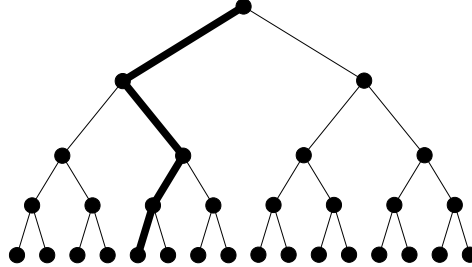


FIG. 3.7. A board with  $2^{n-1}$  winning lines where the first player wins.

The strategy of the first player is following. In the first step he takes the root. In the second step he takes some point from the second level of vertices, then from the third level, and so on. Finally, he takes some leaf. This all is done in such a way that the first player keeps building a connected path from the root downwards. Each vertex  $v \in T$  is a root of some subtree of  $T$ , and every such  $v$  (except the leaves) has two sons  $v_1, v_2$ , which are roots of two subtrees  $T_1, T_2$ .

The strategy is simple. Let us denote the bottom vertex of the actual path by  $v$ . First player waits for the second player to move, i.e. to occupy a vertex  $y$ . If  $y \in T_1$  then first player takes  $v_2$ , otherwise he takes  $v_1$ . Clearly, the second player cannot prevent the first player from taking a whole path from root to some leaf.

### 3.3.3 Sufficient condition for weak win

A straightforward adaptation of the Erdős-Selfridge resource counting gives Weak Win Criterion.

**Theorem 3.5.** (Weak Win Criterion) *Assume that we are playing the weak game on a  $k$ -uniform hypergraph  $\mathcal{F} = (V, F)$ . Assume that, fixing any two distinct points of  $V$ , there are no more than  $\Delta_2 = \Delta_2(\mathcal{F})$  hyperedges  $A \in F$  containing both points.*

- (a) *If  $|F| > 2^{k-3} \cdot \Delta_2 \cdot |V|$ , then the maker has a weak win in  $\mathcal{F}$ .*
- (b) *In particular, for almost disjoint hypergraph  $\mathcal{F}$  we have  $\Delta_2 = 1$  and requirement simplifies to  $|F| > 2^{k-3} \cdot |V|$ .*

*Proof.* The proof is a straightforward adaptation of the Erdős-Selfridge resource counting. Assume we are at the stage of the play where the first player has already occupied the points  $x_1, x_2, \dots, x_i$  and the second player has occupied  $y_1, y_2, \dots, y_i$ . The question is how to choose first player's next point  $x_{i+1}$ . Those winning sets which contain at least one  $y_j$ ,  $j \leq i$ , are useless for the first player. We call them *dead sets*. The winning sets which are not dead (yet) are called *survivors*. The survivors have a chance to be completely occupied by the first player (the weak win). What is the total "chance" of the position?

We evaluate the given position by the following *chance function*:  $C_i = \sum_{s \in S} 2^{-u_s}$  where  $S$  is the set of survivor indices,  $u_s$  is the number of unoccupied points of the survivor  $A_s$ ,  $s \in S$ , and index  $i$  indicates that we are at the stage of choosing the  $(i+1)$ -st point  $x_{i+1}$  of first player. Note that the chance function can be much bigger than 1—this is not a probability, but it is always non-negative.

A natural choice for  $x_{i+1}$  is to maximise the chance  $C_{i+1}$  at the next stage. Let  $x_{i+1}$  and  $y_{i+1}$  denote the next moves of the two players. What is their effect on  $C_{i+1}$ ? First  $x_{i+1}$  doubles the chances for each survivor  $A_s \ni x_{i+1}$ , that is, we have to add the sum  $\sum_{A_s \ni x_{i+1}} 2^{-u_s}$  to  $C_i$ .

On the other hand,  $y_{i+1}$  kills all the survivors  $A_s \ni y_{i+1}$ , which means we have to subtract the sum

$$\sum_{A_s \ni y_{i+1}} 2^{-u_s}$$

from  $C_i$ .

We have to make a correction of those survivors  $A_s$  which contain both  $x_{i+1}$  and  $y_{i+1}$ . These survivors  $A_s$  were doubled first and killed second. So what we have subtract from  $C_i$  is not

$$\sum_{A_s \supseteq \{x_{i+1}, y_{i+1}\}} 2^{-u_s}$$

but the twice as large

$$\sum_{A_s \supseteq \{x_{i+1}, y_{i+1}\}} 2^{-u_s+1}.$$

It follows that

$$C_{i+1} = C_i + \sum_{A_s \ni x_{i+1}} 2^{-u_s} - \sum_{A_s \ni y_{i+1}} 2^{-u_s} - \sum_{A_s \supseteq \{x_{i+1}, y_{i+1}\}} 2^{-u_s}.$$

Now the natural choice for  $x_{i+1}$  is that unoccupied  $z$  for which  $\sum_{A_s \ni z} 2^{-u_s}$  is maximum. Then clearly

$$C_{i+1} \geq C_i - \sum_{A_s \ni \{x_{i+1}, y_{i+1}\}} 2^{-u_s}.$$

We trivially have

$$\sum_{A_s \ni \{x_{i+1}, y_{i+1}\}} 2^{-u_s} \leq \frac{\Delta_2}{4}.$$

Indeed, there are at most  $\Delta_2$  winning sets  $A_s$  containing the given two points  $\{x_{i+1}, y_{i+1}\}$ , and  $2^{-u_s} \leq 2^{-2}$  since  $x_{i+1}$  and  $y_{i+1}$  were definitely unoccupied points at the previous stage. Therefore,

$$C_{i+1} \geq C_i - \frac{\Delta_2}{4}.$$

What happens at the end? Let  $\ell$  denote the number of stages, i.e. the  $\ell$ -th stage is the last one. Clearly  $\ell = |V|/2$ . Inequality  $C_\ell = C_{\text{last}} > 0$  means that the second player could not kill (block) all the winning sets. Indeed, at the last stage all points are occupied, so  $C_\ell = C_{\text{last}} > 0$  means that first player was able to completely occupy a winning set, i.e. weak win.

So all what we have to check is that  $C_\ell = C_{\text{last}} > 0$ . But this is trivial. Indeed,  $C_0 = |F| \cdot 2^{-k}$ , so we have

$$C_{\text{last}} \geq |F| \cdot 2^{-k} - \frac{|V| \Delta_2}{2 \cdot 4}.$$

It follows that  $C_{\text{last}} > 0$  if  $|F| > 2^{k-3} \cdot |V| \cdot \Delta_2$ , which completes the proof of case (a).

Finally, if  $\mathcal{F}$  is almost disjoint, then fixing any two points of the board  $V$ , there is at most one winning set containing both of them. So  $\Delta_2 = \Delta_2(\mathcal{F}) = 1$  and case (b) follows from (a).  $\square$

We should mention that the proofs of Theorem 3.4 and Theorem 3.5 are constructive, that means they give an explicit strategy description.

### 3.4 Degree game

Assume the board is a complete graph  $K_n$  on  $n$  vertices and two players alternately pick previously unselected edges. For a given  $n$ , what is the largest star subgraph of  $K_n$  the first player can build? Formally, let the edges of  $K_n$  are (partially) coloured by two colours, for  $v \in V(K_n)$  denote by  $d_1(v)$  the number of edges incident with  $v$  that are coloured by the first player's colour. Let  $D^*(n) = \max_{v \in V(G)} d_1(v)$ . For a given  $n$ , what is the largest  $D^*(n)$  such that the first player has a winning strategy reaching this number in edge-colouring weak game? This was originally a question of Erdős. We give a close answer to this question.

### 3.4.1 Discrepancy lemma

Let us first formulate and prove a technical lemma due to Beck [Be81]. However, our proof goes into much more details, which are omitted in [Be81].

**Lemma 3.6.** *Given a hypergraph  $\mathcal{F} = (V, F)$  and a real number  $\alpha$ ,  $1/2 < \alpha \leq 1$ , consider the following two-player combinatorial game. The players alternately pick previously unselected points of  $V$  and the second player (maker) wins if he can cover at least  $\alpha|A|$  points of some  $A \in F$ . Otherwise the first player (breaker) wins. If*

$$\sum_{A \in F} (2\alpha^\alpha(1-\alpha)^{1-\alpha})^{-|A|} < 1,$$

*then the first player has a winning strategy.*

*Proof.* The proof is another application of resource counting method, but we shall use slightly different valuation method. Suppose we are at the stage when the  $i$ -th move has just finished and the first and second players have previously picked the points  $X = \{x_1, \dots, x_i\}$  and  $Y = \{y_1, \dots, y_i\}$ , respectively. We wish to choose a good point  $x_{i+1}$  for breaker. Now we give each edge  $A \in F$  a value  $w_i(A)$  which is equal to

$$(1 + \mu)^{|A \cap Y| - \alpha|A|} (1 - \mu)^{|A \cap X| - (1-\alpha)|A|}$$

where  $\mu = 2\alpha - 1$ . Next, we give each point  $x \in V$  a value  $w_i(x)$  where

$$w_i(x) = \sum_{A: x \in A \in F} w_i(A).$$

Finally, we evaluate the *total chance of breaker* of the whole position by

$$W_i = \sum_{A \in F} w_i(A).$$

The natural choice for  $x_{i+1}$  is to maximise the chance  $W_{i+1}$ . Therefore, in the  $(i+1)$ -st move we pick a point  $x$  of the greatest value  $w_i(x)$ . We prove this is the desired winning strategy of breaker.

We claim  $W_{i+1} \leq W_i$ . To do that, we compute how does the total chance change during the  $(i+1)$ -st move. Let us consider the state immediately after breaker has picked the previously mentioned  $x_{i+1}$ . Consider some edge  $A \ni x_{i+1}$ . How does the value of  $A$  change? One point has been picked from  $A$  by breaker, so the difference of the value is

$$(1 + \mu)^{|A \cap Y| - \alpha|A|} (1 - \mu)^{|A \cap X| + 1 - (1-\alpha)|A|} - w_i(A) = (1 - \mu)w_i(A) - w_i(A) = -\mu w_i(A).$$

That gives

$$\sum_{A \ni x_{i+1}} -\mu w_i(A) = -\mu w_i(x_{i+1})$$

contribution by the breaker's move.

Now assume maker has picked some point  $y_{i+1}$ . What is the change in the total chance? Consider some edge  $A \ni y_{i+1}$ . One point of  $A$  has been picked by maker, therefore the difference is

$$(1 + \mu)^{|A \cap Y| + 1 - \alpha|A|} (1 - \mu)^{|A \cap X| - (1 - \alpha)|A|} - w_i(A) = \mu w_i(A),$$

which gives

$$\sum_{A \ni y_{i+1}} \mu w_i(A) = \mu w_i(y_{i+1})$$

in total. However, we have to take care of edges  $A$  such that  $\{x_{i+1}, y_{i+1}\} \subseteq A$ . In this case the difference is

$$(1 + \mu)^{|A \cap Y| + 1 - \alpha|A|} (1 - \mu)^{|A \cap X| + 1 - (1 - \alpha)|A|} - w_i(A) = -\mu^2 w_i(A).$$

In the total change, we have to add this number as a compensation. Therefore, the new total chance is

$$\begin{aligned} W_{i+1} &= W_i - \mu w_i(x_{i+1}) + \mu w_i(y_{i+1}) - \sum_{A \supseteq \{x_{i+1}, y_{i+1}\}} \mu^2 w_i(A) \\ &\leq W_i - \mu w_i(x_{i+1}) + \mu w_i(y_{i+1}) \leq W_i, \end{aligned}$$

since  $w_i(x_{i+1})$  was maximum. This implies  $W_i \leq W_0$  for all  $i$ . By the hypothesis of the lemma,

$$\begin{aligned} W_0 &= \sum_{A \in F} (1 + \mu)^{-\alpha|A|} (1 - \mu)^{-(1 - \alpha)|A|} = \sum_{A \in F} (2\alpha)^{-\alpha|A|} (2(1 - \alpha))^{-(1 - \alpha)|A|} \\ &= \sum_{A \in F} (2\alpha^\alpha (1 - \alpha)^{1 - \alpha})^{-|A|} < 1, \end{aligned}$$

therefore  $W_i < 1$  for all  $i$ .

Now, assume that maker can cover at least  $\alpha|A|$  points of some  $A \in F$ . That means  $|A \cap Y| - \alpha|A| \geq 0$  and  $|A \cap X| - (1 - \alpha)|A| \leq 0$ , so  $w_i(A) \geq 1$  for some (large)  $i$ . From this follows that  $1 \leq w_i(A) \leq W_i$ . Hence, our indirect assumption leads to contradiction.  $\square$



### 3.4.2 Breaker's winning condition

The following technical lemma is a preparation for Theorem 3.8.

**Lemma 3.7.** *There exists a positive constant  $c$  such that the condition*

$$n(2\alpha^\alpha(1-\alpha)^{1-\alpha})^{-(n-1)} < 1$$

holds for  $\alpha = \frac{1}{2} + cn^{-\frac{1}{2}} \log^{\frac{1}{2}} n$ .

*Proof.* Assume we already know the constant  $c$ . After applying logarithm, this means to check

$$\log n < (n-1) \left( 1 + \alpha \log \frac{\alpha}{1-\alpha} + \log(1-\alpha) \right).$$

Therefore, it is sufficient to show that  $1 + \alpha \log \frac{\alpha}{1-\alpha} + \log(1-\alpha)$  is greater than some positive constant. To do this, recall the Taylor expansions

$$\log(1+x) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i} \quad \text{and} \quad \log \frac{1+x}{1-x} = \sum_{i=0}^{\infty} 2 \frac{x^{2i+1}}{2i+1}$$

for  $-1 < x < 1$  (see e.g. [ST]). Let  $\beta = 2cn^{-\frac{1}{2}} \log^{\frac{1}{2}} n$ . Because  $\lim_{n \rightarrow \infty} \beta = 0$  we can set the constant  $c$  such that  $\beta < 1$  (this is in fact the place where we have established  $c$ ). Observe that

$$\log(1-\alpha) = \log(1-\beta) - \log 2 = - \sum_{i=1}^{\infty} \frac{\beta^i}{i}$$

and

$$\log \frac{\alpha}{1-\alpha} = \log \frac{1+\beta}{1-\beta} = \sum_{i=0}^{\infty} 2 \frac{\beta^{2i+1}}{2i+1}.$$

Therefore,  $1 + \alpha \log \frac{\alpha}{1-\alpha} + \log(1-\alpha)$  equals

$$1 - \log 2 + \sum_{i=0}^{\infty} \frac{\beta^{2i+1}}{2i+1} + \sum_{i=0}^{\infty} \frac{\beta^{2i+2}}{2i+1} - \sum_{i=1}^{\infty} \frac{\beta^i}{i} = 1 - \log 2 + \sum_{i=1}^{\infty} \beta^{2i} \frac{1}{2i(2i-1)} > 1 - \log 2 > 0$$

because the sum  $\sum_{i=1}^{\infty} \beta^{2i} \frac{1}{2i(2i-1)}$  is convergent and positive for  $\beta < 1$ .  $\square$

The following theorem is originally a result of Székely [Sz]. However, our proof shows all the necessary details, which are missing from the original paper.

**Theorem 3.8.** *Consider the degree game: the board consists of edges of  $K_n$ , breaker and maker alternately select a previously unselected edge. Breaker marks his edges blue and maker marks his edges red. The goal of maker is to build as large red substar as possible. Then breaker can prevent a maker's substar of size  $\frac{n}{2} + c\sqrt{n \log n}$  where  $c$  is a fixed positive constant.*

*Proof.* We define the hypergraph  $\mathcal{F} = (V, F)$  such that playing the degree game on  $K_n$  is equivalent with the weak game on  $\mathcal{F}$ . This means,  $V = E(K_n)$  and the hyperedges are all  $(n-1)$ -stars in  $K_n$ , that is,  $F = \{S \subseteq E(K_n); |S| = n-1, \bigcap S = v \in V\}$ .

Assume we know  $c$  already, in fact we establish it later. Our goal is to apply Lemma 3.6 here. Note that  $|A| = n-1$  for  $A \in F$ , and  $|F| = n$ . Let

$$\alpha = \frac{1}{2} + cn^{-\frac{1}{2}} \log^{\frac{1}{2}} n.$$

In order to apply Lemma 3.6, we have to check the condition

$$\sum_{A \in F} (2\alpha^\alpha(1-\alpha)^{1-\alpha})^{-|A|} = n (2\alpha^\alpha(1-\alpha)^{1-\alpha})^{-(n-1)} < 1.$$

This holds by Lemma 3.7, which also gives the desired constant  $c$ . Lemma 3.6 therefore yields the desired star size at most  $\alpha n = \frac{n}{2} + c\sqrt{n \log n}$ , which completes the deduction.  $\square$

### 3.4.3 Maker's winning condition

In the other direction, we can prove that maker can achieve a star of size  $\frac{n}{2} + c\sqrt{n}$  with some positive absolute constant  $c$ . We do this by modification of another proof, as suggested by Beck [Be]. Consider the following *row-column game*: two players, maker and breaker, alternately pick previously unselected points of an  $n \times n$  chessboard. Then maker is able to achieve a line (either row or column) containing at least  $\frac{n}{2} + \frac{\sqrt{n}}{32}$  points owned by him. For the original proof see Beck [Be] or [Be93].

**Theorem 3.9.** *Consider the degree game: the board consists of edges of  $K_n$ , breaker and maker alternately select a previously unselected edge. Breaker marks his edges blue and maker marks his edges red. Maker's object is to achieve a red substar of at least  $\frac{n}{2} + k$ ,  $k \geq 1$ , red edges. If  $k \leq \frac{\sqrt{n}}{32}$ , then maker has a winning strategy.*

*Proof.* Consider a play in the degree game on the graph  $K_n$ . In this proof, wherever we write about "substar", we mean the set of edges  $A \subset E(K_n)$  such that they form a star in  $K_n$ . By "star" we mean the maximal substar in  $K_n$ , i.e. on  $n$  vertices. Let  $x_1, x_2, \dots, x_i$  be the blue edges in the graph selected by breaker in his first  $i$  moves, and let  $y_1, y_2, \dots, y_{i-1}$  be the red edges selected by maker in his first  $(i-1)$  moves. The question is how to find maker's optimal  $i$ -th move  $y_i$ . Write

$$X_i = \{x_1, x_2, \dots, x_i\} \quad \text{and} \quad Y_{i-1} = \{y_1, y_2, \dots, y_{i-1}\}.$$

Let  $A$  be a star in the graph  $K_n$ , and introduce the following "weight":

$$w_i(A) = \left\{ |A \cap Y_{i-1}| - |A \cap X_i| + \frac{\sqrt{n}}{4} \right\}^+$$

where

$$\{\alpha\}^+ = \begin{cases} \alpha & \text{if } \alpha > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $y$  be an arbitrary unselected edge, and write

$$w_i(y) = w_i(A) + w_i(B),$$

where  $A$  and  $B$  are the two stars containing  $y$ .

Here is the maker's winning strategy: at his  $i$ -th move he selects that previously unselected edge  $y$  for which the maximum of the "weights"

$$\max_{y \text{ unselected}} w_i(y)$$

is attained.

The following total sum is a sort of "variance":

$$T_i = \sum_{n \text{ stars } A} (w_i(A))^2.$$

The idea of the proof is to study the behaviour of  $T_i$  as  $i = 1, 2, 3, \dots$  and to show that  $T_{\text{end}}$  is "large".

First we compare  $T_i$  and  $T_{i+1}$ , that is, we study the effects of the edges  $y_i$  and  $x_{i+1}$ . We distinguish two cases.

- (1) The edges  $y_i$  and  $x_{i+1}$  determine four different stars.
- (2) The edges  $y_i$  and  $x_{i+1}$  determine three different stars.

In case (1), an easy analysis shows that

$$T_{i+1} \geq T_i + 1, \tag{3.11}$$

except in the "unlikely situation" when  $w_i(y_i) = 0$ . Indeed,

$$w_i(y_i) = w_i(A) + w_i(B) \geq w_i(x_{i+1}) = w_i(C) + w_i(D),$$

and so

$$T_{i+1} = T_i + 2w_i(y_i) - 2w_i(x_{i+1}) + \beta \geq T_i + \beta$$

where

$$\beta = \begin{cases} 2 & \text{if } w_i(A) > 0 \text{ and } w_i(B) > 0, \\ 1 & \text{if } \max\{w_i(A), w_i(B)\} > 0 \text{ and } \min\{w_i(A), w_i(B)\} = 0, \\ 0 & \text{if } w_i(A) = w_i(B) = 0. \end{cases}$$

Even if the "unlikely situation" occurs, we have at least equality:  $T_{i+1} = T_i$ . Because  $y_i$  was the edge of maximum weight, for  $x_{i+1}$  and for every other unselected edge  $x$ ,  $w_i(x) = 0$ .

Similarly, in case (2),

$$T_{i+1} \geq T_i + 1 \quad (3.12)$$

except in the following “unlikely situation”:  $w_i(B) = 0$  where  $A$  is the star containing both  $y_i$  and  $x_{i+1}$ , and  $B$  is the other star containing  $y_i$ . Even if this “unlikely situation” occurs, we have at least equality:  $T_{i+1} = T_i$ . Because  $y_i$  was an edge of maximum weight, it follows that  $w_i(C) = 0$  where  $C$  is the other star containing  $x_{i+1}$ . Similarly, for every other unselected edge  $x$  in star  $A$ ,  $w_i(D_x) = 0$  where  $D_x$  is the other star containing  $x$ .

If  $i$  is an index for which the “unlikely situation” in case (1) occurs, let  $\text{unsel}(i)$  denote the set of all unselected edges after breaker’s  $i$ -th move. Similarly, if  $i$  is an index for which the “unlikely situation” in case (2) occurs, let  $\text{unsel}(i, A)$  denote the set of all unselected edges after breaker’s  $(i + 1)$ -st move in star  $A$  containing both  $y_i$  and  $x_{i+1}$ , including  $y_i$  and  $x_{i+1}$ .

If the “unlikely situation” occurs in less than  $3\binom{n}{2}/10$  moves (i.e. in less than 60% of the total time), then we are done. Indeed, by (3.11) and (3.12),

$$T_{\text{end}} = T_{\binom{n}{2}/2} \geq \frac{\binom{n}{2}}{5}.$$

Since  $T_{\text{end}}$  is a sum of  $n$  terms, we have

$$\max_{n \text{ stars } A} \left( w_{\binom{n}{2}/2}(A) \right)^2 \geq \frac{\binom{n}{2}/5}{n} \geq \frac{n}{12}.$$

Equivalently, for some star  $A$ ,

$$w_{\binom{n}{2}/2}(A) = \left\{ |A \cap Y_{\binom{n}{2}/2-1}| - |A \cap X_{\binom{n}{2}/2}| + \frac{\sqrt{n}}{4} \right\}^+ \geq \sqrt{n/12}.$$

So

$$|A \cap Y_{\binom{n}{2}/2-1}| - |A \cap X_{\binom{n}{2}/2}| \geq \sqrt{n/12} - \frac{\sqrt{n}}{4} > \frac{\sqrt{n}}{30},$$

and Theorem 3.9 follows.

If the “unlikely situation” in case (1) occurs in more than  $\binom{n}{2}/10$  moves (i.e. in more than 20% of the time), then let  $i_0$  be the first time when this happens. Clearly,

$$|\text{unsel}(i_0)| > 2\binom{n}{2}/10 = \binom{n}{2}/5.$$

It follows that there are at least  $\frac{\binom{n}{2}/5}{n-1} = n/10$  stars  $D$  containing (at least one) element of  $\text{unsel}(i_0)$  each. So  $w_i(D) = 0$  for at least  $n/10$  stars  $D$ , that is,

$$|D \cap X_i| - |D \cap Y_{i-1}| \geq \frac{\sqrt{n}}{4}$$

for at least  $n/10$  stars  $D$ . Therefore, after breaker’s  $i_0$ -th move,

$$\sum_{n \text{ stars } D} \{|D \cap X_i| - |D \cap Y_{i-1}|\}^+ > \frac{n}{10} \frac{\sqrt{n}}{4}. \quad (3.13)$$

Since

$$2 + \sum_{n \text{ stars } D} \{|D \cap Y_{i-1}| - |D \cap X_i|\}^+ = \sum_{n \text{ stars } D} \{|D \cap X_i| - |D \cap Y_{i-1}|\}^+,$$

by (3.13),

$$\sum_{n \text{ stars } D} \{|D \cap Y_{i-1}| - |D \cap X_i|\}^+ \geq \frac{n^{3/2}}{20} - 1.$$

Since the number of terms on the left-side is less than  $n - n/5 = 4n/5$ , after breaker's  $i_0$ -th move we have

$$\max_D \{|D \cap Y_{i-1}| - |D \cap X_i|\} > \frac{n^{3/2}/20 - 1}{4n/5} > \frac{\sqrt{n}}{20}.$$

Obviously, maker can keep this advantage of  $\sqrt{n}/20$  for the rest of the game, and again Theorem 3.9 follows.

Finally, we study the case when the “unlikely situation” of case (2) occurs for at least  $\binom{n}{2}/5$  moves (i.e. for at least 40% of the time). Without loss of generality, we can assume there are at least  $\binom{n}{2}/10$  “unlikely” indices  $i$ ; denote the star containing both  $y_i$  and  $x_{i+1}$  by  $A$ . We claim that there is an “unlikely” index  $i_0$  when

$$|\text{unsel}(i_0, A)| \geq (n-1)/5. \quad (3.14)$$

Indeed, by choosing  $y_i$  and  $x_{i+1}$ , in each “unlikely” move the set  $\text{unsel}(i, A)$  is decreasing by 2, and because we have  $n$  stars, the number of “unlikely” indices  $i$  when  $\text{unsel}(i, A) < (n-1)/5$  is altogether less than  $n \frac{(n-1)/5}{2} = \binom{n}{2}/5$ .

Now we can complete the proof just like before. We recall that  $w_{i_0}(D) = 0$  for those stars  $D$  which contain some edge from  $\text{unsel}(i_0, A)$  (here  $A$  is the star containing both  $y_{i_0}$  and  $x_{i_0+1}$ ). So by (3.14),  $w_{i_0}(D) = 0$  for at least  $(n-1)/5$  stars, that is,

$$|D \cap X_i| - |D \cap Y_{i-1}| \geq \frac{\sqrt{n}}{4}$$

for at least  $(n-1)/5$  stars  $D$ . Therefore, after breaker's  $i_0$ -th move

$$\sum_{n \text{ stars } D} \{|D \cap X_i| - |D \cap Y_{i-1}|\}^+ > \frac{n-1}{5} \frac{\sqrt{n}}{4}. \quad (3.15)$$

Since

$$2 + \sum_{n \text{ stars } D} \{|D \cap Y_{i-1}| - |D \cap X_i|\}^+ = \sum_{n \text{ stars } D} \{|D \cap X_i| - |D \cap Y_{i-1}|\}^+,$$

by (3.15),

$$\sum_{n \text{ stars } D} \{|D \cap Y_{i-1}| - |D \cap X_i|\}^+ \geq \frac{(n-1)\sqrt{n}}{20} - 1 \geq \frac{n^{3/2}}{25}$$

for  $n$  sufficiently large. Since the number of terms on the left-side is less than  $n - n/5 = 4n/5$ , after breaker's  $i_0$ -th move we have,

$$\max_D \{|D \cap Y_{i-1}| - |D \cap X_i|\} > \frac{n^{3/2}/25}{4n/5} = \frac{\sqrt{n}}{20}.$$

Obviously maker can keep this advantage of  $\sqrt{n}/20$  for the rest of the game, and again Theorem 3.9 follows. The proof is complete.  $\square$

# 4. Ramsey Theorems and Combinatorial Games

This chapter contains main and original results of this thesis. We study positional games derived from Ramsey-type theorems, establish upper bounds guaranteeing the first player's win, and finally we show large gaps between these game numbers and Ramsey numbers.

## 4.1 Ramsey theorems and positional games

What is the thing that connects Ramsey theory and positional games? Assume we play strong game on a hypergraph  $\mathcal{F}$  and assume that the hypergraph is *Ramsey* for two colours. By *Ramsey* we mean that given arbitrary colouring of points of  $\mathcal{F}$  by two colours, there exists a monochromatic hyperedge in  $\mathcal{F}$ . In other words,  $\chi(\mathcal{F}) > 2$ . On many kinds of hypergraphs (objects, boards) this condition depends only on the number of vertices, i.e. on such a kind of hypergraph there is always a monochromatic edge, provided the number of vertices is (very) large. So what does this all mean for the positional game theory? Suppose we take the board large, large enough to fulfil conditions of the appropriate Ramsey-type theorem. Then the game cannot be draw, someone must win!

Moreover, from the Strategy Stealing argument we know that in any strong game the first player cannot lose. To sum it up, if the first player cannot lose and someone must win, the first player wins. The same holds for the weak game. We formulate it precisely.

**Theorem 4.1.** *Assume  $\mathcal{F}$  is a hypergraph such that  $\chi(\mathcal{F}) > 2$  and assume two players play strong or weak game on  $\mathcal{F}$ . Then the first player wins.*

*Proof.* By Strategy Stealing argument (Theorem 3.2), we know that the first player has at least a drawing strategy  $S$  in the strong game on  $\mathcal{F}$ , that is, he knows how to force draw and maybe also how to win. Assume the first player uses the strategy  $S$  and the game finishes as draw, i.e. there is no monochromatic hyperedge. But this is a contradiction with the condition  $\chi(\mathcal{G}) > 2$ , there must exist such an edge. Therefore, the only alternative is that using the strategy  $S$  the first player wins. Finally, observe that the strategy  $S$  is winning also in the weak game on  $\mathcal{F}$ .  $\square$

We are mostly interested in “playing Ramsey-type theorems”. This means, to take a hypergraph which is studied by a particular Ramsey-type theorem and try to find a winning strategy of the first player. We are interested in the minimum number of vertices such that the first player wins. However, analysis of strong games seems to be hopelessly difficult in many cases. Therefore, in such cases we study the weak game instead.

The crucial point is that the board size providing first player's weak win is usually *enormously lower* than the Ramsey number.

## 4.2 Arithmetic progression games

For two integers  $k \geq 3$  and  $n$ , we define the weak (maker-breaker) *arithmetic progression game* (also called *van der Waerden game*) on the set  $S = \{1, 2, \dots, n\}$  as follows: two players are alternately colouring previously uncoloured points of  $S$  by two colours, both players by their own. The first player (maker) wins if he is able to colour a  $k$ -term subset forming arithmetic progression exclusively by his colour, otherwise the second player (breaker) wins. By  $\mathcal{A}^*(k)$  we denote the smallest  $n$  such that maker has a winning strategy. One can also consider the strong version of the arithmetic progression game, that is, such player wins who is able to colour a  $k$ -term arithmetic progression first.

Analogously, for two integers  $k \geq 3$  and  $n$ , we define the weak (maker-breaker) *arithmetic progression game with difference* on the set  $S = \{1, 2, \dots, n\}$  as follows: The first player (maker) is trying to colour a  $k$ -term arithmetic progression  $P$  as before, but now together with the number  $d$  denoting the difference of  $P$ . That is, maker wins if he colours some  $(k + 1)$ -tuple of  $S$  where  $k$  elements form an arithmetic progression  $P$  and the remaining element  $d$  denotes the difference of  $P$ . If he is unable to colour such set, the second player (breaker) wins. By  $\mathcal{A}_d^*(k)$  we mean the smallest  $n$  such that maker has a winning strategy. Again, one can also consider the strong version of this game.

The arithmetic progression games were investigated by Beck [Be81]. We generalise the proof ideas of Beck to work also on arithmetic progression games with difference.

**Theorem 4.2.** *Let  $k \geq 2$  be an integer. Assume maker and breaker play the  $k$ -term arithmetic progression game with difference (i.e. the weak game). Then the first player has a winning strategy on board of size  $\mathcal{O}(2^k k^3)$ .*

*Proof.* The idea of the proof is to reduce the game on general hypergraph game and then apply results from Chapter 3, particularly Weak Win Criterion (Theorem 3.5). From assumptions of Weak Win Criterion we compute the necessary board size such that maker has a winning strategy.

For a fixed  $n$ , let us define a  $(k + 1)$ -uniform hypergraph  $\mathcal{F} = (\{1, \dots, n\}, F)$ , the board together with the winning lines  $F$ . The set  $F$  contains all  $(k + 1)$ -element subsets  $S \subseteq \{1, \dots, n\}$  such that

- (1) some  $k$  elements of  $S$  form an arithmetic progression  $P$  of length  $k$ ,
- (2) the one remaining element  $d$  of  $S$  denotes the difference  $d$  of  $P$ .

Clearly, playing the weak game on  $\mathcal{F}$  is equivalent with the original weak arithmetic progression game with difference.

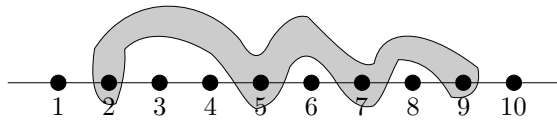


FIG. 4.8. Example of edge in the hypergraph  $\mathcal{F}$  for  $k = 3$ .

We are going to find the smallest  $n$  such that the inequality  $|F| > 2^{k-2} \cdot n \cdot \Delta_2(\mathcal{F})$  from Weak Win Criterion holds (note that  $\mathcal{F}$  is  $(k + 1)$ -uniform), thus proving the existence of



maker's winning strategy. To do this, we find an upper bound  $d(n)$  on  $\Delta_2 = \Delta_2(\mathcal{F})$  and lower bound  $f(n)$  on  $|F|$ , and we solve the inequality  $f(n) > 2^{k-2} \cdot n \cdot d(n)$ . Assume we know  $n$  already, we compute it precisely later.

To establish  $d(n)$ , let us fix two distinct points  $a, b \in \{1, \dots, n\}$ ,  $a < b$ . We consider the roles of  $a$  and  $b$  in a hyperedge  $S \in F$  and we count the maximal number of edges incident both with  $a$  and  $b$ . Three cases are possible:

- (1) The point  $a$  denotes the arithmetic progression difference. Therefore,  $b$  can lay on  $k$  positions of the arithmetic progression, so we get at most  $k$  possibilities.
- (2) The point  $b$  denotes the difference. Similarly, there is at most  $k$  possibilities.
- (3) Both  $a$  and  $b$  are members of the arithmetic progression. The number of possibilities is therefore at most  $\binom{k}{2}$  as this is the number of all positions the two points can occupy in a  $k$ -term progression.

Thus, we have  $\binom{k}{2} + 2k \geq \Delta_2(\mathcal{F})$ .

Let us establish a lower bound on  $|F|$ . For a difference  $d$ , a  $k$ -term arithmetic progression of difference  $d$  spans  $(k-1)d + 1$  elements. Thus for  $d < \frac{n-1}{k-1}$ , there are at least  $n - kd$  positions where to start the arithmetic progression. We have

$$|F| \geq \sum_{d=1}^{\frac{n-1}{k-1}-1} n - kd = n \left( \frac{n-1}{k-1} - 1 \right) + \frac{k(n-1)}{2(k-1)} \left( \frac{n-1}{k-1} - 1 \right) \geq C \frac{n^2}{k}$$

for some fixed constant  $C$ . By solving the inequality

$$C \frac{n^2}{k} > 2^{k-2} \left( \binom{k}{2} + 2k \right) n,$$

we get  $n = \mathcal{O}(2^k k^3)$ . By Weak Win Criterion, maker has a winning strategy on  $\mathcal{F}$  with  $n$  vertices, therefore also in the original game. This completes the proof.  $\square$

We also give a lower bound on the size of board for the arithmetic progression game with difference. This means, we show that for small enough set  $S = \{1, \dots, n\}$  the first player cannot win, both in the strong and weak game. Note that by Strategy Stealing (Theorem 3.2) he also cannot lose, therefore the game is draw.

**Theorem 4.3.** *Let  $k \geq 2$  be an integer. Assume two players play the  $k$ -term arithmetic progression game with difference on  $\{1, \dots, n\}$ . Then the second player can force a draw, both in the strong and weak game for  $n = \Omega(2^{k/2} \sqrt{k})$ .*

*Proof.* Similarly to the proof of Theorem 4.2, we construct an equivalent game on a hypergraph such that we can apply the theorem of Erdős and Selfridge (Theorem 3.4) and thus have the second player's winning strategy.

Assume we already know the size  $n$ , we actually compute it later. Let us define the  $(k+1)$ -uniform hypergraph  $\mathcal{F} = (\{1, \dots, n\}, F)$ . The edge set  $F$  contains all  $(k+1)$ -element subsets  $S \subseteq \{1, \dots, n\}$  such that some  $k$  elements of  $S$  form an arithmetic progression  $P$  of length  $k$  and the remaining element  $d$  denotes the difference of  $P$ . Our

goal is to find the highest possible  $n$  such that  $|F| < 2^k$  (note that  $\mathcal{F}$  is  $(k+1)$ -uniform), i.e.  $n$  such that Theorem 3.4 holds.

To establish an upper bound on  $|F|$ , recall the method used in Theorem 4.2. For difference  $d \leq \lceil n/k \rceil$ , there are at least  $n - kd$  positions where to start a progression, therefore

$$|F| \leq \sum_{d=1}^{\lceil n/k \rceil} n - kd = \frac{n^2}{k} - k \frac{(\lceil n/k \rceil)(\lceil n/k \rceil + 1)}{2} \leq C \frac{n^2}{k}$$

for some fixed constant  $C$ . By solving the inequality  $Cn^2k < 2^k$ , we get  $n = \mathcal{O}(2^{k/2}\sqrt{k})$ . Theorem 3.4 applied on  $\mathcal{F}$  defined only on  $n$  vertices proves the existence of second player's drawing strategy. Note that a drawing strategy works both for the strong and weak version of the game.  $\square$

As we have mentioned already, the method used in Theorem 4.2 is a generalisation of the method used by Beck [Be81]. By small modifications of the proof of Theorem 4.2, one can prove the following upper bound on the van der Waerden game number:  $\mathcal{A}^*(k) < k^3 2^{k-4}$ . For the sake of completeness we briefly sketch the proof.

**Theorem 4.4.** (Beck 1981) *Let  $k \geq 3$  be an integer. Assume maker and breaker play the  $k$ -term arithmetic progression game on  $S = \{1, \dots, n\}$ . If  $n = \mathcal{O}(k^3 2^k)$  then maker has a winning strategy.*

*Proof.* Let maker and breaker play the game on the set  $S$ , we establish the size  $n$  later. Let us define the equivalent game hypergraph  $\mathcal{F} = (S, E)$  where edges are the  $k$ -term subsets of  $S$  forming arithmetic progressions. Similarly to proof of Theorem 4.2, we count the number of arithmetic progression in  $S$ , which is  $\Theta(n^2/k)$ , therefore also  $|E| = \Theta(n^2/k)$ . And again by similar argument,  $\Delta_2(\mathcal{F}) \leq \binom{k}{2}$ .

By satisfying the assumptions of Weak Win Criterion (Theorem 3.5), that is, by solving the inequality  $|F| > 2^{k-3} \cdot n \cdot \Delta_2(\mathcal{F})$ , we estimate the size  $n$  of  $S$  such that maker has a winning strategy.  $\square$

However, the lower bound established by Beck [Be81] is  $2^{k-7k^{7/8}} < \mathcal{A}^*(k)$  by using number theory results, which are inapplicable in our case of arithmetic progression game with difference. Therefore, we skip this proof. We only mention that the method used in Theorem 4.3 works, yielding  $\mathcal{A}^*(k) = \Omega(2^{k/2}\sqrt{k})$ .

To sum the contents of this section, we formulate the following corollary.

**Corollary 4.5.** *For an integer  $k \geq 2$ , we have*

$$\Omega(2^{k/2}\sqrt{k}) = \mathcal{A}_d^*(k) = \mathcal{O}(2^k k^3)$$

and

$$\Omega(2^{k-7k^{7/8}}) = \mathcal{A}^*(k) = \mathcal{O}(2^k k^3).$$

*Proof.* The first statement clearly follows from Theorem 4.2 and Theorem 4.3. The second statement was proved by Beck, see Theorem 4.4 for the upper bound and consult [Be81] for the lower bound.  $\square$

### 4.3 Generalised Tic-Tac-Toe

Tic-Tac-Toe is probably the best known positional game, which was the original motivation for Hales and Jewett to prove Theorem 2.4. Let us first define the terms. By the  $n^d$  game we mean a positional game on board consisting of  $[n]^d$  cube, where each winning line is a combinatorial line in this cube (see Section 2.3 for definitions).

The strong  $n^d$  game is not yet fully solved, therefore it still “remains funny” for human players. However, many partial results have been obtained. We do not wish to list them, as this topic is quite long. For details see Beck [Be].

We focus on the weak  $n^d$  game. Recall the theorem of Hales and Jewett and ask the appropriate game-related question: For a fixed integer  $n$ , what is the minimum dimension  $d$  such that the first player has a winning strategy in the weak  $n^d$  game? The following result has been shown by Beck [Be].

**Theorem 4.6.** *Let  $n$  and  $d$  be two positive integers. If  $d > c \cdot n^2$  ( $c$  is an absolute positive constant), then the first player (maker) has a winning strategy in the weak  $n^d$  game.*

*Proof.* Let us apply Weak Win Criterion (Theorem 3.5) to the  $n^d$  game.

We first need to count the number of winning lines. To do this, note that for each  $j \in \{1, 2, \dots, d\}$ , the sequence  $a_j^1, a_j^2, \dots, a_j^n$  composed of the  $j$ -th coordinates of the points on a winning line is either strictly *increasing* from 1 to  $n$ , or strictly *decreasing* from  $n$  to 1, or a *constant*  $c = c_j \in \{1, 2, \dots, n\}$ . Since for each coordinate we have  $n+2$  possibilities  $\{1, 2, \dots, n, \textit{increasing}, \textit{decreasing}\}$ , this gives  $(n+2)^d$ , but we have to subtract  $n^d$  because it is impossible that each coordinate is *constant*. Finally, we have to divide by 2, since every line has two orientations. This yields the total of  $((n+2)^d - n^d)/2$  possibilities.

An alternative geometric way of getting this number goes as follows. Imagine the board  $n^d$  surrounded by an additional layer of cells, one cell thick. This new object is a cube  $(n+2)^d$ . It is easy to see that every winning line of the  $n^d$  board extends to a uniquely determined pair of cells in the new surface layer.

Another observation is that no two combinatorial lines share more than one common point. Therefore,  $\Delta_2 = 1$  for the  $n^d$  game, i.e. the hypergraph is almost disjoint.

By Weak Win Criterion, it remains to verify the condition

$$\frac{(n+2)^d - n^d}{2} > 2^{n-3} n^d,$$

which is equivalent to

$$\left(1 + \frac{2}{n}\right)^d > 2^{n-2} + 1. \quad (4.1)$$

Finally, note that inequality (4.1) holds for  $n < \sqrt{2d/\log 2}$ , or equivalently, if  $d > (\log 2) \cdot n^2/2$  (provided  $n$  is sufficiently large). The theorem follows.  $\square$

## 4.4 Ramsey games

The original graph Ramsey theorem (Theorem 2.2) or the hypergraph Ramsey theorem (Theorem 2.3) can serve as a basis for the following game. Let  $n$  be a positive integer and consider a complete graph  $K_N$ . Assume two players alternately pick previously unselected edges of  $K_N$ . The goal is to create a monochromatic  $K_n$ -subgraph, whoever creates it first, wins. This is the *strong Ramsey game*. One can also consider the *weak Ramsey game*, first player (maker) is trying to build a  $K_n$ -subgraph, second player (breaker) has to prevent it. Both games can be played on general complete  $k$ -graphs.

In 1981, József Beck published the following famous result [Be81], where he also later (see e.g. [Be]) improved the special case of  $k = 2$ .

**Theorem 4.7.** (Beck, 1981) *Let  $k$  and  $n$  be positive integers. Consider the weak Ramsey game, where the maker is trying to build a  $K_n^k$  subgraph of a complete  $k$ -graph  $K_N^k$ . Then there is positive constant  $c_k$  such that if  $N \geq 2^{c_k \cdot n^k}$  then maker has a winning strategy. Moreover, in the case of  $k = 2$ , maker wins if  $N \geq 2^{n+2}$ .*

For the proof, see [Be81] and [Be]. We only mention that proof of the general case is an application of Weak Win Criterion.

We also mention the *strong Ramsey game on infinite board*. Consider a countable infinite complete graph  $K_\infty$ . Two players alternately select previously unselected edges of  $K_\infty$ . The player wins who first creates a  $K_3$  subgraph. This game is actually very easy, one can easily find a winning strategy of the first player using 3 moves. The real problem arises when we replace  $K_3$  by  $K_4$ , i.e. the goal is to build a  $K_4$  subgraph. This has been long time the open problem of Beck (see e.g. [Be]). Recently, Jelínek, Kára, Šámal and Valla [KG] have found an explicit winning strategy of the first player. The paper is in preparation.

## 4.5 Restricted Ramsey games

We will study positional games on  $k$ -graphs that do not contain some forbidden subgraph. Note that for  $k = 2$  these games reduce to games on ordinary graphs. Let us fix two hypergraphs: the “forbidden”  $k$ -graph  $F$  and the “winning”  $k$ -graph  $W$ . From now we consider only “playgrounds” that do not contain  $F$  as a subgraph. Given such a  $k$ -graph  $G$ , two players alternately colour the edges of  $G$  by two colours, each player by his own. The goal of the game is to colour a subgraph of  $G$  isomorphic to  $W$ .

We can consider the strong or weak game. That means, the player who first colours a  $W$ -subgraph wins, and the first player wins if he is able to colour a  $W$ -subgraph, otherwise the second wins (i.e. the maker-breaker game), respectively. Let us call these games *strong restricted Ramsey game* and *weak restricted Ramsey game*, respectively.

Actually, we understand the strong restricted Ramsey game very little. In this section, we are interested mostly in the weak game. One can also consider the similar game on a  $k$ -graph  $G$  where players colour vertices instead of edges. This game actually turns out to be quite easy for many kinds of winning and forbidden  $k$ -graphs.

We are interested in finding the least number of vertices  $V$ , such that there exists a  $k$ -graph  $G = (V, E)$  where the first player has a winning strategy.

Let us define the weak restricted Ramsey game precisely. Let us fix an integer  $k \geq 2$  and let  $W$  and  $F$  be two fixed  $k$ -graphs. We define the number  $\mathcal{R}_k^*(W, F)$  as the minimum number of vertices  $V$ , such that there exists a  $k$ -graph  $G = (V, E)$  with the following properties:

- (1)  $G$  does not contain  $F$  as a subgraph.
- (2) Consider the weak (maker-breaker) game on  $G$  played by colouring edges  $E$ . The goal of maker is to colour the edges of a subgraph isomorphic to  $W$ , breaker is trying to prevent it. Then maker has a winning strategy on  $G$ .

Analogously, we define  $\tilde{\mathcal{R}}_k^*(W, F)$  as the least number of vertices  $V$ , such that there exists an  $F$ -free  $k$ -graph  $G = (V, E)$ , where maker has a winning strategy in the similar game of finding a  $W$ -subgraph by colouring just the vertices  $V$ .

Compared to arithmetic progression games (see Section 4.2), our task is more difficult because first we have to construct (or somehow show that there exists) an  $F$ -free  $k$ -graph  $G$ , and second prove that the first player has a winning strategy on  $G$ . Moreover, we are trying to find  $G$  and the strategy such that the least possible number of vertices are used.

### 4.5.1 Colouring edges

We start with simpler theorem which we later generalise.

**Theorem 4.8.** *Assume three integers  $k, p, q$  such that  $k \geq 2$ ,  $q > p \geq 2$ . Then there exists a  $K_{q+1}^k$ -free  $k$ -graph  $G$  on  $\mathcal{O}(2^{\binom{p}{2}} p^3)$  vertices such that maker has a winning strategy in weak restricted Ramsey game on  $G$ , trying to colour a  $K_q^k$ -subgraph.*

*Proof.* We first construct a suitable  $k$ -graph  $G$ , the “playground” which does not contain  $K_{q+1}^k$  as a subgraph. Second, we construct a hypergraph  $\mathcal{G}$  such that playing the weak game on  $\mathcal{G}$  is equivalent with the original game. Then we apply Weak Win Criterion (Theorem 3.5) on  $\mathcal{G}$ , proving there exists a maker’s winning strategy, and we estimate the number  $n$  of vertices necessary.

Assume we already know the number of vertices  $n$  (we actually estimate  $n$  at the end of the proof) and without loss of generality assume  $n/p$  is integer. Let us define the  $k$ -graph  $G = (V, E)$  on  $n$  vertices to be the complete  $p$ -partite  $k$ -graph with  $p$  parts of equal size  $s = n/p$ . To be exact,  $V = \bigcup_{i=1}^k V_i$  where  $V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,s}\}$ . There are all  $k$ -edges except those having two vertices from the same part, that is,  $E = \{S \in \binom{V}{k}; |S \cap V_i| \leq 1 \text{ for every } i\}$ . Clearly, there is no  $K_{q+1}^k$  subgraph in  $G$  by pigeonhole principle. Two vertices  $x, y$  of  $K_{q+1}^k$  would have to be in one part  $V_i$ , thus allowing no edge containing both  $x$  and  $y$ .

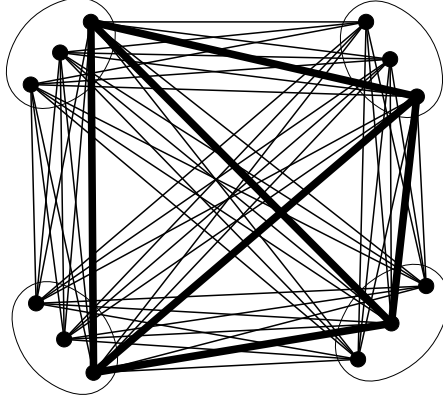


FIG. 4.9. Example of complete 4-partite 2-graph and its  $K_4^2$  subgraph.

Let us define the hypergraph  $\mathcal{G} = (E, F)$ . Its vertex set consists of edges from  $G$ . Each edge in  $F$  corresponds to one  $K_p^k$ -subgraph in  $G$  and contains all the appropriate edges, that is,  $F = \{S \in \binom{E}{\binom{p}{2}}; (\bigcup S, S) \simeq K_p^k\}$ . Observe, that vertex-colouring game on  $\mathcal{G}$  is equivalent with the original edge-colouring game on  $G$ .

The hypergraph  $\mathcal{G}$  is  $\binom{p}{2}$ -uniform,  $|E| = \binom{p}{2}s^2$  and  $|F| = s^p$ , as the number of edges is the same as the number of  $K_p^k$ -subgraphs in  $G$ . In order to use Weak Win Criterion, we need to establish an upper bound on  $\Delta_2(\mathcal{G})$ . Let us consider two distinct points  $u, v \in E$ , i.e. two edges of  $G$ . We count the highest number of ways to extend the edges  $u, v$  into  $K_p^k$  subgraph. There are two cases:  $u \cap v = \emptyset$  and  $u \cap v \neq \emptyset$ . The case  $u \cap v \neq \emptyset$  yields higher number than the second. Three parts of  $G$  are occupied, we have to choose one point from each remaining part, therefore  $\Delta_2(\mathcal{G}) \leq s^{p-3}$ .

By solving the inequality

$$\left(\frac{n}{p}\right)^p > 2^{\binom{p}{2}-3} \binom{p}{2} \left(\frac{n}{p}\right)^2 \left(\frac{n}{p}\right)^{p-3},$$

we get  $n$  large enough to fulfil the assumptions of Weak Win Criterion. After simple calculation,

$$n > 2^{\binom{p}{2}-3} p \binom{p}{2},$$

therefore there exists a  $k$ -graph  $G$  on  $n$  vertices ( $n$  satisfying the previous condition) where maker has a winning strategy.  $\square$

We further generalise the method used in Theorem 4.8 for more combinations of winning and forbidden subgraphs.

**Theorem 4.9.** *Let  $k \geq 2$  be an integer and let  $W$  (winning) and  $F$  (forbidden) be two  $k$ -graphs such that there does not exist a homomorphism  $h: F \rightarrow W$ . Let  $p = |V(W)|$  and  $q = |E(W)|$ . Consider the weak restricted Ramsey game with winning subgraph  $W$  and forbidden subgraph  $F$ . Then there exists a  $k$ -graph  $G$  on  $\mathcal{O}(2^q pq)$  vertices where maker has a winning strategy.*

*Proof.* The proof is a generalisation of method used in Theorem 4.8. This means, we first construct a suitable  $F$ -free game  $k$ -graph  $G$ , second construct a hypergraph  $\mathcal{G}$  such that playing the weak vertex-colouring game on  $\mathcal{G}$  is equivalent with the original game on  $G$ , and using Weak Win Criterion, we prove that maker has a winning strategy on  $\mathcal{G}$ , thus also on  $G$ .

Assume we know  $n$  already (we compute it at the end) and without loss of generality, assume  $n/p$  is integer. For  $W = (V_W, E_W)$ ,  $V_W = \{w_1, \dots, w_p\}$ , let us define the game  $k$ -graph  $G = (V_G, E_G)$  on  $n$  vertices as “inflated”  $W$  in the following way. For  $s = \lfloor n/p \rfloor$  the vertex set  $V_G = V_1 \cup \dots \cup V_p$ , where  $V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,s}\}$ . Edges in  $E_G$  resemble the original edges of  $W$ , that is,

$$E_G = \left\{ \{v_{a_1, b_1}, \dots, v_{a_k, b_k}\} \in \binom{V_G}{k}; \{a_1, \dots, a_k\} \in E_W \right\}.$$

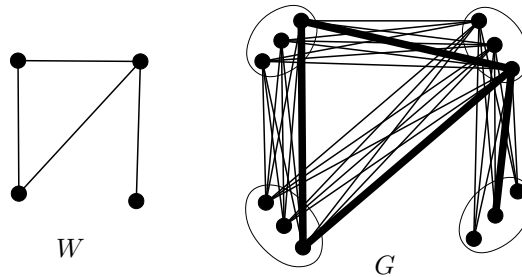


FIG. 4.10. Illustration of inflating the 2-graph  $W$  into  $G$ .

Assume there is a subgraph  $S \subseteq G$ ,  $S \simeq F$ . Let  $m: V(S) \rightarrow \{w_1, \dots, w_p\}$  be a mapping that maps each vertex  $v \in V(S)$  on the vertex  $w_i$  such that  $v \in V_i$ . But the mapping  $m$  is exactly a homomorphism  $(S \simeq F) \rightarrow W$ , which is contradiction. Therefore,  $G$  is  $F$ -free.

Let us define a hypergraph  $\mathcal{G} = (V_G, E_G)$  on the edge set of  $G$ , that is,  $V_G = E_G$ . One edge of  $\mathcal{G}$  contains all edges from  $E_G$  that form a  $W$ -subgraph in  $G$ , i.e.

$$E_G = \left\{ S \in \binom{V_G}{|E_W|}; (\bigcup S, S) \simeq W \right\}.$$

The hypergraph  $\mathcal{G}$  is  $q$ -uniform, on  $|E_G| = qs^k$  vertices and with  $s^p$  edges; to see this, recall the definition of  $E_G$  and observe there are  $s^p$   $W$ -subgraphs in  $G$ . To estimate  $\Delta_2(\mathcal{G})$ , fix two distinct elements  $S, T \in V_G$ , i.e. two edges of  $E_G$ . The edges  $S$  and  $T$  cover at least  $k+1$  points of  $W$ , therefore there are at most  $s^{|V_G|-k-1}$  ways to extend  $S$  and  $T$  into a  $W$ -subgraph. Provided the size  $n$  of  $G$  satisfies

$$\binom{n}{p} > 2^{q-3} q \binom{n}{p}^k \binom{n}{p}^{p-k-1},$$

by Weak Win Criterion (Theorem 3.5) there exists a maker's winning strategy. By simple calculation, taking

$$n > 2^{q-3}qp$$

is sufficient, which finishes the proof.  $\square$

Note that Theorem 4.9 implies Theorem 4.8.

Our understanding of strong restricted Ramsey games is rather limited and it seems to be hopelessly hard to prove results similar to Theorem 4.8 and 4.9. However, we study a special case of strong restricted Ramsey game.

**Example 4.10.** We give an example of small  $K_4$ -free graph, where two players alternately colour the edges, trying to colour their own  $K_3$  subgraph first (i.e. the strong restricted Ramsey edge-colouring game). We show winning strategy of the first player.

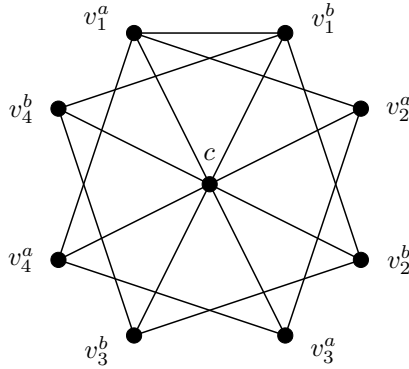


FIG. 4.11. The  $K_4$ -free playground.

The graph on Figure 4.11 does not contain  $K_4$  (easy observation) and there exists an explicit winning strategy of the first player in the strong game. As a first move, the first player takes the edge  $\{c, v_1^a\}$ . Then the second player responds. Let us distinguish two cases:

- (1) Second player's move was one of  $\{c, v_i^a\}$  or  $\{v_i^a, v_j^a\}$ . Then first player in the following 4 moves takes the edges  $\{c, v_1^b\}$ ,  $\{c, v_2^b\}$ ,  $\{c, v_3^b\}$ ,  $\{c, v_4^b\}$ , respectively. The second player is forced to take the edges  $\{v_1^a, v_1^b\}$ ,  $\{v_1^b, v_2^b\}$ ,  $\{v_2^b, v_3^b\}$ ,  $\{v_3^b, v_4^b\}$ , respectively, otherwise first player takes them and wins. After the fourth move, the edge  $\{v_4^b, v_1^b\}$  is left unoccupied, thus allowing first player to win.
- (2) Second player's move was one of  $\{c, v_i^b\}$  or  $\{v_i^b, v_j^b\}$  or  $\{v_1^a, v_1^b\}$ . Then, first player in the following 3 moves takes the edges  $\{c, v_2^a\}$ ,  $\{c, v_3^a\}$ ,  $\{c, v_4^a\}$ , respectively. The second player is forced to take the edges  $\{v_1^a, v_2^a\}$ ,  $\{v_2^a, v_3^a\}$ ,  $\{v_3^a, v_4^a\}$ , respectively, otherwise first player takes them and wins. After the third move, the edge  $\{v_4^a, v_1^a\}$  is left unoccupied, thus allowing first player to win.



## 4.5.2 Colouring vertices

We can easily adapt the construction from the proof of Theorem 4.9 for the vertex-colouring version of the restricted Ramsey game.

**Theorem 4.11.** *Let  $k \geq 2$  be an integer and  $W$  (winning) and  $F$  (forbidden) be two  $k$ -graphs such that there does not exist a homomorphism  $h: F \rightarrow W$ . Let  $p = |V(W)|$  and consider the strong version of the restricted Ramsey vertex-colouring game. Then there exists an  $F$ -free  $k$ -graph  $G$  on  $2p - 1$  vertices, where the first player is able to colour a  $W$ -subgraph first. Moreover, the number  $2p - 1$  is exact, i.e. there does not exist an  $F$ -free graph  $G$  with maker's winning strategy on less than  $2p - 1$  vertices.*

*Proof.* For  $W = (\{w_1, \dots, w_p\}, E_W)$ , let us define the game  $k$ -graph  $G = (V_1 \cup \dots \cup V_p, E_G)$  as the “inflated  $W$ ”, where  $V_1 = \{w_1\}$  and  $V_i = \{w_i, w'_i\}$  for  $i \geq 2$ . Let the edge set be

$$E_G = \left\{ \{v_{a_1, b_1}, \dots, v_{a_k, b_k}\} \in \binom{V_G}{k}; \{a_1, \dots, a_k\} \in E_W \right\}.$$

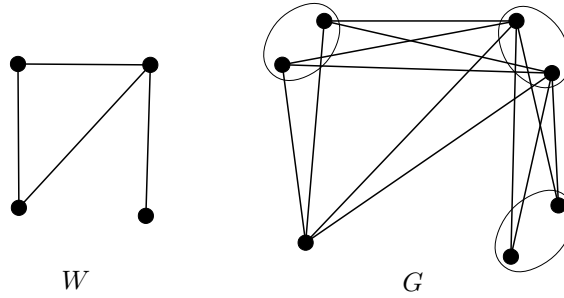


FIG. 4.12. Illustration of the 2-graph  $W$  and the corresponding  $G$ .

Assume there is a subgraph  $S \subseteq G$ ,  $S \simeq F$ . Let  $m: V(S) \rightarrow \{w_1, \dots, w_p\}$  be a mapping that maps each vertex  $v \in V(S)$  on the vertex  $w_i$  such that  $v \in V_i$ . But the mapping  $m$  is exactly a homomorphism  $(S \simeq F) \rightarrow W$ , which is a contradiction. Therefore,  $G$  is  $F$ -free.

The strategy of the first player is following. In the first move occupy  $V_1$ . When opponent takes one point from  $V_i$ , take the remaining point from  $V_i$ . Observe that after  $p$  moves the first player wins, leaving a monochromatic  $W$  subgraph.

Assume there exists a graph  $G$  on less than  $2p - 1$  vertices, where the first player is able to win. Let the play finishes. Then at most  $p - 1$  vertices can be coloured by the first player, which is not enough to find a  $W$ -subgraph.  $\square$

Our previous results, both for vertex-colouring and edge-colouring, did not work for the case when the forbidden  $k$ -graph  $F$  is a cycle, particularly an even cycle. Then the inflation technique is simply not enough. We need some tool for constructing  $C_s$ -free  $k$ -graphs.

Our basic tool is the following lemma, which shows that there exist “dense” hypergraphs without short cycles. The proof is a clever application of probabilistic method, invented by Erdős. See [ES] where the original proof can be found.

**Lemma 4.12.** *For all positive integers  $k$  and  $s$  there exists a  $k$ -graph  $G = (V, E)$ ,  $|V| = n$  without cycles of length  $< s$  and with  $|E| > n^{1+1/s}$  edges for all  $n$  sufficiently large.*

*Proof.* We first sketch the idea of the proof. The idea is to take one  $k$ -graph with approximately  $2n^{1+1/s}$  edges and delete all edges  $E_C$ , which belong to cycles of length  $< s$ . Therefore, if the number of edges  $E_C$  is smaller than  $n^{1+1/s}$ , we have the desired  $k$ -graph  $G$ . The only problem is to find such a  $k$ -graph  $G$ . To do this, we consider the set of all  $k$ -graphs with  $2n^{1+1/s}$  edges. We count the *average number*  $A$  of edges belonging to cycles of length  $< s$ , over all such hypergraphs. However, if  $A < n^{1+1/s}$ , there must exist a  $k$ -graph with at least  $n^{1+1/s}$  edges without short cycles! Let us execute the proof formally.

Let us consider a set  $\mathcal{M}_V$  of all  $k$ -graphs  $(V, E)$ ,  $|V| = n$  with  $m = 2\lceil n^{1+1/s} \rceil$  edges. Then

$$|\mathcal{M}_V| = \binom{\binom{n}{k}}{m}.$$

Consider a cycle  $C_j$  of length  $j$ . There are

$$\binom{n}{j(k-1)}$$

possibilities to place vertices of  $C_j$  on the set  $V$  and there are

$$\binom{\binom{n}{k} - j}{m - j}$$

possibilities how to place the remaining edges on  $V$ , such that the resulting hypergraph belongs to  $\mathcal{M}_V$ . Recall the well-known facts that  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  and that  $\binom{n}{k} = \Theta(n^k) = \frac{n!}{(n-k)!}$  for  $k \ll n$ . On a cycle of length  $j$ , there can be at most

$$\binom{j}{2} + \binom{j}{3} + \cdots + \binom{j}{k} \leq kj^k$$

edges adjacent to at least two vertices. Therefore, the average number of edges contained in cycles of length  $j$  is less than

$$kj^k \binom{n}{j(k-1)} \frac{\binom{\binom{n}{k} - j}{m - j}}{\binom{\binom{n}{k}}{m}}.$$

The average number of edges contained in cycles of length  $< s$  is thus less than

$$\begin{aligned}
\sum_{j=2}^{s-1} k j^k \binom{n}{j(k-1)} \frac{\binom{\binom{n}{k} - j}{m-j}}{\binom{\binom{n}{k}}{m}} &\leq \sum_{j=2}^{s-1} \mathcal{O} \left( n^{j(k-1)} \frac{(\binom{n}{k} - j)! m!}{(\binom{n}{k})! (m-j)!} \right) \\
&\leq \sum_{j=2}^{s-1} \mathcal{O} \left( n^{j(k-1)} \frac{m^j}{\binom{n}{k}^j} \right) \leq \sum_{j=2}^{s-1} \mathcal{O} \left( n^{j(k-1)} \frac{n^{j+j/s}}{n^{kj}} \right) \\
&\leq \sum_{j=2}^{s-1} \mathcal{O}(n^{j/s}) \leq \mathcal{O}(n^{(s-1)/s}).
\end{aligned}$$

Here the numbers  $j$ ,  $k$  and  $s$  are constants, therefore all top-level expressions depending only on  $j, k, s$  are “hidden” by the  $\mathcal{O}$  notation.

Consequently, for all  $n$  sufficiently large (formally this means there is some constant  $n_0$  such that for  $n > n_0$  the following holds), there exists an example of a  $k$ -graph  $G = (V, E)$ ,  $|V| = n$ ,  $|E| = 2 \lceil n^{1+1/s} \rceil$  such that  $G$  contains at most  $\lceil n^{1+1/s} \rceil$  edges contained in cycles of length  $< s$ . After deleting these edges we are left a  $k$ -graph with at least  $\lceil n^{1+1/s} \rceil$  edges without cycles of length  $< s$ .  $\square$

Note that the proof of Lemma 4.12 is not constructive, i.e. it gives the desired  $k$ -graph  $G$  by purely existential argument. The following simple lemma will be necessary.

**Lemma 4.13.** *Let  $\mathcal{F} = (V, F)$  be an arbitrary finite hypergraph. Assume there exists a hypergraph  $\mathcal{F}' = (V, F')$ ,  $F' \subseteq F$  such that the weak game on  $\mathcal{F}'$  is win for the first player. Then the first player has a winning strategy also in the weak game on  $\mathcal{F}$ .*

*Proof.* Consider the weak game on the hypergraph  $\mathcal{F}'$  and the appropriate winning strategy  $S$  of the first player. Then apply  $S$  in the weak game on  $\mathcal{F}$ . Clearly, if we restrict the winning lines on  $F'$  and the first player still wins, the second player is unable to block him on the set  $F$ .  $\square$

Note that Lemma 4.13 does not hold for the class of strong games.

Now we are ready to apply the knowledge of existence of dense hypergraphs without short cycles in restricted Ramsey games.

**Theorem 4.14.** *Let  $k \geq 2$  be an integer and  $W$  (winning) and  $F$  (forbidden) two  $k$ -graphs. Let  $p = |V(W)|$  and let there be an integer  $\ell$  such that  $C_\ell \subseteq F$  and  $C_\ell \not\subseteq W$ . Then there exists an  $F$ -free  $k$ -graph  $G$  on  $\mathcal{O}(2^{p^\ell})$  vertices such that the first player has a winning strategy in the weak restricted vertex-colouring Ramsey game on  $G$  with winning subgraph  $W$  and forbidden subgraph  $F$ .*

*Proof.* Let us first sketch the idea of the proof. Assume we know  $n = |V|$  already, we compute it at the end. First, we use Lemma 4.12 to get a dense  $C_\ell$ -free  $p$ -graph  $G = (V, E)$  on  $n$  vertices. We need the density property for Weak Win Criterion to work. Second,

we “stuff” each hyperedge of  $G$  by  $W$ , that is, we draw a copy of  $W$  on each  $A \in E$ . This means there are now many  $W$ -subgraphs in  $G$ . Third, using Weak Win Criterion we prove the first player wins on such a stuffed graph and we compute the number  $n$ . Let us do the proof precisely.

For the number of vertices  $n$  (we will establish the actual number later) there exists by Lemma 4.12 a  $p$ -graph  $G = (V, E)$  such that  $G$  does not contain a cycle of length  $< \ell + 1$  and  $|E| > n^{1+1/(\ell+1)}$ . Let us define the  $k$ -graph  $G' = (V, E')$  by taking  $G$  and arbitrarily replacing each edge by a copy of  $W$ . That is,  $G' = \bigcup_{S \in E} (S, E_S)$  where  $(S, E_S) \simeq W$  for every  $S$ . The actual game takes place on  $G'$ .

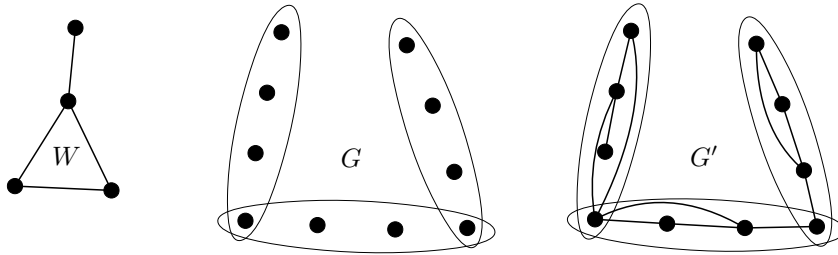


FIG. 4.13. “Stuffing” the  $p$ -graph  $G$  by copies of  $W$  for  $k = 2$ .

Let us show that  $G'$  is  $F$ -free. Assume there is an  $F$ -subgraph in  $G'$ . Two cases can occur:

- (1) The vertices of  $F$  are entirely contained in a single hyperedge of  $G$ . The assumption  $C_\ell \subseteq F$  and  $C_\ell \not\subseteq W$  implies that  $F \not\subseteq W$  ( $W$  is missing the cycle  $C_\ell$ ). Therefore, this case cannot occur.
- (2) The vertices of  $F$  are incident with more than one hyperedge of  $G$ . Restrict ourselves only on such edges  $\{E_1, \dots, E_t\} \subseteq E$  that are incident with the cycle  $C_\ell \subseteq F$ , i.e.  $|E_i \cap C_\ell| \geq 1$  for  $1 \leq i \leq t \leq \ell$ . If  $C_\ell \subseteq G'$  there must exist a sequence  $S = (E_{s_1}, E_{s_2}, \dots, E_{s_\ell})$  consisting of edges from  $\{E_1, \dots, E_t\}$  such that the vertices of  $C_\ell$  lay on edges from  $S$ , respectively. But the hypergraph  $(V(C_\ell), \{E_1, \dots, E_t\})$  is precisely a cycle of length  $\ell$  in the hypergraph  $G$ . Therefore, we get a contradiction with the fact that  $G$  does not contain a cycle of length  $< \ell + 1$ .

Let us construct a hypergraph  $\mathcal{G} = (V_{\mathcal{G}}, E_{\mathcal{G}})$  such that playing weak game on  $\mathcal{G}$  is equivalent with the original game on  $G'$ . That is,  $V_{\mathcal{G}} = V$  and

$$E_{\mathcal{G}} = \left\{ S \in \binom{V_{\mathcal{G}}}{|E(W)|}; \exists T \subseteq E(G') : (S, T) \simeq W \right\},$$

i.e. each edge in  $E_{\mathcal{G}}$  corresponds to a set of vertices on which there is a  $W$ -subgraph in  $G'$ . Observe that  $G \subseteq \mathcal{G}$ ; by the “stuffing” procedure, there are at least the edges of  $G$  in  $\mathcal{G}$  and maybe some more. Due to Lemma 4.13, we can restrict ourselves only to the weak game on  $G$ ; if we show the first player wins on  $G$  then he wins on  $\mathcal{G}$  and therefore also on  $G'$ .

The  $k$ -graph  $\mathcal{G}$  contains  $n$  vertices, at least  $n^{1+1/(c+1)}$  edges, and is almost-disjoint since it does not contain a 2-cycle. Provided the size  $n$  of  $G'$  satisfies

$$n^{1+\frac{1}{c+1}} > 2^{p-3} \cdot n,$$

by Weak Win Criterion (Theorem 3.5) there exists a winning strategy of maker. By simple calculation, taking  $n = \Omega(2^{cp})$  is sufficient, which finishes the proof.  $\square$

Let us note that the proof of Theorem 4.14 is existential because the main tool for finding the game hypergraph, Lemma 4.12, works by existential arguments.

### 4.5.3 Summary

Let us sum the results shown in Section 4.5.

**Corollary 4.15.** *Let  $k \geq 2$  be an integer. We have proved that*

- (1)  $\mathcal{R}_2^*(K_3, K_4) \leq 9$ ,
- (2)  $\mathcal{R}_k^*(K_p^k, K_{p+1}^k) = \mathcal{O}(2^{\binom{p}{2}} p^3)$ ,
- (3)  $\mathcal{R}_k^*(W, F) = \mathcal{O}(2^{|E(W)|} \cdot |V(W)| \cdot |E(W)|)$  for two  $k$ -graphs  $W$  and  $F$  satisfying the condition that there does not exist a homomorphism  $h: W \rightarrow F$ ,
- (4)  $\tilde{\mathcal{R}}_k^*(W, F) = 2|V(W)| - 1$  for two  $k$ -graphs  $W$  and  $F$  satisfying the condition that there does not exist a homomorphism  $h: W \rightarrow F$ ,
- (5)  $\tilde{\mathcal{R}}_k^*(W, F) = \mathcal{O}(2^{c|V(W)|})$  for the  $k$ -graphs  $W$  and  $F$  satisfying the condition that there is a number  $c$  such that  $C_c \subseteq F$  and  $C_c \subseteq W$ .

*Proof.* The results clearly follow from Example 4.10, Theorem 4.8, Theorem 4.9, Theorem 4.11 and Theorem 4.14, respectively.  $\square$

## 4.6 Loebel game

Recall Conjecture 2.13 presented in Section 2.7. As we have already done for some kinds of Ramsey objects, let us consider a weak game with the following rules. Given an arbitrary tree  $T$  on  $n$  vertices, maker and breaker alternately pick edges of a complete graph  $K_N$ . Maker is trying to build a copy of  $T$  entirely by his colour, the goal of breaker is to prevent this. We call it *Loebel game*. The question is, what is the least number  $N$  of vertices of the  $K_N$  such that maker has a winning strategy for arbitrary tree  $T$  on  $n$  vertices? Let us denote by  $\mathcal{L}^*(n)$  the least number of vertices  $N$  such that the first player wins in the weak Loebel game.

**Theorem 4.16.** *Consider a strong Loebel game for trees on  $n$  vertices. Then the first player is able to win on a complete graph  $K_N$  for  $N = 2n - 2$ .*

*Proof.* Let us fix a tree  $T$  and assume the play holds on the complete graph  $K_{2n-2}$ . We describe the simple strategy of the first player: he keeps building a connected subtree of  $T$ , and after  $n - 1$  moves he is left with the complete tree  $T$ .

We define the ordering of edges of  $T$  in which the first player should build them. Let us root the tree  $T$  at an arbitrary vertex. The edges connecting vertices in level  $i$  with level  $i + 1$  are prior to edges connecting levels  $j$  and  $j + 1$  for  $j > i$ . The ordering of edges inside one particular level is arbitrary. Maker keeps picking previously unselected edges such that the tree  $T$  “grows from the root”, i.e. in every step first player’s edges form a connected tree  $T' \subseteq T$ .

Finally, observe that  $2n - 2$  vertices are enough to complete the procedure. Just before the  $i$ -th step, for each vertex at least  $2(n - i) - 1$  edges are unoccupied. Thus for all  $i = 1, \dots, n - 1$  steps, there is at least one edge free for the purposes of the first player. Moreover, the second player does not have time to complete his own tree  $T$  because the game ends exactly after  $n - 1$  moves by the first player’s victory.  $\square$

**Theorem 4.17.** *Consider the weak Loeb game for trees on  $n$  vertices. Then breaker has a winning strategy on boards  $K_N$  for  $N \leq 2n - c\sqrt{n \log n}$  where  $c$  is an absolute positive constant.*

*Proof.* To prove this result, we need to find a suitable tree  $T$  on  $n$  vertices together with breaker’s strategy able to prevent maker from building  $T$  on  $K_N$ . We show that star on  $n$  vertices is the right tree  $T$ . Assume maker has to create such a star  $T$ . But this is precisely the “degree game” as presented in Section 3.4.

By Theorem 3.8, we know breaker is able to win if the goal of maker is to create star of size  $\frac{N}{2} + c\sqrt{N \log N}$  for an absolute positive constant  $c$  (assuming we play on  $K_N$ ). Therefore, it remains to solve the inequality

$$n > \frac{N}{2} + c\sqrt{N \log N},$$

which is satisfied for  $N \leq 2n - 4c\sqrt{n \log n}$ . The theorem follows.  $\square$

**Corollary 4.18.** *There is a positive absolute constant  $c$  such that*

$$2n - c\sqrt{n \log n} \leq \mathcal{L}^*(n) \leq 2n - 2$$

*for every positive integer  $n$ .*

*Proof.* Follows immediately from Theorem 4.17 and Theorem 4.16.  $\square$

## 4.7 Comparison of Ramsey and game numbers

The main goal of our work is to compare the two numbers of each object: its *Ramsey number*  $\mathcal{N}$  and its *game number*  $\mathcal{N}^*$ . This means, the minimum size such that given arbitrary colouring there exists a monochromatic sub-object and the minimum size such that the first player has a winning strategy.

However, establishing the number  $\mathcal{N}^*$  for the case of strong games is immensely complicated. Usually, by Theorem 4.1, our best bound is only  $\mathcal{N}^* \leq \mathcal{N}$ . Therefore, we study the class of weak games, they are easier to analyse.

Often, the gap between  $\mathcal{N}$  and  $\mathcal{N}^*$  turns out to be enormous. In many Ramsey-type theorems, even in the case of two colours, the object size which guarantees their validity is not even primitive recursive (see Section 2.8 for details) and the game number is very small.

We continue by the list of Ramsey-type theorems, both with the appropriate Ramsey number upper bound a game number upper bound.

**Corollary 4.19.** (Arithmetic progression numbers) *Let  $k$  be a positive integer. Then*

$$\mathcal{A}^*(k) = \mathcal{O}(2^k k^3) \quad \text{and} \quad \mathcal{A}(k) \leq 2^{2^{2^{2^{k+9}}}}.$$

*Proof.* Clearly follows from Corollary 4.5 and Theorem 2.15. □

Similar gap arises in the case of arithmetic progression with difference. However, the game number remains asymptotically the same as in arithmetic progression game only.

**Corollary 4.20.** (Arithmetic progression with difference numbers) *Let  $k$  be a positive integer. Then*

$$\mathcal{A}_d^*(k) = \mathcal{O}(2^k k^3) \quad \text{and} \quad \mathcal{A}_d(k) \leq 2^{2^{2^{2^{2^{k^2+k+9}}}}}.$$

*Proof.* See Corollary 4.5 and (2.12) in Section 2.8. □

Next, we compare the bounds on Hales-Jewett numbers. Let  $\mathcal{H}^*(n)$  be the smallest dimension  $d$  such that the first player wins in the weak  $n^d$  game (see Section 4.3).

**Corollary 4.21.** (Tic-Tac-Toe vs. Hales-Jewett) *Let  $n$  be a positive integer. Then*

$$\mathcal{H}^*(n) = \mathcal{O}(n^2) \quad \text{and} \quad \mathcal{H}(n, 2) \leq f_5(c \cdot n),$$

where  $c$  is an absolute positive constant and  $f_5$  is the function from the Ackermann hierarchy.

*Proof.* Follows directly from Theorem 4.6 and Corollary 2.14. □

Let us turn our attention to the original Ramsey theorem (see Section 2.2) and to Ramsey games (see Section 4.4). By  $\mathcal{R}_k^*(n)$  we denote the smallest  $N$  such that first player can build a  $K_n^k$  subgraph of  $K_N^k$  in the weak Ramsey game.

**Corollary 4.22.** (Ramsey theorem vs. Ramsey game) *Let  $k$  and  $n$  be two integers. Then*

$$\mathcal{R}_k^*(n) \leq 2^{\mathcal{O}(n^k)}, \mathcal{R}_2^*(n) \leq 2^{n+2} \quad \text{and} \quad \mathcal{R}_k(n, 2) \leq 2^{2^{\dots^{2^{\mathcal{O}(n)}}}},$$

where there are  $k - 1$  2's on the stack.

*Proof.* See Theorem 4.7 and (2.10), (2.11) in Section 2.8. □

Another topic we have studied are the restricted Ramsey theorems (see Section 2.6) and restricted Ramsey games (see Section 4.5). See Corollary 4.15 for the summary of what we have been able to prove in the case of restricted Ramsey games. Usually, the sufficient size of the game hypergraph is of the form of 2 to the size of the winning subgraph. On the other hand, the upper bound on the restricted Ramsey theorem is not known to be even primitive recursive.

## 4.8 Future work

First thing we believe to be possible to improve is the Loebbl game. Theorem 4.16 gives the upper bound  $2n - 2$ , which works both for the strong and weak game. However, if we consider only the weak game, Theorem 3.8 shows that for the class of stars we can do much better than  $2n - 2$ . This leads to the belief that for more tree classes or even arbitrary trees we can achieve better upper bound.

Another goal is to prove Theorem 4.14 for the edge-colouring game. Moreover, we think it could be possible to simplify and generalise assumptions in Theorem 4.9 and Theorem 4.11. In the ideal case, to the generality level of Theorem 2.11.

There seems to be endless opportunities to work on, as there are many Ramsey-type theorems and almost all can be taken as boards for a positional games and studied.

Finally, we present an open problem, which is unrelated to our other work. It is inspired by the following famous theorem of Erdős and Szekeres.

**Theorem 4.23.** (Erdős, Szekeres) *For every integer  $k$  there exists an integer  $N$  such that any set of  $N$  points in plane in general position (no three points lie on a line) contains  $k$  points forming convex  $k$ -polygon.*

For the proof see e.g. Valla and Matoušek [VM].

Let us now consider the game version of Theorem 4.23. Two players alternately draw points into plane such that no three lie on a line. If after some player's move there is a set forming convex  $k$ -polygon, this player wins. By Theorem 4.23 we know that after finite number of moves someone must win. But who has winning strategy in this game and how long does it take? Using case study one can show that for  $k = 3$  it is win for the first player, for  $k = 4$  and  $k = 5$  the second players wins. And this is all we know. We consider this problem especially challenging.



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