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Professor Zdeněk Němeček
Faculty of Mathematics and Physics
Charles University
Prague, Czech Republic

Dear Professor Němeček,

Please find below my report on the doctoral thesis of Jan Štola entitled

"Representations and Visualizations of Graphs".

The focus of this thesis is on 3-dimensional visibility representations of complete graphs. The main contributions are as follows:

(1) The author proves that the complete graph on 51 vertices does not have a 3-dimensional visibility representation by axis-aligned rectangles. This improves upon the previous best bound of 56. At first glance an improvement from 56 to 51 is modest. However, the nature of the proof is significant.

(2) The proof of (1) is based on a detailed analysis of multi-dimensional unimodal sequences that is of independent interest, and potentially of application to other areas of Ramsey theory. The method employed by the author is highly original and of general applicability. For example, the famous Erdős-Szekeres theorem is a corollary of one of his results. Theorem 4 of the thesis is a real highlight.

(3) The author proves that the maximum order of a complete graph that has a 3-dimensional visibility representation by congruent regular k-gons is $O(k^4)$. This is a huge improvement over the previous best exponential upper bound, and indicates the author's thorough and deep understanding of the field. The author also establishes the best known lower bounds.

The above three results certainly prove the author's ability for creative scientific work.

For example, result (2) with result (1) as an interesting application has the potential for publication in a leading combinatorics journal such as the *Journal of Combinatorial Theory, Series A or B*. Result (3) has the potential for publication in a leading geometry journal such as *Discrete and Computational Geometry*.

What follows are a number of suggestions for improving the thesis. Also

Department of Mathematics and Statistics
The University of Melbourne Victoria 3010 Australia
T: +61 3 8344 5550 F: +61 3 8344 4599 W: www.ms.unimelb.edu.au

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be aware that I have emailed a scanned copy of the thesis containing my notes about it. These notes contain many more suggestions for improvement.

- In places the thesis is overly modest about its contribution. For example, the work in Section 2 describes a general framework that includes the Erdős-Szekeres theorem. This should be highlighted more prominently. Perhaps start Section 2 with the Erdős-Szekeres theorem, and state that this section describes a general framework for proving Ramsey-theoretic results about subsequences that includes the Erdős-Szekeres theorem, as well as the main original results about multi-dimensional unimodal sequences.
- Throughout the thesis, the term "visibility drawing" is used, whereas "visibility representation" is more commonly used in the literature.
- The title of the thesis is vague. A precise title that accurately describes the contents would be more appropriate. For example, "Three-dimensional visibility representations of complete graphs".
- The proof of Theorem 4 mentions a computer program that determines $u_5(n)$. The program code should be included as an appendix to the thesis. More importantly, can the program compute $u_6(n)$ or $u_7(n)$? Was this attempted?
- The conclusion reads: "Our upper bounds are based on Lemma 19(iii) only while all five conditions of this lemma must hold simultaneously. It remains an open problem how to combine these conditions to obtain a better bound." This question seems ripe for solution by computer search. Has this been attempted?
- Erdős and Szekeres proved that any sequence of length at least $(r-1)(s-1)+1$ contains either a monotonically increasing subsequence of length r, or a monotonically decreasing subsequence of length s. This is a little stronger than saying that every sequence of length at least $(r-1)^2+1$ contains a monotone subsequence of length r. Can similar strengthenings be made for multi-dimensional unimodal subsequences?
- The connection to Dedekind numbers should be described in more detail. Can any of the known results about (the asymptotics of) Dedekind numbers be used to estimate $u_k(n)$? Briefly describe the many places in which Dedekind numbers arise. For example, [from Wikipedia] the Dedekind number D_k counts the number of monotonic boolean functions of k variables, the number of antichains of subsets of a k-element set, the number of elements in a free distributive lattice with k generators, and the number of abstract simplicial complexes with k elements.

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- The proofs of Lemmas 20 and 21 should be written in full. In a journal paper, these proofs could be omitted, but not in a thesis.
- Somewhere in the thesis the proof by Fekete et al. that K_{56} has no 3-dimensional visibility representation by axis-aligned rectangles should be described. Highlight where the proof in this thesis improves upon the proof by Fekete et al.
- The thesis is lacking a few key definitions. In particular, "subsequence", "monotone", and "upper set" must be defined (especially given that elementary graph theory definitions are described in detail). I found the definitions in Section 3.2 very difficult to understand.
- Notation such as "$< \preceq_U >$" is highly non-intuitive. Are there alternatives? Avoid double subscripts. Replace b_{i_1} and b_{i_2} by b_i and b_j. These are much easier to read.
- The introduction would be greatly enhanced if it described other models of 3-dimensional graph drawing. For example, have a few sentences on 3-dimensional orthogonal graph drawing with bends, and on 3-dimensional straight-line graph drawing.
- Similarly, the thesis would greatly benefit if a paragraph or two describing results on visibility representations of planar graphs were added. Some relevant papers:
  - Chen, Chieh-Yu; Hung, Ya-Fei; Lu, Hsueh-I *Visibility representations of four-connected plane graphs with near optimal heights.* Comput. Geom. 42 (2009), no. 9, 865–872.
  - He, Xin; Zhang, Huaming *Nearly optimal visibility representations of plane graphs.* SIAM J. Discrete Math. 22 (2008), no. 4, 1364–1380.
  - Thomassen, Carsten *Rectangular and visibility representations of infinite planar graphs.* J. Graph Theory 52 (2006), no. 3, 257–265.
  - Zhang, Huaming; He, Xin *Improved visibility representation of plane graphs.* Comput. Geom. 30 (2005), no. 1, 29–39.
  - Dean, Alice M.; Evans, William; Gethner, Ellen; Laison, Joshua D.; Safari, Mohammad Ali; Trotter, William T. *Bar k-visibility graphs.* J. Graph Algorithms Appl. 11 (2007), no. 1, 45–59
- The bibliography is brief for a thesis concerning a well-studied topic. Also journal papers should be cited rather than preliminary conference papers. Papers to cite:
  - Helmut Alt, Michael Godau, Sue Whitesides. *Universal 3-dimensional visibility representations for graphs.* Comput. Geom.

9 (1998), no. 1-2, 111–125.

- J. Michael Steele. *Variations on the monotone subsequence problem of Erdős and Szekeres.* In Aldous, Diaconis, and Steele (eds.). Discrete Probability and Algorithms, pages 111-132, Springer, 1995. `http://www-stat.wharton.upenn.edu/~steele/Publications/`
- Richard P. Stanley. *Increasing and decreasing subsequences and their variants.* International Congress of Mathematicians. Vol. I, 545–579, Eur. Math. Soc., Zrich, 2007.
- J. Michael Steele. *Long unimodal subsequences: a problem of F. R. K. Chung.* Discrete Math. 33 (1981), no. 2, 223–225.

I recommend that this thesis be accepted.

Yours sincerely,

D Wood

Dr. David R. Wood
Senior Research Fellow

Department of Mathematics and Statistics
The University of Melbourne Victoria 3010 Australia
T: +61 3 8344 5550 F: +61 3 8344 4599 W: www.ms.unimelb