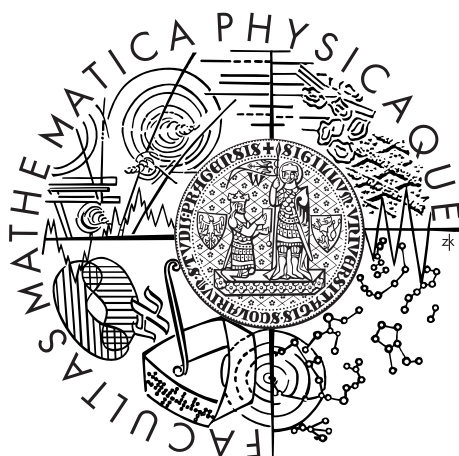


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DOCTORAL THESIS



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Representations and visualization of graphs

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I hereby declare that I have written this thesis on my own and using exclusively the cited sources. I authorize Charles University to lend this document to other institutions and individuals for academic or research purposes.

Jan Štola
Prague, April 12th, 2010

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Abstract

The 3D visibility (graph) drawing is a graph drawing in \mathbb{R}^3 where vertices are represented by 2D sets placed into planes parallel to xy -plane and the edges correspond to z -parallel visibility among these sets. We continue the study of 3D visibility drawing of complete graphs by rectangles and regular polygons.

We show that the maximum size of a complete graph with a 3D visibility drawing by regular n -gons is $O(n^4)$. This polynomial bound improves significantly the previous best known (exponential) bound $\binom{6n-3}{3n-1} - 3 \approx 2^{6n}$.

We also provide several lower bounds. We show that the complete graph K_{2k+3} (resp. K_{4k+6}) has a 3D visibility drawing by regular $2k$ -gons (resp. $(2k+1)$ -gons).

We improve the best known upper bound on the size of a complete graph with a 3D visibility drawing by rectangles from 55 to 50. This result is based on the exploration of unimodal sequences of k -tuples of numbers.

A sequence of numbers is unimodal if it first increases and then decreases. A sequence of k -tuples of numbers is unimodal if it is unimodal in each component. We derive tight bounds on the maximum length of a sequence of k -tuples without a unimodal subsequence of length n . We show a connection between these results and Dedekind numbers, i.e., the numbers of antichains of a power set $\mathcal{P}(\{1, \dots, k\})$ ordered by inclusion.

1 Introduction

1.1 Graph Theory

A (simple) graph G is an ordered pair (V, E) where V is a finite set and E is a set of 2-element subsets of V , i.e., $E \subset \binom{V}{2}$. Members of the set V are called *vertices* and members of the set E are called *edges*. The vertices resp. the edges of a graph G are also denoted by $V(G)$ resp. $E(G)$. The vertices belonging to an edge are called *ends*, *endpoints* or *end vertices* of the edge. An edge $\{u, v\}$ is usually denoted simply by uv . Two vertices are *adjacent* if an edge exists between them. The vertices adjacent to a vertex v are called *neighbors* of v . A *subgraph* H of a graph G is a graph such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$.

A *path* in a graph G is a sequence $(v_i)_{i=1}^k$ of vertices of G such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i < k$. A *cycle* is a path such that the first vertex and the last vertex are the same. A *simple path* is a path with no repeated vertices. A cycle with distinct vertices aside from the necessary repetition of the first and the last vertex is a *simple cycle*. The length of a path is the number of edges on the path, i.e., the length of the path $(v_i)_{i=1}^k$ is $k - 1$.

A *directed graph* G is an ordered pair (V, E) where V is a finite set and E is a set of ordered pairs of V , i.e., $E \subset V^2$. A directed graph is also called a *digraph*. A directed graph can be considered as a simple graph with an additional information (the direction of the edges) provided. Hence, a lot of terms defined for simple graphs apply to directed graphs as well.

A *directed path* is a path $(v_i)_{i=1}^k$ such that $(v_i, v_{i+1}) \in E$, i.e., all edges of the path have the same direction. Similarly, a *directed cycle* is a cycle with all edges having the same direction. A directed graph is *acyclic* if it doesn't contain any directed cycle.

A *tournament* is a directed graph in which each pair of vertices is connected by exactly one edge, i.e., for every vertices $u, v \in V$ there is either $(u, v) \in E$ or $(v, u) \in E$.

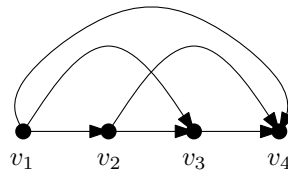


Figure 1. A tournament with 4 vertices

A *complete graph* is a graph in which every vertex is adjacent to every other. A complete graph on n vertices is denoted by K_n . A *bipartite graph* is

a graph whose vertices can be divided into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2 . In general, a graph is k -partite if its vertex set can be divided into k pairwise disjoint sets V_1, \dots, V_k such that every edge connects vertices from the different sets. A *complete k -partite graph* is a k -partite graph that has an edge between every pair of vertices from the different sets. A complete k -partite graph is denoted by K_{n_1, \dots, n_k} where $n_i = |V_i|, 1 \leq i \leq k$.

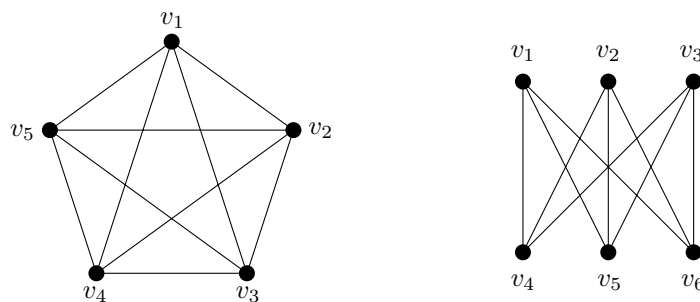


Figure 2. A complete graph K_5 and a complete bipartite graph $K_{3,3}$

A simple path (resp. a simple cycle) that includes every vertex of a graph is known as a *Hamiltonian path* (resp. a *Hamiltonian cycle*). A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*.

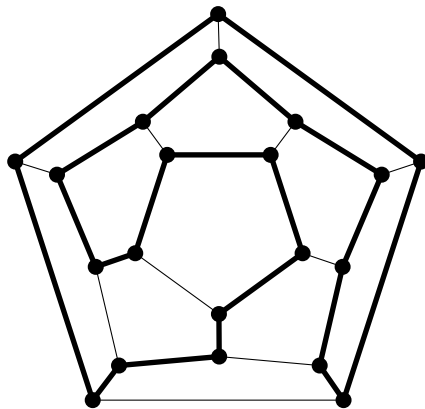


Figure 3. A Hamiltonian graph with a Hamiltonian cycle

If it is possible to establish a path from any vertex to any other vertex of a graph then the graph is said to be *connected*. Otherwise, the graph is *disconnected*. A (*connected*) *component* of a graph G is a maximal connected

subgraph of G . A *cut vertex* of a graph is a vertex whose removal from the graph increases the number of connected components.

A *vertex labeling* is a function from the vertex set to some fixed set. The values assigned to individual vertices are called (*vertex*) *labels*. A (*proper vertex*) *coloring* is a vertex labeling such that no two vertices sharing the same edge have the same label. The labels of a coloring are called *colors*. A coloring using at most k colors is a *k-coloring*. A graph that can be assigned a k -coloring is *k-colorable*. The smallest k such that a graph G is k -colorable is a *chromatic number* of G .

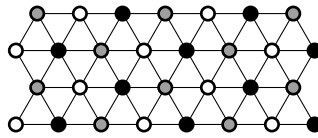


Figure 4. A 3-coloring of a graph

1.2 Graph Drawing

The aim of graph drawing is a creation of a geometric representation of a (combinatorial) graph – often for visualization purposes. Vertices of the graph are represented by geometric objects (points, line segments, rectangles, etc.) and edges are represented either as a different type of geometric objects or as a specific relationship between the objects representing vertices (intersection, visibility, etc.)

Graphs can be found in any area of our life. Vertices of a graph can represent domain entities and edges correspond to a relationship between the entities. Therefore, there is a huge amount of graph drawing applications. We can find them in software engineering (layouts of UML diagrams in CASE tools or ER diagrams in DB systems), electronic engineering (VLSI design, circuit board layouts), biology, chemistry (molecular drawings), cartography, etc.

Given a certain graph it is natural to look for the best drawing of this graph. Unfortunately, there is no best drawing of a graph. One can assess the quality of a drawing of a graph in many ways because the different ways of displaying a graph emphasize different characteristics of the graph.

We can attempt to achieve various aesthetic criteria – minimize the number of edge crossings, minimize the area of a bounding box (the smallest rectangle or box that surrounds the drawing), maximize the angular resolution (the size of the smallest angle between any pair of edges incident to

the same vertex), minimize the total length of edges, maximize the display of symmetries and many other. Usually, there are trade-offs among these criteria, see Figure 5.

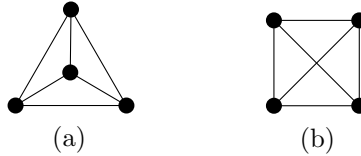


Figure 5. A drawing of K_4 (a) minimizing the number of crossings (b) maximizing the display of symmetries

The most common type of graph drawing is a drawing that represents vertices by points in a plane and edges by simple arcs (homeomorphic images of the interval $[0, 1]$) such that

- the endpoints of the arc corresponding to an edge e are the points associated with the end vertices of e and
- no arc includes points associated with other vertices.

If, in addition, two arcs never intersect at a point which is in an interior of either of the arcs then this drawing is called a *planar embedding* of the graph. A graph that admits such a drawing is called a *planar graph*.

Planar graphs were characterized by Kazimierz Kuratowski [19].

Theorem (Kuratowski's theorem). *A graph is planar if and only if it doesn't contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Let's remind that a *subdivision* of a graph is a graph resulting from subdivisions of edges in G . A subdivision of an edge $\{u, v\}$ is a replacement of this edge by a new vertex w and edges $\{u, w\}$ and $\{w, v\}$.

There are various other styles of graph drawing, see [3, 4] for an overview of graph drawing types and algorithms. We concentrate on visibility drawings in this thesis.

1.3 Visibility Drawing

The visibility drawing of a graph represents vertices by disjoint sets in \mathbb{R}^n and expresses edges as visibility relations among these sets. There are several types of visibility drawings that differ by the (set of) shapes used to represent

vertices (i.e., rectangles, boxes, polygons) and by the direction(s) in which we determine the visibility between vertices.

We say that sets A, B (resp. the corresponding vertices) can *see* each other if there exists $a \in A$ and $b \in B$ such that the line segment ab doesn't intersect other sets (besides A and B) associated with vertices. We say that the sets (resp. the vertices) can *see* each other *in the direction* of a vector \vec{w} (resp. a line h) if the line segment ab is parallel to \vec{w} (resp. h).

The visibility drawings have two subtypes: weak and strong. Two vertices can see each other in a *strong visibility drawing* if and only if there is an edge between these vertices. There is a one-to-one relation between the edges of the graph and the pairs of mutually visible vertices. A *weak visibility drawing* allows visibility between vertices that are not connected by an edge, i.e., if there is an edge between two vertices then these vertices must be able to see each other but they can see each other even when they are not connected by an edge.

The advantage of strong visibility drawings is that the represented graph is specified by the location of (the sets representing) the vertices only, i.e., we don't have to draw the edges. The theory of strong visibility drawings also seems to be broader. On the other hand, weak visibility drawings are probably more practical. For example, there is no problem if two chips on a circuit board can see each other and we don't connect them with a wire. This thesis is focused on drawing of complete graphs where there is no difference between weak and strong drawings. We consider the weak visibility drawing on the few spots where it could make a difference.

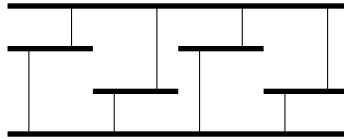


Figure 6. A bar-visibility drawing of $K_{2,4}$

An example of visibility drawing is a *bar-visibility drawing*. The bar-visibility drawing represents vertices by parallel line segments in \mathbb{R}^2 . Two line segments must see each other in the direction orthogonal to the line segments whenever the corresponding vertices are connected by an edge.

Another example of visibility drawing is a *2D rectangle visibility drawing*. This type of drawing represents vertices by axis-aligned rectangles in a plane. Two rectangles must see each other in the direction parallel to some axis (i.e., x -axis or y -axis) whenever the corresponding vertices are connected by an edge.

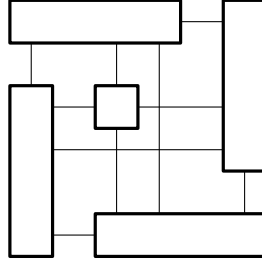


Figure 7. A 2D rectangle visibility drawing of K_5

Two-dimensional variants of visibility drawings received a wide attention due to their applications in CASE tools, circuit board layouts or VLSI design [11, 29].

The increasing popularity of visibility drawings led to an introduction of three-dimensional variants. In fact, a 3D analogy of the 2D rectangle visibility drawing is another well-known type of drawing: the 0-bend 3D orthogonal (box) drawing, i.e., a drawing where vertices are represented by axis-aligned boxes in \mathbb{R}^3 and the edges are axis-parallel lines of visibility among boxes.

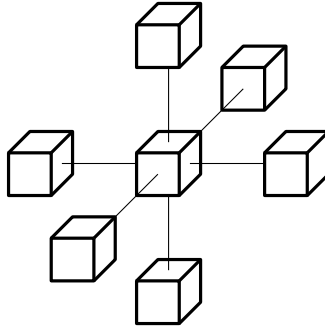


Figure 8. A 0-bend 3D orthogonal drawing of $K_{1,6}$

A 3D analogy of the bar-visibility drawing is the 3D visibility drawing. It represents vertices by two-dimensional sets placed into planes parallel to the xy -plane. Two sets must see each other in the direction of the z -axis whenever the corresponding vertices are connected by an edge. We concentrate on the 3D visibility drawing in this thesis.

There are several subtypes of the 3D visibility drawing. They differ by the allowed shapes of vertices. The most popular subtype is the 3D rectangle visibility drawing that allows only rectangular vertices. We study this

drawing in Section 3.1. Another popular subtype represents vertices by equal regular polygons. We explore this subtype in Section 4.

A natural question regarding any type of graph drawing is ‘What graphs can be represented by this type of drawing?’ Unfortunately, the recognition of visibility graphs turns out to be difficult. Shermer [21] shows that the recognition of graphs with a 2D rectangle visibility drawing is an NP-complete problem. Fekete et al. [15] show that the recognition of graphs with a 3D visibility drawing is NP-hard when the vertices are represented by unit squares and Štola [23] shows the same result for drawings by equal triangles.

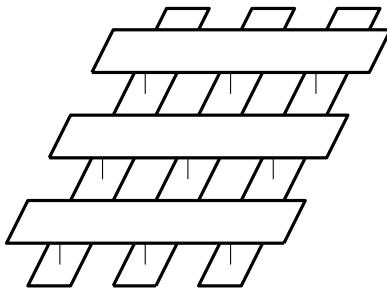


Figure 9. A 3D rectangle visibility drawing of $K_{3,3}$

If we cannot decide effectively whether a graph has a drawing of the given type then it is natural to look for classes of graphs for which this decision is possible. The research in this area has been concentrated on complete graphs [1, 6, 7, 9, 14, 16, 28], complete bipartite graphs [1, 9] and on graphs with the bounded colorability [25, 26]. We continue to study the set of complete graphs, i.e., we attempt to determine the maximum size of a complete graph with the given type of 3D visibility drawing.

The drawing of K_{22} given by Rote and Zelle (included in [7, 14]) provides the best known lower bound on the maximum size of a complete graph with a 3D rectangle visibility drawing. On the other hand, Bose et al. [6] showed that no complete graph with 103 or more vertices has such a drawing. This result was then improved to 56 by Fekete et al. [7, 14]. We further lower this bound to 51 in Section 3.1. This improvement is based on the study of unimodal sequences of k -tuples. A *unimodal sequence* of numbers is, loosely speaking, a sequence that first increases and then decreases. A sequence of k -tuples is unimodal if the sequences of individual components are unimodal. We explore unimodal sequences in Section 2.

If the vertices are represented by unit squares then the largest complete graph with this type of 3D visibility drawing is K_7 according to Fekete et al.

[14]. This is the only exact result known about drawings by equal regular polygons. Only estimates are known for $n \neq 4$. Babilon et al. [2] show that K_{14} can be represented by equal triangles. They also present a lower bound $\lfloor \frac{n+1}{2} \rfloor + 2$ on the maximum size of a complete graph with a 3D visibility drawing by equal regular n -gons. Štola [24] then moved this bound to $n + 1$. We prove new lower bounds in Section 4.2. We show that K_{2k+3} has a 3D visibility drawing by regular $2k$ -gons and K_{4k+6} has a 3D visibility drawing by regular $(2k + 1)$ -gons.

The first upper bound 2^{2^n} (on the maximum size of a complete graph with a 3D visibility drawing by equal regular n -gons) was given by Babilon et al. [2]. This doubly-exponential estimate was improved by Štola [24] to an exponential $\binom{6n-3}{3n-1} - 3 \approx 2^{6n}$. The main result of Section 4.1 is another significant improvement of this bound. We present a polynomial upper bound $O(n^4)$ there.

2 Unimodal Sequences

This section is not devoted to graph drawing directly. It deals with unimodal sequences. It turns out that these sequences play an important role in the analysis of some types of graph drawing.

Definition 1. A finite sequence is unimodal if it first increases and then decreases, i.e., a sequence $(s_i)_{i=1}^n$ is unimodal if there exists $l \in \{1, \dots, n\}$ such that $s_1 \leq s_2 \leq \dots \leq s_l$ and $s_l \geq s_{l+1} \geq \dots \geq s_n$. \square

Some authors (see, for example, [18]) call a sequence with this property an upper unimodal sequence and call a sequence that first decreases and then increases a lower unimodal sequence. Upper unimodal sequences are also called strongly unimodal (in [8]) or unimaximal (in [7]).

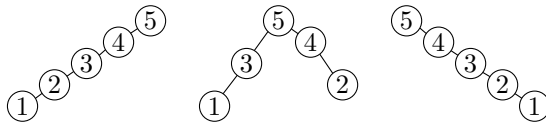


Figure 10. Examples of unimodal sequences

The notion of unimodality can be generalized into higher dimensions.

Definition 2. A sequence $((s_i^j)_{j=1}^k)_i$ of k -tuples of real numbers is unimodal if all sequences $(s_i^j)_i$, $j \in \{1, \dots, k\}$ are unimodal, i.e., a sequence of k -tuples is unimodal if it is unimodal in each component. \square

Unimodal sequences of integers can be found in many areas of combinatorics. Unimodal sequences of k -tuples of real numbers appear, for example, in some types of graph drawing, see [7, 27].

The basic result in this area is attributed (by Chung [8]) to V. Chvátal and J.M. Steele, among others.

Theorem 1. [8] The maximum length of a sequence of distinct integers that doesn't contain a unimodal subsequence of length n is $\binom{n}{2}$.

An upper bound on the maximum length of a sequence of k -tuples without a unimodal subsequence of the given length can be derived from the upper bound for the one-dimensional case. For example, we obtain an upper bound $\binom{\binom{n}{2}+1}{2} \approx \frac{1}{8}n^4$ for sequences of pairs. Unfortunately, the bounds obtained in this way are not tight. A tight upper bound for sequences of pairs is $\frac{1}{12}n^2(n^2 - 1)$ according to Štola [27].

We generalize the approach used in [27] to obtain the tight bound for sequences of pairs. This generalization gives us a tool to determine the maximum length of a sequence without more types of forbidden subsequences (not only unimodal subsequences). For example, we show that Erdős-Szekeres theorem [13] can be obtained as a special case – monotone subsequences of the given length are forbidden in this case. The details about the generalized approach are in Sections 2.1, 2.2 and 2.3.

Let $u_k(n)$ be the maximum length of a sequence of k -tuples of real numbers without a unimodal subsequence of length n . We show that $u_k(n)$ is a polynomial in n and determine its coefficients for $k \leq 5$ in Section 2.4.

Section 2.5 shows a connection between unimodal subsequences and Dedekind numbers. *Dedekind number* D_k is the number of antichains of a power set $\mathcal{P}(\{1, \dots, k\})$ ordered by inclusion. We prove that $D_k = u_k(3)$ and $D_{k+1} = u_k(4)$.

2.1 Preliminaries

Definition 3. Let B be a (base) set and R be a finite set of binary relations on B . We call a sequence $(b_i)_i$ of members of B an R -sequence if for every subsequence b_{i_1}, b_{i_2} of $(b_i)_i$ there exists exactly one relation $r(b_{i_1}, b_{i_2}) \in R$ such that $(b_{i_1}, b_{i_2}) \in r(b_{i_1}, b_{i_2})$. \square

For example, if $B = \mathbb{N}$ and $<, \leq, >, \geq$ are the standard ‘lower/greater than (or equal)’ relations on \mathbb{N} , $R_1 = \{<, >\}$ and $R_2 = \{\leq, \geq\}$ then the sequence $1, 3, 5, 4, 2$ is both an R_1 -sequence and an R_2 -sequence. On the other hand, the sequence $1, 2, 3, 2, 1$ is neither an R_1 -sequence (because $1 \not< 1$ and $1 \not> 1$) nor an R_2 -sequence (because $1 \leq 1$ and $1 \geq 1$, i.e., the relation $r(1, 1)$ is not unique).

Definition 4. Let \preceq be a partial order on R . An R -sequence $(b_i)_{i=1}^n$ is forbidden if $\forall i \in \{1, \dots, n-2\} : r(b_i, b_{i+1}) \preceq r(b_{i+1}, b_{i+2})$. Moreover, we denote by $s(B, R, \preceq, n)$ the maximum length of an R -sequence without a forbidden subsequence of length n . \square

Example 1. Let $<$ (resp. $>$) be the standard ‘lower than’ (resp. ‘greater than’) relations on \mathbb{R} . If $B = \mathbb{R}$, $R_M = \{<, >\}$ and \preceq_M is a discrete order, i.e., $\preceq_M = \{(<, <), (>, >)\}$ then a sequence is an R_M -sequence if and only if its members are distinct. An R_M -sequence is forbidden if and only if it is monotone. Therefore, the number $s(\mathbb{R}, R_M, \preceq_M, n)$ is the maximum length of a sequence of distinct numbers without a monotone subsequence of length n .

Example 2. Let $B = \mathbb{R}$, $R_U = \{<, >\}$ and $< \preceq_U >$, i.e., $\preceq_U = \{(<, <), (<, >), (>, >)\}$. The order \preceq_U ensures that a decreasing sequence may follow an increasing sequence in a forbidden sequence but the opposite is not allowed. Therefore, a sequence is forbidden in this case if and only if it is unimodal. Hence, the number $s(\mathbb{R}, R_U, \preceq_U, n)$ is the maximum length of a sequence of distinct numbers without a unimodal subsequence of length n .

If (R, \preceq) is a partially ordered set then we denote by \preceq^k the partial order on R^k such that $(r_i^1)_{i=1}^k \preceq^k (r_i^2)_{i=1}^k$ if and only if $r_i^1 \preceq r_i^2$ for every $i \in \{1, \dots, k\}$.

Example 3. The previous examples can be generalized into higher dimensions. If we take $B' = B^k$, $R' = R^k$ and $\preceq' = \preceq^k$ then a sequence $((b_i^j)_{j=1}^k)_i$ of k -tuples is (B', R', \preceq') -forbidden if and only if all sequences $(b_i^j)_i$, $1 \leq j \leq k$ are (B, R, \preceq) -forbidden. If we generalize the first (resp. the second) example then a sequence of k -tuples is forbidden if and only if it is monotone (resp. unimodal) in each component.

2.2 Labeled Tournaments

We examine the function $s(B, R, \preceq, n)$ in the following two sections. We start by forgetting the base set B and by looking on the ordered set of relations first.

Definition 5. Let $T_m = G(\{1, \dots, m\}, \{(i, j), i < j\})$ be an acyclic tournament. If ℓ is a mapping of $E(T_m)$ into R then we say that ℓ is an R -labeling of (edges of) T_m and that (T_m, ℓ) is a labeled tournament. \square

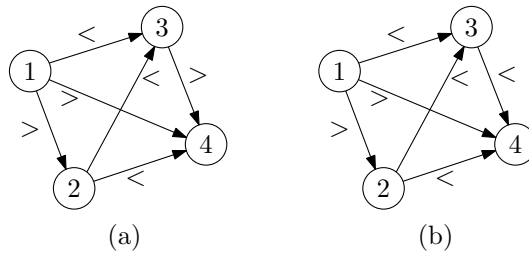


Figure 11. Examples of the $\{<, >\}$ -labeling of the tournament T_4

Definition 6. Let \preceq be a partial order on R . A directed path $(v_i)_{i=1}^n$ in an acyclic R -labeled tournament (T, ℓ) is forbidden if $\ell(v_i, v_{i+1}) \preceq \ell(v_{i+1}, v_{i+2})$ for $1 \leq i \leq n-2$. We say that a labeling ℓ of a tournament T is n -correct if

there is no forbidden path of length n in (T, ℓ) . We denote by $t(R, \preceq, n)$ the maximum size of an acyclic tournament that has an n -correct R -labeling. \square

For example, if $R = \{<, >\}$ and \preceq is a discrete order on R then the R -labeling on Figure 11a is 2-correct. On the other hand, the R -labeling on Figure 11b is not 2-correct because $\ell(1, 3) = \ell(3, 4) = <$, i.e., $\ell(1, 3) \preceq \ell(3, 4)$.

Lemma 1. $s(B, R, \preceq, n) \leq t(R, \preceq, n - 1)$

Proof. Let $(b_i)_{i=1}^m$ be an R -sequence. We define $\ell : E(T_m) \rightarrow R$ by $\ell(i, j) = r(b_i, b_j)$. Clearly, $(b_i)_i$ contains a forbidden subsequence of length n if and only if the R -labeled tournament (T_m, ℓ) contains a forbidden path of length $n - 1$ (i.e., with n vertices). Hence, every R -sequence $(b_i)_{i=1}^{s(B, R, \preceq, n)}$ without a forbidden subsequence of length n defines an R -labeled tournament with $(n - 1)$ -correct labeling. Therefore, $s(B, R, \preceq, n) \leq t(R, \preceq, n - 1)$. \square

The proof of Lemma 1 shows that every R -sequence $(b_i)_{i=1}^m$ defines an R -labeled tournament (T_m, ℓ) . We say that $(b_i)_{i=1}^m$ is a *realization* of the tournament (T_m, ℓ) . If a tournament (T_m, ℓ) has a realization then we say that it is *realizable*.

For example, the $\{<, >\}$ -labeled tournament on Figure 11a is realizable by sequence 3, 1, 4, 2. If b_1, b_2, b_3, b_4 is a realization (in \mathbb{R}) of the tournament on Figure 11b then it must be $b_1 < b_3 < b_4$ and $b_1 > b_4$, i.e., the tournament is not realizable.

The opposite inequality in Lemma 1 may not hold because some tournaments may not be realizable in B .

Lemma 2. If \mathcal{U} is the set of all upper sets of the partially ordered set (R, \preceq) then $t(R, \preceq, n) \leq \sum_{U \in \mathcal{U}} t(R \setminus U, \preceq, n - 1)$ for $n \geq 2$.

Proof. Let $T_m, m = t(R, \preceq, n)$ be an acyclic tournament with an n -correct R -labeling ℓ . We show that the vertex set $V(T_m)$ can be partitioned into pairwise disjoint sets V^U , $U \in \mathcal{U}$ such that the subtournament induced by V^U is $(n - 1)$ -correctly $(R \setminus U)$ -labeled.

For any vertex $v \in V(T_m)$ we denote by R^v the set of the last labels on the forbidden paths of length $n - 1$ ending in v , i.e., $R^v = \{r \in R; \exists \text{ a forbidden path } (v_i)_{i=1}^n : \ell(v_{n-1}, v_n) = r \text{ and } v_n = v\}$.

Let $V^U = \{v \in V(T_m) : \uparrow R^v = U\}$, $U \in \mathcal{U}$, i.e., we group the vertices of T_m according to $\uparrow R^v$ (the smallest upper set containing R^v). Clearly, $\bigcup_{U \in \mathcal{U}} V^U = V(T_m)$ and $V^{U_1} \cap V^{U_2} = \emptyset$ for $U_1 \neq U_2$, $U_1, U_2 \in \mathcal{U}$.

We claim that $\ell(v, w) \notin U$ for any $v, w \in V^U$. Let's assume that $\ell(v, w) \in U$. We know that $\uparrow R^v = U$. Therefore, there exists $r \in R^v, r \preceq \ell(v, w)$ and a forbidden path $(v_i)_{i=1}^n$ such that $v_n = v$ and $\ell(v_{n-1}, v_n) = r$. We can append

w to this path to obtain a forbidden path $v_1, \dots, v_n = v, w$ of length n . This is in contradiction with the definition of T_m .

If $(v_i)_{i=1}^n$ is a forbidden path with all vertices in V^U then $\ell(v_{n-1}, v_n) \in R^{v_n} \subseteq \uparrow R^{v_n} = U$. This is not possible by the previous paragraph. Hence, the labeled subtournament T^U induced by the vertex set V^U doesn't contain a forbidden path of length $n - 1$.

The last two paragraphs show that T^U is $(n - 1)$ -correctly $(R \setminus U)$ -labeled tournament. Therefore, we have $|V^U| \leq t(R \setminus U, \preceq, n - 1)$ and $t(R, \preceq, n) = |V(T_m)| = \sum_{U \in \mathcal{U}} |V(T^U)| \leq \sum_{U \in \mathcal{U}} t(R \setminus U, \preceq, n - 1)$. \square

The following lemma shows that the estimate of $t(R, \preceq, n)$ in Lemma 2 is tight.

Lemma 3. *If \mathcal{U} is the set of all upper sets of the partially ordered set (R, \preceq) then $t(R, \preceq, n) \geq \sum_{U \in \mathcal{U}} t(R \setminus U, \preceq, n - 1)$ for $n \geq 2$.*

Proof. Let T^U be an acyclic tournament with $t(R \setminus U, \preceq, n - 1)$ vertices and $(n - 1)$ -correct $(R \setminus U)$ -labeling ℓ_U . Let $(U_i)_i$ be an ordering of \mathcal{U} such that $U_k \setminus U_j \neq \emptyset$ for $j < k$. An ordering with this property always exists. We can, for example, order the upper sets according to their cardinality.

Let T be an acyclic tournament with the vertices $\bigcup_{U \in \mathcal{U}} V(T^U)$ and edges $\bigcup_{U \in \mathcal{U}} E(T^U) \cup \{(v, w); v \in T^{U_j}, w \in T^{U_k}, j < k\}$, i.e., the tournament T is a 'concatenation' of tournaments T_U . We define an R -labeling ℓ on the edges of T . We keep the labeling of the edges of subtournaments T^U and label the edges between subtournaments T^{U_j} and $T^{U_k}, j < k$ by members of $U_k \setminus U_j$ arbitrarily. Formally, if $(v, w) \in E(T)$, $v \in T^{U_j}$ and $w \in T^{U_k}$ then

- $\ell(v, w) = \ell_{U_j}(v, w)$ ($= \ell_{U_k}(v, w)$) for $j = k$,
- $\ell(v, w) \in U_k \setminus U_j$ for $j < k$.

We claim that ℓ is an n -correct R -labeling of T . Let's assume that $(v_i)_i$ is a forbidden path in T . If the whole path is contained in some subtournament T^U then its length is at most $n - 2$ by the definition of T^U . Therefore, we can assume that the path $(v_i)_i$ visits at least two subtournaments. Let T^{U_j} be the subtournament where the path starts and $v_x \in T^{U_k}$ be the first vertex on the path $(v_i)_i$ that is not in T^{U_j} .

If v_x is not the last vertex of the path then either $v_{x+1} \in T^{U_k}$ or $v_{x+1} \in T^{U_l}, k < l$. We have $\ell(v_x, v_{x+1}) \in R \setminus U_k$ in the first case and $\ell(v_x, v_{x+1}) \in U_l \setminus U_k$ in the second case. Therefore, $\ell(v_x, v_{x+1}) \notin U_k$. On the other hand, $\ell(v_x, v_{x+1}) \succeq \ell(v_{x-1}, v_x) \in U_k \setminus U_j$ and $\ell(v_x, v_{x+1}) \in U_k$ because U_k is an upper set. Hence, v_x must be the last vertex of the path.

We know that a forbidden path in T^{U_j} has length at most $n-2$. Therefore, $x \leq n$ and the length of a forbidden path in T is at most $n-1$. This proves that ℓ is an n -correct R -labeling of T . Hence, $t(R, \preceq, n) \geq |V(T)| = \sum_{U \in \mathcal{U}} |V(T^U)| = \sum_{U \in \mathcal{U}} t(R \setminus U, \preceq, n-1)$. \square

If we combine Lemma 2 and Lemma 3 then we obtain the following theorem.

Theorem 2. *If \mathcal{U} is the set of all upper sets of the partially ordered set (R, \preceq) then $t(R, \preceq, n) = \sum_{U \in \mathcal{U}} t(R \setminus U, \preceq, n-1)$ for $n \geq 2$.*

2.3 Realization of Tournaments

We extend the results of the previous section from tournaments to sequences. The following lemma is an analogy of Lemma 2.

Lemma 4. *If \mathcal{U} is the set of all upper sets of the partially ordered set (R, \preceq) then $s(B, R, \preceq, n) \leq \sum_{U \in \mathcal{U}} s(B, R \setminus U, \preceq, n-1)$ for $n \geq 3$.*

Proof. Let $(b_i)_{i=1}^{s(B, R, \preceq, n)}$ be an R -sequence without a forbidden subsequence of length n and (T, ℓ) be an $(n-1)$ -correctly R -labeled acyclic tournament realized by $(b_i)_i$.

We know from the proof of Lemma 2 that the tournament (T, ℓ) can be partitioned into subtournaments $(T^U, \ell_U), U \in \mathcal{U}$. Every subtournament (T^U, ℓ_U) has a realization by a subsequence of $(b_i)_i$. We denote this subsequence by s^U . The sequences $s^U, U \in \mathcal{U}$ form a partitioning of $(b_i)_i$, i.e., for any $i \in \{1, \dots, s(B, R, \preceq, n)\}$ there is exactly one subsequence s^U containing b_i .

The proof of Lemma 2 shows that ℓ_U is an $(n-2)$ -correct $(R \setminus U)$ -labeling of T^U . Therefore, $r(b_i, b_j) = \ell^U(i, j) \in R \setminus U$ for any $b_i, b_j \in s^U, i < j$, i.e., s^U is $(R \setminus U)$ -sequence. Moreover, s^U doesn't have a forbidden subsequence of length $n-1$. Hence, $s(B, R, \preceq, n) = \sum_{U \in \mathcal{U}} \text{length}(s^U) \leq \sum_{U \in \mathcal{U}} s(B, R \setminus U, \preceq, n-1)$. \square

The opposite inequality in Lemma 4 may not hold because the tournament T constructed in the proof of Lemma 3 may not have a realization in (B, R) .

Definition 7. *Let R be a finite set of binary relations on a set B . We say that (B, R) has a realization property if for every $r \in R$ and every pair of realizable R -labeled tournaments (T_1, ℓ_1) and (T_2, ℓ_2) there exists a realization of an R -labeled tournament $T_1 \overset{r}{\oplus} T_2$ such that*

- $V(T_1 \overset{r}{\oplus} T_2) = V(T_1) \cup V(T_2),$

- $E(T_1 \overset{r}{\oplus} T_2) = E(T_1) \cup E(T_2) \cup \{(v, w) : v \in V(T_1), w \in V(T_2)\},$
- $\ell(v, w) = \ell_1(v, w)$ for $(v, w) \in E(T_1),$
- $\ell(v, w) = \ell_2(v, w)$ for $(v, w) \in E(T_2),$
- $\ell(v, w) = r$ for $v \in V(T_1)$ and $w \in V(T_2).$ □

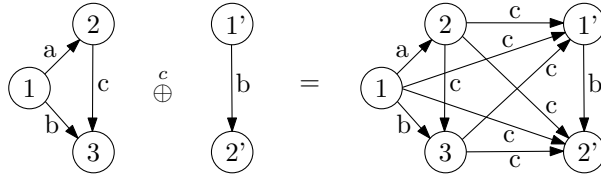


Figure 12. The construction of the tournament $T_1 \overset{c}{\oplus} T_2$

If (B, R) has a realization property then the estimate in Lemma 4 is tight.

Lemma 5. *If (B, R) has a realization property and \mathcal{U} is the set of all upper sets of the partially ordered set (R, \preceq) then $s(B, R, \preceq, n) \geq \sum_{U \in \mathcal{U}} s(B, R \setminus U, \preceq, n-1)$ for $n \geq 3$.*

Proof. Let $s^U = (b_i^U)_i$ be an $(R \setminus U)$ -sequence of length $s(B, R \setminus U, \preceq, n-1)$ that doesn't contain a forbidden subsequence of length $n-1$. Let (T^U, ℓ_U) be the $(n-2)$ -correctly $(R \setminus U)$ -labeled acyclic tournament realized by s^U .

We proceed in the same way as in the proof of Lemma 3. We define an R -labeled tournament T again but we have to specify the order on \mathcal{U} and the labels of the edges between subtournaments more carefully (to ensure that the resulting tournament has a realization).

Let $<_R$ be an arbitrary linear order on R . We define a linear order $<_{\mathcal{U}}$ on \mathcal{U} . If $U, V \in \mathcal{U}$, $U \subsetneq V$ then $U <_{\mathcal{U}} V$. If neither $U \subset V$ nor $V \subset U$ then $U <_{\mathcal{U}} V$ if and only if $\max_{<_R}(U \setminus V) <_R \max_{<_R}(V \setminus U)$.

We define an acyclic tournament T with vertices $\bigcup_{U \in \mathcal{U}} V(T^U)$ and edges $\bigcup_{U \in \mathcal{U}} E(T^U) \cup \{(u, v); u \in T^U, v \in T^V, U <_{\mathcal{U}} V\}$. Moreover, we define an R -labeling ℓ of T . If $(u, v) \in E(T)$, $u \in T^U$ and $v \in T^V$ then

- $\ell(u, v) = \ell_U(u, v)$ ($= \ell_V(u, v)$) for $U = V$,
- $\ell(u, v) = \max_{<_R}(V \setminus U)$ for $U <_{\mathcal{U}} V$.

We know from the proof of Lemma 3 that ℓ is $(n-1)$ -correct R -labeling of T . We claim that (T, ℓ) has a realization in (B, R) .

Let $(r_j)_{j=1}^{|R|}$ be the set R ordered according to $<_R$. We proceed by induction on i and show that for any set $S \subseteq \{r_j : j \geq i\}$ the labeled subtournament T_i^S induced by the vertex set $\bigcup_{U \in \mathcal{U}_i^S} V(T^U)$, $\mathcal{U}_i^S = \{U \in \mathcal{U} : U \cap \{r_j : j \geq i\} = S\}$ has a realization.

If $i = 1$ then we have $\mathcal{U}_1^S \neq \emptyset$ if and only if $S \in \mathcal{U}$. Hence, $V(T_1^S) = \bigcup_{U \in \mathcal{U}_1^S = \{S\}} V(T^U) = V(T^S)$ and we know that s^S is a realization of the corresponding subtournament $T_1^S = T^S$.

If $1 < i \leq |R| + 1$ and $S' = S \cup \{r_{i-1}\}$ then

$$V(T_i^S) = \bigcup_{U \in \mathcal{U}_i^S} V(T^U) = \bigcup_{U \in \mathcal{U}_{i-1}^S} V(T^U) \cup \bigcup_{U \in \mathcal{U}_{i-1}^{S'}} V(T^U) = V(T_{i-1}^S) \cup V(T_{i-1}^{S'}).$$

We have $r_{i-1} \notin U$ (resp. $r_{i-1} \in V$) for any $U \in \mathcal{U}_{i-1}^S$ (resp. $V \in \mathcal{U}_{i-1}^{S'}$). Moreover, for any r_j , $j \geq i$ it is $r_j \in U$ if and only if $r_j \in V$ (if and only if $r_j \in S$). Therefore, $U <_{\mathcal{U}} V$ and $\ell(u, v) = r_{i-1}$ for any $u \in V(T^U)$ and $v \in V(T^V)$. In other words, $T_i^S = T_{i-1}^S \oplus^{r_{i-1}} T_{i-1}^{S'}$.

We know from the previous step of the induction that both T_{i-1}^S and $T_{i-1}^{S'}$ have a realization. Hence, $T_i^S = T_{i-1}^S \oplus^{r_{i-1}} T_{i-1}^{S'}$ has a realization because (B, R) has a realization property.

We proved that every T_i^S has a realization. For $S = \emptyset$ and $i = |R| + 1$ we have $\mathcal{U}_{|R|+1}^\emptyset = \{U \in \mathcal{U} : U \cap \emptyset = \emptyset\} = \mathcal{U}$. Therefore, $T_{|R|+1}^\emptyset = T$ and T has a realization.

Let $(b_i)_{i=1}^m$ be an R -sequence that realizes (T, ℓ) . This sequence doesn't contain a forbidden subsequence of length n because ℓ is $(n-1)$ -correct R -labeling. Therefore, $s(B, R, \preceq, n) \geq m = |V(T)| = \sum_{U \in \mathcal{U}} |V(T^U)| = \sum_{U \in \mathcal{U}} \text{length}(s^U) = \sum_{U \in \mathcal{U}} s(B, R \setminus U, \preceq, n-1)$. \square

If we combine Lemma 4 and Lemma 5 then we obtain an analogy of Theorem 2.

Theorem 3. *If (B, R) has a realization property and \mathcal{U} is the set of all upper sets of the poset (R, \preceq) then $s(B, R, \preceq, n) = \sum_{U \in \mathcal{U}} s(B, R \setminus U, \preceq, n-1)$ for $n \geq 3$.*

Corollary 1. *If (B, R) has a realization property and $n \geq 1$ then $s(B, R, \preceq, n) = t(R, \preceq, n-1)$.*

Proof. We proceed by induction on n and show that for any $M \subseteq R$ it is $s(B, M, \preceq, n) = t(M, \preceq, n-1)$.

For $n \in \{1, 2\}$ and $M \subseteq R$ we have $s(B, M, \preceq, 1) = t(M, \preceq, 0) = 0$ and $s(B, M, \preceq, 2) = t(M, \preceq, 1) = 1$.

We know that (B, M) has a realization property because (B, R) has a realization property. Therefore, if $n \geq 3$ then according to Theorem 3, the previous step of the induction and Theorem 2 we have

$$\begin{aligned} s(B, M, \preceq, n) &= \sum_{U \in \mathcal{U}_M} s(B, M \setminus U, \preceq, n-1) = \\ &= \sum_{U \in \mathcal{U}_M} t(M \setminus U, \preceq, n-2) = t(M, \preceq, n-1) \end{aligned}$$

where \mathcal{U}_M is the set of all upper sets of the poset (M, \preceq) . \square

2.4 Unimodal and Monotone Subsequences

In this section we determine the maximum length of a sequence of k -tuples without a unimodal subsequence of length n . At first we show that we can depend on the realization property in this case.

Lemma 6. *If $R \subseteq \{<, >\}^n$ then (\mathbb{R}^n, R) has a realization property.*

Proof. Let $r = (r_i)_{i=1}^n \in R$ and T_1 (resp. T_2) be a tournament with a realization $(s_i^1)_{i=1}^{m_1}$ (resp. $(s_i^2)_{i=1}^{m_2}$). We define a realization $(s_i)_{i=1}^{m_1+m_2}$ of $T_1 \overset{r}{\oplus} T_2$ in the following way. We set $s_i = s_i^1$ for $i \leq m_1$ and $s_i = s_i^2 + D$, $D \in \mathbb{R}^n$ for $i > m_1$. We choose D such that $(s_i, s_j) \in r$ for every $i \leq m_1$ and $j > m_1$.

Let $(e_i)_{i=1}^n$ be the standard orthonormal basis of \mathbb{R}^n , i.e., $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, etc. If $D = C \sum_{i=1}^n c_i e_i$ where $c_i = 1$ (resp. $c_i = -1$) for $r_i = <$ (resp. $r_i = >$) and $C \in \mathbb{R}^+$ is sufficiently large then the sequence $(s_i)_{i=1}^{m_1+m_2}$ is a realization of $T_1 \overset{r}{\oplus} T_2$. \square

If n is small and the set of all supersets has either small cardinality or its structure is simple then Theorem 2 (resp. Theorem 3) can be used directly to determine $t(R, \preceq, n)$ (resp. $s(B, R, \preceq, n)$). It turns out that these theorems can be used easily to determine $t(R, \preceq, n)$ and $s(B, R, \preceq, n)$ even for large values of n . The following lemma shows how to do it.

Lemma 7. *If (R, \preceq) is a partially ordered set then $t(R, \preceq, n)$ is a polynomial with a positive leading coefficient and the degree $|R|$. Moreover, if \mathcal{U} is the set of all upper sets of (R, \preceq) then $t(R, \preceq, n) = 1 + \sum_{k=1}^{n-1} \sum_{U \in \mathcal{U} \setminus \{\emptyset\}} t(R \setminus U, \preceq, k)$ for $n \geq 1$.*

Proof. We proceed by induction on $|R|$. If $|R| = 0$ then $R = \emptyset$, $t(\emptyset, \preceq, n) = 1$ and $1 + \sum_{k=1}^{n-1} \sum_{U \in \emptyset} t(\emptyset, \preceq, k) = 1 + \sum_{k=1}^{n-1} 0 = 1$ for $n \geq 1$.

If $|R| > 0$ and $n \geq 2$ then according to Theorem 2 we have

$$\begin{aligned} t(R, \preceq, n) &= t(R, \preceq, n-1) + \sum_{U \in \mathcal{U} \setminus \{\emptyset\}} t(R \setminus U, \preceq, n-1) = \\ &= t(R, \preceq, 1) + \sum_{k=1}^{n-1} \sum_{U \in \mathcal{U} \setminus \{\emptyset\}} t(R \setminus U, \preceq, k) = 1 + \sum_{k=1}^{n-1} \sum_{U \in \mathcal{U} \setminus \{\emptyset\}} t(R \setminus U, \preceq, k). \end{aligned}$$

If $n = 1$ then $t(R, \preceq, 1) = 1$ and $1 + \sum_{k=1}^0 \sum_{U \in \mathcal{U}} t(R \setminus U, \preceq, k) = 1$. Therefore, the identity $t(R, \preceq, n) = 1 + \sum_{k=1}^{n-1} \sum_{U \in \mathcal{U} \setminus \{\emptyset\}} t(R \setminus U, \preceq, k)$ holds for $n \geq 1$.

We know from the previous step of the induction that $t(R \setminus U, \preceq, k)$ is a polynomial with the degree $|R \setminus U|$ and a positive leading coefficient. Hence, $\sum_{U \in \mathcal{U} \setminus \{\emptyset\}} t(R \setminus U, \preceq, k)$ is a polynomial with the degree $\max_{U \in \mathcal{U} \setminus \{\emptyset\}} |R \setminus U|$. Therefore, $t(R, \preceq, n)$ is a polynomial with the degree $1 + \max_{U \in \mathcal{U} \setminus \{\emptyset\}} |R \setminus U|$. The leading coefficient of the polynomial $t(R, \preceq, n)$ is positive because all polynomials $t(R \setminus U, \preceq, k)$, $U \in \mathcal{U} \setminus \{\emptyset\}$ have positive leading coefficients.

The poset (R, \preceq) has at least one maximum r_{max} because R is finite. Therefore, $\max_{U \in \mathcal{U} \setminus \{\emptyset\}} |R \setminus U| = |R| - 1$ because $\{r_{max}\}$ is an upper set. Hence, the degree of the polynomial $t(R, \preceq, n)$ is $|R|$. \square

Corollary 2. *If (R, \preceq) is a partially ordered set and (B, R) has a realization property then $s(B, R, \preceq, n)$ is a polynomial with a positive leading coefficient and the degree $|R|$. Moreover, if \mathcal{U} is the set of all upper sets of (R, \preceq) then $s(B, R, \preceq, n) = 1 + \sum_{k=2}^{n-1} \sum_{U \in \mathcal{U} \setminus \{\emptyset\}} s(B, R \setminus U, \preceq, k)$ for $n \geq 2$.*

Proof. This corollary is a simple consequence of Lemma 7 and Corollary 1. \square

Corollary 3. *If (R, \preceq) is a discrete poset then $t(R, \preceq, n) = n^{|R|}$ for $n \geq 1$.*

Proof. We proceed by induction on $|R|$. If $|R| = 0$ then $R = \emptyset$ and $t(\emptyset, \preceq, n) = 1$. If $|R| > 0$ then every subset of R is an upper set in the discrete poset (R, \preceq) . Therefore, according to Lemma 7 and the previous step of the induction we have

$$\begin{aligned} t(R, \preceq, n) &= 1 + \sum_{k=1}^{n-1} \sum_{m=0}^{|R|-1} \sum_{U \in \mathcal{U}, |U|=|R|-m} t(R \setminus U, \preceq, k) = \\ &= 1 + \sum_{k=1}^{n-1} \sum_{m=0}^{|R|-1} \binom{|R|}{|R|-m} k^m = 1 + \sum_{k=1}^{n-1} ((k+1)^{|R|} - k^{|R|}) = n^{|R|}. \end{aligned}$$

\square

If (R_M, \preceq_M) is the partially ordered set from Example 1 in Section 2.1 then (\mathbb{R}, R_M) has a realization property due to Lemma 6. Therefore, $s(\mathbb{R}, R_M, \preceq_M, n) = t(R_M, \preceq_M, n-1) = (n-1)^2$ by Corollary 1 and Corollary 3. This means that the maximum length of a sequence (of distinct numbers) that doesn't contain a monotone subsequence of length n is $(n-1)^2$. In other words, Erdős-Szekeres theorem [13] is a special case of Corollary 3.

Lemma 8. *The maximum length of a sequence of k -tuples of real numbers that doesn't contain a unimodal subsequence of length n is $s(\mathbb{R}^k, R_U^k, \preceq_U^k, n)$ where $R_U = \{<, >\}$ and $\preceq_U = \{(<, <), (<, >), (>, >)\}$.*

Proof. We know (from Example 2 and Example 3) that the maximum length of an R_U^k -sequence of k -tuples that doesn't contain a unimodal sequence of length n is $s(\mathbb{R}^k, R_U^k, \preceq_U^k, n)$. We have to show that this bound holds also for sequences that are not R_U^k -sequences. We claim that every sequence $s = ((b_i^j)_{j=1}^k)_i$ of k -tuples (i.e., not necessarily R_U^k -sequence) longer than $s(\mathbb{R}^k, R_U^k, \preceq_U^k, n)$ contains a unimodal subsequence of length n .

We repeat the following process until we obtain an R_U^k -sequence. If the sequence s contains members $(b_x^j)_{j=1}^k$ and $(b_y^j)_{j=1}^k$, $x \neq y$ such that $b_x^{j'} = b_y^{j'}$ for some $j' \in \{1, \dots, k\}$ then we replace $b_x^{j'}$ by $b_x^{j'} + \varepsilon$. We choose $\varepsilon > 0$ such that $b_x^{j'} + \varepsilon > b_i^{j'}$ (resp. $b_x^{j'} + \varepsilon < b_i^{j'}$) whenever $b_x^{j'} > b_i^{j'}$ (resp. $b_x^{j'} < b_i^{j'}$).

The resulting R_U^k -sequence $s' = ((c_i^j)_{j=1}^k)_i$ has the same length as the original sequence s (i.e., is longer than $s(\mathbb{R}^k, R_U^k, \preceq_U^k, n)$). Therefore, it contains a unimodal subsequence $((c_{i_l}^j)_{j=1}^k)_{l=1}^n$. We performed the replacements in s such that $c_x^j < c_y^j$ (resp. $c_x^j > c_y^j$) only if $b_x^j \leq b_y^j$ (resp. $b_x^j \geq b_y^j$). Hence, the subsequence $((b_{i_l}^j)_{j=1}^k)_{l=1}^n$ of s is also unimodal and has length n . \square

Now we can proceed to the main result of this section.

Theorem 4. *If $u_k(n)$ is the maximum length of a sequence of k -tuples of real numbers that doesn't contain a unimodal subsequence of length $n \geq 2$ then*

$$\begin{aligned}
u_1(n) &= (n-1)n/2 \\
u_2(n) &= (n-1)n^2(n+1)/12 \\
u_3(n) &= (n-1)n(n+1)(n+2)(2n^4 + 4n^3 + n^2 - n + 4)/1680 \\
u_4(n) &= (n-1)n(n+1)^2(n+2)(n+3)(2188n^{10} + 21880n^9 + 81000n^8 \\
&\quad + 122880n^7 + 106689n^6 + 390150n^5 + 857015n^4 + 320180n^3 \\
&\quad - 1778862n^2 - 2788020n + 2872800)/27243216000 \\
u_5(n) &= (n-1)n(n+1)(n+2)(n+3)(n+4)(482024870952388n^{26} + 18798969967143132n^{25} \\
&\quad + 331613261704350160n^{24} + 3478540936142196360n^{23}
\end{aligned}$$

$$\begin{aligned}
& +24245249756669384305n^{22} + 122065775931899836905n^{21} \\
& +498970836617766995500n^{20} + 1898007921240632301585n^{19} \\
& +6719775479513894172100n^{18} + 18902955809181381835185n^{17} \\
& +36331230260439507728350n^{16} + 30739296553709404093710n^{15} \\
& -160517480691164408211635n^{14} - 774895455899521661518305n^{13} \\
& +1307253531304994398188100n^{12} + 13993216097643044201045385n^{11} \\
& -3149148469074212091802190n^{10} - 145226381189665260254119725n^9 \\
& -25378478176587338599016550n^8 + 935650325100713964444939960n^7 \\
& +160293815795710365537370032n^6 - 3467393139664210783067322192n^5 \\
& +403013636559535706828929440n^4 + 6753480934722715830642048000n^3 \\
& -8342695734133534267314240000n^2 - 22545812692304999988871680000n \\
& +60321249146355912658944000000)/8565456931435336268464128000000
\end{aligned}$$

Proof. It is sufficient to determine $s(\mathbb{R}^k, R_U^k, \preceq_U^k, n)$ because it is equal to $u_k(n)$ by Lemma 8.

Let $u_k(R, n) = s(\mathbb{R}^k, R, \preceq_U^k, n)$, $R \subseteq R_U^k$. If $k = 1$ then according to Corollary 2 we have

$$\begin{aligned}
u_1(\emptyset, n) &= 1, \\
u_1(\{<\}, n) &= 1 + \sum_{m=2}^{n-1} u_1(\emptyset, m) = n - 1, \\
u_1(n) = u_1(R_U, n) &= 1 + \sum_{m=2}^{n-1} \left(u_1(\{<\}, m) + u_1(\emptyset, m) \right) = \\
&= 1 + \sum_{m=2}^{n-1} m = \binom{n}{2}.
\end{aligned}$$

If $k > 1$ then we proceed in the same way, i.e., we use the formula from Corollary 2 until we calculate $u_k(n)$. The length of the calculation increases significantly with the increasing k . It is easy to determine $u_2(n)$ using a paper and a pencil but for $k > 2$ we used a simple Java program. The program computed u_3 and u_4 in a fraction of a second. The computation of u_5 took 8 minutes on an average laptop. \square

2.5 Dedekind Numbers

The results presented in this section don't have direct applications in graph drawing but they allow us to explain why we weren't able to find a general formula for $u_k(n)$ in Theorem 4.

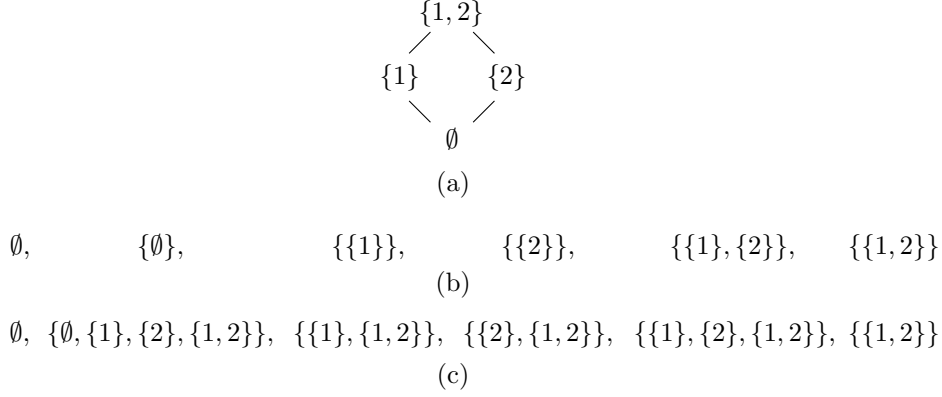


Figure 13. (a) The Hasse diagram of the poset $(\mathcal{P}(\{1,2\}), \subseteq)$, (b) the antichains of this poset and (c) the corresponding upper sets

Let's remind the definition of Dedekind number.

Definition 8. Dedekind number D_k is the number of antichains of the power set $\mathcal{P}(\{1, \dots, k\})$ ordered by inclusion. \square

We show a connection between Dedekind numbers and unimodal subsequences in this section.

Lemma 9. $u_k(3) = D_k$

Proof. Let $f : (R_U^k, \preceq_U^k) \rightarrow (\mathcal{P}(\{1, \dots, k\}), \subseteq)$ be a mapping defined by

$$f((r_i)_{i=1}^k) = \{i, i \in \{1, \dots, k\} : r_i = >\}.$$

The mapping f is a bijection. Moreover, f is an isomorphism because

$$\begin{aligned} (r_i^1)_{i=1}^k \preceq_U^k (r_i^2)_{i=1}^k &\equiv \forall j \in \{1, \dots, k\} \ r_j^1 \preceq_U r_j^2 \equiv \\ &\equiv \forall j \in \{1, \dots, k\} \ (r_j^1, r_j^2) \in \{(<, <), (<, >), (>, >)\} \equiv \\ &\equiv \forall j \in \{1, \dots, k\} \ (j \in f((r_i^1)_i) \Rightarrow j \in f((r_i^2)_i)) \equiv \\ &\equiv f((r_i^1)_i) \subseteq f((r_i^2)_i). \end{aligned}$$

Therefore, the number of antichains in (R_U^k, \preceq_U^k) is D_k .

Let \mathcal{U} be the set of all upper sets of the partially ordered set (R_U^k, \preceq_U^k) . We have $D_k = |\mathcal{U}|$ because antichains correspond to upper sets in finite posets. Finally, according to Theorem 3 we have

$$u_k(3) = s(\mathbb{R}^k, R_U^k, \preceq_U^k, 3) = \sum_{V \in \mathcal{U}} s(\mathbb{R}^k, R_U^k \setminus V, \preceq_U^k, 2) = \sum_{V \in \mathcal{U}} 1 = |\mathcal{U}| = D_k.$$

\square

Lemma 10. $u_{a+b}(n) \leq u_a(u_b(n) + 1)$

Proof. Let $s = ((b_i^j)_{j=1}^{a+b})_i$ be a sequence of $(a+b)$ -tuples of length $u_a(u_b(n) + 1) + 1$. This sequence contains a subsequence $s' = ((b_{i_l}^j)_{j=1}^{a+b})_{l=1}^{u_b(n)+1}$ unimodal in the first a components, i.e., $(b_{i_l}^j)_{l=1}^{u_b(n)+1}$ is unimodal for any $j \in \{1, \dots, a\}$. Similarly, the sequence s' contains a subsequence s'' of length n unimodal in the last b components. Moreover, the sequence s'' is unimodal in the first a components because every subsequence of a unimodal sequence is unimodal. In other words, the subsequence s'' of s is unimodal and has length n . Therefore, $u_{a+b}(n) < u_a(u_b(n) + 1) + 1$. \square

Corollary 4. $D_{k+m} \leq u_k(D_m + 1)$ for $m \geq 0$

Proof. If $m = 0$ then $D_0 = 2$ and $D_k = u_k(3) = u_k(D_0 + 1)$ by Lemma 9. If $m > 0$ then according to Lemma 9 and Lemma 10 we have $D_{k+m} = u_{k+m}(3) \leq u_k(u_m(3) + 1) = u_k(D_m + 1)$. \square

Lemma 9 shows that the estimate in Corollary 4 is tight for $m = 0$. The following lemma shows that it is tight for $m = 1$ as well.

Lemma 11. $u_k(4) = D_{k+1}$

Proof. We denote R_U^k (resp. \preceq_U^k) simply by R (resp. \preceq) in this proof. Let \mathcal{U}_S , $S \subseteq R$ be the set of all upper sets of the partially ordered set (S, \preceq) . According to Lemma 6 and Theorem 3 we have

$$\begin{aligned} u_k(4) &= s(\mathbb{R}^k, R, \preceq, 4) = \sum_{V \in \mathcal{U}_R} s(\mathbb{R}^k, R \setminus V, \preceq, 3) = \\ &= \sum_{V \in \mathcal{U}_R} \sum_{W \in \mathcal{U}_{R \setminus V}} s(\mathbb{R}^k, (R \setminus V) \setminus W, \preceq, 2) = \sum_{V \in \mathcal{U}_R} \sum_{W \in \mathcal{U}_{R \setminus V}} 1 = |\mathcal{M}| \end{aligned}$$

where $\mathcal{M} = \{(V, W) : V \in \mathcal{U}_R, W \in \mathcal{U}_{R \setminus V}\}$.

Let f be the isomorphism from the proof of Lemma 9. Let \mathcal{U}_{k+1} be the set of all upper sets of the partially ordered set $(\mathcal{P}(\{1, \dots, k+1\}), \subseteq)$. We define a mapping $g : \mathcal{M} \rightarrow \mathcal{U}_{k+1}$ by

$$\begin{aligned} g(V, W) &= \left\{ f(v) : v \in V \right\} \cup \left\{ f(v) \cup \{k+1\} : v \in V \cup W \right\} = \\ &= f(V) \cup \left\{ x \cup \{k+1\} : x \in f(V \cup W) \right\}. \end{aligned}$$

We claim that g is a bijection.

At first we show that $V \cup W$ is an upper set in (R, \preceq) for any $(V, W) \in \mathcal{M}$. Let $x \in V \cup W$ and $x \leq y$, $y \in R$. If $y \in V$ then $y \in V \cup W$ obviously. If

$y \notin V$ then $x \notin V$ because V is an upper set. Finally, if $x \in W$ and $y \in R \setminus V$ then $y \in W \subset V \cup W$ because W is an upper set in $(R \setminus V, \preceq)$.

If $(V, W) \in \mathcal{M}$ then $f(V)$ (resp. $f(V \cup W)$) is an upper set in $(\mathcal{P}(\{1, \dots, k\}), \subseteq)$ because f is an isomorphism and V (resp. $V \cup W$) is an upper set in (R, \preceq) .

We have to show that g is defined correctly, i.e., that $g(V, W)$ is an upper set for any $(V, W) \in \mathcal{M}$. Let $x \in g(V, W)$ and $x \subseteq y$, $y \in \mathcal{P}(\{1, \dots, k+1\})$. We have $x \setminus \{k+1\} \in f(V \cup W)$ and $x \setminus \{k+1\} \subseteq y \setminus \{k+1\} \in f(V \cup W)$ because $f(V \cup W)$ is an upper set. Therefore, if $k+1 \in y$ then $y \in g(V, W)$. If $k+1 \notin y$ then $x \in f(V)$ and $y \in f(V) \subseteq g(V, W)$ because $f(V)$ is an upper set.

g is injective because f is injective. It remains to show that g is surjective. Let U be an upper set in $(\mathcal{P}(\{1, \dots, k+1\}), \subseteq)$. Let $V = f^{-1}(\{x \in U : \{k+1\} \notin x\})$ and $W = f^{-1}(\{x \setminus \{k+1\} : x \in U\}) \setminus V$. Clearly, $g(V, W) = U$ but we have to show that $(U, V) \in \mathcal{M}$.

V is an upper set in (R, \preceq) because f is an isomorphism and $\{x \in U : \{k+1\} \notin x\}$ is an upper set in $(\mathcal{P}(\{1, \dots, k\}), \subseteq)$. Similarly, $V \cup W = f^{-1}(\{x \setminus \{k+1\} : x \in U\})$ is an upper set in (R, \preceq) because $\{x \setminus \{k+1\} : x \in U\}$ is an upper set in $(\mathcal{P}(\{1, \dots, k\}), \subseteq)$. Hence, $W = (V \cup W) \setminus V$ is an upper set in $(R \setminus V, \preceq)$ and $(V, W) \in \mathcal{M}$.

We proved that g is a bijection. Therefore, $D_{k+1} = |\mathcal{U}_{k+1}| = |\mathcal{M}| = u_k(4)$. \square

It might be tempting to conjecture that the estimate in Corollary 4 is tight for any m . Unfortunately, this is not true. The estimate is not tight for $m = 2$ and $k = 1$ already: $D_3 = 20 < 21 = u_1(7) = u_1(D_2 + 1)$.

D_0	2
D_1	3
D_2	6
D_3	20
D_4	168
D_5	7581
D_6	7828354
D_7	2414682040998
D_8	56130437228687557907788

Table 1. The only known Dedekind numbers [22]

Let's recall that Dedekind's problem [12] is a longstanding problem requiring to provide a closed formula for Dedekind numbers. Nobody has been able to provide such a formula so far. In fact, there are only nine Dedekind

numbers (D_0, D_1, \dots, D_8) known by now, see Table 1. Hence, it is not surprising that we are not able to find a closed formula for $u_k(n)$ for a general k . Lemma 9 and Lemma 11 would give a solution to Dedekind's problem otherwise.

3 Rectangle Visibility Drawing

We study the rectangle visibility drawing in this section.

Definition 9. *A graph is representable as a rectangle visibility graph in \mathbb{R}^n if each vertex can be mapped to a hyper-rectangle in \mathbb{R}^n (where the hyper-rectangle is a cartesian product of $n - 1$ intervals in \mathbb{R} and a number in the last coordinate, i.e., $[a_1, b_1] \times \dots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$) such that two hyper-rectangles see each other in the direction orthogonal to the hyper-rectangles whenever the corresponding vertices are connected by an edge. \square*

Graph drawing terminology is slightly confusing regarding rectangle visibility drawings. Graphs representable as rectangle visibility graphs in \mathbb{R}^3 are also called 3D rectangle visibility graphs. On the other hand, 2D rectangle visibility graphs are not necessarily representable as rectangle visibility graphs in \mathbb{R}^2 . Graphs representable as rectangle visibility graphs in \mathbb{R}^2 are those graphs that admit a visibility drawing with vertices represented by line segments and edges orthogonal to these line segments. On the other hand, the 2D rectangle visibility drawing represents vertices by axis-aligned rectangles and edges are parallel either to the x -axis or to the y -axis.

Cobos et al. [9] show that every graph is representable as a rectangle visibility graph.

Lemma 12. *[9] Given a graph G , there exists $n \in \mathbb{N}$ such that G is representable as a rectangle visibility graph in \mathbb{R}^n .*

It is interesting to look for the minimum dimension in which the graph is representable.

Definition 10. *A graph G has a representation index equal to n if it is representable as a rectangle visibility graph in \mathbb{R}^n and is not representable in \mathbb{R}^m , $m < n$. We denote the representation index of G by $RI(G)$. \square*

Lemma 12 ensures that the representation index is defined for any graph. Cobos et al. [9] also show that the set of graphs representable in \mathbb{R}^n grows with n .

Lemma 13. *[9] Every graph representable as a rectangle visibility graph in \mathbb{R}^n is representable in \mathbb{R}^{n+1} .*

Graphs with the representation index at most 2 are called *bar-visibility graphs*. Tamassia, Tollis [29] and Wismath [30] give the following characterization of bar-visibility graphs.

Lemma 14. [29, 30] *A graph G is a bar-visibility graph (i.e., $RI(G) \leq 2$) if and only if there is a planar embedding of G with all cut vertices on the exterior face.*

The characterization of graphs with the representation index at most 3 is an open problem. All planar graphs are in this category according to Bose et al. [7].

It turns out to be useful to refine the definition of the representation index.

Definition 11. *A graph G of the (old) representation index equal to n has a (fractional) representation index equal to $(n - 1) + 1/2$ if it admits a representation in \mathbb{R}^n such that any hyper-rectangle (representing a vertex) is of the form $[a_1, b_1] \times \cdots \times [a_{n-2}, b_{n-2}] \times [a_{n-1}, 0] \times \{a_n\}$. \square*

We use the term representation index for the fractional representation index in the sequel.

Cobos et al. [9] characterize the graphs with the representation index lower or equal to $1+1/2$.

Lemma 15. [9] *A graph G has a representation index at most $1+1/2$ if and only if it has a Hamiltonian path such that there exists a planar embedding of G with all edges of this path on the exterior face.*

The characterization of graphs with the representation index lower or equal to $k/2$, $k \in \mathbb{N}$, $k > 4$ remains an open problem.

3.1 3D Rectangle Visibility Drawing

This section is focused on complete graphs with the representation index lower or equal to 3. These graphs are called 3D rectangle visibility graphs. In other words, a graph is a *3D rectangle visibility graph* if it has a visibility drawing in \mathbb{R}^3 where vertices are represented by axis-aligned rectangles placed in planes parallel to xy -plane. If two vertices are connected by an edge then the corresponding rectangles must see each other in the direction parallel to z -axis. A drawing with these properties is called a *3D rectangle visibility drawing*.

Let's assume that we have a 3D rectangle visibility drawing of a complete graph and consider orthogonal projections of all rectangles into xy -plane. Every pair of projections must intersect because every pair of rectangles can see each other (because they represent a complete graph). Therefore, all projections have a common intersection due to a Helly-type theorem for axis-aligned rectangles in a plane (see [10]).

Let O be the common intersection of the projections. We introduce a coordinate system with the origin O and four axes p_1 , p_2 , p_3 and $p_4 = p_0$ orthogonal to individual sides of the rectangles, see Figure 14. Every projection can be described using a 4-tuple of orthogonal distances of individual sides of the projection from the origin O . We call these 4-tuples *rectangle coordinates*. Hence, the location of any rectangle in the drawing can be described using its rectangle coordinates and its z -coordinate.

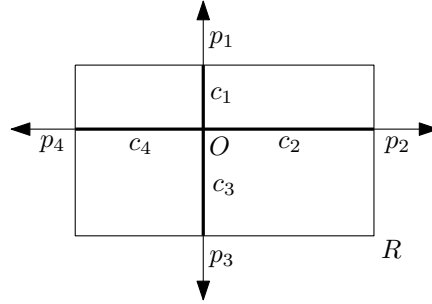


Figure 14. A rectangle R with the coordinates (c_1, c_2, c_3, c_4) .

We don't include the z -coordinate in rectangle coordinates because its exact value is unimportant. It is sufficient to know the order of the rectangles according to the z -coordinate only. If we sort the rectangles coordinates according to the z -coordinate of the corresponding rectangle then we obtain a sequence of 4-tuples that fully describes the drawing.

Fekete et al. [7, 14] found a sufficient and necessary condition for the visibility between two rectangles.

Lemma 16. [7, 14] *Let rectangles $(R_k)_{k=1}^m$ (ordered according to the z -coordinate) form a 3D rectangle visibility drawing of a complete graph K_m and let $(c_i^k)_{i=1}^4$ be rectangle coordinates of R_k . The rectangles R_a and R_b , $a < b$ can see each other in the quadrant $\widehat{p_{l-1}p_l}$, $l \in \{1, 2, 3, 4\}$ if and only if for every rectangle R_x , $a < x < b$ it is $c_{l-1}^x < \min(c_{l-1}^a, c_{l-1}^b)$ or $c_l^x < \min(c_l^a, c_l^b)$.*

Proof. Let R'_k be the orthogonal projection of R_k into the xy -plane. Any potential line of visibility (in the quadrant $\widehat{p_{l-1}p_l}$) between rectangles R_a and R_b must intersect $Q = R'_a \cap R'_b \cap \widehat{p_{l-1}p_l}$.

If there exists a rectangle R_x , $a < x < b$ such that $c_{l-1}^x \geq \min(c_{l-1}^a, c_{l-1}^b)$ and $c_l^x \geq \min(c_l^a, c_l^b)$ then all lines of visibility going through Q are blocked by R_x , see Figure 15a. Therefore, the rectangles R_a and R_b cannot see each other in the quadrant $\widehat{p_{l-1}p_l}$.

On the other hand, if $c_{l-1}^x < \min(c_{l-1}^a, c_{l-1}^b)$ or $c_l^x < \min(c_l^a, c_l^b)$ for every $x : a < x < b$ then the z -parallel lines of visibility in the neighborhood of the

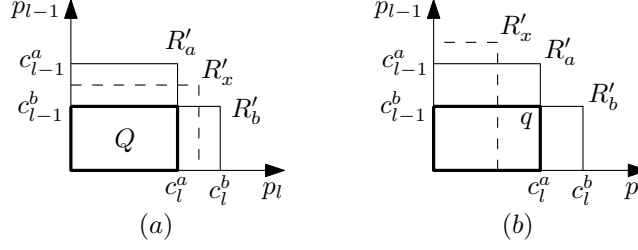


Figure 15

point q are not blocked, see Figure 15b. Therefore, the rectangles R_a and R_b can see each other in the quadrant $\widehat{p_{l-1}p_l}$. \square

The following lemma provides an important property of 3D rectangle visibility drawings of complete graphs. It is inspired by the proof of Lemma 2.3 in [7].

Lemma 17. *Let rectangles $(R_k)_{k=1}^m$ (ordered according to the z -coordinate) form a 3D rectangle visibility drawing of a complete graph K_m and let $(c_1^k, c_2^k, c_3^k, c_4^k)$ be rectangle coordinates of R_k . If $(c_1^k)_{k=1}^m$ is a unimodal sequence then the rectangles $(R'_k)_{k=1}^m$ with the coordinates $(0, c_2^k, c_3^k, c_4^k)$ also form a 3D rectangle visibility drawing of K_m .*

Proof. Let $(\bar{c}_i^k)_{i=1}^4$ be rectangle coordinates of R'_k . The rectangles R'_a and R'_b see each other in the quadrant $\widehat{p_{l-1}p_l}$, $l \in \{1, 2, 3, 4\}$ if and only if for every rectangle R'_x , $a < x < b$ it is $\bar{c}_{l-1}^x < \min(\bar{c}_{l-1}^a, \bar{c}_{l-1}^b)$ or $\bar{c}_l^x < \min(\bar{c}_l^a, \bar{c}_l^b)$ according to Lemma 16.

If $l \neq 1$ then $\bar{c}_l^x < \min(\bar{c}_l^a, \bar{c}_l^b)$ if and only if $c_l^x < \min(c_l^a, c_l^b)$ because $c_l^k = \bar{c}_l^k$ for any $k \in \{1, \dots, m\}$.

If $l = 1$ then $c_l^x < \min(c_l^a, c_l^b)$ doesn't hold because the sequence $(c_1^k)_{k=1}^m$ is unimodal. The inequality $0 = \bar{c}_1^x < \min(\bar{c}_1^a, \bar{c}_1^b) = \min(0, 0) = 0$ also doesn't hold.

Therefore, the rectangles R'_a and R'_b see each other in the quadrant $\widehat{p_{l-1}p_l}$, $l \in \{1, 2, 3, 4\}$ if and only if the rectangles R_a and R_b can see each other in the same quadrant. Hence, the rectangles $(R'_k)_{k=1}^m$ also form a drawing of a complete graph. \square

Lemma 17 motivates the study of 3D rectangle visibility drawings of complete graphs where some coordinates of all rectangles are equal to zero.

Lemma 18. *Let rectangles $(R_k)_{k=1}^m$ (ordered according to the z -coordinate) form a 3D rectangle visibility drawing of a complete graph K_m and let $(c_1^k, c_2^k, c_3^k, c_4^k)$ be rectangle coordinates of R_k .*

- (i) If $c_1^k = c_2^k = c_3^k = c_4^k = 0, k \in \{1, \dots, m\}$ then $m \leq 2$.
- (ii) If $c_1^k = c_2^k = c_3^k = 0, k \in \{1, \dots, m\}$ then $m \leq 3$.
- (iii) If $c_1^k = c_3^k = 0, k \in \{1, \dots, m\}$ then $m \leq 4$.
- (iv) If $c_1^k = c_2^k = 0, k \in \{1, \dots, m\}$ then $m \leq 6$.
- (v) If $c_1^k = 0, k \in \{1, \dots, m\}$ then $m \leq 10$.

Proof. (i) The rectangles degenerate to points in this case. If $m > 2$ then any point/rectangle $R_k, k = 2, \dots, m-1$ blocks the visibility between R_1 and R_m . Hence, $m \leq 2$.

(ii) The rectangles degenerate to line segments in this case. All the line segments lie in the xz -plane. Moreover, one endpoint of each line segment lies on the z -axis and all line segments lie in one half-plane determined by the z -axis. In other words, these line segments form a rectangle visibility drawing that shows that the representation index of the corresponding graph is at most $1 + 1/2$. Hence, it must be $m \leq 3$ because $K_m, m > 3$ has a representation index bigger than $1 + 1/2$ according to Lemma 15.

(iii) This case is similar to the previous one. The rectangles are again degenerated to line segments and lie in the xz -plane. They form a bar-visibility drawing there. All bar-visibility graphs are planar according to Lemma 14. Therefore, $m \leq 4$ because $K_m, m > 4$ is not planar.

(iv) If $m \geq 7$ then the sequence $((c_3^k, c_4^k))_{k=1}^m$ contains a unimodal subsequence $((c_3^{k_i}, c_4^{k_i}))_{i=1}^3$ according to Theorem 4. If we apply Lemma 17 on rectangles $R_{k_1}, R_{k_2}, R_{k_3}$ (we apply the lemma twice - at first we use it on the 3rd coordinates and then on the 4th coordinates) then we obtain a 3D rectangle visibility drawing of K_3 that is in contradiction with (i). Hence, $m \leq 6$.

(v) This case is similar to the previous one. If $m \geq 11$ then the sequence $(c_3^k)_{k=1}^m$ contains a unimodal subsequence $(c_3^{k_i})_{i=1}^5$ according to Theorem 1. If we apply Lemma 17 on rectangles $(R_{k_i})_{i=1}^5$ then we obtain a 3D rectangle visibility drawing of K_5 that is in contradiction with (iii). Hence, $m \leq 10$. \square

Figure 16 shows that all upper bounds in Lemma 18 are tight. Figure 16(iv) shows 4 rectangles that can be seen from the back and front. We obtain a drawing of K_6 by adding two large rectangles into this drawing of K_4 . We add one rectangle behind and one rectangle in front of all displayed rectangles. There is a similar situation on Figure 16(v).

The following lemma is a simple consequence of Lemma 17 and Lemma 18 but it provides a lot of information about structure of 3D rectangle visibility drawings of complete graphs.

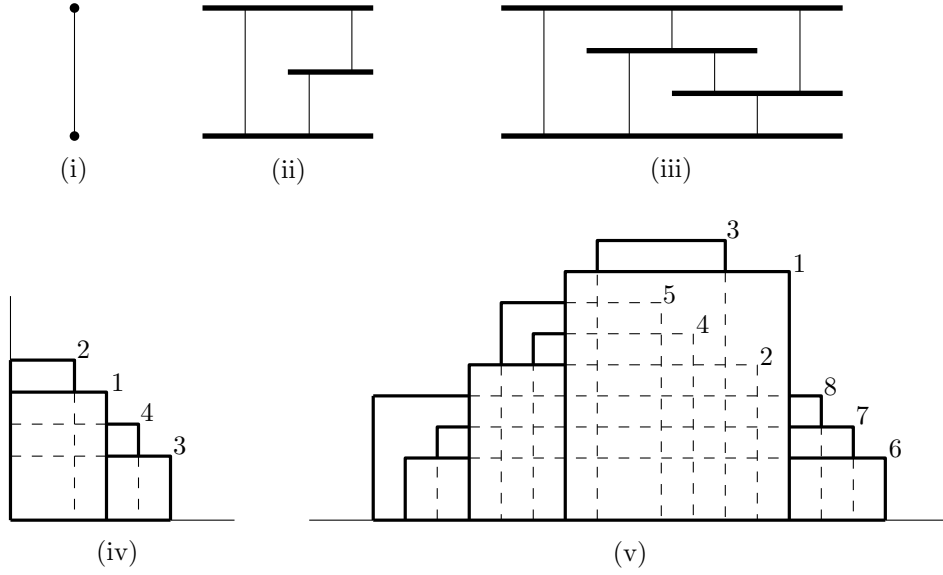


Figure 16. 3D rectangle visibility drawings of complete graphs showing that the upper bounds given by Lemma 18 are tight (the numbers represent the z -coordinates of the rectangles)

Lemma 19. Let rectangles $(R_k)_{k=1}^m$ (ordered according to the z -coordinate) form a 3D rectangle visibility drawing of a complete graph K_m and let $(c_1^k, c_2^k, c_3^k, c_4^k)$ be rectangle coordinates of R_k .

- (i) The sequence $((c_1^k, c_2^k, c_3^k, c_4^k))_{k=1}^m$ doesn't contain a unimodal subsequence of length 3.
- (ii) The sequence $((c_1^k, c_2^k, c_3^k))_{k=1}^m$ doesn't contain a unimodal subsequence of length 4.
- (iii) The sequence $((c_1^k, c_3^k))_{k=1}^m$ doesn't contain a unimodal subsequence of length 5.
- (iv) The sequence $((c_1^k, c_2^k))_{k=1}^m$ doesn't contain a unimodal subsequence of length 7.
- (v) The sequence $(c_1^k)_{k=1}^m$ doesn't contain a unimodal subsequence of length 11.

Proof. (i) If the sequence $((c_1^k, c_2^k, c_3^k, c_4^k))_{k=1}^m$ contains a unimodal subsequence $((c_1^{k_i}, c_2^{k_i}, c_3^{k_i}, c_4^{k_i}))_{i=1}^3$ then we can apply Lemma 17 on rectangles $(R_{k_i})_{i=1}^3$ to

obtain a 3D rectangle visibility drawing of K_3 that is in contradiction with Lemma 18(i).

The proofs of other cases are analogous. \square

Now we are ready to improve the best known upper bound on the maximum size of a complete graph with a 3D rectangle visibility drawing.

Theorem 5. *If $(R_k)_{k=1}^m$ is a 3D rectangle visibility drawing of a complete graph then $m \leq 50$.*

Proof. Let $(c_i^k)_{i=1}^4$ be the rectangle coordinates of R_k . If $m > u_2(5) = 50$ then the sequence $((c_1^k, c_3^k))_{k=1}^m$ of pairs contains a unimodal subsequence $((c_1^{k_i}, c_3^{k_i}))_{i=1}^5$ according to Theorem 4. This is in contradiction with Lemma 19 (iii). Hence, $m \leq 50$. \square

The proof of Theorem 5 utilizes the part (iii) of Lemma 19. The proof can be rephrased to use other parts of Lemma 19 but we would obtain weaker bounds. On the other hand, it is important to note that the proof uses the part (iii) only while the conditions from all parts of Lemma 19 must be satisfied. It remains an open question how to combine these conditions to obtain a better bound. There still remains a big gap between our upper bound and the largest known complete graph with a 3D rectangle visibility drawing (K_{22}), see Table 2.

z	1	2	3	4	5	6	7	8	9	10	11
c_1^z	22	11	9	8	7	5	1	17	16	4	15
c_2^z	22	15	18	12	9	8	7	5	4	1	3
c_3^z	22	13	6	2	19	17	18	14	12	20	15
c_4^z	22	16	15	20	8	11	12	1	2	14	3

z	12	13	14	15	16	17	18	19	20	21	22
c_1^z	19	14	6	3	2	20	13	10	18	12	21
c_2^z	2	19	6	10	11	16	14	20	13	17	21
c_3^z	16	1	3	4	5	7	8	9	10	11	21
c_4^z	4	6	18	17	19	5	7	9	10	13	21

Table 2. Rectangle coordinates of the 3D rectangle visibility drawing of K_{22} by Rote and Zelle (included in [7, 14])

3.2 Rectangle Visibility Drawing in Higher Dimensions

The ideas from the previous section can be used also in Euclidean spaces with more dimensions.

Let's assume that some complete graph is representable as a (hyper-) rectangle visibility graph in \mathbb{R}^n . We consider orthogonal projections of all hyper-rectangles into the hyperplane $x_n = 0$. Every pair of projections must intersect because every pair of hyper-rectangles can see each other. Therefore, all projections have a common intersection according to Helly-type theorem for axis-aligned hyper-rectangles, see [10].

We can assume (without loss of generality) that the common intersection is the origin of the coordinate system, i.e., for every hyper-rectangle $[a_1, b_1] \times \dots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$ we have $a_i \leq 0$ and $b_i \geq 0$, $i \in \{1, \dots, n-1\}$.

We define hyper-rectangle coordinates similar to rectangle coordinates.

Definition 12. Let R be a hyper-rectangle $[a_1, b_1] \times \dots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$. We call the $(n-1)$ -tuple of pairs $((-a_i, b_i))_{i=1}^{n-1}$ hyper-rectangle coordinates of R . \square

The following lemmas generalize Lemma 16 and Lemma 17.

Lemma 20. Let hyper-rectangles $(R_k)_{k=1}^m$ (ordered according to the x_n -coordinate) form a rectangle visibility drawing of a complete graph K_m in \mathbb{R}^n and let $((c_i^k, d_i^k))_{i=1}^{n-1}$ be hyper-rectangle coordinates of R_k . Let $Q = Q_1 \times \dots \times Q_{n-1} \times \mathbb{R}$ where each Q_i is either \mathbb{R}_0^+ or \mathbb{R}_0^- .

The hyper-rectangles R_a and R_b , $a < b$ can see each other in the 'hyper-quadrant' Q if and only if for every hyper-rectangle R_x , $a < x < b$ there exists $i \in \{1, \dots, n-1\}$ such that

- $c_i^x < \min(c_i^a, c_i^b)$ and $Q_i = \mathbb{R}_0^-$ or
- $d_i^x < \min(d_i^a, d_i^b)$ and $Q_i = \mathbb{R}_0^+$.

Proof. The proof is analogous to the proof of Lemma 16. \square

Lemma 21. Let hyper-rectangles $(R_k)_{k=1}^m$ (ordered according to the x_n -coordinate) form a rectangle visibility drawing of a complete graph K_m in \mathbb{R}^n and let $((c_i^k, d_i^k))_{i=1}^{n-1}$ be hyper-rectangle coordinates of R_k . Let $(d_{n-1}^k)_{k=1}^m$ be a unimodal sequence. Finally, let R'_k be a hyper-rectangle with hyper-rectangle coordinates $((c_i^k, \bar{d}_i^k))_{i=1}^{n-1}$ where $\bar{d}_i^k = d_i^k$, $i \in \{1, \dots, n-2\}$ and $\bar{d}_{n-1}^k = 0$. The hyper-rectangles R'_k also form a rectangle visibility drawing of K_m in \mathbb{R}^n .

Proof. The proof is analogous to the proof of Lemma 17. \square

Lemma 21 allows us to derive an estimate of $RI(K_m)$.

Theorem 6. $RI(K_{u_k(m)+1}) \geq RI(K_m) + k/2$

Proof. Let n be a representation index of $K_{u_k(m)+1}$, i.e., $n = RI(K_{u_k(m)+1})$. Let $(R_j)_{j=1}^{u_k(m)+1}$ be a (hyper-)rectangle visibility drawing of $K_{u_k(m)+1}$ in \mathbb{R}^n (with rectangles ordered according to the x_n -coordinate) and $((c_i^j, d_i^j))_{i=1}^{n-1}$ be hyper-rectangle coordinates of R_j .

We assume that $RI(K_{u_k(m)+1})$ is an integer. The proof is analogous when $RI(K_{u_k(m)+1})$ is not an integer.

We consider a sequence $((e_i^j)_{i=1}^{2n-2})_{j=1}^{u_k(m)+1}$ where $e_{2i-1}^j = c_i^j$ and $e_{2i}^j = d_i^j$, $i \in \{1, \dots, n-1\}$. This sequence contains a subsequence $((e_i^{j_l})_{i=1}^{2n-2})_{l=1}^m$ that is unimodal in the last k components.

We use Lemma 21 on rectangles $(R_{j_l})_{l=1}^m$ repeatedly to obtain a rectangle visibility drawing of K_m by hyper-rectangles that have the last k hyper-rectangle coordinates equal to zero. In other words, we obtain a rectangle visibility drawing of K_m showing that $RI(K_m) \leq n - k/2$, i.e., $RI(K_m) + k/2 \leq RI(K_{u_k(m)+1})$. \square

Theorem 6 generalizes the estimate $RI(K_{\binom{m}{2}+1}) > RI(K_m)$ given by Cobos et al. [9]. They also derived the following estimate.

Theorem. [9] $RI(K_{2m}) \leq RI(K_m) + 1/2$

Corollary. If $n \geq 6$ and $m \leq 11 \cdot 2^{n-5}$ then $RI(K_m) \leq n/2$.

Proof. The corollary is an immediate consequence of the previous theorem and the fact that $RI(K_{22}) \leq 3$ (according to Table 2). \square

4 3D Visibility Drawing by Polygons

We concentrate on 3D visibility drawings by regular polygons in this section. These drawings represent vertices by shifted copies of a regular polygon. The rotation of the polygons is not allowed because otherwise any complete graph can be represented in a trivial way, see Figure 17.

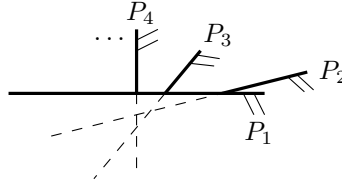


Figure 17. A sketch of a visibility drawing of K_n by rotated copies of (one side of) a polygon, see [2] for the details of this drawing

The results of Section 3.1 are based on Lemma 16. The lemma holds because the system of rectangle coordinates has the origin in the common intersection of the rectangles and because the axes of the coordinate system intersect the corresponding sides of the rectangles. Unfortunately, a 3D visibility drawing by regular polygons doesn't have to meet these criteria. For example, Figure 18a shows a drawing of a complete graph by triangles that don't have a common intersection.

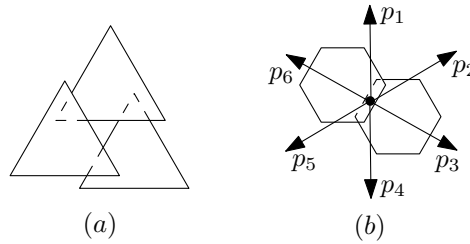


Figure 18. 3D visibility drawings by polygons that do not form a short-distance set

Even if the polygons in a drawing have a common intersection and we place the origin of our coordinate system there then the axes of the coordinate system don't have to intersect the corresponding sides of polygons, see Figure 18b. It turns out that these problems don't occur if the polygons are close to each other.

Definition 13. Let $\{P_i, P_i = P + \vec{w}_i\}$ be a set of shifted copies of a regular n -gon P (inscribed in a unit circle). We say that this set is a short-distance set if $\forall i : |\vec{w}_i| < \min(\sin(\pi/n), \cos(\pi/n))$. \square

Let P be a regular n -gon inscribed in a unit circle (with the center c). Let $v_0, v_1, \dots, v_n = v_0$ be the vertices of P , $s_0 = \overline{v_0 v_1}, \dots, s_{n-1} = \overline{v_{n-1} v_n}, s_n = s_0$ the sides of P , m_j the center of s_j and p_j the half-line $\overrightarrow{cm_j}$. If P_i is a copy of P (shifted by a vector \vec{w}_i) then we denote its vertices by v_j^i and the sides by s_j^i .

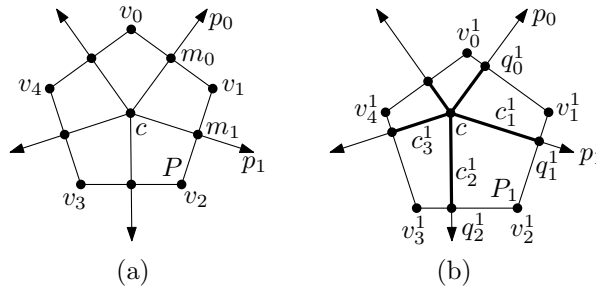


Figure 19

The distance of v_j and p_j is $\sin(\pi/n)$, similarly $\text{dist}(v_j, p_{j-1}) = \sin(\pi/n)$ and $\text{dist}(s_j, c) = \cos(\pi/n)$. Hence, if $|\vec{w}_i| < \cos(\pi/n)$ then c lies in the shifted polygon P_i . If, in addition, $|\vec{w}_i| < \sin(\pi/n)$ then v_j^i (the shifted copy of v_j) remains in the angle $\widehat{p_{j-1} p_j}$ and s_j^i intersects p_j . In other words, if $\{P_i, P_i = P + \vec{w}_i\}$ is a short-distance set then $c \in \bigcap P_i$ and the half-line p_j intersects j -th sides of polygons from the set.

The definition of a short-distance set requires a reference polygon P that is close to every polygon from the set. If the polygons $P_i = P + \vec{w}_i$ are far from P but close to each other, i.e., $\forall i, j : |\vec{w}_i - \vec{w}_j| < \min(\sin(\pi/n), \cos(\pi/n))$ then they also form a short-distance set because we can take any P_i as the reference polygon in this case.

For a polygon P_i from a short-distance set we can define $q_j^i = p_j \cap s_j^i$ and $c_j^i = \text{dist}(c, q_j^i)$, see Figure 19b. We call the n -tuple $(c_j^i)_{j=1}^n$ *polygon coordinates* of P_i .

Every polygon can be reconstructed from its coordinates, see Figure 20. If H_j^i is the half-plane with its boundary line h_j^i such that $c \in H_j^i$, $h_j^i \perp p_j$ and $\text{dist}(h_j^i, c) = c_j^i$ then $P_i = \bigcap_{j=1}^n H_j^i$. Therefore, the intersection $P_i \cap P_k = \bigcap_{j=1}^n (H_j^i \cap H_j^k)$ can be described by coordinates $(\min(c_j^i, c_j^k))_{j=1}^n$.

We assume in the rest of Section 4 that P is a regular n -gon inscribed in a unit circle and $\{P_i = P + \vec{w}_i, i = 1, \dots, m\}$ is a 3D visibility representation

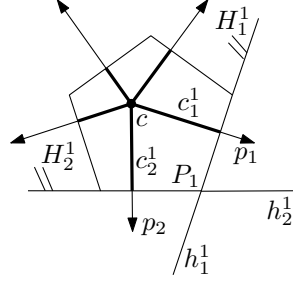


Figure 20. The reconstruction of the polygon P_1 from its coordinates

of a complete graph K_m . We assume that the z -coordinate of P_i is i but we use it to identify polygons that can block the visibility between other polygons only. Otherwise, we ignore the z -coordinate and work with the polygons as if they were in the same xy -parallel plane. Formally, these operations represent operations over orthogonal projections of the relevant objects (points, lines, polygons) into a common xy -parallel plane and the projection of the results (for example, intersection points) into individual planes of the polygons.

The following lemma is a polygonal analogy of Lemma 16.

Lemma 22. *Let $\{P_i, i = 1, \dots, m\}$ be a short-distance set of regular n -gons and $(c_j^i)_{j=1}^n$ be polygon coordinates of P_i . The polygons P_i and P_k can see each other if and only if there exists l such that $\forall j, i < j < k : (c_l^j < \min(c_l^i, c_l^k)$ or $c_{l+1}^j < \min(c_{l+1}^i, c_{l+1}^k))$.*

Proof. $Q = P_i \cap P_k$ is a polygon given by coordinates $(\min(c_j^i, c_j^k))_{j=1}^n$. Let Q_l be the intersection of Q with the angle $\widehat{p_l p_{l+1}}$ and q_l be the (only) vertex of Q in $\widehat{p_l p_{l+1}}$.

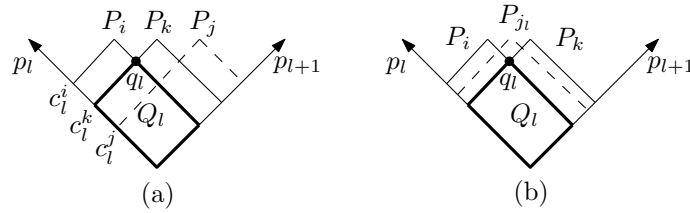


Figure 21

If $c_l^j < \min(c_l^i, c_l^k)$ or $c_{l+1}^j < \min(c_{l+1}^i, c_{l+1}^k)$ then P_j doesn't block the visibility of P_i and P_k in the neighborhood of q_l , see Figure 21a. Hence, if for a fixed l this condition holds for all polygons P_j between P_i and P_k then P_i and P_k can see each other in the neighborhood of q_l .

On the other hand, if $\forall l \exists j_l : i < j_l < k, c_l^i \geq \min(c_l^i, c_l^k)$ and $c_{l+1}^i \geq \min(c_{l+1}^i, c_{l+1}^k)$ then P_{j_l} blocks the visibility between P_i and P_k in the angle $\widehat{p_l p_{l+1}}$, see Figure 21b. Therefore, P_i cannot see P_k . \square

Lemma 22 describes a sufficient and necessary condition for the visibility between two polygons from a short-distance set. If we shift the polygon P_i by a sufficiently small vector then we don't break any of the strict inequalities in Lemma 22. In other words, the shifted polygon can see all polygons that the original polygon can see. Therefore, we can replace the original polygon P_i by the shifted one without breaking the completeness of the represented graph. This observation allows us to assume in the sequel that j -th coordinates of polygons are distinct, i.e., $\forall i, j, k, i \neq k : c_j^i \neq c_j^k$.

Lemma 23. *Let P_i be a regular n -gon with coordinates $(c_j^i)_{j=1}^n$ and $P_k = P_i + \vec{w}$ be a shifted copy of P_i with coordinates $(c_j^k)_{j=1}^n$. If n is even then there are exactly $n/2$ adjacent coordinates with $\text{sgn}(c_j^k - c_j^i) = 1$ and $n/2$ adjacent coordinates with the opposite signum. If n is odd then there are $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ adjacent coordinates with $\text{sgn}(c_j^k - c_j^i) = 1$ and the rest with the opposite signum.*

Proof. The length of the orthogonal projection of \vec{w} into the line containing p_j is $|c_j^k - c_j^i|$. The difference $c_j^k - c_j^i$ is positive (resp. negative) if this projection of \vec{w} has the same (resp. the opposite) orientation as p_j .

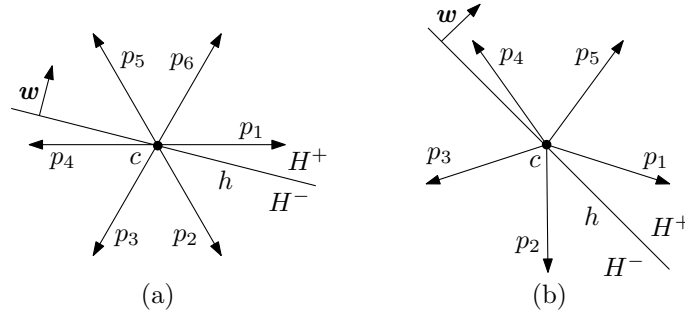


Figure 22

Let h be a line such that $h \perp \vec{w}$ and $c \in h$. h divides the plane into half-planes H^+ and H^- . Let H^+ be the half-plane in the direction of the vector \vec{w} . p_j lies in H^+ resp. H^- if $c_j^k > c_j^i$ resp. $c_j^i > c_j^k$.

If n is even then exactly $n/2$ adjacent half-lines from $(p_j)_{j=1}^n$ lie in H^+ and $n/2$ adjacent half-lines lie in H^- , see Figure 22a. If n is odd then $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ adjacent half-lines lie in H^+ and the rest of them lie in H^- , see Figure 22b. \square

4.1 Upper Bounds

The next lemma shows that every 3D visibility drawing (by regular polygons) of a complete graph contains a large short-distance subset. The following sections focus on these subsets.

Lemma 24. *Let $\{P_i = P + \vec{w}_i, i = 1, \dots, m\}$ be a set of regular n -gons. If $(P_i)_i$ is a 3D visibility drawing of a complete graph K_m then $(P_i)_i$ contains a short-distance subset with at least $\lceil m/16n^2 \rceil$ polygons.*

Proof. Every two polygons P_j, P_k from the drawing have to intersect (to see each other). The polygons $(P_i)_i$ are shifted copies of P (a polygon inscribed into a unit circle). Hence, P_j can intersect P_k only if the distance of their centers is at most 2. Therefore, the set C of the centers of the polygons from $(P_i)_i$ has the diameter at most 2.

Let S be a square that contains all points from C and whose side-length is 2. We can divide this square into $4n \times 4n = 16n^2$ sub-squares with the side-length $1/2n$. At least one of these sub-squares must contain at least $\lceil m/16n^2 \rceil$ points of C . We claim that the polygons with the center in this sub-square form a short-distance set.

It is sufficient to show that two points in one sub-square have the distance lower than $\min(\sin(\pi/n), \cos(\pi/n))$. For $x \in (0, \pi/3)$ we have $\frac{x}{\sqrt{2}\pi} < \min(\sin x, \cos x)$. Hence, for $n \geq 3$ we have $\frac{1}{\sqrt{2}n} < \min(\sin(\pi/n), \cos(\pi/n))$ and $\frac{1}{\sqrt{2}n}$ is the maximum distance of two points in one sub-square. \square

4.1.1 Regular $2k$ -gons

The goal of this section is a polynomial upper bound on the maximum size of a complete graph with a 3D visibility drawing by regular $2k$ -gons. We start with a lemma that points out an important forbidden configuration of three polygons.

Lemma 25. *Let $\{P_1, P_2, P_3\}$ be a short-distance set of regular $2k$ -gons. If $\{P_1, P_2, P_3\}$ is a 3D visibility drawing of a complete graph K_3 then it cannot happen that $c_1^1 < c_1^2 < c_1^3$ and $c_2^1 > c_2^2 > c_2^3$ (where $(c_j^i)_{j=1}^{2k}$ are coordinates of P_i).*

Proof. If $c_1^1 < c_1^2 < c_1^3$ and $c_2^1 > c_2^2 > c_2^3$ then $c_l^1 > c_l^2 > c_l^3$ for $l \in \{2, \dots, k+1\}$ and $c_l^1 < c_l^2 < c_l^3$ for $l \in \{k+2, \dots, 2k\} \cup \{1\}$ by Lemma 23. Therefore, $c_l^2 > \min(c_l^1, c_l^3)$ for $l \in \{1, \dots, 2k\}$ and P_1 cannot see P_3 according to Lemma 22 but this is a contradiction. \square

The following lemma shows that if the sequence $(c_1^i)_i$ of the first coordinates is monotone then the size of the represented graph is small.

Lemma 26. *Let $\{P_i, i = 1, \dots, m\}$ be a short-distance set of regular $2k$ -gons. If $(P_i)_i$ is a 3D visibility drawing of a complete graph K_m and $(c_1^i)_{i=1}^m$ is a monotone sequence (where $(c_j^i)_{j=1}^{2k}$ are coordinates of P_i) then $m \leq k+1$.*

Proof. We assume that the sequence $(c_1^i)_{i=1}^m$ is increasing. The proof for a decreasing sequence is similar. Let $I = \{\{i, j\} : i < j, c_2^i > c_2^j\}$, i.e., the pairs of polygons whose boundaries intersect in $\widehat{p_1 p_2}$. We claim that $I = \emptyset$ or $\bigcap I \neq \emptyset$.

We proceed by contradiction. Let's assume that $I \neq \emptyset$ and $\bigcap I = \emptyset$. At first we show that there must be (at least) two disjoint pairs in I . Let's assume that there aren't two disjoint pairs in I . If $\{a, \bar{a} : a < \bar{a}\} \in I$ then there exist $B = \{b, \bar{b} : b < \bar{b}\}$ and $C = \{c, \bar{c} : c < \bar{c}\}$ in I such that $a \notin B$ and $\bar{a} \notin C$ (because $a, \bar{a} \notin \bigcap I$). Moreover, $\bar{a} \in B$ and $a \in C$ because the pairs $\{a, \bar{a}\}$ and B (resp. C) are not disjoint. If $\bar{a} = b$ then $c_1^a < c_1^{\bar{a}} = c_1^b < c_1^{\bar{b}}$ and $c_2^a > c_2^{\bar{a}} = c_2^b > c_2^{\bar{b}}$ which is in contradiction with Lemma 25. Therefore, $\bar{a} = \bar{b}$ and $B = \{b, \bar{a}\}$. An analogous argument shows that $a = c$ and $C = \{a, \bar{c}\}$. The pairs B and C are not disjoint according to our assumption. This can happen only if $\bar{c} = b$ but then $c_1^a < c_1^{\bar{c}} = c_1^b < c_1^{\bar{a}}$ and $c_2^a > c_2^{\bar{c}} = c_2^b > c_2^{\bar{a}}$ which is in contradiction with Lemma 25 again. This means that there must be two disjoint pairs in I .

Let $\{a, \bar{a} : a < \bar{a}\}$ and $\{b, \bar{b} : b < \bar{b}\}$ be disjoint pairs in I . We can assume without loss of generality that $a < b$.

Let's assume that $\bar{a} < \bar{b}$ (see Figure 23):

$$\begin{aligned} a < \bar{a} < \bar{b}, a < b < \bar{b}, (c_1^i)_i \text{ increasing} &\Rightarrow c_1^a < c_1^{\bar{a}} < c_1^{\bar{b}}, c_1^a < c_1^b < c_1^{\bar{b}} \\ \{a, \bar{a} : a < \bar{a}\}, \{b, \bar{b} : b < \bar{b}\} \in I &\Rightarrow c_2^a > c_2^{\bar{a}}, c_2^b > c_2^{\bar{b}} \\ c_1^b < c_1^{\bar{b}}, c_2^b > c_2^{\bar{b}} &\Rightarrow c_l^b > c_l^{\bar{b}}, l \in \{2, \dots, k+1\} \text{ by Lemma 23} \\ c_1^a < c_1^{\bar{a}}, c_2^a > c_2^{\bar{a}} &\Rightarrow c_l^a < c_l^{\bar{a}}, l \in \{k+2, \dots, 2k\} \cup \{1\} \text{ by Lemma 23} \\ c_1^{\bar{a}} < c_1^{\bar{b}} &\Rightarrow c_{k+1}^{\bar{b}} < c_{k+1}^{\bar{a}} \text{ by Lemma 23} \end{aligned}$$

We can see that $c_1^a < c_1^b$ and $c_l^b > c_l^{\bar{b}}, l \in \{2, \dots, k+1\}$. Therefore, $c_l^b > \min(c_l^a, c_l^{\bar{b}})$, $l \in \{1, \dots, k+1\}$. Similarly, $c_{k+1}^{\bar{b}} < c_{k+1}^{\bar{a}}$ and $c_l^a < c_l^{\bar{a}}, l \in \{k+2, \dots, 2k\} \cup \{1\}$, i.e., $c_l^{\bar{a}} > \min(c_l^a, c_l^{\bar{b}})$, $l \in \{k+1, \dots, 2k\} \cup \{1\}$. Hence, P_a cannot see $P_{\bar{b}}$ according to Lemma 22 but this cannot happen because $(P_i)_i$ is a drawing of a complete graph. Therefore, it cannot be $\bar{a} < \bar{b}$.

If $\bar{b} < \bar{a}$ then $a < b < \bar{b} < \bar{a}$ and $c_1^a < c_1^b < c_1^{\bar{b}} < c_1^{\bar{a}}$ because $(c_1^i)_i$ is increasing. $c_2^{\bar{a}} < c_2^a$ and $c_2^{\bar{b}} < c_2^b$ because $\{a, \bar{a} : a < \bar{a}\}, \{b, \bar{b} : b < \bar{b}\} \in I$. If $c_2^{\bar{a}} < c_2^{\bar{b}}$ then $P_b, P_{\bar{b}}$ and $P_{\bar{a}}$ are in contradiction with Lemma 25. Similarly, if $c_2^{\bar{b}} < c_2^a$ then P_a, P_b and $P_{\bar{b}}$ are in contradiction with Lemma 25. Therefore, it

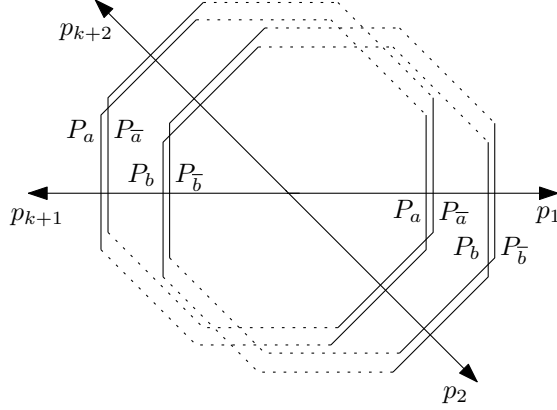


Figure 23

must be $\bar{c}_2^b < \bar{c}_2^a < c_2^a < c_2^b$ but this means that the disjoint pairs $\{a, \bar{b} : a < \bar{b}\}$, $\{b, \bar{a} : b < \bar{a}\}$ satisfy the assumptions of the previous paragraph and we again have a contradiction with the completeness of the represented graph.

We know that $\bar{a} \neq \bar{b}$ because $\{a, \bar{a}\}$ and $\{b, \bar{b}\}$ are disjoint. On the other hand, both possibilities $\bar{a} < \bar{b}$ and $\bar{b} < \bar{a}$ lead to a contradiction. Hence, the original assumption that $I \neq \emptyset$ and $\bigcap I = \emptyset$ cannot be satisfied. It must be either $I = \emptyset$ or $\bigcap I \neq \emptyset$.

If $I = \emptyset$ then $(c_2^i)_i$ is increasing. If $I \neq \emptyset$ then there exists $a \in \bigcap I$. This means that if $i < j$ and $c_i^2 > c_j^2$ then $i = a$ or $j = a$. In other words, the sequence $(c_2^i)_{i \in \{1, \dots, m\} \setminus \{a\}}$ is increasing.

We can repeat this proof with c_2, c_3, \dots, c_k subsequently and show that there is a set A such that $|A| \leq k$ and $(c_{k+1}^i)_{i \in \{1, \dots, m\} \setminus A}$ is increasing. On the other hand, this sequence is also decreasing by Lemma 23 because the sequence $(c_1^i)_{i \in \{1, \dots, m\} \setminus A}$ is increasing. Therefore, the sequence $(c_{k+1}^i)_{i \in \{1, \dots, m\} \setminus A}$ has length at most 1 and $1 \geq |\{1, \dots, m\} \setminus A| \geq m - k$. \square

Now we are ready to prove the main theorem of this section.

Theorem 7. *If $(P_i)_{i=1}^m$ is a 3D visibility drawing of a complete graph K_m by equal regular n -gons (where $n = 2k$) then $m \leq 4n^2(n+2)^2$.*

Proof. The set $\{P_i, i = 1, \dots, m\}$ contains a short-distance subset $\{P_{i_l}, l = 1, \dots, \lceil m/16n^2 \rceil\}$ according to Lemma 24. Let $(c_j^l)_{j=1}^n$ be the coordinates of P_{i_l} . If $\lceil m/16n^2 \rceil \geq (k+1)^2 + 1$ then due to Erdős-Szekeres theorem [13] the sequence $(c_1^l)_{l=1}^{\lceil m/16n^2 \rceil}$ contains a monotone subsequence of length $k+2$ which is in contradiction with Lemma 26. Therefore, $m/16n^2 \leq \lceil m/16n^2 \rceil \leq (k+1)^2$. \square

4.1.2 Regular $(2k + 1)$ -gons

We focus on regular $(2k + 1)$ -gons in this section. We prove a theorem analogous to Theorem 7. Unfortunately, Lemma 25 doesn't hold for $(2k + 1)$ -gons. We have to use a more complicated version.

Lemma 27. *Let $\{P_1, P_2, P_3, P_4\}$ be a short-distance set of regular $(2k + 1)$ -gons. If $(P_i)_i$ is a 3D visibility drawing of a complete graph K_4 then it cannot happen that $c_1^1 < c_1^2 < c_1^3 < c_1^4$ and $c_2^1 > c_2^2 > c_2^3 > c_2^4$ (where $(c_j^i)_{j=1}^{2k+1}$ are coordinates of P_i).*

Proof. If $c_1^1 < c_1^2 < c_1^3 < c_1^4$ and $c_2^1 > c_2^2 > c_2^3 > c_2^4$ then $c_l^1 > c_l^2 > c_l^3 > c_l^4$ for $l \in \{2, \dots, k + 1\}$ and $c_l^1 < c_l^2 < c_l^3 < c_l^4$ for $l \in \{k + 3, \dots, 2k + 1\} \cup \{1\}$ by Lemma 23. In other words, $c_l^2 > \min(c_l^1, c_l^3)$ and $c_l^3 > \min(c_l^2, c_l^4)$ for $l \in \{1, \dots, 2k + 1\} \setminus \{k + 2\}$.

P_1 and P_3 can see each other. Therefore, $c_{k+2}^2 < \min(c_{k+2}^1, c_{k+2}^3)$ according to Lemma 22. Similarly, $c_{k+2}^3 < \min(c_{k+2}^2, c_{k+2}^4)$ because P_2 and P_4 can see each other. But this is a contradiction because the first inequality gives us $c_{k+2}^2 < c_{k+2}^3$ while $c_{k+2}^3 < c_{k+2}^2$ by the second inequality. \square

We need the following consequence of Lemma 27 several times in this section.

Corollary 5. *Let $\{P_1, P_2, P_3, P_4\}$ be a short-distance set of regular $(2k + 1)$ -gons. If $(P_i)_i$ is a 3D visibility drawing of a complete graph K_4 then it cannot happen that $c_1^1 < c_1^2 < c_1^3 < c_1^4$ and $c_{k+1}^1 < c_{k+1}^2 < c_{k+1}^3 < c_{k+1}^4$ (or $c_{k+2}^1 < c_{k+2}^2 < c_{k+2}^3 < c_{k+2}^4$).*

Proof. If $c_1^1 < c_1^2 < c_1^3 < c_1^4$ and $c_{k+1}^1 < c_{k+1}^2 < c_{k+1}^3 < c_{k+1}^4$ then $c_{k+2}^1 > c_{k+2}^2 > c_{k+2}^3 > c_{k+2}^4$ by Lemma 23 but this is in contradiction with Lemma 27 for coordinates $k + 1$ and $k + 2$ (Lemma 27 holds for any pair of adjacent coordinates).

Similarly, if $c_1^1 < c_1^2 < c_1^3 < c_1^4$ and $c_{k+2}^1 < c_{k+2}^2 < c_{k+2}^3 < c_{k+2}^4$ then $c_{k+1}^1 > c_{k+1}^2 > c_{k+1}^3 > c_{k+1}^4$ by Lemma 23 and we have a contradiction again. \square

The next lemma is an analogy of Lemma 26. The proof of this lemma is more complicated because the drawings by $(2k + 1)$ -gons are more complicated but the main ideas of both proofs (of Lemma 26 and Lemma 28) are the same.

Lemma 28. *Let $\{P_i, i = 1, \dots, m\}$ be a short-distance set of regular $(2k + 1)$ -gons. There exists $c > 0$ independent of k such that if $(P_i)_i$ is a 3D visibility drawing of a complete graph K_m and $(c_1^i)_{i=1}^m$ is a monotone sequence (where $(c_j^i)_{j=1}^{2k+1}$ are coordinates of P_i) then $m \leq ck$.*

Proof. We assume that the sequence $(c_1^i)_{i=1}^m$ is increasing. The proof for a decreasing sequence is similar. Let $I = \{\{i, j\} : i < j, c_2^i > c_2^j\}$. We claim that there exists $n_0 \in \mathbb{N}$ (independent of k) such that I doesn't contain n_0 pairwise disjoint pairs.

Let's assume that $J \subseteq I : \forall A, B \in J, A \neq B \Rightarrow A \cap B = \emptyset$. Consider a complete graph on the vertex set J . We color the edge $\{\{a, \bar{a} : a < \bar{a}\}, \{b, \bar{b} : b < \bar{b}\} : a < b\}$ by

- color 1 when $\bar{a} < \bar{b}$ and $c_{k+2}^{\bar{a}} < \min(c_{k+2}^a, c_{k+2}^{\bar{b}})$
- color 2 when $\bar{a} < \bar{b}$ and $c_{k+2}^{\bar{a}} > \min(c_{k+2}^a, c_{k+2}^{\bar{b}})$
- color 3 when $\bar{b} < \bar{a}, c_2^a < c_2^b, c_2^{\bar{b}} < c_2^{\bar{a}}$ and $c_{k+2}^{\bar{b}} < \min(c_{k+2}^a, c_{k+2}^{\bar{a}})$
- color 4 when $\bar{b} < \bar{a}, c_2^a < c_2^b, c_2^{\bar{b}} < c_2^{\bar{a}}$ and $c_{k+2}^{\bar{b}} > \min(c_{k+2}^a, c_{k+2}^{\bar{a}})$
- color 5 when $\bar{b} < \bar{a}, c_2^a < c_2^b$ and $c_2^{\bar{a}} < c_2^{\bar{b}}$
- color 6 when $\bar{b} < \bar{a}, c_2^b < c_2^a$ and $c_2^{\bar{b}} < c_2^{\bar{a}}$
- color 7 when $\bar{b} < \bar{a}, c_2^b < c_2^a$ and $c_2^{\bar{a}} < c_2^{\bar{b}}$

If $\{\{a, \bar{a} : a < \bar{a}\}, \{b, \bar{b} : b < \bar{b}\} : a < b\}$ has the 7th color then $c_1^a < c_1^b < c_1^{\bar{b}} < c_1^{\bar{a}}$ because $a < b < \bar{b} < \bar{a}$ and $(c_1^i)_i$ is increasing. $c_2^{\bar{b}} < c_2^b$ because $\{b, \bar{b} : b < \bar{b}\} \in I$. Therefore, $c_2^{\bar{a}} < c_2^{\bar{b}} < c_2^b < c_2^a$ and $P_a, P_b, P_{\bar{b}}, P_{\bar{a}}$ are in contradiction with Lemma 27. Hence, the 7th color is not used and every edge of K_J has one of the first six colors.

According to Ramsey's theorem [17, 20] there exists n_0 such that if $|J| \geq n_0$ then K_J contains a monochromatic subgraph K_S , $S = \{\{a, \bar{a} : a < \bar{a}\}, \{b, \bar{b} : b < \bar{b}\}, \{c, \bar{c} : c < \bar{c}\}, \{d, \bar{d} : d < \bar{d}\} : a < b < c < d\}$.

If K_S has color 1 then $c_{k+2}^{\bar{a}} < c_{k+2}^{\bar{b}} < c_{k+2}^{\bar{c}} < c_{k+2}^{\bar{d}}$, $\bar{a} < \bar{b} < \bar{c} < \bar{d}$ and $c_1^{\bar{a}} < c_1^{\bar{b}} < c_1^{\bar{c}} < c_1^{\bar{d}}$ (because $(c_1^i)_i$ is increasing). This is in contradiction with Corollary 5.

If K_S has color 2 then we have

$$\begin{aligned} a < b < \bar{b}, a < \bar{a} < \bar{b}, (c_1^i)_i \text{ increasing} &\Rightarrow c_1^a < c_1^b < c_1^{\bar{b}}, c_1^a < c_1^{\bar{a}} < c_1^{\bar{b}} \\ \{a, \bar{a} : a < \bar{a}\}, \{b, \bar{b} : b < \bar{b}\} \in I &\Rightarrow c_2^{\bar{a}} < c_2^a, c_2^{\bar{b}} < c_2^b \\ c_1^b < c_1^{\bar{b}}, c_2^{\bar{b}} < c_2^b &\Rightarrow c_l^{\bar{b}} < c_l^b, l \in \{2, \dots, k+1\} \text{ by Lemma 23} \\ c_1^a < c_1^{\bar{a}}, c_2^{\bar{a}} < c_2^a &\Rightarrow c_l^{\bar{a}} < c_l^a, l \in \{k+3, \dots, 2k+1\} \cup \{1\} \text{ by Lemma 23} \end{aligned}$$

We can see that $c_1^a < c_1^b$ and $c_l^{\bar{b}} < c_l^b, l \in \{2, \dots, k+1\}$. Hence, $c_l^b > \min(c_l^a, c_l^{\bar{b}})$ for $l \in \{1, \dots, k+1\}$. Similarly, $c_l^a < c_l^{\bar{a}}, l \in \{k+3, \dots, 2k+1\} \cup \{1\}$ and

$c_{k+2}^{\bar{a}} > \min(c_{k+2}^a, c_{k+2}^{\bar{b}})$. Therefore, $c_l^{\bar{a}} > \min(c_l^a, c_l^{\bar{b}})$ for $l \in \{k+2, \dots, 2k+1\} \cup \{1\}$. If $c_{k+1}^{\bar{a}} > \min(c_{k+1}^a, c_{k+1}^{\bar{b}})$ then P_a cannot see $P_{\bar{b}}$ according to Lemma 22. It must be $c_{k+1}^{\bar{a}} < \min(c_{k+1}^a, c_{k+1}^{\bar{b}})$, namely $c_{k+1}^{\bar{a}} < c_{k+1}^{\bar{b}}$. The same argument shows that also $c_{k+1}^{\bar{b}} < c_{k+1}^{\bar{c}} < c_{k+1}^{\bar{d}}$. On the other hand, $c_1^{\bar{a}} < c_1^{\bar{b}} < c_1^{\bar{c}} < c_1^{\bar{d}}$ (because $\bar{a} < \bar{b} < \bar{c} < \bar{d}$) which is in contradiction with Corollary 5.

If K_S has color 3 then $c_{k+2}^{\bar{d}} < c_{k+2}^{\bar{c}} < c_{k+2}^{\bar{b}} < c_{k+2}^{\bar{a}}$, $\bar{d} < \bar{c} < \bar{b} < \bar{a}$ and $c_1^{\bar{d}} < c_1^{\bar{c}} < c_1^{\bar{b}} < c_1^{\bar{a}}$ (because $(c_1^i)_i$ is increasing) and we have a contradiction again.

If K_S has color 4 then we proceed in a similar way as with the second color. We have

$$\begin{aligned} c_2^a &< c_2^b, c_2^{\bar{b}} < c_2^{\bar{a}} \\ a &< b < \bar{b} < \bar{a}, (c_1^i)_i \text{ increasing} \Rightarrow c_1^a < c_1^b < c_1^{\bar{b}} < c_1^{\bar{a}} \\ \{a, \bar{a} : a < \bar{a}\} &\in I \Rightarrow c_2^{\bar{a}} < c_2^a \\ c_1^b &< c_1^{\bar{a}}, c_2^{\bar{a}} < c_2^a < c_2^b \Rightarrow c_l^{\bar{a}} < c_l^b, l \in \{2, \dots, k+1\} \text{ by Lemma 23} \\ c_1^a &< c_1^{\bar{b}}, c_2^{\bar{b}} < c_2^{\bar{a}} < c_2^a \Rightarrow c_l^a < c_l^{\bar{b}}, l \in \{k+3, \dots, 2k+1\} \cup \{1\} \text{ by Lemma 23} \end{aligned}$$

We can see that $c_1^a < c_1^b$ and $c_l^{\bar{a}} < c_l^b, l \in \{2, \dots, k+1\}$. Hence, $c_l^b > \min(c_l^a, c_l^{\bar{a}})$ for $l \in \{1, \dots, k+1\}$. Similarly, $c_l^a < c_l^{\bar{b}}, l \in \{k+3, \dots, 2k+1\} \cup \{1\}$ and $c_{k+2}^{\bar{b}} > \min(c_{k+2}^a, c_{k+2}^{\bar{a}})$. Therefore, $c_l^{\bar{b}} > \min(c_l^a, c_l^{\bar{a}})$ for $l \in \{k+2, \dots, 2k+1\} \cup \{1\}$. If $c_{k+1}^{\bar{b}} > \min(c_{k+1}^a, c_{k+1}^{\bar{a}})$ then P_a cannot see $P_{\bar{a}}$ according to Lemma 22. It must be $c_{k+1}^{\bar{b}} < \min(c_{k+1}^a, c_{k+1}^{\bar{a}})$, namely $c_{k+1}^{\bar{b}} < c_{k+1}^{\bar{a}}$. The same argument shows that also $c_{k+1}^{\bar{d}} < c_{k+1}^{\bar{c}} < c_{k+1}^{\bar{b}}$. On the other hand, $c_1^{\bar{d}} < c_1^{\bar{c}} < c_1^{\bar{b}} < c_1^{\bar{a}}$ (because $\bar{d} < \bar{c} < \bar{b} < \bar{a}$) which is in contradiction with Corollary 5.

If K_S has color 5 then $c_2^{\bar{a}} < c_2^{\bar{b}} < c_2^{\bar{c}} < c_2^{\bar{d}}$, $\bar{d} < \bar{c} < \bar{b} < \bar{a}$ and $c_1^{\bar{d}} < c_1^{\bar{c}} < c_1^{\bar{b}} < c_1^{\bar{a}}$ (because $(c_1^i)_i$ is increasing). This is in contradiction with Lemma 27.

If K_S has color 6 then $c_2^d < c_2^c < c_2^b < c_2^a$, $a < b < c < d$ and $c_1^a < c_1^b < c_1^c < c_1^d$ (because $(c_1^i)_i$ is increasing) and we have a contradiction with Lemma 27 again.

We can see that K_J cannot contain a monochromatic subgraph K_S . Therefore, $|J| \leq n_0 - 1$, i.e., I doesn't contain n_0 pairwise disjoint pairs.

Let $J_{max} \subseteq I$ be a maximal subset of pairwise disjoint pairs. We know that $|\bigcup J_{max}| = 2|J_{max}| \leq 2(n_0 - 1)$. For any $A \in I$ there exists $B \in J_{max}$ such that $A \cap B \neq \emptyset$. Hence, the sequence $(c_2^i)_{i \in \{1, \dots, m\} \cup J_{max}}$ is increasing.

We can repeat this proof with c_2, c_3, \dots, c_k subsequently and show that there is a set J' such that $|J'| \leq 2(n_0 - 1)k$ and $(c_{k+1}^i)_{i \in \{1, \dots, m\} \setminus J'}$ is increasing. The sequence $(c_1^i)_{i \in \{1, \dots, m\} \setminus J'}$ is also increasing. Therefore, its length is less than 4 by Corollary 5, i.e., $4 > |\{1, \dots, m\} \setminus J'| \geq m - 2(n_0 - 1)k$. \square

Lemma 28 allows us to prove an analogy of Theorem 7 for regular $(2k+1)$ -gons.

Theorem 8. *There exists $c > 0$ such that if $\{P_i, i = 1, \dots, m\}$ is a 3D visibility drawing of a complete graph K_m by equal regular n -gons (where $n = 2k + 1$) then $m \leq cn^4$.*

Proof. The proof is the same as the proof of Theorem 7 (using Lemma 28 instead of Lemma 26). \square

If we combine Theorem 7 and Theorem 8 then we obtain the following result.

Theorem 9. *If $s(n)$ is the maximum size of a complete graph with a 3D visibility drawing by equal regular n -gons then $s(n) = O(n^4)$.*

Proof. Theorem 7 if n is even and Theorem 8 if n is odd. \square

4.2 Lower Bounds

The previous section presents several upper bounds on the maximum size of a complete graph with a 3D visibility drawing by equal regular polygons. We concentrate on lower bounds in this section. The following definition helps us to describe the construction of the drawings used in proofs of our lower bounds.

Definition 14. *Let P_1 (resp. P_2) be a regular n -gon with the polygon coordinates $(c_i^1)_{i=1}^n$ (resp. $(c_i^2)_{i=1}^n$). We say that P_1 has an (r, s) -relation to P_2*

- *if $1 \leq r \leq s \leq n$ and*
 - $c_i^1 < c_i^2$ *for* $i \in \{r, \dots, s\}$ *and*
 - $c_i^1 > c_i^2$ *otherwise*
- or*
- *if $1 \leq s < r \leq n$ and*
 - $c_i^1 < c_i^2$ *for* $i \in \{1, \dots, s\} \cup \{r, \dots, n\}$
 - $c_i^1 > c_i^2$ *otherwise.*

If P_1 has an (r, s) -relation to P_2 then we write $P_1 \xrightarrow{(r,s)} P_2$. \square

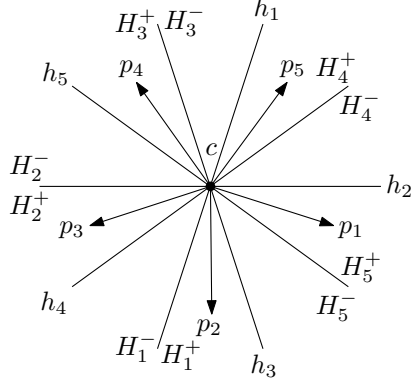


Figure 24. The definition of the lines h_i and the half-planes H_i for drawings by regular pentagons

Lemma 23 shows that there are only n possible relations between two polygons when n is even and there are only $2n$ possible relations when n is odd. There are only $(1, k), (2, k+1), \dots, (n, k-1)$ relations for $n = 2k$ and $(1, k), (1, k+1), (2, k+1), (2, k+2), \dots, (n, k)$ relations for $n = 2k+1$.

Let's remind that we use a coordinate system that has an origin c and n axes p_1, p_2, \dots, p_n . Let h_i be a line such that $c \in h_i$ and $h_i \perp p_i$. The line h_i divides the plane into half-planes H_i^+ and H_i^- . Let H_i^+ be the half-plane in the direction of the i -th axis, i.e., H_i^+ is the half-plane that contains the half-line p_i , see Figure 24.

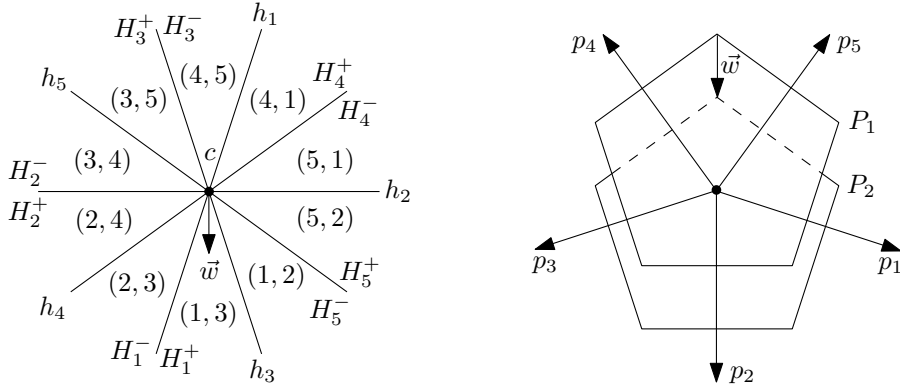


Figure 25. The correspondence between shift vectors and (r, s) -relations

Let P_1 (resp. P_2) be a regular n -gon with the polygon coordinates $(c_i^1)_{i=1}^n$ (resp. $(c_i^2)_{i=1}^n$) and \vec{w} be the shift vector between P_1 and P_2 , i.e., $P_2 = P_1 + \vec{w}$. We have $c_i^2 > c_i^1$ (resp. $c_i^2 < c_i^1$) if the vector \vec{w} points into H_i^+ (resp. H_i^-).

Therefore, the polygon P_1 has an (r, s) -relation to P_2 , $r \leq s$ if and only if $\vec{w} \in H_i^+$ for $i \in \{r, \dots, s\}$ and $\vec{w} \in H_i^-$ for $i \notin \{r, \dots, s\}$, see Figure 25. There is an analogous condition for (r, s) -relations with $r > s$.

In other words, the lines h_i , $i \in \{1, \dots, n\}$ divide the plane into sections that represent individual relations between polygons. The polygon P_1 has an (r, s) -relation to P_2 if the shift vector \vec{w} points into the corresponding section.

The relation between polygons is given by the direction of the shift vector only. Therefore, a ‘compression’ of the centers of polygons doesn’t change the relations between polygons.

Lemma 29. *Let P_i , $i \in \{1, \dots, m\}$ be a regular polygon with the center (x_i, y_i) . If P'_i , $i \in \{1, \dots, m\}$ is a regular polygon with the center (cx_i, cy_i) (where $c > 0$ independent of i) then the relation of P_i to P_j is the same as the relation of P'_i to P'_j , $i, j \in \{1, \dots, m\}$, $i \neq j$.*

Proof. The relation of the polygon P_i to the polygon P_j is given by the direction of the vector $(x_j - x_i, y_j - y_i)$. Similarly, the relation of the polygon P'_i to the polygon P'_j is given by the direction of the vector $(cx_j - cx_i, cy_j - cy_i)$. The directions of the vectors $(x_j - x_i, y_j - y_i)$ and $(cx_j - cx_i, cy_j - cy_i)$ are the same. Therefore, $P_i \xrightarrow{(r,s)} P_j$ if and only if $P'_i \xrightarrow{(r,s)} P'_j$. \square

We present several 3D visibility drawings of complete graphs in this section. It happens frequently (during construction of these drawings) that we need to add a polygon that has a specified relation to all polygons present already in the drawing. The following lemma shows that a polygon with such relations always exists.

Lemma 30. *Let $(P_i)_{i=1}^m$ be a short-distance set of regular n -gons. Let (r, s) be a relation that can occur between two n -gons. There exists a short-distance set $(P'_i)_{i=0}^m$ such that*

- *the relation of P_i to P_j is the same as the relation of P'_i to P'_j , $i, j \in \{1, \dots, m\}$, $i \neq j$ and*
- *P'_0 has the (r, s) -relation to P'_i , $i \in \{1, \dots, m\}$.*

Proof. We prove this lemma for n even. The proof is similar for n odd.

Let c_i be the center of the regular n -gon P_i . Let \overline{H}_r^+ be a shifted copy of H_r^+ such that it contains all points of $S = \{c_i, i \in \{1, \dots, m\}\}$ and at least one of these points lies on its boundary line \overline{h}_r . Let \overline{H}_{r-1}^- be a copy of H_{r-1}^- shifted in the same way, i.e., $S \subset \overline{H}_{r-1}^-$ and $S \cap \overline{h}_{r-1} \neq \emptyset$. Finally, let O be

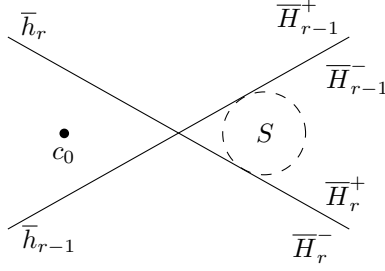


Figure 26

the interior of $\overline{H}_r^- \cap \overline{H}_{r-1}^+$, c_0 be a point in O and P_0 be the regular n -gon with the center c_0 , see Figure 26.

If $(c_j^i)_{j=1}^n$ are the polygon coordinates of P_i then $c_r^0 < c_r^i$ and $c_{r-1}^0 > c_{r-1}^i$, $i \in \{1, \dots, m\}$ clearly. Therefore, P_0 has the (r, s) -relation to P_i , $i \in \{1, \dots, m\}$ according to Lemma 23.

It may happen that c_0 is far from S , i.e., that $(P_i)_{i=0}^m$ don't form a short-distance set. We can use Lemma 29 in this case to move the centers of polygons close to each other sufficiently. Let $\{P'_i, i \in \{0, \dots, m\}\}$ be the resulting short-distance set of polygons. Obviously, $P'_0 \xrightarrow{(r,s)} P'_i$, $i \in \{1, \dots, m\}$ and $P_i \xrightarrow{(x,y)} P_j$ if and only if $P'_i \xrightarrow{(x,y)} P'_j$, $i, j \in \{1, \dots, m\}$, $i \neq j$. \square

Lemma 30 holds also in the opposite direction, i.e., when we request a polygon P'_0 such that all polygons $(P'_i)_{i=1}^m$ have an (r, s) -relation to P'_0 . It is a consequence of the fact that $P'_i \xrightarrow{(r,s)} P'_0$ if and only if $P'_0 \xrightarrow{(s+1, r-1)} P'_i$.

The next lemma allows us to verify the visibility among polygons in one special polygon configuration. We use this lemma several times in the proofs of completeness of the graphs represented by 3D visibility drawings described further in this section.

Lemma 31. *Let $\{P_i, i = 1, \dots, m\}$ be a short-distance set of n -gons. Let $(c_j^i)_{j=1}^n$ be the polygon coordinates of P_i . If $c_1^m > c_1^1 > c_1^2 > \dots > c_1^{m-1}$ then the polygon P_m can see all polygons P_1, P_2, \dots, P_{m-1} .*

Proof. We have $c_1^m > c_1^k$ and $c_1^k > c_1^i$, i.e., $c_1^i < \min(c_1^k, c_1^m)$, for $1 \leq k < i < m$. Therefore, the polygon P_m can see the polygon P_k according to Lemma 22. \square

4.2.1 Regular 2k-gons

The following lemma shows that the upper bound given by Lemma 26 is tight.

Lemma 32. *There exists a short-distance set $\{P_i, i = 1, \dots, k+1\}$ of equal regular $2k$ -gons such that $(P_i)_{i=1}^{k+1}$ is a 3D visibility drawing of a complete graph K_{k+1} and $(c_1^i)_{i=1}^{k+1}$ is a monotone sequence (where $(c_j^i)_{j=1}^{2k}$ are polygon coordinates of P_i).*

Proof. We start the construction of the required short-distance set with the polygon P_1 and add polygons P_2, P_3, \dots, P_{k+1} subsequently such that the polygons P_1, P_2, \dots, P_{i-1} (the polygons added already) have the $(i, i+k-1)$ -relation to P_i , i.e.,

$$P_1, \dots, P_{i-1} \xrightarrow{(i, i+k-1)} P_i, i \in \{2, \dots, k+1\}.$$

A short distance set with these relations exists by Lemma 30. We claim that this set forms a 3D visibility drawing of a complete graph K_{k+1} .

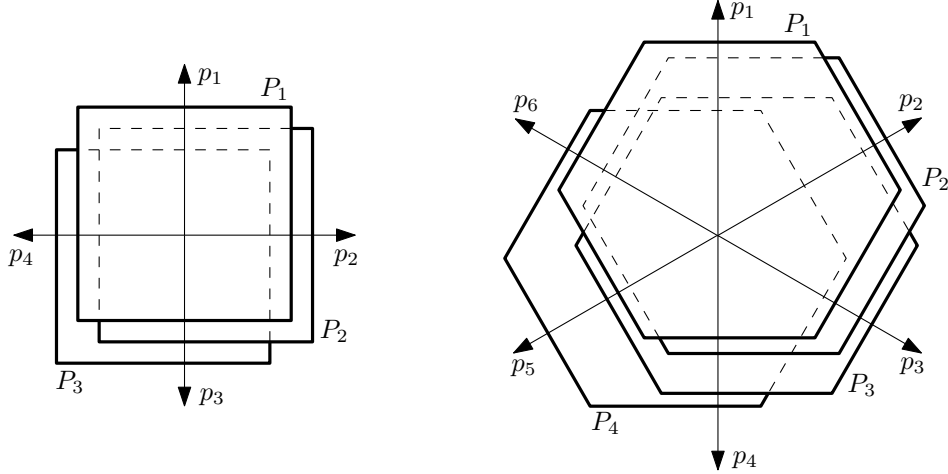


Figure 27. A 3D visibility drawings of K_3 by squares and K_4 by regular hexagons based on Lemma 32

We have

$$P_{i-1} \xrightarrow{(i, i+k-1)} P_i \Rightarrow c_x^{i-1} > c_x^i, x \in \{i+k, \dots, 2k\}, i \in \{2, \dots, k+1\}, \quad (*)$$

$$P_1 \xrightarrow{(i, i+k-1)} P_i \Rightarrow c_{i+k-1}^i > c_{i+k-1}^1, i \in \{2, \dots, k+1\}. \quad (**)$$

If we fix $x \in \{k+1, \dots, 2k\}$ then $c_x^{i-1} > c_x^i$ for $i \in \{2, \dots, x-k\}$ by (*). If we set $y = x - k + 1$ then we obtain $c_{y+k-1}^{i-1} > c_{y+k-1}^i$ for $y \in \{2, \dots, k+1\}$ and $i \in \{2, \dots, y-1\}$.

We can combine the last inequality with (**) to get

$$c_{y+k-1}^y > c_{y+k-1}^1 > c_{y+k-1}^2 > \dots > c_{y+k-1}^{y-1}, y \in \{2, \dots, k+1\}.$$

Therefore, the polygon P_y can see the polygons P_1, \dots, P_{y-1} by Lemma 31. In other words, $(P_i)_{i=1}^{k+1}$ form a 3D visibility drawing of a complete graph K_{k+1} .

Finally, we have

$$P_{i-1} \xrightarrow{(i,i+k-1)} P_i \Rightarrow c_1^{i-1} > c_1^i, i \in \{2, \dots, k+1\}.$$

Hence, the sequence $(c_1^i)_{i=1}^{k+1}$ is monotone. \square

We can combine two drawings provided by Lemma 32 to obtain a 3D visibility drawing of K_{2k+2} .

Lemma 33. *There exists a 3D visibility drawing by equal regular $2k$ -gons of a complete graph K_{2k+2} .*

Proof. Let $(P_i)_{i=1}^{k+1}$ be the short-distance set from Lemma 32 and $(c_j^i)_{j=1}^{2k}$ be the polygon coordinates of P_i .

We know that the sequence $(c_1^i)_{i=1}^{k+1}$ is monotone. This ensures that the first sides of polygons $(P_i)_i$ are visible from one side of the drawing. The first sides form a stair-like configuration with all stairs visible from above or all stairs visible from below (with respect to the z -axis).

Let $(P'_i)_{i=1}^{k+1}$ be a copy of $(P_i)_{i=1}^{k+1}$ rotated by π/k and turned upside down (w.r. to the z -axis). The polygons $(P'_i)_{i=1}^{k+1}$ also form a 3D visibility drawing of K_{k+1} . Moreover, if $(\bar{c}_j^i)_{j=1}^{2k}$ are polygon coordinates of P'_i then the sequence $(\bar{c}_2^i)_{i=1}^{k+1}$ is monotone.

If the original stair configuration in $(P_i)_{i=1}^{k+1}$ is turned upward (resp. downward) then the corresponding stair configuration in $(P'_i)_{i=1}^{k+1}$ is turned downward (resp. upward).

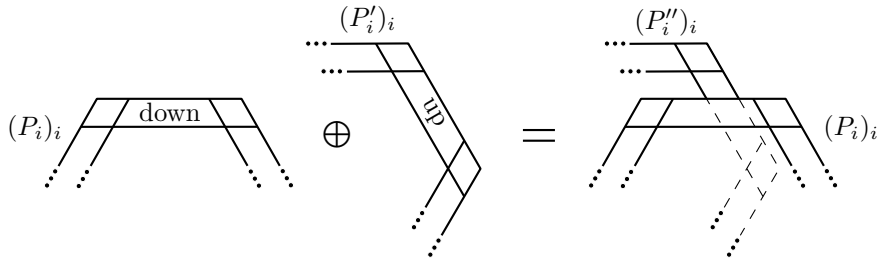


Figure 28. The construction of a 3D visibility drawing of K_{2k+2} from two drawings of K_{k+1}

Let $(P''_i)_{i=1}^{k+1}$ be a copy of $(P'_i)_{i=1}^{k+1}$ shifted such that the stair configurations in $(P_i)_i$ and $(P''_i)_i$ cross and face each other, i.e., all the first sides of $(P_i)_i$ cross all the second sides of $(P''_i)_i$, see Figure 28.

We claim that $\{P_i, i \in \{1, \dots, k+1\}\} \cup \{P''_i, i \in \{1, \dots, k+1\}\}$ form a 3D visibility drawing of K_{2k+2} . The polygons P_i and P_j (resp. P''_i and P''_j), $i, j \in \{1, \dots, k+1\}$, $i \neq j$ can see each other because $(P_i)_i$ (resp. $(P''_i)_i$) is a 3D visibility drawing of K_{k+1} . The polygons P_i and P''_j , $i, j \in \{1, \dots, k+1\}$ can see each other in the intersection of the stair configurations. \square

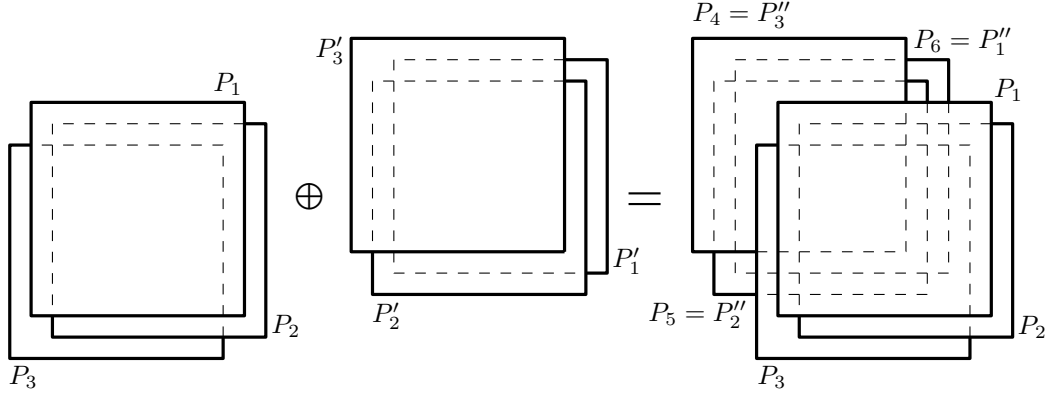


Figure 29. A 3D visibility drawing of K_6 by squares based on Lemma 33

We can use the ideas from the proof of Lemma 33 in a more inventive way to obtain a drawing of a complete graph with one more vertex.

Theorem 10. *There exists a 3D visibility drawing by equal regular $2k$ -gons of a complete graph K_{2k+3} , $k \geq 3$.*

Proof. We construct the 3D visibility drawing of K_{2k+3} in a similar way as we constructed the drawing of K_{2k+2} in the previous proof.

Let $(P_i)_{i=1}^k$ be the polygons from the proof of Lemma 32 and $(c_j^i)_{j=1}^{2k}$ be polygon coordinates of P_i . We know that the sequence $(c_1^i)_{i=1}^k$ is decreasing. The proof of Lemma 32 shows that also the sequence $(c_{2k}^i)_{i=1}^k$ is decreasing (note that we omitted the polygon P_{k+1} from our consideration).

Let $(P'_i)_{i=1}^k$ be a copy of $(P_i)_{i=1}^k$ rotated by π/k and turned upside down (with respect to the z -axis). The first and the second sides of $(P'_i)_i$ form stair-like configurations oriented in the opposite direction (w.r. to the z -axis) than the corresponding stair configurations formed by the $2k$ -th and the first sides of $(P_i)_{i=1}^k$.

Finally, let P''_1, P''_2, P''_3 be regular $2k$ -gons such that $P''_1 \xrightarrow{(1,k)} P''_2$ and $P''_1, P''_2 \xrightarrow{(2,k+1)} P''_3$. Polygons with these properties exist by Lemma 30. Let $(\bar{c}_j^i)_{j=1}^{2k}$ be polygons coordinates of P''_i . Clearly, $\bar{c}_k^1 < \bar{c}_k^2 < \bar{c}_k^3$ and $\bar{c}_{k+2}^1 >$

$\bar{c}_{k+2}^2 > \bar{c}_{k+2}^3$. These polygons form a 3D visibility drawing of K_3 (P_1'' can see P_3 according to Lemma 31 because $\bar{c}_{k+1}^3 > \bar{c}_{k+1}^1 > \bar{c}_{k+1}^2$).

We shift $(P_i')_{i=1}^k$ such that the stairs formed by the second sides of $(P_i')_{i=1}^k$ and the stairs formed by the $2k$ -th sides of $(P_i)_{i=1}^k$ cross and face each other.

We put the polygons $(P_i'')_{i=1}^3$ between (w.r. to the z -axis) the polygons $(P_i)_{i=1}^k$ and $(P_i')_{i=1}^k$. We shift the polygons $(P_i'')_{i=1}^3$ such that the stairs formed by the k -th sides of $(P_i'')_{i=1}^3$ and the stairs formed by the first sides of $(P_i)_{i=1}^k$ cross and face each other. Moreover, we shift the polygons $(P_i'')_{i=1}^3$ such that also the stairs formed by the $(k+2)$ -nd sides of $(P_i'')_{i=1}^3$ and the stairs formed by the first sides of $(P_i')_{i=1}^k$ cross and face each other. Finally, the polygons $(P_i'')_{i=1}^3$ should be shifted such that the $(k+1)$ -st sides of $(P_i'')_{i=1}^3$ do not block the visibility among polygons $(P_i)_{i=1}^k$ and $(P_i')_{i=1}^k$, see Figure 30.

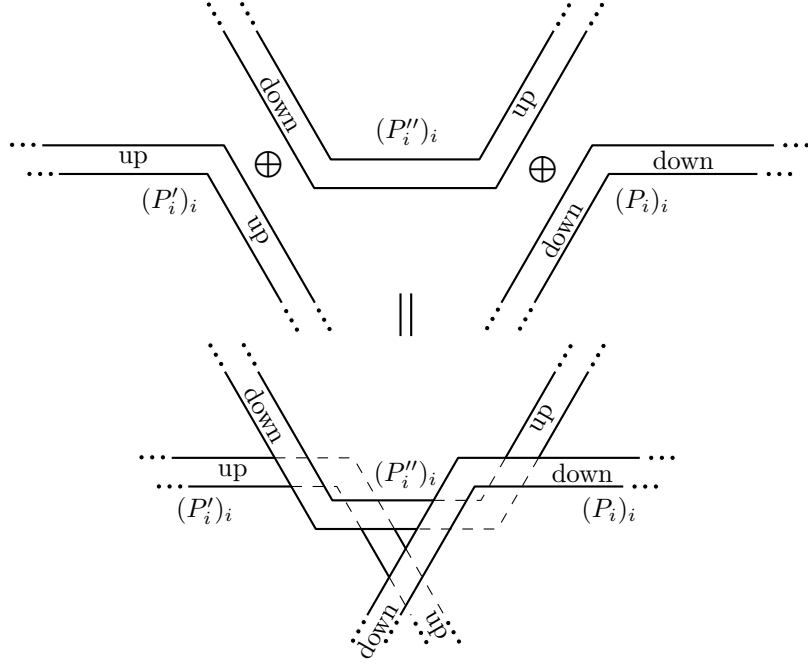


Figure 30. A sketch of the construction of the 3D visibility drawing according to the proof of Theorem 10

The resulting drawing is a 3D visibility drawing of K_{2k+3} . The polygons from the same group ($(P_i)_{i=1}^k$, $(P_i')_{i=1}^k$ or $(P_i'')_{i=1}^3$) see each other because the individual groups form 3D visibility drawings of complete graphs. The polygons from the different groups can see each other in the area where the corresponding stair configurations cross.

This construction works for $k \geq 3$ only because for $k = 2$ the second sides of $(P_i')_{i=1}^k$ and the 4th sides of $(P_i)_{i=1}^k$ are parallel, i.e., they cannot cross. \square

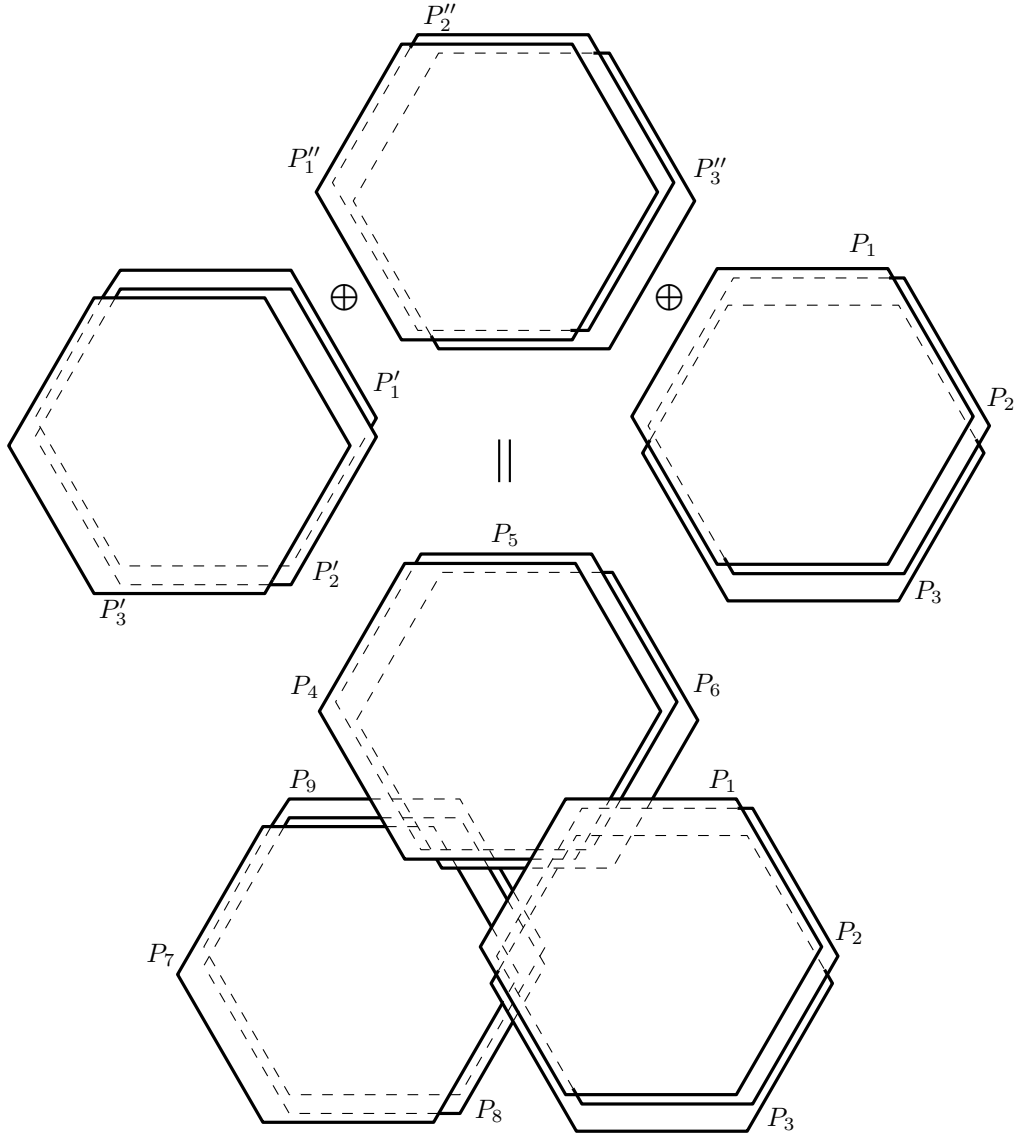


Figure 31. A 3D visibility drawing of K_9 by regular hexagons based on Theorem 10

The construction in the proof of Theorem 10 works for $k > 2$ only, but Fekete et al. [14] show that this theorem holds for $k = 2$ as well, i.e., they describe a 3D visibility drawing of K_7 by squares, see Figure 32.

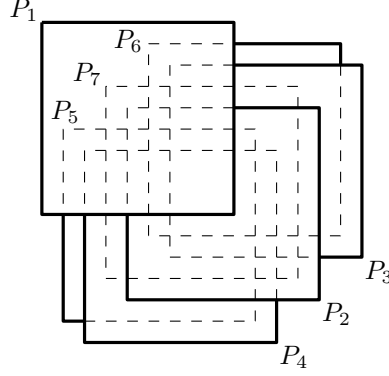


Figure 32. A 3D visibility drawing of K_7 by squares

Theorem 10 improves the best known lower bound on the maximum size of a complete graph with a 3D visibility drawing by regular n -gons (for n even). Štola [24] describes a drawing of K_{n+1} while Theorem 10 provides a drawing of K_{n+3} .

4.2.2 Regular $(2k+1)$ -gons

We have seen that drawings by $(2k+1)$ -gons can be more complex than drawings by $2k$ -gons. Therefore, we are able to construct drawings of larger complete graphs by $(2k+1)$ -gons than by $2k$ -gons.

Lemma 34. *There exists a short-distance set $\{P_i, i = 1, \dots, 2k+3\}$ of equal regular $(2k+1)$ -gons such that $(P_i)_{i=1}^{2k+3}$ is a 3D visibility drawing of a complete graph K_{2k+3} and $(c_{k+1}^i)_{i=1}^{2k+3}$ is a monotone sequence (where $(c_j^i)_{j=1}^{2k+1}$ are polygon coordinates of P_i).*

Proof. We construct a 3D visibility drawing of K_{2k+2} at first and we describe how to add the last polygon later.

We start our construction of the required short-distance set with the polygon P_{k+1} and add polygons $P_{k+2}, P_k, P_{k+3}, P_{k-1}, P_{k+4}, \dots, P_1, P_{2k+2}$ subsequently such that the added polygon has the same relation to all polygons already added:

$$P_i \xrightarrow{(k+2-i, 2k+1-i)} P_{i+1}, \dots, P_{2k+2-i}, i \in \{1, \dots, k\}, \quad (\text{R1})$$

$$P_{2k+3-i}, \dots, P_{i-1} \xrightarrow{(i-k-1, i-1)} P_i, i \in \{k+2, \dots, 2k+2\}. \quad (\text{R2})$$

A short-distance set with these relations exists by Lemma 30. We claim that this set forms a 3D visibility drawing of a complete graph K_{2k+2} .

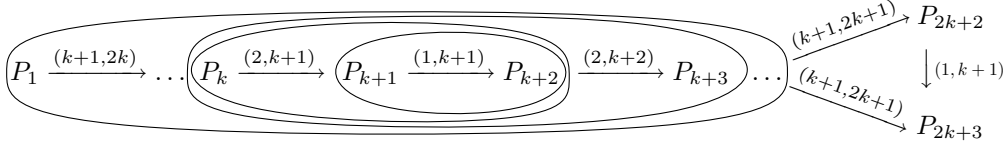


Figure 33. A schema of the relations among the polygons

We have

$$P_i \xrightarrow{(k+2-i, 2k+1-i)} P_{i+1} \Rightarrow c_x^i < c_x^{i+1}, x \in \{k+2-i, \dots, 2k+1-i\}, i \in \{1, \dots, k\},$$

$$P_{i-1} \xrightarrow{(i-k-1, i-1)} P_i \Rightarrow c_x^{i-1} < c_x^i, x \in \{i-k-1, \dots, i-1\}, i \in \{k+2, \dots, 2k+2\}.$$

If we fix $x \in \{1, \dots, k+1\}$ then $c_x^i < c_x^{i+1}$ for $i \in \{k+2-x, \dots, k\}$ and $c_x^{i-1} < c_x^i$ for $i \in \{k+2, \dots, x+k+1\}$. In other words,

$$c_x^{k+2-x} < c_x^{k+3-x} < \dots < c_x^{x+k+1}, x \in \{1, \dots, k+1\}. \quad (*)$$

Moreover, it is

$$P_i \xrightarrow{(k+2-i, 2k+1-i)} P_{2k+2-i} \Rightarrow c_{k+1-i}^i > c_{k+1-i}^{2k+2-i}, i \in \{1, \dots, k\}.$$

If we set $x = k+1-i$ then $c_x^{k+1-x} > c_x^{2k+2-x}$, $x \in \{1, \dots, k\}$. Lemma 31, the last inequality and (*) prove that the polygon P_{k+1-x} can see all polygons $P_{k+2-x}, \dots, P_{x+k+1}$, $x \in \{1, \dots, k\}$. If we set $i = k+1-x$ then the last sentence shows that P_i can see the polygons $P_{i+1}, \dots, P_{2k+2-i}$, i.e., if we add a polygon according to the rule (R1) then the new polygon can see all previously added polygons.

We proceed in an analogous way to show that the same holds for the rule (R2), too. We have

$$P_i \xrightarrow{(k+2-i, 2k+1-i)} P_{i+1} \Rightarrow c_x^i > c_x^{i+1}, x \in \{2k+2-i, \dots, 2k+1\}, i \in \{1, \dots, k\},$$

$$P_{i-1} \xrightarrow{(i-k-1, i-1)} P_i \Rightarrow c_x^{i-1} > c_x^i, x \in \{i, \dots, 2k+1\}, i \in \{k+2, \dots, 2k+2\}.$$

If we fix $x \in \{k+1, \dots, 2k+1\}$ then $c_x^i > c_x^{i+1}$ for $i \in \{2k+2-x, \dots, k\}$ and $c_x^{i-1} > c_x^i$ for $i \in \{k+2, \dots, x\}$. In other words,

$$c_x^{2k+2-x} > c_x^{2k+3-x} > \dots > c_x^x, x \in \{k+1, \dots, 2k+1\}. \quad (**)$$

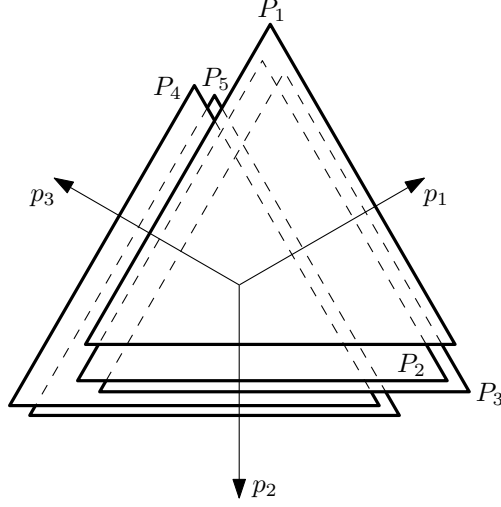


Figure 34. A 3D visibility drawing of K_5 by triangles based on Lemma 34

Moreover, it is

$$P_{2k+3-i} \xrightarrow{(i-k-1, i-1)} P_i \Rightarrow c_{i-1}^{2k+3-i} < c_{i-1}^i, i \in \{k+2, \dots, 2k+2\}.$$

If we set $x = i - 1$ then $c_x^{2k+2-x} < c_x^{x+1}$, $x \in \{k+1, \dots, 2k+1\}$. Lemma 31, the last inequality and $(**)$ prove that the polygon P_{x+1} can see all polygons P_{2k+2-x}, \dots, P_x , $x \in \{k+1, \dots, 2k+1\}$, i.e., if we add a polygon according to the rule (R2) then the new polygon can see all previously added polygons.

We start our construction with a single polygon and add subsequent polygons such that the added polygon can see all existing polygons. Hence, the resulting drawing is a drawing of a complete graph K_{2k+2} . Finally, the sequence $(c_{k+1}^i)_{i=1}^{2k+2}$ is increasing according to $(*)$.

It remains to add one more polygon. We add a polygon P_{2k+3} such that $P_{2k+2} \xrightarrow{(1, k+1)} P_{2k+3}$ and $P_1, \dots, P_{2k+1} \xrightarrow{(k+1, 2k+1)} P_{2k+3}$, i.e., the polygon P_{2k+3} has the same relation to polygons P_1, \dots, P_{2k+1} as the polygon P_{2k+2} .

We must show that it is possible to add P_{2k+3} in this way. The addition of the polygon P_{2k+2} is based on Lemma 30. The proof of this lemma shows that it is possible to place the center of P_{2k+2} anywhere in the set O (described in the proof) to keep the prescribed relation of P_{2k+2} to all other polygons. If we place the center of P_{2k+3} into O then the polygon P_{2k+3} has the same relation as P_{2k+2} to other polygons. Additionally, we know that the relation between two polygons is given by the direction of the shift vector between these polygons. Let \vec{w} be the vector that corresponds to the $(1, k+1)$ -relation. We place the center of P_{2k+3} into O in the direction of the vector \vec{w} from the

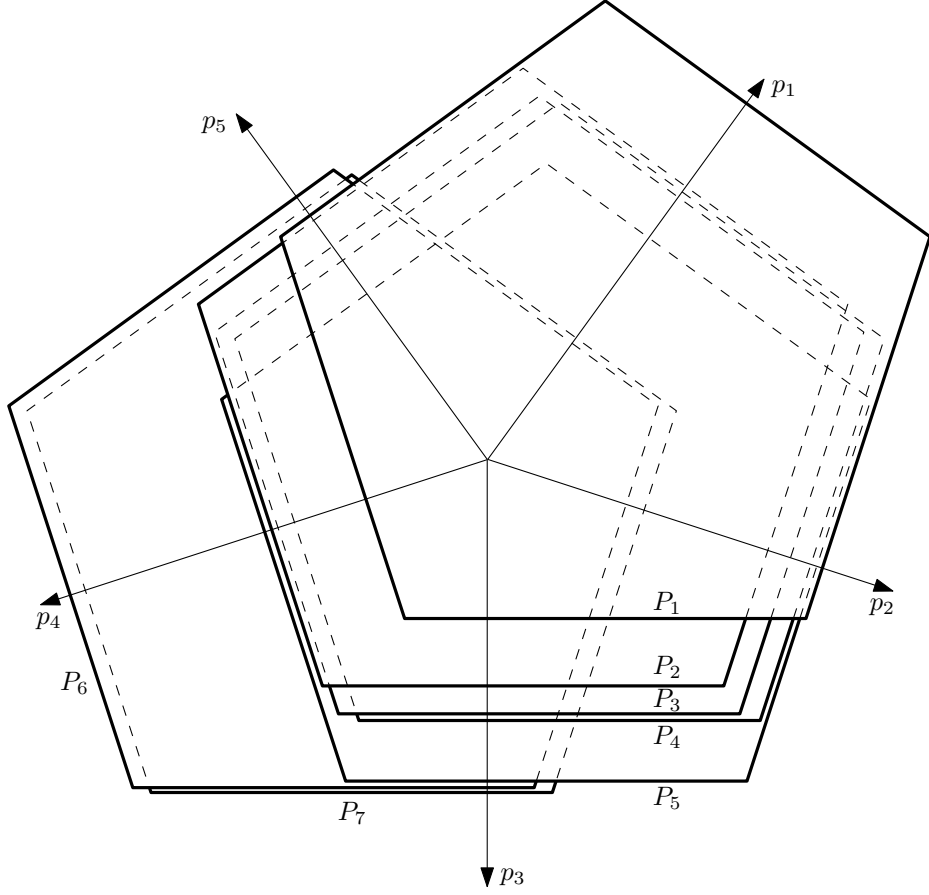


Figure 35. A 3D visibility drawing of K_7 by regular pentagons based on Lemma 34

center of P_{2k+2} . This is always possible because the set O is open (it is an interior of an intersection of two half-planes).

We have to show that P_{2k+3} can see all other polygons. We have $c_{2k+1}^1 > c_{2k+1}^2 > \dots > c_{2k+1}^{2k+1}$ according to $(**)$ and

$$P_1, \dots, P_{2k+1} \xrightarrow{(k+1, 2k+1)} P_{2k+3} \Rightarrow c_{2k+1}^1 < c_{2k+1}^{2k+3}, c_1^i > c_1^{2k+3}, i \in \{1, \dots, 2k+1\}$$

$$P_{2k+2} \xrightarrow{(1, k+1)} P_{2k+3} \Rightarrow c_1^{2k+2} < c_1^{2k+3}.$$

The polygon P_{2k+3} can see the polygons P_1, \dots, P_{2k+2} (in $\widehat{p_{2k+1}p_1}$) according to these inequalities and Lemma 22.

Finally, $c_{k+1}^{2k+2} < c_{k+1}^{2k+3}$ because the polygon P_{2k+2} has the $(1, k+1)$ -relation to P_{2k+3} . Therefore, the addition of the last polygon doesn't break the monotonicity of the sequence $(c_{k+1}^i)_{i=1}^{2k+3}$. \square

Two drawings provided by Lemma 34 can be combined to obtain a 3D visibility drawing of K_{4k+6} by regular $(2k+1)$ -gons.

Theorem 11. *There exists a 3D visibility drawing by equal regular $(2k+1)$ -gons of a complete graph K_{4k+6} .*

Proof. The proof of this lemma is the same as the proof of Lemma 33. The only difference is that we use the drawing provided by Lemma 34 instead of the drawing given by Lemma 32. \square

Theorem 11 improves the best known lower bound on the maximum size of a complete graph with a 3D visibility drawing by regular n -gons (for n odd). It provides a 3D visibility drawing of K_{2n+4} while Štola [24] gives a drawing of K_{n+1} only.

5 Related Results

This section contains several results indirectly related to the main topic of this thesis (i.e., 3D visibility drawings of complete graphs). We relax our requirements on the type of the drawing or the class of the represented graphs. Section 5.1 concentrates on complete graphs in drawings related to the 3D visibility drawing. The other sections deal with the visibility drawing of other classes of graphs, i.e., graphs that are not necessarily complete graphs.

5.1 Complete Graphs

5.1.1 Orthogonal Drawing

Let's remind the definition of a d -dimensional b -bend orthogonal drawing.

Definition 15. A d -dimensional b -bend orthogonal drawing of a graph G is a graph drawing where

- $v \in V(G)$ is represented by an axis-aligned box B_v
- $\forall v, w \in V(G), v \neq w \Rightarrow B_v \cap B_w = \emptyset$
- $\{v, w\} \in E(G)$ is represented by an axis-aligned polyline (with $b+1$ line segments) connecting points on the surface of B_v and B_w
- edges (with the exception of their endpoints) don't intersect vertices
- if $d > 2$ then the edges don't intersect other edges
- if $d = 2$ then the edges can have a finite number of intersections. \square

Theorem 5 allows us to improve the best known upper bound on the size of a complete graph with a 0-bend 3D orthogonal drawing.

Fekete and Meijer [16] show that no complete graph with more than 183 vertices has a 0-bend 3D orthogonal drawing. The following lemma can be distilled from their proof.

Lemma 35. [16] *If K_m is a complete graph with a 0-bend 3D orthogonal drawing then $m \leq 3k + 18$ where k is the maximum size of a complete graph with a 3D rectangle visibility drawing.*

We can combine this lemma with Theorem 5 to obtain the improved upper bound.

Theorem 12. *No complete graph with more than 168 vertices has a 0-bend 3D orthogonal drawing.*

Proof. The maximum size of a complete graph with a 3D rectangle visibility drawing is at most 50 according to Theorem 5. Therefore, the size of a complete graph with a 0-bend 3D orthogonal drawing is at most $3 \times 50 + 18 = 168$ by Lemma 35. \square

The largest complete graph that is known to have a 0-bend 3D orthogonal drawing is K_{56} by [16]. Fekete and Meijer [16] also studied 3D orthogonal drawings where vertices are represented by unit cubes. They show that K_8 has and K_{10} doesn't have such a drawing. Their paper provides a lot of information about any potential drawing of K_9 but they weren't able to construct a drawing of this graph or prove that it doesn't exist. They conjectured that K_9 doesn't have a 0-bend 3D orthogonal drawing by cubes.

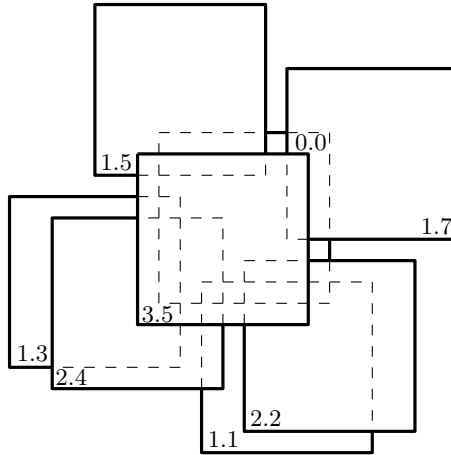


Figure 36. A 0-bend 3D orthogonal drawing of K_8 by unit cubes (the number in each cube is the z -coordinate of the bottom face of the cube)

The horizontal (resp. vertical) edges of any 0-bend 2D orthogonal drawing induce a planar subgraph. Therefore, every graph with a 0-bend 2D orthogonal drawing is a union of at most two planar graphs. Beineke [5] proved that K_9 is not a union of at most two planar graphs. Hence, the largest complete graph with a 0-bend 2D orthogonal drawing has at most 8 vertices. The matching upper bound (a 0-bend 2D orthogonal drawing of K_8) was given by Dean and Hutchinson [11], see Figure 37.

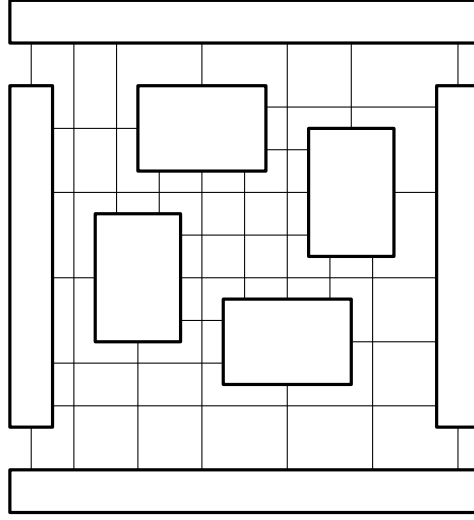


Figure 37. A 0-bend 2D orthogonal drawing of K_8

5.1.2 Limited Number of Shapes

Graphs that have a 3D rectangle visibility drawing or a 0-bend 3D orthogonal drawing appear in some 3D packing algorithms, see [32]. The number of types of objects being packed is usually limited. This is the reason why Fekete and Meijer [16] started to study drawings of complete graphs by rectangles/boxes with a limited number of shapes. They considered two ways in which two bodies can be considered equal:

- They can be made identical by translations only.
- They can be made identical by translations and rotations.

They say that two objects have the same *size* (resp. the same *shape*) if the first (resp. the second) condition holds. For example, an axis-aligned rectangle with the sides a and b (i.e., with the given shape) can have two sizes: $a \times b$ and $b \times a$.

It was shown by Fekete et al. [14] that K_8 does not have a 3D visibility drawing by unit squares. This implies an upper bound on the maximal size of a complete graph K_n with a 3D visibility drawing by rectangles of r different sizes. Any subset consisting of rectangles of the same size can be converted into a set of unit squares by scaling the coordinates appropriately. Therefore, $n \leq 7r$. The same argument shows that $n \leq 14s$ for drawings by rectangles of s different shapes. Moreover, $n \leq 50$ by Theorem 5.

shapes	min	max	sizes	min	max
1	12	14	1	7	7
2	18	28	2	12	14
3	20	42	3	18	21
4	20	50	4	20	28
5	20	50	5	20	35
6	22	50	6	20	42
\vdots	\vdots	\vdots	7	20	49
			8	20	50
			9	20	50
			10	20	50
			11	22	50
			\vdots	\vdots	\vdots

Table 3. Lower and upper bounds on the maximum size of a complete graph with a 3D visibility drawing by rectangles with a limited number of shapes or sizes

shapes	min	max	sizes	min	max	sizes	min	max
1	30	54	1	8	9	14	52	126
2	44	108	2	14	18	15	52	135
3	50	162	3	20	27	16	52	144
4	52	168	4	25	36	17	52	153
5	52	168	5	31	45	18	53	162
6	56	168	6	36	54	19	54	168
\vdots	\vdots	\vdots	7	42	63	\vdots	\vdots	\vdots
			8	46	72			
			9	48	81	24	54	168
			10	50	90	25	55	168
			11	51	99	26	56	168
			12	52	108	\vdots	\vdots	\vdots
			13	52	117			

Table 4. Lower and upper bounds on the maximum size of a complete graph with a 0-bend 3D orthogonal drawing by boxes with a limited number of shapes or sizes

As for the lower bounds, Fekete and Meijer [16] provided several drawings of complete graphs with a limited number of sizes and shapes. Table 3 summarizes these results.

We can use analogous arguments for the 0-bend 3D orthogonal drawing. We know that K_{10} doesn't have a 0-bend 3D orthogonal drawing by unit cubes according to Fekete and Meijer [16]. Therefore, if K_n is a complete graph with a 0-bend 3D orthogonal drawing by boxes of r different sizes then $n \leq 9r$. An axis-aligned box of a certain shape can have 6 different sizes. Hence, $n \leq 54s$ for a drawing by boxes of s different shapes. Moreover, $n \leq 168$ by Theorem 12.

Several lower bounds were given by Fekete and Meijer [16]. Table 4 provides a summary of these results.

5.2 Bipartite Graphs

5.2.1 0-bend 2D Orthogonal Drawing

We say that a graph G has a *thickness* t if t is the minimum number of planar subgraphs G_i of G such that G is a union of $(G_i)_i$, i.e., $E(G) = \bigcup_{i=1}^t E(G_i)$.

The thickness of a graph with a 0-bend 2D orthogonal drawing is at most two because the vertical (resp. horizontal) edges of the drawing induce a planar graph (a bar-visibility graph).

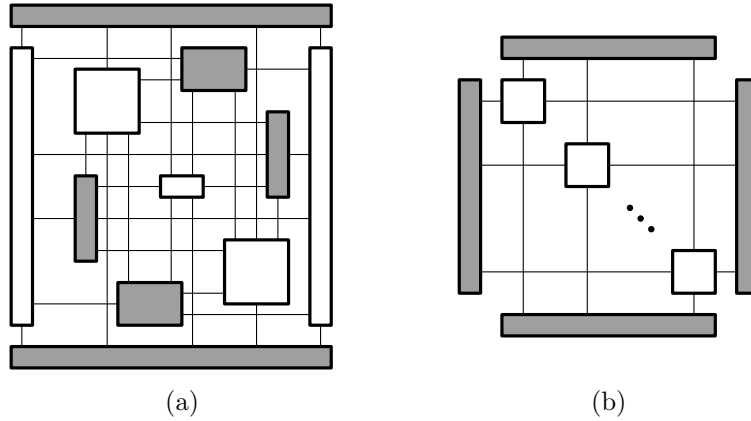


Figure 38. A 0-bend 2D orthogonal drawing of (a) $K_{5,6}$ (by Wood [31]) and (b) $K_{4,n}$

The bipartite graphs $K_{4,n}$ and $K_{5,6}$ have a 0-bend 2D orthogonal drawing, see Figure 38. Even though the graphs $K_{5,n}$, $n \in \{7, \dots, 12\}$ and $K_{6,n}$, $n \in \{6, 7, 8\}$ have thickness two according to Beineke [5] it is unknown if

any of these graphs admits a 0-bend 2D orthogonal drawing. Wood [31] conjectured that $K_{5,7}$ and $K_{6,6}$ do not admit such drawings.

5.2.2 Representation Index of Bipartite Graphs

The representation index of all complete bipartite graphs is known. We have $RI(K_{1,2}) = RI(K_{2,2}) = 1 + 1/2$ and $RI(K_{2,n}) = 2$, $n > 2$ by Lemma 14 and Lemma 15.

The complete bipartite graphs $K_{3,n}$, $n \geq 3$ are not planar. Therefore, $RI(K_{3,n}) > 2$ by Lemma 14. Cobos et al. [9] prove that $RI(K_{3,n}) \leq 2 + 1/2$, see Figure 39a. Hence, $RI(K_{3,n}) = 2 + 1/2$.

Cobos et al. [9] also show that $RI(K_{4,4}) > 2 + 1/2$. Hence, $RI(K_{m,n}) = 3$, $m \geq n \geq 4$ because every bipartite graph has a 3D rectangle visibility drawing, see Figure 39b.

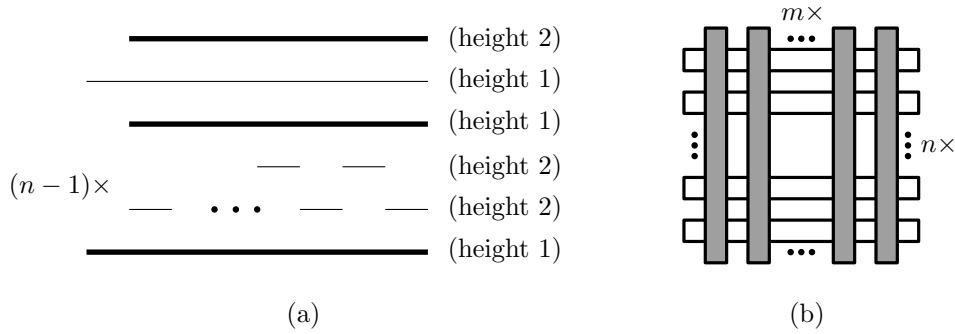


Figure 39. (a) $RI(K_{3,n}) \leq 2 + 1/2$ (b) $RI(K_{m,n}) \leq 3$

5.3 Multipartite Graphs

If a complete graph K_n has a visibility drawing then any graph with at most n vertices has a weak visibility drawing of the same type. This is the reason why the visibility drawings of complete graphs have received a wide attention. The results proved for complete graphs can be applied on all sufficiently small graphs. Unfortunately, these results don't give us much information about drawing of large graphs.

We must study other classes of graphs if we are interested in drawing of large graphs. The previous section presents several results regarding bipartite graphs. Štola [26] generalizes this approach and focuses on multipartite graphs. He studies the following question: What is the maximum integer k such that every k -partite graph has a drawing of the specified type? He introduces a multipartite number of a type of drawing.

Definition 16. A multipartite number of the given type of drawing is the maximum $k \in \mathbb{N}$ such that every k -partite graph has a drawing of that type. We say that the multipartite number is infinite when every multipartite graph has such a drawing. \square

Štola [26] determines the multipartite number for several types of drawing, see Table 5.

v	d	b	multipartite number
1	1	0	1
	2	≥ 2	∞
		1	2
		0	1
	≥ 3	≥ 1	∞
2	2	0	3
		≥ 1	∞
	3	0	1
3	3	≥ 1	$\in [22, 42]$
		0	∞
			$\in [22, 42]$
rectangle visibility drawing			8

Table 5. The multipartite number of the d -dimensional b -bend orthogonal drawing by v -dimensional boxes

Every k -colorable graph is a k -partite graph. Hence, if k is a multipartite number of some drawing and G is a graph then it is sufficient to find a k -coloring of G to show that this graph has a drawing of this type. For example, it is sufficient to find a 22-coloring of a graph to show that this graph has a 0-bend 3D orthogonal drawing.

6 Conclusion

Section 2 describes a general approach that can be used to determine the maximum length of a sequence without (some type of) a forbidden subsequence of the specified length. We show that this approach is applicable on monotone and unimodal subsequences. Moreover, we derive a formula $u_k(n)$, $k \leq 5$ for the maximum length of a sequence of k -tuples without a unimodal subsequence of length n .

It is not surprising that we are not able to provide a closed formula for $u_k(n)$ for a general k . Lemma 9 and Lemma 11 show that we would solve the longstanding Dedekind problem otherwise. Nobody has been able to provide a closed formula for $u_k(3)$ or $u_k(4)$. Even the value of $u_9(3) = u_8(4) = D_9$ is not known yet.

The theory of unimodal subsequences has applications in the 3D visibility drawing by rectangles. It provides a foundation for the improvement of the upper bound on the size of a complete graph with a 3D visibility drawing by rectangles. Theorem 5 moves this bound from 55 to 50. This result allows us to lower also the upper bound on the size of a complete graph with a 0-bend 3D orthogonal drawing. Fekete and Meijer [16] prove that no complete graph with more than 183 vertices admits such a drawing while Theorem 12 shows that the largest complete graph with a 0-bend 3D orthogonal drawing has at most 168 vertices.

We believe that Lemma 19 can be used for further improvement of these bounds. Our upper bounds are based on Lemma 19(iii) only while all five conditions of this lemma must hold simultaneously. It remains an open problem how to combine these conditions to obtain a better bound.

We prove several upper bounds on the size of a complete graph with (some type of) a 3D visibility drawing by equal regular n -gons in Section 4.1. If

- the polygons in the drawing form a short-distance set and
- the sequence of their first coordinates is monotone

then the complete graph has at most $n/2 + 1$ (resp. cn) vertices for n even (resp. odd) according to Lemma 26 (resp. Lemma 28). If we start to remove these conditions then the upper bound increases quadratically (in each step) by Erdős-Szekeres theorem [13] and Lemma 24. Therefore, the maximum size of a complete graph with a 3D visibility drawing by regular n -gons is $O(n^4)$, see Theorem 9. This result is a significant improvement of the previously known exponential bound $\binom{6n-3}{3n-1} - 3 \approx 2^{6n}$ from [24].

The initial upper bound (with the both mentioned conditions satisfied) is tight for n even according to Lemma 32 and has the correct (linear) order

for n odd by Lemma 34. On the other hand, we believe that the quadratic increase of the upper bound during removal of these conditions is rough. There is probably a potential for further improvement.

Section 4.2 provides the best known lower bounds on the maximum size of a complete graph with a 3D visibility drawing by equal regular n -gons. Theorem 10 increases the lower bound for $2k$ -gons from $2k + 1$ to $2k + 3$ and Theorem 11 moves the lower bound for $(2k + 1)$ -gons from $2k + 2$ to $4k + 6$. These bounds are based on a simple combination of stair-like configurations of polygons. It would be interesting to find out whether there are more complex combinations that lead to better lower bounds.

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