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Étienne MATHERON
Professeur
Université d'Artois
Laboratoire de Mathématiques de Lens
Rue Jean Souvraz S.P.18
62307 LENS CEDEX (France)
e-mail : etienne.matheron@euler.univ-artois.fr

Report on Antonín Procházka's Doctoral Dissertation
Analysis in Banach spaces

A. Procházka's doctoral dissertation adresses four different topics from Banach space theory: the Radon-Nikodym property, $\mathcal{C}(K)$ spaces for scattered compact spaces K, renorming theory, and variational principles. It consists in 4 independent chapters, preceded by a general introduction.

Chapter 1 (the general introduction), is a detailed description of the results obtained by the author. Here, the necessary background, the underlying motivations and the results themselves are very accurately presented.

In Chapter 2, the author studies a geometrical game originally introduced by J. Malý and M. Zelený. Here is a general version of the game. Let (K,ρ) be a pseudo-metric space, and let $\mathcal{A}$ be a collection of non-empty subsets of K covering K. The game $\mathbf{G}(K,\rho,\mathcal{A})$ has two players, **I** and **II**. Player **I** starts the game by choosing a point $x_0 \in K$, and **II** answers by choosing a set $A_0 \in \mathcal{A}$ containing x_0. Then **I** plays a second point x_1 with $x_1 \in A_0$, **II** answers with a set $A_1 \in \mathcal{A}$ containing x_1, and so on. Player **II** wins the game if the sequence $(x_n)_{n>0}$ is ρ-Cauchy.

Especially interesting is the case when K is a bounded subset of a real Banach space X (with the metric ρ induced by the norm) and $\mathcal{A}$ is one of the following families: the family $\mathcal{H}(K)$ of all hyperplane sections of K, the family $\mathcal{S}_c(K)$ of all closed slices of K (sets of the form $H \cap K$, where H is a closed half-space of X) and the family $\mathcal{S}_o(K)$ of all open slices of K. The original game of Malý and Zelený is $\mathbf{G}(K,\mathcal{H}(K))$, where K is the open euclidean ball in the plane. It is easy to see that the game $\mathbf{G}(K,\mathcal{S}_o(K))$ is harder to win for player **II** than the game $\mathbf{G}(K,\mathcal{S}_c(K))$, which is in turn harder to win than $\mathbf{G}(K,\mathcal{H}(K))$. In a joint work with R. Deville ([DM]), we have shown that if K is closed and convex, then player **II** has a winning strategy in the game with open slices $\mathbf{G}(K,\mathcal{S}_o(K))$ if and only if K has the Radon-Nikodym property; and that if the underlying Banach space X has a uniformly convex enorming, then player **II** has a winning *tactic* in the game with closed slices

$\mathbf{G}(K, \mathcal{S}_c(K))$, that is, a winning strategy where at each step $n \geq 0$, the slice A_n played by **II** depends only on the last point x_n played by **I**.

It was not at all clear for us whether player **II** still has a winning tactic in $\mathbf{G}(K, \mathcal{S}_c(K))$ assuming only that K has the RNP. A. Procházka was able to show that this is indeed the case. This is the main result of the chapter (Theorem 2.15), whose proof is far from being just a routine elaboration on the methods of [DM]. Indeed, a key new idea is introduced, namely a notion of *stability* for "approximatively winning" tactics which allows to define a truly winning tactic by an inductive procedure. A very interesting complement to this result is that player **II** never has a winning tactic in the game with *open* slices $\mathbf{G}(K, \mathcal{S}_o(K))$ (Theorem 2.12), in strong contrast with the existence of a winning strategy when K has the RNP. Moreover, a game characterization of uniformly convex spaces is obtained (Theorem 2.31): a Banach space X has a uniformly convex renorming if and only if, for any $\varepsilon \in (0,1)$, player **II** has a (winning) tactic t in the game $\mathbf{G}(B_X, \mathcal{S}_c)$ with "uniformy short ε-separated runs"; that is, any finite run $(x_0, \ldots, x_m)$ of the game where **II** follows the tactic t and $\|x_{i+1} - x_i\| \geq \varepsilon$ for all $i < m$, has length m less than some positive number $m(\varepsilon)$. Here, the nice geometrical proof uses the main theorem and G. Lancien's characterization of uniform convexity in terms of the dentability index of the space. Finally, the chapter also contains a game characterization of Baire 1 functions (Theorem 2.35): if (E, τ) is a completely metrizable topological space and Z is a normed space, then a function $f : E \to Z$ is Baire 1 iff player **II** has a winning strategy in the game $\mathbf{G}(E, \rho_f, \tau)$, where ρ_f is the pseudo-metric defined by $\rho_f(x, y) = \|f(x) - f(y)\|$; and if so, it is even possible to define a winning strategy such that, for any outcome (x_n) of the game, the sequence (x_n) is convergent (wrt τ) and $f(\lim x_n) = \lim f(x_n)$.

The results of this chapter have been published in the Proceedings of the AMS (2009).

Chapter 3 is based on a joint paper with P. Hájek and G. Lancien (J. Math. Anal. Appl. 2009). The main result (Theorem 3.10) is the exact computation of the w^* dentability index of $C(K)$ spaces, where K is a countable compact space. Let us recall the definition of this index. Let X be a real Banach space, and for any $\varepsilon > 0$, let us denote by d_ε the set-derivation on w^*-compact subsets of X^* defined as follows: $d_\varepsilon(K)$ is obtained from K by removing all w^*-open slices of diameter less than ε. Then define inductively the sets $d_\varepsilon^{(\alpha)}(K)$ for any ordinal α in the usual way, and let $D^*(X, \varepsilon)$ be the least ordinal α such that $d_\varepsilon^{(\alpha)}(B_{X^*}) = \emptyset$, if there is any; otherwise, put $D^*(X, \varepsilon) = \infty$. The *$w^*$-dentability index* of X is $D^*(X) := \sup_{\varepsilon > 0} D^*(X, \varepsilon)$ (with the convention that $\alpha < \infty$ for any ordinal α). If w^*-open slices are replaced by w^*-open sets in the set-derivation process, the resulting ordinal index is the classical *Szlenk index* of X, usually denoted by $Sz(X)$. These indices are isomorphic invariants, and their usefulness is due to the fact that their "finitness" characterizes *Asplund* spaces (the spaces on which every continuous convex function has a dense G_δ set of differentiability points): $D^*(X) < \infty$ iff $Sz(X) < \infty$, iff X is an Asplund space.

For a countable compact space K, the Szlenk index of $\mathcal{C}(K)$ has been computed by C. Samuel in the 1980's. By Bessaga-Pelczynski's classification, the space $\mathcal{C}(K)$ is isomorphic to $\mathcal{C}([0,\omega^{\omega^\alpha}])$, where α is the unique ordinal such that $\omega^\alpha < CB(K) \leq \omega^{\alpha+1}$ (here, $CB(K)$ is the Cantor-Bendixon index of K), and Samuel has shown that $Sz(\mathcal{C}[0,\omega^{\omega^\alpha}]) = \omega^{\alpha+1}$.

In the present chapter, it is shown that $D^*(C([0,\omega^{\omega^\alpha}])) = \omega^{1+\alpha+1}$ for any countable ordinal α; in other words, $D^*(C([0,\omega^{\omega^\alpha}])) = \omega^{\alpha+2}$ if α is a natural number and $D^*(C([0,\omega^{\omega^\alpha}])) = \omega^{\alpha+1}$ if $\alpha \geq \omega$. The upper estimate $D^*(C([0,\omega^{\omega^\alpha}])) \leq \omega^{\alpha+1}$ is obtained by combining the inequalities $D^*(X) \leq Sz(L_2(X))$ (a result due to G. Lancien) and $Sz(L_2(C([0,\omega^{\omega^\alpha}]))) \leq \omega^{1+\alpha+1}$, the second one being proved by elaborating on the ideas of a recent paper by Hájek and Lancien. Using Samuel's result and the trivial inequality $D^*(X) \geq Sz(X)$, it follows easily that $D^*(\mathcal{C}[0,\omega^{\omega^\alpha}]) = \omega^{\alpha+1}$ when $\alpha \geq \omega$. The last part of the proof consists in treating separately the case $\alpha < \omega$. Here, nice geometrical arguments are used.

Using the main theorem and a "separable reduction argument", a more general result is in fact proved, namely the computation of $D^*(\mathcal{C}(K))$, where K is a compact scattered topological space with countable Cantor-Bendixon index.

Finally, it should be added that, besides their intrinsic interest, the results of this chapter are also a step towards a better understanding of the links between the Szlenk index and the w^* dentability index of Asplund spaces. More precisely, they are related to the following general problem: to find the optimal function $\psi_0 : \omega_1 \to \omega_1$ such that $D^*(X) \leq \psi_0(\xi)$ for every Banach space X satisfying $Sz(X) = \xi$. The existence of such a function ψ_0 was established B. Bossard and G. Lancien using descriptive set-theoretic tools, and M. Raja has obtained the explicit upper estimate $\psi_0(\xi) \leq \omega^\xi$. The main result of this chapter gives the *lower* estimate $\psi_0(\xi) \geq \omega \cdot \xi$ when ξ has the form $\omega^{\alpha+1}$.

Chapter 4 is based on a joint paper with P. Hájek (submitted). It is partly motivated by two open (and hard) problems in renorming theory: (1) if X is a Banach space with a $\mathcal{C}^k$-smooth renorming ($k \in \mathbb{N} \cup \{\infty\}$), is it possible to approximate any equivalent norm on X by $\mathcal{C}^k$-smooth norms, uniformly on bounded sets? (2) if X is a Banach space with a $\mathcal{C}^1$-smooth renorming and also with a locally uniformly rotund (LUR) renorming, does it follow that X has an equivalent norm which is both $\mathcal{C}^1$-smooth and LUR? Recall that a norm $\|\cdot\|$ is said to be *LUR* if, whenever a sequence (x_n) in the unit sphere S_X and a point $x \in S_X$ satisfy $\|\frac{x_n+x}{2}\| \to 1$, it follows that $\|x_n - x\| \to 0$. A much weaker form of Problem (1) is also open: (1') if X is Banach space with both a $\mathcal{C}^k$-smooth renorming and a LUR renorming, is it possible to find an equivalent LUR norm on X which can be approximated by $\mathcal{C}^k$-smooth norms?

Problems (1') and (2) are purely non-separable in essence; it is well-known that they both have a positive answer in the separable case. In fact, a theorem due to D. McLaughlin, R. Poliquin, J. Vanderwerff and V. Zizler answers both problems simultaneously. Let us denote by $(\mathcal{P}_k)$ the following property of a Banach space

X: there is an equivalent norm on X which is both LUR and $\mathcal{C}^1$-smooth and can be approximated by $\mathcal{C}^k$-smooth norms. Then, the theorem of McL-P-V-Z reads as follows : *if X is a separable Banach space with a $\mathcal{C}^k$-smooth norm, then X has $(\mathcal{P}_k)$.*

The main result of the chapter (Theorem 4.17) is the following nonseparable version of this theorem : *Let X be a Banach space. Assume that X has a $\mathcal{C}^k$-smooth norm and admits a projectional resolution of identity $(P_\alpha)_{\omega<\alpha\leq\mu}$ such that each space $(P_{\alpha+1} - P_\alpha)X$ has property $(\mathcal{P}_k)$. Then X itself has $(\mathcal{P}_k)$.* (One may note here an obvious formal similarity with a classical "glueing" result due to V. Zizler concerning LUR renormings). The proof of this result is by far the most technical part of the thesis, and it is indeed an impressive piece of work.

This theorem applies in particular to $\mathcal{C}([0,\alpha])$ for any ordinal α and $k=\infty$ (thanks to well-known results of M. Talagrand and R. Haydon): thus, any space $\mathcal{C}([0,\alpha])$ has property $(\mathcal{P}_\infty)$ (Corollary 4.18). Another consequence is the following "abstract" result (Theorem 4.19): *Let $(\mathcal{P})$ be a class of Banach spaces such that each $X\in(\mathcal{P})$ has a $\mathcal{C}^k$-smooth norm and admits a projectional resolution of identity (P_α) with $(P_{\alpha+1}-P_\alpha)X\in(\mathcal{P})$. Then each $X\in(\mathcal{P})$ has property $(\mathcal{P}_k)$.* For example if X is a Banach space with a $\mathcal{C}^k$-smooth norm and either X is a WCD space, or a WLD space, or a $\mathcal{C}(K)$-space where K is a Valdivia compact, then X has $(\mathcal{P}_k)$ (Corollary 4.20).

The final chapter of the thesis is based on a joint paper with R. Deville (to appear in J. Funct. Anal.). It is centred around a "parametrized" version of the Deville-Godefroy-Zizler variational principle.

Typically, a variational principle is a statement of the following form: given a reasonable real-valued f defined e.g. on a Banach space X, it is possible to find a "small perturbation" $f+\Delta$ which attains its infimum at some point $v\in X$ (and in fact a strong minimum: any minimizing sequence is convergent). When the function f depends on a parameter p, it is quite natural to ask whether the small perturbation $f+\Delta_p$ and the minimum v_p can be chosen in a continuous way. Positive answers were given recently by P. Georgiev ([G]) and L. Veselý ([V]) who were able to produce parametrized versions of the well-known Borwein-Preiss variational principle (and also gave several nice applications of these results). Here, the same work is done with the D-G-Z principle.

The general setting is the following. One is given a Banach space X, a function $f: X\times\Pi\to(-\infty,+\infty]$, where Π is a topological space (the parameter space), and a space $\mathcal{D}$ of real-valued functions on X endowed with a norm $\|\cdot\|_{\mathcal{D}}$ (the admissible "perturbing functions"). As already said, the objective is to find continuous maps $\Delta:\Pi\to\mathcal{D}$ and $v:\Pi\to X$ such that, for each $p\in\Pi$, the function $f+\Delta_p$ attains its strong minimum at the point v_p.

Of course, some "minimal" assumptions are needed. For each $p\in\Pi$, the function $f(\cdot,p)$ should be proper, lower semi-continuous and bounded below (the usual assumptions in non-parametrized variational principles). The parameter space Π

"should" be at least paracompact, since one is in fact looking for a "selection" result and paracompactness is a natural assumption in that area. For each $x \in X$, the function $f(\cdot, x)$ should certainly be continuous on Π. Finally, the space $\mathcal{D}$ should have the same kind of properties as the one appearing in the D-G-Z variational principle. In fact, $\mathcal{D}$ will be a convex cone of convex, nonnegative Lipschitz functions on X which is complete under the natural Lip norm, and rich enough (in a sense that needs not be made more precise here).

Now, two other assumptions are added: (1) for each $p \in \Pi$, the function $f \mapsto f(\cdot, p)$ should be *convex*; (2) for every bounded set $B \subset X$, the family of functions $\{f(\cdot, x);\ x \in B\}$ should be *equi-lsc* on Π (the obvious analogue of equicontinuity). Under these assumptions (the minimal ones plus (1) and (2)), the main result of the chapter (Theorem 5.18) reads as follows : *Let $\mathcal{M}$ be the set of all continuous maps $\Delta : \Pi \to \mathcal{D}$ such that, for some continuous map $v : \Pi \to X$, the function $f + \Delta_p$ attains its strong minimum at v_p for every $p \in \Pi$. Then $\mathcal{M}$ is residual in $\mathcal{C}(\Pi, \mathcal{D})$, the space of all continuous maps from Π to $\mathcal{D}$ equipped with the* fine topology.

Recall that the fine topology on $\mathcal{C}(\Pi, \mathcal{D})$ is generated by all "balls" of the form $\mathcal{B}(\Delta, \varepsilon) = \{\phi \in \mathcal{C}(\Pi, \mathcal{D});\ \|\phi_p - \Delta_p\| < \varepsilon_p \text{ for all } p \in \Pi\}$, where ε is a positive continuous function on Π, and that $\mathcal{C}(\Pi, \mathcal{D})$ is a Baire space when endowed with this topology. Needless to say, the proof of the main theorem is based on a Baire category argument; but this argument is far from being straightforward.

The chapter does not contain applications of the main theorem. On the other hand, there is a very interesting discussion concerning assumptions (1) and (2), which are also present in [G] and [V] (Section 5.4). At first sight, the very restrictive convexity condition (1) is quite surprising in this context. However, it becomes natural if one thinks of parametrized variational principles as selection theorems. In fact, both (1) and (2) are needed in order to prove a variant of Michael's selection theorem (Lemma 5.8), which is a basic step towards the main theorem. Convexity is needed in any Michael-type theorem, and (2) appears to be more ore less necessary to ensure that the function $p \mapsto \inf_{x \in X} f(x, p)$ is lsc. Moreover, convexity also appears to be technically crucial in the proof of several preliminary results. Still, one could argue that these are not good reasons, since after all the proof of the main theorem may not be the right one. However, several illuminating examples are given to show that neither (1) nor (2) can be simply removed from the assumptions. Particularly interesting is the observation that (2) cannot be weakened by considering *compact* sets B only, relying as it does on a nice result about poinwise convergent sequences of convex functions (Lemma 5.16).

Finally, some comments are also offered concerning the respective ranges of application of the main theorem given here and those from [G] and [V]. The general idea is that the main theorem "essentially contains" the others, albeit not formally. In particular, in both [G] and [V] it is required that the function $p \mapsto \inf_{x \in X} f(x, p)$ is locally lower bounded, whereas this assumption is entirely removed from Theorem 5.18.

The conclusion of this report should be obvious: this is definitely an excellent PhD thesis, containing a lot of highly nontrivial interesting results. The proofs combine an impressive technical ability with original and quite clever ideas, which make this thesis a very fine contribution to Banach space theory. Finally, the level of exposition is extremely high. A Prohǻzka is already a mature mathematician, and in light of this manuscript, I have no doubt that he would also be an outstanding teacher. Therefore, I am glad to strongly recommend the defense of his thesis.

É. Matheron