Charles University in Prague
Faculty of Mathematics and Physics
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Université Bordeaux 1
École doctorale de Mathématiques et Informatique

Doctoral Thesis

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Analysis in Banach spaces
Analyse dans les espaces de Banach

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Prague, 2009

To Blanca

Acknowledgment

This thesis has been written within the "cotutelle" program under the supervision of Petr Hájek on the Czech side and Robert Deville on the French side. I am sincerely and deeply grateful to both my advisors for many things. They have introduced many beautiful areas of modern analysis to me, they have shared their outstanding knowledge with me, guided me through my research, always pointing me in the right direction. With both Robert and Petr we have spent a lot of time discussing mathematics or just friendly talking. They have been ready to meet me and answer my questions virtually any time. Thanks are due to Petr for putting me in contact with Robert and thanks are due to Robert for the warm welcome he gave me when I first came to France. Also, I thank Robert for his selfless help with the translation of the introduction of this thesis to French.

I would like to thank Gilles Lancien for his kind invitation to Besançon to collaborate on the topic which now forms Chapter 3 of this thesis. I have been honored by this opportunity, I took great pleasure in working with Gilles, and I am grateful for the inspiring discussions and the time he dedicated to me.

For friendly encouragement and excellent working conditions I thank the members of the Department of Topology and Functional Analysis of the Academy of Sciences of the Czech Republic, the Department of Mathematical Analysis of Charles University and the Institut de Mathématiques de Bordeaux.

I would like to thank the French Government and the Fond Mobility of Charles University for the financial support of my stay in France. The research presented in this thesis was also supported by the grants: Institutional Research Plan AV0Z10190503, A100190502, A100190801, GA ČR 201/07/0394.

I thank my parents, sister and grandparents for always encouraging and supporting me no matter how far from them I was.

The one who deserves my sincerest thanks is Blanca who believed in me and inspired me during all my studies. For your love and patience, this work is dedicated to you.

Tony Procházka
Bordeaux, April 2009

Contents

Chapter 1

Introduction

1.1 Introduction (English)

We are going to deal here with four different topics in the theory of Banach spaces and it is the aim of the introductory chapter to show which place in the theory they occupy.

Here and throughout, X is a real Banach space with a closed unit ball B_X and with the dual space X^*.

1.1.1 The Radon-Nikodým property

We start with a definition of an elementary (yet fundamental) concept. Let A be a set in a Banach space X. Let $f \in X^* \setminus \{0\}$ and $a \in \mathbb{R}$. If the set $S = \{x \in A : f(x) > a\}$ is nonempty, it is called an *open slice* of A (or just a *slice* of A when no confusion may arise). We denote $\mathcal{S}_o(A)$ the set of all slices of A.

With the notion of slice in hand we may define an important class of a Banach spaces which we will meet constantly throughout this text. A Banach space X is said to have the *Radon-Nikodým property* (RNP) if every bounded non-empty subset of X has slices (nonempty by the definition) of arbitrarily small diameter. More precisely, X has the RNP if for every bounded non-empty subset A of X and every $\varepsilon > 0$ there is a slice $S \in \mathcal{S}_o(A)$ such that $\operatorname{diam}(S) < \varepsilon$.

Because of the universal quantifier in the definition, the RNP is naturally an isomorphic property. Its importance dwells in the fact that many familiar constructions on the real line can be translated to the spaces with the RNP. An example of our claim is the original, measure theoretic, definition of the RNP which also explains the name of the property: *Let $\mathcal{B}$ be the Borel sets over $[0,1]$, λ be the Lebesgue measure on $[0,1]$. A Banach space X has the RNP if and only if every X-valued measure m on the probability space $([0,1], \mathcal{B}, \lambda)$ which is of finite total variation and absolutely continuous with respect to λ, is represented by a mapping $f \in L^1([0,1], X)$ by means of the equality $m(A) = \int_A f(x) d\lambda(x)$.*

To see how various mathematicians contributed to this result we refer the reader to the excellent monograph [Bou83]. The above theorem is a (rather heavy) means to see that e.g. $\mathbb{R}^n$ enjoys the RNP.

Our first result is a generalization of the elementary fact that bounded monotone sequences of real numbers converge.

Theorem A. *For a Banach space X it is equivalent:*

(i) *X has the RNP,*

(ii) *there exists a mapping $F : X \to X^*$ such that every bounded sequence $(x_n) \subset X$ is convergent whenever it satisfies*

$$\langle F(x_n), x_n \rangle \leq \langle F(x_n), x_{n+1} \rangle \text{ for every } n \in \mathbb{N}.$$

Already in $X = \mathbb{R}^2$, it is not obvious that (ii) holds, so this theorem is not really a good tool to positively determine that some space has the RNP. On the other hand, it is very useful as a sufficient condition for the convergence of sequences in X with the RNP. The implication (ii) $\Rightarrow$ (i) is due to R. Deville and É. Matheron [DM07] who also proved that *if X admits a uniformly rotund norm, then* (ii) *holds.* The implication (i) $\Rightarrow$ (ii) was proved in full generality in [Pro09] and we will see the proof in Chapter 2.

Let $\mathcal{S}_c(X)$ be the set of all *closed halfspaces* of X, i.e. the sets of the form $\{x \in X : f(x) \geq a\}$ for some $f \in X^* \setminus \{0\}$ and $a \in \mathbb{R}$. In fact, the above theorem is a reformulation of a theorem operating with the notion of the *point-closed halfspace game* $\mathbf{G}(X, \mathcal{S}_c(X))$ which we will now describe. There are two players – Player I and Player II. Player I starts the game by choosing arbitrarily a point $x_1 \in X$. Player II then plays a closed halfspace H_1 containing the point x_1; then Player I picks a point $x_2 \in H_1$ and Player's II answer is a closed halfspace H_2 which contains x_2 (but not necessarily x_1); then Player I chooses a point x_3 in H_2 (but not necessarily in H_1); and so on. The above is called a *run* of the game $\mathbf{G}(X, \mathcal{S}_c(X))$. Player II wins a run if the resulting sequence (x_n) is either Cauchy or unbounded. A *winning tactic* for Player II is a mapping $t : X \to \mathcal{S}_c(X)$ such that it respects the rules of the game, i.e. $x \in t(x)$, and such that Player II wins a run in which he or she always chooses $H_n := t(x_n)$. It is easily seen (cf. Proposition 2.4) that (ii) in Theorem A is equivalent to saying that Player II has a winning tactic in the game $\mathbf{G}(X, \mathcal{S}_c(X))$.

In a more abstract setting, if K is any set in X and $\mathcal{A}$ is a collection of subsets of K such that $K = \bigcup \mathcal{A}$, we define the *point-set game* $\mathbf{G}(K, \mathcal{A})$ verbatim (see Definition 2.1).

The game design is due to J. Malý and M. Zelený [MZ06] who also proved, for the game $\mathbf{G}(B_{\mathbb{R}^2}, \{\text{lines}\})$, that Player II has a *winning strategy* – a decision rule represented by a sequence of mappings $t_n : K^n \to \mathcal{A}$ whose application, i.e. the choice $H_n := t_n(x_1, \dots, x_n)$, insures the victory of Player II. Note that any winning tactic of Player II is automatically a winning strategy of Player II but the converse does not hold. Further, it is proved in [DM07] that *if X enjoys the RNP, then Player II has a winning strategy in $\mathbf{G}(B_X, \mathcal{S}_o(B_X))$.* Surprisingly enough, we will show here the following:

Theorem B. *Let* $\dim X > 0$ *and* $\mathcal{A}$ *be a subcollection of open sets of* B_X *such that* $B_X = \bigcup \mathcal{A}$*. Then Player II has* ***never*** *a winning tactic in the game* $\mathbf{G}(B_X, \mathcal{A})$*. In particular, Player II has* ***never*** *a winning tactic in the game* $\mathbf{G}(B_X, \mathcal{S}_o(B_X))$*.*

Deville and Matheron construct their winning strategy in [DM07] using an abstract lemma which they also employ to get the following characterization: *a Banach space* X *has the point of continuity property if and only if there exists a winning strategy for Player II in the game* $\mathbf{G}(B_X, \{\textit{weakly open sets}\})$. Recall that X has the *point of continuity property* (PCP) if every non-empty bounded subset of X has non-empty relatively weakly open subsets of arbitrarily small diameter. This is just an other way of saying that (B_X, w) is norm fragmented. In the same spirit, in particular using the same lemma, we prove here another similar characterization:

Theorem C. *Let* (E, τ) *be a completely metrizable topological space and* $f : E \to X$ *be a function from* E *to a normed linear space* X*. Let* $\mathbf{G}_f(E, \tau)$ *be defined as* $\mathbf{G}(E, \tau)$ *with the difference that Player II wins a run if the sequence* $(f(x_n))_n \subset X$ *is Cauchy, where* $(x_n)_n \subset E$ *is the sequence of points played by Player I during the run. It is equivalent*

(i) f *is Baire one, i.e.* f *is the pointwise limit of a sequence of continuous functions,*

(ii) *Player II has a winning strategy in the game* $\mathbf{G}_f(E, \tau)$*.*

1.1.2 Quantitative aspects of the Radon-Nikodým property

Let us describe a general peeling scheme which is frequently used to assign some (isomorphically invariant) ordinal index to a given Banach space X. Assume that $\mathcal{A}$ is a subcollection of open sets of X and let C be a subset of X. For $\varepsilon > 0$ we define the set derivation

$$[C]'_\varepsilon = C \setminus \bigcup \{A \in \mathcal{A} : \operatorname{diam}(A \cap C) < \varepsilon\}$$

and we put

$$[C]^0_\varepsilon := C, \quad [C]^{\alpha+1}_\varepsilon := [[C]^\alpha_\varepsilon]'_\varepsilon \quad \text{and} \quad [C]^\beta_\varepsilon := \bigcap_{\alpha<\beta} [C]^\alpha_\varepsilon$$

for every ordinal α and every limit ordinal β. Further we define

$$\iota_{\mathcal{A}}(X, \varepsilon) := \inf \{\alpha : [B_X]^\alpha_\varepsilon = \emptyset\} \text{ and } \iota_{\mathcal{A}}(X) := \sup_{\varepsilon>0} \iota_{\mathcal{A}}(X, \varepsilon)$$

adopting the convention that $\inf \emptyset = \infty$ and $\alpha < \infty$ for every ordinal α. The choice $\mathcal{A} = \mathcal{S}_o(X)$ results in the definition of the *dentability index* $\mathrm{D}(X)$ of X. So $\mathrm{D}(X) = \iota_{\mathcal{S}_o(X)}(X)$ and in this case we denote $wd^\alpha_\varepsilon(C) := [C]^\alpha_\varepsilon$.

It is evident that if X has the RNP, then for every $\varepsilon > 0$ and every C non-empty, closed, convex and bounded subset of X one has $wd^1_\varepsilon(C) \subsetneq C$. On the other hand if X does not have the RNP, then there is $\varepsilon > 0$ and a non-empty set $A \subset B_X$ without any slices smaller than ε, thus clearly $A \subset wd^\alpha_\varepsilon(B_X)$ for every ordinal α. It follows that

X has the RNP if and only if $\mathrm{D}(X) < \infty$. As a matter of fact, if X is separable it follows from a cardinality argument (Theorem I.6.9 in [Kec95]) that X has the RNP if and only if $\mathrm{D}(X) < \omega_1$, where ω_1 is the first uncountable ordinal. A theorem due to G. Lancien [Lan95] claims that *a Banach space X admits a uniformly rotund norm if and only if* $\mathrm{D}(X) \leq \omega$, *where ω is the first infinite ordinal.* We have here a small comment to Lancien's theorem – a quantitatively more refined version of Theorem A.

Theorem D. *For a Banach space X it is equivalent:*

(i) *X admits a uniformly rotund norm,*

(ii) *for each $0 < \varepsilon < 1$ there exist a mapping $F : B_X \to X^*$ and a natural number $k \in \mathbb{N}$ such that whenever $(x_n)_{n=1}^m \subset B_X$ satisfies*

$$\|x_n - x_{n+1}\| > \varepsilon \text{ and } \langle F(x_n), x_n\rangle \leq \langle F(x_n), x_{n+1}\rangle$$

for all $n = 1, \ldots, m-1$, then $m < k$.

We remark that M. Zelený has proved in [Zel08] that if $\dim X < \infty$, then the quantifiers in (ii) may be exchanged: *there exists a mapping $F : B_X \to X^*$ such that for each $0 < \varepsilon < 1$, there exists $k \in \mathbb{N}$ …* It is not clear whether this can be done in general for any space X with a uniformly rotund norm.

As before, Theorem D is a consequence of a version (Theorem 2.31) where (ii) is expressed in terms of the game $\mathbf{G}(B_X, \mathcal{S}_c(B_X))$. We will later state similar theorems with (i) expressed in terms of the weak Szlenk index, resp. the oscillation index, and (ii) expressed in terms of the game $\mathbf{G}(B_X, \{\text{weakly open sets}\})$, resp. $\mathbf{G}_f(M, \tau)$ (see the remarks 2.34 and 2.38).

The *weak Szlenk index* $\mathrm{Sz}_w(X)$ of X is obtained by letting $\mathcal{A} = \sigma(X, X^*)$ (where $\sigma(X, X^*)$ stands for the weakly open subsets of X) in the peeling scheme above, i.e. $\mathrm{Sz}_w(X) = \iota_{\sigma(X,X^*)}(X)$. It is clear, by our discussion of the dentability index above, that a Banach space X enjoys the PCP if and only if $\mathrm{Sz}_w(X) < \infty$, and that for separable Banach space with the PCP, one has $\mathrm{Sz}_w(X) < \omega_1$. It also obvious that $\mathrm{Sz}_w(X) \leq \mathrm{D}(X)$, yet we remark that the predual of the James tree space is a separable space with $\mathrm{Sz}_w(X) < \omega_1$ and $\mathrm{D}(X) = \infty$ since it is, as is well known [FGdB97], a PCP space without the RNP.

1.1.3 The Radon-Nikodým property in dual spaces

A Banach space X is an *Asplund space* if every continuous convex function defined on a nonempty open convex subset D of X is Fréchet differentiable at each point of some dense G_δ part of D. The Asplund space theory is very broad and well developed [DGZ93, Phe93, HMSVZ08]. It is connected to the theory of the RNP spaces via the following duality characterization: *A Banach space X is an Asplund space if and only if every non-empty bounded set in the dual X^* has arbitrarily small weak* open slices if and only if X^* has the RNP.* Fragments of this result are due to E. Asplund [Asp68], I. Namioka and R. Phelps

[NP75], and C .Stegall [Ste75]. One of the building blocks for this characterization is the following result: *Let X be a separable Banach space. Then X^* has the RNP if and only if X^* is separable.* For details see e.g. [Phe93]. The possibility of working either with weakly open or with weak* open slices has many consequences. One of them is that we may state a weak* version of our Theorem A with the mapping F having the values in the predual X of X^*. Another is that *a dual Banach space X^* has the RNP if and only if (B_{X^*}, w^*) is norm fragmented.* Note that JT^*, the dual of the James tree space, has the PCP but not the RNP, hence (B_{JT^*}, w) is norm fragmented even though (B_{JT^*}, w^*) is not (see e.g. [FGdB97]).

Let $\mathcal{S}_o^*(X^*)$, resp. $\sigma(X^*, X)$, be the weak* open halfspaces, resp. weak* open sets in X^*. Returning to the general peeling scheme above we define for any Banach space X the indices $\mathrm{Dz}(X) := \iota_{\mathcal{S}_o^*(X^*)}(X^*)$ and $\mathrm{Sz}(X) := \iota_{\sigma(X^*,X)}(X^*)$. The former is called the *weak* dentability index* of X while the latter is the *Szlenk index* of X. Similarly as in the case of the dentability index of X, resp. the weak Szlenk index of X, the space X^* has the RNP if and only if $\mathrm{Dz}(X) < \infty$, resp. (B_{X^*}, w^*) is norm fragmented if and only if $\mathrm{Sz}(X) < \infty$. Gathering all that has been said, one can see that for a Banach space X it is equivalent:

(o) X is Asplund

(i) X^* has the RNP,

(ii) $\mathrm{Dz}(X) < \infty$,

(iii) $\mathrm{Sz}(X) < \infty$,

(iv) (B_{X^*}, w^*) is norm fragmented.

Again, if X is separable, (ii) may be replaced by $\mathrm{Dz}(X) < \omega_1$ while (iii) may be replaced by $\mathrm{Sz}(X) < \omega_1$. In reality, much stronger result due to B. Bossard [Bos02] and G. Lancien [Lan96] claims that *there exists a function $\Psi : (0, \omega_1) \to (0, \omega_1)$ such that $\mathrm{Dz}(X) \leq \Psi(\mathrm{Sz}(X))$ for any Banach space X with $\mathrm{Sz}(X) < \omega_1$.* This result, proved by methods of the descriptive set theory, was recently improved by M. Raja [Raj07] who used purely geometric means in a transfinite induction argument to obtain: $\mathrm{Dz}(X) \leq \omega^{\mathrm{Sz}(X)}$ *for any Banach space X*, thus showing that the function Ψ may be chosen as $\Psi(\alpha) = \omega^\alpha$. In this context it is interesting to ask what are the optimal values for the function Ψ. In other words, what are the explicit values of the function

$$\Psi_o(\alpha) := \inf \{\beta : \mathrm{Dz}(X) \leq \beta \text{ for every } X \text{ with } \mathrm{Sz}(X) = \alpha\}.$$

Thus in this notation, Raja's theorem may be interpreted as $\Psi_o(\alpha) \leq \omega^\alpha$. Of course, the function Ψ_o only makes sense at the points α for which there exists a Banach space X with $\mathrm{Sz}(X) = \alpha$. For example, it is well known (see Lemma 3.3) that *for any Asplund space X, there is an ordinal α such that $\mathrm{Sz}(X) = \omega^\alpha$.* Note that it was shown by P. Hájek and G. Lancien in [HL07] that *if $\mathrm{Sz}(X) \leq \omega$ then $\mathrm{Dz}(X) \leq \omega^2$, with the space c_0 having*

$\mathrm{Sz}(c_0) = \omega$ *and* $\mathrm{Dz}(c_0) = \omega^2$. In particular $\Psi_o(\omega) = \omega^2$. In general, knowing exact values of $\mathrm{Sz}(X)$ and $\mathrm{Dz}(X)$ for some space X gives a lower estimate for Ψ_o at the point $\mathrm{Sz}(X)$. We are going to determine here the exact values of the weak* dentability index for the class of spaces $C([0, \alpha])$ of continuous functions on compact ordinal intervals $[0, \alpha]$ with α countable which will provide the following lower estimate of Ψ_o.

Theorem E (with P. Hájek and G. Lancien)**.** *Let* $0 \leq \alpha < \omega_1$*. Then*

$$\omega \cdot \omega^{\alpha+1} \leq \Psi_o(\omega^{\alpha+1}).$$

1.1.4 Spaces $C(K)$, $K^{(\omega_1)} = \emptyset$

It follows from the general theory of Apslund spaces that *a* $C(K)$ *space, i.e. the space of continuous functions on the compact* K*, is an Asplund space if and only if* K *is scattered.* A compact K is called *scattered* if there is some ordinal α such that the α^{th} Cantor derived set $K^{(\alpha)}$ of K is empty (see Definition 3.8). We will temporarily restrict our attention to the case when $K^{(\omega_1)} = \emptyset$.

Recall the fundamental isomorphic classification of the above spaces $C([0, \alpha])$ by C. Bessaga and A. Pełczyński [BP60, HMSVZ08]: *Let* $\omega \leq \alpha \leq \beta < \omega_1$*. Then* $C([0, \alpha])$ *is isomorphic to* $C([0, \beta])$ *if and only if* $\beta < \alpha^\omega$*.* In fact, the "only if" part was later reproved by C. Samuel [Sam84] by evaluating exactly the Szlenk index of these spaces: *Let* $0 \leq \alpha < \omega_1$*. Then* $\mathrm{Sz}(C([0, \omega^{\omega^\alpha}])) = \omega^{\alpha+1}$*. In particular* $C([0, \beta_1])$ *is isomorphic to* $C([0, \beta_2])$ *if and only if* $\mathrm{Sz}(C([0, \beta_1])) = \mathrm{Sz}(C([0, \beta_2]))$*.* An easy geometrical proof of this was given in [HL07]. Further elaboration on this proof in a joint paper of P. Hájek, G. Lancien and the author [HLP09] leads to the following

Theorem F (with P. Hájek and G. Lancien)**.** *Let* $0 \leq \alpha < \omega_1$*. Then*

$$\mathrm{Dz}(C([0, \omega^{\omega^\alpha}])) = \omega^{1+\alpha+1}.$$

(Keep in mind that $1 + \alpha + 1 = \alpha + 2$ *if* $\alpha < \omega$*, and* $1 + \alpha + 1 = \alpha + 1$ *otherwise.)*

In particular $C([0, \beta_1])$ *is isomorphic to* $C([0, \beta_2])$ *if and only if* $\mathrm{Dz}(C([0, \beta_1])) = \mathrm{Dz}(C([0, \beta_2]))$*.*

It is a well known topological fact, the theorem of Mazurkiewicz and Sierpiński [HMSVZ08, Theorem 2.56], that *every countable compact space* K *is homeomorphic to some ordinal interval* $[0, \alpha]$ *(*α *countable)*, so $C(K)$ is isomorphic to $C([0, \alpha])$ and one may easily compute the weak* dentability index of such $C(K)$. It is possible to make one more step, though. Using a separable reduction argument we get

Theorem G (with P. Hájek and G. Lancien)**.** *Let* $0 \leq \alpha < \omega_1$*. Let* K *be a compact space whose Cantor derived sets satisfy* $K^{(\omega^\alpha)} \neq \emptyset$ *and* $K^{(\omega^{\alpha+1})} = \emptyset$*. Then*

$$\mathrm{Dz}(C(K)) = \omega^{1+\alpha+1}.$$

1.1.5 Norms with good properties

The main results of this section originate in a joint work of P. Hájek and the author ([HP09]) on renormings which are simultaneously LUR, Fréchet smooth, and approximated by norms of higher smoothness. The higher smoothness is meant in the Fréchet sense, too. Let us recall that a norm $\|\cdot\|$ on a Banach space X is *locally uniformly rotund* (LUR) if $\lim_n \|x_n - x\| = 0$ whenever $\lim_n(2\|x_n\|^2 + 2\|x\|^2 - \|x_n + x\|^2) = 0$. Consider the following prototype of the kind of theorems we have in mind.

Theorem H (with P. Hájek)**.** *Let α be an ordinal. Then the space $C([0,\alpha])$ admits an equivalent norm which is C^1-smooth, LUR and a limit (uniform on bounded sets) of C^∞-smooth norms.*

This particular theorem provides a positive solution of Problem 8.2 (c) in [FMZ06]. It follows from a more general theorem (Theorem I). Before stating the general theorem, we are going to survey briefly on the rôle played by the Fréchet differentiability and the LUR property in the Banach space theory.

Fréchet differentiable norms

It is well known that a Banach space which admits a Fréchet differentiable norm is necessarily an Asplund space. By a famous counterexample of R. Haydon [Hay90], the converse does not hold true in general. Recall that a norm which is Fréchet differentiable in $X \setminus \{0\}$ is automatically continuously Fréchet differentiable in $X \setminus \{0\}$. So the notions of C^1-smooth norms and of Fréchet differentiable norms coincide. A short list of examples of the spaces which do admit a Fréchet differentiable norm would contain:

a) spaces with separable dual (M. I. Kadec),

b) spaces for which $\mathrm{Sz}(X) < \omega_1$ (G. Lancien),

c) reflexive spaces (S. Troyanski),

d) $C(K)$ spaces when $K^{(\omega_1)} = \emptyset$ (R. Deville),

e) $c_0(\Gamma)$ for an arbitrary set Γ (N. H. Kuiper),

f) $C([0,\alpha])$ for an arbitrary ordinal α (M. Talagrand, R. Haydon).

As a matter of fact, the spaces in the examples a)–e) admit a norm whose dual norm is LUR, while those in the examples d)–f) admit even a C^∞-smooth norm. It is well known and rather straightforward to prove that *if a dual norm $\|\cdot\|^*$ of a norm $\|\cdot\|$ is LUR, then $\|\cdot\|$ is Fréchet differentiable*, and this is how the examples a)–d) were proved in the first place. Again, the converse does not hold true since M. Talagrand [Tal86] proved that *$C([0,\omega_1])$ admits an equivalent C^∞-Fréchet smooth norm, although it admits no norm whose dual norm is LUR.* The C^∞-smooth norms in the examples d)–f) are constructed using a different approach, namely the notion of functions *locally dependent on finitely many coordinates* (see Definition 4.1), since the LUR property of a dual norm $\|\cdot\|^*$ does not imply anything about the higher smoothness of the norm $\|\cdot\|$. Recently though, P. Hájek and R. Haydon [HH07] proved an important result claiming that *if a $C(K)$ space admits a norm whose dual norm is LUR, then $C(K)$ admits a C^∞-smooth norm.*

Locally uniformly rotund norms

As is now evident, the notion of LUR is of fundamental importance for renorming theory, and we refer to [DGZ93] and the more recent [MOTV09] for an extensive list of authors and results. In a connection with the previous part of this thesis, we mention that it is an open problem whether a Banach space with the RNP has an equivalent LUR norm. It is noteworthy that there are important subclasses of the spaces with the RNP which do admit a LUR norm. In [Lan93], G. Lancien gave an ingenious formula for a LUR norm (resp. a norm whose dual norm is LUR) on the spaces with $\mathrm{D}(X) < \omega_1$ (resp. $\mathrm{Dz}(X) < \omega_1$). Another example is the class of the duals to the Asplund spaces: *a dual Banach space with the RNP admits an equivalent LUR norm.* This is a result of M. Fabian and G. Godefroy [FG88] ([DGZ93, Corollary VII.1.12]). By what we have already said, in some cases this LUR norm cannot be a dual norm. Anyway, this and many other LUR renorming results (see Chapter VII in [DGZ93]) are based on a method originally due to Troyanski [Tro71]. We state here a "glueing" version due to V. Zizler [Ziz84] ([DGZ93, Proposition VII.1.6]): *If a Banach space X admits a projectional resolution of identity $\{P_\alpha\}_{\omega\le\alpha\le\mu}$ such that the spaces $(P_{\alpha+1} - P_\alpha)X$ admit a LUR norm, then X itself admits a LUR norm.* We content ourselves now with saying that a projectional resolution of identity (PRI) $\{P_\alpha\}_{\omega\le\alpha\le\mu}$ is a generalization to non-separable spaces of the sequence of projections associated to a Schauder basis, and we postpone the exact definition for a later chapter (see Definition 4.12).

Combination of the Fréchet differentiability and the LUR property

In many situations one is interested in obtaining a new norm on a given space X which shares good smoothness and rotundity properties. A classical result in this direction is a method generally known as *Asplund averaging*: *A Banach space X which admits a LUR norm $\|\cdot\|_1$ and a norm $\|\cdot\|_2$ the dual norm of which is LUR, admits also a norm $\|\cdot\|_3$ which is LUR and whose dual norm is LUR, too.* In particular $\|\cdot\|_3$ is LUR and Fréchet differentiable simultaneously. The Asplund averaging as stated here was proved by M. Fabian, L. Zajíček and V. Zizler in [FZZ82] ([DGZ93, II.4.3]) using a Baire category argument. It is proved in the cited paper that the set of equivalent LUR norms is either empty or residual in the space of all equivalent norms on X. An analogous result holds for the set of the equivalent norms whose dual norms on X^* are LUR. The Asplund averaging took a surprising twist with the recent deep result of R. Haydon [Hay08]: *if X admits a norm the dual of which is LUR, then X admits also a LUR norm.*

It is necessary to remark in this context, that one cannot combine the LUR property with a higher degree of differentiability in one norm without posing strong structural restrictions on the space. Indeed, by [FWZ83] ([DGZ93, Proposititon V.1.3]), *a space admitting a LUR and simultaneously C^2-smooth norm is superreflexive.* On the other hand, in the superreflexive spaces one can get even a uniformly rotund and uniformly Fréchet differentiable norm by the Asplund averaging.

It follows from our discussion above, that the Asplund averaging is not always available, for instance in the case of $C([0,\omega_1])$ – we should emphasize that it is unknown whether the set of Fréchet smooth norms is residual, or even dense, in the space of all equivalent norms on $C([0,\alpha])$.

An analog of our Theorem H for $X = c_0(\Gamma)$ was proved in [PWZ81] ([DGZ93, Theorem V.1.5]). Yet another similar result, which we will generalize to the non-separable setting (and which serves as a starting point for the generalization), is that of [MPVZ93]: *Any separable C^k-smooth space admits a LUR and C^1-smooth norm which is a limit of C^k-smooth norms.* A *C^k-smooth space* is a space which admits an equivalent C^k-smooth norm.

We are ready to state the main theorem. Note the similarities with the Zizler's version of the Troyanski's renorming stated above. One could say that our theorem is a smoothening of that result.

Theorem I (with P. Hájek). *Let $k \in \mathbb{N} \cup \{\infty\}$. Let X be a Banach space with a PRI $\{P_\alpha\}_{\omega\leq\alpha\leq\mu}$ such that each $(P_{\gamma+1} - P_\gamma)X$ admits a C^1-smooth, LUR equivalent norm which is a limit (uniform on bounded sets) of C^k-smooth norms. Assume moreover that X admits an equivalent C^k-smooth norm.*

Then X admits an equivalent C^1-smooth, LUR norm which is a limit (uniform on bounded sets) of C^k-smooth norms.

This theorem is in fact the inductive step in an argument leading to Theorem H above and also to the following theorem.

Theorem J (with P. Hájek). *Let $k \in \mathbb{N} \cup \{\infty\}$. Let $\mathcal{P}$ be a class of Banach spaces such that every X in $\mathcal{P}$*

(i) *admits a PRI $\{P_\alpha\}_{\omega\leq\alpha\leq\mu}$ such that $(P_{\alpha+1} - P_\alpha)X \in \mathcal{P}$,*

(ii) *admits a C^k-smooth equivalent norm.*

Then each X in $\mathcal{P}$ admits an equivalent, LUR, C^1-smooth norm which is a limit (uniform on bounded sets) of C^k-smooth norms.

Without entering into further details we remark that the condition (i) is satisfied for example for the weakly compactly generated spaces (WCG), i.e. such spaces which contain a linearly norm-dense weak compact. Similarly (i) holds for the weakly countably determined (Vašák) spaces, $C(K)$ spaces where K is a Valdivia compact, weakly Lindelöf determined (WLD) spaces, etc. For the proofs of these nontrivial facts, as well as for the references, we recommend Chapter VI in [DGZ93] and Chapter 5 in [HMSVZ08]. Now it is clear that one can obtain a class $\mathcal{P}$ which satisfies both (i) and (ii) by taking any of the above classes intersected with the class of C^k-smooth spaces.

Finally let us comment on the fact that the new norm is approximated by norms which are C^k-smooth. Such a result is closely related to the question whether in a C^k-smooth space X the set of C^k-smooth norms on X is dense in the space of all equivalent norms on X. Even in the separable case, the answer is not known in full generality, although the positive results in [DFH96] and [DFH98] are quite strong, and apply to most classical Banach spaces, for instance: *Let $X = C(K)$ where K is a countable compact. Then every equivalent norm on X can be approximated by norms which are analytic in $X \setminus \{0\}$.* In a similar vein goes the next result due to M. Fabian, P .Hájek and V. Zizler [FHZ97]: *Any equivalent strongly lattice norm on $(c_0(\Gamma), \|\cdot\|_\infty)$ can be approximated (uniformly on bounded sets) by C^∞-smooth norms.* See also Theorem 4.15 where we prove that the approximating norms in this theorem have some properties which we will need in the proof of Theorem I. Apart from this theorem, no general results are known in the nonseparable setting. In particular, one of the open problems in [DGZ93] is whether on a given WCG Banach space with an equivalent C^k-smooth norm, there exists an equivalent LUR norm which is a limit (uniform on bounded sets) of C^k-smooth norms. Theorem J together with the structural results about the WCG spaces above provide a positive solution to this open problem.

Notice that *if X is Vašák, resp. X is a $C(K)$ space, the existence of C^k-smooth norm (or more generally of a C^k-smooth bump) implies that any continuous function on X can be approximated uniformly by C^k-smooth functions* (see [DGZ93, Chapter VIII], resp. [HH07]) but when one tries to approximate this way a Lipschitz or a convex function (or a norm, for that matter) one has in general no information about the Lipschitzness or convexity of the approximating function.

In the last chapter we are going to treat convex approximations of a convex function f which do not necessarily improve the differentiability but do posses a point of strong minimum. Of course such a topic is already well explored in the work of M. Fabian, P. Hájek and J. Vanderwerff [FHV96]. So we are going to investigate the situation when f depends continuously on a parameter, looking for a possibility of getting these approximations as well as the minimizers in a continuous way with respect to the parameter.

1.1.6 A parametric variational principle

The results in this section come from a joint work of R. Deville and the author ([DP]). What is a *variational principle* can be perhaps best seen on the concrete example of the Ekeland variational principle: *Let X be a Banach space, $f : X \to (-\infty, +\infty]$ be a lower bounded lower semicontinuous (l.s.c.) function and let $\varepsilon > 0$. Then there is a point $v \in X$ such that $x \mapsto f(x) + \varepsilon \|x - v\|$ attains its minimum in v.* Such an assertion serves as a replacement for the compactness of the set in which one looks for the minimum of f. Geometrically speaking, the graph of the function $x \mapsto f(v) - \varepsilon \|x - v\|$

touches the epigraph of the function f at the point v from below. In this case, the function $x \mapsto \varepsilon \|x - v\|$ is called a *perturbation*. The variational principles of J. Borwein and D. Preiss, resp. R. Deville, G. Godefroy and V. Zizler (DGZ), generalize the above theorem by claiming that it is possible to take C^k-smooth perturbations, resp. C^k-smooth and Lipschitz perturbations, provided the space X admits a C^k-smooth norm, resp. a C^k-smooth bump with bounded k^{th} derivative. Apart from being a strengthening of the Borwein-Preiss variational principle, the DGZ variational principle is proved by an elegant Baire cathegory argument.

A *parametric variational principle* has at its input a system $\{x \mapsto f(p, x)\}$ of l.s.c. lower bounded functions which depend continuously on the parameter p in some topological space Π. The aim is to perturb for each $p \in \Pi$ the function $f(p, \cdot)$ by a function $\Delta(p) : X \to \mathbb{R}$ in such a way that $f(p, \cdot) + \Delta(p)$ attains its minimum at some point $v(p)$ such that both $v(p)$ and $\Delta(p)$ depend continuously on the parameter p. A parametric smooth variational principle of Borwein-Preiss kind was introduced by P. Georgiev [Geo05]. Recently L. Veselý [Ves09] modified the proof in order to achieve a parametric smooth variational principle with constraints. More precisely, let $\Pi_0 \subset \Pi$ be such that, for every $p \in \Pi_0$, the function $f(p, \cdot)$ attains its minimum at $v_0(p)$. Veselý constructs a minimizer v which extends v_0. The main theorem of this chapter (Theorem 5.18) is a parametrized version of the DGZ variational principle and its method of proof. Our main theorem implies in particular

Theorem K (with R. Deville)**.** *Let Π be a paracompact Hausdorff topological space, X be a Banach space with a Fréchet smooth norm, $\mathcal{Y}$ be the cone of all convex, positive, Lipschitz, Fréchet smooth functions on X. The cone $\mathcal{Y}$ is endowed with the natural norm $\|g\|_{\mathcal{Y}} = |g(0)| + \|g\|_{\text{Lip}}$. Let $f : \Pi \times X \to \mathbb{R}$ satisfy*

(i) *for all $p \in \Pi$, $f(p, \cdot)$ is convex, continuous, bounded below,*

(ii) *for all $x \in X$, $f(\cdot, x)$ is continuous,*

(iii) *for all $p_0 \in \Pi$, $(f(p_0, \cdot) - f(p, \cdot))^+ \to 0$ uniformly on bounded sets of X as $p \to p_0$.*

Then for every $\varepsilon > 0$ there exist $\Delta \in C(\Pi, \mathcal{Y})$ and $v \in C(\Pi, X)$ such that $\|\Delta(p)\|_{\mathcal{Y}} < \varepsilon$ and $f(x, \cdot) + \Delta(p)$ attains its strong minimum at $v(p)$ for all $p \in \Pi$. Moreover $p \mapsto f(p, v(p)) + \Delta(p)(v(p))$ is continuous.

A major flaw, which restricts severely the ring of possible applications and occurs for all parametric variational principles, is that all the functions $f(p, \cdot)$ have to be convex. This is not just a mere technical difficulty since easy examples (Example 5.26) show that in the nonconvex case there is no hope in general for a continuous minimizer even after the perturbation.

We show in Section 5.4 that the other unexpected assumption of Theorem K, i.e. the assumption (iii), cannot be dropped without replacement either. On the other hand, the condition (iii) follows from (i) and (ii) if we assume that Π is metrizable and the dimension of X is finite. Indeed, this is deduced immediately from the following theorem.

Theorem L (with R. Deville). *Let f and f_n, $n \in \mathbb{N}$, be continuous convex functions from a Banach space X to $\mathbb{R}$ such that $f_n \to f$ pointwise in X. Then $f_n \to f$ uniformly on compact subsets of X.*

Observe finally, that even if $\dim X < \infty$ and $p \mapsto f(p, \cdot)$ attains its minimum at $v(p)$ for every $p \in \Pi$, the function v does not have to be continuous. Indeed, the existence of a *continuous* minimizer for the original function is not granted even in the most simple setting (see Problem 5.1 and Section 5.4).

1.2 General notions

Our notation is rather standard, as could be found in many textbooks (e.g. [FHH+01]). In particular, if $(X, \|\cdot\|)$ is a Banach space, then its dual is denoted $(X^*, \|\cdot\|)$. In some cases, when there is a need for more precision, we add a subscript to the norm to stress its domain of definition: $(X, \|\cdot\|_X)$, and $(X^*, \|\cdot\|_{X^*})$ for the dual.

If $x \in X$ and $r > 0$, we denote $B_{(X,\|\cdot\|)}(x, r) = \{y \in X : \|y - x\| \le r\}$, $B^O_{(X,\|\cdot\|)}(x, r) = \{y \in X : \|y - x\| < r\}$ and $S_{(X,\|\cdot\|)}(x, r) = \{y \in X : \|y - x\| = r\}$. When there is no risk of confusion, we abbreviate $B_X(x, r) := B_{(X,\|\cdot\|)}(x, r)$, etc. Further we denote $B_X = B_X(0, 1)$, the *closed unit ball* of X, and $S_X = S_X(0, 1)$, the *unit sphere* of X.

For $A \subset X$, the *diameter* of A is defined as $\operatorname{diam}(A) = \sup\{\|x - y\| : x, y \in A\}$.

If $(X, \|\cdot\|_X)$ and Y are Banach spaces, then the *support* of a mapping $f : X \to Y$ is defined as $\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}^{\|\cdot\|_X}$. A *bump* b on X is function from X to $\mathbb{R}$ for which $\operatorname{supp}(b)$ is nonempty and bounded.

The weak topology on X is denoted either by w or by $\sigma(X, X^*)$. The weak* topology on X^* is denoted by w^* or by $\sigma(X^*, X)$. The duality pairing between an element x of X and an element $x^* \in X$ is $\langle x, x^* \rangle = \langle x^*, x \rangle = x(x^*) = x^*(x)$ and we shall use all of these possible notations in a seemingly random fashion.

If A is a set, we denote its cardinality $|A|$. We identify a cardinal κ with the smallest ordinal of cardinality κ, silently turning to the axiom of choice.

We denote ω the first infinite ordinal and ω_1 the first uncountable ordinal.

1.3 Úvod (česky)

V této práci se chystáme podrobně rozebrat čtveřici různých otázek v teorii Banachových prostorů. V úvodní kapitole se pokusíme zasadit je na místo, které jim v této teorii náleží.

Písmenem X budeme značit reálný Banachův prostor s uzavřenou jednotkovou koulí B_X a s duálem X^*.

1.3.1 Radonova-Nikodýmova vlastnost

Začněme definicí elementárního, nicméně zásadního, pojmu. Nechť A je podmnožinou Banachova prostoru X. Nechť $f \in X^* \setminus \{0\}$ a $a \in \mathbb{R}$. Je-li množina $S = \{x \in A : f(x) > a\}$ neprázdná, nazývá se *otevřený plátek* množiny A (nebo prostě *plátek* množiny A, je-li vyloučena možnost nedorozumění). Množinu všech plátků množiny A budeme značit $\mathcal{S}_o(A)$.

Vybaveni znalostí plátku můžeme definovat důležitou třídu Banachových prostorů, s níž se budeme v tomto textu setkávat na každém kroku. Banachův prostor X má *Radonovu-Nikodýmovu vlastnost* (RNP), jestliže každá omezená, neprázdná podmnožina prostoru X má plátky (z definice neprázdné) libovolně malého průměru. Přesněji řečeno, X má RNP, jestliže pro každou omezenou, neprázdnou podmnožinu A prostoru X a pro každé $\varepsilon > 0$ existuje takový plátek $S \in \mathcal{S}_o(A)$, že $\operatorname{diam}(S) < \varepsilon$.

Díky univerzálnímu kvantifikátoru v definici je RNP přirozeně izomorfní vlastností. Její význam spočívá ve skutečnosti, že mnohé konstrukce známé na reálné ose mohou být přeneseny do prostorů s RNP. Toto tvrzení můžeme doložit například na původní, z teorie míry pocházející, definici RNP, která také vysvětluje, proč se vlastně RNP jmenuje RNP: *Buď $\mathcal{B}$ borelovská σ-algebra na intervalu $[0,1]$, λ Lebesgueova míra na $[0,1]$. Banachův prostor X má RNP právě tehdy, jestliže každá míra m na pravděpodobnostním prostoru $([0,1], \mathcal{B}, \lambda)$ s hodnotami v X, která má konečnou totální variaci a je absolutně spojitá vzhledem k λ, je reprezentována zobrazením $f \in L^1([0,1], X)$ prostřednictvím rovnosti $m(A) = \int_A f(x) d\lambda(x)$.*

Pro důkaz tohoto tvrzení a přehled matematiků, kteří se na něm podíleli, doporučujeme vynikající monografii [Bou83]. S pomocí zmiňované věty lze snadno nahlédnout, že například $\mathbb{R}^n$ má RNP.

V naší první Větě přeneseme další známou konstrukci do prostorů s RNP. Jedná se o zobecnění základního faktu, že omezené monotónní posloupnosti reálných čísel konvergují.

Věta A. *Následující tvrzení o Banachově prostoru X jsou ekvivalentní:*

(i) *X má RNP,*

(ii) *na X existuje takové zobrazení $F : X \to X^*$, že každá omezená posloupnost $(x_n) \subset X$ konverguje, pokud splňuje*

$$\langle F(x_n), x_n \rangle \leq \langle F(x_n), x_{n+1} \rangle \text{ pro každé } n \in \mathbb{N}.$$

Již v případě $X = \mathbb{R}^2$ není snadné dokázat platnost podmínky (ii), což svědčí spíše o tom, že uvedená věta není nejlepším nástrojem k ověřování, že nějaký zadaný prostor má RNP. Na druhé straně, tato věta je velmi užitečnou postačující podmínkou pro konvergenci posloupností v prostoru X s RNP. Za implikaci (ii) $\Rightarrow$ (i) vděčíme R. Devillovi a É. Matheronovi [DM07], kteří také dokázali, že *pokud lze na X definovat ekvivalentní uniformně konvexní normu, pak platí podmínka* (ii). Implikace (i) $\Rightarrow$ (ii) byl dokázána v [Pro09] a její důkaz uvidíme v Kapitole 2.

Označme $\mathcal{S}_c(X)$ množinu všech *uzavřených poloprostorů* v X, tedy množin tvaru $\{x \in X : f(x) \geq a\}$, kde $f \in X^* \setminus \{0\}$ a $a \in \mathbb{R}$. Jak uvidíme, Věta A je reformulací věty pracující s pojmem hry *bod-uzavřený poloprostor* $\mathbf{G}(X, \mathcal{S}_c(X))$, kterou nyní popíšeme. Hru hrají dva hráči – hráč I a hráč II. Začíná hráč I, který umístí libovolně bod x_1 do X. Pokračuje hráč II, který zvolí uzavřený poloprostor H_1 obsahující bod x_1; následně hráč I zvolí $x_2 \in H_1$ a hráč II mu odpoví uzavřeným poloprostorem H_2, který obsahuje x_2 (ale nemusí obsahovat x_1); načež hráč I volí bod x_3 v H_2 (ale ne nutně v H_1); a tak dále. Právě popsaný proces se nazývá *partie* hry $\mathbf{G}(X, \mathcal{S}_c(X))$. Hráč II vyhrává partii, pokud výsledná posloupnost (x_n) je buď cauchyovská, nebo neomezená. *Vítězná taktika* hráče II je takové zobrazení $t : X \to \mathcal{S}_c(X)$, které respektuje pravidla hry, t.j. $x \in t(x)$, a má tu vlastnost, že hráč II vyhraje partii, ve které vždy volil $H_n := t(x_n)$. Je snadné nahlédnout (viz. Proposition 2.4), že (ii) ve Větě A je ekvivalentní tvrzení "hráč II má vítěznou taktiku ve hře $\mathbf{G}(X, \mathcal{S}_c(X))$".

Poněkud obecněji, je-li K podmnožinou X a $\mathcal{A}$ je kolekcí podmnožin množiny K, která pokrývá K, můžeme definovat hru *bod-množina* $\mathbf{G}(K, \mathcal{A})$ stejně jako množinu bod-poloprostor (viz. Definition 2.1).

Tento druh her pochází od J. Malého a M. Zeleného [MZ06], kteří ukázali, že hráč II má vítěznou strategii ve hře $\mathbf{G}(B_{\mathbb{R}^2}, \{\text{přímky}\})$. Vítězná strategie je pravidlo reprezentované posloupností zobrazení $t_n : K^n \to \mathcal{A}$, jehož použití, totiž volba $H_n := t_n(x_1, \ldots, x_n)$, zaručí hráči II vítězství. Povšimněme si, že každá vítězná taktika hráče II je také jeho vítěznou strategií, což ovšem naopak neplatí. Následně v [DM07] je dokázáno, že *pokud X má RNP, pak hráč II má vítěznou strategii ve hře* $\mathbf{G}(B_X, \mathcal{S}_o(B_X))$. V této souvislosti se může zdát překvapující náš další výsledek:

Věta B. *Nechť* $\dim X > 0$ *a budiž $\mathcal{A}$ tvořena otevřenými podmnožinami B_X tak, že $B_X = \bigcup \mathcal{A}$. Potom hráč II **nemůže** mít vítěznou taktiku ve hře $\mathbf{G}(B_X, \mathcal{A})$. Speciálně, hráč II **nemůže** mít vítěznou taktiku ve hře $\mathbf{G}(B_X, \mathcal{S}_o(B_X))$.*

Deville a Matheron v [DM07] konstruují svou vítěznou strategii za použití abstraktního lemmatu, které jim také posloužilo k důkazu následující charakterizace: *Banachův prostor X má vlastnost bodů spojitosti (PCP) právě tehdy, když hráč II má vítěznou strategii ve hře* $\mathbf{G}(B_X, \{\textit{weakly open sets}\})$. Připomeňme si, že X má *má vlastnost bodů spojitosti* (PCP), pokud každá neprázdná, omezená podmnožina prostoru X má neprázdné, relativně slabě otevřené podmnožiny libovolně malého průměru. To je v podstatě to samé, jako že (B_X, w) je fragmentována normou. Ve stejném duchu zde ukážeme další podobnou charakterizaci:

Věta C. *Nechť (E, τ) je úplně metrizovatelný topologický prostor a $f : E \to X$ budiž* *funkce z E do normovaného lineárního prostoru X. Definujme hru $\mathbf{G}_f(E,\tau)$ stejně jako $\mathbf{G}(E,\tau)$ s tím rozdílem, že hráč II vyhrává partii, pokud posloupnost $(f(x_n))_n \subset X$ je cauchyovská, kde $(x_n)_n \subset E$ je posloupnost tahů hráče I v průběhu partie. Následující výroky jsou ekvivalentní:*

(i) *f je první Baireovy třídy, tj. f je bodovou limitou posloupnosti spojitých funkcí,*

(ii) *hráč II má vítěznou strategii ve hře $\mathbf{G}_f(E,\tau)$.*

1.3.2 Kvantitativní rysy Radonovy-Nikodýmovy vlastnosti

Popišme obecné oškrabávací schéma, které se běžně používá k přiřazení nějakého (izomorfně invariantního) ordinálního indexu k zadanému Banachovu prostoru X. Předpokládejme, že $\mathcal{A}$ je tvořena otevřenými podmnožinami prostoru X a že C je podmnožinou X. Pro $\varepsilon > 0$ definujeme množinovou derivaci

$$[C]'_\varepsilon = C \setminus \bigcup \{A \in \mathcal{A} : \operatorname{diam}(A \cap C) < \varepsilon\}$$

a položíme

$$[C]^0_\varepsilon := C, \quad [C]^{\alpha+1}_\varepsilon := [[C]^\alpha_\varepsilon]'_\varepsilon \quad \text{a} \quad [C]^\beta_\varepsilon := \bigcap_{\alpha<\beta} [C]^\alpha_\varepsilon$$

pro každý ordinál α a každý limitní ordinál β. Dále definujeme

$$\iota_\mathcal{A}(X,\varepsilon) := \inf\{\alpha : [B_X]^\alpha_\varepsilon = \emptyset\} \text{ a } \iota_\mathcal{A}(X) := \sup_{\varepsilon>0} \iota_\mathcal{A}(X,\varepsilon),$$

přičemž se držíme následující konvence: $\inf \emptyset = \infty$ a $\alpha < \infty$ pro každý ordinál α. Volba $\mathcal{A} = \mathcal{S}_o(X)$ vede k definici *zářezového indexu* $\mathrm{D}(X)$ prostoru X. Takže $\mathrm{D}(X) = \iota_{\mathcal{S}_o(X)}(X)$ a v tomto případě značíme $wd^\alpha_\varepsilon(C) := [C]^\alpha_\varepsilon$.

Je zřejmé, že když X má RNP, pak pro každé $\varepsilon > 0$ a každou neprázdnou, uzavřenou, konvexní podmnožinu C prostoru X platí $wd^1_\varepsilon(C) \subsetneq C$. Na druhé straně, pokud X nemá RNP a $\emptyset \neq A \subset B_X$ je nějaká množina bez plátků menších než ε, pak zjevně $A \subset wd^\alpha_\varepsilon(B_X)$ pro každý ordinál α. Z toho plyne, že X má RNP tehdy a jen tehdy, když $\mathrm{D}(X) < \infty$. Podotkněme, že když X je separabilní, pak lze snadno ukázat pomocí argumentu, založeném na kardinalitě báze otevřených množin (Theorem I.6.9 v [Kec95]), že X má RNP právě tehdy, když $\mathrm{D}(X) < \omega_1$, kde ω_1 je první nespočetný ordinál. Věta G. Lanciena [Lan95] tvrdí, že *Banachův prostor X lze uniformně konvexně přenormovat právě tehdy, když $\mathrm{D}(X) \leq \omega$, kde ω je první nekonečný ordinál.* Přidejme k Lancienově větě drobný komentář – kvantitativně přesnější verzi Věty A.

Věta D. *Pro Banachův prostor X je ekvivalentní:*

(i) *na X lze zavést ekvivalentní uniformně konvexní normu,*

(ii) *pro každé* $0 < \varepsilon < 1$ *existuje takové zobrazení* $F : B_X \to X^*$ *a takové přirozené číslo* $k \in \mathbb{N}$*, že kdykoliv* $(x_n)_{n=1}^m \subset B_X$ *splňuje*

$$\|x_n - x_{n+1}\| > \varepsilon \text{ a } \langle F(x_n), x_n\rangle \leq \langle F(x_n), x_{n+1}\rangle$$

pro všechna $n = 1, \dots, m-1$*, pak* $m < k$*.*

Poznamenejme, že M. Zelený dokázal v [Zel08], že pokud $\dim X < \infty$, pak mohou být kvantifikátory v bodu (ii) zaměněny: *existuje takové zobrazení* $F : B_X \to X^*$*, že pro každé* $0 < \varepsilon < 1$ *existuje* $k \in \mathbb{N}$ ... Není jasné, zde lze to samé provést obecně v libovolném prostoru X s uniformně konvexní normou.

Podobně jako v předchozím, Věta D je důsledkem tvrzení (Theorem 2.31), ve kterém je podmínka (ii) vyjádřena pomocí hry $\mathbf{G}(B_X, \mathcal{S}_c(B_X))$. Později vyslovíme obdobné věty, ve kterých bude podmínka (i) vyjádřena pomocí slabého Szlenkova indexu, resp. oscilačního indexu, a podmínka (ii) pomocí hry $\mathbf{G}(B_X, \{\text{slabě otevřené množiny}\})$, resp. $\mathbf{G}_f(M, \tau)$ (viz. poznámky 2.34 a 2.38).

Slabý Szlenkův index $\mathrm{Sz}_w(X)$ prostoru X se získá tak, že položíme $\mathcal{A} = \sigma(X, X^*)$ (kde $\sigma(X, X^*)$ jsou slabě otevřené podmnožiny prostoru X) ve výše zmíněném oškrabávacím schématu, tj. $\mathrm{Sz}_w(X) = \iota_{\sigma(X,X^*)}(X)$. Je jasné, podobně jako v případě zářezového indexu, že Banachův prostor X má PCP právě tehdy, když $\mathrm{Sz}_w(X) < \infty$. V případě separabilního prostoru s PCP máme opět $\mathrm{Sz}_w(X) < \omega_1$. Je také jasné, že $\mathrm{Sz}_w(X) \leq \mathrm{D}(X)$, ale poznamenejme, že preduál B Jamesova prostoru JT je separabilní prostor, pro který $\mathrm{Sz}_w(X) < \omega_1$ a $\mathrm{D}(X) = \infty$, protože jak je známo [FGdB97], B má PCP ale ne RNP.

1.3.3 Radonova-Nikodýmova vlastnost v duálech

Banachův prostor X je *Asplundův*, jestliže každá spojitá konvexní funkce, která je definována v neprázdné, otevřené, konvexní podmnožině D prostoru X, je fréchetovsky diferencovatelná v každém bodě nějaké husté G_δ části množiny D. Teorie Asplundových prostorů je dobře rozvinutá [DGZ93, Phe93, HMSVZ08]. Je neoddělitelná od teorie prostorů s RNP díky následující duální charakterizaci: *Banachův prostor* X *je Asplundův tehdy a jen tehdy, když každá neprázdná omezená množina v duálu* X^* *má libovolně malé* w^* *otevřené plátky. Nebo také tehdy a jen tehdy, když* X^* *má RNP.* Fragmenty tohoto výsledku pocházejí od E. Asplunda [Asp68], I. Namioky a R. Phelpse [NP75], a od C .Stegalla [Ste75]. Jedním z pilířů, podpírajících tuto charakterizaci, je výsledek: *Nechť* X *je separabilní Banachův prostor. Pak* X^* *má RNP právě tehdy, když* X^* *je separabilní.* Detaily jsou k nalezení například v [Phe93]. To, že můžeme pracovat buď se slabě otevřenými nebo w* otevřenými plátky má několik důsledků. Jedním z nich je, že můžeme vyslovit w* verzi Věty A, kde má zobrazení F hodnoty v preduálu X prostoru X^*. Dalším důsledkem je fakt, že *duální prostor* X^* *má RNP právě tehdy, když* (B_{X^*}, w^*) *je fragmentována normou.* Povšimněme si, že JT^*, tedy duál Jamesova prostoru JT, má PCP, ale ne RNP, takže (B_{JT^*}, w) je fragmentována normou, zatímco (B_{JT^*}, w^*) není (viz. např. [FGdB97]).

Budiž $\mathcal{S}_o^*(X^*)$, resp. $\sigma(X^*, X)$, w* otevřené poloprostory, resp. w* otevřené množiny v X^*. Vraťme se k našemu obecnému oškrabávacímu schématu a definujme, pro libovolný

Banachův prostor X, indexy $\mathrm{Dz}(X) := \iota_{\mathcal{S}_o^*(X^*)}(X^*)$ a $\mathrm{Sz}(X) := \iota_{\sigma(X^*,X)}(X^*)$. První z nich se nazývá *w* zářezový index* prostoru X, zatímco druhý je *Szlenkův index* prostoru X. Podobně jako v případě zářezového indexu, resp. slabého Szlenkova indexu, prostor X^* má RNP tehdy a jen tehdy, když $\mathrm{Dz}(X) < \infty$, resp. (B_{X^*}, w^*) je fragmenována normou tehdy a jen tehdy, když $\mathrm{Sz}(X) < \infty$. Shrňme si přehledně, co bylo řečeno. Pro Banachův prostor X jsou následující výroky ekvivalentní:

(o) X je Asplundův

(i) X^* má RNP,

(ii) $\mathrm{Dz}(X) < \infty$,

(iii) $\mathrm{Sz}(X) < \infty$,

(iv) (B_{X^*}, w^*) je fragmentována normou.

A opět, je-li X separabilní, bod (ii) lze nahradit podmínkou $\mathrm{Dz}(X) < \omega_1$, zatímco bod (iii) podmínkou $\mathrm{Sz}(X) < \omega_1$. Ve skutečnosti platí mnohem silnější výsledek B. Bossarda [Bos02] a G. Lanciena [Lan96], který tvrdí, že *existuje taková funkce* $\Psi : (0, \omega_1) \to (0, \omega_1)$*, pro kterou* $\mathrm{Dz}(X) \leq \Psi(\mathrm{Sz}(X))$ *pro libovolný Banachův prostor* X *splňující* $\mathrm{Sz}(X) < \omega_1$. Tento výsledek, dokázaný metodami deskriptivní teorie množin, byl nedávno zesílen M. Rajou [Raj07], který zapojil čistě geometrické prostředky do transfinitní indukce a získal: $\mathrm{Dz}(X) \leq \omega^{\mathrm{Sz}(X)}$ *pro libovolný Banachův prostor* X, čímž ukázal, že funkce Ψ může být volena jako $\Psi(\alpha) = \omega^\alpha$. V této souvislosti je zajímavá otázka, jaké jsou optimální hodnoty funkce Ψ. Jinak řečeno, jaké jsou explicitní hodnoty funkce

$$\Psi_o(\alpha) := \inf \{\beta : \mathrm{Dz}(X) \leq \beta \text{ pro každý } X \text{ splňující } \mathrm{Sz}(X) = \alpha\}.$$

Takže například Rajův výsledek lze reformulovat: $\Psi_o(\alpha) \leq \omega^\alpha$. Abychom učinili přesnosti zadost, musíme dodat, že funkce Ψ_o má smysl pouze v bodech α, pro které existuje nějaký Banachův prostor X tak, že $\mathrm{Sz}(X) = \alpha$. Například, je známo (viz. Lemma 3.3), že *pro každý Banachův prostor* X*, existuje takový ordinál* α*, že* $\mathrm{Sz}(X) = \omega^\alpha$. Zmiňme, že P. Hájek a G. Lancien v [HL07] dokázali, že *je-li* $\mathrm{Sz}(X) \leq \omega$*, pak* $\mathrm{Dz}(X) \leq \omega^2$*. Navíc prostor* c_0 *má* $\mathrm{Sz}(c_0) = \omega$ *a* $\mathrm{Dz}(c_0) = \omega^2$. Speciálně $\Psi_o(\omega) = \omega^2$. Obecně vzato, znalost přesných hodnot indexů $\mathrm{Sz}(X)$ a $\mathrm{Dz}(X)$ pro nějaký prostor X dává dolní odhad pro Ψ_o v bodě $\mathrm{Sz}(X)$. My zde vypočítáme přesné hodnoty w* zářezového indexu pro třídu prostorů $C([0,\alpha])$, tedy spojitých funkcí na kompaktních ordinálních intervalech $[0,\alpha]$, kde α je spočetný. To nám dodá následující dolní odhad funkce Ψ_o.

Věta E (s P. Hájkem a G. Lancienem)**.** *Nechť* $0 \leq \alpha < \omega_1$*. Pak*

$$\omega \cdot \omega^{\alpha+1} \leq \Psi_o(\omega^{\alpha+1}).$$

1.3.4 Prostory $C(K)$, $K^{(\omega_1)} = \emptyset$

Z obecné teorie Asplundových prostorů plyne, že *prostor $C(K)$, tedy prostor spojitých funkcí na kompaktu K, je Asplundův právě tehdy, když K je roztroušený.* Kompakt K nazveme *roztroušeným*, existuje-li nějaký ordinál α tak, že Cantorova derivace $K^{(\alpha)}$ řádu α kompaktu K je prázdná (viz. Definition 3.8). Dočasně se budeme věnovat situaci, kdy $K^{(\omega_1)} = \emptyset$.

Připomeňme si zásadní izomorfní klasifikaci výše zmíněných prostorů $C([0,\alpha])$ pocházející od C. Bessagy a A. Pełczyńského [BP60, HMSVZ08]: *Nechť $\omega \leq \alpha \leq \beta < \omega_1$. Pak $C([0,\alpha])$ je izomorfní prostoru $C([0,\beta])$ tehdy a jen tehdy, když $\beta < \alpha^\omega$.* C. Samuel [Sam84] dodal později jiný důkaz části "jen tehdy", který spočívá v přesném vyhodnocení Szlenkova indexu těchto prostorů: *Nechť $0 \leq \alpha < \omega_1$. Pak $\mathrm{Sz}(C([0,\omega^{\omega^\alpha}])) = \omega^{\alpha+1}$. Speciálně, $C([0,\beta_1])$ je izomorfní s $C([0,\beta_2])$ právě tehdy, když $\mathrm{Sz}(C([0,\beta_1])) = \mathrm{Sz}(C([0,\beta_2]))$.* Snadný geometrický důkaz tohoto faktu je podán v [HL07]. Další rozpracování tohoto důkazu ve společném článku P. Hájka, G. Lanciena a pisatele této práce [HLP09] vedlo k následujícímu.

Věta F (s P. Hájkem a G. Lancienem)**.** *Nechť $0 \leq \alpha < \omega_1$. Pak*

$$\mathrm{Dz}(C([0,\omega^{\omega^\alpha}])) = \omega^{1+\alpha+1}.$$

(osvěžme si, že $1+\alpha+1 = \alpha+2$ pokud $\alpha < \omega$, a $1+\alpha+1 = \alpha+1$ v ostatních případech.)

Speciálně, $C([0,\beta_1])$ je izomorfní s $C([0,\beta_2])$ právě tehdy, když

$$\mathrm{Dz}(C([0,\beta_1])) = \mathrm{Dz}(C([0,\beta_2])).$$

Z topologie je známo, jedná se o větu Mazurkiewicze a Sierpińského [HMSVZ08, Theorem 2.56], že *každý spočetný kompakt K je homeomorfní nějakému ordinálnímu intervalu $[0,\alpha]$ (α spočetné).* Takže $C(K)$ je izomorfní prostoru $C([0,\alpha])$ a tudíž můžeme snadno spočítat w* zářezový index takového $C(K)$. Dá se však zajít ještě dál. Za použití separabilní redukce ukážeme:

Věta G (s P. Hájkem a G. Lancienem)**.** *Nechť $0 \leq \alpha < \omega_1$. Nechť K je kompakt, jehož Cantorovy derivované množiny splňují $K^{(\omega^\alpha)} \neq \emptyset$ a $K^{(\omega^{\alpha+1})} = \emptyset$. Pak*

$$\mathrm{Dz}(C(K)) = \omega^{1+\alpha+1}.$$

1.3.5 Normy s dobrými vlastnostmi

Hlavní výsledky v této sekci pocházejí ze společného článku P. Hájka a autora této práce ([HP09]) o renormacích, které jsou zároveň LUR, fréchetovsky diferencovatelné, a které jsou aproximovány normami s vyšší hladkostí. Připomeňme si, že norma $\|\cdot\|$ na Banachově prostoru X je *lokálně uniformně konvexní* (LUR), pokud $\lim_n \|x_n - x\| = 0$, kdykoliv $\lim_n(2\|x_n\|^2 + 2\|x\|^2 - \|x_n + x\|^2) = 0$. Podívejme se na prototyp výsledků, které jsme získali.

Věta H (s P. Hájkem)**.** *Buď α ordinál. Pak lze na prostoru $C([0,\alpha])$ zavést ekvivalentní normu, která je fréchetovsky diferencovatelná, LUR a limitou (uniformní na omezených množinách) C^∞-hladkých norem.*

Tato věta sama o sobě je pozitivním řešením problému 8.2 (c) v [FMZ06], plyne však z obecnějšího tvrzení (Věta I). Než se do tohoto tvrzení pustíme, podívejme se, jakou roli vlastně hrají fréchetovská diferencovatelnost a vlastnost LUR v teorii Banachových prostorů.

Fréchet diferencovatelné normy

Je všeobecně známo, že Banachův prostor, na kterém existuje ekvivalentní Fréchet diferencovatelná norma, je nutně Asplundův. Je také již dlouho známo, díky slavnému protipříkladu R. Haydona [Hay90], že opačné tvrzení neplatí. Sestavili jsme zde krátký seznam příkladů prostorů, na kterých existuje ekvivalentní Fréchet diferencovatelná norma:

a) prostory se separabilním duálem (M. I. Kadec),

b) prostory, pro něž $\mathrm{Sz}(X) < \omega_1$ (G. Lancien),

c) reflexivní prostory (S. Troyanski),

d) $C(K)$ prostory, pro které $K^{(\omega_1)} = \emptyset$ (R. Deville),

e) $c_0(\Gamma)$ pro libovolnou množinu Γ (N. H. Kuiper),

f) $C([0,\alpha])$ pro libovolný ordinál α (M. Talagrand, R. Haydon).

Dodejme, že na prostorech z příkladů a)–e) existuje norma, jejíž duální norma je LUR, zatímco na prostorech z příkladů d)–f) existuje dokonce C^∞-hladká norma. Je dobře známo, že *pokud je norma $\|\cdot\|^*$ (duální k normě $\|\cdot\|$) LUR, potom norma $\|\cdot\|$ je Fréchet diferencovatelná*. Tímto způsobem byla poprvé dokázána existence Fréchet diferncovatelných norem v příkladech a)–d). Opět podotkněme, že opečné tvrzení neplatí, protože M. Talagrand [Tal86] dokázal, že $C([0,\omega_1])$ *má ekvivalentní C^∞-hladkou normu, ale nemá žádnou normu, jejíž duální norma by byla LUR.* Již zmíněné C^∞-hladké normy z příkladů d)–f) jsou konstruovány pomocí jiné techniky, konkrétně pomocí funkcí, které *lokálně závisejí na konečně mnoha souřadnicích* (viz. Definition 4.1). To proto, že LUR vlastnost duální normy $\|\cdot\|^*$ nepřináší žádnou informaci o vyšší hladkosti normy $\|\cdot\|$. Nezapomeňme však, že P. Hájek a R. Haydon [HH07] nedávno dokázali následující důležité tvrzení: *jestliže na prostoru $C(K)$ existuje norma, jejíž duální norma je LUR, pak na $C(K)$ existuje také C^∞-hladká norma.*

Lokálně uniformně konvexní normy

Nyní je již zřejmé, že pojem LUR má v teorii renormací zásadní význam. Pro rozsáhlý přehled známých výsledků s referencemi doporučujeme [DGZ93] a nebo nedávnou monografii [MOTV09]. V návaznosti na předchozí část této práce zmiňme, že je otevřeným problémem, zda na prostoru s RNP existuje nutně také ekvivalentní LUR norma. Za povšimnutí stojí, že pro významné podtřídy prostorů s RNP tomu tak skutečně je. V článku [Lan93], G. Lancien sestavil důmyslný vzorec pro LUR normu (resp. normu, jejíž duální norma je LUR) na prostorech s $\mathrm{D}(X) < \omega_1$ (resp. $\mathrm{Dz}(X) < \omega_1$). Dalším příkladem je třída duálů Asplundových prostorů: *na duálním Banachově prostoru s RNP existuje ekvivalentní LUR norma.* Tento výsledek pochází od M. Fabiana a G. Godefroye [FG88] ([DGZ93, Corollary VII.1.12]). Z výše zmíněných faktů však plyne, že v některých případech tato LUR norma, nemůže být duální. Tak či onak, tento a mnoho dalších výsledků o LUR renormacích (viz. kapitola VII in [DGZ93]) se opírají o metodu pocházející od S. Troyanského [Tro71]. Za "slepovací" verzi, kterou hned uvedeme, vděčíme V. Zizlerovi [Ziz84] ([DGZ93, Proposition VII.1.6]): *Existuje-li na Banachově prostoru X takový projekční rozklad identity $\{P_\alpha\}_{\omega\leq\alpha\leq\mu}$, že na prostorech $(P_{\alpha+1}-P_\alpha)X$ existuje ekvivalentní LUR norma, pak na X samotném existuje také ekvivalentní LUR norma.* Zatím se spokojme s vysvětlením, že projekční rozklad identity je neseparabilním zobecněním pojmu posloupnosti projekcí přidružených k Schauderově bázi, a odložme její přesnou definici na později (viz. Definition 4.12).

Kombinace fréchetovské diferencovatelnosti a LUR

V mnoha situacích je žádoucí mít na zadaném prostoru X ekvivalentní normu, která má dobré vlastnosti jak po stránce hladkosti, tak po stránce konvexity. Klasickým výsledkem v této oblasti je metoda obecně známá jako *Asplundovo průměrování*: *Existuje-li na Banachově prostoru X LUR norma $\|\cdot\|_1$ a také norma $\|\cdot\|_2$, jejíž duální norma je LUR, potom na X existuje norma $\|\cdot\|_3$, která je LUR a jejíž duální norma je LUR.* Speciálně, norma $\|\cdot\|_3$ je zároveň LUR a fréchetovsky diferencovatelná. Asplundovo průměrování v této formě bylo dokázáno M. Fabianem, L. Zajíčkem a V. Zizlerem v [FZZ82] ([DGZ93, II.4.3]) za pomocí Baireových kategorií. Ve zmiňovaném článku je dokázáno, že množina ekvivalentních LUR norem je buď prázdná, nebo reziduální v prostoru všech ekvivalentních norem na X. Nápodobně pro množinu všech ekvivalentních norem, jejichž duální norma na X^* je LUR. Asplundovo průměrování zažilo nedávno nečekaný zvrat, když R. Haydon [Hay08] dokázal následující hluboký výsledek: *existuje-li na X ekvivalentní norma, jejíž duální norma je LUR, pak na X existuje také ekvivalentní LUR norma.*

V této souvislosti musíme podotknout, že bez silnějších strukturálních požadavků na prostor X nelze očekávat existenci normy, která je zároveň LUR a hladká vyššího řádu. Skutečně, podle [FWZ83] ([DGZ93, Propostiton V.1.3]) platí, že *existuje-li na X norma, jež je LUR a zároveň C^2-hladká, pak X je superreflexivní.* Na druhé straně, v superreflexivních prostorech lze pomocí Asplundova průměrování vyrobit normu, která je uniformně konvexní a zároveň uniformně Fréchet diferencovatelná.

Z výše uvedených úvah plyne, že předpoklady pro Asplundovo průměrování nemusejí být vždy splněny, například v prostoru $C([0,\omega_1])$. V tomto případě bychom měli zdůraznit, že není známo, jestli množina všech Fréchet diferencovatelných norem je reziduální nebo aspoň hustá v prostoru všech ekvivalentních norem na $C([0,\alpha])$.

Věta analogická naší větě H pro případ $X = c_0(\Gamma)$ byla dokázána v [PWZ81] ([DGZ93, Theorem V.1.5]). Separabilním vzorem pro náš hlavní výsledek je následující věta (viz. [MPVZ93]): *Na každém C^k-hladkém Banachově prostoru existuje ekvivalentní LUR a zároveň Fréchet diferencovatelná norma, která je navíc limitou C^k-hladkých norem.*

Nyní vyslovíme náši hlavní větu. Povšimněte si podobnosti s výše uvedenou Zizlerovou verzí Troyanského renormace. Dalo by se říci, že naše věta je jejím zhlazením.

Věta I (s P. Hájkem). *Nechť $k \in \mathbb{N} \cup \{\infty\}$. Nechť X je Banachův prostor s takovou PRI $\{P_\alpha\}_{\omega\le\alpha\le\mu}$, že na každém z prostorů $(P_{\gamma+1} - P_\gamma)X$ existuje ekvivalentní fréchetovsky diferencovatelná, LUR norma, která je navíc limitou (uniformní na omezených množinách) C^k-hladkých norem. Předpokládejme navíc, že na X existuje ekvivalentní C^k-hladká norma.*

Pak na X existuje ekvivalentní fréchetovsky diferencovatelná, LUR norma, která je navíc limitou (uniformní na omezených množinách) C^k-hladkých norem.

Tato věta je ve skutečnosti indukčním krokem v důkazu vedoucím k výše zmíněné větě H a také k následujícímu tvrzení.

Věta J (s P. Hájkem). *Nechť $k \in \mathbb{N} \cup \{\infty\}$. Budiž $\mathcal{P}$ nějaká třída Banachových prostorů, jejíž každý prvek X*

(i) *má PRI $\{P_\alpha\}_{\omega\le\alpha\le\mu}$ s vlastností, že $(P_{\alpha+1} - P_\alpha)X \in \mathcal{P}$,*

(ii) *má ekvivalentní C^k hladkou normu.*

Potom na každém prostoru $X \in \mathcal{P}$ existuje ekvivalentní fréchetovsky diferencovatelná, LUR norma, která je navíc limitou (uniformní na omezených množinách) C^k-hladkých norem.

Aniž bychom zabíhali do detailů, podotýkáme, že podmínka (i) je splněna například pro slabě kompaktně generované prostory (WCG), tedy takové prostory, které obsahují totální (normově lineárně hustý) slabý kompakt. Dále podmínka (i) platí pro Vašákovy prostory, $C(K)$ prostory, kde K je Valdiviův kompakt, atd. Pro důkaz těchto netriviálních tvrzení včetně referencí doporučujeme kapitolu VI v [DGZ93] a kapitolu 5 v [HMSVZ08]. Nyní je zřejmé, že lze získat třídu $\mathcal{P}$, která splňuje obě podmínky (i) i (ii) prostým průnikem jedné z výše uvedených tříd s třídou C^k-hladkých prostorů.

Nakonec pár slov o významu faktu, že naše nově definovaná norma je aproximována normami, které jsou C^k-hladké. Takový výsledek je úzce spjat s otázkou, zda pro C^k-hladký prostoru X jsou C^k-hladké ekvivalentní normy na X husté v prostoru všech ekvivalentních norem na X. Odpověď není známa v plné obecnosti dokonce ani v separabilním případě, ačkoliv pozitivní výsledky z článků [DFH96] a [DFH98] jsou velmi silné a uplatní se pro většinu klasických Banachovýh prostorů. Například: *Nechť* $X = C(K)$, *kde* K *je spočetný kompakt. Pak lze každou ekvivalentní normu na* X *aproximovat analytickými normami.* V podobném duchu se nese výsledek M. Fabiana, P .Hájka a V. Zizlera [FHZ97]: *Libovolná silně svazová norma na* $(c_0(\Gamma), \|\cdot\|_\infty)$ *je aproximovatelná (uniformně na omezených množinách)* C^∞*-hladkými normami.* Podívejte se také na větu 4.15, kde dokazujeme, že tyto aproximující normy mají jisté vlastnosti, které nám přijdou vhod při důkazu věty I. Kromě tohoto tvrzení nejsou v neseparabilní situaci známy žádné obecné výsledky. Speciálně, jeden z otevřených problémů v [DGZ93] klade otázku, zda na C^k hladkém WCG Banachově prostoru musí nutně existovat ekvivalentní LUR norma, která je limitou C^k-hladkých norem. Věta J spolu s výše uvedenými strukturálními výsledky o WCG prostorech dává pozitivní řešení tohoto problému.

Zajímavé je, že *je-li* X *Vašákův, resp.* X *je prostor* $C(K)$*, pak existence* C^k*-hladké normy (obecněji* C^k*-hladkého bumpu) implikuje, že každou spojitou funkci lze aproximovat uniformně* C^k*-hladkými funkcemi* (viz. [DGZ93, Chapter VIII], resp. [HH07]). Když se však tímto způsobem pokusíme aproximovat lipschitzovskou nebo konvexní funkci (nebo třeba normu), obecně nebudeme mít žádnou informaci o lipschitzovském chování či konvexitě aproximující funkce.

V poslední kapitole se budeme zabývat konvexními aproximacemi konvexní funkce f, které sice nezlepšují diferencovatelnost, ale zato se pyšní bodem silného minima. Ano, toto téma je již dobře prozkoumáno v práci M. Fabiana, P. Hájka a J. Vanderwerffa [FHV96], a proto se zaměříme na situaci, kdy f záleží spojitě na nějakém parametru, a budeme se snažit získat tyto aproximace i minimizéry tak, aby také spojitě závisely na daném parametru.

1.3.6 Parametrický variační princip

Výsledky v této sekci pocházejí ze společného článku R. Devilla a autora této práce ([DP]). Co je *variační princip* je snad nejlépe vidět na konkrétním příkladu Ekelandova variačního principu: *Nechť* X *je Banachův prostor,* $f : X \to (-\infty, +\infty]$ *zdola omezená, zdola polospojitá (l.s.c.) funkce a nechť* $\varepsilon > 0$*. Pak existuje bod* $v \in X$ *tak, že* $x \mapsto f(x) + \varepsilon \|x - v\|$ *nabývá svého minima v bodě* v*.* Takovéto tvrzení obvykle slouží jako náhražka za kompaktnost množiny, ve které hledáme minimum f. Řečeno geometricky, graf funkce $x \mapsto f(v) - \varepsilon \|x - v\|$ se zdola dotýká epigrafu funkce f v bodě v. Funkce $x \mapsto \varepsilon \|x - v\|$ se nazývá *perturbace.* Variační principy J. Borweina a D. Preisse, resp. R. Devilla, G. Godefroye a V. Zizlera (DGZ), zobecňují výše zmíněnou větu. Konkrétně tvrdí, že je možné nalézt C^k-hladké perturbace, resp. C^k-hladké a lipschitzovské perturbace, za předpokladu že na prostoru X existuje C^k-hladká norma, resp. C^k-hladký bump s omezenými derivacemi řádu k. DGZ variační princip je tedy zesílením Borweinova-Preissova variačního principu, a navíc je dokázán elegantním argumentem za pomoci Baireových kategorií.

Parametrický variační princip má na svém vstupu systém $\{x \mapsto f(p, x)\}$ zdola polospojitých, zdola omezených funkcí, které závisejí spojitě na parametru p v nějakém topologickém prostoru Π. Cílem je pro každé $p \in \Pi$ perturbovat funkci $f(p,\cdot)$ pomocí funkce $\Delta(p) : X \to \mathbb{R}$ tak, že $f(p,\cdot) + \Delta(p)$ nabývá svého minima v nějakém bodě $v(p)$ a tak, že $v(p)$ i $\Delta(p)$ závisejí spojitě na parametru p. Parametrický hladký variační princip v duchu principu Borwein-Preiss pochází od P. Georgieva [Geo05]. L. Veselý [Ves09] nedávno modifikoval jeho důkaz, aby získal parametrický hladký variační princip s vazbami. Přesněji řečeno: nechť $\Pi_0 \subset \Pi$ je taková podmnožina, že pro každé $p \in \Pi_0$ funkce $f(p,\cdot)$ nabývá minima v bodě $v_0(p)$. Veselý konstruuje minimizér v, který rozšiřuje zobrazení v_0. Hlavní věta této kapitoly (Theorem 5.18) je parametrizovaná verze variačního principu DGZ a jeho důkazu. Naše hlavní věta implikuje například:

Věta K (s R. Devillem)**.** *Nechť Π je parakompaktní Hausdorffův topologický prostor, X je Banachův prostor s Fréchet diferencovatelnou normou a $\mathcal{Y}$ buď kužel všech konvexních, nezáporných, lipschitzovských, Fréchet diferencovatelných funkcí na X. Kužel $\mathcal{Y}$ je vybaven svojí přirozenou normou $\|g\|_{\mathcal{Y}} = |g(0)| + \|g\|_{\mathrm{Lip}}$. Nechť $f : \Pi \times X \to \mathbb{R}$ splňuje*

(i) *pro všechna $p \in \Pi$, funkce $f(p,\cdot)$ je konvexní, spojitá, zdola omezená,*

(ii) *pro všechna $x \in X$, funkce $f(\cdot, x)$ je spojitá,*

(iii) *pro všehcna $p_0 \in \Pi$, $(f(p_0,\cdot)-f(p,\cdot))^+ \to 0$ uniformně na omezených podmnožinách prostoru X, když $p \to p_0$.*

Pak pro každé $\varepsilon > 0$ existují zobrazení $\Delta \in C(\Pi, \mathcal{Y})$ a $v \in C(\Pi, X)$ taková, že $\|\Delta(p)\|_{\mathcal{Y}} < \varepsilon$ a $f(x,\cdot) + \Delta(p)$ nabývá silného minima v bodě $v(p)$ pro všechna $p \in \Pi$. Navíc $p \mapsto f(p, v(p)) + \Delta(p)(v(p))$ je spojité.

Hlavní nedostatek, který je průvodním jevem všech parametrických variačních principů a který výrazně zmenšuje pole jejich působnosti, je fakt, že všechny funkce $f(p,\cdot)$ musejí být být konvexní. Že se nejedná o pouhou technickou obtíž plyne z jednoduchých příkladu (Example 5.26). Ty ukazují, že v nekonvexním případě můžeme zapomenout na spojitý minimizér dokonce i po perturbaci funkce f.

Dále ukážeme v sekci 5.4, že ani druhá z neočekávaných podmínek věty K, tedy podmínka (iii), nemůže být bez náhrady vyškrtnuta. Na druhé straně, podmínka (iii) plyne z podmínek (i) a (ii), předpokládáme-li navíc, že Π je metrizovatelný a že dimenze prostoru X je konečná. To plyne okamžitě z následující věty.

Věta L (s R. Devillem)**.** *Buďte f a f_n ($n \in \mathbb{N}$) spojité a konvexní funkce z Banachova prostoru X do $\mathbb{R}$ a předpokládejme, že $f_n \to f$ bodově v X. Pak $f_n \to f$ uniformně na kompaktních podmnožinách prostoru X.*

Na závěr si uvědomme, že i když $\dim X < \infty$ a $p \mapsto f(p,\cdot)$ nabývá svého minima v nějakém bodě $v(p)$ pro každé $p \in \Pi$, zobrazení v nemusí být spojité. Skutečně, existence *spojitého* minimizéru pro původní funkci není zaručena ani v nejjednoduších situacích (viz. Problem 5.1 a sekce 5.4).

1.4 Introduction (version française)

Nous allons traiter ici quatre sujets différents de la théorie des espaces de Banach et c'est le but de la section d'introduction de montrer la place qu'ils occupent dans la théorie.

Ici et dans toute la suite, X est un espace de Banach réel avec une boule unité fermée B_X et avec l'espace dual X^*.

1.4.1 La propriété de Radon-Nikodým

Nous commençons par une définition d'un concept élémentaire. Soit A un sous-ensemble d'un espace de Banach X. Soit $f \in X^* \setminus \{0\}$ et $a \in \mathbb{R}$. Si l'ensemble $S = \{x \in A : f(x) > a\}$ est non vide, il est appelé *tranche ouverte* de A (ou seulement *tranche* de A quand il n'y a pas de risque de confusion). Nous notons $\mathcal{S}_o(A)$ l'ensemble de toutes les tranches de A.

Avec la notion de tranche à disposition, nous pouvons définir une classe importante d'espaces de Banach que nous rencontrerons constamment dans tout ce texte. On dit qu'un espace de Banach X a propriété de *Radon-Nikodým* (RNP) si chaque sous-ensemble non vide borné de X a des tranches (non vides par la définition) arbitrairement petites. Plus précisément, X a RNP si pour chaque sous-ensemble non vide borné A de X et pour chaque $\varepsilon > 0$, il existe une tranche $S \in \mathcal{S}_o(A)$ tels que $\operatorname{diam}(S) < \varepsilon$.

En raison du quantificateur universel dans la définition, RNP est naturellement une propriété isomorphe. Son importance demeure dans le fait que beaucoup de constructions familières sur les réels peuvent être traduites aux espaces avec RNP. Un exemple de notre affirmation est la définition originale de RNP issue de la théorie de la mesure, qui explique également le nom de la propriété : *Soit $\mathcal{B}$ la tribu des boréliens sur $[0,1]$, soit λ la mesure de Lebesgue sur $[0,1]$. Un espace de Banach X a RNP si et seulement si chaque mesure m à valeurs dans X et définie sur l'espace de probabilité $([0,1], \mathcal{B}, \lambda)$ qui est de variation totale finie et absolument continu par rapport à λ, est représentée par un $f \in L^1([0,1], X)$ au moyen de l'égalité $m(A) = \int_A f(x) d\lambda(x)$.*

Pour voir comment les divers mathématiciens ont contribués à ce résultat, nous renvoyons le lecteur à l'excellente monographie [Bou83]. Le théorème ci-dessus est un moyen (plutôt lourd) de voir que par exemple $\mathbb{R}^n$ a RNP.

Notre premier résultat énonce que les espaces de Banach possédant RNP sont exactement ceux pour lesquels on a une extension du fait élémentaire que les suites monotones bornés de réels convergent.

Théorème A. *Pour un espace de Banach X, les assertions suivantes sont équivalentes :*

(i) *X a RNP,*

(ii) *il existe une application $F : X \to X^*$ telle que toute suite bornée $(x_n) \subset X$ converge si elle satisfait*

$$\langle F(x_n), x_n \rangle \leq \langle F(x_n), x_{n+1} \rangle \text{ pour tout } n \in \mathbb{N}.$$

L'assertion (ii) n'est pas facile à etablir même lorsque $X = \mathbb{R}^2$, ainsi ce théorème n'est pas vraiment un bon outil pour prouver qu'un espace a RNP. Par contre, il est très utile comme une condition suffisante pour la convergence des suites dans X. L'implication (ii) $\Rightarrow$ (i) est due à R. Deville et É. Matheron [DM07] qui ont également montré que *si X possède une norme uniformément convexe, alors* (ii) *est verifié.* L'implication (i) $\Rightarrow$ (ii) a été montrée en toute généralité dans [Pro09] et nous verrons la preuve au Chapitre 2.

Soit $\mathcal{S}_c(X)$ l'ensemble de tous les *demi-espaces fermés* de X, c.-à-d. les ensembles de la forme $\{x \in X : f(x) \geq a\}$ pour un $f \in X^* \setminus \{0\}$ et $a \in \mathbb{R}$. En fait, le théorème ci-dessus peut se refotmuler á l'aide du jeu des points et demi-espaces fermés noté $\mathbf{G}(X, \mathcal{S}_c(X))$ que nous décrivons maintenant. Il y a deux joueurs – le joueur I et joueur II. Le joueur I commence le jeu en choisissant arbitrairement un point $x_1 \in X$. Le joueur II joue puis un demi-espace fermé H_1 contenant le point x_1 ; alors le joueur I sélectionne un point $x_2 \in H_1$ et la réponse du joueur II est un demi-espace fermé H_2 qui contient x_2 (mais pas nécessairement x_1) ; alors le joueur I choisit un point x_3 dans H_2 (mais pas nécessairement dans H_1) ; et ainsi de suite. Ce qui précède s'appelle une *partie* du jeu $\mathbf{G}(X, \mathcal{S}_c(X))$. Le joueur II gagne la partie si la suite (x_n) ainsi formée est de Cauchy ou non bornée. Une *tactique gagnante* pour le joueur II est une application $t : X \to \mathcal{S}_c(X)$ telle que $x \in t(x)$ pour tout x, et telle que si le joueur II choisit toujours $H_n := t(x_n)$, ça va assurer sa victoire. On voit facilement (cf. Proposition 2.4) que (ii) dans Théorème A est équivalent à dire que le joueur II a une tactique gagnante dans le jeu $\mathbf{G}(X, \mathcal{S}_c(X))$.

Dans un cadre plus abstrait, si X est remplacé par un ensemble K et $\mathcal{A}$ est une collection de sous-ensembles de K tels que $K = \bigcup \mathcal{A}$, nous définissent le *jeu point-ensemble* $\mathbf{G}(K, \mathcal{A})$ avec les mêmes règles que ci-dessus (voir Définition 2.1).

La conception de jeu est due à J. Malý et M. Zelený [MZ06] qui ont également montré que le joueur II a une *stratégie gagnante* pour le jeu $\mathbf{G}(B_{\mathbb{R}^2}, \{\text{droites}\})$. Une stratégie gagnante est une règle de décision représentée par une suite des applications $t_n : K^n \to \mathcal{A}$ où le choix $H_n := t_n(x_1, \ldots, x_n)$ assure la victoire du joueur II.

Notez que toute tactique gagnante du joueur II est automatiquement une stratégie gagnante du joueur II. Nous allons illustrer ici le fait que la réciproque est fausse. Il est prouvé dans [DM07] que *si X possède RNP, alors le joueur II a une stratégie gagnante dans* $\mathbf{G}(B_X, \mathcal{S}_o(B_X))$. Par contre, nous prouvons ici :

Théorème B. *Si* $\dim X > 0$ *et si $\mathcal{A}$ est une sous-collection d'ouverts de B_X telle que $B_X = \bigcup \mathcal{A}$. Alors le joueur II n'a **jamais** une tactique gagnante dans le jeu* $\mathbf{G}(B_X, \mathcal{A})$. *En particulier, le joueur II n'a **jamais** de tactique gagnante dans le jeu* $\mathbf{G}(B_X, \mathcal{S}_o(B_X))$.

Deville et Matheron en [DM07] construisent leur stratégie gagnante en utilisant un lemme abstrait qu'ils utilisent également pour obtenir la caractérisation suivante : *un espace de Banach X a la propriété des points de continuité si et seulement s'il existe une stratégie gagnante pour le joueur II dans le jeu* $\mathbf{G}(B_X, \{$*ouverts faibles*$\})$. Rappelons que X a la *propriété des points de continuité* (PCP) si chaque sous-ensemble borné non vide de X a des sous-ensembles relativement faiblement ouverts non vides de diamètre arbitrairement petit. C'est juste une autre manière de dire que (B_X, w) est fragmentée par la norme. Dans le même esprit, et en utilisant le même lemme, nous prouvons ici une caractérisation des fonctions de première classe de Baire :

Théorème C. *Soit (E,τ) une espace topologique complètement métrisable et soit $f : E \to X$ une application de E vers un espace vectoriel normé X. Le jeu $\mathbf{G}_f(E,\tau)$ est défini comme $\mathbf{G}(E,\tau)$ avec la différence que le joueur II gagne une partie si la suite $(f(x_n))_n \subset X$ est de Cauchy, où $(x_n)_n \subset E$ est la suite des points joués par le joueur I pendant la partie. Les assertions suivantes sont équivalentes*

(i) *f est de première classe de Baire, c.-à-d. f est limite ponctuelle d'une suite d'applications continues,*

(ii) *le joueur II a une stratégie gagnante dans le jeu $\mathbf{G}_f(E,\tau)$.*

1.4.2 Aspects quantitatifs de la propriété de Radon-Nikodým

Décrivons un procédé général d'épluchage qui est fréquemment employé pour assigner un certain indice ordinal (invariant par isomorphisme) à un espace de Banach X donné. Supposons que $\mathcal{A}$ est une sous-collection d'ouverts de X et soit C un sous-ensemble de X. Pour $\varepsilon > 0$ nous définissons la dérivée de C

$$[C]'_\varepsilon = C \setminus \bigcup \{A \in \mathcal{A} : \operatorname{diam}(A \cap C) < \varepsilon\}$$

et nous posons

$$[C]^0_\varepsilon := C, \quad [C]^{\alpha+1}_\varepsilon := [[C]^\alpha_\varepsilon]'_\varepsilon \quad \text{et} \quad [C]^\beta_\varepsilon := \bigcap_{\alpha<\beta} [C]^\alpha_\varepsilon$$

pour chaque ordinal α et chaque ordinal limite β. Puis nous définissons

$$\iota_{\mathcal{A}}(X,\varepsilon) := \inf\{\alpha : [B_X]^\alpha_\varepsilon = \emptyset\} \text{ et } \iota_{\mathcal{A}}(X) := \sup_{\varepsilon>0} \iota_{\mathcal{A}}(X,\varepsilon)$$

en adoptant la convention que $\inf\emptyset = \infty$ et $\alpha < \infty$ pour chaque ordinal α. Le choix $\mathcal{A} = \mathcal{S}_o(X)$, a comme conséquence la définition de *l'indice de dentabilité* $\mathrm{D}(X)$ de X. Donc $\mathrm{D}(X) = \iota_{\mathcal{S}_o(X)}(X)$ et nous noterons $wd^\alpha_\varepsilon(C) := [C]^\alpha_\varepsilon$ dans ce cas.

C'est évident que si X possède RNP, alors $wd^1_\varepsilon(C) \subsetneq C$ pour tout $\varepsilon > 0$ et C sous-ensemble de X non vide, fermé, convexe et borné. Par contre si X n'a pas RNP, il existe $\varepsilon > 0$ et $A \subset B_X$ non vide sans aucune tranche plus petite que ε, puis clairement $A \subset wd^\alpha_\varepsilon(B_X)$ pour chaque ordinal α. Il suit que X a RNP si et seulement si $\mathrm{D}(X) < \infty$. En fait, si X est séparable il découle d'un argument de cardinalité (Theorem I.6.9 de [Kec95]) que X a RNP si et seulement si $\mathrm{D}(X) < \omega_1$, où ω_1 est le premier ordinal non dénombrable. Un théorème dû à G. Lancien [Lan95] dit que *un espace de Banach X une norme uniformément convexe si et seulement si* $\mathrm{D}(X) \le \omega$, *où ω est le premier nombre ordinal infini.* Nous avons ici un petit commentaire au théorème de Lancien – une version quantitative du Théorème A.

Théorème D. *Soit X un espace de Banach. Le assertions suivantes sont équivalentes :*

(i) *X possède une norme uniformément convexe,*

(ii) *pour chaque $0 < \varepsilon < 1$, il existe une application $F : B_X \to X^*$ et un nombre naturel $k \in \mathbb{N}$ tels que si $(x_n)_{n=1}^m \subset B_X$ est une suite finie telle que*

$$\|x_n - x_{n+1}\| > \varepsilon \text{ et } \langle F(x_n), x_n \rangle \le \langle F(x_n), x_{n+1} \rangle$$

pour $n = 1, \ldots, m-1$, alors $m < k$.

Nous remarquons que M. Zelený a montré dans [Zel08] que si $\dim X < \infty$, alors les quantificateurs dans (ii) peuvent être échangés : *il existe une application $F : B_X \to X^*$ telle que pour tout $0 < \varepsilon < 1$, il existe $k \in \mathbb{N}$ ...* Il n'est pas clair que ceci puisse être fait en général pour n'importe quel espace X avec une norme uniformément convexe.

Comme avant, le Théorème D est une conséquence d'une version (Theorem 2.31) où (ii) est exprimée en termes du jeu $\mathbf{G}(B_X, \mathcal{S}_c(B_X))$. Nous énoncerons ultérieurement des théorèmes semblables avec (i) exprimé en termes d'indice faible de Szlenk, resp. d'indice d'oscillation, et (ii) exprimé en termes du jeu $\mathbf{G}(B_X, \{\text{ouverts faibles}\})$, resp. $\mathbf{G}_f(M, \tau)$ (voir les remarques 2.34 et 2.38).

L'indice faible de Szlenk $\mathrm{Sz}_w(X)$ de X est obtenu en prenant $\mathcal{A} = \sigma(X, X^*)$ (où $\sigma(X, X^*)$ sont les ouverts faibles de X) dans le procédé général d'épluchage ci-dessus, c.-à-d. $\mathrm{Sz}_w(X) = \iota_{\sigma(X,X^*)}(X)$. C'est clair, par notre discussion de l'indice de dentabilité ci-dessus, qu'un espace de Banach X possède PCP si et seulement si $\mathrm{Sz}_w(X) < \infty$, et que pour un espace de Banach séparable avec PCP, on a $\mathrm{Sz}_w(X) < \omega_1$. C'est aussi évident que $\mathrm{Sz}_w(X) \le \mathrm{D}(X)$, mais nous remarquons que le prédual de l'espace de James JT est un espace séparable avec $\mathrm{Sz}_w(X) < \omega_1$ et $\mathrm{D}(X) = \infty$ parce que cet espace a PCP mais pas RNP [FGdB97].

1.4.3 La propriété de Radon-Nikodým dans les duaux

Un espace de Banach X est dit *espace d'Asplund* si chaque fonction continue convexe définie dans un ouvert $D \subset X$ convexe non vide est Fréchet différentiable en chaque point d'un sous-ensemble dense et G_δ de D. La théorie des espaces d'Asplund est bien développée [DGZ93, Phe93, HMSVZ08]. Elle est reliée à la théorie des espaces avec RNP via la caractérisation suivante de dualité : *Un espace de Banach X est d'Asplund si et seulement si chaque sous-ensemble borné non vide du dual X^* a des tranches préfaiblement ouvertes arbitrairement petites si et seulement si X^* a RNP.* Des fragments de ce résultat sont dûs à E. Asplund [Asp68], I. Namioka and R. Phelps [NP75], et C. Stegall [Ste75]. Un des blocs constitutifs pour cette caractérisation est le résultat suivant : *Soit X un espace de Banach séparable. Alors X^* a RNP si et seulement si X^* est séparable.* Pour des détails voyez par exemple [Phe93]. La possibilité de travailler soit avec les tranches faiblement ouvertes soit avec les tranches préfaiblement ouvertes a beaucoup de conséquences. L'une d'entre eux est que nous pouvons énoncer une version préfaible de notre Théorème A avec l'application F ayant des valeurs dans le prédual X de X^*. Une autre est qu'*un espace* *dual X^* a RNP si et seulement si (B_{X^*}, w^*) est fragmentée par la norme.* Notez que JT^*, le dual de l'espace de James JT, a PCP mais pas RNP, par conséquent (B_{JT^*}, w) est fragmentée par la norme bien que (B_{JT^*}, w^*) ne le soit (voir par exemple [FGdB97]).

Soit $\mathcal{S}_o^*(X^*)$, resp. $\sigma(X^*, X)$, les demi-espaces, resp. les ensembles préfaiblement ouverts dans X^*. Par analogie au procédé général d'épluchage ci-dessus, nous définissons pour n'importe quel espace de Banach X les indices $\mathrm{Dz}(X) := \iota_{\mathcal{S}_o^*(X^*)}(X^*)$ et $\mathrm{Sz}(X) := \iota_{\sigma(X^*,X)}(X^*)$. Le premier s'appelle *l'indice de dentabilité préfaible* de X tandis que le second est *l'indice de Szlenk* de X. Notons qu'il est évident que pour tout espace de Banach X on a $\mathrm{Sz}(X) \leq \mathrm{Dz}(X)$. De même, comme dans le cas de l'indice de dentabilité de X, resp. indice de Szlenk faible, l'espace X^* a RNP si et seulement si $\mathrm{Dz}(X) < \infty$, resp. (B_{X^*}, w^*) est fragmentée par la norme si et seulement si $\mathrm{Sz}(X) < \infty$. Recueillant tout ce qui a été dit, on peut voir que pour un espace de Banach X, les assertions suivantes sont équivalentes :

(o) X est d'Asplund

(i) X^* a RNP,

(ii) $\mathrm{Dz}(X) < \infty$,

(iii) $\mathrm{Sz}(X) < \infty$,

(iv) (B_{X^*}, w^*) est fragmentée par la norme.

Enfin, si X est séparable, (ii) peut être remplacé par $\mathrm{Dz}(X) < \omega_1$ et (iii) peut être remplacé par $\mathrm{Sz}(X) < \omega_1$. En réalité, un résultat beaucoup plus fort dû à B. Bossard [Bos02] et G. Lancien [Lan96] dit qu'*il existe une fonction $\Psi : (0, \omega_1) \to (0, \omega_1)$ telle que $\mathrm{Dz}(X) \leq \Psi(\mathrm{Sz}(X))$ pour n'importe quel espace de Banach X avec $\mathrm{Sz}(X) < \omega_1$.* Ce résultat, prouvé par des méthodes de théorie descriptive des ensembles, a été récemment amélioré par M. Raja [Raj07] qui a employé des moyens purement géométriques et un argument d'induction transfinie pour obtenir : $\mathrm{Dz}(X) \leq \omega^{\mathrm{Sz}(X)}$ *pour tout espace de Banach X*, de ce fait prouvant que le théorème de Bossard et Lancien est vérifié si $\Psi(\alpha) = \omega^\alpha$. Dans ce contexte il est intéressant de demander quelles sont les valeurs optimales pour la fonction Ψ. En d'autres termes, quelles sont les valeurs explicites de la fonction

$$\Psi_o(\alpha) := \inf \{\beta : \mathrm{Dz}(X) \leq \beta \text{ pour chaque } X \text{ avec } \mathrm{Sz}(X) = \alpha\}.$$

Ainsi le théorème de Raja signifie $\Psi_o(\alpha) \leq \omega^\alpha$. Pour être exacts, nous devons ajouter que la fonction Ψ_o n'a d'intérêt qu'aux points α pour lesquels il existe un espace de Banach X avec $\mathrm{Sz}(X) = \alpha$. Par exemple, il est bien connu (voir Lemma 3.3) que *pour tout espace X d'Asplund il existe un ordinal α tel que* $\mathrm{Sz}(X) = \omega^\alpha$. Notez qu'il a été montré par P. Hájek et G. Lancien dans [HL07] que *si* $\mathrm{Sz}(X) \leq \omega$, *alors* $\mathrm{Dz}(X) \leq \omega^2$. *L'espace c_0 verifie* $\mathrm{Sz}(c_0) = \omega$ *et* $\mathrm{Dz}(c_0) = \omega^2$. En particulier $\Psi_o(\omega) = \omega^2$. Généralement, savoir des valeurs exactes de $\mathrm{Sz}(X)$ et $\mathrm{Dz}(X)$ pour un certain espace X donne une estimation inférieure pour Ψ_o au point $\mathrm{Sz}(X)$. Nous allons déterminer ici les valeurs exactes de l'indice de dentabilité préfaible pour la classe des espaces $C([0,\alpha])$ des fonctions continues sur des intervalles ordinaux compacts $[0,\alpha]$ avec α dénombrable. Ceci fournira l'estimation inférieure suivante de Ψ_o.

Théorème E (avec P. Hájek et G. Lancien). *Soit* $0 \leq \alpha < \omega_1$. *Alors*

$$\omega \cdot \omega^{\alpha+1} \leq \Psi_o(\omega^{\alpha+1}).$$

1.4.4 Les espaces $C(K)$, $K^{(\omega_1)} = \emptyset$

Il découle de la théorie générale des espaces d'Apslund que *l'espace* $C(K)$ *des fonctions continues sur le compact* K*, est un espace d'Asplund si et seulement si* K *est dispersé.* Un compact K s'appelle *dispersé* s'il existe un ordinal α tel que le dérivé de Cantor $K^{(\alpha)}$ d'ordre α est vide (voir Definition 3.8). Nous limiterons temporairement notre attention au cas $K^{(\omega_1)} = \emptyset$.

Rappelons la classification isomorphe fondamentale des espaces ci-dessus $C([0,\alpha])$ par C. Bessaga et A. Pełczyński [BP60, HMSVZ08] : *Soit* $\omega \leq \alpha \leq \beta < \omega_1$. *Alors* $C([0,\alpha])$ *est isomorphe à* $C([0,\beta])$ *si et seulement si* $\beta < \alpha^\omega$. En fait, la partie "seulement si" a été réprouvée plus tard par C. Samuel [Sam84] en évaluant l'indice de Szlenk de ces espaces de façon exacte : *Soit* $0 \leq \alpha < \omega_1$. *Alors* $\mathrm{Sz}(C([0,\omega^{\omega^\alpha}])) = \omega^{\alpha+1}$. *En particulier* $C([0,\beta_1])$ *est isomorphe à* $C([0,\beta_2])$ *si et seulement si* $\mathrm{Sz}(C([0,\beta_1])) = \mathrm{Sz}(C([0,\beta_2]))$. Une preuve facile géométrique de ce fait est apportée dans [HL07]. Davantage d'élaboration sur cette preuve dans un papier commun de P. Hájek, G. Lancien et de l'auteur [HLP09] mène au résultat suivant :

Théorème F (avec P. Hájek et G. Lancien). *Soit* $0 \leq \alpha < \omega_1$. *Alors*

$$\mathrm{Dz}(C([0,\omega^{\omega^\alpha}])) = \omega^{1+\alpha+1}.$$

(N'oublions pas que $1+\alpha+1 = \alpha+2$ *si* $\alpha < \omega$*, et* $1+\alpha+1 = \alpha+1$ *sinon.)*

En particulier $C([0,\beta_1])$ *est isomorphe à* $C([0,\beta_2])$ *si et seulement si* $\mathrm{Dz}(C([0,\beta_1])) = \mathrm{Dz}(C([0,\beta_2]))$.

Un fait topologique bien connu, dû à Mazurkiewicz et Sierpiński [HMSVZ08, Theorem 2.56], est que *tout compact* K *dénombrable est homéomorphe à quelque intervalle ordinal* $[0,\alpha]$ *(*α *dénombrable)*, donc $C(K)$ est isomorphe à $C([0,\alpha])$ et on peut facilement calculer l'indice de dentabilité d'un tel $C(K)$. C'est possible de faire une étape de plus, cependant. En utilisant un argument de réduction séparable nous obtenons

Théorème G (avec P. Hájek et G. Lancien). *Soit* $0 \leq \alpha < \omega_1$. *Soit* K *un espace compact dont les dérivées de Cantor satisfont* $K^{(\omega^\alpha)} \neq \emptyset$ *et* $K^{(\omega^{\alpha+1})} = \emptyset$. *Alors*

$$\mathrm{Dz}(C(K)) = \omega^{1+\alpha+1}.$$

1.4.5 Normes avec de bonnes propriétés

Les résultats principaux de cette section proviennent d'un travail commun de P. Hájek et l'auteur ([HP09]) sur les renormages qui sont simultanément LUC, lisses, et approximables par des normes d'une lissité plus élevée. Rappelons qu'une norme $\|\cdot\|$ sur un espace de Banach X est *localement uniformément convexe* (LUC) si $\lim_n \|x_n - x\| = 0$ lorsque $\lim_n(2\|x_n\|^2 + 2\|x\|^2 - \|x_n + x\|^2) = 0$. Voici un prototype du genre de théorèmes que nous obtiendrons.

Théorème H (avec P. Hájek)**.** *Soit α un ordinal. Alors l'espace $C([0,\alpha])$ possède une norme équivalente qui est C^1-lisse, LUC et limite (uniforme sur les sous-ensembles bornés de X) de normes C^∞-lisses.*

Ce théorème particulier fournit une solution positive du Problème 8.2 (c) dans [FMZ06]. Il découle d'un théorème plus général (Théorème I). Avant d'énoncer le théorème général, nous allons examiner brièvement le rôle joué par la différentiabilité de Fréchet et la propriété LUC dans la théorie des espaces de Banach.

Les normes Fréchet-différentiables

Il est bien connu qu'un espace de Banach qui admet une norme équivalente Fréchet-différentiable est nécessairement un espace d'Asplund. Un contre-exemple célèbre de R. Haydon [Hay90] montre que l'inverse n'est pas vrai en général. En tant que une fonction convexe, si une norme $\|\cdot\|$ est différentiable en tout point de $X \setminus \{0\}$, alors elle est C^1 sur $X \setminus \{0\}$, et nous dirons alors que $\|\cdot\|$ est Fréchet différentiable ou C^1-lisse. Voici une courte liste d'exemples d'espaces qui admettent une norme Fréchet-différentiable :

a) les espaces à dual séparable (M. I. Kadec),

b) les espaces avec $\mathrm{Sz}(X) < \omega_1$ (G. Lancien),

c) les espaces réflexifs (S. Troyanski),

d) les espaces $C(K)$ lorsque $K^{(\omega_1)} = \emptyset$ (R. Deville),

e) $c_0(\Gamma)$ pour tout ensemble Γ (N. H. Kuiper),

f) $C([0,\alpha])$ pour tout ordinal α (M. Talagrand, R. Haydon).

En fait, les exemples a)–e) admettent une norme dont la norme duale est LUC tandis que les exemples d)–f) admettent même une norme C^∞-lisse. Il est bien connu que *si une norme duale $\|\cdot\|^*$ par rapport à $\|\cdot\|$ est LUC, alors $\|\cdot\|$ est Fréchet-différentiable*, et c'est en utilisant ce résultat que les exemples a)–d) ont été prouvés en premier lieu. Encore, l'inverse n'est pas vrai puisque M. Talagrand [Tal86] a prouvé que $C([0,\omega_1])$ *admet une norme équivalente C^∞-Fréchet lisse, bien qu'il n'admet aucune norme dont la norme duale est LUC.* Les normes C^∞ lisses dans les exemples d)–f) sont construites en utilisant une approche différente, à savoir la notion de fonctions localement dépendantes d'un nombre fini de coordonnées (voir Définition 4.1), puisque la propriété LUC d'une norme duale $\|\cdot\|^*$ n'implique rien au sujet de la lissité d'ordre≥ 2 de la norme $\|\cdot\|$. Récemment cependant, P. Hájek et R. Haydon [HH07] ont prouvé un résultat important : *si un espace $C(K)$ admet une norme dont la norme duale est LUC, alors $C(K)$ admet une norme C^∞-lisse.*

Les normes localement uniformément convexes

C'est maintenant évident que la notion de LUC est d'importance fondamentale pour la théorie de rénormages, et nous nous référons à [DGZ93] et au [MOTV09] plus récent pour une liste étendue de résultats avec leur références. Dans un raccordement avec la partie précédente de cette thèse, nous mentionnons que c'est un problème non résolu de savoir si un espace de Banach avec RNP a une norme équivalente LUC. Il est remarquable qu'il y ait des sous-classes importantes des espaces avec RNP qui admettent une norme LUC. Dans [Lan93], G. Lancien a construit de façon ingénieuse pour une norme LUC (resp. norme dont la norme duale est LUC) sur les espaces avec $\mathrm{D}(X) < \omega_1$ (resp. $\mathrm{Dz}(X) < \omega_1$). Un autre exemple est la classe de duaux d'espaces d'Asplund : *un espace de Banach dual avec RNP admet une norme équivalente LUC.* C'est un résultat de M. Fabian et G. Godefroy [FG88] ([DGZ93, Corollary VII.1.12]). Par ce que nous avons déjà dit, dans certains cas cette norme LUC ne peut pas être une norme duale. Quoi qu'il en soit, ceci et beaucoup d'autres rénormages LUC (voir chapitre VII de [DGZ93]) sont basés sur une méthode à l'origine due à Troyanski [Tro71]. Nous énonçons ici une version "collage" due à V. Zizler [Ziz84] ([DGZ93, Proposition VII.1.6]) : *Si un espace de Banach X admet une longue suite de projections $\{P_\alpha\}_{\omega\leq\alpha\leq\mu}$ telle que les espaces $(P_{\alpha+1} - P_\alpha)X$ admettent des normes LUC, alors X lui-même admet une norme LUC.* Nous nous contentons maintenant de dire qu'une longue suite de projections (PRI) $\{P_\alpha\}_{\omega\leq\alpha\leq\mu}$ est une généralisation de la notion de suite de projections associées à une base de Schauder aux espaces non séparables et nous donnerons la définition exacte dans un chapitre postérieur (voir Définition 4.12).

Combinaison de la differentiabilité de Fréchet et de la propriété LUC

Dans beaucoup de situations on est intéressé d'obtenir une nouvelle norme sur un espace X qui partage de bonnes propriétés de lissité et de convexité. Un résultat classique dans cette direction est une méthode généralement connue sous le nom de *technique de moyennage d'Asplund* : *Si un espace de Banach X possède une norme LUC $\|\cdot\|_1$ et une (autre) norme $\|\cdot\|_2$ dont la norme duale est LUC, alors X admet aussi une norme $\|\cdot\|_3$ qui est LUC et dont la norme duale est LUC.* En particulier $\|\cdot\|_3$ est LUC et Fréchet différentiable simultanément. La technique de moyennage d'Asplund comme indiqué ici a été prouvé par M. Fabian, L. Zajíček et V. Zizler dans [FZZ82] ([DGZ93, II.4.3]) en utilisant un argument de catégories de Baire. Ils démontrent en fait que l'ensemble de normes équivalentes LUC est soit vide soit résiduel dans l'espace de toutes les normes équivalentes sur X muni de la topologie de la convergence uniforme sur les bornés. Un résultat analogue est vérifié pour l'ensemble de normes équivalentes dont les normes duales sur X^* sont LUC. La technique de moyennage d'Asplund a pris une tournure surprenante avec le résultat profond récent de R. Haydon [Hay08] : *si X a une norme dont la norme duale est LUC, alors X a aussi une norme LUC.*

Il est nécessaire de remarquer dans ce contexte, qu'il n'est pas possible de combiner la propriété LUC avec un degré plus élevé de differentiabilité dans une seule norme sans poser des restrictions structurelles fortes sur l'espace. En effet, par [FWZ83] ([DGZ93, Proposititon V.1.3]), *un espace admettant une norme LUC et C^2-lisse simultanément est déjà superreflexif.* Rappelons que dans les espaces superreflexifs, on peut obtenir une norme uniformément convexe et uniformément Fréchet-différentiable en utilisant la technique de moyennage d'Asplund.

Il découle de notre discussion que la technique de moyennage d'Asplund n'est pas toujours disponible, par exemple dans le cas de $C([0,\omega_1])$. Il est inconnu si l'ensemble des normes Fréchet différentiables est résiduel, ou même dense, dans l'espace de toutes les normes équivalentes sur $C([0,\alpha])$.

Un analogue de notre Théorème H pour $X = c_0(\Gamma)$ a été prouvé dans [PWZ81] ([DGZ93, Theorem V.1.5]). Nous allons généraliser le théorme suivant [MPVZ93] : *Tout espace séparable C^k-lisse admet une norme LUC et C^1-lisse qui est limite de normes C^k-lisses.*

Nous sommes prêts à énoncer le théorème principal. Notez les similitudes avec la version de Zizler de la renormage de Troyanski indiqué ci-dessus. On pourrait dire que notre théorème est une version lisse de ce résultat.

Théorème I (avec P. Hájek)**.** *Soit $k \in \mathbb{N} \cup \{\infty\}$. Soit $(X, |\cdot|)$ un espace de Banach avec une PRI $\{P_\alpha\}_{\omega \le \alpha \le \mu}$ telle que tout $(P_{\gamma+1} - P_\gamma)X$ admet une norme équivalente C^1-lisse, LUC qui est limite (uniforme sur les ensembles bornés) de normes C^k-lisses. Supposons de plus que X admet une norme équivalente $\|\cdot\|$ C^k-lisse.*

Alors X admet une norme équivalente $|||\cdot|||$ C^1-lisse, LUC, qui est limite (uniforme sur les ensembles bornés) de normes C^k-lisses.

Ce théorème est en fait l'étape inductive dans un argument menant au Théorème H ci-dessus et également au théorème suivant.

Théorème J (avec P. Hájek)**.** *Soit $k \in \mathbb{N} \cup \{\infty\}$. Soit $\mathcal{P}$ une classe d'espaces de Banach telle que tout X dans $\mathcal{P}$*

(i) *admet une PRI $\{P_\alpha\}_{\omega \le \alpha \le \mu}$ telle que $(P_{\alpha+1} - P_\alpha)X \in \mathcal{P}$ pour tout α,*

(ii) *admet une norme équivalente C^k-lisse.*

Alors tout X dans $\mathcal{P}$ admet une norme équivalente C^1-lisse, LUC $|||\cdot|||$ qui est limite (uniforme sur les ensembles bornś) de normes C^k-lisses.

Sans trop rentrer dans les détails, nous remarquons que la condition (i) est satisfaite par exemple pour les espaces WCG (weakly compactly generated), c.-à-d. les espaces qui contiennent un compact faible total. De même (i) est verifié pour les espaces de Vašák, les espaces $C(K)$ où K est un compact de Valdivia, les espaces WLD, etc. Pour les preuves de ces faits non triviaux, aussi bien que pour les références, nous recommandons le chapitre VI en [DGZ93] et le chapitre 5 en [HMSVZ08]. Maintenant il est clair qu'on peut obtenir une classe $\mathcal{P}$ qui satisfait (i) et (ii) en prenant l'une des classes ci-dessus intersectée avec la classe des espaces C^k-lisses.

Enfin commentons le fait que la nouvelle norme est approximable par des normes qui sont C^k-lisses. Un tel résultat est étroitement lié à la question de savoir si dans un espace C^k-lisse X, les normes C^k-lisses sur X sont denses dans l'espace de toutes les normes équivalentes sur X. Même dans le cas séparable, la réponse n'est pas connue en toute généralité. Des cas particuliers importants de cette conjecture ont été démontrés dans [DFH96] et [DFH98], et s'appliquent à la plupart des espaces de Banach classiques, par exemple : *Soit $X = C(K)$ où K est un compact dénombrable. Alors toute norme équivalente sur X peut être approximée par des normes analytiques dans $X\setminus\{0\}$*. Dans une direction semblable, on a le résultat dû à M. Fabian, P .Hájek et V. Zizler [FHZ97] : *Tout norme équivalente fortement treillis sur $(c_0(\Gamma), \|\cdot\|_\infty)$ peut être approximée (uniformément sur les ensembles bornés) par des normes C^∞-lisses.* Voir aussi le théorème 4.15 où on montre que les normes d'approximation dans ce théorème ont des propriétés utiles dans la preuve de Théorème I. Mis à part ce thórème, aucun résultat général n'est connu dans le cas non séparable. En particulier, un des problèmes posés dans [DGZ93] est de savoir si sur un espace de Banach WCG avec une norme équivalente C^k-lisse, il existe une norme équivalente LUC qui est limite uniforme sur les ensembles bornés de normes C^k-lisses. Le Théorème J ainsi que les résultats structuraux au sujet des espaces WCG fournissent une solution positive à ce problème.

Notez que *si X est Vašák, resp. X est un espace $C(K)$, l'existence de la norme C^k-lisse (ou plus généralement d'une fonction bosse C^k-lisse) implique que n'importe quelle fonction continue sur X peut être approchée uniformément par des fonctions C^k-lisses* (voir [DGZ93, chapitre VIII], resp. [HH07]) mais quand on essaye d'approcher de cette façon une fonction Lipschitzienne ou convexe (ou une norme) on n'a en général aucune information sur le comportement Lipschitz ou la convexité de la fonction approximante.

Dans le dernier chapitre nous allons traiter des approximations convexes d'une fonction convexe f qui n'améliorent pas nécessairement le differentiabilité mais qui possèdent un point de minimum fort. Naturellement un tel sujet est déjà bien exploré dans le travail de M. Fabian, P. Hájek et J. Vanderwerff [FHV96]. Ainsi nous allons étudier la situation quand f dépend de façon continue d'un paramètre, recherchant une possibilité d'obtenir ces approximations aussi bien que les minimiseurs d'une manière continue par rapport au paramètre.

1.4.6 Un principe variationnel paramétrique

Les résultats dans cette section viennent d'un travail commun de R. Deville et de l'auteur([DP]). Ce qui est un *principe variationnel* peut être mieux vu sur l'exemple concret du principe variationnel d'Ekeland : *Soient X un espace de Banach, $f : X \to (-\infty, +\infty]$ une fonction minorée semicontinue inférieurement (s.c.i.) et soit $\varepsilon > 0$. Alors il existe un point $v \in X$ tel que $x \mapsto f(x)+\varepsilon\|x-v\|$ a un minimum en v.* Une telle affirmation sert de remplacement à la compacité de l'ensemble dans lesquels on recherche le minimum de f. Géométriquement parlant, le graphe de la fonction $x \mapsto f(v)-\varepsilon\|x-v\|$ touche l'épigraphe de la fonction f au point v de dessous. Dans ce cas la fonction $x \mapsto \varepsilon\|x-v\|$ est appelée *perturbation.* Les principes variationnels de J. Borwein et D. Preiss, resp. R. Deville, G. Godefroy et V. Zizler (DGZ), généralisent le théorème ci-dessous en disant qu'il est possible de prendre des perturbations C^k-lisses, resp. C^k-lisses et Lipschitziennes, si l'espace X admet une norme C^k-lisse, resp. une fonction bosse C^k-lisse dont la dérivée d'ordre k est bornée. Indépendamment d'être un renforcement du principe variationnel de Borwein-Preiss, le principe variationnel de DGZ est prouvé par un argument élégant de catégorie de Baire.

Dans un *principe variationnel paramétrique*, on dispose d'un système $\{x \mapsto f(p,x)\}$ de fonctions s.c.i. minorées qui dépendent de façon continue du paramètre p dans un espace topologique Π. Le but est de perturber pour chaque $p \in \Pi$ la fonction $f(p,\cdot)$ par une fonction $\Delta(p) : X \to \mathbb{R}$ de telle manière que $f(p,\cdot) + \Delta(p)$ atteigne son minimum en un certain point $v(p)$ et que $v(p)$ et $\Delta(p)$ dépendent de façon continue du paramètre p. Un principe variationnel paramétrique lisse de style Borwein-Preiss a été introduit par P. Georgiev [Geo05]. Récemment L. Veselý [Ves09] a modifié la preuve afin de réaliser un principe variationnel lisse paramétrique avec des contraintes. Plus précisément, soit $\Pi_0 \subset \Pi$ tel que pour tout $p \in \Pi_0$, la fonction $f(p,\cdot)$ atteint son minimum en $v_0(p)$. Veselý construit un minimiseur v qui est une extension de v_0. Le théorème principal de ce chapitre (Théorème 5.18) est une version paramétrisée du principe variationnel de DGZ et de sa méthode de preuve. Notre théorème principal implique en particulier :

Théorème K (avec R. Deville)**.** *Soient Π un espace topologique paracompact séparé, X un espace de Banach avec une norme Fréchet lisse, $\mathcal{Y}$ le cône de toutes les fonctions convexes, positives, Lipschitz, Fréchet lisses sur X. Le cône $\mathcal{Y}$ est équipé par la norme naturelle $\|g\|_{\mathcal{Y}} = |g(0)| + \|g\|_{\mathrm{Lip}}$. Supposons que $f : \Pi \times X \to \mathbb{R}$ satisfait*

(i) *pour tout $p \in \Pi$, la fonction $f(p,\cdot)$ est convexe, continue, minorée,*
(ii) *pour tout $x \in X$, la fonction $f(\cdot,x)$ est continue,*
(iii) *pour tout $p_0 \in \Pi$, $(f(p_0,\cdot) - f(p,\cdot))^+ \to 0$ uniformément sur les ensembles bornés de X quand $p \to p_0$.*

Alors pour tout $\varepsilon > 0$, il existe $\Delta \in C(\Pi,\mathcal{Y})$ et $v \in C(\Pi,X)$ tels que $\|\Delta(p)\|_{\mathcal{Y}} < \varepsilon$ et $f(x,\cdot) + \Delta(p)$ a son minimum fort en $v(p)$ pour tout $p \in \Pi$. De plus $p \mapsto f(p,v(p)) + \Delta(p)(v(p))$ est continue.

Un fait important, qui limite sévèrement les applications possibles et se produit pour tous les principes variationnels paramétriques, est que toutes les fonctions $f(p,\cdot)$ doivent être convexes. Ce n'est pas simplement une difficulté technique puisque des exemples faciles (Exemple 5.26) prouvent que dans le cas non convexe, il n'y a aucun espoir en général de trouver un minimiseur continu même après perturbation.

Nous prouvons ici (voir Section 5.4) que l'autre hypothèse inattendue, c.-à-d. l'hypothèse (iii), ne peut pas être abandonnée sans remplacement non plus.

Par contre, la condition (iii) est une conséquence de (i) et (ii) si nous supposons que Π est métrisable et que la dimension de X est finie. En effet, ceci se déduit immédiatement du théorème suivant.

Théorème L (avec R. Deville)**.** *Soient f et f_n, $n \in \mathbb{N}$, des fonctions continues convexes d'un espace de Banach X à valeurs dans $\mathbb{R}$ telles que $f_n \to f$ ponctuellement sur X. Alors $f_n \to f$ uniformément sur les compacts de X.*

Enfin observons que même si l'espace X est de dimension finie et si pour tout $p \in \Pi$, $p \mapsto f(p, \cdot)$ atteint son minimum en $v(p)$, la fonction v n'est pas nécessairement continue. En effet, on n'a pas l'existence d'un minimiseur *continu* pour la fonction originale même dans le cas le plus simple (voir Problem 5.1 et Section 5.4).

Chapter 2

Games

The central topic of this chapter is the notion of a point-set game. Using concrete versions of this general concept we prove a characterization of the spaces with the Radon-Nikodým property (Section 2.3), a characterization of the superreflexive spaces (Section 2.4) and a characterization of Baire one functions (Section 2.5).

2.1 Preliminaries

2.1.1 Games, tactics, strategies

Definition 2.1. Let K be a set in a real Banach space X. Let $\mathcal{A}$ be a collection of subsets of K such that for every $x \in K$ there exists $A \in \mathcal{A}$ which has $x \in A$. We define a game $\mathbf{G}(K, \mathcal{A})$ as follows. There are two players, Player I and Player II. Player I starts the game by choosing an arbitrary point $x_1 \in K$. Player II then chooses a set $A_1 \in \mathcal{A}$ so that $x_1 \in A_1$; then Player I chooses a point $x_2 \in A_1$ and Player II chooses a set $A_2 \in \mathcal{A}$ so that $x_2 \in A_2$; and so on. Summing up, the rules are:

- Player I starts by playing $x_1 \in K$ arbitrarily;
- after x_n has been played, Player II must choose A_n so that $A_n \in \mathcal{A}$ and $x_n \in A_n$;
- after A_n has been played, Player I must play x_{n+1} so that $x_{n+1} \in A_n$.

Formally

$$\mathbf{G}(K, \mathcal{A}) = \{(x, A) \in K^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} : x = (x_n),\ A = (A_n) \text{ and } x_n \in A_n \ni x_{n+1} \text{ for all } n \in \mathbb{N}\}$$

and each element of this set is called a *run* of the game $\mathbf{G}(K, \mathcal{A})$. Player II *wins* the run (x, A) if the sequence $(x_n)_{n=1}^{\infty}$ is Cauchy or unbounded. Otherwise Player I wins.

In plain life, there is hardly any difference between the words ‘tactic’ and ‘strategy’. Mathematically, they stand for different concepts. If Player II plays according to a tactic, he decides his next move A_n only taking into account the last move x_n of Player I. If Player II plays according to a strategy, he considers the whole history of Player’s I moves $(x_i)_{i=1}^n$ before playing A_n. Let us express this more formally.

Definition 2.2. We say that a function $t : K \to \mathcal{A}$ is a *tactic for Player II* if $x \in t(x)$ for all $x \in K$. We say that a tactic $t : K \to \mathcal{A}$ for Player II is *winning* (WT) if any bounded sequence $(x_n) \subset K$ which satisfies $x_{n+1} \in t(x_n)$ for all $n \in \mathbb{N}$ is necessarily Cauchy.

A *strategy for Player II* is a sequence $(t_n)_{n\in\mathbb{N}}$ where $t_n : D_n \to \mathcal{A}$. The domains D_n are defined inductively by $D_1 = K$ and

$$D_{n+1} = \left\{(x_i)_{i=1}^{n+1} : (x_i)_{i=1}^n \in D_n, x_{n+1} \in t_n(x_1, \dots, x_n)\right\}.$$

Each t_n must satisfy $x_n \in t_n(x_1, \dots, x_n)$ for all $(x_i)_{i=1}^n \in D_n$. A strategy (t_n) for Player II is *winning* if every $(x_n) \subset K$ is Cauchy whenever it satisfies $x_{n+1} \in t_n(x_1, \dots, x_n)$ for all $n \in \mathbb{N}$.

Winning tactics for Player II in $\mathbf{G}(K, \mathcal{A})$ are obviously a subset of winning strategies for Player II in $\mathbf{G}(K, \mathcal{A})$.

We will only deal with the tactics and winning tactics for Player II so for ecological reasons we will not usually mention it. There is not much to be said about winning tactics for Player I anyway: they simply do not exist. Indeed, if they existed, what would happen if Player II repeated always the same move (i.e. $A_n := A$ for all $n \in \mathbb{N}$)?

2.1.2 Point-slice games

Definition 2.3. For $f \in X^* \setminus \{0\}$, $a \in \mathbb{R}$ we define the *open halfspace* $H(f, a) = \{x \in X : f(x) > a\}$ and *closed halfspace* $\overline{H(f, a)} = \{x \in X : f(x) \geq a\}$. Let K be a subset of X and assume that $K \cap \overline{H(f, a)} \neq \emptyset$. Then we call this nonempty intersction a *closed slice* of K given by f and a. Similarly, if $K \cap H(f, a) \neq \emptyset$, then it is called an *open slice* of K given by f and a. Finally, we denote $\mathcal{S}_c(K)$ all the closed slices of K and $\mathcal{S}_o(K)$ all the open slices of K.

We are going to abbreviate $\mathbf{G}(K, \mathcal{S}_c) := \mathbf{G}(K, \mathcal{S}_c(K))$ and $\mathbf{G}(K, \mathcal{S}_o) := \mathbf{G}(K, \mathcal{S}_o(K))$. The following observation, inspired by [DM07], is the link between the convergence of bounded “monotone” sequences and the existence of a winning tactic in $\mathbf{G}(K, \mathcal{S}_c)$, in particular it is a link between Theorem A and Corollary 2.17 (resp. Theorem D and Theorem 2.31).

Proposition 2.4. *Let K be a subset of X. Then it is equivalent*

(a) *Player II has a winning tactic in the game $\mathbf{G}(K, \mathcal{S}_c)$,*

(b) *there exists a function $F : K \to X^*$ such that if a bounded sequence $(x_n) \subset K$ satisfies $\langle F(x_n), x_{n+1}\rangle \geq \langle F(x_n), x_n\rangle$ for all $n \in \mathbb{N}$, then (x_n) is Cauchy.*

Proof. The condition (b) is obviously equivalent to $t'(x) := K \cap \overline{H(F(x), F(x)x)}$ being a WT.

On the other hand, a general WT t in $\mathbf{G}(K, \mathcal{S}_c)$ is determined by functions $F : K \to X^*$ and $a : K \to \mathbb{R}$ by means of the equality $t(x) = K \cap \overline{H(F(x), a(x))}$ for all $x \in K$. The function F then satisfies (b). Indeed, we may define $t'(x) := K \cap \overline{H(F(x), F(x)x)}$. Then $x \in t'(x) \subset t(x)$ for all $x \in K$ and thus t' is a WT in $\mathbf{G}(K, \mathcal{S}_c)$, so we may exploit the first part of the proof. □

Later we will be using tactics of the general form but because of the above proposition we will never be interested in the function a as much as in the function F.

2.1.3 (Small) slices inside slices

The following well known lemma says that a slice of a bounded set L given by a functional f contains slices of L given by the functionals in some neighborhood of f.

Lemma 2.5. *Let L be a nonempty bounded subset of X and suppose that $L \cap H(f, a) \neq \emptyset$ for some $a \in \mathbb{R}$ and $f \in X^* \setminus \{0\}$. Then there is an $r > 0$ such that for every $g \in X^*$ such that $\|f - g\| < r$ there is $\alpha \in \mathbb{R}$ with*

$$\emptyset \neq L \cap H(g, \alpha) \subset L \cap H(f, a).$$

The above lemma is a special case of the following

Lemma 2.6. *Let L be a nonempty bounded subset of X and let $f \in X^* \setminus \{0\}$ and $a < b$ such that $S_2 := L \cap H(f, b) \subset L \cap H(f, a) =: S_1$. We suppose that S_1 is a slice of L (in particular nonempty) and S_2 is either a slice of L or the empty set. Then there exists $r > 0$ with the property that for every $g \in B_{X^*}(f, r)$ there exists $\alpha \in \mathbb{R}$ such that*

$$L \cap H(f, b) \subset L \cap H(g, \alpha) \subset L \cap H(f, a).$$

Proof. We may push the scene (i.e. $x \mapsto x - y$ for some $y \in X$) in order to have $a = -b$. Also, since L is bounded, we may suppose without loss of generality that $L \subset B_X$ (see Figure 2.1). Let $M = \{x \in B_X : |f(x)| = |a|\}$ (by our assumptions $M \neq \emptyset$) and let $\|f - g\| < |a|$. Then $M \cap \ker g = \emptyset$. Indeed, let $x \in M \cap \ker g$. Then $|a| = |f(x)| = |(f - g)(x)| \leq \|f - g\| < |a|$. We see that the hyperplane $\{g = 0\}$ separates S_2 from $L \setminus H(f, a)$. So we may set $r := |a|/2$. Finally, we push the scene back so $\alpha := g(y)$. □

Definition 2.7. Let C be a bounded closed convex set in X. We say that C has the *Radon-Nikodým property* (RNP) if for every $\varepsilon > 0$ and every $A \subset C$ there exists a slice $S \in \mathcal{S}_o(A)$ such that $\operatorname{diam} S < \varepsilon$. A Banach space X has the RNP if B_X has the RNP.

Definition 2.8. Let C be a convex subset of X. We say that a functional $f \in X^*$ *strongly exposes* a point $x \in C$ if $\sup f(C) = f(x)$ and $\|x_n - x\| \to 0$ whenever $(x_n) \subset C$ is a sequence in C such that $f(x_n) \to f(x)$. Such a functional is then called a *strongly exposing functional* of C. We denote the set of all strongly exposing functionals of C by $\mathcal{SE}(C)$.

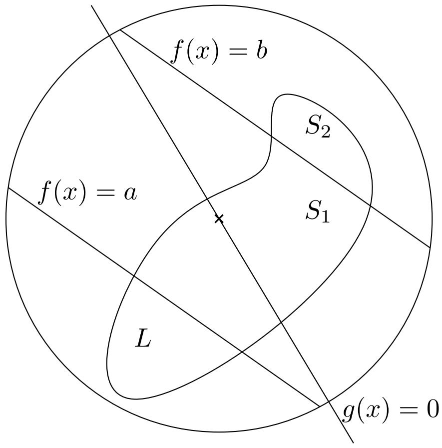

Figure 2.1: The situation of Lemma 2.6

One of the fundamental results about the convex sets with the RNP is the following Bourgain-Phelps theorem (see [Phe93, Theorem 5.20] or [Bou83, Theorem 3.5.4]).

Theorem 2.9. *Let C be a closed convex and bounded subset of X. If C has the RNP, then the set $\mathcal{SE}(C)$ is a dense G_δ subset of X^*.*

The key to the definition of all the strategies or tactics in the sets with the RNP is the well following known fact.

Lemma 2.10. *Let $L \subset X$ be a closed convex bounded set with the RNP and $L \cap H(f, a)$ be a slice of L. For any $\varepsilon > 0$, $r > 0$ there is a slice $L \cap H(g, \beta) \subset L \cap H(f, a)$ with the diameter less than ε and $\|g - f\| < r$.*

Proof. By Lemma 2.5, there is $\tilde{r} > 0$ such that for all $g \in B_{X^*}(f, \tilde{r})$ there exists $\alpha \in \mathbb{R}$ such that $\emptyset \neq L \cap H(g, \alpha) \subset L \cap H(f, a)$. By Theorem 2.9 we have that $\mathcal{SE}(L)$ is dense in X^*, so we choose some $g \in \mathcal{SE}(L) \cap B_{X^*}(f, \min\{r, r'\})$. Since $g \in \mathcal{SE}(L)$, there is some $\beta \in \mathbb{R}$ such that $L \cap H(g, \beta) \neq \emptyset$ and $\operatorname{diam}(L \cap H(g, \beta)) < \varepsilon$. □

Next we combine the above lemma with the Hahn-Banach theorem.

Lemma 2.11. *Let $C_1 \subset C_2$ be bounded closed convex nonempty sets in X. Let C_2 have the RNP and $\varepsilon > 0$. If $C_2 \setminus C_1 \neq \emptyset$, then there exists a slice $S \in \mathcal{S}_o(C_2)$ such that* $\operatorname{diam} S < \varepsilon$ *and $S \cap C_1 = \emptyset$.*

Proof. Pick a point x in $C_2 \setminus C_1$. Separate it from C_1 by a Hahn-Banach functional f, i.e. $f(x) > a > \sup f(C_1)$ for some $a \in \mathbb{R}$. We have that $C_2 \cap H(f, a)$ is a nonempty (contains x) open slice of C_2, so we may use Lemma 2.10 to get the wanted small slice $S = H(g, \beta)$. □

2.2 No open winning tactics

The difference between the concept of winning strategies and winning tactics is illustrated also by the fact that there are no winning tactics in the case when Player II uses open sets.

Theorem 2.12. *Let (E, d) be a non-scattered (see Definition 3.8) complete metric space. Let $\mathcal{A} \subset \{\text{open sets}\}$ such that $\bigcup \mathcal{A} = E$. Then there is no winning tactic for Player II in the game $\mathbf{G}(E, \mathcal{A})$.*

Remark 2.13. In particular, Player II has no winning tactic in the game $\mathbf{G}(K, \mathcal{S}_o)$ whenever K is non-scattered. This is contrasting with the result of Deville and Matheron that Player II has a winning strategy in $\mathbf{G}(K, \mathcal{S}_o)$ provided K has the RNP.

Proof. Since E is non-scattered, it has a nonempty perfect part $F \subset E$. We continue by contradiction. Let $t : E \to \mathcal{A}$ be a WT. For $n \in \mathbb{N}$, we denote

$$D_n = \{x \in F : B_E(x, 1/n) \subset t(x)\}.$$

Then $\bigcup D_n = F$ and so, by the Baire category theorem, for some index n the relative (with respect to F) interior of $\overline{D_n}$ is nonempty. Hence there is a relatively open set $G \subset F$ such that $G \cap D_n$ is dense in G and $\operatorname{diam} G < 1/n$. For any $x \in G \cap D_n$ one has $G \subset t(x)$ by the definition of D_n. Also, $G \cap D_n$ is in fact infinite since F is perfect. Player I is therefore recommended to stay in the set $G \cap D_n$ switching there merely between two different points to produce a divergent sequence and the contradiction. □

Our next observation is really simple. Yet it deserves to be mentioned as a counterpart of the above theorem.

Exercise 2.14. Let X be a Banach space. If $\delta > 0$ and $f \in X^* \setminus \{0\}$, we denote $K_\delta(f)$ the cone $\{x \in X : \|x\| \leq \delta f(x)\}$. Let

$$\mathcal{A}_\delta = \{A \subset B_X : A = B_X \cap (x + K_\delta(f)) \text{ for some } x \in X, \text{ and } 0 \neq f \in X^*\}.$$

Then for every $\delta > 0$, Player II has a winning tactic in the game $\mathbf{G}(B_X, \mathcal{A}_\delta)$.

Hint. Let $f \in X^* \setminus \{0\}$ be fixed. Then $t(x) := B_X \cap (x + K_\delta(f))$ is a winning tactic. Indeed, it is clear that t is a tactic. To see that it is winning consider a sequence (x_n) satisfying $x_{n+1} \in t(x_n)$. Then $(f(x_n))_n$ is a nondecreasing and bounded sequence of real numbers. Also $\|x_{n+1} - x_n\| \leq \delta(f(x_{n+1}) - f(x_n))$ showing that (x_n) is a Cauchy sequence. □

2.3 Characterization of the RNP

The main result of this chapter is the following.

Theorem 2.15. *Let K be a closed convex bounded subset of X and let K have the RNP. Then there exists a winning tactic $t : K \to \mathcal{S}_c(K)$ for Player II in the game $\mathbf{G}(K, \mathcal{S}_c K)$.*

Moreover, our particular construction yields a tactic of the form

$$t(x) = K \cap \overline{H(F(x), F(x)x)}$$

where $F : K \to X^*$ is a Baire one mapping. Optionally, given a closed convex subset A of K, we may construct the WT t so that for every $x \in K \setminus A$ it holds $t(x) \cap A = \emptyset$.

Corollary 2.16. *If the Banach space X has the RNP, then, for any bounded set $K \subset X$, Player II has a winning tactic in the point-closed slice game $\mathbf{G}(K, \mathcal{S}_c)$.*

Proof. We may suppose that $K \subset B_X$ while Theorem 2.15 provides a WT t for $\mathbf{G}(B_X, \mathcal{S}_c)$. Clearly the restriction $t\restriction_K$ (more precisely $x \in K \mapsto t(x) \cap K$) is a WT for $\mathbf{G}(K, \mathcal{S}_c)$. □

Since Deville and Matheron have proved that, for $\Omega \subset X$ with $\operatorname{int} \Omega \neq \emptyset$, the existence of a winning strategy for Player II in the point-hyperplane game in Ω implies the RNP for Ω, we may restate their [DM07, Theorem 3.4].

Corollary 2.17. *The following are equivalent:*

(i) *X has the Radon-Nikodým property;*

(ii) *Player II has a winning tactic in the point-closed slice game $\mathbf{G}(B_X, \mathcal{S}_c)$;*

(iii) *Player II has a winning tactic in the point-closed slice game $\mathbf{G}(X, \mathcal{S}_c)$.*

Proof. The implication (iii) ⇒ (ii) is trivial. The implication (ii) ⇒ (i) was proved in [DM07] so we only have to prove the implication (i) ⇒ (iii). So let us suppose that X has the RNP. Then for every $n \in \mathbb{N}$ the multiple nB_X has the RNP. For every $n \in \mathbb{N}$, let t_n be the WT in $\mathbf{G}((n+1)B_X, \mathcal{S}_c)$ which comes from Theorem 2.15 with the set $A = nB_X$. We assume that $t_n(x) = \overline{H(f_n(x), a_n(x))} \cap (n+1)B_X$. For $x \in X$ with $\|x\| > 1$ let $N(x) \in \mathbb{N}$ be the uniquely determined number N such that $x \in (N+1)B_X \setminus NB_X$. We define a mapping $t : X \to \mathcal{S}_c(X)$ by

$$t(x) := \begin{cases} \overline{H(f_{N(x)}(x), a_{N(x)}(x))} & \text{if } \|x\| > 1, \\ \overline{H(f_1(x), a_1(x))} & \text{otherwise,} \end{cases}$$

and we claim that t is a WT in $\mathbf{G}(X, \mathcal{S}_c)$. Indeed, let (x_n) be a sequence which satisfies $x_{n+1} \in t(x_n)$ for ever $n \in \mathbb{N}$. If (x_n) is unbounded, Player II has won his run. We proceed by assuming that (x_n) is bounded. Without loss of generality, let $\sup_n \|x_n\| > 1$. Let $N \in \mathbb{N}$ be the smallest integer such that $N \geq \sup_n \|x_n\|$. Then for all but finitely many indices $n \in \mathbb{N}$ we have $x_n \in NB_X \setminus (N-1)B_X$ because of the optional property of t_{N-1}. So $x_{n+1} \in t_{N-1}(x_n)$ for all but finitely many indices n thus (x_n) converges as t_{N-1} is a WT in $\mathbf{G}(NB_X, \mathcal{S}_c)$. □

The subsequent definitions and lemmata are needed in order to prove Theorem 2.15. From now on, $K \subset X$ will always be a closed convex bounded set with the RNP even though much of the following would make sense also in more generality.

2.3.1 ε-slicings, ε-tactics

For $\varepsilon > 0$, we will consider the game ε-$\mathbf{G}(K, \mathcal{S}_o)$ with Player's II objective to make the sequence (x_n) ε-Cauchy (i.e. $\|x_n - x_m\| < \varepsilon$ for n, m large enough). A WT in this game will be simply an *ε-winning tactic* (ε-WT). The WT in $\mathbf{G}(K, \mathcal{S}_c)$ from Theorem 2.15 will be constructed as a limit of a sequence of 2^{-n}-winning tactics.

Definition 2.18. A *slicing* $\mathcal{Z}$ of a convex bounded set $L \subset X$ given by the halfspaces $(H(f_\xi, a_\xi))_{\xi \leq \eta}$ is a family $(Z_\xi)_{\xi \leq \eta}$ of relatively closed convex subsets of L, where η is an ordinal, satisfying:

(a) $Z_{\xi+1} = Z_\xi \setminus H(f_\xi, a_\xi)$;

(b) for each limit ordinal $\lambda \leq \eta$, $Z_\lambda = \bigcap_{\xi < \lambda} Z_\xi$;

(c) $Z_0 = L$ and $Z_\eta = \emptyset$.

For $x \in L$, let $\Gamma_\mathcal{Z}(x)$ be the unique ordinal $\gamma < \eta$ such that $x \in Z_\gamma \setminus Z_{\gamma+1}$. Also notice that if $\alpha \leq \beta$, then $Z_\alpha \supset Z_\beta$.

If moreover $\mathcal{Z}$ has small difference sets, i.e. it satisfies $\operatorname{diam} Z_\xi \setminus Z_{\xi+1} < \varepsilon$ for some $\varepsilon > 0$ and all $\xi < \eta$, we shall call it *ε-slicing*.

The following proposition shows that there is a canonical way of defining an ε-WT once we have an ε-slicing.

Proposition 2.19. *Let $\mathcal{Z}$ be an ε-slicing of K given by the halfspaces $(H(f_\xi, a_\xi))_{\xi \leq \eta}$. Then $t_\mathcal{Z} : K \to \mathcal{S}_o$ defined as $t_\mathcal{Z}(x) := K \cap H(f_{\Gamma_\mathcal{Z}(x)}, a_{\Gamma_\mathcal{Z}(x)})$ is an ε-winning tactic in ε-$\mathbf{G}(K, \mathcal{S}_o)$.*

Proof. Let x_n be the last move of Player I. Then $H(f_{\Gamma_\mathcal{Z}(x_n)}, a_{\Gamma_\mathcal{Z}(x_n)}) \cap Z_\beta = \emptyset$ for any $\beta > \Gamma_\mathcal{Z}(x_n)$ which shows that $(\Gamma_\mathcal{Z}(x_n))_{n=1}^\infty$ is a nonincreasing sequence of ordinals if Player II sticks to the tactic $t_\mathcal{Z}$. Hence $(\Gamma_\mathcal{Z}(x_n))_{n=1}^\infty$ must be eventually constant or, equivalently, x_n stays eventually in $Z_\xi \setminus Z_{\xi+1}$ for some particular $\xi < \eta$. This difference set has its diameter smaller than ε as $\mathcal{Z}$ is an ε-slicing. Thus (x_n) is ε-Cauchy. $\square$

It is useful to notice that if $t_\mathcal{Z}(x) = K \cap H(F(x), a(x))$ is a tactic obtained from a slicing $\mathcal{Z}$ as in the previous proposition, then $F : K \to X^*$ is constant on difference sets $Z_\xi \setminus Z_{\xi+1}$ (we say it is a *slice constant mapping*).

2.3.2 Refining ε-slicings

Let us treat the space of all mappings from K to X^* as the power $(X^*)^K$. We recall that the *box topology* [Mun00, page 114] on the product $(X^*)^K$ is the one generated by the basis of open sets of the form $\prod_{x \in K} B^O_{X^*}(f_x, r_x)$ where $f_x \in X^*$ and $r_x > 0$ for all $x \in K$. We call these basis sets *open boxes*. The closure of an open box admits the representation $\overline{\prod_{x \in K} U_x} = \prod_{x \in K} \overline{U_x}$.

We will be interested in boxes which relate to slicings in a special way.

Definition 2.20. Let $\mathcal{Z} = (Z_\xi)_{\xi\leq\eta}$ be a slicing of K and let $t_{\mathcal{Z}}(x) = K \cap H(F(x), a(x))$ be the canonically corresponding tactic. We say that a box U is a *box around* $\mathcal{Z}$ if

$$U = \prod_{x\in K} B^O_{X^*}(F(x), r(x))$$

and if $r : K \to (0, +\infty)$ is constant on the difference sets $Z_\xi \setminus Z_{\xi+1}$, i.e. there exists some transfinite sequence $(r_\xi)_{\xi\leq\eta}$ of positive numbers such that $r(x) = r_{\Gamma_{\mathcal{Z}}(x)}$ for every $x \in K$. Remember that, by definition, $F(x) = f_{\Gamma_{\mathcal{Z}}(x)}$ so we may view the box U around $\mathcal{Z}$ as a set-valued mapping that is constant on difference sets $Z_\xi \setminus Z_{\xi+1}$. We will use the term *selection* known from this context.

Definition 2.21. Let $\mathcal{Z} = (Z_\xi)_{\xi\leq\eta}$ and $\mathcal{Y} = (Y_\mu)_{\mu\leq\lambda}$ be slicings of K. If $\mathcal{Z} \subset \mathcal{Y}$, then we say that $\mathcal{Y}$ is a *refinement* of $\mathcal{Z}$. Further if $\mathcal{Z}$ is a slicing of K and U is a box around $\mathcal{Z}$, then we say that $\mathcal{Y}$ is a *U-refinement* of $\mathcal{Z}$ if $\mathcal{Y}$ is a refinement of $\mathcal{Z}$ and $t_{\mathcal{Y}}(x) = K \cap H(G(x), b(x))$ satisfies $G \in U$.

One can build up refinements in the following manner.

Lemma 2.22. *Let $\mathcal{Z} = (Z_\xi)_{\xi\leq\eta}$ be a slicing of K and for every $\xi < \eta$ let $\mathcal{Y}_\xi = (Y_{(\xi,\mu)})_{\mu\leq\eta_\xi}$ be a slicing of the difference set $Z_\xi \setminus Z_{\xi+1}$ given by the halfspaces $(H(f_{(\xi,\mu)}, a_{(\xi,\mu)}))_{\mu\leq\eta_\xi}$. If*

$$H(f_{(\xi,\mu)}, a_{(\xi,\mu)}) \cap Z_{\xi+1} = \emptyset \text{ for all } \xi < \eta \text{ and } \mu \leq \eta_\xi, \tag{2.1}$$

then $\mathcal{Y} = (Y_{(\xi,\mu)} \cup Z_{\xi+1})_{(\xi,\mu)}$ with the lexicographical order on the doubles (ξ, μ) is a refinement of $\mathcal{Z}$.

Proof. A straightforward verification of the definition of slicing. It is exactly the condition (2.1) that makes it possible to verify the property (a) of Definition 2.18. □

We will use the refinements in order to achieve two things. The first of them is to make the ε of an ε-slicing smaller. This is the moment when we start making use of the RNP of the set K.

Proposition 2.23. *Let $\mathcal{Z}$ be a slicing of K and let U be a box around $\mathcal{Z}$. Then for any $\varepsilon > 0$ there is a U-refinement $\mathcal{Y}$ of $\mathcal{Z}$ which is an ε-slicing.*

Proof. Suppose that $\mathcal{Z} = (Z_\xi)_{\xi\leq\eta}$ is given by the halfspaces $(H(f_\xi, a_\xi))_{\xi\leq\eta}$ and suppose that the box U is given by the positive numbers $(r_\xi)_{\xi\leq\eta}$. For $\xi < \eta$ fixed, an ε-slicing $(Y_{(\xi,\mu)})_\mu)_{\mu\leq\eta_\xi}$ of $Z_\xi \setminus Z_{\xi+1}$ and the corresponding $(H(g_{(\xi,\mu)}, b_{(\xi,\mu)}))_{\mu\leq\eta_\xi}$ with $\|g_{(\xi,\mu)} - f_\xi\| < r_\xi$ are obtained by the iterated use of Lemma 2.10 in the following way. Put $Y_{(\xi,0)} := Z_\xi \setminus Z_{\xi+1}$. With $Y_{(\xi,\mu)}$ defined put $L := Y_{(\xi,\mu)} \cup Z_{\xi+1}$. Then $Y_{(\xi,\mu)} = L \cap H(f_\xi, a_\xi)$ is a slice of L. So we use Lemma 2.10 with $r = r_\xi$ to get a slice $S := L \cap H(g_{(\xi,\mu)}, b_{(\xi,\mu)}) \subset Y_{(\xi,\mu)}$. In particular $H(g_{(\xi,\mu)}, b_{(\xi,\mu)}) \cap Z_{\xi+1} = \emptyset$, which guarantees the condition (2.1). We put $Y_{(\xi,\mu+1)} := Y_{(\xi,\mu)} \setminus S$. We take intersections to get $Y_{(\xi,\lambda)}$ for limit ordinals λ. Observe that for cardinality reasons there indeed does exist an ordinal η_ξ such that the condition (c) of Definition 2.18 is satisfied. The proof is completed using Lemma 2.22. □

2.3.3 Stability

The second thing which we get through the refining of slicings is an additional stability property of the corresponding ε-WT. Roughly speaking, the next defined *stable ε-winning tactic* is such an ε-WT whose suitable perturbations are again ε-WT's.

Definition 2.24. An ε-winning tactic $t : K \to \mathcal{S}_o$, $t(x) = K \cap H(F(x), a(x))$ is *stable* if the mapping $F : K \to X^*$ is an interior point of the set

$$W = \Big\{ G \in (X^*)^K : x \mapsto K \cap H(G(x), b(x)) \text{ is an } \varepsilon\text{-WT for some } b : K \to \mathbb{R} \Big\}$$

in the box topology on the product X^{*K}.

Let $\mathcal{Z}$ be an ε-slicing of K and let $t_{\mathcal{Z}}$ be the corresponding ε-WT. If $t_{\mathcal{Z}}$ is stable and there exists a box U around $\mathcal{Z}$ such that $F \in U \subset \overline{U} \subset W$, we say that $t_{\mathcal{Z}}$ is *U-stable* and $\mathcal{Z}$ is a *U-stable ε-slicing*. In this case we also call $\overline{U}$ a *stability box* of $t_{\mathcal{Z}}$. This terminology is motivated by the important fact, that any selection G from $\overline{U}$ then gives rise to an ε-WT.

Clearly, if $U' \subset U$ are boxes around $\mathcal{Z}$ and $\mathcal{Z}$ is a U-stable ε-slicing, then it is also U'-stable.

We observe that the ε-winning tactics that arise from ε-slicings are close to being stable. In fact, to any ε-slicing there exists a stable refinement.

Proposition 2.25. *Let $\mathcal{Z}$ be an ε-slicing of K. Then there exists an ε-slicing $\mathcal{Y}$ of K which is a V-refinement of $\mathcal{Z}$ for every box V around $\mathcal{Z}$ (i.e. we use the same functionals). Moreover, there exists a box U around $\mathcal{Y}$ such that $\mathcal{Y}$ is U-stable.*

Proof. Suppose that $\mathcal{Z} = (Z_\xi)_{\xi \le \eta}$ is given by halfspaces $(H(f_\xi, a_\xi))_{\xi \le \eta}$. Let $\xi < \eta$ be fixed. We will slice up the difference set $Z_\xi \setminus Z_{\xi+1}$ by countably many hyperplanes parallel to $\{f_\xi = a_\xi\}$.

Denote $A = \sup \{f_\xi(x) : x \in Z_\xi\}$ and define a slicing $(Y_{(\xi,n)})_n$ of $Z_\xi \setminus Z_{\xi+1}$ by

$$Y_{(\xi,n)} := Z_\xi \setminus H(f_\xi, \frac{1}{n}A + (1 - \frac{1}{n})a_\xi).$$

So $g_{(\xi,n)} = f_\xi$ and obviously $H(g_{(\xi,n)}, a_{(\xi,n)}) \cap Z_{\xi+1} = \emptyset$. This tells us (using Lemma 2.22) that $(Y_{(\xi,n)})$ with the lexicographical order on the doubles (ξ, n) is an ε-slicing of K. It is of course a V-refinement of $\mathcal{Z}$ for every box V around $\mathcal{Z}$ since $g_{(\xi,n)} = f_\xi$ for all $\xi < \eta$ and $n \in \mathbb{N}$.

In order to prove the stability claim we will show that it is possible to perturb the ε-WT $t_{\mathcal{Y}}$ corresponding to $\mathcal{Y}$ and still get an ε-WT. We start by defining $r_{(\xi,n)} > 0$ using Lemma 2.6 with $L = Y_{(\xi,n)}$, $S_2 = L \setminus Y_{(\xi,n+1)}$, and $S_1 = L \setminus Y_{(\xi,n+2)}$. We claim that $\mathcal{Y}$ is U-stable with

$$U = \prod_{x \in K} B^O_{X^*}(t_{\mathcal{Y}}(x), r_{\Gamma_{\mathcal{Y}}(x)}).$$

Indeed, suppose that $F : K \to X^*$ is any selection from $\overline{U}$ and consider $x \in K$ such that it is in the difference set $Y_{(\xi,i)} \setminus Y_{(\xi,i+1)}$. Lemma 2.6 insures existence of $\alpha(x)$ such that the hyperplane $\{y : F(x)y = \alpha(x)\}$ separates $Y_{(\xi,i)} \setminus Y_{(\xi,i+1)}$ from $Y_{(\xi,i+2)}$. Thus, what is more important, $x \in H(F(x), \alpha(x))$ and $H(F(x), \alpha(x)) \cap Z_{\xi+1} = \emptyset$ (cf. Figure 2.2). We

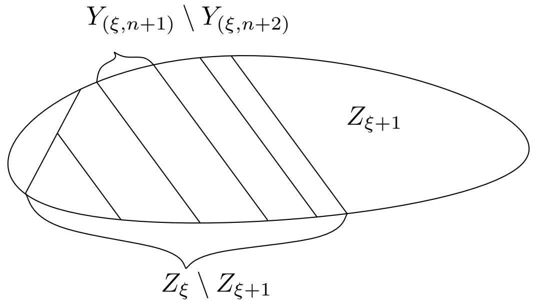

Figure 2.2:

may therefore define $t(x) := K \cap H(F(x), \alpha(x))$ and it will satisfy $x \in t(x) \subset t_{\mathcal{Z}}(x)$. This of course implies that t is an ε-WT since $t_{\mathcal{Z}}$ was. That means that $\mathcal{Y}$ is U-stable. □

2.3.4 Induction

The proof of Theorem 2.15 has an inductive character. Let us isolate the main ingredient of the induction step in the following corollary.

Corollary 2.26. *Let $\mathcal{Z}_1$ be a U_1-stable ε-slicing of K for some box U_1 around $\mathcal{Z}_1$ and for some $\varepsilon > 0$. Then there exists an $\frac{\varepsilon}{2}$-slicing $\mathcal{Z}_2$ of K which is U_1-refinement of $\mathcal{Z}_1$. Moreover, $\mathcal{Z}_2$ is U_2-stable for some box U_2 around $\mathcal{Z}_2$ and $U_2 \subset U_1$. (See Figure 2.3.)*

Proof. We may apply Proposition 2.23 to get a U_1-refinement $\mathcal{Y}$ of $\mathcal{Z}_1$ which is an $\frac{\varepsilon}{2}$-slicing of K. Then we refine $\mathcal{Y}$ (using Proposition 2.25) in order to get $\mathcal{Z}_2$ which is U_2-stable $\frac{\varepsilon}{2}$-slicing of K for some box U_2 around $\mathcal{Z}_2$. Since $\mathcal{Z}_2$ is a V-refinement of $\mathcal{Y}$ for every box V around $\mathcal{Y}$ (says Proposition 2.25), it is a U_1-refinement of the original $\mathcal{Z}_1$. Of course, U_2 may be chosen to satisfy $U_2 \subset U_1$. □

We are ready now to complete the proof of Theorem 2.15.

Proof of Theorem 2.15. First we construct an 1-slicing $\mathcal{Y} = (Y_\xi)_{\xi \le \eta}$ of K such that $Y_\gamma = A$ for some ordinal γ. This is easy since we put $Y_0 = K$ and then, when Y_ξ is already defined, we use Lemma 2.11 with $C_1 = A$ and $C_2 = Y_\xi$ to obtain $Y_{\xi+1}$; and we take intersections to obtain Y_λ when λ is a limit ordinal. This way, we arrive sooner or later (for cardinality reason) to the ordinal γ for which $Y_\gamma = A$. Then we just continue subtracting small slices from A until we are left with $Y_\eta = \emptyset$ for some ordinal η.

We use Proposition 2.25 to get a refinement $\mathcal{Z}_0$ of $\mathcal{Y}$ which is 1-slicing and for which there exists a box $U_0 = \prod_{x \in K} U_0(x)$ around $\mathcal{Z}_0$ such that $\mathcal{Z}_0$ is U_0-stable. We may suppose that $\operatorname{diam} U_0(x) < 1$ for all $x \in K$.

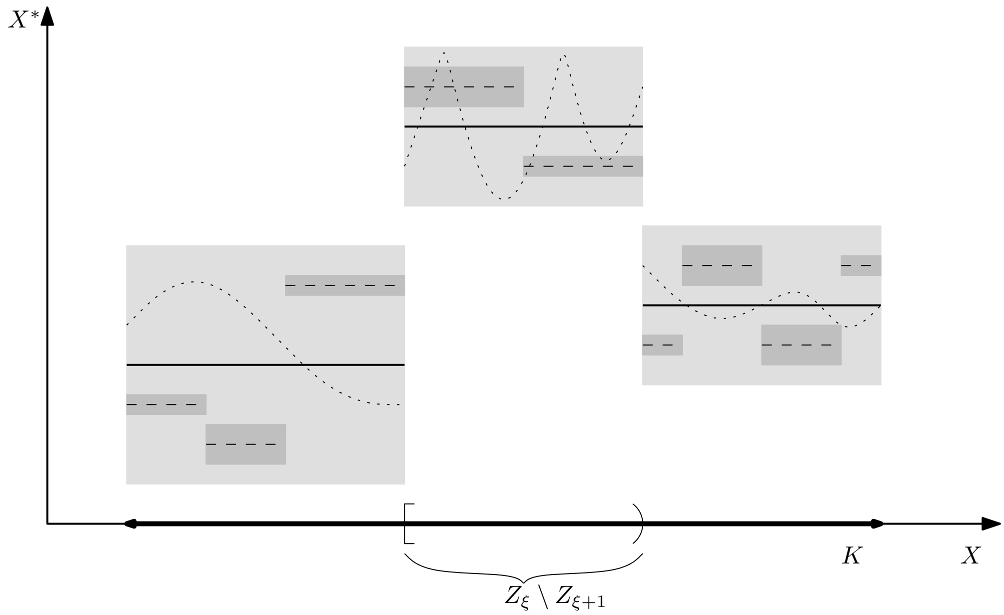

Figure 2.3: The solid line is $t_{\mathcal{Z}_1}$ and the light grey region is its stability box U_1. Any selection (dotted line) from U_1 would correspond to an ε-winning tactic. The dashed line is $t_{\mathcal{Z}_2}$ from Corollary 2.26 with its stability box U_2 drawn in dark grey.

Let $\mathcal{Z}_n$ be a U_n-stable 2^{-n}-slicing with $\operatorname{diam} U_n(x) < 2^{-n}$ for all $x \in K$. Then we may apply Corollary 2.26 to get a U_n-refinement $\mathcal{Z}_{n+1}$ of $\mathcal{Z}_n$ which is a U_{n+1}-stable 2^{-n-1}-slicing for which the box U_{n+1} satisfies $U_{n+1} \subset U_n$. Moreover, we may suppose that $\overline{\operatorname{diam} U_{n+1}(x)} < 2^{-(n+1)}$ for all $x \in K$. It is therefore possible to define $t(x) = K \cap \overline{H(F(x), a(x))}$ where $F(x)$ is the unique member of the intersection $\bigcap_{n=1}^{\infty} \overline{U_n(x)}$ and $a(x) = F(x)x$. Now for every $n \in \mathbb{N}$, $F : K \to X^*$ is a selection from the stability box $\overline{U_n}$ of the 2^{-n}-WT $t_{\mathcal{Z}_n}$, thus the mapping $t : x \mapsto K \cap \overline{H(F(x), a(x))}$ is a 2^{-n}-WT itself (in 2^{-n}-$\mathbf{G}(K, \mathcal{S}_c)$). That obviously implies that t is a winning tactic in $\mathbf{G}(K, \mathcal{S}_c)$.

For every $n \in \mathbb{N} \cup \{0\}$, let $t_{\mathcal{Z}_n} = K \cap H(F_n, a_n)$ be the 2^{-n}-WT canonically corresponding to the slicing $\mathcal{Z}_n$. Since $F \in \overline{U_n}$, one has $\sup_{x \in K} \|F(x) - F_n(x)\| \leq 2^{-n}$ for every $n \geq 0$. Thus F is a uniform limit of *slice constant* mappings. By [DGZ93, Proposition I.4.5] these mappings are Baire one so F is a Baire one mapping, too. □

Remark 2.27. It is not difficult to observe that our tactic t is continuous with respect to the game, i.e. if (x_n) satisfies $x_{n+1} \in t(x_n)$ and $t(x_n) = K \cap \overline{H(f_n, a_n)}$, then we have both (x_n) and (f_n) convergent.

Remark 2.28. Let us recall the following weak* version of Theorem 2.9 (see Theorem 3.5.4 (w*) in [Bou83]): *Assume that $C \subset X^*$ is weak* compact, convex, and every $A \subset C$ has arbitrarily small weak* open slices. Then the set $\mathcal{SE}(C) \cap X$ is a dense G_δ subset of X.*

The assumptions of this theorem are satisfied in particular when X^* is a dual space with the RNP and $C = B_{X^*}$ as has been pointed out in Chapter 1. Let $\mathcal{S}_c^*(B_{X^*})$ be the closed slices of B_{X^*} given by functionals from X. Then we claim that Player II has a winning tactic in the game $\mathbf{G}(B_{X^*}, \mathcal{S}_c^*)$. Indeed, the whole proof may be easily rephrased in terms of functionals from X provided one uses the above mentioned weak* version of Theorem 2.9

2.4 Winning tactics and uniformly rotund norms

Let t be a winning tactic in $\mathbf{G}(B_X, \mathcal{S}_c)$ and let $\varepsilon > 0$. Since t is winning, there clearly does not exist any infinite sequence $(x_i) \subset B_X$ satisfying

$$x_{i+1} \in t(x_i) \text{ and } \|x_i - x_{i+1}\| \geq \varepsilon \tag{2.2}$$

for all $i \in \mathbb{N}$ but one may ask whether there exists some uniform bound on the length of sequences that satisfy the above condition. One result in this direction was obtained by Zelený [Zel08] who constructed a WT t in $\mathbf{G}(B_{\mathbb{R}^N}, \mathcal{H})$ (where $\mathcal{H}$ are the hyperplane sections of $B_{\mathbb{R}^N}$) with the property that for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that no sequence $(x_i)_{i=1}^m \subset B_{\mathbb{R}^N}$ satisfies (2.2) for all $i < m$.

Definition 2.29. Let $t : B_X \to \mathcal{S}_c(B_X)$ be a tactic (winning or not) in the game $\mathbf{G}(B_X, \mathcal{S}_c)$. Let $\varepsilon > 0$. We say that t has *uniformly short ε-separated runs* if the following holds: there exists $m \in \mathbb{N}$ such that whenever $(x_i)_{i=1}^n \subset B_X$ satisfies (2.2) for all $i < n$, then $n < m$.

Zelený's result therefore reads: there is a winning tactic t in $\mathbf{G}(B_{\mathbb{R}^N}, \mathcal{H})$ which has uniformly short ε-separated runs for every $\varepsilon > 0$.

Our next theorem shows, in particular, that this is not possible in spaces that are not superreflexive. (Superreflexive spaces are those that admit an equivalent uniformly rotund norm.)

Definition 2.30. A norm $\|\cdot\|$ on a Banach space X is *uniformly rotund* if its *modulus of convexity*

$$\delta(t) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_{(X,\|\cdot\|)},\ \|x-y\| \geq t\right\}$$

is strictly positive for every $t \in (0, 2]$.

Theorem 2.31. *Let $(X, |\cdot|)$ be a Banach spaces. It is equivalent*

(i) *X admits a uniformly rotund norm $\|\cdot\|$;*

(ii) *for every $0 < \varepsilon < 1$ there exists a winning tactic t for Player II in $\mathbf{G}(B_{(X,|\cdot|)}, \mathcal{S}_c)$ which has uniformly short ε-separated runs.*

(iii) *for every $0 < \varepsilon < 1$ there exists a tactic t for Player II in $\mathbf{G}(B_{(X,|\cdot|)}, \mathcal{S}_c)$ which has uniformly short ε-separated runs.*

Before proving this theorem, let us recall the definition of the dentability index and a renorming result of G. Lancien [Lan95].

Definition 2.32. Let C be a convex, closed and bounded subset of X and $\varepsilon > 0$. We define the following set derivation

$$wd_\varepsilon(C) = C \setminus \bigcup \{S \in \mathcal{S}_o(C) : \operatorname{diam}(S) < \varepsilon\}$$

and we put $wd^0_\varepsilon(C) := C$ and $wd^{\alpha+1}_\varepsilon(C) := wd_\varepsilon(wd^\alpha_\varepsilon(C))$ for every ordinal α. Finally $wd^\beta_\varepsilon(C) := \bigcap_{\alpha<\beta} wd^\alpha_\varepsilon(C)$ for every limit ordinal β. We define *dentability index* $\mathrm{D}(X)$ of X as $\mathrm{D}(X) := \sup_{\varepsilon>0} \mathrm{D}(X,\varepsilon)$ where $\mathrm{D}(X,\varepsilon) := \inf\{\alpha : wd^\alpha_\varepsilon(B_X) = \emptyset\}$. Here we use the convention that $\inf \emptyset = \infty$ and $\alpha < \infty$ for every ordinal α.

Theorem 2.33. *A Banach space X admits a uniformly rotund norm if and only if $\mathrm{D}(X) \leq \omega$.*

Proof of Theorem 2.31. (i) $\Rightarrow$ (ii) Let $\varepsilon > 0$ be given. We may and do assume that $\|x\| \leq |x|$ for every $x \in X$. We will construct the desired WT t on the larger set $B_{(X,\|\cdot\|)}$. Let $\delta(\cdot)$ be the modulus of convexity of $\|\cdot\|$. Any slice S (open or closed) of $B_{(X,\|\cdot\|)}$ which does not intersect $(1-\delta(\varepsilon))B_{(X,\|\cdot\|)}$ has diameter smaller than ε. Similarly, for $n \in \mathbb{N}$,

$$\operatorname{diam}\left((1-\delta(\varepsilon))^n B_{(X,\|\cdot\|)} \cap S\right) < \varepsilon$$

whenever $(1-\delta(\varepsilon))^{n+1} B_{(X,\|\cdot\|)} \cap S = \emptyset$. On the other hand, there exists $m \in \mathbb{N}$ such that $(1-\delta(\varepsilon))^m < \varepsilon/2$ so all slices of $(1-\delta(\varepsilon))^m B_{(X,\|\cdot\|)}$ have automatically diameter smaller than ε.

We proceed similarly as in the proof of Corollary 2.17. For $n = 1,\dots,m$, let t_n be the WT in $\mathbf{G}(B_{(X,\|\cdot\|)}, \mathcal{S}_c)$ which comes from Theorem 2.15 with the set $A = (1-\delta(\varepsilon))^n B_{(X,\|\cdot\|)}$. For $x \in X$ with let $N(x) \in \mathbb{N}$ be the uniquely determined number N such that $x \in (1-\delta(\varepsilon))^{N-1} B_X \setminus (1-\delta(\varepsilon))^N B_X$. We define a mapping $t : X \to \mathcal{S}_c(X)$ by

$$t(x) := \begin{cases} t_{N(x)}(x) \text{ if } \|x\| > (1-\delta(\varepsilon))^m, \\ t_m(x) \text{ otherwise.} \end{cases}$$

Now let Player II play according to the tactic t and let x_i, x_{i+1} be consecutive moves of Player I such that $\|x_i - x_{i+1}\| \geq \varepsilon$. If $x_i \in (1-\delta(\varepsilon))^m B_{(X,\|\cdot\|)}$, then clearly $x_{i+1} \notin (1-\delta(\varepsilon))^m B_{(X,\|\cdot\|)}$. If $x_i \in (1-\delta(\varepsilon))^n B_{(X,\|\cdot\|)} \setminus (1-\delta(\varepsilon))^{n+1} B_{(X,\|\cdot\|)}$ for some $n < m$, then $t(x_i) \cap (1-\delta(\varepsilon))^{n+1} B_{(X,\|\cdot\|)} = \emptyset$. Since $x_{i+1} \in t(x_i)$ and $\operatorname{diam}\left((1-\delta(\varepsilon))^n B_{(X,\|\cdot\|)} \cap t(x_i)\right) < \varepsilon$, we conclude that $x_{i+1} \notin (1-\delta(\varepsilon))^n B_{(X,\|\cdot\|)}$. Thus $N(x_i) > N(x_{i+1})$ and one can see that ε-separated runs of the game cannot be longer than $m+1$ steps.

(ii) $\Rightarrow$ (iii) is trivial so it remains to prove (iii) $\Rightarrow$ (i). Let X be a Banach space without any uniformly rotund norm. Let $\|\cdot\|$ be any equivalent norm in X. We will show that no tactic t defined in $B_X = B_{(X,\|\cdot\|)}$ satisfies (iii). The next claim and its proof is inspired by the proof of Theorem 3.4 in [DM07].

Claim. There exist an $\varepsilon > 0$ and a sequence (V_n) of nonempty open subsets of B_X such that the following hold for every $n \in \mathbb{N}$: a) $V_{n+1} \subset V_n$, and b) for any open or closed slice S of B_X one has: if $S \cap V_{n+1} \neq \emptyset$, then $\operatorname{diam}(S \cap V_n) > 2\varepsilon$.

By Theorem 2.33, $D(X) > \omega$. It means that there is some $\varepsilon \in (0,1)$ such that $wd^{\omega}_{6\varepsilon}(B_X) \neq \emptyset$. Let $B_n := \frac{1}{2} wd^{n}_{6\varepsilon}(B_X)$. Then (B_n) is a sequence of nonempty closed convex subsets of $\frac{1}{2}B_X$ such that for every open slice S of B_X one has by homogeneity: if $S \cap B_{n+1} \neq \emptyset$, then $\operatorname{diam}(S \cap B_n) \geq 3\varepsilon$. We chose some strictly decreasing sequence (η_n) of positive numbers converging to 0 such that $\eta_1 < \varepsilon/2$; and we define $V_n := \{x \in X : \operatorname{dist}(x, B_n) < \eta_n\}$. Now each V_n is an open subset of B_X which contains B_n and it is easily seen that the property a) holds. We will prove the assertion b). Since each V_n is an open set, it is enough to prove only the case when S is an open slice. Indeed, if $\overline{H(f,a)} \cap V_{n+1} \neq \emptyset$, then $H(f,a) \cap V_{n+1} \neq \emptyset$ and $\operatorname{diam}\left(\overline{H(f,a)} \cap V_n\right) \geq \operatorname{diam}\left(H(f,a) \cap V_n\right)$. So let $n \in \mathbb{N}$ be fixed and let $x \in H(f,a) \cap V_{n+1}$ for some $x \in X$, $f \in X^* \setminus \{0\}$ and $a \in \mathbb{R}$. Then there exists an $x' \in B_{n+1}$ such that $\|x - x'\| < \eta_{n+1}$. It follows that, for $a' = a - (f(x) - f(x'))$, one has $x' \in H(f,a') \cap B_{n+1}$ and so there exist $y_1', y_2' \in H(f,a') \cap B_n$ such that $\|y_1' - y_2'\| \geq 3\varepsilon$. We define $y_i := y_i' + (x - x')$ for $i = 1, 2$. Evidently $\|y_i - y_i'\| < \eta_{n+1} < \eta_n$ so $y_i \in V_n$ for $i = 1, 2$. Further $f(y_i) > a$ for $i = 1, 2$, too and finally, by the triangle inequality, $\|y_1 - y_2\| \geq 3\varepsilon - 2\eta_{n+1} > 2\varepsilon$. So we have proved that $\operatorname{diam}(H(f,a) \cap V_n) > 2\varepsilon$, which finishes the proof of the claim.

Now let t be any tactic (winning or not) in $\mathbf{G}(B_X, \mathcal{S}_c)$. It is a consequence of the above claim, that if the last move x_i of Player I is in the set V_{n+1} for some $n \in \mathbb{N}$, then Player I can always choose some $x_{i+1} \in t(x_i) \cap V_n$ such that $\|x_i - x_{i+1}\| > \varepsilon$. It follows, that t does not have uniformly short ε-separated runs. □

Remark 2.34. After having seen the above proof, the next easily proved claim should be no surprise: For a Banach space X it is equivalent

(i) $\operatorname{Sz}_w(X) \leq \omega$;

(ii) for every $0 < \varepsilon < 1$ there exists a tactic t for Player II in $\mathbf{G}(B_X, \sigma(X, X^*))$ which has uniformly short ε-separated runs.

This should be compared with the result of R. Deville and É. Matheron [DM07] who proved that *a Banach space X has the PCP if and only if Player II has a winning strategy in the game $\mathbf{G}(B_X, \sigma(X, X^*))$.*

2.5 Baire one functions

In this section we are going to gather few observations which lead to a characterization of Baire one functions using games. A mapping f from a topological space (E, τ) to a normed linear space $(X, \|\cdot\|)$ is called *Baire one* if f is a pointwise limit of a sequence (f_n) of continuous mappings from (E, τ) to $(X, \|\cdot\|)$.

If ρ is a pseudometric on E, let $\mathbf{G}((E,\rho),\tau)$ be the usual game in E where Player II wins if the sequence (x_n) of the moves of Player I is ρ-Cauchy. Observe that if f is a mapping from E to a normed linear space $(X,\|\cdot\|)$, then $\rho_f(x,y)=\|f(x)-f(y)\|$ defines a pseudometric on E.

Theorem 2.35. *Let (E,τ) be a completely metrizable space, $(X,\|\cdot\|)$ a normed linear space and f be a mapping from (E,τ) to $(X,\|\cdot\|)$. Let d be a compatible complete metric on (E,τ). Then it is equivalent*

(i) *f is Baire one;*

(ii) *Player II has a winning strategy in the game $\mathbf{G}((E,\rho_f),\tau)$.*

(iii) *Player II has a winning strategy in the game $\mathbf{G}((E,\rho_f+d),\tau)$.*

Moreover, if (i) – (iii) is satisfied, then it is possible to construct a winning strategy $t=(t_n)$ in the game $\mathbf{G}((E,\rho_f+d),\tau)$ in such a way that if $(x_n)\subset E$ satisfies $x_{n+1}\in t_n(x_1,\dots,x_n)$, then

$$f(\lim_n x_n)=\lim_n f(x_n). \tag{2.3}$$

We will say that a topology τ on E is *fragmented* by a pseudometric ρ if for every $\varepsilon>0$ and every subset A of E there exists a relatively τ-open subset B of A such that $\rho-\operatorname{diam}(B)<\varepsilon$.

Let us recall Theorem 3.7 from [DM07].

Theorem 2.36. *The topological space (E,τ) is fragmented by the pseudometric ρ if and only if Player II has a winning strategy by in $\mathbf{G}((E,\rho),\tau)$.*

As a matter of fact, this theorem is formulated in [DM07] for a *metric* ρ. But the proof works also for a *pseudometric* ρ.

Proof of the equivalence (i)–(iii) in Theorem 2.35. We will show that (E,τ) is fragmented by ρ_f if and only if f is Baire one. Then applying Theorem 2.36 the equivalence of (i) and (ii) will be established. As far as the equivalence of (iii) is concerned, it is enough to observe that (E,τ) is fragmented by the sum $\rho_1+\rho_2$ of two pseudometrics ρ_1,ρ_2 if and only if (E,τ) is fragmented by ρ_i for $i=1,2$. Thus, since d is compatible with τ, we conclude that (E,τ) is fragmented by ρ_f if and only if it is fragmented by ρ_f+d.

We say that a mapping f is *barely continous* from (E,τ) to $(X,\|\cdot\|)$ if for very closed subset F of E there exists $x\in F$ such that $f\restriction_F$ is continuous at x.

Baire's Great Theorem (Theorem I.4.1 in [DGZ93]) asserts that the sets of Baire one and barely continuous mappings from (E,τ) to $(X,\|\cdot\|)$ coincide provided (E,τ) is completely metrizable.

It is easy to see that if $f:(E,\tau)\to(X,\|\cdot\|)$ is barely continuous, then τ is fragmented by ρ_f. Indeed, let $\varepsilon>0$ and A be any subset of E. Then there is a τ-open set U such that $\overline{A}^\tau\cap U\neq\emptyset$ and $\rho_f-\operatorname{diam}(\overline{A}^\tau\cap U)<\varepsilon$. Of course the same properties hold for $A\cap U$.

If (E,τ) is hereditarily Baire, i.e. each closed subset of E is a Baire space, in particular if (E,τ) is completely metrizable, then f is barely continuous provided τ is fragmented by ρ_f. Suppose that it is not true. Then there exists a closed subset F of E such that $f\restriction_F$ is discontinuous everywhere. We denote $F_n = \left\{x \in F : \operatorname{osc}_f(x) \geq \frac{1}{n}\right\}$ where $\operatorname{osc}_f(x) = \inf\{\rho_f - \operatorname{diam}(U)\}$ with the infimum taken over all relatively τ-open neighborhoods of x in F. It follows that each F_n is closed. The discontinuity assumption yields that $F = \bigcup F_n$. Since F is Baire, there exists $n \in \mathbb{N}$ such that F_n has nonempty interior $\operatorname{int} F_n$ relative to F. If U is any nonempty relatively open subset of F_n, then $\emptyset \neq \operatorname{int} F_n \cap U \subset U$ and by the definition of F_n we have $\rho_f - \operatorname{diam}(\operatorname{int} F_n \cap U) \geq \frac{1}{n}$.
This finishes the proof of the equivalence of the points (i), (ii) and (iii). □

For the moreover part we have to put our hands inside the construction of the winning strategy. In order to do so we first extend the terminology defined in Section 2.3.1 Let (E,τ) be a topological space. A *slicing* $\mathcal{Z}$ of E is a family $(Z_\xi)_{\xi\leq\eta}$ of τ-closed subsets of E, where η is an ordinal, satisfying:

(a) If $\alpha \leq \beta$ then $Z_\beta \subset Z_\alpha$;

(b) for each limit ordinal $\lambda \leq \eta$, $Z_\lambda = \bigcap_{\xi<\lambda} Z_\xi$;

(c) $Z_0 = F$ and $Z_\eta = \emptyset$.

For $x \in E$, let $\Gamma_{\mathcal{Z}}(x)$ be the unique ordinal $\gamma < \eta$ such that $x \in Z_\gamma \setminus Z_{\gamma+1}$.

Let ρ be some (pseudo)metric on E. If $\rho - \operatorname{diam}(Z_\xi \setminus Z_{\xi+1}) < \varepsilon$ for some $\varepsilon > 0$ and all $\xi < \eta$, then $\mathcal{Z}$ is called *ε-slicing.* As before, a slicing $\mathcal{Y}$ of E is a *refinement* of a slicing $\mathcal{Z}$, if $\mathcal{Z} \subset \mathcal{Y}$.

If (E,τ) is fragmented by a pseudometric ρ, it is easily seen, that there exists a sequence $(\mathcal{Z}^n)$ of slicings of E such that $\mathcal{Z}^{n+1}$ is a refinement of $\mathcal{Z}^n$, and $\mathcal{Z}^n$ is a $\frac{1}{n}$-slicing for each $n \in \mathbb{N}$. This occurs, in particular, in the setting of Theorem 2.35, i.e. if (E,d) is a complete metric space (with the topology τ), $f : (E,d) \to (X, \|\cdot\|)$ is a Baire one function and $\rho := \rho_f + d$. In this situation, we may finally define the winning strategy satisfying (2.3).

Proposition 2.37. *Let $t_n : D_n \to \tau$ be defined as*

$$t_n(x_1,\ldots,x_n) := t_1(x_1) \cap \ldots \cap t_{n-1}(x_1,\ldots,x_{n-1}) \cap B^O_{(E,d)}\left(x, \frac{d - \operatorname{dist}(x, Z^n_{\Gamma_{\mathcal{Z}^n}(x)+1})}{2}\right).$$

Then $t = (t_n)$ is a winning strategy for Player II in the game $\mathbf{G}((E,\rho),\tau)$ such that the validity of

$$x_{n+1} \in t_n(x_1,\ldots,x_n) \tag{2.4}$$

for every $n \in \mathbb{N}$ implies $f(\lim x_n) = \lim f(x_n)$.

(See Definition 2.2 to refresh the notions used in the above proposition.)

Proof. Let $(x_n) \subset E$ be a sequence which satisfies (2.4) for every $n \in \mathbb{N}$. Let $\varepsilon > 0$. Let us fix $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$. By the definition of t we have that $(\Gamma_{\mathcal{Z}^k}(x_n))_n$ is a nonincreasing sequence of ordinals. Thus this sequence is eventually constant, say $\Gamma_{\mathcal{Z}^k}(x_n) = \Gamma_{\mathcal{Z}^k}(x_j)$ for all $n \geq j$ for some $j > k$. So, the definition t_j yields for every $n \geq j$ that

$$x_n \in Z^k_{\Gamma_{\mathcal{Z}^k}(x_j)} \cap B^O_{(E,d)}\left(x_j, \frac{d - \operatorname{dist}(x_j, Z^j_{\Gamma_{\mathcal{Z}^j}(x_j)+1})}{2}\right). \tag{2.5}$$

In particular, since $\mathcal{Z}^j$ is a refinement of $\mathcal{Z}^k$,

$$(x_n)_{n \geq j} \subset Z^k_{\Gamma_{\mathcal{Z}^k}(x_j)} \setminus Z^k_{\Gamma_{\mathcal{Z}^k}(x_j)+1} =: M.$$

Because $\rho - \operatorname{diam}(M) < \varepsilon$ we have that (x_n) is ε-Cauchy (with respect to the metric ρ). Moreover, if (x_n) d-converges to some $x \in E$, then (using d-closedness of $Z^k_{\Gamma_{\mathcal{Z}^k}(x_j)}$ and (2.5)) necessarily $x \in M$, and therefore $\rho(x, x_n) < \varepsilon$ for all $n \geq j$. In this case it follows that

$$\|f(x) - f(x_n)\| < \varepsilon \tag{2.6}$$

for all $n \geq j$.

The proof is now finished by noting that ε was arbitrarily chosen. So (x_n) is Cauchy with respect to $\rho = d + \rho_f$, a fortiori (x_n) is Cauchy with respect to d which is a complete metric. Therefore (x_n) really d-converges to some $x \in E$ and so (2.6) holds for any $\varepsilon > 0$. This completes the proof of the moreover part of Theorem 2.35. □

Remark 2.38. Let (E, τ) be a completely metrizable space, $(X, \|\cdot\|)$ be a normed linear space and let f be a mapping from E to X. The *oscillation index* $\beta(f)$ of f is defined by the general peeling scheme described in the section 1.1.2 using the set derivation

$$[A]'_\varepsilon := A \setminus \bigcup \{U \in \tau : \rho_f(U \cap A) < \varepsilon\}$$

for $A \subset E$ (see [KL90]). It is clear, that the following version of Remark 2.34 holds true. It is equivalent:

(i) $\beta(f) \leq \omega$;

(ii) for every $0 < \varepsilon < 1$ there exists a tactic t for Player II in $\mathbf{G}((E, \rho_f), \tau)$ which has uniformly short ε-separated runs.

Chapter 3

Weak* dentability index of $C(K)$, $K^{(\omega_1)} = \emptyset$

The main result of this chapter, Theorem 3.18, is a computation of the weak* dentability index of the spaces $C(K)$ where K is a scattered compact of countable height. In fact, this theorem is obtained (by a separable reduction argument) from Theorem 3.10, which is a computation of the weak* dentability index of the space $C([0,\alpha])$, where α is a countable ordinal. To see how these results fit into a broader context, see Section 1.1.3.

3.1 Preliminaries

Definition 3.1. Let A be a weak* compact subset of X^*. Let $\mathcal{S}_o^*(A)$ be the weak* open slices of A. Recall that a *weak* open slice* S of A is a non-empty subset of A given as $S = A \cap H(x,a)$ for some $x \in X$ and some $a \in \mathbb{R}$, where $H(x,a) = \{x^* \in X^* : x^*(x) > a\}$ is a *weak* open halfspace*. We define the set derivation

$$d_\varepsilon(A) = A \setminus \bigcup \{S \in \mathcal{S}_o^*(A) : \operatorname{diam}(S) < \varepsilon\}$$

and we put

$$d_\varepsilon^0(C) := C, \quad d_\varepsilon^{\alpha+1}(C) := d_\varepsilon(d_\varepsilon^\alpha(C)) \text{ and } d_\varepsilon^\beta(C) := \bigcap_{\alpha<\beta} d_\varepsilon^\alpha(C)$$

for every ordinal α and every limit ordinal β. Further we define

$$\operatorname{Dz}(X,\varepsilon) := \inf\{\alpha : d_\varepsilon^\alpha(B_X) = \emptyset\} \text{ and } \operatorname{Dz}(X) := \sup_{\varepsilon>0} \operatorname{Dz}(X,\varepsilon)$$

adopting the convention that $\inf\emptyset = \infty$ and $\alpha < \infty$ for every ordinal α. The quantity $\operatorname{Dz}(X)$ is called the *weak* dentability index* of X. In a similar way, we define the *Szlenk index* of X, $\operatorname{Sz}(X) = \sup_{\varepsilon>0} \operatorname{Sz}(X,\varepsilon)$, based on the set derivation

$$s_\varepsilon(A) = A \setminus \bigcup \{V \in \sigma(X^*, X) : \operatorname{diam}(V \cap A) < \varepsilon\}.$$

The notation $s_\varepsilon^\alpha(A)$ has the obvious meaning.

It is obvious from the definition that these indices are isomorphically invariant. The general idea of assigning an isomorphically invariant ordinal index to a class of Banach spaces proved to be extremely fruitful in many situations. We suggest the articles [Lan06, Ode04] for a survey of the numerous applications to the geometry of Banach spaces.

Let us recapitulate here some useful facts about the weak* dentability index.

Lemma 3.2. *If Y is a closed subspace of a Banach space X, then* $\mathrm{Dz}(Y) \leq \mathrm{Dz}(X)$ *and* $\mathrm{Sz}(Y) \leq \mathrm{Sz}(X)$.

We shall use the above well known lemma frequently and without further reference. We will prove the case of the weak* dentability index. A similar proof for the case of the Szlenk index is given in [HMSVZ08, Lemma 2.39].

Proof. Let us denote $i : Y \to X$ the embedding operator and let $\varepsilon > 0$ be fixed. Suppose that $P \subset B_{X^*}$, $S \subset B_{Y^*}$ are weak* compact and convex sets such that $S \subset i^*(P)$. We claim that $d^1_\varepsilon(S) \subset i^*(d^1_{\varepsilon/3}(P))$. Assume therefore that $s \in d^1_\varepsilon(S)$. Then

$$s \in \overline{\mathrm{co}}^{w^*}\Big(S \setminus B_{Y^*}(s, \frac{\varepsilon}{3})\Big),$$

where $\overline{\mathrm{co}}^{w^*}(C)$ stands for the weak* closed convex hull of a set C. So there is a net $(\bar{s}_\xi)_\xi$ weak* converging to s such that for each ξ there are $n_\xi \in \mathbb{N}$, $\lambda_\xi \in \mathbb{R}^{n_\xi}$ and $s_\xi \in S^{n_\xi}$ such that $\bar{s}_\xi = \sum_i \lambda_\xi(i) s_\xi(i)$ and $1 = \sum_i \lambda_\xi(i)$ for all ξ, and $\|s - s_\xi(i)\| > \varepsilon/3$ for all ξ and all $i = 1, \dots, n_\xi$. Let us choose some $p_\xi(i) \in P$ such that $i^*(p_\xi(i)) = s_\xi(i)$ for all ξ and i. By the weak* compactness of P we may assume that $\bar{p}_\xi$ defined as $\bar{p}_\xi = \sum_i \lambda_\xi(i) s_\xi(i)$ converges weak* to some $p \in P$. Now $i^*(p) = s$ by the weak*–weak* continuity of i^*. Thus $\|p - p_\xi(i)\| > \varepsilon/3$ and we conclude that $p \in d^1_{\varepsilon/3}(P)$. A transfinite induction argument now yields

$$d^\alpha_{\varepsilon/3}(P) = \emptyset \Rightarrow d^\alpha_\varepsilon(S) = \emptyset.$$

In particular, taking $S = B_{Y^*}$ and $P = B_{X^*}$, implies $\mathrm{Dz}(Y) \leq \mathrm{Dz}(X)$. □

Lemma 3.3. *If* $\mathrm{Dz}(X) < \infty$, *then there exists an ordinal α such that* $\mathrm{Dz}(X) = \omega^\alpha$. *An analogous result is true about the Szlenk index.*

For reader's convenience, we present here the proof for the case of weak* dentability index. Almost identical proof for the case of the Szlenk index may be found in [HMSVZ08, Theorem 2.43].

Proof. The proof is based on the following observation: for every ordinal α,

$$\frac{1}{2} d^\alpha_\varepsilon(B_{X^*}) + \frac{1}{2} B_{X^*} \subset d^\alpha_{\varepsilon/2}(B_{X^*}). \tag{3.1}$$

This is proved by a transfinite induction, which is trivial for $\alpha = 0$ and in the case when α is a limit ordinal. Let us assume that $\alpha = \beta + 1$, that (3.1) is true for β, and that $x \in d^\alpha_\varepsilon(B_{X^*})$. Let $x' := \frac{1}{2}x + \frac{1}{2}z$ for some $z \in B_{X^*}$ and let $x' \in H(f, a')$ for some $f \in X$, $a \in \mathbb{R}$. Put $a := 2a' - f(z)$. Then $x \in H(f,a)$. It follows that there are $y_1, y_2 \in d_\varepsilon^\beta(B_{X^*}) \cap H(f,a)$ such that $\|y_1 - y_2\| \geq \varepsilon$. By the inductive hypothesis, $y_i' := \frac{1}{2}y_i + \frac{1}{2}z \in d_{\varepsilon/2}^\beta(B_{X^*})$ for $i = 1,2$. It is evident that $\|y_1' - y_2'\| \geq \varepsilon/2$ and $y_i' \in H(f,a')$ for $i = 1,2$, which finishes the proof of the observation.

Now we will show that $\mathrm{Dz}(X) > \omega^\alpha$ implies $\mathrm{Dz}(X) \geq \omega^{\alpha+1}$. The conclusion then easily follows. So let us assume that $\mathrm{Dz}(X) > \omega^\alpha$. This means that for some $\varepsilon > 0$ one has $0 \in d_\varepsilon^{\omega^\alpha}(B_{X^*})$. An application of (3.1) yields $\frac{1}{2}B_{X^*} \subset d_{\varepsilon/2}^{\omega^\alpha}(B_{X^*})$. Also, by homogeneity, $0 \in d_{\varepsilon/2}^{\omega^\alpha}(\frac{1}{2}B_{X^*})$ so

$$0 \in d_{\varepsilon/2}^{\omega^\alpha}(\frac{1}{2}B_{X^*}) \subset d_{\varepsilon/2}^{\omega^\alpha \cdot 2}(B_{X^*}).$$

Proceeding this way, one gets $0 \in d_{\varepsilon/2^n}^{\omega^\alpha \cdot 2^n}(B_{X^*})$ for every $n \in \mathbb{N}$. Thus $\mathrm{Dz}(X) \geq \omega^{n+1}$. $\square$

Lemma 3.4. *Let X be a Banach space and let α be an ordinal. Assume that for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$d_\varepsilon^\alpha(B_{X^*}) \subset (1 - \delta(\varepsilon))B_{X^*}.$$

Then $\mathrm{Dz}(X) \leq \alpha \cdot \omega$.

For an analog of this lemma in terms of the Szlenk index see Proposition 2.40 in [HMSVZ08].

Proof. Let $\varepsilon > 0$. By homogeneity, it is clear that

$$d_\varepsilon^{\alpha \cdot n}(B_{X^*}) \subset (1 - \delta(\varepsilon))^n B_{X^*} \text{ for every } n \in \mathbb{N}.$$

On the other hand, $(1-\delta(\varepsilon))^N < \varepsilon/2$ for some $N \in \mathbb{N}$, and so $d_\varepsilon^{\alpha \cdot N+1}(B_{X^*}) = \emptyset$. It follows that $\mathrm{Dz}(X) \leq \alpha \cdot \omega$. $\square$

Let X be a Banach space and $L_2(X)$ be the Bochner space $L_2([0,1], X)$. The following is well known (cf. [DU77, Theorem IV.1.1])

Theorem 3.5. *When X^* has the RNP then $(L_2(X))^*$ is isomorphically isometric to $L_2(X^*)$. In particular the duality pairing of $x \in L_2(X)$ and $f \in L^2(X^*)$ is given by*

$$\langle f, x \rangle = \int_0^1 \langle f(t), x(t) \rangle \, dt.$$

We shall also use the following lemma due to G. Lancien [Lan06, Lemma 1].

Lemma 3.6. *For every Banach space X it holds*

$$\mathrm{Dz}(X) \leq \mathrm{Sz}(L_2(X)).$$

The main result of our note, Theorem 3.10, is a precise evaluation of the weak* dentability index for the class of $C([0,\alpha])$, α countable. These spaces have been classified isomorphically by C. Bessaga and A. Pełczyński [BP60] in the following way.

Theorem 3.7. *Let $\omega \le \alpha \le \beta < \omega_1$. Then $C([0,\alpha])$ is isomorphic to $C([0,\beta])$ if and only if $\beta < \alpha^\omega$.*

Moreover, $C(K)$ is isomorphic to $C([0,\omega^{\omega^\alpha}])$ for every countable compact space K whose height $\eta(K)$ satisfies $\omega^\alpha < \eta(K) \le \omega^{\alpha+1}$.

Definition 3.8. Let K be a compact space. Then we define the *Cantor derived set* of K as

$$K' = K \setminus \{p \in K : p \text{ is an isolated point of } K\}$$

and the Cantor derived set of order α by the usual iteration: $K^{(0)} = K$, $K^{(\alpha+1)} = (K^{(\alpha)})'$ for every ordinal α, and $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}$ if α is a limit ordinal. The compact space K is *scattered* if there is some ordinal α such that $K^{(\alpha)} = \emptyset$. The *height* $\eta(K)$ of a scattered compact K is the least ordinal α for which $K^{(\alpha)} = \emptyset$.

The "only if" part of Theorem 3.7 is seen immediately from the following result of C. Samuel [Sam84] (see also [HL07]).

Theorem 3.9. *Let $0 \le \alpha < \omega_1$. Then* $\mathrm{Sz}(C([0,\omega^{\omega^\alpha}])) = \omega^{\alpha+1}$.

It is also well-known and easy to show that for $\alpha \ge \omega$, $C([0,\alpha])$ is isomorphic to $C_0([0,\alpha])$ where $C_0([0,\alpha]) = \{f \in C([0,\alpha]) : f(\alpha) = 0\}$.

3.2 Weak* dentability index of $C([0,\alpha])$

Theorem 3.10. *Let $0 \le \alpha < \omega_1$. Then* $\mathrm{Dz}(C([0,\omega^{\omega^\alpha}])) = \omega^{1+\alpha+1}$.

3.2.1 The upper estimate

We start by proving the upper estimate

$$\mathrm{Dz}(C([0,\omega^{\omega^\alpha}])) \le \omega^{1+\alpha+1}, \tag{3.2}$$

Having in mind Lemma 3.6, it is sufficient to prove the following.

Proposition 3.11. *Let $0 \le \alpha < \omega_1$. Then* $\mathrm{Sz}(L_2(C([0,\omega^{\omega^\alpha}]))) \le \omega^{1+\alpha+1}$.

The method of the proof is similar to [HL07], where a short and direct computation of the Szlenk index of the spaces $C([0,\alpha])$ is presented.

Proof. For a fixed $\alpha < \omega_1$ and $\gamma < \omega^{\omega^\alpha}$, let us put $Z = L_2(\ell_1([0,\omega^{\omega^\alpha}]))$ and $Z_\gamma = L_2(\ell_1([0,\gamma]))$. Since $\ell_1([0,\lambda])$ is separable for λ countable, we may, using Theorem 3.5, identify $Z = \left[L_2(C_0([0,\omega^{\omega^\alpha}]))\right]^*$ and $Z_\gamma = [L_2(C([0,\gamma]))]^*$.

Let P_γ be the canonical projection from $\ell_1([0,\omega^{\omega^\alpha}))$ onto $\ell_1([0,\gamma])$. Then, for $f \in Z$ and $t \in [0,1]$, we define $(\Pi_\gamma f)(t) = P_\gamma(f(t))$. Clearly, Π_γ is a norm one projection from Z onto Z_γ (viewed as a subspace of Z). We also have that for any $f \in Z$, $\|\Pi_\gamma f - f\|$ tends to 0 as γ tends to ω^{ω^α}.

Next is a variant of Lemma 3.3 in [HL07].

Lemma 3.12. *Let $\alpha < \omega_1$, $\gamma < \omega^{\omega^\alpha}$, $\beta < \omega_1$ and $\varepsilon > 0$. If $z \in s^\beta_{3\varepsilon}(B_Z)$ and $\|\Pi_\gamma z\|^2 > 1-\varepsilon^2$, then $\Pi_\gamma z \in s^\beta_\varepsilon(B_{Z_\gamma})$.*

Proof. We will proceed by transfinite induction in β. The cases $\beta = 0$ and β a limit ordinal are clear. Next we assume that $\beta = \mu + 1$ and the statement has been proved for all ordinals less than or equal to μ. Consider $f \in B_Z$ with $\|\Pi_\gamma f\|^2 > 1-\varepsilon^2$ and $\Pi_\gamma f \notin s^\beta_\varepsilon(B_{Z_\gamma})$. Assuming $f \notin s^\mu_{3\varepsilon}(B_Z) \supset s^\beta_{3\varepsilon}(B_Z)$ finishes the proof, so we may suppose that $f \in s^\mu_{3\varepsilon}(B_Z)$. By the inductive hypothesis, $\Pi_\gamma f \in s^\mu_\varepsilon(B_{Z_\gamma})$. Thus there exists a weak*-neighborhood V of f such that the diameter of $V \cap s^\mu_\varepsilon(B_{Z_\gamma})$ is less than ε. We may assume that V can be written $V = \bigcap_{i=1}^k H(\varphi_i, a_i)$, where $a_i \in \mathbb{R}$ and $\varphi_i \in L_2(C([0,\gamma]))$. We may also assume, using Hahn-Banach theorem, that $V \cap (1-\varepsilon^2)^{1/2} B_{Z_\gamma} = \emptyset$.

Define $\Phi_i \in L_2(C_0([0,\omega^{\omega^\alpha}))$ by $\Phi_i(t)(\sigma) = \varphi_i(t)(\sigma)$ if $\sigma \le \gamma$ and $\Phi_i(t)(\sigma) = 0$ otherwise. Then define $W = \bigcap_{i=1}^k H(\Phi_i, a_i)$. Note that for f in Z, $f \in W$ if and only if $\Pi_\gamma f \in V$. In particular W is a weak*-neighborhood of f. Consider now $g, g' \in W \cap s^\mu_{3\varepsilon}(B_Z)$. Then $\Pi_\gamma g$ and $\Pi_\gamma g'$ belong to V and therefore they have norms greater than $(1-\varepsilon^2)^{1/2}$. It follows from the induction hypothesis that $\Pi_\gamma g, \Pi_\gamma g' \in s^\mu_\varepsilon(B_{Z_\gamma})$ thus $\|\Pi_\gamma g - \Pi_\gamma g'\| \le \varepsilon$. Since $\|\Pi_\gamma g\|^2 > 1-\varepsilon^2$ and $\|g\| \le 1$, we also have $\|g - \Pi_\gamma g\| < \varepsilon$. The same is true for g' and therefore $\|g - g'\| < 3\varepsilon$. This finishes the proof of the Lemma. □

We are now in position to prove Proposition 3.11. For that purpose it is enough to show that for all $\alpha < \omega_1$:

$$\forall \gamma < \omega^{\omega^\alpha} \quad \forall \varepsilon > 0 \quad s_\varepsilon^{\omega^{1+\alpha}}(B_{Z_\gamma}) = \emptyset. \tag{3.3}$$

We will prove this by transfinite induction on $\alpha < \omega_1$.

For $\alpha = 0$, γ is finite and the space Z_γ is isomorphic to L_2 and therefore, according to Theorem 2.33, $s^\omega_\varepsilon(B_{Z_\gamma})$ is empty. So (3.3) is true for $\alpha = 0$.

Assume that (3.3) holds for $\alpha < \omega_1$. Let $Z = L_2(C_0([0,\omega^{\omega^\alpha}]))$. It follows from Lemma 3.12 and the fact that (for all $f \in Z$) $\|\Pi_\gamma f - f\|$ tends to 0 as γ tends to ω^{ω^α}, that

$$\forall \varepsilon > 0 \quad s_\varepsilon^{\omega^{1+\alpha}}(B_Z) \subset (1-\varepsilon^2)^{1/2} B_Z.$$

From this and Lemma 3.4 it follows that

$$\forall \varepsilon > 0 \quad s_\varepsilon^{\omega^{1+\alpha+1}}(B_Z) = \emptyset.$$

By Theorem 3.7 we know that the spaces $C([0,\gamma])$, $C([0,\omega^{\omega^\alpha}])$, and also $C_0([0,\omega^{\omega^\alpha}])$ are isomorphic, whenever $\omega^{\omega^\alpha} \le \gamma < \omega^{\omega^{\alpha+1}}$. Thus $s_\varepsilon^{\omega^{1+\alpha+1}}(B_{Z_\gamma}) = \emptyset$ for any $\varepsilon > 0$ and $\gamma < \omega^{\omega^{\alpha+1}}$, i.e. (3.3) holds for $\alpha + 1$.

Finally, the induction is clear for limit ordinals. □

3.2.2 The lower estimate

In order to complete the proof of Theorem 3.10 we have to focus on the converse inequality

$$\mathrm{Dz}(C([0,\omega^{\omega^\alpha}])) \ge \omega^{1+\alpha+1}. \tag{3.4}$$

Note that it suffices to deal with the spaces $C([0,\omega^{\omega^\alpha}])$ where $\alpha < \omega$. Indeed, in case $\alpha \ge \omega$, the inequality (3.2) and Theorem 3.9 imply that

$$\omega^{\alpha+1} = \omega^{1+\alpha+1} \ge \mathrm{Dz}(C([0,\omega^{\omega^\alpha}])) \ge \mathrm{Sz}(C([0,\omega^{\omega^\alpha}])) = \omega^{\alpha+1}.$$

Proposition 3.13. *Let X, Z be Banach spaces and let $Y \subset X^*$ be a closed subspace. Let there be $T \in \mathcal{B}(X,Z)$ such that T^* is an isometric isomorphism from Z^* onto Y. Let $\varepsilon > 0$, α be an ordinal such that $B_{X^*} \cap Y \subset d_\varepsilon^\alpha(B_{X^*})$, and $z \in Z^*$. If $z \in d_\varepsilon^\beta(B_{Z^*})$, then $T^*z \in d_\varepsilon^{\alpha+\beta}(B_{X^*})$.*

Proof. By induction with respect to β. The cases when $\beta = 0$ or β is a limit ordinal are clear. Let $\beta = \mu + 1$ and suppose that $T^*z \notin d_\varepsilon^{\alpha+\beta}(B_{X^*})$. If $z \notin d_\varepsilon^\mu(B_{Z^*})$, then the proof is finished. So we proceed assuming that $z \in d_\varepsilon^\mu(B_{Z^*})$, which by the inductive hypothesis implies that $T^*z \in d_\varepsilon^{\alpha+\mu}(B_{X^*})$. There exist $x \in X$, $t \in \mathbb{R}$, such that $T^*z \in H(x,t) \cap d_\varepsilon^{\alpha+\mu}(B_{X^*}) =: S$ and $\operatorname{diam} S < \varepsilon$. Consider the slice $S' = H(Tx,t) \cap d_\varepsilon^\mu(B_{Z^*})$. We have $\langle Tx, z\rangle = \langle x, T^*z\rangle$, so $z \in S'$. Also, $\operatorname{diam} S' \le \operatorname{diam} S < \varepsilon$ as T^* is an isometry. We conclude that $z \notin d_\varepsilon^\beta(B_{Z^*})$, which finishes the argument. □

Definition 3.14. Let α be an ordinal and $\varepsilon > 0$. We will say that a subset M of X^* is an *ε-α-obstacle* for $f \in B_{X^*}$ if

(i) $\operatorname{dist}(f, M) \ge \varepsilon$,

(ii) for every $\beta < \alpha$ and every w^*-open slice S of $d_\varepsilon^\beta(B_{X^*})$ with $f \in S$ we have $S \cap M \ne \emptyset$.

It follows by transfinite induction that if f has an ε-α-obstacle, then $f \in d_\varepsilon^\alpha(B_{X^*})$.

An *(n,ε)-tree* in a Banach space X is a finite sequence $(x_i)_{i=0}^{2^{n+1}-2} \subset X$ such that

$$x_i = \frac{x_{2i+1} + x_{2i+2}}{2} \text{ and } \|x_{2i+1} - x_{2i+2}\| \ge \varepsilon$$

for $i = 0, \dots, 2^n - 2$. The element x_0 is called the *root* of the tree $(x_i)_{i=0}^{2^{n+1}-2}$. Note that if $(h_i)_{i=0}^{2^{n+1}-2} \subset B_{X^*}$ is an (n,ε)-tree in X^*, then $h_0 \in d_\varepsilon^n(B_{X^*})$.

Define $f_\beta \in \ell_1([0,\alpha])$, for $\alpha \ge \beta$, by $f_\beta(\xi) = 1$ if $\xi = \beta$ and $f_\beta(\xi) = 0$ otherwise.

Lemma 3.15.
$$f_\omega \in d^{\omega}_{1/2}(B_{\ell_1([0,\omega])})$$

Proof. Let $\mathcal{P} = \big\{h \in B_{\ell_1([0,\omega])} : \|h\|_1 = 1,\ h(n) \geq 0,\ h(\omega) = 0\big\}$. Let A be a finite subset of $\mathbb{N}$. We denote $\sigma_A \in \ell_1([0,\omega])$ the normalized characteristic function of A, i.e. $\sigma_A(k) = \frac{1}{|A|}$ if $k \in A$, and $\sigma_A(k) = 0$ otherwise. Observe that then $\sigma_A \in \mathcal{P}$.

We claim that for every $n \in \mathbb{N} \cup \{0\}$ and every A of cardinality $|A| = 2^n$, there is a $(1,n)$-tree $(x_i)_{i=0}^{2^{n+1}-2} \subset \mathcal{P}$ such that $\sigma_A = x_0$.

We prove the claim by induction on n. If $n = 0$, the conclusion is trivially satisfied. Let us assume that the assertion has been proved for every $m \leq n$ and let $A \subset \mathbb{N}$ be of cardinality $|A| = 2^{n+1}$. We assume that $A = \{k_1, \dots, k_{2^{n+1}}\}$. We define elements $x_1, x_2 \in \mathcal{P}$ as follows:

$$x_1(k) := \begin{cases} 2\sigma_A(k) & \text{if } k = k_{2i}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad x_2(k) := \begin{cases} 2\sigma_A(k) & \text{if } k = k_{2i-1}, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\|x_1 - x_2\|_1 = 1$ and that $\sigma_A = \frac{x_1+x_2}{2}$. The induction is finished by noting that $x_i = \sigma_{A_i}$ for some $A_i \subset \mathbb{N}$ for which $|A_i| = 2^n$ for both $i = 1, 2$. (see also [FHH⁺01, Exercise 9.20]).

We have $\sigma_A \in d^n_{1/2}(B_{\ell_1([0,\omega])})$ whenever $|A| = 2^n$, and $\operatorname{dist}(f_\omega, \mathcal{P}) = 2$. Finally, for every $n \in \mathbb{N}$, every $x \in C([0,\omega])$ and every $t \in \mathbb{R}$ such that $f_\omega \in H(x,t)$, there exists $A \subset \mathbb{N}$, $|A| = 2^n$ such that $\sigma_A \in H(x,t)$. Therefore the set $\{\sigma_A : A \subset \mathbb{N}, |A| < \infty\}$ is an $\frac{1}{2}$-ω-obstacle for f_ω. Thus $f_\omega \in d^{\omega}_{1/2}(B_{\ell_1([0,\omega])})$. □

Proposition 3.16. *For every $\alpha < \omega$,*

$$f_{\omega^{\omega^\alpha}} \in d^{\omega^{1+\alpha}}_{1/2}(B_{\ell_1([0,\omega^{\omega^\alpha}])}) \tag{3.5}$$

Proof. The case $\alpha = 0$ is contained in Lemma 3.15. Let us suppose that we have proved the assertion (3.5) for all ordinals (natural numbers, in fact) less than or equal to α. It is enough to show, for every $n \in \mathbb{N}$, that

$$f_{\left(\omega^{\omega^\alpha}\right)^n} \in d^{\omega^{1+\alpha}n}_{1/2}(B_{\ell_1([0,\left(\omega^{\omega^\alpha}\right)^n])}). \tag{3.6}$$

Indeed, using Proposition 3.13, the equation (3.6) implies

$$f_{\left(\omega^{\omega^\alpha}\right)^n} \in d^{\omega^{1+\alpha}n}_{1/2}(B_{\ell_1([0,\omega^{\omega^{\alpha+1}}])}).$$

Since $f_{(\omega^{\omega^\alpha})^n} \xrightarrow{w^*} f_{\omega^{\omega^{\alpha+1}}}$ and $\left\|f_{(\omega^{\omega^\alpha})^n} - f_{\omega^{\omega^{\alpha+1}}}\right\| = 2$, we see that $\{f_{\left(\omega^{\omega^\alpha}\right)^n} : n \in \mathbb{N}\}$ is an $\frac{1}{2}$-$\omega^{1+\alpha+1}$-obstacle for $f_{\omega^{\omega^{\alpha+1}}}$. That implies (3.5) for $\alpha + 1$.

In order to prove (3.6) we will proceed by induction. The case $n = 1$ follows from the inductive hypothesis as indicated above, so let us suppose that $n = m+1$ and (3.6) holds for m.

Define a mapping $T : C([0, (\omega^{\omega^\alpha})^n]) \to C([0, \omega^{\omega^\alpha}])$ by

$$(Tx)(\gamma) = x((\omega^{\omega^\alpha})^m(1+\gamma)),\ \gamma \le \omega^{\omega^\alpha}.$$

A simple computation shows that the dual map T^* is given by

$$(T^*g)(\gamma) = \begin{cases} g(\xi), \text{ if } \gamma = (\omega^{\omega^\alpha})^m(1+\xi),\ \xi \le \omega^{\omega^\alpha} \\ 0 \text{ otherwise.} \end{cases}$$

Clearly, T^* is an isometric isomorphism of $\ell_1([0, \omega^{\omega^\alpha}])$ onto $\operatorname{rng} T^*$. We claim that

$$B_{\ell_1([0,(\omega^{\omega^\alpha})^n])} \cap \operatorname{rng} T^* \subset d_{1/2}^{\omega^{1+\alpha}m}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])}). \tag{3.7}$$

Note that the set of extremal points of $B_{\ell_1([0,(\omega^{\omega^\alpha})^n])} \cap \operatorname{rng} T^*$ satisfies

$$\operatorname{Ext}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])} \cap \operatorname{rng} T^*) \subset \left\{f_\gamma, -f_\gamma : \gamma = (\omega^{\omega^\alpha})^m(1+\xi),\ \xi \le \omega^{\omega^\alpha}\right\}.$$

By the inductive assumption and by symmetry, both $f_{(\omega^{\omega^\alpha})^m}$ and $-f_{(\omega^{\omega^\alpha})^m}$ belong to $d_{1/2}^{\omega^{1+\alpha}m}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])})$. It is easy to see that more generally, both f_γ and $-f_\gamma$ belong to $d_{1/2}^{\omega^{1+\alpha}m}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])})$, whenever $\gamma = (\omega^{\omega^\alpha})^m(1+\xi)$, $\xi \le \omega^{\omega^\alpha}$. Thus we have verified that

$$\operatorname{Ext}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])} \cap \operatorname{rng} T^*) \subset d_{1/2}^{\omega^{1+\alpha}m}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])}),$$

and the claim (3.7) follows using the Krein-Milman theorem.

This together with the inductive assumption (3.5) allows us to apply Proposition 3.13 (with $\ell_1([0, (\omega^{\omega^\alpha})^n])$ as X^*, $C([0, \omega^{\omega^\alpha}])$ as Z, and $\operatorname{rng} T^*$ as Y) to get

$$f_{(\omega^{\omega^\alpha})^n} = T^* f_{\omega^{\omega^\alpha}} \in d_{1/2}^{\omega^{1+\alpha}n}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])}).$$

□

Proof of Theorem 3.10. A combination of Lemma 3.3 and Proposition 3.16 yields the lower estimate (3.4) for $\alpha < \omega$. Together with (3.2) we obtain

$$\operatorname{Dz}(C([0, \omega^{\omega^\alpha}])) = \omega^{1+\alpha+1}$$

for $\alpha < \omega$. For $\omega \le \alpha < \omega_1$, we use that $\omega^{1+\alpha+1} = \omega^{\alpha+1} = \operatorname{Sz}(C([0, \omega^{\omega^\alpha}])) = \operatorname{Dz}(C([0, \omega^{\omega^\alpha}]))$, which finishes the proof. □

Our next proposition is a direct consequence of Theorem 3.10, Lemma 3.6 and Proposition 3.11.

Proposition 3.17. *Let $0 \le \alpha < \omega_1$. Then $\operatorname{Sz}(L_2(C([0, \omega^{\omega^\alpha}]))) = \omega^{1+\alpha+1}$.*

3.3 The general case

Our main result can be extended to the non separable case as follows.

Theorem 3.18. *Let $0 \leq \alpha < \omega_1$. Let K be a compact space whose Cantor derived sets satisfy $K^{\omega^\alpha} \neq \emptyset$ and $K^{\omega^{\alpha+1}} = \emptyset$. Then $\mathrm{Dz}(C(K)) = \omega^{1+\alpha+1}$.*

Before proving the theorem we have to recall a theorem due to G. Lancien [Lan96] which claims that the Szlenk and dentability index are separably determined, provided they are countable.

Theorem 3.19. *Let X be an Asplund space and $\alpha < \omega_1$. If $\mathrm{Dz}(X) > \alpha$, then there exists a separable subspace Y of X with $\mathrm{Dz}(Y) > \alpha$.*

An analogous assertion is true about the Szlenk index of X.

It follows immediately that if $\mathrm{Dz}(X) \geq \alpha$, then there exists a separable subspace Y of X with $\mathrm{Dz}(Y) \geq \alpha$.

Proof of Theorem 3.18. A straightforward transfinite induction shows that the Dirac functional f_t (as defined on the page 68) belongs to $s_1^\beta(B_{C(K)^*})$ provided $t \in K^{(\beta)}$. Thus $\mathrm{Sz}(C(K)) > \omega^\alpha$, and the Szlenk index version of Lemma 3.3 implies that $\mathrm{Sz}(C(K)) \geq \omega^{\alpha+1}$. Using Theorem 3.19, we obtain a separable subspace X of $C(K)$ such that $\mathrm{Sz}(X) \geq \omega^{\alpha+1}$. Let Y be the closed subalgebra of $C(K)$ generated by X. Then Y is separable and $\mathrm{Sz}(Y) \geq \omega^{\alpha+1}$ (as $X \subset Y$). By the Arens theorem [Lac74, Theorem 3.9], Y is isometrically isomorphic to some $C(L)$. Since Y is separable, L is necessarily metrizable [FHH+01, Lemma 3.23]. Since Y is an Asplund space, it follows that L is scattered [DGZ93, Lemma VI.8.3]. Being metrizable and scattered, L is a countable compact [DGZ93, Lemma VI.8.2]. By the moreover part of Theorem 3.7 and by Theorem 3.9, $C(L)$ is isomorphic to $C([0,\omega^{\omega^\beta}])$ for some $\beta \geq \alpha$. Now Theorem 3.10 implies that $\mathrm{Dz}(Y) = \mathrm{Dz}(C(L)) \geq \omega^{1+\alpha+1}$. This completes the lower estimate of $\mathrm{Dz}(C(K))$.

In order to prove the upper estimate, we follow in the footsteps of [Lan06, Proposition 7]. Let X be any separable subspace of $C(K)$. For $t \in K$ we define $\phi(t) := f_t \restriction_X$, i.e. the restriction of the Dirac functional f_t to the subspace X. It is easily seen that ϕ is a continuous mapping from K to $(B_{X^*}, \sigma(X^*, X))$. Since the latter is metrizable, it follows that $L := \phi(K)$ is a metrizable compact. Also, X embeds isometrically into $C(L)$ by the canonical embedding i defined as $[i(x)](l) := x(\phi^{-1}(l))$ where the choice of a particular element in $\phi^{-1}(l)$ is irrelevant by the definition of ϕ. So $\mathrm{Dz}(X) \leq \mathrm{Dz}(C(L))$. Further, [DGZ93, Lemma VI.8.1] implies that $L^{(\beta)} \subset \phi(K^{(\beta)})$ for any ordinal β. Therefore $L^{(\omega^{\alpha+1})} = \emptyset$, and L is countable. Now it follows from Theorem 3.7 that $C(L)$ is isomorphic to $C([0,\omega^{\omega^\alpha}])$. So our Theorem 3.10 yields that $\mathrm{Dz}(X) \leq \mathrm{Dz}(C(L)) \leq \omega^{1+\alpha+1}$. Now, since X was arbitrary, Theorem 3.19 finishes the proof of the upper estimate of $\mathrm{Dz}(C(K))$. □

Chapter 4

C^k-smooth approximation of locally uniformly rotund norms

The main result (Theorem 4.17) of this chapter claims that for a variety of non-separable C^k-smooth Banach spaces it is possible to construct an equivalent norm which is at the same time C^1-smooth, LUR and a limit of C^k-smooth norms. For the background of this result, see Section 1.1.5. We start this chapter with an overview of the tools we will be using in the proof of the main result. The main result and its corollaries are gathered in Section 4.2. The corollaries are proved on the spot. The proof of Theorem 4.17 then spreads through sections 4.3 and 4.4 until the end of the chapter. Here and throughout, the smoothness and the higher smoothness is meant in the Fréchet sense.

4.1 Preliminaries

4.1.1 Functions that locally depend on finitely many coordinates

We use Γ to denote an index set. The space $\ell^\infty(\Gamma)$ is the space of all bounded functions from Γ to $\mathbb{R}$ together with the supremum norm. The space $c_0(\Gamma)$ is the closed subspace of $\ell^\infty(\Gamma)$ for which $x \in c_0(\Gamma)$ if and only if $\{\gamma \in \Gamma : |x(\gamma)| > \delta\}$ is finite for every $\delta > 0$.

Definition 4.1. Let $U \subset \ell^\infty(\Gamma)$. Let $M = \{\gamma_1, \dots, \gamma_n\}$ be a finite subset of Γ. We say that a function $f : \ell^\infty(\Gamma) \to \mathbb{R}$ *depends in U only on coordinates from M* if there exists a function $g : \mathbb{R}^{|M|} \to \mathbb{R}$ such that $f(x) = g(x(\gamma_1), \dots, x(\gamma_n))$ for each $x \in U$.

Let $A \subset \ell^\infty(\Gamma)$. We say that a function $f : \ell^\infty(\Gamma) \to \mathbb{R}$ in A *locally depends on finitely many coordinates* (LFC) if for each $x \in A$ there exist a neighborhood U of x and a finite $M \subset \Gamma$ such that f depends in U only on coordinates from M.

Definition 4.2. Let X be a vector space. A function $g : X \to \ell^\infty(\Gamma)$ is said to be *coordinatewise convex* if, for each $\gamma \in \Gamma$, the function $x \mapsto g_\gamma(x)$ is convex. We use the terms as *coordinatewise non-negative* or *coordinatewise C^k-smooth* in a similar way.

Lemma 4.3. *Let X be a Banach space and let $h : X \to \ell^\infty(\Gamma)$ be a continuous function which is coordinatewise C^k-smooth, $k \in \mathbb{N} \cup \{\infty\}$. Let $f : \ell^\infty(\Gamma) \to \mathbb{R}$ be a C^k-smooth function which locally depends on finitely many coordinates. Then $f \circ h$ is C^k-smooth.*

Proof. Let $x \in X$ be fixed. Since f is LFC, there is a neighborhood U of $h(x)$, $M = \{\gamma_1, \dots, \gamma_n\} \subset \Gamma$ and $g : \mathbb{R}^{|M|} \to \mathbb{R}$ as in Definition 4.1. The function g is C^k-smooth, because f is C^k-smooth. As h is continuous, there exists a neighborhood V of x such that $h(V) \subset U$. Since h is coordinatewise C^k-smooth, it follows that

$$h(\cdot) \restriction_M := (h(\cdot)(\gamma_1), \dots, h(\cdot)(\gamma_n))$$

is C^k-smooth from X to $\mathbb{R}^{|M|}$. Finally, we have for each $y \in V$ that $f(h(y)) = g(h(y) \restriction_M)$ and the claim follows. □

Lemma 4.4. *Let $\Phi : \ell^\infty(\Gamma) \to \mathbb{R}$ and let $x \in \ell^\infty(\Gamma)$ be such that*

a) *Φ depends in some neighborhood V of x only on coordinates from some finite subset M of Γ,*

b) *$\Phi'(x)x \neq 0$,*

c) *$\Phi(\cdot)$ and $\Phi'(\cdot)$ are continuous at x.*

Then there is a neighborhood U of x and a unique function $F : U \to \mathbb{R}$ which is continuous at x and satisfies $F(x) = 1$ and $\Phi(\frac{y}{F(y)}) = 1$ for all $y \in U$. Moreover there is a neighborhood U' of x such that F depends in U' only on coordinates from M.

Proof. The first part of the assertion follows immediately from the Implicit Function Theorem. We will show that F is LFC at x. From the assumption a) we know that there are a neighborhood V of x, $M = \{\gamma_1, \dots, \gamma_n\} \subset \Gamma$, and $g : \mathbb{R}^n \to \mathbb{R}$ such that $\Phi(y) = g(y \restriction_M)$ for all $y \in V$. It is obvious that $g(\cdot)$ and $g'(\cdot)$ are continuous at $x \restriction_M$, and that $g'(x \restriction_M)x \restriction_M = \Phi'(x)x$. Thus it is possible to apply the Implicit Function Theorem to the equation

$$g\left(\frac{y}{h(y)}\right) = 1 \tag{4.1}$$

to obtain some neighborhood V' of $x \restriction_M$ and a function $h : V' \to \mathbb{R}$ such that $h(x \restriction_M) = 1$, h is continuous at $x \restriction_M$ and (4.1) is satisfied in V'.

There is a neighborhood $U' \subset U \cap V$ of x such that we may define $H : U' \to \mathbb{R}$ by $H(y) := h(y \restriction_M)$ for $y \in U'$. Then $H(x) = 1$ and H is continuous at x. Also, $\Phi\left(\frac{y}{H(y)}\right) = g\left(\frac{y\restriction_M}{h(y\restriction_M)}\right) = 1$ for all $y \in U'$. The uniqueness of F implies that $F = H$ in U', so F depends in U' only on coordinates from M. □

4.1.2 Facts about convexity

Definition 4.5. The *Minkowski functional* μ_C of a convex subset C of a Banach space X is defined for every $x \in X$ as

$$\mu_C(x) := \inf\{\lambda > 0 : x \in \lambda C\}.$$

If C is moreover bounded, symmetric and contains the origin in its interior, then μ_C is an equivalent norm on X.

Definition 4.6. A norm $\|\cdot\|$ in a Banach space X is *locally uniformly rotund* (LUR) if $\lim_r \|x_r - x\| = 0$ whenever $x \in X$ and $(x_r)_{r\in\mathbb{N}}$ is a sequence of points in X such that $\lim_r \left(2\|x_r\|^2 + 2\|x\|^2 - \|x_r + x\|^2\right) = 0$.

The following lemma is a variant of Fact II.2.3(i) in [DGZ93].

Lemma 4.7. *Let $\varphi : X \to \mathbb{R}$ be a convex non-negative function, $x_r, x \in X$ for $r \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) $\frac{\varphi^2(x_r)+\varphi^2(x)}{2} - \varphi^2(\frac{x+x_r}{2}) \to 0$,

(ii) $\lim \varphi(x_r) = \lim \varphi(\frac{x+x_r}{2}) = \varphi(x)$.

If φ is homogeneous, the above conditions are also equivalent to

(iii) $2\varphi^2(x_r) + 2\varphi^2(x) - \varphi^2(x + x_r) \to 0$.

Proof. Since φ is convex and non-negative, and $y \mapsto y^2$ is increasing for $y \in [0, +\infty)$, it holds

$$\frac{\varphi^2(x_r)+\varphi^2(x)}{2} - \varphi^2\left(\frac{x+x_r}{2}\right) \geq \frac{\varphi^2(x_r)+\varphi^2(x)}{2} - \left(\frac{\varphi(x)+\varphi(x_r)}{2}\right)^2 = \left(\frac{\varphi(x)-\varphi(x_r)}{2}\right)^2$$

which proves (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (i) is trivial and so is the equivalence (i) $\Leftrightarrow$ (iii). □

Lemma 4.8. *Let f, g be twice differentiable, convex, non-negative, real functions of one real variable. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be given as $F(x, y) := f(x)g(y)$. For F to be convex in $\mathbb{R}^2$, it is sufficient that g is convex and*

$$(f'(x))^2(g'(y))^2 \leq f''(x)f(x)g''(y)g(y). \tag{4.2}$$

for all $(x, y) \in \mathbb{R}^2$.

Proof. Let $(x, y) \in \mathbb{R}^2$ be fixed. Since g is convex, the function F is convex when restricted to the vertical line going through (x, y). Let $s = at + b$ $(a, b \in \mathbb{R})$ be a line going through (x, y), i.e. $y = ax + b$. The second derivative at a point (x, y) of F restricted to this line is given as:

$$f(x)g''(y)a^2 + 2f'(x)g'(y)a + f''(x)g(y).$$

In order for the second derivative to be non-negative for all $a \in \mathbb{R}$, it is sufficient that the discriminant $(2f'(x)g'(y))^2 - 4f(x)g''(y)f''(x)g(y)$ of the above quadratic term be non-positive, which occurs exactly when our condition (4.2) holds for (x, y). □

Definition 4.9. We say that a function $f : \ell^\infty(\Gamma) \to \mathbb{R}$ is *strongly lattice* if $f(x) \leq f(y)$ whenever $|x(\gamma)| \leq |y(\gamma)|$ for all $\gamma \in \Gamma$.

Lemma 4.10. *Let $f : \ell^\infty(\Gamma) \to \mathbb{R}$ be convex and strongly lattice. Let $g : X \to \ell^\infty(\Gamma)$ be coordinatewise convex and coordinatewise non-negative. Then $f \circ g : X \to \mathbb{R}$ is convex.*

Proof. Let $a, b \geq 0$ and $a + b = 1$. Since g is coordinatewise convex and non-negative, we have

$$0 \leq g_\gamma(ax + by) \leq a g_\gamma(x) + b g_\gamma(y)$$

for each $\gamma \in \Gamma$. The strongly lattice property and the convexity of f yield

$$f(g(ax + by)) \leq f(ag(x) + bg(y)) \leq af(g(x)) + bf(g(y))$$

so $f \circ g$ is convex. □

Definition 4.11. Let us define $\lceil\cdot\rceil : \ell^\infty(\Gamma) \to \mathbb{R}$ by $\lceil x \rceil = \inf\{t; \{\gamma; |x(\gamma)| > t\} \text{ is finite}\}$. Then $\lceil\cdot\rceil$ is 1-Lipschitz, strongly lattice seminorm on $(\ell^\infty(\Gamma), \|\cdot\|_\infty)$.

Proof. In fact $\lceil x \rceil = \|q(x)\|_{\ell^\infty/c_0}$, where $q : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma)/c_0(\Gamma)$ is the quotient map and $\|\cdot\|_{\ell^\infty/c_0}$ the canonical norm on the quotient $\ell^\infty(\Gamma)/c_0(\Gamma)$. Clearly, $\lceil x \rceil = 0$ if and only if $x \in c_0(\Gamma)$. Let us assume that $\lceil x \rceil = t > 0$. Then, for every $0 < s < t$, there are infinitely many $\gamma \in \Gamma$ such that $|x(\gamma)| > s$. It follows that $\|x - y\|_\infty > s$ for every $y \in c_0(\Gamma)$ and consequently $\|q(x)\|_{\ell^\infty/c_0} \geq t$. On the other hand, we may define $y \in c_0(\Gamma)$ as

$$y(\gamma) := \begin{cases} x(\gamma) - t \text{ if } x(\gamma) > t, \\ x(\gamma) + t \text{ if } x(\gamma) < -t, \\ 0 \text{ otherwise.} \end{cases}$$

Obviously $\|x - y\|_\infty \leq t$, so $\|q(x)\|_{\ell^\infty/c_0} \leq t$. The strongly lattice property of $\lceil\cdot\rceil$ follows directly from the definition. □

4.1.3 Projectional resolution of identity

Definition 4.12. Let $(X, \|\cdot\|)$ be a Banach space and let μ be the smallest ordinal such that $|\mu| = \text{dens}(X)$. A system $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$ of projections from X into X is called a *projectional resolution of identity* (PRI) provided that, for every $\alpha \in [\omega, \mu]$, the following conditions hold true

(a) $\|P_\alpha\| = 1$,

(b) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for $\omega \leq \alpha \leq \beta \leq \mu$,

(c) $\text{dens}(P_\alpha X) \leq |\alpha|$,

(d) $\bigcup\{P_{\beta+1}X : \beta < \alpha\}$ is norm-dense in $P_\alpha X$,

(e) $P_\mu = Id_X$, the identity on X.

If $\{P_\alpha\}_{\omega\le\alpha\le\mu}$ is a PRI on a Banach space X, we use the following notation: $\Lambda := \{0\}\cup[\omega,\mu)$, $Q_\gamma := P_{\gamma+1}-P_\gamma$ for all $\gamma\in[\omega,\mu)$ while $Q_0 := P_\omega$, and $P_A := \sum_{\gamma\in A} Q_\gamma$ for any finite subset A of Λ.

We gather for reader's convenience the most important properties of a PRI which we will need.

Lemma 4.13. *Let X be a Banach space and $\{P_\alpha\}_{\omega\le\alpha\le\mu}$ be a PRI on X. Then*

(i) *For every $x\in X$, the map $\alpha\mapsto P_\alpha x$ is norm continuous from $[\omega,\mu]$ to X.*

(ii) *For every $x\in X$, the transfinite sequence $(\|Q_\gamma x\|)_{\gamma\in\Lambda}$ belongs to $c_0(\Lambda)$.*

For the proof see [DGZ93, Lemma VI.1.2].

Lemma 4.14. *Let X be a Banach space with a PRI $\{P_\alpha\}_{\omega\le\alpha\le\mu}$. Then for each $x\in X$, $\varepsilon>0$, $\alpha\in[\omega,\mu]$ there is a finite set $A^\alpha_\varepsilon(x)\subset\Lambda$ such that*

$$\left\|P_{A^\alpha_\varepsilon(x)}x - P_\alpha x\right\| < \varepsilon.$$

We may choose $A = A^\alpha_\varepsilon(x)$ in such a way that $Q_\beta x\neq 0$ for $\beta\in A$ since $P_A = \sum_{\gamma\in A} Q_\gamma$.

Proof. We will proceed by a transfinite induction on α. If $\alpha=\omega$, then $A^\omega_\varepsilon(x) := \{0\}$ for any $\varepsilon>0$. If $\alpha=\beta+1$ for some ordinal β, then $A^\alpha_\varepsilon(x) := A^\beta_\varepsilon(x)\cup\{\beta\}$ for all $\varepsilon>0$. Finally, if α is a limit ordinal, we will use the continuity of the mapping $\gamma\mapsto P_\gamma x$ at α (Lemma 4.13 (i)) to find $\beta<\alpha$ such that $\|P_\beta x - P_\alpha x\|<\varepsilon/2$. Thus it is possible to set $A^\alpha_\varepsilon(x) := A^\beta_{\varepsilon/2}(x)$. □

4.1.4 Approximation of norms

We shall need the following more detailed version of the theorem of M. Fabian, P. Hájek and V. Zizler [FHZ97]. Here e_α denotes the vector in $c_0(\Gamma)$ such that $e_\alpha(\beta)=\delta_{\alpha\beta}$ where $\delta_{\alpha\beta}$ is the Kronecker delta.

Theorem 4.15. *For every equivalent strongly lattice norm $\|\cdot\|$ on $(c_0(\Gamma),\|\cdot\|_\infty)$ and every $p>0$ there is a C^∞-smooth norm W such that $(1-p)W(x)\le\|x\|\le W(x)$ for all $x\in c_0(\Gamma)$. Moreover*

(a) *the norm W locally depends on finitely many non-zero coordinates, i.e. for every $x=\sum_{\gamma\in\Gamma} x(\gamma)e_\gamma$ in $c_0(\Gamma)$ there is a $\delta>0$ and a neighborhood U of x such that for every $y\in U$,*

$$W\left(\sum_{\gamma\in\Gamma} y(\gamma)e_\gamma\right) = W\left(\sum_{\gamma\in\Gamma(x,\delta)} y(\gamma)e_\gamma\right)$$

where $\Gamma(x,\delta) := \{\gamma\in\Gamma : |x(\gamma)|>\delta\}$;

(b) *the norm W is strongly lattice for positive vectors in $c_0(\Gamma)$, i.e. if $0 \le x \le y$ in the lattice $c_0(\Gamma)$, then $W(x) \le W(y)$.*

Proof. We are going to repeat the construction of [FHZ97]. So let $\|\cdot\|$ be a strongly lattice equivalent norm on $c_0(\Gamma)$. For $0 < \Delta < 1$ define $f_\Delta : c_0(\Gamma) \to \mathbb{R}$ by

$$f_\Delta(\sum_{\gamma\in\Gamma} x(\gamma)e_\gamma) = \sup\left\{ \left\| \sum_{\gamma\in\Gamma} y(\gamma)e_\gamma \right\|, \right.$$
$$\left. \text{where } y(\gamma) = x(\gamma) \text{ if } |x(\gamma)| > \Delta \text{ and } |y(\gamma)| \le \Delta \text{ if } |x(\gamma)| \le \Delta \right\}.$$

Further, put $F_\Delta(x) = f_\Delta^2(x)$ and let C_Δ be the lower convex envelope of F_Δ, i.e.

$$C_\Delta(x) = \inf\left\{ \sum_{i=1}^n \lambda_i F_\Delta(x_i) : x = \sum_{i=1}^n \lambda_i x_i, \sum_{i=1}^n \lambda_i = 1, \lambda_i > 0 \right\}.$$

The strongly lattice property of $\|\cdot\|$ passes easily to C_Δ. It is shown in [FHZ97, pp. 267–269] that for every $0 < \Delta < 1$ there exists a $\delta > 0$ such that

$$C_\Delta\Big(\sum_{\gamma\in\Gamma} x(\gamma)e_\gamma\Big) = C_\Delta\Big(\sum_{\gamma\in\Gamma(x,\delta)} x(\gamma)e_\gamma\Big) \tag{4.3}$$

for every $x = \sum_{\gamma\in\Gamma} x(\gamma)e_\gamma \in c_0(\Gamma)$ with $\|x\| \le 2$.

Let b be a C^∞-smooth bump function on $\mathbb{R}$, such that $0 \le b(t) = b(-t)$, $\operatorname{supp}(b) \subset [-\delta/4, \delta/4]$ and $\int_{-\infty}^{\infty} b(t)dt = 1$. We define, for arbitrary $\gamma_0 \in \Gamma$, the function $C_\Delta^{\gamma_0}$ as

$$C_\Delta^{\gamma_0}\Big(\sum_{\gamma\in\Gamma} x(\gamma)e_\gamma\Big) = \int_{-\infty}^{\infty} C_\Delta\Big(\sum_{\gamma\ne\gamma_0} x(\gamma)e_\gamma + te_{\gamma_0}\Big) b(x(\gamma_0) - t)dt.$$

Roughly speaking, this operation adds the smoothness with respect to the coordinate γ_0. Also $C_\Delta^{\gamma_0}$ is convex and $C_\Delta^{\gamma_0} \ge C_\Delta$ (see [FHZ97]). Note that we omit to indicate the dependence of $C_\Delta^{\gamma_0}$ on the choice of δ in order to stay consistent with the notation of [FHZ97].

Claim. If $0 \le x \le y$ in the lattice $c_0(\Gamma)$ and $\|y\| \le 2$, then $C_\Delta^{\gamma_0}(x) \le C_\Delta^{\gamma_0}(y)$.

By the change of variable we have for any $z \in c_0(\Gamma)$,

$$C_\Delta^{\gamma_0}\Big(\sum_{\gamma\in\Gamma} z(\gamma)e_\gamma\Big) = \int_{-\infty}^{\infty} C_\Delta\Big(\sum_{\gamma\ne\gamma_0} z(\gamma)e_\gamma + (z(\gamma_0) - t)e_{\gamma_0}\Big) b(t)dt. \tag{4.4}$$

To prove the claim, we would like to use that C_Δ is strongly lattice so we are interested when $|x(\gamma_0) - t| \le |y(\gamma_0) - t|$. Let us assume $|x(\gamma_0) - t| > |y(\gamma_0) - t|$. Since $0 \le x(\gamma_0) \le$

$y(\gamma_0)$, this may happen only if $t > x_0$. But if $t \geq \delta/4$, we get that $b(t) = 0$. If $t < \delta/4$ is the case, then $|x_0 - t| < \delta/4$ and

$$\begin{aligned} C_\Delta\Big(\sum_{\gamma\neq\gamma_0} x(\gamma)e_\gamma + (x(\gamma_0)-t)e_{\gamma_0}\Big) &\leq C_\Delta\Big(\sum_{\gamma\neq\gamma_0} y(\gamma)e_\gamma + (x(\gamma_0)-t)e_{\gamma_0}\Big) \\ &\leq C_\Delta\Big(\sum_{\gamma\neq\gamma_0} y(\gamma)e_\gamma + (y(\gamma_0)-t)e_{\gamma_0}\Big) \end{aligned}$$

where the first inequality comes from the strongly lattice property of C_Δ while the second one comes from (4.3) and again the strongly lattice property of C_Δ. Put together, for all $t \in \mathbb{R}$ it is satisfied

$$C_\Delta\Big(\sum_{\gamma\neq\gamma_0} x(\gamma)e_\gamma + (x(\gamma_0)-t)e_{\gamma_0}\Big)b(t) \leq C_\Delta\Big(\sum_{\gamma\neq\gamma_0} y(\gamma)e_\gamma + (y(\gamma_0)-t)e_{\gamma_0}\Big)b(t), \quad (4.5)$$

so, keeping in mind (4.4), the claim is proved.

Let Π be the set of all finite subsets of Γ. For $\pi = \{\gamma_1, \dots, \gamma_n\} \in \Pi$ define

$$\begin{aligned} &C^\pi_\Delta\Big(\sum_{\gamma\in\Gamma} x(\gamma)e_\gamma\Big) = \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} C_\Delta\Big(\sum_{\gamma\notin\pi} x(\gamma)e_\gamma + \sum_{i=1}^{n} t_i e_{\gamma_i}\Big)b(x(\gamma_1)-t_1)\dots b(x(\gamma_n)-t_n)dt_1\dots dt_n. \end{aligned}$$

It is clear that iterating (4.5) one obtains that, for $\|y\| \leq 2$,

$$0 \leq x \leq y \Rightarrow C^\pi_\Delta(x) \leq C^\pi_\Delta(y).$$

Let us define $\tilde{C}_\Delta(x) = \sup\{C^\pi_\Delta(x) : \pi \in \Pi\}$. Let $\|x\| \leq 2 - \frac{\delta}{2}$ and denote $\Gamma_2 = \Gamma(x, \frac{\delta}{4})$. By [FHZ97, p. 270], $\tilde{C}_\Delta(y) = C^{\Gamma_2}_\Delta(y)$ for all $y \in B_{(c_0,\|\cdot\|_\infty)}(x, \frac{\delta}{4})$. Thus $\tilde{C}_\Delta$ depends in $B_{(c_0,\|\cdot\|_\infty)}(x, \frac{\delta}{4})$ only on the finitely many coordinates from $\Gamma(x, \frac{\delta}{4})$. It follows that $\tilde{C}_\Delta$ is C^∞-smooth for $\|x\| < 2 - \frac{\delta}{2}$.

Since for each $x \in B_{(c_0(\Gamma),\|\cdot\|)}(x, 2-\frac{\delta}{2})$ there are a neighborhood U of x and a finite subset π of Γ such that $\tilde{C}_\Delta = C^\pi_\Delta$ on U, we claim that $\tilde{C}_\Delta$ satisfies $\tilde{C}_\Delta(x) \leq \tilde{C}_\Delta(y)$ whenever $0 \leq x \leq y$ and $\|y\| \leq 2 - \frac{\delta}{2}$. Indeed, let us consider such x and y. Then the segment $[x, y]$ is a connected compact so we may find $\pi_1, \dots, \pi_n \in \Pi$, open sets $U_1, \dots, U_n$ and points $x_0, \dots, x_n \in [x, y]$ such that $\tilde{C}_\Delta = C^{\pi_i}_\Delta$ on U_i for $i = 1, \dots, n$, $x_0 = x \in U_1$, $x_n = y \in U_n$, $x_i \in U_i \cap U_{i+1}$ for $i = 1, \dots, n-1$, and $x_i \leq x_{i+1}$ in the lattice $c_0(\Gamma)$ for all $i = 0, \dots, n$. It is now clear how we obtain our claim.

It is also shown in [FHZ97, p. 264] that, given $\varepsilon > 0$, there exists $\Delta \in (0, 1)$ such that when $\delta > 0$ is chosen as above, it is satisfied

$$\|x\|^2 \leq \tilde{C}_\Delta(X) \leq \sup\left\{(\|x+v\|+\varepsilon)^2 : \|v\|_\infty < \frac{\delta}{2}\right\}$$

for $\|x\| \le 2 - \frac{\delta}{2}$. Remember that $\tilde{C}_\Delta$ depends also on our choice of δ. It follows that if $p > 0$ is given, there are $\Delta > 0$ and $0 < \delta < \frac{1}{2}$ such that $\|x\|^2 \le \tilde{C}_\Delta(X) \le (\|x\| + p)^2$ for $\|x\| \le \frac{3}{2}$.

The norm W is defined as the Minkowski functional of the set $\left\{x \in c_0(\Gamma) : \tilde{C}_\Delta(x) \le 1\right\}$. It is immediate that $(1-p)W(x) \le \|x\| \le W(x)$. By the Implicit Function Theorem, W is C^∞-smooth. By Lemma 4.4, for every $x \in c_0(\Gamma)$ such that $\tilde{C}_\Delta(x) = 1$, there is a neighborhood U such that W depends in U only on coordinates from $\Gamma(x, \frac{\delta}{4})$. The assertion (a) now follows by the homogeneity of W.

In order to finish the proof of the assertion (b) we assume that $0 \le x \le y$ in the lattice $c_0(\Gamma)$. Then, for some $a > 0$ one has that $W(ay) = 1$. It follows that $\tilde{C}_\Delta(ay) = 1$ and, by what we have already proved, $\tilde{C}_\Delta(ax) \le \tilde{C}_\Delta(ay)$. We may conclude that $W(x) \le \frac{1}{a} = W(y)$. □

Finally, let us recall the separable result of D. McLaughlin, R. Poliquin, J. Vanderwerff and V. Zizler [MPVZ93] (see also [DGZ93, Theorem V.1.7] which we will need as the first step in our upcoming inductive arguments.

Theorem 4.16. *Let $k \in \mathbb{N} \cup \{\infty\}$. Let X be a separable Banach space that admits a C^k-smooth norm. Then X admits an LUR and C^1-smooth norm which is a limit (uniform on bounded sets) of C^k-smooth norms.*

4.2 Main result

Theorem 4.17. *Let $k \in \mathbb{N} \cup \{\infty\}$. Let $(X, |\cdot|)$ be a Banach space with a PRI $\{P_\alpha\}_{\omega \le \alpha \le \mu}$ such that, for every $\gamma \in [\omega, \mu)$, the space $(P_{\gamma+1} - P_\gamma)X$ (resp. the space $P_\omega X$) admits a C^1-smooth, LUR equivalent norm which is a limit (uniform on bounded sets) of C^k-smooth norms. Let X admit an equivalent C^k-smooth norm $\|\cdot\|$.*

Then X admits an equivalent C^1-smooth, LUR norm $|||\cdot|||$ which is a limit (uniform on bounded sets) of C^k-smooth norms.

Our first corollary provides a positive solution of Problem 8.2 (c) in [FMZ06].

Corollary 4.18. *Let α be an ordinal. Then the space $C([0, \alpha])$ admits an equivalent norm which is C^1-smooth, LUR and a limit of C^∞-smooth norms.*

Proof of Corollary 4.18. By a result of Talagrand [Tal86] and Haydon [Hay96], $C([0, \alpha])$ admits an equivalent C^∞-smooth norm. On the other hand, assuming without loss of generality that $|\alpha| = \alpha$, there is a natural PRI on $C([0, \alpha])$ defined as

$$(P_\gamma x)(\beta) = \begin{cases} x(\beta) \text{ if } \beta \le \gamma, \\ x(\gamma) \text{ if } \beta \ge \gamma. \end{cases}$$

Now $Q_0 X = P_\omega X$ is (by the definition of PRI) separable. As a closed subspace of $C([0, \alpha])$, $Q_0 X$ clearly admits a C^∞-smooth norm. Applying Theorem 4.16, we see that there is a

C^1-smooth LUR norm on Q_0X which is a limit of C^∞-smooth norms. Finally, the space $Q_\gamma X = (P_{\gamma+1} - P_\gamma)X$ is one-dimensional for each $\gamma \in [\omega, \alpha)$, so Theorem 4.17 yields the conclusion. □

Theorem 4.19. *Let $k \in \mathbb{N} \cup \{\infty\}$. Let $\mathcal{P}$ be a class of Banach spaces such that every X in $\mathcal{P}$*

- *admits a PRI $\{P_\alpha\}_{\omega\le\alpha\le\mu}$ such that $(P_{\alpha+1} - P_\alpha)X \in \mathcal{P}$,*
- *admits a C^k-smooth equivalent norm.*

Then each X in $\mathcal{P}$ admits an equivalent, LUR, C^1-smooth norm which is a limit (uniform on bounded sets) of C^k-smooth norms.

Proof. We will carry out induction on the density of X. Let $X \in \mathcal{P}$ be separable, i.e. $\operatorname{dens}(X) = \omega$. Then we get the result from Theorem 4.16.

Next, we assume for $X \in \mathcal{P}$ that $\operatorname{dens}(X) = \mu$ and that every Banach space $Y \in \mathcal{P}$ with $\operatorname{dens}(Y) < \mu$ admits a C^1-smooth, LUR norm which is a limit of C^k-smooth norms. Let $\{P_\alpha\}_{\omega\le\alpha\le\mu}$ be a PRI on X such that $Q_\alpha X \in \mathcal{P}$ for each $\alpha \in \Lambda$. Then $\operatorname{dens}(Q_\alpha X) \le |\alpha + 1| = |\alpha| < \mu$. Thus the inductive hypothesis enables the use of Theorem 4.17 which finishes the proof. □

The above theorem has immediate corollaries for each $\mathcal{P}$-class (see [HMSVZ08] for this notion). The following Corollary 4.20 solves in the affirmative Problem 8.8 (s) in [FMZ06] (see also Problem VIII.4 in [DGZ93]).

Corollary 4.20. *Let X admit a C^k-smooth norm for some $k \in \mathbb{N} \cup \{\infty\}$. If X is Vašák (i.e. WCD) or WLD or $C(K)$ where K is a Valdivia compact, then X admits a C^1-smooth, LUR equivalent norm which is a limit (uniform on bounded sets) of C^k-smooth norms.*

Proof of Theorem 4.17. Let $0 < c < 1$. It follows from the hypothesis that, for each $\gamma \in \Lambda$, there are a C^1-smooth, LUR norm $\|\cdot\|_\gamma$ on $Q_\gamma X$ and C^k-smooth norms $(\|\cdot\|_{\gamma,i})_{i\in\mathbb{N}}$ on $Q_\gamma X$ such that

$$c\,\|x\| \le \|x\|_\gamma \le \|x\| \tag{4.6}$$

for all $x \in Q_\gamma X$ and such that $(1 - \frac{1}{i^2})\,\|x\|_\gamma \le \|x\|_{\gamma,i} \le \|x\|_\gamma$ for all $x \in Q_\gamma X$.

We seek the new norm on X in the form

$$|||x|||^2 := N(x)^2 + J(x)^2 + \|x\|^2\,.$$

We will insure during the construction that both N and J are C^1-smooth and approximated by C^k-smooth (semi)norms. In order to see that $|||\cdot|||$ is LUR, we are going to show that $\|x - x_r\| \to 0$ provided that

$$2\,|||x_r|||^2 + 2\,|||x|||^2 - |||x + x_r|||^2 \to 0 \text{ as } r \to \infty. \tag{4.7}$$

Consider the following two statements:
a) $\|P_A x_r - P_A x\| \to 0$ for each finite $A \subset \Lambda$ with $0 \notin \{Q_\gamma x : \gamma \in A\}$,
b) for every $\varepsilon > 0$ there exists a finite $A \subset \Lambda$ with $0 \notin \{Q_\gamma x : \gamma \in A\}$ and such that $\|P_A x - x\| < \varepsilon$ and $\|P_A x_r - x_r\| < \varepsilon$ for all but finitely many $r \in \mathbb{N}$.

Clearly, the simultaneous validity of **a)** and **b)** would imply that $\|x - x_r\| \to 0$ since

$$\|x - x_r\| \leq \|P_A x_r - P_A x\| + \|P_A x - x\| + \|P_A x_r - x_r\|.$$

We construct N in such a way that we can prove in Lemma 4.21 that (4.7) implies **a)**. Consequently, we construct J in such a way that we can prove in Lemma 4.29 that (4.7) implies **b)**. □

4.3 About N

We may and do assume that the equivalent norms $|\cdot|$ and $\|\cdot\|$ satisfy

$$|\cdot| \leq \|\cdot\| \leq C\,|\cdot|$$

for some $C \geq 1$. Lemma 4.13 (ii) and the above equivalence of norms yield that $(\|Q_\gamma x\|)_{\gamma\in\Lambda} \in c_0(\Lambda)$. Using $\|Q_\gamma\| \leq 2C$ and the second inequality of (4.6), it follows that

$$T : x \in (X, \|\cdot\|) \mapsto (\|Q_\gamma x\|_\gamma)_{\gamma\in\Lambda} \in (c_0(\Lambda), \|\cdot\|_\infty)$$

is a $2C$-Lipschitz mapping. Similarly for $T_i : x \in X \mapsto (\|Q_\gamma x\|_{\gamma,i})_{\gamma\in\Lambda} \in c_0(\Lambda)$.

For each $n \in \mathbb{N}$, we will consider an equivalent norm on $c_0(\Lambda)$ given as

$$\zeta_n(x) := \sup_{M\in\Lambda_n} \sqrt{\sum_{\gamma\in M} x(\gamma)^2}$$

where $\Lambda_n := \left\{M \in 2^\Lambda : |M| = n\right\}$. It is easily seen that ζ_n is n-Lipschitz with respect to the usual norm on $c_0(\Lambda)$. Also, ζ_n is obviously strongly lattice, so by Theorem 4.15 for each $\varepsilon > 0$ there is a C^∞-smooth equivalent norm $N_{n,\varepsilon}$ on $c_0(\Lambda)$ such that $(1-\varepsilon)N_{n,\varepsilon}(x) \leq \zeta_n(x) \leq N_{n,\varepsilon}(x)$ for all $x \in c_0(\Lambda)$. Note that $N_{n,\varepsilon}(T(\cdot))$ is a C^k-smooth seminorm on X. Indeed, the homogeneity is clear. Further, by the property (b) of Theorem 4.15 and by Lemma 4.10, $N_{n,\varepsilon}(T(\cdot))$ is also convex, so we have established that it is a seminorm. At last, by Theorem 4.15 (a), $N_{n,\varepsilon}(T(\cdot))$ is C^k-smooth as a composition of a C^∞-smooth mapping which depends on finitely many non-zero coordinates and of the mapping T which satisfies that each coordinate map $x \mapsto T(x)(\gamma)$ is C^k-smooth on the set where it is not zero (cf. also Remark on page 461 in [Hay96]). Finally, we define

$$N(x)^2 := \sum_{m,n\in\mathbb{N}} \frac{1}{2^{n+m}} N^2_{n,\frac{1}{m}}(T(x))$$

Now the seminorm $N(\cdot)$ is C^1-smooth since each $N_{n,\frac{1}{m}} \circ T$ is $2nC$-Lipschitz and C^k-smooth.

We may define the approximating seminorms as

$$N_i(x)^2 := \sum_{m,n=1}^{i} \frac{1}{2^{n+m}} N^2_{n,\frac{1}{m}}(T_i(x)).$$

As a finite sum of C^k-smooth seminorms, N_i is a C^k-smooth seminorm. Using $\frac{2C}{i^2} \geq \|T_i(x) - T(x)\|_\infty$ for $\|x\| \leq 1$, it is standard to check that $N_i(x) \to N(x)$ uniformly for $\|x\| \leq 1$. We carry out some similar considerations in more detail on page 95 when we demonstrate that J is approximated by C^k-smooth norms.

Lemma 4.21. *Let us assume that* (4.7) *holds for* $x, x_r \in X$, $r \in \mathbb{N}$, *and let* $\tilde{A} \subset \Lambda$ *be a finite set such that* $Q_\gamma x \neq 0$ *for* $\gamma \in \tilde{A}$. *Then* $\|P_{\tilde{A}}x - P_{\tilde{A}}x_r\| \to 0$ *as* $r \to \infty$.

Proof. Let $A := \left\{\gamma \in \Lambda : \|Q_\gamma x\|_\gamma \geq \min_{\alpha\in\tilde{A}} \|Q_\alpha x\|_\alpha\right\}$. Let $n := |A|$. We may assume that $|||x||| \leq 1$ which implies $\|Tx\|_\infty \leq 2C$. Using (4.7) and Lemma 4.7 we may assume that $|||x_r||| \leq 2$ thus $\|Tx_r\|_\infty \leq 4C$ and $\|T(x + x_r)\| \leq 6C$. The convergence (4.7) and convexity (see Fact II.2.3 in [DGZ93]) imply that

$$2N^2_{n,\frac{1}{m}}(T(x_r)) + 2N^2_{n,\frac{1}{m}}(T(x)) - N^2_{n,\frac{1}{m}}(T(x + x_r)) \xrightarrow{r} 0$$

for all $m \in \mathbb{N}$. This further yields that

$$2\zeta_n^2(T(x_r)) + 2\zeta_n^2(T(x)) - \zeta_n^2(T(x + x_r)) \xrightarrow{r} 0$$

as well. Indeed, let $\varepsilon > 0$ be given. We use that $N_{n,\frac{1}{m}} \to \zeta_n$ uniformly on bounded sets of $c_0(\Lambda)$ to find $m_0 \in \mathbb{N}$ such that $\left|N^2_{n,\frac{1}{m}}(y) - \zeta_n^2(y)\right| < \varepsilon/6$ for all $y \in 6CB_{c_0(\Lambda)}$ and all $m \geq m_0$. Now let $r_0 \in \mathbb{N}$ satisfy that for all $r \geq r_0$ it holds $2N^2_{n,\frac{1}{m_0}}(T(x_r)) + 2N^2_{n,\frac{1}{m_0}}(T(x)) - N^2_{n,\frac{1}{m_0}}(T(x + x_r)) < \varepsilon/6$. For each $r \geq r_0$ we obtain $2\zeta_n^2(T(x_r)) + 2\zeta_n^2(T(x)) - \zeta_n^2(T(x + x_r)) < \varepsilon$.

Let $B \in \Lambda_n$ be arbitrary and let $A_r \in \Lambda_n$ such that

$$\sqrt{\sum_{\gamma\in A_r} \|Q_\gamma(x + x_r)\|_\gamma^2} = \zeta_n(x + x_r).$$

Then

$$\begin{aligned}
2\zeta_n^2(T(x_r)) + 2\zeta_n^2(T(x)) - \zeta_n^2(T(x + x_r)) \geq \\
\geq 2\sum_{\gamma\in B} \|Q_\gamma x\|_\gamma^2 + 2\sum_{\gamma\in A_r} \|Q_\gamma x_r\|_\gamma^2 - \sum_{\gamma\in A_r} \|Q_\gamma(x + x_r)\|_\gamma^2 \\
= 2\sum_{\gamma\in A_r} \|Q_\gamma x\|_\gamma^2 + 2\sum_{\gamma\in A_r} \|Q_\gamma x_r\|_\gamma^2 - \sum_{\gamma\in A_r} \|Q_\gamma(x + x_r)\|_\gamma^2 \\
+ 2\left(\sum_{\gamma\in B} \|Q_\gamma x\|_\gamma^2 - \sum_{\gamma\in A_r} \|Q_\gamma x\|_\gamma^2\right) \quad (4.8)
\end{aligned}$$

Since

$$2\sum_{\gamma\in A_r}\|Q_\gamma x\|_\gamma^2+2\sum_{\gamma\in A_r}\|Q_\gamma x_r\|_\gamma^2-\sum_{\gamma\in A_r}\|Q_\gamma(x+x_r)\|_\gamma^2\geq 0$$

we get from (4.8) that

$$\liminf_r \sum_{\gamma\in A_r}\|Q_\gamma x\|_\gamma^2 \geq \sup\left\{\sum_{\gamma\in B}\|Q_\gamma x\|_\gamma^2 : B\in\Lambda_n\right\} = \zeta_n(T(x)) = \sum_{\gamma\in A}\|Q_\gamma x\|_\gamma^2 \tag{4.9}$$

where the last equality follows from the definition of A. The equation (4.9) together with the definition of A show that $A = A_r$ for all r sufficiently large. We continue with such r and we choose $B := A$ in (4.8) to get that

$$2\sum_{\gamma\in A}\|Q_\gamma x\|_\gamma^2+2\sum_{\gamma\in A}\|Q_\gamma x_r\|_\gamma^2-\sum_{\gamma\in A}\|Q_\gamma(x+x_r)\|_\gamma^2\xrightarrow{r} 0.$$

Since $x\mapsto\sqrt{\sum_{\gamma\in A}\|Q_\gamma x\|_\gamma^2}$ is an equivalent LUR norm on $P_A X$, we infer that $\|P_A(x-x_r)\|$ converges to 0 and, by continuity of $P_{\tilde{A}}$, we obtain the claim of the lemma. □

4.4 About J

The upcoming section is rather technical so we are going to spend some time with a motivation. Recall that $|||x|||^2 = \|x\|^2 + N^2(x) + J^2(x)$ and that our main concern is to construct a norm J so that the properties we equip J with will enable us to prove: if

$$2\,|||x_r|||^2 + 2\,|||x|||^2 - |||x+x_r|||^2 \to 0 \text{ as } r\to\infty \tag{4.7}$$

holds for some $x, x_r \in X$, then for every $\varepsilon > 0$ there exists a finite $A\subset\Lambda$ with $0\notin\{Q_\gamma x : \gamma\in A\}$ and such that $\|P_A x - x\| < \varepsilon$ and $\|P_A x_r - x_r\| < \varepsilon$ for all but finitely many $r\in\mathbb{N}$.

By lemma 4.14, it is easy to find a set $A\subset\Lambda$ such that the first estimate above holds. Clearly, if we prove that for this A also $\|P_A x_r - x_r\| \xrightarrow{r} \|P_A x - x\|$, we are done.

Now, let $g(\cdot,\cdot)$ be a continuous function from $[0,+\infty)\times[0,+\infty)$ to $\mathbb{R}$ such that $g(t,\cdot)$ is increasing for every $t\in[0,+\infty)$. Then, as is easily seen (cf. Lemma 4.25), the condition $g(t_r,s_r)\to g(t,s)$ and $t_r\to t$ implies that $s_r\to s$. Let us consider the following

$$Hy(A) := g\Big(\sum_{\gamma\in A}\|Q_\gamma y\|_\gamma, \|P_A y - y\|\Big).$$

By Lemma 4.21, the convergence (4.7) implies that $\sum_{\gamma\in B}\|Q_\gamma x_r\|_\gamma \xrightarrow{r} \sum_{\gamma\in B}\|Q_\gamma x\|_\gamma$ for every finite $B\subset\Lambda$. So if we show that $Hx_r(A)\xrightarrow{r} Hx(A)$ we are done. This last requirement may be achieved by showing that the mapping $H : x\mapsto (Hx(B))_{B\in F}$, where F are the finite subsets of Λ, has its range in some "nice" target space. Should this space be, for example, the $c_0(F)$, we could work with some C^∞-smooth, LUR, strongly lattice norm Z of $c_0(F)$. In this hypothetical case we define $J(x) := Z(Hx)$ (everything can be arranged in such a way that $J \circ H$ is a seminorm) and it is easily seen that (4.7) implies $Hx_r(A) \xrightarrow{r} Hx(A)$ as needed. Also, by considering suitable smoothenings in the definition of H, we would be able to produce J with the required smoothness properties.

It will become apparent that the above motivation is a shameless simplification. We still hope that it does provide a rough guidance in what follows.

4.4.1 A "nice" target space

Let $\{\phi_\eta\}_{0<\eta<1}$ be a system of functions satisfying

(i) $\phi_\eta : [0, +\infty) \to [0, +\infty)$, for $0 < \eta < 1$, is a convex C^∞-smooth function such that ϕ_η is strictly convex on $[1-\eta, +\infty)$, $\phi_\eta([0, 1-\eta]) = \{0\}$ and $\phi_\eta(1) = 1$.

(ii) If $0 < \eta_1 \le \eta_2 < 1$ then $\phi_{\eta_1}(x) \le \phi_{\eta_2}(x)$ for any $x \in [0, 1]$.

One example of such a system can be constructed as follows: let $\phi : \mathbb{R} \to \mathbb{R}$ be C^∞-smooth such that $\phi(x) = 0$ if $x \le 0$, $\phi(1) = 1$ and ϕ is increasing and strictly convex on $[0, +\infty)$. We define $\phi_\eta(x) := \phi(\frac{x-(1-\eta)}{\eta})$ for all $x \in [0, 1]$. Now the system $\{\phi_\eta\}$ satisfies (ii) since $\eta \mapsto \frac{x-(1-\eta)}{\eta}$ is increasing for every $x \in [0, 1)$ while the validity of (i) follows from the properties of ϕ.

We define a function $\Phi_\eta : \ell^\infty(\Gamma) \to (-\infty, +\infty]$ by

$$\Phi_\eta(x) = \sum_{\gamma \in \Gamma} \phi_\eta(|x(\gamma)|).$$

Let us define $Z_\eta : \ell^\infty(\Gamma) \to \mathbb{R}$ as the Minkowski functional of the set $C = \{x \in \ell^\infty(\Gamma); \Phi_\eta(x) < 1/2\}$.

Lemma 4.22. *Let $0 < \eta < 1$ be fixed. Then Z_η is a strongly lattice seminorm such that $(1-\eta)Z_\eta(x) \le \|x\|_\infty$ and Z_η is LFC, C^∞-smooth and strictly positive in the set*

$$A_\eta(\Gamma) := \{x \in \ell^\infty(\Gamma) : \lceil x \rceil < (1-\eta)\|x\|_\infty\}.$$

Moreover $(1-\eta)Z_\eta(x) < \|x\|_\infty$ for all $x \in A_\eta(\Gamma)$.

Proof. The set C is symmetric convex with zero as interior point (indeed, $(1-\eta)B_{\ell^\infty(\Gamma)} \subset C$) so Z_η is $\frac{1}{1-\eta}$-Lipschitz and convex.

Let $A'_\eta(\Gamma) := \{x \in \ell^\infty(\Gamma) : \lceil x \rceil < 1-\eta\}$. This set is convex and open since $\lceil \cdot \rceil$ is continuous and convex. The function Φ_η is in $A'_\eta(\Gamma)$ a locally finite sum of convex C^∞-smooth functions, thus it is a convex function which is LFC and C^∞-smooth in $A'_\eta(\Gamma)$.

Let us fix $x_0 \in A'_\eta(\Gamma)$ such that $\Phi_\eta(x_0) = 1/2$. Then, since ϕ_η is increasing at the points where it is not zero, we get $Z_\eta(x_0) = 1$ and $\Phi'_\eta(x_0)x_0 > 0$. As is usual, we consider the equation $\Phi_\eta\left(\frac{x}{Z_\eta(x)}\right) = \frac{1}{2}$. By the Implicit Function Theorem, this equation locally redefines Z_η and proves that Z_η is C^∞-smooth on some neighborhood U of x_0 since Φ_η is. Moreover by application of Lemma 4.4 we get that Z_η is LFC in $\{x_0\}$.

To prove that Z_η is LFC, strictly positive and C^∞-smooth in $A_\eta(\Gamma)$ it is enough to show that for each $x \in A_\eta(\Gamma)$ there is $\lambda > 0$ such that $\lambda x \in A'_\eta(\Gamma)$ and $\Phi_\eta(\lambda \cdot x) = 1/2$ and then use the homogeneity of Z_η.

Let $x \in A_\eta(\Gamma)$. Then $\left\lceil \frac{x}{\|x\|_\infty} \right\rceil < 1 - \eta$ and since $A'_\eta(\Gamma)$ is convex, it follows that $[0, \frac{x}{\|x\|_\infty}] \subset A'_\eta(\Gamma)$. We have for such x that $\Phi_\eta(\frac{x}{\|x\|_\infty}) \geq 1$, $\Phi_\eta(0 \cdot x) = 0$ and the mapping $\lambda \mapsto \Phi_\eta(\lambda x)$ is continuous for $\lambda \in [0, \frac{1}{\|x\|_\infty}]$. Hence there must exist $\lambda \in (0, \frac{1}{\|x\|_\infty})$ such that $\lambda x \in A'_\eta(\Gamma)$ and $\Phi_\eta(\lambda \cdot x) = 1/2$.

We continue showing that Z_η is strongly lattice. First observe that Φ_η is strongly lattice as ϕ_η is nondecreasing. Let $|x| \leq |y|$ and $Z_\eta(x) = 1$. Then $x \in \partial C$ which implies that $\lceil x \rceil = 1 - \eta$ or $\Phi_\eta(x) = 1/2$. Since both functions $\lceil \cdot \rceil$ and Φ_η are strongly lattice, we conclude that $\lceil y \rceil \geq 1 - \eta$ or $\Phi_\eta(y) \geq 1/2$ which in turn implies that $Z_\eta(y) \geq 1$. For a general x we employ the homogeneity of Z_η, so Z_η is strongly lattice.

Finally, if $x \in A_\eta(\Gamma)$, then the above considerations imply that $\Phi_\eta\left(\frac{x}{Z_\eta(x)}\right) = 1/2$. This is possible only if there is some $\gamma \in \Gamma$ such that $\frac{x(\gamma)}{Z_\eta(x)} > 1 - \eta$, and the moreover claim follows. □

Lemma 4.23. *Let $0 < \eta_1 \leq \eta_2 < 1$. Then $Z_{\eta_1}(x) \leq Z_{\eta_2}(x)$ for every $x \in A_{\eta_2}(\Gamma)$.*

Proof. First of all, if $x \in A_{\eta_2}(\Gamma)$, then $x \in A_{\eta_1}(\Gamma)$. So the equivalence $Z_{\eta_i}(\lambda x) = 1 \Leftrightarrow \Phi_{\eta_i}(\lambda x) = 1/2$ holds for both $i = 1, 2$. Let us assume that $Z_{\eta_1}(\lambda x) = 1$ for some $\lambda > 0$. Then the ordering of functions ϕ_η yields $1/2 = \Phi_{\eta_1}(\lambda x) \leq \Phi_{\eta_2}(\lambda x)$ which results in $Z_{\eta_2}(\lambda x) \geq 1$. □

The following lemma shows that the seminorm Z_η satisfies a weaker version of the LUR property for some of the vectors in $\ell^\infty(\Gamma)$. The proof is inspired by that of Theorem V.1.5 in [DGZ93].

Lemma 4.24. *Let $0 < \eta < 1$ be given and let $x_r, x \in A_\eta(\Gamma)$ ($r \in \mathbb{N}$) be non-negative (in the lattice $\ell^\infty(\Gamma)$) such that*

$$2Z_\eta^2(x) + 2Z_\eta^2(x_r) - Z_\eta^2(x + x_r) \to 0 \text{ as } r \to \infty.$$

Then $x_r(\gamma) \to x(\gamma)$ for any $\gamma \in \Gamma$ such that $x(\gamma) > Z_\eta(x)(1 - \eta)$.

Proof. The assumption and Lemma 4.7 yield

$$Z_\eta(x_r) \xrightarrow{r} Z_\eta(x) \quad \text{and} \quad Z_\eta\left(\frac{x + x_r}{2}\right) \xrightarrow{r} Z_\eta(x). \tag{4.10}$$

Let us put $\tilde{x} := \frac{x}{Z_\eta(x)}$ and $\tilde{x}_r := \frac{x_r}{Z_\eta(x_r)}$. We get from (4.10) that

$$2Z_\eta^2(\tilde{x}) + 2Z_\eta^2(\tilde{x}_r) - Z_\eta^2(\tilde{x} + \tilde{x}_r) \xrightarrow{r} 0.$$

Since $Z_\eta(\tilde{x}) = Z_\eta(\tilde{x}_r) = 1$, the above implies that

$$\lambda_r := Z_\eta(\tilde{x} + \tilde{x}_r) \xrightarrow{r} 2.$$

We may deduce from $x, x_r \in A_\eta(\Gamma)$ that $\Phi_\eta(\tilde{x}) = 1/2 = \Phi_\eta(\tilde{x}_r)$ for all $r \in \mathbb{N}$. Also, $\Phi_\eta(\lambda_r^{-1}(\tilde{x} + \tilde{x}_r)) = 1/2$ for all but finitely many $r \in \mathbb{N}$. Indeed, if $\Phi_\eta(\lambda_r^{-1}(\tilde{x} + \tilde{x}_r)) \neq 1/2$, then $\lambda_r^{-1}(\tilde{x} + \tilde{x}_r) \in \partial A'_\eta(\Gamma)$. Then in fact $\lceil \lambda_r^{-1}(\tilde{x} + \tilde{x}_r) \rceil = 1 - \eta$. As $\tilde{x} \in A'_\eta(\Gamma)$, there is $\xi > 0$ such that $\lceil \tilde{x} \rceil + \xi < 1 - \eta$. By the same reasoning $\lceil \tilde{x}_r \rceil < 1 - \eta$. By the subaditivity of $\lceil \cdot \rceil$ and these estimates one has

$$\lceil \tilde{x} + \tilde{x}_r \rceil \leq \lceil \tilde{x} \rceil + \lceil \tilde{x}_r \rceil < 2(1 - \eta) - \xi.$$

Finally, $\lambda_r < \frac{2(1-\eta)-\xi}{1-\eta}$ which can happen only for finitely many r since $\lambda_r \to 2$.

As Φ_η is continuous at $\tilde{x}$ and $\lambda_r \to 2$, it follows

$$\Phi_\eta((\lambda_r - 1)^{-1}\tilde{x}) \xrightarrow{r} 1/2.$$

Consequently

$$(1 - \lambda_r^{-1})\Phi_\eta\left((\lambda_r - 1)^{-1}\tilde{x}\right) + \lambda_r^{-1}\Phi_\eta(\tilde{x}_r) - \Phi_\eta\left(\lambda_r^{-1}(\tilde{x} + \tilde{x}_r)\right) \xrightarrow{r} 0. \tag{4.11}$$

Let $a > 1 - \eta$. The definition of ϕ_η and a compactness argument imply that for each $\varepsilon > 0$ there exists $\Delta > 0$ such that if for the reals q, s, α it holds

(1) $0 \leq q \leq 4 \max\{\|\tilde{x}\|_\infty, \sup_r \|\tilde{x}_r\|_\infty\}$,

(2) $a \leq s \leq 4 \max\{\|\tilde{x}\|_\infty, \sup_r \|\tilde{x}_r\|_\infty\}$,

(3) $\frac{1}{4} \leq \alpha \leq \frac{3}{4}$, and

(4) $\alpha\phi_\eta(q) + (1 - \alpha)\phi_\eta(s) - \phi_\eta(\alpha q + (1 - \alpha)s) < \Delta$,

then $|q - s| < \varepsilon$. Indeed, otherwise there is $\varepsilon > 0$ such that for every $\Delta > 0$ there exist $q_\Delta, s_\Delta, \alpha_\Delta$ satisfying (1)-(4) and at the same time $|q_\Delta - s_\Delta| \geq \varepsilon$. By compactness, we may assume that $q_\Delta \to q$, $s_\Delta \to s$ and $\alpha_\Delta \to \alpha$. Observe that $|q - s| \geq \varepsilon$, $a \leq s$, $\alpha \in [\frac{1}{4}, \frac{3}{4}]$ and

$$\alpha\phi_\eta(q) + (1 - \alpha)\phi_\eta(s) - \phi_\eta(\alpha q + (1 - \alpha)s) = 0.$$

It is now clear that we are contradicting the strict convexity of the function ϕ_η at the point s.

In particular, let $a > 1 - \eta$ be such that $\{\gamma \in \Gamma; \tilde{x}(\gamma) > 1 - \eta\} = \{\gamma \in \Gamma; \tilde{x}(\gamma) > a\}$ and let $\gamma \in \Gamma$ be such that $\tilde{x}(\gamma) > a$. Then for r large enough we have $(\lambda_r - 1)^{-1}\tilde{x}(\gamma) > a$ so we may substitute $r := \tilde{x}_r(\gamma)$, $s := (\lambda_r - 1)^{-1}\tilde{x}(\gamma)$ and $\alpha := \lambda_r^{-1}$. It follows from (4.11) that one has $|(\lambda_r - 1)^{-1}\tilde{x}(\gamma) - \tilde{x}_r(\gamma)| \to 0$ as $r \to \infty$. Since $\lambda_r \to 2$ and using (4.11), we finally get that $x_r(\gamma) \to x(\gamma)$ as $r \to \infty$. □

4.4.2 Mapping X into the "nice" space

The following system of convex functions is at the heart of our construction. We recall that $C \geq 1$ is the constant of equivalence between the norms $|\cdot|$ and $\|\cdot\|$, which was introduced in Section 4.3.

Lemma 4.25. *There exist*

- *a decreasing sequence of positive numbers* $\delta_n \searrow 0$*;* $\delta_1 < 2C$*;*
- *a decreasing sequence of positive numbers* $\rho_n \searrow 0$*;*
- *positive numbers* $\kappa_{n,m} > 0$ *such that for each* $n \in \mathbb{N}$ *the sequence* $(\kappa_{n,m})_m$ *is decreasing and* $\kappa_{n,m} \xrightarrow{m} 0$*; for each* $n, m \in \mathbb{N}$ *one has* $\rho_n > 2\kappa_{n,m}$*;*
- *an equi-Lipschitz system of non-negative,* C^∞*-smooth,* 1*-bounded, convex functions*

$$\{g_{n,m,l} : \mathcal{D}_{n,l} \to \mathbb{R} : n, m \in \mathbb{N}, l = 1, \dots, n\},$$

where $\mathcal{D}_{n,l} := [0, 2nC - \delta_n(n-l)] \times [0, 1 + 2nC]$*, satisfying (with* $n, m, l \in \mathbb{N}$*,* $l \leq n$*, resp.* $l < n$ *in (A2),(A5))*

(A1) $g_{n,m,l}(t,s) = 0$ *iff* $(t,s) \in [0, l\delta_n] \times [0, 1 + 2nC] =: \mathcal{N}_{n,l}$*;*

(A2) $g_{n,m,l}(t,s) \geq g_{n,m,l+1}(t,s) + \rho_n$ *whenever* $(t,s) \in \mathcal{D}_{n,l} \setminus \mathcal{N}_{n,l+1}$*;*

(A3) *if* $(t,0) \in \mathcal{D}_{n,l} \setminus \mathcal{N}_{n,l}$*, then* $s \mapsto g_{n,m,l}(t,s)$ *is increasing on* $[0, 1+2nC]$ *and*

$$g_{n,m,l}(t, 1+2nC) - g_{n,m,l}(t,0) \leq \kappa_{n,m};$$

(A4) *if* $(t,0) \in \mathcal{D}_{n,l}$*, then* $g_{n,m,l}(t,0) = g_{n,m+1,l}(t,0)$*;*

(A5) *for all* $(t,s) \in \mathcal{D}_{n,l} \setminus \mathcal{N}_{n,l}$ *it holds* $g_{n,m,l}(t,s) < g_{n,m,l+1}(t+r,s)$ *provided* $r > \delta_n$*.*

(A6) *Let* $(t,s) \in \mathcal{D}_{n,l} \setminus \mathcal{N}_{n,l}$*. If* $(t_r, s_r) \in \mathcal{D}_{n,l}$ *and* $t_r \to t$ *and* $g_{n,m,l}(t_r,s_r) \to g_{n,m,l}(t,s)$ *as* $r \to \infty$*, then* $s_r \to s$*.*

(A7) *The mapping* $(t,s) \mapsto g_{n,m,l}(|t|, |s|)$ *is strongly lattice in* $\mathcal{D}_{n,l}$*.*

Proof. Let $f : \mathbb{R} \to [0, +\infty)$ be defined as

$$f(t) := \begin{cases} 0, & \text{for } t \leq 0, \\ \exp(-\frac{1}{t^2}) & \text{for } t > 0. \end{cases}$$

It is elementary (one may use Lemma 4.8) to check that $f(t) \cdot (s^2 + s + 1)$ is convex in the strip $(-\infty, 10^{-1}] \times [0, 10^{-1}]$ so the function

$$g(t,s) := f(10^{-1}t) \cdot ((10^{-1}s)^2 + 10^{-1}s + 1)$$

is convex in the strip $(-\infty, 1] \times [0, 1]$. We take for $(\delta_n)_n$ just any decreasing null sequence of positive numbers such that $\delta_1 < 2C$, and we define

$$g_{n,m,l}(t, s) := g\left(\frac{t - \delta_n l}{(2C - \delta_n)n}, \theta_{n,m}\frac{s}{1 + 2nC}\right).$$

where $\theta_{n,m} \in (0, 1)$ will be chosen later. Now since our functions $g_{n,m,l}$ are just shifts and stretches of one non-negative, C^∞-smooth, 1-bounded, Lipschitz, convex function, it follows that all $g_{n,m,l}$ share these properties (with the same Lipschitz constant).

Properties (A1), (A4) and (A5) are straightforward, see also Figure 4.1. Notice that,

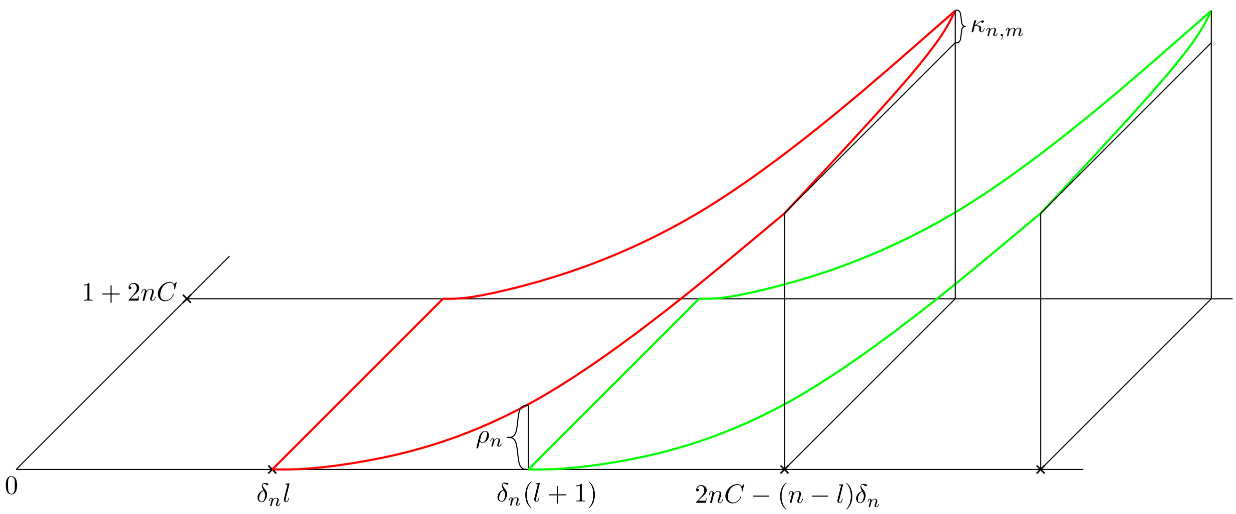

Figure 4.1:

when $t > 0$, the function $s \mapsto g(t, s)$ is increasing on $[0, 1]$. This implies the first part of (A3). In order to satisfy (A2), we may define ρ_n as

$$\rho_n := \inf \{g_{n,m,l}(t, s) - g_{n,m,l+1}(t, s) : l, m \in \mathbb{N}, l < n, (t, s) \in \mathcal{D}_{n,l} \setminus \mathcal{N}_{n,l+1}\}$$

which evaluates as $\rho_n = g_{n,1,1}(2\delta_n, 0) = f\left(\frac{\delta_n}{(2-\delta_n)n}\right) \searrow 0$ as $n \to \infty$. Notice that this ρ_n does not depend on the choice of $\theta_{n,m}$. On the other hand, in order to fulfill (A3), $\kappa_{n,m}$ may be defined as

$$\kappa_{n,m} := \sup \{g_{n,m,l}(t, 1 + 2nC) - g_{n,m,l}(t, 0) : l \leq n, (t, 0) \in \mathcal{D}_{n,l}\}$$

which evaluates as $\kappa_{n,m} = g_{n,m,n}(2nC, 1 + nC) - g_{n,m,n}(2nC, 0)$. We see that, by an appropriate choice of $\theta_{n,m}$ (in particular, for each $n \in \mathbb{N}$, the sequence $(\theta_{n,m})_m$ should be decreasing to zero), one may satisfy the requirements $\rho_n > 2\kappa_{n,m}$ and $\kappa_{n,m} \searrow 0$ as $m \to \infty$.

For the proof of (A6) let us assume that $s_r \not\to s$. The fact that $g_{n,m,l}(t_r, \cdot) \to g_{n,m,l}(t, \cdot)$ uniformly on $[0, 1 + 2nC]$ leads quickly to a contradiction.

Finally (A7) follows since g is non-decreasing in $\mathcal{D}_{n,l}$ in each variable. □

Let us fix, for each $\delta > 0$, some C^∞-smooth, convex mapping ξ_δ from $[0, +\infty)$ to $[0, +\infty)$ which satisfies $\xi_\delta([0, \delta]) = \{0\}$, $\xi_\delta(t) > 0$ for $t > \delta$ and $\xi_\delta(t) = t - 2\delta$ for $t \geq 3\delta$. Such a mapping can be constructed e.g. by integrating twice a C^∞-smooth, non-negative bump.

Recall that $B^O_{(X,\|\cdot\|)}$ stands for the open unit ball of $(X, \|\cdot\|)$.

Lemma 4.26. *Let $n, m \in \mathbb{N}$ be fixed and let us define a mapping $H_{n,m} : B^O_{(X,\|\cdot\|)} \to \ell^\infty(F_n)$ where $F_n = \left\{(A, B) \in 2^\Lambda \times 2^\Lambda : |A| \leq n, B \subset A, A \neq \emptyset \neq B\right\}$ by*

$$H_{n,m}x(A, B) := g_{n,m,|A|}\left(\sum_{\gamma \in A} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), \xi_{\delta_n}(\|P_B x - x\|)\right).$$

Then $H_{n,m}$ is a continuous, coordinatewise convex and coordinatewise C^1-smooth mapping, and for each $x \in X$ such that $\|x\| < 1$ it holds $H_{n,m}x \in A_{\rho_n/2-\kappa_{n,m}}(F_n) \cup \{0\}$ (see the definition of the set $A_{\rho_n/2-\kappa_{n,m}}(F_n)$ in Lemma 4.22).

Notice that, by the definition of $\kappa_{n,m}$ in Lemma 4.25, we have always $\rho_n/2 - \kappa_{n,m} > 0$. We will use the notation $\eta_{n,m} := \rho_n/2 - \kappa_{n,m}$.

Proof. When $\|x\| < 1$, then (4.6) yields $\left(\sum_{\gamma \in A} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), \xi_{\delta_n}(\|P_B x - x\|)\right) \in [0, 1 + 2\,|A|\,C) \times [0, 2\,|A|\,C) \subset \mathcal{D}_{n,|A|}$. So for each $(A, B) \in F_n$ the mapping $x \mapsto H_{n,m}x(A, B)$ is C^1-smooth as a composition of such mappings (remember that $\|\cdot\|_\gamma$ is supposed to be C^1-smooth). Also, $\{x \mapsto H_{n,m}x(A, B) : (A, B) \in F_n\}$ is equi-Lipschitz thus $H_{n,m}$ is continuous. Each $x \mapsto H_{n,m}x(A, B)$ is convex by application of Lemma 4.10 since $g_{n,m,l}$ is convex and strongly lattice. Because $\sup g_{n,m,l}(\mathcal{D}_{n,|A|}) < 1$ for each $l \leq n$, we get that $\|H_{n,m}x\|_\infty < 1$.

We are going to prove that $\lceil H_{n,m}x \rceil < \|H_{n,m}x\|_\infty (1 - \rho_n/2 + \kappa_{n,m})$ or $\|H_{n,m}x\|_\infty = 0$. For any $x \in X$ and $\delta > 0$, let $\Lambda(x, \delta) := \left\{\gamma \in \Lambda : \|Q_\gamma x\|_\gamma > \delta\right\}$. Let $x \in B^O_{(X,\|\cdot\|)}$ be fixed and let us define a set $E \subset F_n$ as $E := \{(A, B) \in F_n : A \subset \Lambda(x, \delta_n)\}$. Since E is finite, it holds

$$\lceil H_{n,m}x \rceil = \lceil H_{n,m}x \restriction_{F_n \setminus E} \rceil \leq \sup\{H_{n,m}x(A, B) : (A, B) \in F_n \setminus E\}. \tag{4.12}$$

If there is no $(A, B) \in F_n \setminus E$ such that $H_{n,m}(A, B) > 0$, then $\lceil H_{n,m}x \rceil = 0$ and our claim is trivially true. We proceed assuming that $H_{n,m}x(A, B) > 0$ for some $(A, B) \in F_n \setminus E$. Then

$$\left(\sum_{\gamma \in A} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), \xi_{\delta_n}(\|P_B x - x\|)\right) \notin \mathcal{N}_{n,|A|}$$

which, by (A1) in Lemma 4.25, can happen only if $C := A \cap \Lambda(x, \delta_n) \neq \emptyset$. Since $(A, B) \notin$

E, we have $|C| < |A|$. It follows from Lemma 4.25 (A2) and (A3) that

$$g_{n,m,|A|}\Big(\sum_{\gamma\in A}\xi_{\delta_n}(\|Q_\gamma x\|_\gamma),\xi_{\delta_n}(\|P_B x - x\|)\Big) \leq$$

$$\leq g_{n,m,|C|}\left(\sum_{\gamma\in C}\xi_{\delta_n}(\|Q_\gamma x\|_\gamma),\xi_{\delta_n}(\|P_B x - x\|)\right) - \rho_n$$

$$\leq g_{n,m,|C|}\left(\sum_{\gamma\in C}\xi_{\delta_n}(\|Q_\gamma x\|_\gamma),\xi_{\delta_n}(\|P_D x - x\|)\right) - \rho_n + \kappa_{n,m}$$

for any $D \subset C$. Of course, since $\Lambda(x, \delta_n)$ is finite, there are only finitely many couples (C, D) such that $D \subset C \subset \Lambda(x, \delta_n)$. We may therefore write

$$H_{n,m}x(A, B) \leq \max_{D\subset C\subset\Lambda(x,\delta_n)} H_{n,m}x(C, D) - \rho_n + \kappa_{n,m} \leq \|H_{n,m}\|_\infty (1 - \rho_n + \kappa_{n,m})$$

for any $(A, B) \in F_n \setminus E$. This together with (4.12) gives $\lceil H_{n,m}x\rceil < \|H_{n,m}\|_\infty (1-(\rho_n/2 - \kappa_{n,m}))$. □

Lemma 4.27. *Let $0 \neq x \in B^O_{(X,\|\cdot\|)}$ and let A be a finite subset of Λ such that $Q_\gamma x \neq 0$ when $\gamma \in A$. We claim that, for all $n, m \in \mathbb{N}$ sufficiently large, there exists a finite $C_{n,m} \subset \Lambda$ such that*

- *$A \subset C_{n,m}$, and*
- *$H_{n,m}x(C_{n,m}, A) > (1 - \eta_{n,m})Z_{\eta_{n,m}}(H_{n,m}x)$.*

Proof. We start by defining $A^* := \Big\{\gamma \in \Lambda : \|Q_\gamma x\|_\gamma \geq \min_{\alpha\in A} \|Q_\alpha x\|_\alpha\Big\}$ and we set out for finding $C_{n,m}$ so that in fact $A^* \subset C_{n,m}$.

Let us investigate the mapping $L_n : B^O_{(X,\|\cdot\|)} \to \ell^\infty(F_n)$ defined as

$$L_n y(D, E) := g_{n,1,|D|}\left(\sum_{\gamma\in D}\xi_{\delta_n}(\|Q_\gamma y\|_\gamma), 0\right).$$

By the same argument as in the proof of Lemma 4.26, we get that $\lceil L_n x\rceil \leq (1-\rho_n)\|L_n x\|_\infty$ or $L_n x = 0$. Hence $L_n x \in A_{\rho_n/2} \cup \{0\}$. If n is large enough, necessarily $L_n x \neq 0$. It follows that $L_n x$ attains a nonzero maximum. For $n \in \mathbb{N}$, let C_n be such that $L_n x(C_n, D) = \|L_n x\|_\infty$ for some (and all) non-empty $D \subset C_n$. We claim that, for n sufficiently large, $A^* \subset C_n$.

Let us denote $b := \min\Big\{\|Q_\gamma x\|_\gamma : \gamma \in A^*\Big\} - \max\Big\{\|Q_\gamma x\|_\gamma : \gamma \in \Lambda \setminus A^*\Big\}$. Since $Q_\gamma x \neq 0$ for all $\gamma \in A$, and for the c_0-nature of $(\|Q_\gamma x\|_\gamma)_{\gamma\in\Lambda}$, it follows that $b > 0$. Notice that

$$b_n := \xi_{\delta_n}\left(\min\Big\{\|Q_\gamma x\|_\gamma : \gamma \in A^*\Big\}\right) - \xi_{\delta_n}\left(\max\Big\{\|Q_\gamma x\|_\gamma : \gamma \in \Lambda \setminus A^*\Big\}\right) \to b \text{ as } n \to \infty.$$

Let $n \geq |A^*|$ be so large that $\delta_n < \xi_{\delta_n}\left(\min\left\{\|Q_\gamma x\|_\gamma : \gamma \in A^*\right\}\right)$ and $\delta_n < b_n$.

If $A^* \not\subseteq C_n$, there exists $\gamma_1 \in A^* \setminus C_n$. If $|C_n| < n$, then we define $\tilde{C}_n := \{\gamma_1\} \cup C_n$. By our choice of n, we have that $\xi_{\delta_n}(\|Q_{\gamma_1} x\|_{\gamma_1}) > \delta_n$ and so by the property (A5) in Lemma 4.25 we get that

$$g_{n,1,|C_n|}\left(\sum_{\gamma\in C_n} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), 0\right) < g_{n,1,|\tilde{C}_n|}\left(\sum_{\gamma\in \tilde{C}_n} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), 0\right)$$

contradicting that any couple $(C_n, D) \in F_n$ maximizes $L_n x$.

If $|C_n| = n$, then there exists $\gamma_2 \in C_n \setminus A^*$ and we define $\tilde{C}_n := \{\gamma_1\} \cup C_n \setminus \gamma_2$. Our choice of n yields that $\xi_{\delta_n}(\|Q_{\gamma_1} x\|_{\gamma_1}) - \xi_{\delta_n}(\|Q_{\gamma_2} x\|_{\gamma_2}) > \delta_n$ so (A5) in Lemma 4.25 implies

$$g_{n,1,n}\left(\sum_{\gamma\in C_n} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), 0\right) < g_{n,1,n}\left(\sum_{\gamma\in \tilde{C}_n} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), 0\right)$$

once again contradicting that any couple $(C_n, D) \in F_n$ maximizes $L_n x$. So $A^* \subset C_n$.

At this moment, we leave n fixed according to the choices above and we start tuning m. First of all, let us observe that $L_n x(C_n, A) > Z_{\rho_n/2}(L_n x)(1 - \rho_n/2)$ by the moreover part of Lemma 4.22. Since $\eta_{n,m} \nearrow \rho_n/2$ as $m \to \infty$, we deduce that there is some $p \in \mathbb{N}$ such that $L_n x(C_n, A) > Z_{\rho_n/2}(L_n x)(1 - \eta_{n,p})$. We will work, for $\gamma \in F_n$, with the set $M_\gamma = \left\{u \in \ell^\infty(F_n) : |u(\gamma)| > Z_{\rho_n/2}(u)(1 - \eta_{n,p})\right\}$. The set M_γ is open and, in particular, $L_n x \in M_{(C_n,A)}$.

Using (A3) and (A4) in Lemma 4.25 we may see that $H_{n,m} x \to L_n x$ in $(\ell^\infty(F_n), \|\cdot\|_\infty)$ as $m \to \infty$. Since $L_n x$ is a member of the open set $A_{\rho_n/2}(F_n)$, so will be $H_{n,m} x$ for m large enough. Similarly, the openness of $M_{(C_n,A)}$ insures that $H_{n,m} x \in M_{(C_n,A)}$ for $m \geq p$ and large enough. This means that

$$H_{n,m} x(C_n, A) > Z_{\rho_n/2}(H_{n,m} x)(1 - \eta_{n,p}) \geq Z_{\eta_{n,m}}(H_{n,m} x)(1 - \eta_{n,m})$$

where the second inequality follows from Lemma 4.23 as $\rho_n/2 \geq \eta_{n,m}$ and $\eta_{n,m} \geq \eta_{n,p}$ for all $m \geq p$. So we may define $C_{n,m} := C_n$ for m sufficiently large. □

4.4.3 The definition of J

We came close to the definition of the norm J. First, we choose some decreasing sequence of positive numbers $\sigma_j \searrow 0$ and we define $J_{j,n,m} : B^O_{(X,\|\cdot\|)} \to \mathbb{R}$ as

$$J_{j,n,m}(x) := \xi_{\sigma_j}(Z_{\eta_{n,m}}(H_{n,m} x)).$$

Next, let $\tilde{J} : B^O_{(X,\|\cdot\|)} \to \mathbb{R}$ be defined as

$$\tilde{J}^2(x) := \|x\|^2 + \sum_{j,n,m\in\mathbb{N}} \frac{1}{2^{j+n+m}} J^2_{j,n,m}(x)$$

and finally let $J : X \to \mathbb{R}$ be defined as the Minkowski functional of $\left\{x \in X : \tilde{J}(x) \leq 1/2\right\}$.

Lemma 4.28. *The function J is an equivalent norm on X which is C^1-smooth away from the origin.*

Proof. To see the differentiability it is sufficient, in the light of the Implicit Function Theorem, to show that for each $x \in X$ such that $J(x) = 1$, the function $\tilde{J}$ is Fréchet differentiable on some neighborhood of x with $\tilde{J}'(x)x \neq 0$.

First of all let us observe that, for each $\sigma > 0$ and $\eta > 0$, the composed function $\xi_\sigma \circ Z_\eta : \ell^\infty(F_n) \to \mathbb{R}$ is C^∞-smooth and LFC in $A_\eta(F_n) \cup \{0\}$. Of course it is – we know it already for points in $A_\eta(F_n)$ and clearly, there is a neighborhood U of $0 \in F_n$ such that $\xi_\sigma \circ Z_\eta$ is constant in U.

First assume that $\|x\| < 1$. It follows from Lemma 4.26 and from Lemma 4.3 that each $J_{j,n,m}$ is C^1-smooth at x. Further we claim that there is a constant $K > 0$ such that each $J_{j,n,m}$ is nK-Lipschitz. Indeed, there is a constant $K' > 0$ such that $H_{n,m}$ is $(1+2nC)K'$-Lipschitz for all $n, m \in \mathbb{N}$; Z_η is 2-Lipschitz for each $0 < \eta < 1/2$ and ξ_σ is 1-Lipschitz for each $\sigma > 0$. It follows that $\tilde{J}$ is K''-Lipschitz for some $K'' > 0$. The calculus rules now lead to the conclusion that $\tilde{J}$ is Fréchet differentiable on a neighborhood of any $x \in X$ such that $\|x\| < 1$. Let $x \in X$ such that $J(x) = 1$. Then $\|x\| \leq 1/2$, so $\tilde{J}$ is Fréchet differentiable at x. Also, the convexity of $\tilde{J}$ and the fact that $\tilde{J}(0) = 0$ imply that $\tilde{J}'(x)x > 0$.

Finally, $2\|x\| \leq J(x) \leq 2K''\|x\|$ where the second inequality follows from the K''-Lipschitzness of $\tilde{J}$. □

Lemma 4.29. *If $x \in X$ and $(x_r)_{r\in\mathbb{N}} \subset X$ are such that*

$$2\,|||x_r|||^2 + 2\,|||x|||^2 - |||x + x_r|||^2 \to 0 \text{ as } r \to \infty \tag{4.7}$$

is satisfied, then for each $\varepsilon > 0$ there is a finite subset A of Λ such that $Q_\gamma x \neq 0$ for $\gamma \in A$, $\|P_A x - x\| < \varepsilon$ and $\|P_A x_r - x_r\| < \varepsilon$ for all r sufficiently large.

Recall that $|||x|||^2 = \|x\|^2 + N^2(x) + J^2(x)$.

Proof. We may assume, that $J(x) = 1$. We start by finding a finite $A \subset \Lambda$ such that $\|P_A x - x\| < \varepsilon/2$ and such that $Q_\gamma x \neq 0$ for $\gamma \in A$. This is possible by Lemma 4.14. Now we will just show that $\|P_A x_r - x_r\| \to \|P_A x - x\|$.

It follows from (4.7) and from the uniform continuity of $\tilde{J}$ on bounded sets that

$$\frac{\tilde{J}^2(x_r) + \tilde{J}^2(x)}{2} - \tilde{J}^2\left(\frac{x + x_r}{2}\right) \to 0 \text{ as } k \to \infty. \tag{4.13}$$

By the convexity of the terms in the definition of $\tilde{J}$, we get that

$$\frac{J^2_{j,n,m}(x_r) + J^2_{j,n,m}(x)}{2} - J^2_{j,n,m}\left(\frac{x + x_r}{2}\right) \to 0 \text{ as } k \to \infty. \tag{4.14}$$

for each $j, n, m \in \mathbb{N}$.

Let us borrow the notation $L_n x$ from the proof of Lemma 4.27. Let us recall that $H_{n,m}x \geq L_n x \geq 0$ (in the lattice $\ell^\infty(F_n)$) for all $m \in \mathbb{N}$. There is some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have that $L_n x \neq 0$. Hence $Z_{\eta_{n,m}}(H_{n,m}x) \geq Z_{\eta_{n,m}}(L_n x) \geq Z_{\eta_{n,1}}(L_n x) > 0$ for $n \geq n_n$ and $m \in \mathbb{N}$. Therefore for each $n \geq n_0$ there exists $j_n \in \mathbb{N}$ such that for all $j \geq j_n$ and all $m \in \mathbb{N}$ one has $J_{j,n,m} > 0$. Since, for $n \geq n_0, m \in \mathbb{N}$ and $j \geq j_n$, (4.14) is equivalent to

$$\lim_r J_{j,n,m}(x_r) = J_{j,n,m}(x) = \lim_r J_{j,n,m}\left(\frac{x_r + x}{2}\right)$$

and $\xi_\sigma \restriction_{(\sigma,+\infty)}$ has a continuous inverse, it follows that

$$\frac{Z^2_{\eta_{n,m}}(H_{n,m}x_r) + Z^2_{\eta_{n,m}}(H_{n,m}x)}{2} - Z^2_{\eta_{n,m}}\left(H_{n,m}\left(\frac{x_r + x}{2}\right)\right) \xrightarrow{r} 0$$

for all $n \geq n_0$ and $m \in \mathbb{N}$. Since $x \mapsto H_{n,m}x(A,B)$ is convex and non-negative for each $(A,B) \in F_n$ and since $Z_{\eta_{n,m}}$ is strongly lattice and convex it follows

$$\begin{aligned} 0 \leftarrow \frac{Z^2_{\eta_{n,m}}(H_{n,m}x_r) + Z^2_{\eta_{n,m}}(H_{n,m}x)}{2} - Z^2_{\eta_{n,m}}\left(H_{n,m}\left(\frac{x_r + x}{2}\right)\right) \geq \\ \geq \frac{Z^2_{\eta_{n,m}}(H_{n,m}x_r) + Z^2_{\eta_{n,m}}(H_{n,m}x)}{2} - Z^2_{\eta_{n,m}}\left(\frac{H_{n,m}x_r + H_{n,m}x}{2}\right) \geq 0 \end{aligned}$$

for every $n \geq n_0$ and $m \in \mathbb{N}$. Let us fix $n \geq n_0$ and $m \in \mathbb{N}$ both large enough in the sense of Lemma 4.27. We also require that $\delta_n < \|P_A x - x\|$. By application of Lemma 4.27, we obtain a set $C_{n,m}$ such that $\gamma := (C_{n,m}, A) \in F_n$ satisfies the assumptions of Lemma 4.24. Thus, using this last mentioned lemma, we may conclude that $H_{n,m}x_r(C_{n,m}, A) \to H_{n,m}x(C_{n,m}, A)$ as $r \to \infty$.

To finish the argument, we employ Lemma 4.21 to see that

$$\sum_{\gamma \in C_{n,m}} \xi_{\delta_n}(\|Q_\gamma x_r\|_\gamma) \to \sum_{\gamma \in C_{n,m}} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma) \text{ as } r \to \infty$$

and we apply Lemma 4.25 (A6) on the function $g_{n,m,|C_{n,m}|}$. This leads to the convergence $\xi_{\delta_n}(\|P_A x_r - x_r\|) \to \xi_{\delta_n}(\|P_A x - x\|)$ which in turn means that $\|P_A x_r - x_r\| \to \|P_A x - x\|$ because of our choice of n. □

4.4.4 J is a limit of C^k-smooth norms

In the end of all we are going to show that J is a limit of C^k-smooth norms. A self-evident choice for the approximating norms J_i is as follows. Let us define

$$H^i_{n,m}x(A,B) := g_{n,m,l}\left(\sum_{\gamma \in A} \xi_{\delta_n}(\|Q_\gamma x\|_{\gamma,i}), \xi_{\delta_n}(\|P_B x - x\|)\right),$$

$$J_{j,n,m,i}(x) := \xi_{\sigma_j}(Z_{\eta_{n,m}}(H^i_{n,m}x)),$$
$$\tilde{J}_i^2(x) := \|x\|^2 + \sum_{1\le j,n,m\le i} \frac{1}{2^{j+n+m}} J^2_{j,n,m,i}(x)$$

and J_i as the Minkowski functional of $\left\{x \in X : \tilde{J}_i(x) \le 1/2\right\}$. As a finite sum of C^k-smooth functions, $\tilde{J}_i$ is C^k-smooth. The Implicit Function Theorem implies the same about J_i. Moreover $2\|x\| \le J_i(x) \le 2K'' \|x\|$ as in the proof of Lemma 4.28. Let $\varepsilon > 0$ be given. We will show that there is an index $i_0 \in \mathbb{N}$ such that $\left|\tilde{J}_i^2(x) - \tilde{J}^2(x)\right| < \varepsilon$ whenever $\|x\| < 1$ and $i \ge i_0$. For this it is sufficient that $\left(\frac{2C}{i_0}\right)^2 < \varepsilon/2$ and

$$\sum_{\max\{j,n,m\}\ge i_0} \frac{2}{2^{j+n+m}} < \varepsilon/2$$

because then, for each $i \ge i_0$,

$$\begin{aligned}\left|\tilde{J}_i^2(x) - \tilde{J}^2(x)\right| &\le \sum_{1\le j,n,m\le i} \frac{1}{2^{j+n+m}}(J^2_{j,n,m,i}(x) - J^2_{j,n,m}(x)) + \sum_{\max\{j,n,m\}\ge i_0} \frac{1}{2^{j+n+m}} J^2_{j,n,m}(x) \\ &< \sum_{1\le j,n,m\le i} \frac{1}{2^{j+n+m}} \left(\frac{2Ci}{i^2}\right)^2 + \varepsilon/2 < \varepsilon\end{aligned}$$

where in the second inequality we are using (4.6) and $(1-\frac{1}{i^2})\|x\|_\gamma \le \|x\|_{\gamma,i} \le \|x\|_\gamma$ to estimate the first term and $J_{j,n,m}(x) \le 2$ for $\|x\| < 1$. This proves that $\tilde{J}_i \to \tilde{J}$ uniformly on $B^O_{(X,\|\cdot\|)}$.

Now let us observe that, since $\tilde{J}(0) = 0$, we have the estimate

$$\frac{1}{2}|\lambda - 1| \le \left|\frac{1}{2} - \tilde{J}(\lambda x)\right| \tag{4.15}$$

for all $x \in X$ such that $\tilde{J}(x) = \frac{1}{2}$, or equivalently such that $J(x) = 1$.

We assume that there is a sequence $(x_i) \subset B^O_{(X,\|\cdot\|)}$ such that $J_i(x_i) - J(x_i) \nrightarrow 0$. Let $c_i > 0$, resp. $d_i > 0$, be such that $J(c_i x_i) = 1$, resp. $J_i(d_i x_i) = 1$. It follows that $\lambda_i := \frac{d_i}{c_i} \nrightarrow 1$ so we may and do assume that there is some $\varepsilon > 0$ such that $|\lambda_i - 1| > 2\varepsilon$ for all $i \in \mathbb{N}$. On the other hand, since $\|d_i x_i\| \le \frac{1}{2}$ and since $\tilde{J}_i \to \tilde{J}$ uniformly on $B^O_{(X,\|\cdot\|)}$, we get that $\left|\tilde{J}(\lambda_i c_i x_i) - \frac{1}{2}\right| = \left|\tilde{J}(\lambda_i c_i x_i) - \tilde{J}_i(d_i x_i)\right| \le \varepsilon$ for i large enough. Thus, having in mind (4.15), we obtain $|\lambda_i - 1| \le 2\varepsilon$. As a result of this contradiction we see immediatelly that $J_i \to J$ uniformly on bounded sets.

Chapter 5

A parametric variational principle

In this chapter we investigate a possibility of parametrizing the following theorem.

Deville-Godefroy-Zizler variational principle ([DGZ93])**.** *Let $(X, \|\cdot\|_X)$ be a Banach space. Let $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be a Banach space of bounded real functions on X satisfying*

(i) $\|\cdot\|_\infty \leq \|\cdot\|_{\mathcal{Y}}$,

(ii) *$\mathcal{Y}$ contains a bump,*

(iii) *if $g \in \mathcal{Y}$ then $g(a\cdot) \in \mathcal{Y}$ for all $a > 0$, $\tau_y g \in \mathcal{Y}$ and $\|\tau_y g\|_{\mathcal{Y}} = \|g\|_{\mathcal{Y}}$ for all $x \in X$ where $\tau_y g(x) = g(x - y)$.*

Let $f : X \to (-\infty, +\infty]$ be lower bounded, l.s.c., proper function. Then the set of functions $g \in \mathcal{Y}$ such that $f + g$ attains its strong minimum is a dense G_δ subset of $\mathcal{Y}$.

In order to have a good starting point we consider the following problem.

Problem 5.1. Let Π be a topological space, X be a Banach space, and let $\mathcal{Y}$ be a fixed space of continuous functions, we call them *perturbations*, from X to $\mathbb{R}$. Given a function $f : \Pi \times X \to (-\infty, +\infty]$ which satisfies the following, we call them *minimal*, conditions:

(M1) for every $p \in \Pi$ the function $f(p, \cdot)$ is proper, l.s.c., lower bounded,

(M2) for every $x \in X$ the function $f(\cdot, x)$ is a continuous function from Π to $(-\infty, +\infty]$ with its usual topology,

is it possible to find $\Delta : \Pi \to \mathcal{Y}$ and $v : \Pi \to X$ continuous such that

$$f(p, v(p)) + \Delta(p)(v(p)) = \inf \{f(p, x) + \Delta(p)(x) : x \in X\} \tag{5.1}$$

for every $p \in \Pi$?

First let us comment on the minimal conditions. These requirements are quite natural. Indeed, (M1) is a usual condition of nonparametrized variational principles (Ekeland, Borwein-Preiss, DGZ) and there is no reason why a parametrized version should hold under more general assumptions. In fact, the solution of Problem 5.1 for the case Π=singleton is exactly a nonparametrized variational principle. The necessity of (M2) is obvious, consider e.g. $\Pi = \mathbb{R} = X$, $f(p, x) = \delta_{\{\mathrm{sign}(p)\}}(x)$ where $\mathrm{sign}(p) = \frac{p}{|p|}$ if $p \neq 0$ and $\mathrm{sign}(0) = 0$; δ_A is the indicator function of a set $A \subset X$, i.e. $\delta_A(x) = 0$ if $x \in A$ otherwise $\delta_A(x) = +\infty$.

Nevertheless, Problem 5.1 has, in its general form, a negative solution (see Section 5.4). The main theorem of this chapter (Theorem 5.18) gives a positive answer to the problem when Π is a paracompact space, $\mathcal{Y}$ is a certain (see Notation 5.4) cone of Lipschitz functions and provided f (apart from obeying (M1), (M2)) is convex in the second variable and satisfies an equi-lower semicontinuity condition (see (A2) in Theorem 5.18) which leads essentially to the lower semicontinuity of $\inf f(\cdot, X)$ (see Proposition 5.11). In this situation we show that the set of the functions Δ which satisfy (5.1) is residual in $C(\Pi, \mathcal{Y})$ equipped with the *fine* topology (see Definition 5.6). This topology enjoys two important properties, it is finer than the uniform topology on $C(\Pi, \mathcal{Y})$ and it is Baire. The latter makes it possible to prove the main theorem in the spirit of the proof of the DGZ variational principle, replacing the points in X by continuous functions from Π to X. The additional assumptions appear due to the use of a variant of Michael's selection theorem in the proof (see Section 5.2). This includes, apart from the requirements mentioned above, the requirement of Π being paracompact. On the other hand, we demonstrate in Section 5.4 that none of the additional assumptions (convexity, equi-lower semicontinuity) can be dropped without replacement.

The organization of this chapter is the following. In Section 5.1 we describe general conditions on the space of perturbations $\mathcal{Y}$, we give some concrete examples of spaces which meet these requirements. We also define and examine the fine topology on the space $C(\Pi, \mathcal{Y})$. In Section 5.2 we present a version of Michael's selection theorem (Lemma 5.8) and some lemmata involving the equi-lower semicontinuity (Lemma 5.16 might be of independent interest). In Section 5.3 we state and prove the main theorem. We state a corollary which can be understood as a localized version of the main theorem and essentially includes as special cases the theorems of Georgiev [Geo05] and Veselý [Ves09]. Section 5.4 gives some examples to illustrate the limits of the main theorem.

Throughout this chapter, $(X, \|\cdot\|_X)$ will be a Banach space. For a function $g : X \to (-\infty, +\infty]$ we denote $\mathrm{dom}(g)$ its *effective domain*, i.e. $\mathrm{dom}(g) = \{x \in X : g(x) < +\infty\}$. We say that g is *proper* if $\mathrm{dom}(g) \neq \emptyset$.

5.1 Space of perturbations

Definition 5.2. Let X be a Banach space. A nonnegative convex function $b : X \to \mathbb{R}$ is called a *convex separating function* if for some $\varepsilon > 0$ the set $\{x \in X : b(x) < \varepsilon\}$ is nonempty and bounded. Observe that then $\{x \in X : b(x) < \varepsilon'\}$ is nonempty and bounded for all $\varepsilon' > \varepsilon$.

One of the important properties of a convex separating function is described next.

Lemma 5.3. *Let $b : X \to \mathbb{R}$ be a convex separating function. Then for every $x_0 \in X$ there exist $c_{x_0} > 0$ and $C_{x_0} > 0$ such that*

$$b(x) - b(x_0) \geq c_{x_0} \|x - x_0\|_X$$

for all $x \in X \setminus B_X(x_0, C_{x_0})$.

Proof. Since b is a convex separating function it is easily seen that there exist $z \in X$, $c > 0$ and $C > 0$ such that $b(x) - b(z) \geq c\|x - z\|_X$ for $x \in X \setminus B_X(z, C)$. Now let $x_0 \in X$ be given and let us estimate

$$\begin{aligned} b(x) - b(x_0) &= (b(x) - b(z)) + (b(z) - b(x_0)) \geq c\|x - z\|_X + (b(z) - b(x_0)) \\ &\geq c\|x - x_0\|_X - c\|x_0 - z\|_X + b(z) - b(x_0) \\ &\geq \frac{c}{2}\|x - x_0\|_X + \left\{\frac{c}{2}\|x - x_0\|_X - c\|x_0 - z\|_X + b(z) - b(x_0)\right\} \end{aligned}$$

when $x \in X \setminus B_X(z, C)$. Observe that the term in curly braces becomes positive when $\|x - x_0\|_X$ is sufficiently large, say larger than some $D > 0$. We therefore put $c_{x_0} := \frac{c}{2}$ and $C_{x_0} > D$ so large that $B_X(z, C) \subset B_X(x_0, C_{x_0})$. □

Notation 5.4. We will denote $\mathcal{Y}$ some set of convex, Lipschitz functions from X to $[0, +\infty)$ such that

(i) $\mathcal{Y}$ is a complete positive cone under the norm

$$\|g\|_{\mathcal{Y}} = g(0) + \sup\left\{\frac{|g(x) - g(y)|}{\|x - y\|_X} : x, y \in X,\, x \neq y\right\}$$

(ii) $\mathcal{Y}$ contains some convex separating function b

(iii) if $g \in \mathcal{Y}$ then $g(a\cdot) \in \mathcal{Y}$ for all $a > 0$, $g - \inf g(X) \in \mathcal{Y}$, and $\tau_y g \in \mathcal{Y}$ for all $y \in X$ where $\tau_y g(x) = g(x - y)$.

Traditionally, for a norm $\|\cdot\|$ on X with a certain smoothness, $\sqrt{1 + \|\cdot\|^2}$ is a Lipschitz convex separating function with the same smoothness. Thus we may take for $\mathcal{Y}$

(1) all convex, Lipschitz functions from X to $[0, +\infty)$,

(2) all convex, Lipschitz functions from X to $[0, +\infty)$ which are moreover Gâteaux differentiable, provided $\|\cdot\|_Y$ is Gâteaux differentiable

(3) all convex, Lipschitz functions from X to $[0, +\infty)$ which are moreover Fréchet differentiable, provided $\|\cdot\|_Y$ is Fréchet differentiable.

Remark 5.5. In fact, if there exists a Fréchet (resp. continuous and Gâteaux) differentiable convex separating function b defined in X, then X admits a Fréchet (resp. Gâteaux) differentiable norm.

Indeed, let us assume without loss of generality that $C = \{x \in X : b(x) < 1\}$ is nonempty, bounded and symmetric. In particular C contains 0. Since b is continuous, the set C is open. Let $|||\cdot|||$ be defined as the Minkowski functional of C (see Definition 4.5). It follows that $|||\cdot|||$ is an equivalent norm on X. Let us denote $b'_x \in X^*$ the Gâteaux derivative of b at $x \in X$. Let $x \in X$ be such that $b(x) = 1$. First observe, b'_x separates $\overline{C}$ from x. This is easy since $\{b'_x\} = \partial b(x)$ and by the definition of the subdifferential one has $b(y) - b(x) \geq b'_x(y) - b'_x(x)$ for all $y \in X$. So $b'_x(x) \geq b'_x(y) + b(x) - b(y) \geq b'_x(y)$ for all $y \in \overline{C}$. It is immediate that $t_x := \frac{b'_x}{b'_x(x)}$ is a tangent to $B_{(X,|||\cdot|||)} = C$ at such a point x, i.e. $t_x \in \partial\, |||x|||$. Really, $1 = t_x(x) = \sup\left\{\left|\frac{b'_x(y)}{b'_x(x)}\right| : y \in C\right\}$. We claim that t_x is the Fréchet (resp. Gâteaux) derivative of $|||\cdot|||$ at any x such that $|||x||| = 1$. In order to prove the claim we further define t_x for all $x \in X \setminus \{0\}$ by $t_x := |||x|||\, t_{x/|||x|||}$. By the homogeneity, t_x is a tangent to $|||x|||C$ at x. If b is Fréchet (resp. Gáteaux) differentiable, then the mapping $x \mapsto b'_x$ is norm-to-norm (resp. norm-to-weak*) continuous. It follows that the mapping $x \mapsto t_x$ is a norm-to-norm (resp. norm-to-weak*) continuous selection from the subdifferential $\partial\, |||\cdot|||$ so Proposition 2.8 in [Phe93] gives the conclusion.

If Π is a Hausdorff topological space, we denote $C(\Pi, \mathcal{Y})$ the positive cone of all continuous mappings from Π to $\mathcal{Y}$ together with the *fine* topology – the definition follows.

Definition 5.6 (cf. [Mun00])**.** The *fine topology* on $C(\Pi, \mathcal{Y})$ is the one generated by the neighborhoods of the form

$$B_{\text{fine}}(f, \delta) = \left\{g \in C(\Pi, \mathcal{Y}) : \|f(p) - g(p)\|_{\mathcal{Y}} < \delta(p) \text{ for every } p \in \Pi\right\}$$

where $f \in C(\Pi, \mathcal{Y})$, $\delta \in C(\Pi, (0, +\infty))$.

Lemma 5.7. *The fine topology on* $C(\Pi, \mathcal{Y})$ *is Baire.*

We include the standard proof for the sake of completeness.

Proof. In fact, the assertion is true whenever $\mathcal{Y}$ is a complete metric space. Let (G_n) be a sequence of dense open sets in $C(\Pi, \mathcal{Y})$ and let V by any open set in $C(\Pi, \mathcal{Y})$. We claim that there exist sequences $(f_n) \subset C(\Pi, \mathcal{Y})$ and $(\delta_n) \subset C(\Pi, (0, +\infty))$ such that

- $\overline{B_{\text{fine}}(f_{n+1}, \delta_{n+1})} \subset B_{\text{fine}}(f_n, \delta_n)$,
- $\sup_{p\in\Pi} \delta_n(p) \leq 1/n$
- and $\overline{B_{\text{fine}}(f_n, \delta_n)} \subset V \cap \bigcap_{i=1}^n G_i$.

Indeed, let us assume that we have constructed $f_1, \dots, f_n$ and $\delta_1, \dots, \delta_n$ with the above properties. Since G_{n+1} is dense and open we have $B_{\text{fine}}(f_n, \delta_n) \cap G_{n+1} \supset B_{\text{fine}}(f, 2\delta) \supset \overline{B_{\text{fine}}(f, \delta)} \supset B_{\text{fine}}(f, \delta)$ for some $f \in C(\Pi, \mathcal{Y})$, $\delta \in C(\Pi, (0, +\infty))$. Without loss of generality we may assume that $\sup \delta(\Pi) \le 1/(n+1)$ so clearly the above conditions are satisfied for $f_{n+1} := f$ and $\delta_{n+1} := \delta$.

Since $\mathcal{Y}$ is complete, this yields that $\lim f_n(p) = f(p)$ exists for every $p \in \Pi$. Moreover, f is a uniform limit of continuous functions f_n which makes it continuous itself and last but not least $f \in \overline{B_{\text{fine}}(f_n, \delta_n)} \subset \bigcap_{i=1}^{n} G_i \cap V$ for every $n \in \mathbb{N}$. Thus $\bigcap_{i=1}^{\infty} G_i$ is dense in $C(\Pi, \mathcal{Y})$. □

5.2 The existence of an approximate minimum

A principal step, common in the proof of all parametrized variational principles (cf. [Geo05, Ves09]), is the use of some variant of Michael's selection theorem.

Lemma 5.8 (Selection Lemma)**.** *Let Π be a paracompact Hausdorff topological space and $\varepsilon \in C(\Pi, (0, +\infty))$. Let $f : \Pi \times X \to (-\infty, +\infty]$ satisfy*

(a) *for every $p \in \Pi$, the function $f(p, \cdot)$ is proper, lower bounded and convex,*

(b) *for every $x \in X$, the function $f(\cdot, x)$ is u.s.c. from Π to $(-\infty, +\infty]$,*

(c) *the function $\inf f(\cdot, X)$ is l.s.c. from Π to $\mathbb{R}$.*

Then there is a continuous function $\varphi \in C(\Pi, X)$ such that $f(p, \varphi(p)) < \inf f(p, X) + \varepsilon(p)$.

Proof. For each $x \in X$ we define $U_x = \{p \in \Pi : f(p, x) < \inf f(p, X) + \varepsilon(p)\}$. By the assumptions (b) and (c), U_x is open. By the lower boundedness of $f(p, \cdot)$, the system $\{U_x\}_{x \in X}$ covers Π. Let $\{\psi_s\}_{s \in S}$ be a locally finite partition of unity subordinated to $\{U_x\}_{x \in X}$. For every $s \in S$ we find some U_x such that $\operatorname{supp} \psi_s \subset U_x$ and we define $x_s := x$. Now $\varphi(p) := \sum_{s \in S} \psi_s(p) x_s$ satisfies the required property. Indeed, $f(p, \varphi(p)) \le \sum_{s \in S} \psi_s(p) f(p, x_s) < \sum_{s \in S} \psi_s(p)(\inf f(p, X) + \varepsilon(p))$ where the first inequality follows from the convexity of $f(p, \cdot)$ and the second one from the fact that $p \in U_{x_s}$ if $\psi_s(p) \neq 0$. □

In order to verify the condition (c) of the previous lemma we look for certain sufficient conditions (see Proposition 5.11). One of them is the equi-lower semicontinuity which we define in such a fashion that allows us to handle the functions with extended values.

Definition 5.9. We say that a system $\{f_s : s \in S\}$ of functions from a topological space Π to $(-\infty, +\infty]$ is *equi-l.s.c. at* $p_0 \in \Pi$ if for every $a > 0$ and every $K > 0$ there exists an open neighborhood U of p_0 such that for all $p \in U$ either $f_s(p_0) - a < f_s(p)$, when $s \in S$ satisfies $f_s(p_0) < +\infty$, or $K < f_s(p)$, when $s \in S$ satisfies $f_s(p_0) = +\infty$. A system $\{f_s : s \in S\}$ is *equi-l.s.c.* if it is equi-l.s.c. at every $p_0 \in \Pi$.

Observe that when $\{f_s : s \in S\}$ is equi-l.s.c. at p_0, $\{g_s : s \in S\}$ is equi-l.s.c. at p_0, $-\infty < \inf_{s\in S} f_s(p_0)$ and $-\infty < \inf_{s\in S} g_s(p_0)$, then $\{f_s + g_s : s \in S\}$ is equi-l.s.c. at p_0.

Observe that, when all f_s are real-valued, we may equivalently say that $\{f_s : s \in S\}$ are equi-l.s.c. if for every p_0, $(f_s(p_0) - f_s(p))^+ \to 0$ uniformly with respect to $s \in S$ as $p \to p_0$.

Lemma 5.10. *Let $\{f_s : s \in S\}$ be an equi-l.s.c. system of functions from a topological space Π to $(-\infty, +\infty]$. Then $\inf_{s\in S} f_s$ is l.s.c.*

Proof. Let us fix $p_0 \in \Pi$. If $\inf_{s\in S} f_s(p_0) = +\infty$, the conclusion follows immediately from the definition, so we suppose that $\inf_{s\in S} f_s(p_0) < +\infty$. We choose $K > \inf_{s\in S} f_s(p_0)$ and $\varepsilon > 0$ arbitrarily. The equi-l.s.c. property provides an open neighborhood U of p_0 such that, for all $p \in U$, $f_s(p_0) - \varepsilon \le f_s(p)$ if $s \in S$ is such that $f_s(p_0) < +\infty$ and $\inf_{t\in S} f_t(p_0) < K < f_s(p)$ if $f_s(p_0) = +\infty$. Consequently, $\inf_{s\in S} f_s(p_0) - \varepsilon \le \inf_{s\in S} f_s(p)$. □

Let us abbreviate $f_\Delta(p, x)$ for $f(p, x) + \Delta(p)(x)$, when $f : \Pi \times X \to (-\infty, +\infty]$, $\Delta \in C(\Pi, \mathcal{Y})$, $p \in \Pi$ and $x \in X$.

Proposition 5.11. *Let Π be a Hausdorff topological space and let $f : \Pi \times X \to (-\infty, +\infty]$ satisfy* (M1), (M2), *i.e.*

(M1) *for every $p \in \Pi$ the function $f(p, \cdot)$ is proper, l.s.c., lower bounded,*

(M2) *for every $x \in X$ the function $f(\cdot, x)$ is a continuous function from Π to $(-\infty, +\infty]$ with its usual topology,*

and moreover

(A1) *for every $p \in \Pi$, $f(p, \cdot)$ is convex,*

(A2) *$\{f(\cdot, x) : x \in D\}$ is equi-l.s.c. whenever $D \subset X$ is bounded.*

Let us fix $\Delta \in C(\Pi, \mathcal{Y})$ and consider the following assertions

(i) *the set-valued mapping $D_\Delta(p) := \{x \in X : f_\Delta(p, x) < \inf(f_\Delta(p, X)) + 1\}$ is locally bounded;*

(ii) *the mapping $p \mapsto \inf f_\Delta(p, X)$ is continuous from Π to $\mathbb{R}$.*

Then (i) *implies* (ii), *and there is a dense set $A \subset C(\Pi, \mathcal{Y})$ such that* (i) *holds for every $\Delta \in A$.*

A key observation permitting to prove the proposition is in the following lemma.

Lemma 5.12. *Let $f : \Pi \times X \to (-\infty, +\infty]$ satisfy* (M1), (M2), (A1) *and* (A2). *Then for all $p_0 \in \Pi$, all $x_0 \in \mathrm{dom}(f(p_0, \cdot))$ and all $\varepsilon > 0$ there are an open neighborhood $V_\varepsilon^{x_0}(p_0)$ of p_0 and $r_\varepsilon^{x_0}(p_0) > 0$ with*

$$f(p, x) - f(p, x_0) \ge -\varepsilon \|x - x_0\|_X$$

for all $p \in V_\varepsilon^{x_0}(p_0)$ and all $\|x - x_0\|_X > r_\varepsilon^{x_0}(p_0)$.

Proof. Let $p_0 \in \Pi$, $x_0 \in \operatorname{dom}(f(p_0,\cdot))$ and $\varepsilon > 0$ be fixed. By (M2), in particular by the continuity of $f(\cdot,x_0)$, there is an open neighborhood U of p_0 such that $s := f(p_0,x_0)+1 > f(p,x_0)$ for all $p \in U$. Let us denote $q := \inf f(p_0,X)-1$ and let us choose a positive $R > 0$ such that $(s-q)/R < \varepsilon$. By the assumption (A2) there exists an open neighborhood V of p_0, $V \subset U$, such that $f(p,x) > q$ for all $p \in V$ and all $x \in B_X(x_0,R)$. Let $z \in X$, $\|z-x_0\|_X > R$. Set $x := x_0 + R\frac{z-x_0}{\|z-x_0\|_X}$. For $p \in V$ we have

$$\frac{f(p,z)-f(p,x_0)}{\|z-x_0\|_X} \geq \frac{f(p,x)-f(p,x_0)}{\|x-x_0\|_X} \geq \frac{q-s}{R} \geq -\varepsilon$$

where the first inequality follows from the convexity of $f(p,\cdot)$. Thus we set $V_\varepsilon^{x_0}(p_0) := V$ and $r_\varepsilon^{x_0}(p_0) := R$. □

Lemma 5.13. *Let $\Delta \in C(\Pi,\mathcal{Y})$ and D be a bounded subset of X. Then $\{\Delta(\cdot)(x) : x \in D\}$ are equi-continuous functions from Π to $\mathbb{R}$.*

Proof. Let us fix $p_0 \in \Pi$ and $\varepsilon > 0$. Since $\Delta \in C(\Pi,\mathcal{Y})$ there is an open neighborhood U of p_0 such that

$$\|\Delta(p_0)-\Delta(p)\|_{\mathcal{Y}} < \varepsilon \text{ for every } p \in U.$$

It follows that

$$|\Delta(p_0)(x)-\Delta(p)(x)| \leq |\Delta(p_0)(0)-\Delta(p)(0)| + \varepsilon\|x\|_X \leq (1+\|x\|_X)\varepsilon$$

so the set in question is equi-continuous at p_0. □

Proof of Proposition 5.11. We first prove that in fact (i) implies (ii). Let $p_0 \in \Pi$ be fixed. Then there is a neighborhood U of p_0 and a bounded set $E \subset X$ such that $D_\Delta(p) \in E$ for all $p \in U$. By the definition of D_Δ we have

$$\inf\{f_\Delta(p,x) : x \in X\} = \inf\{f_\Delta(p,x) : x \in E\} \tag{5.2}$$

for all $p \in U$. By Lemma 5.13 and by the assumption (A2), the functions $\{f_\Delta(\cdot,x) : x \in E\}$ are equi-l.s.c. Using Lemma 5.10 and (5.2), we conclude that $\inf f_\Delta(\cdot,X)$ is l.s.c. Clearly, $\inf f_\Delta(\cdot,X)$ is u.s.c. as an infimum of continuous functions.

We will now show that there are densely many $\Delta \in C(\Pi,\mathcal{Y})$ satisfying (i). Let $\Delta \in C(\Pi,\mathcal{Y})$ and $\varepsilon \in C(\Pi,(0,+\infty))$ be given. We put $h(p)(x) := b(x)\cdot\varepsilon(p)$ and $\Delta' := \Delta + h$ where $b \in \mathcal{Y}$ is a convex separating function, $\|b\|_{\mathcal{Y}} < 1$. It follows that $\|h(p)\|_{\mathcal{Y}} = \varepsilon(p)\|b\|_{\mathcal{Y}} < \varepsilon(p)$ for all $p \in \Pi$ thus $\Delta' \in B_{\text{fine}}(\Delta,\varepsilon)$. Recall that Lemma 5.3 insures that for each $x_0 \in X$ there exist $c_{x_0} > 0$ and $C_{x_0} > 0$ such that $b(x)-b(x_0) \geq c_{x_0}\|x-x_0\|_X$ for all $x \in X \setminus B_X(x_0,C_{x_0})$. We therefore have

$$h(p)(x)-h(p)(x_0) \geq \varepsilon(p)c_{x_0}\|x-x_0\|_X \tag{5.3}$$

for all $p \in \Pi$ and $\|x-x_0\|_X > C_{x_0}$. Now $D_{\Delta'}$ is locally bounded.

Indeed, let us fix $p_0 \in \Pi$ and $x_0 \in \operatorname{dom} f(p_0, \cdot)$. Then there are an open neighborhood U of p_0 and $\eta > 0$ such that $\inf \varepsilon(U) > 2\eta/c_{x_0}$. Since f_Δ satisfies (M1), (M2), (A1) and (A2), we apply Lemma 5.12 for f_Δ in order to obtain $V_\eta^{x_0}(p_0)$ and $r_\eta^{x_0}(p_0)$. Without loss of generality $V_\eta^{x_0}(p_0) \subset V$ and $r_\eta^{x_0}(p_0) > C_{x_0}$. This and (5.3) imply the estimate

$$f_{\Delta+h}(p, x) - f_{\Delta+h}(p, x_0) \geq \varepsilon(p) c_{x_0} \|x - x_0\|_X - \eta \|x - x_0\|_X \geq \eta \|x - x_0\|_X$$

for $p \in V_\eta^{x_0}(p_0)$ and $\|x - x_0\|_X > r_\eta^{x_0}(p_0)$. It follows that $D_{\Delta'}(p) \subset B_X(x_0, \max\{r_\eta^{x_0}(p_0), \frac{1}{\eta}\})$ when $p \in V_\eta^{x_0}(p_0)$. □

5.2.1 Interesting facts about convex functions

In the final part of this section we collect some interesting observations which help to understand the effect of the assumptions (M1), (M2), (A1) and (A2). These observations are not used in the proof of the variational principle but we will use them in Section 5.4. First is a corollary of Proposition 5.11.

Corollary 5.14. *Let* Π, X *and* f *be as in* Proposition 5.11 *and moreover* $\dim X < \infty$. *Let* $p_0 \in \Pi$ *be such that* $A = \{x \in X : f(p_0, x) = \inf f(p_0, X)\}$ *is bounded. Then the function* $\inf f(\cdot, X)$ *is continuous at* p_0.

The corollary, even if A is just a singleton, does not hold in the case $\dim X = \infty$. This can be seen in Example 5.25.

Proof. We may assume that $0 = \inf f(p_0, X)$ and $0 \in A \subset \frac{1}{2} B_X$. The lower semicontinuity of $f(p_0, \cdot)$ and the compactness of S_X yield that $\inf f(p_0, S_X) > a$ for some $a > 0$. We may find, using (M2) and (A2), a neighborhood U of p_0 such that $f(p, 0) < a/3$ and $\inf f(p, S_X) > 2a/3$ for all $p \in U$. The assumption (A1) then implies that $f(p, x) \geq \frac{a}{3} \|x\|_X$ for each $p \in U$ and each $\|x\|_X \geq 1$. In particular, the set-valued mapping $D_0(p) = \{x \in X : f(p, x) < \inf f(p, X) + 1\}$ is bounded at U. Applying Proposition 5.11 we get that $\inf f(\cdot, X)$ is continuous at p_0. □

Remark 5.15. Let Π be a metrizable space. If we suppose that f maps $\Pi \times X$ into $\mathbb{R}$, i.e. *it has no infinite value*, and satisfies (M1), (M2) and (A1), then $\{f(\cdot, x) : x \in K\}$ is equi-l.s.c. for any compact $K \subset X$. In particular, f satisfies (A2) automatically provided $\dim X < \infty$ and Π is metrizable.

Indeed, for any $p \in \Pi$, the function $f(p, \cdot)$ is continuous as it is convex, l.s.c. and with finite values (see [Phe93, Proposition 3.3]). Assume that $\{f(\cdot, x) : x \in K\}$ is not equi-l.s.c. for some compact $K \subset X$. Then, since Π is metrizable, there are $p \in \Pi$ and $(p_n) \subset \Pi$, $p_n \to p$, so that $f(p_n, \cdot)$ does not tend uniformly on K to $f(p, \cdot)$. But $f(p_n, \cdot) \to f(p, \cdot)$ pointwise on X by (M2) so we get a contradiction with the lemma below.

Lemma 5.16. *Let* f *and* f_n, $n \in \mathbb{N}$, *be real continuous convex functions defined on an open convex subset* V *of a Banach space* X *such that* $f_n \to f$ *pointwise on* V. *Then* $f_n \to f$ *uniformly on compact subsets of* V.

Proof. Let $K \subset V$ be a fixed compact. First we will show that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $x \in K$ one has $f(x) - \varepsilon < f_n(x)$, i.e. $(f - f_n)^+ \to 0$ uniformly on K, i.e. $\{f(\cdot, x) : x \in K\}$ is equi-l.s.c. where $f : (\mathbb{N} \cup \{\infty\}) \times V \to \mathbb{R}$ such that $f(n, x) = f_n(x)$ and $f(\infty, x) = f(x)$.

Let us assume that it is not true. Then there exist $\varepsilon > 0$ and a sequence $(x_n) \subset K$ such that, without loss of generality, for all $n \in \mathbb{N}$,

$$f_n(x_n) \leq f(x_n) - \varepsilon. \tag{5.4}$$

We may assume, by the compactness of K that $x_n \to x \in K$. Since $f_n \to f$ pointwise in V, we may use a Baire category argument to get a nonempty open set $U_{p,\varepsilon/4} \subset V$ such that for all $z \in U_{p,\varepsilon/4}$ and all $n > p$ we have

$$|f_n(z) - f(z)| < \frac{\varepsilon}{4}. \tag{5.5}$$

Further, by the compactness of K and the continuity of f, there exist $\lambda \in (\frac{3}{4}, 1)$ and a nonempty open subset U of $U_{p,\varepsilon/4}$ such that for all $a \in K$, $b \in U$,

$$f(\lambda a + (1 - \lambda)b) - (\lambda f(a) + (1 - \lambda)f(b)) > -\frac{\varepsilon}{4}. \tag{5.6}$$

This requires a proof: fix any $b' \in U_{p,\varepsilon/4}$ and observe that $F(x, y, \lambda) := f(\lambda x - (1 - \lambda)y) - (\lambda f(x) + (1 - \lambda)f(y))$ is continuous on $K \times V \times [0, 1]$. Further $F(a, b', 1) = 0$ for all $a \in K$. So for each $a \in K$ there are a neighborhood U_a of a, a neighborhood V_a of b' and an interval $I_a = (\lambda_a, 1]$ such that $F(x, y, \lambda) > -\varepsilon/4$ for all $(x, y, \lambda) \in U_a \times V_a \times I_a$. The system $\{U_a : a \in K\}$ is an open cover for K. We find its finite open subcover $\{U_{a_1}, \dots, U_{a_k}\}$ and define $U := V_{a_1} \cap \dots \cap V_{a_k}$ and $\lambda \in (\frac{3}{4}, 1)$ such that $\lambda > \lambda_{a_1}, \dots, \lambda_{a_k}$. The set U is nonempty and open as a finite intersection of open neighborhoods of b' and obviously satisfies our claim.

It is possible to find $\tilde{x} \in V$ such that for any n sufficiently large there are $z_n \in U$ such that

$$\tilde{x} = \lambda x_n + (1 - \lambda)z_n.$$

Indeed, choose any $z \in U$ and set $\tilde{x} := \lambda x + (1 - \lambda)z$, $z_n := \frac{\tilde{x} - \lambda x_n}{1 - \lambda}$. Hence by (5.6) we have $f(\tilde{x}) + \frac{\varepsilon}{4} > \lambda f(x_n) + (1 - \lambda)f(z_n)$. It follows

$$\begin{aligned} f_n(\tilde{x}) &\leq \lambda f_n(x_n) + (1 - \lambda)f_n(z_n) \quad \text{by the convexity of } f_n \\ &\leq \lambda(f(x_n) - \varepsilon) + (1 - \lambda)(f(z_n) + \frac{\varepsilon}{4}) \quad \text{from (5.4) and (5.5)} \\ &\leq f(\tilde{x}) + \frac{\varepsilon}{4} - \lambda\varepsilon + (1 - \lambda)\frac{\varepsilon}{4} \leq f(\tilde{x}) - \frac{\varepsilon}{4} \end{aligned}$$

which contradicts $f_n(\tilde{x}) \to f(\tilde{x})$. So we have $(f - f_n)^+ \to 0$ uniformly on K.

On the other hand, if we set $F_n(x) := \sup\{f_m(x) : m \geq n\}$ for $y \in V$, we have that F_n is a convex, lower semicontinuous function as the supremum of such functions and $F_n \searrow f$ pointwise. Hence F_n is real-valued. We may use Proposition 3.3 in [Phe93] to see that F_n is in fact continuous on V. By Dini's theorem, $F_n \to f$ uniformly on K thus $(f_n - f)^+ \to 0$ uniformly on K. □

Remark 5.17. Let us weaken the assumptions of the lemma in the following way. Let $K \subset X$ be a convex compact set. Let f and f_n, for $n \in \mathbb{N}$, be continuous and convex on K such that $f_n \to f$ pointwise on K. Then these assumptions are not enough to prove that $f_n \to f$ uniformly on K. Indeed, let $X = \ell_1$, $K = \overline{\text{co}}\left\{\frac{e_n}{n}\right\}$ (where (e_n) is the unit vector basis) and $f_n(x) = -nx(n)$. Then $f_n \to 0 =: f$ pointwise, but not even $(f - f_n)^+$ tends to 0 uniformly on K. To be sure, let $x \in K$ and let us prove that $f_n(x) \to 0$. By Choquet's theorem [FHH+01] there exists a probability measure μ_x on K with $\mu_x(\text{Ext}K) = 1$ such that for all $f \in (\ell_1)^*$ one has $f(x) = \int_K f(z) d\mu_x(z)$. Since $\frac{e_n}{n} \to 0$, one may see (with the help of Milman's theorem [FHH+01]) that $\text{Ext}K = \left\{\frac{e_n}{n}\right\} \cup \{0\}$. Let us denote $z_i := \frac{e_i}{i}$ for $i \in \mathbb{N}$ and $z_0 := 0$. The probability measure μ_x is therefore nothing else than a sequence $(\lambda_i)_{i=0}^\infty$ of positive numbers such that $\sum \lambda_i = 1$ where $\mu_x(z_i) = \lambda_i$ for $i \in \mathbb{N} \cup \{0\}$. We evaluate $f_n(x) = \int_K f_n(z) d\mu_x(z) = \sum_{i=0}^\infty \lambda_i f_n(z_i) = -\lambda_n \to 0$ as $n \to \infty$.

5.3 Parametric variational principle

Recall that a function $h : X \to (-\infty, +\infty]$ attains a *strong minimum* at a point $x \in X$ if it attains minimum at the point x and every minimizing sequence converges to x, i.e. for any sequence $(x_n) \subset X$ one has that $h(x_n) \to h(x)$ implies $x_n \to x$.

Theorem 5.18. *Let Π be a paracompact Hausdorff topological space. Let $f : \Pi \times X \to (-\infty, +\infty]$ satisfy* (M1), (M2), (A1) *and* (A2), *i.e.*

(M1) *for every $p \in \Pi$ the function $f(p, \cdot)$ is proper, l.s.c., lower bounded,*

(M2) *for every $x \in X$ the function $f(\cdot, x)$ is a continuous function from Π to $(-\infty, +\infty]$ with its usual topology,*

(A1) *for every $p \in \Pi$, $f(p, \cdot)$ is convex,*

(A2) *$\{f(\cdot, x) : x \in D\}$ is equi-l.s.c. whenever $D \subset X$ is bounded.*

Then the set

$$\mathcal{M} = \Big\{ \Delta \in C(\Pi, \mathcal{Y}) : \text{ there is } v \in C(\Pi, X) \text{ such that } f(p, \cdot) + \Delta(p) \text{ attains its strong minimum at } v(p) \text{ for all } p \in \Pi \Big\}$$

is residual in $C(\Pi, \mathcal{Y})$. Moreover, if $\Delta \in \mathcal{M}$, then $p \mapsto \inf f_\Delta(p, X)$ is continuous.

In particular, for every $\varepsilon \in C(\Pi, (0, +\infty))$, there are $\Delta \in C(\Pi, \mathcal{Y})$ and $v \in C(\Pi, X)$ such that $\|\Delta(p)\|_{\mathcal{Y}} < \varepsilon(p)$ and $f(p, \cdot) + \Delta(p)$ attains its strong minimum at $v(p)$ for all $p \in \Pi$.

We remind that we are abbreviating $f_\Delta(p, x) := f(p, x) + \Delta(p)(x)$ whenever $\Delta \in C(\Pi, \mathcal{Y})$, $p \in \Pi$ and $x \in X$. For the proof we will need one last elementary lemma.

Lemma 5.19. *Let* Π *be a paracompact topological space,* $\phi : \Pi \to \mathbb{R}$ *be locally bounded. Then there exists a continuous function* $\varphi : \Pi \to (0, +\infty)$ *such that* $|\phi(p)| < \varphi(p)$ *for all* $p \in \Pi$.

Proof. For every $p \in \Pi$ we find an open set $U_p \ni p$ and a constant c_p such that $\phi(U_p) \subset (-c_p, c_p)$. Since Π is paracompact we may find a locally finite partition of unity $\{\psi_s\}_{s\in S}$ subordinated to the open cover $\{U_p\}$ of Π. For every $s \in S$ we define $c_s := c_p$ for some $p \in \Pi$ such that $\operatorname{supp} \psi_s \subset U_p$. The function $\varphi(p) := \sum_{s\in S} \psi_s(p) c_s$ then satisfies the required property. □

Proof of Theorem 5.18. Let us consider, for every $n \in \mathbb{N}$, the set

$$U_n = \Bigg\{ \Delta \in C(\Pi, \mathcal{Y}) : \text{ there are } v_n \in C(\Pi, X) \text{ and } \delta \in C(\Pi, (0, +\infty)) \text{ such that} \\ f_\Delta(p, v_n(p)) + \delta(p) < \inf \left\{ f_\Delta(p, x) : \|x - v_n(p)\|_X \geq \frac{1}{n} \right\} \text{ for all } p \in \Pi \Bigg\}.$$

Claim. U_n is open in $C(\Pi, \mathcal{Y})$.

Let $p \in \Pi$ be fixed and let us abbreviate $f = f(p, \cdot)$. Let $g_1 \in \mathcal{Y}$ satisfy

$$f(v_n) + g_1(v_n) + \delta < \inf \left\{ f(x) + g_1(x) : \|x - v_n\|_X \geq \frac{1}{n} \right\}$$

for some $v_n \in X$ and for some $\delta > 0$. Let $g_2 \in \mathcal{Y}$ such that $\|g_1 - g_2\|_{\mathcal{Y}} \leq \frac{\delta \cdot n}{2}$. Then $g_1(v_n) - g_2(v_n) - g_1(z) + g_2(z) \geq -\frac{\delta \cdot n}{2} \cdot \frac{1}{n}$ for any $z \in S_X(v_n, \frac{1}{n})$. Hence

$$\begin{aligned} f(z) + g_2(z) &\geq f(z) + g_1(z) + g_2(v_n) - g_1(v_n) - \frac{\delta}{2} \\ &\geq f(v_n) + g_1(v_n) + \delta + g_2(v_n) - g_1(v_n) - \frac{\delta}{2} \\ &\geq f(v_n) + g_2(v_n) + \frac{\delta}{2} \end{aligned}$$

Since $f + g_2$ is convex, it follows that for any $z \in X$, $\|z - v_n\|_X \geq \frac{1}{n}$

$$f(z) + g_2(z) \geq f(v_n) + g_2(v_n) + \frac{\delta}{2}.$$

Now if $\Delta_1 \in U_n$ with $v_n \in C(\Pi, X)$ and $\delta \in C(\Pi, (0, +\infty))$ and $\Delta_2 \in B_{\text{fine}}(\Delta_1, \frac{n\delta}{2})$ then $\Delta_2 \in U_n$ (with the same v_n and with $\delta/2$), so U_n is open.

Claim. The set U_n is dense in $C(\Pi, \mathcal{Y})$.

Let $\Delta \in C(\Pi, \mathcal{Y})$ and $\varepsilon \in C(\Pi, (0,1))$. We need to find $\Delta' \in C(\Pi, \mathcal{Y})$, $\Delta' \in B_{\text{fine}}(\Delta, \varepsilon)$, $\delta \in C(\Pi, (0, +\infty))$ and $v_n \in C(\Pi, X)$ such that

$$f_{\Delta'}(p, v_n(p)) + \delta(p) < \inf \left\{ f_{\Delta'}(p, x) : \|x - v_n(p)\|_X \geq \frac{1}{n} \right\}$$

for every $p \in \Pi$. Thanks to Proposition 5.11 it is enough to consider such Δ that the function $p \mapsto \inf f_\Delta(p, X)$ is l.s.c. and

$$D_\Delta(p) = \{x \in X : f_\Delta(p, x) < \inf f_\Delta(p, X) + 1\}$$

is locally bounded. Let $b \in \mathcal{Y}$ be a convex separating function such that

(B1) $\|b\|_{\mathcal{Y}} < 2$ and for all $y \in X$, $\|\tau_y b\|_{\mathcal{Y}} \geq 1$,

(B2) $b(0) + \frac{1}{n} \leq \inf \left\{ b(x) : \|x\|_X \geq \frac{1}{n} \right\}$.

Let $T : \Pi \to (0, +\infty)$ be defined as $T(p) := \sup \left\{ \|\tau_y b\|_{\mathcal{Y}} : y \in D_\Delta(p) \right\}$. Then T is locally bounded. Let φ be the continuous function that comes from Lemma 5.19 and satisfies $T(p) \leq \varphi(p)$. We may assume that $\varphi \geq 1$ and use Selection Lemma 5.8 to find $v_n \in C(\Pi, X)$ such that

$$f_\Delta(p, v_n(p)) < \inf \{ f_\Delta(p, x) : x \in X \} + \frac{\varepsilon(p)}{4n\varphi(p)}$$

for every $p \in \Pi$. Since the fraction above is smaller than 1, we infer that $v_n(p) \in D_\Delta(p)$ and consequently $\left\|\tau_{v_n(p)} b\right\|_{\mathcal{Y}} \leq \varphi(p)$. We define

$$h(p)(x) := \frac{b(x - v_n(p)) \cdot \varepsilon(p)}{2 \left\|\tau_{v_n(p)} b\right\|_{\mathcal{Y}}}.$$

It is obvious that $h \in C(\Pi, \mathcal{Y})$ and $\|h(p)\|_{\mathcal{Y}} < \varepsilon(p)$ for all $p \in \Pi$, thus $\Delta' \in B_{\text{fine}}(\Delta, \varepsilon)$ for Δ' defined as $\Delta' := \Delta + h$. It follows that for every $p \in \Pi$

$$\begin{aligned} f(p, v_n(p)) + \Delta'(p)(v_n(p)) &= f_\Delta(p, v_n(p)) + h(p)(v_n(p)) \\ &< \inf f_\Delta(p, X) + h(p)(v_n(p)) + \frac{\varepsilon(p)}{4n\varphi(p)}. \end{aligned} \tag{5.7}$$

If $p \in \Pi$, $x \in X$ and $\|x - v_n(p)\|_X \geq \frac{1}{n}$, then by (B2) and the definition of h we get

$$h(p)(x) \geq h(p)(v_n(p)) + \frac{\varepsilon(p)}{2n \left\|\tau_{v_n(p)} b\right\|_{\mathcal{Y}}}$$

which we use immediately in the following estimate

$$\begin{aligned} f(p, x) + \Delta'(p)(x) &= f_\Delta(p, x) + h(p)(x) \\ &\geq \inf f_\Delta(p, X) + h(p)(x) \\ &\geq \inf f_\Delta(p, X) + h(p)(v_n(p)) + \frac{\varepsilon(p)}{4n\varphi(p)} + \delta(p) \end{aligned} \tag{5.8}$$

where

$$\delta(p) = \frac{\varepsilon(p)}{2n}\left(\frac{1}{\left\|\tau_{v_n(p)}b\right\|_{\mathcal{Y}}} - \frac{1}{2\varphi(p)}\right) > 0.$$

Combining (5.7) and (5.8) we conclude that $\Delta' \in U_n$ which shows that U_n is a dense part of $C(\Pi, \mathcal{Y})$ and the proof of the claim is finished.

Consequently, by Lemma 5.7, $\bigcap_{n\in\mathbb{N}} U_n$ is a dense G_δ-subset of $C(\Pi, \mathcal{Y})$.

Claim. $\bigcap U_n \subset \mathcal{M}$; i.e. if $\Delta \in \bigcap U_n$, then there is $v \in C(\Pi, X)$ such that $f_\Delta(p, \cdot)$ attains its strong minimum at $v(p)$ for every $p \in \Pi$.

Indeed, for each $n \in \mathbb{N}$, let $v_n \in C(\Pi, X)$ be such that

$$f_\Delta(p, v_n(p)) < \inf\left\{f_\Delta(p, x) : \|x - v_n(p)\|_X \geq \frac{1}{n}\right\}.$$

Clearly for every $p \in \Pi$, $\|v_m(p) - v_n(x)\|_X < \frac{1}{n}$ if $m \geq n$ (otherwise, by the choice of v_n, we would have $f_\Delta(p, v_m(p)) > f_\Delta(p, v_n(p))$ as well as, by the choice of v_m, we have the opposite strict inequality for every $p \in \Pi$, which is a contradiction). Therefore (v_n) is Cauchy in $C(\Pi, X)$ (with the norm $\|\cdot\|_\infty$ from $C_b(\Pi, X)$) and it converges to some $v \in C(\Pi, X)$. Let us fix $p \in \Pi$ and use the lower semicontinuity of $f(p, \cdot)$

$$\begin{aligned} f_\Delta(p, v(p)) &\leq \liminf_{n\to\infty} f_\Delta(p, v_n(p)) \\ &\leq \liminf_{n\to\infty} \left[\inf\left\{f_\Delta(p, x) : \|x - v_n(p)\|_X \geq \frac{1}{n}\right\}\right] \\ &\leq \inf\left\{f_\Delta(p, x) : x \in X \setminus \{v(p)\}\right\}. \end{aligned}$$

So $v(p)$ is a point of minimum for $f_\Delta(p, \cdot)$. To see that the minimum attained at $v(p)$ is strong, assume that $(z_n) \subset X$ is a sequence in X such that $f_\Delta(p, z_n) \to f_\Delta(p, v(p))$ but $z_n \nrightarrow v(p)$. For some subsequence of (z_n) which we will call again (z_n) and for some $p \in \mathbb{N}$ we have $\|z_n - v_p(p)\|_X \geq 1/p$ for all $n \in \mathbb{N}$. Consequently

$$\begin{aligned} f_\Delta(p, v(p)) \leq f_\Delta(p, v_p(p)) &< \inf\left\{f_\Delta(p, x) : \|x - v_p(p)\|_X \geq 1/p\right\} \\ &\leq f_\Delta(p, z_n) \end{aligned}$$

for all $n \in \mathbb{N}$ which is contradictory to $f_\Delta(p, z_n) \to f_\Delta(p, v(p))$. So the proof of the claim is finished.

Finally, let $\Delta \in \mathcal{M}$ and $p_0 \in \Pi$ be fixed. We will show that $p \mapsto \inf f_\Delta(p, X)$ is continuous at p_0. Indeed, let $v \in C(\Pi, X)$ such that $f_\Delta(p, v(p)) = \inf f_\Delta(p, X)$. There are an open neighborhood U of p_0 and $k > 0$ such that $v(p) \in kB_X$ for all $p \in U$. By (A2), $\{f_\Delta(\cdot, x) : x \in kB_X\}$ is equi-l.s.c. so by Lemma 5.10 $\inf f_\Delta(\cdot, kB_X) = \inf f_\Delta(\cdot, X)$ is l.s.c. at p_0. It is obviously also u.s.c. as the infimum of u.s.c. functions. □

The following corollary shows that it is possible to localize the points where the minimum is attained. We also include the possibility of not perturbing the function $f(p, \cdot)$

for p in a certain closed subspace Π_0 of Π. So the corollary actually generalizes a result of Veselý [Ves09, Theorem 4.1] (see also Remark 5.21 below) since it applies in particular when Π is metrizable and Π_0 is its closed subspace.

Corollary 5.20. *Let Π be a paracompact Hausdorff topological space and Π_0 its closed subspace so that $\Pi \setminus \Pi_0$ is paracompact. Let $f : \Pi \times X \to (-\infty, +\infty]$ be like in Theorem 5.18. Then for any continuous $\varepsilon : \Pi \to [0, 1)$ such that $\varepsilon^{-1}(0) = \Pi_0$ and any continuous mapping $v_0 : \Pi \to X$ with*

$$f(p, v_0(p)) \leq \inf f(p, X) + \varepsilon(p)^2 \text{ when } p \in \Pi \tag{5.9}$$

there are

$$v \in C(\Pi, X) \text{ and } \Delta \in C(\Pi, \mathcal{Y})$$

such that

(i) *$f(p, \cdot) + \Delta(p)$ attains its minimum at $v(p)$ for every $p \in \Pi$,*

(ii) *$\|v(p) - v_0(p)\|_X \leq \varepsilon(p)$ for every $p \in \Pi$,*

(iii) *$\|\Delta(p)\|_{\mathcal{Y}} \leq \frac{2\varepsilon(p)}{c}(\|v_0(p)\|_X + 1) + \varepsilon(p)^2$ for some constant $c > 0$ which depends only on $\mathcal{Y}$.*

Proof. Let $b \in \mathcal{Y}$ be a separating convex function which moreover satisfies $b(0) = 0$, $\|b\|_{\mathcal{Y}} = 1$ and $b \geq c$ outside B_Y for some $c > 0$. The existence of such b is an immediate consequence of conditions posed on $\mathcal{Y}$ and Theorem 5.18 with $\Pi = \{1\}$ and $f(1, \cdot)$ any convex separating function (possibly without minimum). The assumptions on b imply $c \leq 1$ and $\|b_r\|_{\mathcal{Y}} = \frac{1}{r}$ for $r > 0$ where b_r is defined by $b_r(z) := b(z/r)$.

Let us work only on the paracompact space $\Pi \setminus \Pi_0$. Observe that

$$\left\|b_{\varepsilon(p)}\right\|_{\mathcal{Y}} = \frac{1}{\varepsilon(p)}.$$

We define

$$h(p)(x) := \frac{2\varepsilon(p)^2}{c} b_{\varepsilon(p)}(x - v_0(p))$$

so $\|h(p)\|_{\mathcal{Y}} \leq \frac{2\varepsilon(p)}{c}(\|v_0(p)\|_X + 1)$. By Theorem 5.18, there exist $k \in C(\Pi \setminus \Pi_0, \mathcal{Y})$ and $v \in C(\Pi \setminus \Pi_0, X)$ such that $\|k(p)\|_{\mathcal{Y}} < \varepsilon(p)^2$ and $f(p, \cdot) + h(p) + k(p)$ attains its minimum at $v(p)$. We define $\Delta = h + k$. The condition (i) is satisfied. Further we have $\|\Delta(p)\|_{\mathcal{Y}} \leq \|h(p)\|_{\mathcal{Y}} + \|k(p)\|_{\mathcal{Y}} \leq \frac{2\varepsilon(p)}{c}(\|v_0(p)\|_X + 1) + \varepsilon(p)^2$. Further, since $h(x, v_0(p)) = 0$ and from (5.9),

$$\begin{aligned} f(p, v_0(p)) + \Delta(p)(v_0(p)) &= f(p, v_0(p)) + k(p)(v_0(p)) \\ &\leq \inf f(p, X) + \varepsilon(p)^2 + k(p)(v_0(p)) \end{aligned}$$

while

$$\begin{aligned} f(p,x) + \Delta(p)(x) &\geq \inf f(p,X) + \frac{2\varepsilon(p)^2 \cdot c}{c} + k(p)(v_0(p)) - \varepsilon(p)\varepsilon(p)^2 \\ &> \inf f(p,X) + \varepsilon(p)^2 + k(p)(v_0(p)) \end{aligned}$$

for $p \in \Pi \setminus \Pi_0$ and $x \in X$ such that $\|x - v_0(p)\|_X = \varepsilon(p)$. From the convexity of $f(p,\cdot) + \Delta(p)$ it follows that $\|v(p) - v_0(p)\|_X < \varepsilon(p)$.

We define $\Delta \restriction_{\Pi_0} = 0$ and $v \restriction_{\Pi_0} = v_0$. □

Remark 5.21. Let us point out the most important differences with respect to the parametrized variational principles [Geo05, Theorem 3.1] and [Ves09, Theorem 1.3].
a) In both of the cited theorems, the function f has to satisfy the following condition:

$$\inf f(\cdot, X) \text{ is locally lower bounded.}$$

We have suppressed completely this assumption in our theorem. Examples 5.24 and 5.25 show functions f which do not meet the above condition, but our Theorem 5.18 still applies for them. We remark that using the respective part of our proof (Lemma 5.12), this condition could be removed also from both cited theorems.
b) If $\mathrm{dom}(f(p,\cdot)) = D$ for all $p \in \Pi$, we are in the setting of Veselý. We show in Example 5.29 that Thoeorem 5.18 goes beyond this setting.
c) We use Lipschitz functions as perturbations while [Geo05, Ves09] use functions of the form $\sum \nu_n(p) \|x - x_n(p)\|_X^2$, therefore we are "perturbing less" the original function f.

Observe that our main theorem stays valid, if we assume the following alternative to Notation 5.4. The space of the above functions from [Geo05, Ves09] already fits in this more general framework.

The set $\mathcal{Y}$ is a complete (with respect to some norm $\|\cdot\|_{\mathcal{Y}}$) cone of convex continuous functions from X to $[0, +\infty)$ which satisfies:

(i) for every bounded subset C of X there exists a constant $M_C > 0$ such that $\sup g(C) \leq M_C \|g\|_{\mathcal{Y}}$ for all $g \in \mathcal{Y}$;

(ii) $\mathcal{Y}$ contains some convex separating function b;

(iii) if $g \in \mathcal{Y}$, then $g(a\cdot) \in \mathcal{Y}$ for all $a > 0$, $g - \inf g(X) \in \mathcal{Y}$, and $\tau_y g \in \mathcal{Y}$ for all $x \in X$ with $y \mapsto \|\tau_y g\|_{\mathcal{Y}}$ continuous.

d) Corollary 5.20 does not cover the situation $\Pi = [0, \omega_1]$, $\Pi_0 = \{\omega_1\}$ covered by Theorem 1.3 of [Ves09]. A version of the corollary where we replace the assumption "$\Pi \setminus \Pi_0$ is paracompact" by the assumption "Π_0 is discrete" is needed. In order to prove such a version, we can use Lemma 1.2 of [Ves09] in the proof of Theorem 5.18 instead of our Lemma 5.8 and replace the space $C(\Pi, \mathcal{Y})$ by the space $C_{\Pi_0}(\Pi, \mathcal{Y}) = \{\Delta \in C(\Pi, \mathcal{Y}) : \Delta(p)$ has a minimum at $v_0(p)$ for every $p \in \Pi_0\}$.
e) Let us say that $f : \Pi \times X \to (-\infty, +\infty]$ attains a locally uniformly strong minimum (l.u.s.m.) at $v \in C(\Pi, X)$ if a) $f(p, v(p)) = \inf f(p, X)$ for all $p \in \Pi$, and b) for each

$p_0 \in \Pi$ and each $\varepsilon > 0$ there are $\delta > 0$ and an open neighborhood U of p_0 such that for all $p \in U$ and all $x \in X$ the following implication holds true

$$f(p, x) - \inf f(p, X) < \delta \Rightarrow \|x - v(p_0)\| < \varepsilon.$$

A closer inspection of the proof of Theorem 5.18 reveals that the G_δ dense set $\bigcap U_n$ equals $\{\Delta \in C(\Pi, \mathcal{Y}) : \text{there is } v \in C(\Pi, X) \text{ such that } f_\Delta \text{ attains a l.u.s.m. at } v\}$.

5.4 Examples

The functions described in the following proposition will be a prototype for some of our examples.

Proposition 5.22. *Let $\Pi = X^*$ and let us consider $f : \Pi \times X \to (-\infty, +\infty]$ of the form $f(p, x) = g(x) - p(x)$ for $x \in X$ and $p \in \Pi$, where g is a proper, l.s.c. and lower bounded function from X to $(-\infty, +\infty]$ which satisfies*

$$\lim_{x\to\infty} \frac{g(x)}{\|x\|_X} = +\infty. \tag{5.10}$$

Then f satisfies (M1), (M2) *and* (A2). *If g is convex, then f satisfies also* (A1). *In contrast with* (A2), *$\{f(\cdot, x) : x \in X\}$ is equi-l.s.c. if and only if* $\operatorname{dom}(g)$ *is bounded.*

Of course, if we assume that g is proper, l.s.c., convex and satisfies (5.10), then g is automatically lower bounded.

Proof. The lower boundedness of $f(p, \cdot)$ is implied by (5.10). Everything else is trivial. □

In the first two examples, we will show that the parametric variational principle is still needed even in the spaces which are notorious for having no lower bounded, l.s.c., convex and coercive functions without a minimum, such as the reflexive, Hilbert and finite dimensional spaces. In other words, even if $f(p, \cdot)$ attains its minimum for every $p \in \Pi$, there does not have to necessarily exist $v \in C(\Pi, X)$ such that $f(p, \cdot)$ attains the minimum at $v(p)$. This shows that $\mathcal{Y}$ should be reasonably rich.

Example 5.23. Let X be reflexive, $\Pi = X^*$, and let us define

$$g(x) = \begin{cases} 0, & \|x\|_X \le 1, \\ \|x\|_X^2 - 1, & \|x\|_X > 1 \end{cases}$$

and $f(p, x) = g(x) - p(x)$. Then g is convex and it satisfies (5.10) so f satisfies (M1), (M2), (A1) and (A2) but every function $v : \Pi \to X$ with $f(p, v(p)) = \inf f(p, X)$ is discontinuous at 0.

Proof. Let $p \in B_{X^*} \setminus \{0\}$. If we denote $M_p = \{x \in X : f(p,x) = \inf f(p,X)\}$, then $\emptyset \neq M_p$ is a closed subset of S_X and it is not difficult to see that $M_p \cap M_{-p} = \emptyset$ and $M_p = M_{\lambda p}$, for $0 < \lambda < 1/\|p\|_{X^*}$. This obviously contradicts continuity of v at 0. $\square$

In the next example, we examine the continuity of the function $p \mapsto \inf f(p,X)$.

Example 5.24. Let $\Pi = [0,+\infty)$, $X = \mathbb{R}$ and

$$f(p,x) = \begin{cases} 0, & p = 0 \\ \left|px - \frac{1}{p}\right| - \frac{1}{p}, & p \neq 0. \end{cases}$$

Then f satisfies (M1), (M2), (A1) and (A2), but

$$\inf f(p,X) = \begin{cases} 0, & p = 0 \\ -\frac{1}{p}, & p \neq 0 \text{ (attained uniquely at } x = \frac{1}{p^2}). \end{cases}$$

Obviously $p \mapsto \inf f(p,X)$ is not locally lower bounded at $p = 0$ (cf. Remark 5.21 **a)**) so the theorems of Georgiev and Veselý do not apply in this case. Observe, that after application of Theorem 5.18, $\inf f_\Delta(\cdot,X)$ is already continuous.

The previous example may be modified in such a way that $f(p,\cdot)$ attains a strict minimum for each $p \in \Pi$. Since Corollary 5.14 makes it impossible to construct such an example if $\dim X < \infty$, we construct it in an infinite dimensional setting.

Example 5.25. Let $\Pi = [0,1]$ and $X = L^\infty[0,1]$. Let χ_A be the characteristic function of a set A, i.e. $\chi_A(z) = 1$ if $z \in A$, otherwise $\chi_A(z) = 0$. Let us define functions

$$m : [0,1] \times \mathbb{R} \to \mathbb{R} \qquad\qquad n : [0,1] \times \mathbb{R} \to \mathbb{R}$$
$$m(t,x) := t\,|x| \qquad\qquad n(t,x) := t\left(\left|x - t^{-3}\right| - t^{-3}\right)$$

and operators

$$M : X \to X \qquad\qquad N : X \to X$$
$$M(x) := m(\cdot, x(\cdot)) \qquad\qquad N(x) := n(\cdot, x(\cdot)).$$

Further, we define mappings

$$F : \Pi \to L^1[0,1] \qquad\qquad G : \Pi \to L^1[0,1]$$
$$F(p) := \chi_{[0,\frac{p}{2}] \cup [p,1]} \qquad\qquad G(p) := \chi_{[\frac{p}{2},p]}$$

and finally

$$f(p,x) = \langle F(p), M(x) \rangle + \langle G(p), N(x) \rangle\,.$$

We claim that

(a) f satisfies (M1), (M2), (A1) and (A2);

(b) for each $p \in \Pi$ the function $f(p, \cdot)$ attains its strict minimum at some $v(p)$ and both $v(\cdot)$ and $\inf f(\cdot, X)$ are discontinuous at 0;

(c) the function $\inf f(\cdot, X)$ is not locally lower bounded at 0.

Proof. Since, for each $t \in [0,1]$, $m(t, \cdot)$ and $n(t, \cdot)$ are 1-Lipschitz, we have that M and N are continuous contractions from $L^\infty[0,1]$ to $L^\infty[0,1]$. On the other hand F and G are continuous. It follows, due to the duality $(L^1[0,1])^* = L^\infty[0,1]$, that $f(p, \cdot)$ is continuous for every $p \in \Pi$ and $\{f(\cdot, x) : x \in D\}$ is equi-continuous for every bounded $D \subset X$. Since $m(t, \cdot)$ and $n(t, \cdot)$ are convex for each $t \in [0,1]$, the function $f(p, \cdot)$ is convex for each $p \in \Pi$. This proves the claim (a) with the exception of the lower boundedness of $f(p, \cdot)$ – it will follow once we prove the claim (b).

Now, $\langle F(p), M(\cdot)\rangle$ attains a minimum at $x \in X$ if and only if $x(t) = 0$ for almost all $t \in [0, \frac{p}{2}] \cup [p, 1]$. Similarly $\langle G(p), N(\cdot)\rangle$ attains its minimum at $x \in X$ if and only if $x(t) = -t^{-3}$ for almost all $t \in [\frac{p}{2}, p]$. It follows that $f(p, \cdot)$ attains its minimum at x if and only if the two above conditions hold simultaneously – this identifies x uniquely, so the minimum is strict. In particular, $f(0, \cdot)$ attains the strict minimum at $v(0) = 0 \in X$ and the value of the minimum is 0 while $f(p, \cdot)$ attains its minimum at the uniquely determined $v(p)$ of the norm $\|v(p)\|_X = 8x^{-3}$ and $f(p, v(p)) = -p^{-1}$ which proves the claims (b) and (c). □

Our fourth example shows that Theorem 5.18 need not hold if we drop the convexity assumption (A1) on $f(p, \cdot)$.

Example 5.26. Let X be a Banach space and we put again $\Pi = X^*$. Let $a \in S_X$ be fixed. We define

$$g(x) := \inf\left\{\|x-a\|^2, \|x+a\|^2\right\}$$

and $f(p, x) = g(x) - p(x)$. We will show that given $0 < \varepsilon < 1/16$ there are no $\Delta \in C(\Pi, \mathcal{Y})$, $v \in C(\Pi, X)$ such that $\|\Delta(p)\|_{\mathcal{Y}} < \varepsilon$ and $f_\Delta(p, v(p)) = \inf f_\Delta(p, X)$ for all $p \in \Pi$.

Proof. We start with the following observation:

Claim. Let $0 < \varepsilon < 1/4$. Let ϕ be any Lipschitz function from X to $\mathbb{R}$ with $\|\phi\|_{\mathcal{Y}} < \varepsilon$. If $g + \phi$ attains its minimum at $m \in X$, then $\|m - a\|_X < \sqrt{3\varepsilon}$ or $\|m + a\|_X < \sqrt{3\varepsilon}$.

Notice that for every $0 < \delta < 1$, one has that

$$g(x) < \delta \Rightarrow \|x-a\|_X < \sqrt{\delta} \text{ or } \|x+a\|_X < \sqrt{\delta}. \tag{5.11}$$

Without loss of generality, let $\phi(0) = 0$. If $\|x\|_X \geq 2$ then $g(x) \geq (\|x\|_X - 1)^2$. It follows that

$$\begin{aligned} g(x) + \phi(x) &\geq g(x) - \varepsilon \|x\|_X \\ &\geq (\|x\|_X - 1)^2 - \varepsilon \|x\|_X \geq 1 - 2\varepsilon. \end{aligned}$$

Further

$$g(m) + \phi(m) = \min(g + \phi) \leq g(a) + \phi(a) \leq 0 + \varepsilon$$

hence $\|m\|_X < 2$ thus $g(m) - 2\varepsilon \leq g(m) + \phi(m) \leq \varepsilon$ and finally $g(m) \leq 3\varepsilon$. The claim then follows from (5.11).

To finish the proof of the example fix $0 < \varepsilon < 1/16$ and suppose there are $\Delta \in C(\Pi, \mathcal{Y})$, $v \in C(\Pi, X)$ such that $\|\Delta(p)\|_{\mathcal{Y}} < \varepsilon$ and $f(p, v(p)) + \Delta(p)(v(p)) = \inf\{f(p,x) + \Delta(p)(x) : x \in X\}$. For every $p \in \Pi$ such that $\|p\|_{X^*} \leq 1/8$ we have $\|\Delta(p) + p\|_{\mathcal{Y}} < 3/16$ because $\|p\|_{X^*} = \|p\|_{\mathcal{Y}}$. As $f(p,\cdot) + \Delta(p)(\cdot) = g(\cdot) - p(\cdot) + \Delta(p)(\cdot)$ attains its minimum at $v(p)$, it follows by the claim that $\|v(p) - a\|_X < 3/4$ or $\|v(p) + a\|_X < 3/4$. Now let $p \in \frac{1}{8}S_{X^*}$ such that $p(a) = \frac{1}{8}$, i.e. p is $\frac{1}{8}$-times tangent to a. It is not difficult to see that then only $\|v(p) - a\|_X < 3/4$ holds. Similarly, only $\|v(-x) + a\|_X < 3/4$. This shows that the v-image of the connected set $\frac{1}{8}B_{X^*}$ is contained in the disjoint union $B_X(a, \frac{3}{4}) \cup B_X(-a, \frac{3}{4})$ and both $v(\frac{1}{8}B_{X^*}) \cap B_X(\pm a, \frac{3}{4})$ are nonempty – so v has to have a point of discontinuity in $\frac{1}{8}B_{X^*}$. □

The next two examples show that the equi-lower semicontinuity of $\{f(\cdot, D)\}$ for any bounded $D \subset X$ cannot be dropped. In fact, Example 5.28 shows that even if $\{f(\cdot, K)\}$ is equi-l.s.c. for every compact $K \subset X$, Theorem 5.18 may still fail.

Example 5.27. Let $\Pi = [0,1]$ and $X = \mathbb{R}^2$ with the supremum norm $\|\cdot\|_\infty$. We define $g(x) := \max\{x_1 - 3x_2 + 2, -x_1 - x_2 + 2, 2x_2 - 2\}$ (see Figure 5.1)and

$$f(p,x) = \begin{cases} g(x_1, x_2/p), & p \neq 0 \\ +\infty, & p = 0, x_2 \neq 0 \\ g(x_1, 0), & p = 0, x_2 = 0 \end{cases}$$

Then f satisfies (M1), (M2), (A1) but not (A2). Given $0 < \varepsilon < \frac{1}{5}$ and $\Delta \in C(\Pi, \mathcal{Y})$, $\|\Delta(p)\|_{\mathcal{Y}} < \varepsilon$ for all $p \in \Pi$, there is no $v \in C(\Pi, X)$ such that $f_\Delta(p, v(p)) = \inf f_\Delta(p, X)$ for all $p \in \Pi$.

Proof. Observe that g enjoys the following properties:

a) $g(1,1) = 0$ is a strong minimum of g such that $g(x) - g(1,1) \geq \frac{2}{5}\|x - (1,1)\|_\infty$ for all $x \in X$,

b) $g(0,0) = 2$ is a strong minimum of $g(\cdot, 0)$ such that $g(x_1, 0) - g(0,0) \geq |x_1|$ for all $x_1 \in \mathbb{R}$.

Let $U = [0, 2p)$ be a neighborhood of 0 in Π. Then for $x = (1, p)$ we have $\|x\|_\infty = 1$ and $f(p,x) = 0$. On the other hand $f(0,x) \geq 2$ for all $x \in X$. This shows that $\{f(\cdot, x) : \|x\|_\infty \leq 1\}$ is not equi-l.s.c. at 0.

Further, let $\|\Delta(p)\|_{\mathcal{Y}} < \frac{1}{5}$ for a $\Delta \in C(\Pi, \mathcal{Y})$. It follows from a) that, for all $p \in (0,1]$, $f_\Delta(p, \cdot)$ attains its strong minimum at $(1, 1/p)$ and it follows from b) that $f_\Delta(0, \cdot)$ attains its strong minimum at $(0,0)$. So the uniquely determined v is discontinuous. This shows the breakdown of Theorem 5.18. □

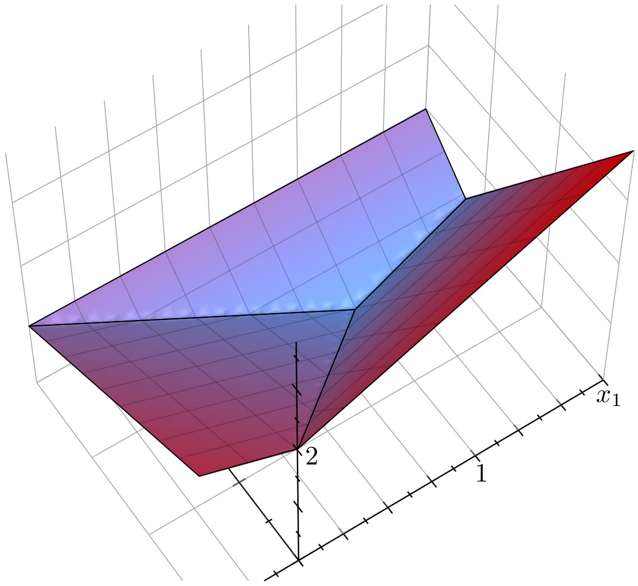

Figure 5.1: The function g.

Example 5.28. Let $\Pi = [0,1]$ and $X = L^2[0,+\infty)$ with the usual inner product $\langle\cdot,\cdot\rangle$. Let $F : \Pi \to X$ be defined as $F(p) = \chi_{[1/p,1/p+1]}$, where $\chi_A(z) = 1$ if $z \in A$, otherwise $\chi_A(z) = 0$. We define

$$f(p,x) := \begin{cases} \langle F(p), x\rangle + \|x\|_X^2\,, & p \neq 0 \\ \|x\|_X^2\,, & p = 0 \end{cases}$$

Then f is real-valued and meets the conditions (M1), (M2), (A1) but not (A2). By Remark 5.15, $\{f(\cdot,x) : x \in K\}$ is equi-l.s.c. for any compact subset K of X. However, Theorem 5.18 fails for f.

Proof. It is obvious that $f(p,\cdot)$ is real-valued, convex, lower bounded and l.s.c. for every $p \in \Pi$. It is also standard that $f(\cdot,x)$ is continuous for every $x \in X$. An easy computation yields that, for each $p \in (0,1]$, $f(p,\cdot)$ attains its minimum at $-\frac{F(p)}{2}$, in fact $f(p,-\frac{F(p)}{2}) = -\frac{1}{4}$. Moreover, since $\left\|x + \frac{F(p)}{2}\right\|_X^2 = f(p,x) + \frac{1}{4}$, one has for every $\varepsilon > 0$,

$$f(p,x) + \frac{1}{4} \leq \varepsilon^2 \Rightarrow \left\|x + \frac{F(p)}{2}\right\|_X \leq \varepsilon. \tag{5.12}$$

Let $\varepsilon > 0$ be sufficiently small. And let $\Delta \in C(\Pi,\mathcal{Y})$ such that $\|\Delta(p)\|_{\mathcal{Y}} < \varepsilon$. Let us assume temporarily that $\|x\| \geq 2$. Then $f(p,x) \geq \|x\|_X$. It follows that $f_\Delta(p,x) \geq f(p,x) - \varepsilon(\|x\|_X + 1) \geq 2 - 3\varepsilon$. So if $f_\Delta(p,\cdot)$ attains its minimum at $v(p)$, we get

$$f_\Delta(p,v(p)) = \min f_\Delta(p,X) \leq f(p,-\frac{F(p)}{2}) + \Delta(p)(-\frac{F(p)}{2}) \leq -\frac{1}{4} + \varepsilon + \frac{\varepsilon}{2}.$$

Thus $\|v(p)\|_X < 2$ and

$$f(p, v(p)) - 3\varepsilon \le f(p, v(p)) - \varepsilon - \varepsilon \|v(p)\|_X \le f_\Delta(p, v(p)) \le -\frac{1}{4} + \frac{3\varepsilon}{2}$$

whence $f(p, v(p)) + \frac{1}{4} \le \frac{9\varepsilon}{2}$. Using (5.12) we get $\left\| v(p) + \frac{F(p)}{2} \right\|_X \le \sqrt{\frac{9\varepsilon}{2}}$ and finally $\|v(p)\|_X \ge \frac{1}{2} - \sqrt{\frac{9\varepsilon}{2}}$.

Similarly, if $\|\Delta(0)\|_{\mathcal{Y}} < \varepsilon$ and $f_\Delta(0, \cdot)$ attains its minimum at $v(0)$, we get that $\|v(0)\|_X < \sqrt{\varepsilon}$. This contradicts the continuity of v at 0 whenever ε is sufficiently small. □

Example 5.29. Let $\Pi = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, $X = \mathbb{R}$ and let us define

$$f(p, x) = \begin{cases} \frac{1}{x-p}, & x - p > 0 \\ +\infty, & x - p \le 0. \end{cases}$$

Then f satisfies (M1), (M2), (A1) and (A2). Obviously $\operatorname{dom} f(p, \cdot) = \operatorname{dom} f(q, \cdot)$ if and only if $p = q$. Our variational principle applies in this situation while the theorems of Georgiev and Veselý do not.

Index

List of Symbols

$|A|$ the cardinality of the set A, page 20

$\mathcal{Y}$ a cone of convex, Lipschitz functions, page 99

$B_{(X,\|\cdot\|)}(x,r)$ the closed ball in X of center x and radius r, page 20

B_X the closed unit ball of X, page 20

$\mathcal{S}_c(K)$ the set of all closed slices of K, page 46

$\mathrm{Dz}(X)$ weak* dentability index of X, page 13

$\mathrm{diam}(A)$ diameter of A, page 20

$\mathrm{dom}(g)$ the effective domain of g, page 98

$\langle x, x^* \rangle$ the duality pairing of x and x^*, page 20

$B_{\mathrm{fine}}(f,\delta)$ a fine neighborhood of f, page 100

$\mathbf{G}(K,\mathcal{A})$ a point-set game, page 45

Λ an index set of the form $\Lambda = \{0\} \cup [\omega, \mu)$, page 77

ω the first infinite ordinal, page 12

ω_1 the first uncountable ordinal, page 12

$B^O_{(X,\|\cdot\|)}(x,r)$ the open ball in X of center x and radius r, page 20

$\mathrm{osc}_f(x)$ the oscillation of f at x, page 60

$\mathcal{S}_o(A)$ the set of the open slices of A, page 9

$\mathcal{S}^*_o(A)$ weak* open slices of A, page 63

$\overline{\mathrm{co}}^{w^*}(C)$ the weak* closed convex hull of a set C, page 64

$\overline{H(f,a)}$ a closed halfspace, page 46

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