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Generalized subdifferentials and Darboux property of Fréchet derivatives.

Doctoral dissertation thesis

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Hereby I declare that I have written this thesis on my own and that I cited all used sources of information. I agree with public availablility and lending of the thesis.

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Contents

Introduction

This thesis consists of two papers,

- *The approximate and the Clarke subdifferentials can be different everywhere* (J. Math. Anal. Appl. **347** (2008) 652–658)
- *Porosity and the Darboux property of Fréchet derivatives* (to appear in Real Anal. Exchange) which is joint work with O. Kurka.

The paper *The approximate and the Clarke subdifferentials can be different everywhere* deals with two important objects in nonsmooth analysis, the approximate subdifferential and the Clarke subdifferential. For $n \geq 2$ we construct a Lipschitz function on $\mathbb{R}^n$ for which these two subdifferentials are different at every point $x \in \mathbb{R}^n$. This completely answers question by A.D. Ioffe whether the approximate and the Clarke subdifferentials must generically coincide (see [3]), which was partially answered by G. Katriel in [4](difference on a set of positive measure) and by D. Borwein, J.M. Borwein and X. Wang in [1](difference almost everywhere).

For f a Lipschitz function on an open set $U \subset \mathbb{R}^n$ the lower Dini derivative of f at $x \in U$ in a direction $v \in \mathbb{R}^n$ is defined by

$$D_v^- f(x) = \liminf_{t \to 0^+} \frac{f(x+tv) - f(x)}{t}.$$

The Dini subdifferential of f at x is

$$\partial^- f(x) = \{x^* \in \mathbb{R}^n : \langle x^*, v\rangle \leq D_v^- f(x) \text{ for all } v \in \mathbb{R}^n\}.$$

The approximate subdifferential of f at x is

$$\partial_a f(x) = \limsup_{z \to x} \partial^- f(z) = \bigcap_{r>0} \overline{\bigcup_{z \in B(x,r)} \partial^- f(z)}.$$

And finally, the Clarke subdifferential of f at x is

$$\partial_c f(x) = \mathrm{conv}(\partial_a f(x)).$$

This means that we are actually looking for a function, for which $\partial_a f(x)$ is nonconvex. We work on a special set open U which is interior of a equilateral triangle and prove the following:

There is a Lipschitz function f on U such that
(A) for any $x \in U$ there is a direction v such that $D_v^- f(x) \leq -\frac{1}{4\sqrt{3}}$,
(B) for any $x \in U$ we have $\{2v_1, 2v_2, 2v_3\} \subset \partial_a f(x)$.

Where v_1, v_2 and v_3 are nonzero vectors such that the set $\mathrm{conv}(\{v_1, v_2, v_3\})$ contains 0. This means that $\partial_c f(x)$ contains 0 for every $x \in U$, but any point in $\partial_a f(x)$ must have norm at least $\frac{1}{4\sqrt{3}}$.

To obtain function $F : \mathbb{R}^2 \to \mathbb{R}$ with $\partial_a F(x) \neq \partial_c F(x)$ for every $x \in \mathbb{R}^2$ we just take a diffeomorphism $\varphi : \mathbb{R}^2 \to U$ and then put $F = f \circ \varphi$. And finally, for such function F_n on $\mathbb{R}^n, n > 2$ we put $F_n(x_1, x_2, \ldots, x_n) = F(x_1, x_2)$.

The main idea of the construction is to take supremum of suitable system of functions with pyramid shaped graph (multipliers of distance function to equilateral triangles) which have the property that have great directional derivative in some direction at every point and convex hull of possible derivatives in the points of differentiability contains 0.

In the second paper *Porosity and the Darboux property of Fréchet derivatives* we prove one implication of characterization of sets $M \subset \mathbb{R}^d$ without isolated points and with connected interior on which for every (relative) Fréchet derivative f the set $f(M)$ is connected. We prove necessarity of the condition which was proved as sufficient by P. Holický, C. E. Weil and L. Zajíček in general Banach space in [2].

By Fréchet derivative of a function $f : B \to \mathbb{R}$, where B is a subset of a Banach space X with no isolated points, we mean a function $g : B \to X^*$ for which

$$\lim_{x \to a,\, x \in B} \frac{f(x) - f(a) - g(a)(x-a)}{\|x-a\|} = 0$$

for each $a \in B$. A set A in a real Banach space X is said to be porous at $a \in X$ if there are $c > 0$ and $x_n \in X$, $x_n \neq a$, with $x_n \to a$ such that $x \notin A$ whenever $n \in \mathbb{N}$ and $\|x - x_n\| < c\|a - x_n\|$.

The result is based on the observation how can the Fréchet derivatives behave in the boundary points of the set. P. Holický, C. E. Weil and L. Zajíček proved the following lemma:

Lemma. *Let X be a real Banach space, $G \subset X$ open, $a \in \partial G$ and let $X \setminus G$ be porous at a. Let $M := G \cup \{a\}$ and suppose that $g : M \to X^*$ is a Fréchet derivative of a function $f : M \to \mathbb{R}$ on M. Then $(a, g(a))$ belongs to the closure of the graph of $g|_G$ in $X \times X^*$. In particular, $g(a) \in \overline{g(G)}$.*

Our key result is to prove the reverse of this fact in euclidean spaces. The proof is divided into three main steps. First of all we construct just Lipschitz function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with good behaving directional derivatives on the complement of some special class of discrete sets which are not porous at the origin and for which $\varphi'(0)$ exists and is equal to 0. These sets are of form

$$D = \bigcup_{i \in \mathbb{N}} r_i D_{p_i},$$

where

$$D_p = \Big\{(x_1, \ldots, x_d) \in \partial([-1,1]^d) : 2px_1, \ldots, 2px_d \in \mathbb{Z}\Big\}, \quad p \in \mathbb{N},$$

for suitable sequences $p_i \nearrow \infty$ and $r_i \searrow 0$. The statement we need is the following:

For every $x \in \mathbb{R}^d \setminus (D \cup \{0\})$, there is a direction $\nu \in \mathbb{R}^d, |\nu| = 1$, and a neighbourhood U_x of x such that $\frac{\partial \varphi}{\partial \nu}(y) \geq 1/2$ for almost every $y \in U_x$.

Main idea of the construction of such function is, roughly speaking, to make φ oscillating on the faces of cubes $r_i[-1,1]^d$ with big derivative in the directions of the face, and use the points in $r_i D_{p_i}$ as possible points of nondifferentiability to avoid small derivatives in possible stationary points. Between sets $r_i \partial[-1,1]^d$ we make φ oscillating with big derivative in the "radius" direction.

In the second step, we use mollifications of these Lipschitz functions to obtain counterexample for our special class of sets. More presicely, we prove prove the following statement

There is a Lipschitz function $F : \mathbb{R}^d \to \mathbb{R}$ with properties

1. *$F'(0) = 0$,*
2. *$F'(x)$ exists and $|F'(x)| \geq 1/(4\sqrt{d})$ whenever $x \in \mathbb{R}^d \setminus (D \cup \{0\})$.*

The function F is obtained simply by formula

$$F(x) = \int_{\mathbb{R}^d} \varphi(x + \delta(x)y)\psi(y)dy, \quad x \in \mathbb{R}^d \setminus (D \cup \{0\})$$

for suitable $\delta \in \mathcal{C}^1(\mathbb{R}^d \setminus (D \cup \{0\}))$, where $\delta(x)$ depends on the diameter of the corresponding neighborhood U_x and ψ is usual mollification kernel.

And finally, for a general set M which is not porous at some point, we construct a suitable diffeomorphism which maps some of our special sets into M and using composition of that diffeomorphism and the counterexample on that special set, we obtain counterexample for M.

The question whether the condition is necessary in every Banach space remains still open.

References

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The approximate and the Clarke subdifferentials can be different everywhere

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Abstract

We prove that, for a Lipschitz function on $\mathbb{R}^n$, $n \geq 2$, the approximate and the Clarke subdifferentials can differ everywhere. This completely answers a question by A.D. Ioffe, which was partially answered by G. Katriel.

1 Introduction

The approximate subdifferential and the Clarke subdifferential are two important objects in nonsmooth analysis. In [3], A.D. Ioffe posed the question, whether the approximate and the Clarke subdifferentials must generically coincide.

Let us recall the definition of the approximate and the Clarke subdifferentials for a Lipschitz function on $\mathbb{R}^n$ (more general definitions for more general functions will not be needed here).

Let f be a Lipschitz function on an open set $U \subset \mathbb{R}^n$. The lower Dini derivative of f at $x \in U$ in a direction $v \in \mathbb{R}^n$ is defined by

$$D_v^- f(x) = \liminf_{t \to 0^+} \frac{f(x+tv) - f(x)}{t} .$$

The Dini subdifferential of f at x is

$$\partial^- f(x) = \{x^* \in \mathbb{R}^n : \langle x^*, v\rangle \leq D_v^- f(x) \text{ for all } v \in \mathbb{R}^n\}.$$

The approximate subdifferential of f at x is

$$\partial_a f(x) = \limsup_{z \to x} \partial^- f(z) = \bigcap_{r>0} \overline{\bigcup_{z \in B(x,r)} \partial^- f(z)}.$$

Finally, the Clarke subdifferential of f at x is

$$\partial_c f(x) = \mathrm{conv}(\partial_a f(x)).$$

In a measure sense, Ioffe's question was answered by G. Katriel in [4] (positively in $\mathbb{R}$ and negatively in the higher dimensions). In their paper [1], D. Borwein, J.M. Borwein and X. Wang improved this result proving that there is a Lipschitz function on $\mathbb{R}^n$ ($n \geq 2$) such that these two subdifferentials are different almost everywhere. Katriel also asked, if the approximate and the Clarke subdifferentials must be equal on a dense G_δ set. In our paper we will construct a Lipschitz function f on $\mathbb{R}^n, n \geq 2$, such that $\partial_a f(x) \neq \partial_c f(x)$ for each $x \in \mathbb{R}^n$. This gives a negative answer to Katriel's question and also a definitive negative answer to Ioffe's question.

2 Preliminaries

We will use the following standard notation.

We denote by $\operatorname{conv} A$ the convex hull of a set A. For $x \in \mathbb{R}^n$ and $r > 0$ we use $B(x,r)$ for an open ball with a center x and a radius r. We say that a function is K-Lipschitz if it is Lipschitz with a constant K. We use $\langle x, y\rangle$ for the inner product of $x, y \in \mathbb{R}^n$.

We will also need the following special definitions and notions.

We denote by $\tilde{T}$ the closed triangle in $\mathbb{R}^2$ with the vertices $(1,0)$, $(-1,0)$ and $(0,\sqrt{3})$ and U will be the interior of $\tilde{T}$. Put $v_1 = (0,1)$, $v_2 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $v_3 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$. For $i \in \{1,2,3\}$ and $n \in \mathbb{N}$ put $v_i^n = (2 - \frac{1}{n})v_i$. $\mathcal{T}$ will be the system of all closed triangles $T \subset U$ with the sides parallel to the sides of the triangle $\tilde{T}$.

For $T \in \mathcal{T}$ or $T = \tilde{T}$ and $i \in \{1,2,3\}$ we denote by s_T^i the (closed) side of the triangle T which is orthogonal to the vector v_i. The vertex which is opposite to the side s_T^i will be denoted by q_T^i and c_T will be the centroid of the triangle T. Put $t_T^i = \operatorname{conv}\{c_T, q_T^i\}$ and $t_T = t_T^1 \cup t_T^2 \cup t_T^3$.

For $n \in \mathbb{N}$, $\tilde{J}^n$ will be the function on $\tilde{T}$ defined by

$$\tilde{J}^n(x) = (2 - 1/n)\operatorname{dist}(x, \partial U).$$

Let f be a function on U, let V be an open subset of U, $n \in \mathbb{N}$ and $i \in \{1,2,3\}$. We will say that f is of the type $\mathcal{A}_i^n$ on V, if it is positive on V and there is an $a \in \mathbb{R}$ such that $f(x) = a + \langle x, v_i^n\rangle$ for each $x \in V$. The function f is said to be of the type $\mathcal{A}^n$ on V if it is of the type $\mathcal{A}_i^n$ on V for some $i \in \{1,2,3\}$.

Let f be a function on U, let G be an open subset of U and $n \in \mathbb{N}$. We will say that f is of the type $\mathcal{V}^n$ on the set G if for each $x \in G$ there is an open set $V \subset G$ with $x \in V$ such that f is of the type $\mathcal{A}^n$ on V.

For $T \in \mathcal{T}$ or $T = \tilde{T}$ and $n \in \mathbb{N}$ define the function $J_T^n : U \to \mathbb{R}$ by

$$J_T^n(x) = \begin{cases} (2 - 1/n)\operatorname{dist}(x, \partial T) & \text{for } x \in T \cap U \\ 0 & \text{for } x \in U \setminus T. \end{cases}$$

We will need the following easy geometrical facts.

(G1) For $T \in \mathcal{T}$ and $n \in \mathbb{N}$, the function J_T^n is of the type $\mathcal{A}_1^n$ ($\mathcal{A}_2^n$ or $\mathcal{A}_3^n$) on the interior of the triangle with the sides s_T^1, t_T^2, t_T^3, (s_T^2, t_T^1, t_T^3 or s_T^3, t_T^1, t_T^2). The set

$$\{(x,y) \in T \times \mathbb{R} : 0 \le y \le J_T^n(x)\}$$

is a closed pyramid with a base $T \times \{0\}$. Similar facts hold for the functions $\tilde{J}^n$ and $J_{\tilde{T}}^n$.

(G2) By a simple geometrical argument, choosing $x \in U$, $n \in \mathbb{N}$ and putting $T_x = \{T \in \mathcal{T} : c_T = x\}$, we have

$$\tilde{J}^n(x) = J_{\tilde{T}}^n(x) = \sup_{T \in \mathcal{T}_x} J_T^n(x).$$

(G3) Defining $f^n(x) = \langle v_1^n, x \rangle$, an easy computation shows that the set

$$\{x \in \tilde{T} : f^n(x) \le \tilde{J}^{n+1}(x)\}$$

is a triangle T_0 with the vertices $p^1 = (0, b_n)$, $p^2 = (1,0)$, $p^3 = (-1,0)$, where

$$b_n = \frac{\sqrt{3}}{2} \cdot \frac{2 - \frac{1}{n+1}}{3 - (\frac{1}{2(n+1)} + \frac{1}{n})} .$$

Moreover, if $T \in \mathcal{T}$ and $\alpha \in \mathbb{R}$, define

$$F_\alpha = \{x \in T : f^n(x) + \alpha \le J_T^{n+1}(x)\}.$$

(G3a) In the case $f^n(c_T) + \alpha = J_T^{n+1}(c_T)$ we have

$$\frac{f^n(x) - f^n(c_T)}{|x - c_T|} > \frac{J^{n+1}(x) - J^{n+1}(c_T)}{|x - c_T|}$$

for each $x \in T \setminus \{c_T\}$. So $f^n(x) > J^{n+1}(x)$ for each $x \in T \setminus \{c_T\}$. In particuar, the set F_α consists exactly of the one point c_T.

(G3b) If $f^n(c_T) + \alpha < J_T^{n+1}(c_T)$ and $f^n + \alpha \ge 0$ on T the set F_α is homothetic to T_0. Denote this homothety by S (the one with $S(F_\alpha) = T_0$). It has the property that for each $x, y \in F_\alpha$ we have

$$\frac{J_T^{n+1}(x) - J_T^{n+1}(y)}{|x - y|} = \frac{\tilde{J}^{n+1}(S(x)) - \tilde{J}^{n+1}(S(y))}{|S(x) - S(y)|} .$$

We will need the following two easy lemmas.

Lemma 1. *Let f be a Lipschitz function on an open set $V \subset \mathbb{R}^2$, $x \in V$ and $\alpha \in \mathbb{R}$. Suppose that for any $n \in \mathbb{N}$ there is an $x_n \in V$ such that $|x - x_n| < \frac{1}{n}$ and*

$$\frac{f(x_n) - f(x)}{|x_n - x|} \le \alpha.$$

Then there is a direction $v \in \mathbb{R}^2$ with $|v| = 1$ and such that $D_v^- f(x) \le \alpha$.

Proof. Put $v_n = \frac{x_n - x}{|x_n - x|}$. Due to the compactness of the unit sphere in $\mathbb{R}^2$, there is a unit vector $v \in \mathbb{R}^2$ and a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{v_n\}_{n=1}^{\infty}$ such that

$$\lim_{k \to \infty} v_{n_k} = v.$$

So it is sufficient to use the well known fact that (since f is Lipschitz) we have

$$D_v^- f(x) = \liminf_{t \to 0^+,\ u \to v} \frac{f(x + tu) - f(x)}{t} .$$

□

Lemma 2. *Let $n \in \mathbb{N}$ and let f and g be two Lipschitz functions on an open set $V \subset \mathbb{R}^n$, let $f \leq g$ on V and $x \in V$. Suppose that $f(x) = g(x)$. Then we have $\partial^- f(x) \subset \partial^- g(x)$.*

Proof. The Lemma directly follows from the obvious fact that $D_v^- f(x) \leq D_v^- g(x)$ for any direction $v \in \mathbb{R}^n$. □

3 Main result

Lemma 3. *Let f be a 2-Lipschitz function on U, let $r > 0$, $w \in U$ and $n \in \mathbb{N}$. Suppose that f is of the type $\mathcal{A}^n$ on $V = B(w, r)$ and $f \leq J^n$ on U. Then there is a 2-Lipschitz function $\tilde{f} : U \to \mathbb{R}$ and an open set $G \subset V$ such that, denoting $W = \{x \in V : \tilde{f}(x) > f(x)\}$, the following conditions hold:*

(i) $w \in W$,

(ii) $\overline{W} \subset V$,

(iii) $f \leq \tilde{f} \leq J^{n+1}$,

(iv) $G \subset W$ and G is dense in W,

(v) $\tilde{f}$ is of the type $\mathcal{V}^{n+1}$ on G,

(vi) for each $x \in \partial G$ there is a $y \in V \setminus G$, $y \neq x$, such that $\operatorname{conv}\{x, y\} \cap G = \emptyset$, $\operatorname{conv}\{x, y\} \subset V$ *and for each $z \in \operatorname{conv}\{x, y\} \setminus \{x\}$ we have*

$$\frac{\tilde{f}(z) - \tilde{f}(x)}{|z - x|} \leq -\frac{1}{4\sqrt{3}} ,$$

(vii) for each $x \in G$ there is a $y \in \partial G$ such that

$$\frac{\tilde{f}(y) - \tilde{f}(x)}{|y - x|} \leq -\frac{1}{4\sqrt{3}} ,$$

(viii) there are $y_1, y_2, y_3 \in \partial G$ such that $v_i^{n+1} \in \partial^- \tilde{f}(y_i)$.

Proof. By the symmetry, we can suppose that f is of the type $\mathcal{A}_1^n$ on V. So for some $\alpha \in \mathbb{R}$, we have $f(x) = \langle v_1^n, x\rangle + \alpha$ for each $x \in V$. Let $\mathcal{T}_w$ be the system of the triangles from $\mathcal{T}$ with the centroid w. Due to fact (G2) we have

$$f(w) \leq J^n(w) < J^{n+1}(w) = \sup_{T \in \mathcal{T}_w} J_T^{n+1}(w).$$

So there is a triangle $T^* \in T_w$ such that $J_{T^*}^{n+1}(w) > f(w)$ and so we can choose $T \in \mathcal{T}_w$ with $J_T^{n+1}(w) = f(w)$. For $l \in \mathbb{N}$, let T^l be the triangle with the vertices $q_i^l = w + (1 + \frac{1}{l})(q_T^i - w)$. By geometrical fact (G3a), the inequality $\langle v_1^n, y\rangle + \alpha > J_T^{n+1}(y)$ holds for each $y \in T \setminus \{w\}$. So there is some $l_0 \in \mathbb{N}$ such that for each $l \geq l_0$ we have $T^l \in \mathcal{T}_w$ and $\langle v_1^n, y\rangle + \alpha \geq 0$ for all $y \in T^l$. Obviously, $J_{T^l}^{n+1} \searrow J_T^{n+1}$ uniformly on U and $f(w) = J_T^{n+1}(w) < J_{T^l}^{n+1}(w)$. This implies that the sets $W_l = \{z \in V : J_{T^l}^{n+1}(z) > \langle v_1^n, z\rangle + \alpha\}$ are nonempty and $\bigcap \overline{W}_l = \{w\}$. We used the fact that $\overline{W}_l = \{z \in V : J_{T^l}^{n+1}(z) \geq \langle v_1^n, z\rangle + \alpha\}$. Thus we can choose $l_1 \geq l_0$ with $\overline{W}_{l_1} \subset V$. Put

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in U \setminus W_{l_1} \\ J_{T^{l_1}}^{n+1}(x) & \text{for } x \in W_{l_1} \end{cases}$$

and $G = W_{l_1} \setminus t_{T^{l_1}}$. Then for any $x \in U$ we have

$$\limsup_{y \to x} \frac{|\tilde{f}(x) - \tilde{f}(y)|}{|x - y|} \leq 2$$

and so $\tilde{f}$ is 2-Lipschitz on U (see [2], 2.2.7).

Properties (i)-(iv) are clear and (v) holds by geometrical fact (G1). Put

$$b_n = \frac{\sqrt{3}}{2} \cdot \frac{2 - \frac{1}{n+1}}{3 - (\frac{1}{2(n+1)} + \frac{1}{n})}, \quad p^1 = (0, b_n), \quad p^2 = (1, 0), \quad p^3 = (-1, 0),$$

$s^1 = \operatorname{conv}\{p^2, p^3\}$, $s^2 = \operatorname{conv}\{p^1, p^3\}$, $s^3 = \operatorname{conv}\{p^1, p^2\}$ and for $i \in \{1, 2, 3\}$ set $t^i = \operatorname{conv}\{p^i, c_{\tilde{T}}\}$. Denote by T_0 the closed triangle with vertices p^1, p^2, p^3. Note that

$$\overline{G} = \overline{W}_{l_1} = \{z \in V : J_{T^{l_1}}^{n+1}(z) \geq f(z)\}.$$

By geometrical fact (G3b), there is a homothety $S : \overline{G} \to T_0$ with the property that for each $x, y \in \overline{G}$ we have

$$\frac{\tilde{f}(x) - \tilde{f}(y)}{|x - y|} = \frac{\tilde{J}^{n+1}(S(x)) - \tilde{J}^{n+1}(S(y))}{|S(x) - S(y)|}.$$

Turn to (vi). Choose $x \in \partial G$. We have

$$S(\partial G) = s^1 \cup (s^2 \setminus \{p^3\}) \cup (s^3 \setminus \{p^2\}) \cup (t^1 \setminus \{p^1\}) \cup (t^2 \setminus \{p^2\}) \cup (t^3 \setminus \{p^3\}).$$

Suppose that $S(x) \in s^1$. There is an $\alpha > 0$ such that $x - \alpha v_1 \in V$. Put $y = x - \alpha v_1$. Then for any $z \in \mathrm{conv}\{x, y\} \setminus \{x\}$ we have

$$\frac{\tilde f(z) - \tilde f(x)}{|z - x|} = -2 + \frac{1}{n} \le -\frac{1}{4\sqrt{3}}\,.$$

In the case $S(x) \in s^2 \setminus \{p^3\}$ put $y = S^{-1}(p^3)$. For $z \in \mathrm{conv}\{x, y\} \setminus \{x\}$ we have

$$\begin{aligned}\frac{\tilde f(z) - \tilde f(x)}{|z - x|} &= \frac{\langle S(z), v_1^n\rangle - \langle S(x), v_1^n\rangle}{|S(z) - S(x)|} = \frac{\langle p^3, v_1^n\rangle - \langle p^1, v_1^n\rangle}{|p^3 - p^1|}\\ &= -\frac{\big(2 - \frac{1}{n}\big)b_n}{\sqrt{b_n^2 + 1}} \le -\frac{1}{4\sqrt{3}}\,.\end{aligned}$$

We use the fact that $\frac{\sqrt{3}}{6} \le b_n \le \sqrt{3}$. If $S(x) \in s^3 \setminus \{p^2\}$ put $y = S^{-1}(p^2)$. Just as in the previous case, for each $z \in \mathrm{conv}\{x, y\} \setminus \{x\}$ we have

$$\begin{aligned}\frac{\tilde f(z) - \tilde f(x)}{|z - x|} &= \frac{\langle S(z), v_1^n\rangle - \langle S(x), v_1^n\rangle}{|S(z) - S(x)|} = \frac{\langle p^2, v_1^n\rangle - \langle p^1, v_1^n\rangle}{|p^2 - p^1|}\\ &= -\frac{\big(2 - \frac{1}{n}\big)b_n}{\sqrt{b_n^2 + 1}} \le -\frac{1}{4\sqrt{3}}\,.\end{aligned}$$

If $S(x) \in t^i \setminus \{p^i\}$ for $i \in \{1, 2, 3\}$ put $y = S(p^i)$. In these cases we have

$$\frac{\tilde f(z) - \tilde f(x)}{|z - x|} = -\Big(2 - \frac{1}{n+1}\Big)\sin\frac{\pi}{6} \le -\frac{1}{4\sqrt{3}},$$

provided $z \in \mathrm{conv}\{x, y\} \setminus \{x\}$.

Turn to (vii). To prove it, choose $x \in G$. Using geometrical observation (G1), we have $G = G_1 \cup G_2 \cup G_3$ such that $\tilde f$ is of the type $\mathcal{A}_i^{n+1}$ on G_i and the set G_1, (G_2 or G_3) is the interior of the triangle with sides $S^{-1}(s^1), S^{-1}(t^2), S^{-1}(t^3)$, ($S^{-1}(s^2), S^{-1}(t^1), S^{-1}(t^3)$ or $S^{-1}(s^3), S^{-1}(t^1), S^{-1}(t^2)$). For $x \in G_i$, with $i \in \{1, 2, 3\}$, choose $y \in s^i \subset \partial G$ with $x - y$ parallel to the vector v_i. Then we have

$$\frac{\tilde f(y) - \tilde f(x)}{|y - x|} = -2 + \frac{1}{n+1} \le -\frac{1}{4\sqrt{3}}\,.$$

It remains to verify (viii). Put

$$y_1 = S^{-1}\Big(\frac{p^2 + p^3}{2}\Big),\quad y_2 = S^{-1}\Big(\frac{p^1 + p^3}{2}\Big),\quad \text{and}\quad y_3 = S^{-1}\Big(\frac{p^1 + p^2}{2}\Big)\,.$$

To complete the proof, it is sufficient to use Lemma 2 and the fact that $v_i^{n+1} \in \partial^- J_{T^{l_1}}^{n+1}(y_i)$. $\square$

Lemma 4. *There is a sequence $\{f_n\}_{n=1}^\infty$ of 2-Lipschitz functions on U and a sequence $\{G_n\}_{n=1}^\infty$ of open sets in U such that the following conditions hold:*

(1) G_n *is dense in* U*,*

(2) for $n > 1$ *we have* $G_n \subset G_{n-1}$*,*

(3) for $n > 1$ *we have* $f_n = f_{n-1}$ *on* $U \setminus G_{n-1}$*,*

(4) f_n *is of the type* $\mathcal{V}^n$ *on the set* G_n*,*

(5) for $n > 1$ *we have* $f_{n-1} \leq f_n$ *on* U*,*

(6) $0 < f_n \leq J^n$ *on* U*,*

(7) if $n > 1$ *then for any* $x \in G_n$ *there is an* $x_n \in U \setminus G_n$ *such that* $|x - x_n| < \frac{1}{n}$ *and*

$$\frac{f_n(x_n) - f_n(x)}{|x_n - x|} \leq -\frac{1}{4\sqrt{3}},$$

(8) for $x \in U \setminus G_n$ *and* $l \in \mathbb{N}$ *there is an* $x_l^n \in U \setminus G_n$ *such that* $|x - x_l^n| < \frac{1}{l}$ *and*

$$\frac{f_n(x_l^n) - f_n(x)}{|x_l^n - x|} \leq -\frac{1}{4\sqrt{3}},$$

(9) if $n > 1$ *then for any* $x \in U$ *there are* $y_1^n, y_2^n, y_3^n \in U \setminus G_n$ *such that* $|x - y_i^n| < \frac{1}{n}$ *and* $v_i^n \in \partial^- f_n(y_i^n)$ *for* $i \in \{1, 2, 3\}$*.*

Proof. The sequences $\{f_n\}_{n=1}^{\infty}$ and $\{G_n\}_{n=1}^{\infty}$ will be constructed inductively.

Put $f_1 = J_{\tilde{T}}^1$ and $G_1 = U \setminus t_{\tilde{T}}$. In the case $n = 1$ conditions (1) and (6) are clear, the condition (4) holds by (G1). To prove (8), choose $x \in U \setminus G_1 = t_{\tilde{T}} \cap U$ and $l \in \mathbb{N}$. By the symmetry we can suppose that $x \in t_{\tilde{T}}^1 \cap U$. Choose $x_l \in t_{\tilde{T}}^1 \cap U$ with $f(x) > f(x_l)$ and $|x - x_l| < \frac{1}{l}$. Since an easy computation shows that

$$\frac{f_1(x_l) - f_1(x)}{|x_l - x|} = -\sin\frac{\pi}{6} = -\frac{1}{2} \leq -\frac{1}{4\sqrt{3}},$$

we are done.

Now suppose that, for some $q > 1$, the functions $f_1, \ldots, f_{q-1}$ and the sets $G_1, \ldots, G_{q-1}$ have been constructed. The function f_q and the set G_q will be obtained by the following inductive procedure using Lemma 3.

Choose a dense sequence $\{a_m\}_{m=1}^{\infty}$ in G_{q-1}. We will construct 2-Lipschitz functions $f_q^k : U \to \mathbb{R}$ and open sets G_q^k for $k = 0, 1, \ldots$.

Put $f_q^0 = f_{q-1}$ and $G_q^0 = \emptyset$. Suppose that we already have the functions $f_q^0, \ldots, f_q^{k-1}$ and the sets $G_q^0, \ldots, G_q^{k-1}$ such that the set $H^k := G_{q-1} \setminus \bigcup_{l=0}^{k-1} \overline{G_q^l}$ is nonempty and the sets $\overline{G_q^1}, ..., \overline{G_q^{k-1}}$ are pairwise disjoint. Choose the minimal m such that $a_m \in H^k$ and find

$$0 < r_k < \min\Big(\operatorname{dist}\big(a_m, \partial H^k\big), \frac{1}{4q}\Big).$$

Then f_{q-1} is of the type $\mathcal{A}^{q-1}$ on $B(a_m, r_k)$ and so we can use Lemma 3 for $f = f_{q-1}$, $w = a_m$, $r = r_k$ and $n = q - 1$. Lemma 3 provides us with a function $\tilde{f}$ and a set G such that conditions (i)-(viii) hold. Put $f_q^k = \tilde{f}$ and $G_q^k = G$.

Define

$$f_q = \sup_{k \in \mathbb{N}} f_q^k \quad \text{and} \quad G_q = \bigcup_{k=1}^{\infty} G_q^k.$$

Then f_q is 2-Lipschitz as a supremum of 2-Lipschitz functions. Note that the sets $\{\overline{G_q^k}\}_{k=1}^\infty$ are pairwise disjoint and by (iv) we have $\overline{G_q^k} \supset W_q^k \supset G_q^k$, where $W_q^k = \{x \in U : f_q^k(x) > f_{q-1}(x)\}$ for $k = 1, 2, \dots$. In particular, the sets $\{W_q^k\}_{k=1}^\infty$ are pairwise disjoint as well and for any $l \in \mathbb{N}$ we have

$$(*) \quad \{x \in U : f_q^l(x) = f_q(x)\} = U \setminus \bigcup_{k \neq l} W_q^k \,.$$

Moreover, we have

$$(**) \quad \{x \in U : f_{q-1}(x) = f_q(x)\} = U \setminus \bigcup_{k \in \mathbb{N}} W_q^k \,.$$

It remains to verify the validity of conditions (1)-(9) for $n = q$. Property (1) holds due to the fact that

$$\overline{G}_n \supset \overline{\bigcup \overline{G_n^k}} \supset \overline{\bigcup \{a_m\}} \supset \overline{G_n} \supset \overline{U}.$$

Conditions (2) and (3) are clear. Condition (4) holds by (v), condition (5) by (iii) and condition (6) by the induction hypothesis and (iii). To prove (7), choose $x \in G_n$. There is some $k \in \mathbb{N}$ such that $x \in G_n^k$. Due to (vii) from Lemma 3, there is an $x_n \in \partial G_n^k \subset \partial G_n$ such that

$$\frac{f_n(x_n) - f_n(x)}{|x_n - x|} = \frac{f_n^k(x_n) - f_n^k(x)}{|x_n - x|} \leq -\frac{1}{4\sqrt{3}} \,.$$

So we are done because

$$|x_n - x| \leq \operatorname{diam} G_n^k \leq \frac{1}{2n} < \frac{1}{n} \,.$$

Now turn to (8). Choose $x \in U \setminus G_n$ and $l \in \mathbb{N}$. If $x \in U \setminus G_{n-1}$ we are done due to the induction hypothesis. So we can suppose $x \in G_{n-1} \setminus G_n$. There are two possibilities, either there is a $k_0 \in \mathbb{N}$ such that $x \in \overline{G_n^{k_0}}$ or $x \notin \overline{G_n^k}$ for each $k \in \mathbb{N}$.

In the first case, using condition (vi) from Lemma 3, we can find $y \in U$, $y \neq x$, such that $\operatorname{conv}\{x, y\} \subset G_{n-1} \setminus G_n^{k_0}$ and $|x - y| < \frac{1}{l}$, and we have

$$\frac{f_n^{k_0}(z) - f_n^{k_0}(x)}{|z - x|} \leq -\frac{1}{4\sqrt{3}} \quad \text{for each } z \in \operatorname{conv}\{x, y\} \setminus \{x\}.$$

Now, by $(*)$, it is sufficient to find an $x_l^n \in \operatorname{conv}\{x, y\} \setminus (\{x\} \cup \bigcup_{k \neq k_0} W_n^k)$. But this is clearly possible, since the set $\operatorname{conv}\{x, y\} \setminus \{x\}$ cannot be covered by the pairwise disjoint open sets W_n^k, $k \neq k_0$, unless there is some $k_1 \neq k_0$ such that $\operatorname{conv}\{x, y\} \setminus \{x\} \subset W_n^{k_1}$. This contradicts the fact that the sets $\overline{G_n^{k_0}} = \overline{W_n^{k_0}}$ and $\overline{G_n^{k_1}} = \overline{W_n^{k_1}}$ are disjoint.

In the second case there is some $0 < \alpha < \frac{1}{l}$ and some $i \in \{1,2,3\}$ such that $\operatorname{conv}\{x, x-\alpha v_i\} \subset G_{n-1}$ and for any $0 < \beta \leq \alpha$ we have

$$\frac{f_{n-1}(x-\beta v_i) - f_{n-1}(x)}{\beta} = -2 + \frac{1}{n-1} \leq -\frac{1}{4\sqrt{3}} \, .$$

So, by $(**)$, it is sufficient to find $x_l^n \in \operatorname{conv}\{x, x-\alpha v_i\} \setminus (\{x\} \cup \bigcup_{k\in\mathbb{N}} W_n^k)$. But just as in the previous case, the set $\operatorname{conv}\{x,y\} \setminus \{x\}$ cannot be covered by the pairwise disjoint open sets W_n^k unless there is some k_1 such that $\operatorname{conv}\{x,y\} \setminus \{x\} \subset W_n^{k_1}$. This is a contradiction with the fact that $x \notin \overline{G_n^{k_1}} = \overline{W_n^{k_1}}$.

To complete the proof, it remains to verify (9). Choose $x \in U$ and find $m \in \mathbb{N}$ such that $|x - a_m| < \frac{1}{2n}$. There is $k \in \mathbb{N}$ such that $a_m \in \overline{G_n^k}$. Due to the fact that $\operatorname{diam} \overline{G_n^k} < \frac{1}{2n}$, we have $\partial G_n^k \subset B(x, \frac{1}{n})$ and it suffices to use (viii) from Lemma 3. □

Proposition 5. *There is a Lipschitz function f on U such that*
(A) for any $x \in U$ there is a direction v such that $D_v^- f(x) \leq -\frac{1}{4\sqrt{3}}$,
(B) for any $x \in U$ we have $\{2v_1, 2v_2, 2v_3\} \subset \partial_a f(x)$.

Proof. Let $\{f_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$ be the sequences from Lemma 4. Put $f = \sup f_n$. Then f is 2-Lipschitz as a supremum of 2-Lipschitz functions (it is finite by (6)). Let $x \in U$. To prove (A), it is sufficient (by Lemma 1) to show that for any $l \in \mathbb{N}$, $l > 1$, there is a $\xi_l \in B(x, \frac{1}{l})$ with

$$\frac{f(\xi_l) - f(x)}{|\xi_l - x|} \leq -\frac{1}{4\sqrt{3}} \, .$$

Condition (7) (for $x \in G_l$) or (8) (for $x \in U \setminus G_l$) implies that there is a $\xi_l \in U \setminus G_l$ ($\xi_l = x_l$ or $\xi_l = x_l^l$ respectively) such that $|\xi_l - x| < \frac{1}{l}$ and

$$\frac{f_l(\xi_l) - f_l(x)}{|\xi_l - x|} \leq -\frac{1}{4\sqrt{3}} \, .$$

Properties (3) and (5) give $f(\xi_l) = f_l(\xi_l)$ and $f(x) \geq f_l(x)$ and so we are done. Turn to (B). Choose $m \in \mathbb{N}$. Condition (9) provides us with $y_1^m, y_2^m, y_3^m \in B(x, \frac{1}{m}) \cap (U \setminus G_m)$ such that $v_i^m \in \partial^- f_m(y_i^m)$ for $i \in \{1,2,3\}$. Using (3), we have $f(y_i^m) = f_m(y_i^m)$ for $i \in \{1,2,3\}$. To finish the proof it suffices to use Lemma 2 and the fact that $v_i^m \to 2v_i$ for $m \to \infty$ and $i \in \{1,2,3\}$. □

Theorem 6. *There is a Lipschitz function f on U such that $\partial_a f(x) \neq \partial_c f(x)$ for each $x \in U$.*

Proof. Let f be the function from Proposition 5. Firstly, observe that (A) implies that $0 \notin \partial_a f(x)$. On the other hand, (B) implies that $0 \in \operatorname{conv}(\partial_a f(x)) = \partial_c f(x)$. □

Corollary 7. *For $n \geq 2$ there is a Lipschitz function F_n on $\mathbb{R}^n$ such that $\partial_a f(x) \neq \partial_c f(x)$ for each $x \in \mathbb{R}^n$.*

Proof. Let f be the function from Theorem 6. Choose some $\Phi : \mathbb{R}^2 \to U$ $\mathcal{C}^1-$diffeomorphism and put $F_2 = f \circ \Phi$. Using chain rule for the Dini subdifferential (see for example [5], 3.20) we obtain that for any $x \in \mathbb{R}^2$

$$\partial^- F_2(x) = \bigcup_{y \in \partial^- f(\Phi(x))} y \cdot \Phi'(x).$$

So (by continuity of Φ') the same equality holds for the approximate subdifferential and we are done for $n = 2$. If $n > 2$ it is sufficient to put $F_n(x_1, x_2, \ldots, x_n) = F_2(x_1, x_2)$. □

Remark 8. In fact, the Proposition 5. gives that for some constant $C > 0$ the (Hausdorff) distance of $\partial_a f(x)$ and $\partial_c f(x)$ is greater than C for each $x \in U$. In the proof of Corollary 7. this property fails because of application of the diffeomorphism. If we want to obtain such function on the whole space, one way to do it is to begin the construction with a suitable affine function (for emample $x \to \langle x, v_1 \rangle$) instead of $J^1_{\tilde{T}}$ and then make simillar piramide procedure on the whole space.

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Porosity and the Darboux property of Fréchet derivatives

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Abstract

We study a relation between the porosity of sets in Euclidean spaces and the Darboux property of (relative) Fréchet derivatives.

1 Introduction and main result

A set A in a real Banach space X is said to be porous at $a \in X$ if there are $c > 0$ and $x_n \in X$, $x_n \neq a$, with $x_n \to a$ such that $x \notin A$ whenever $n \in \mathbb{N}$ and $\|x - x_n\| < c\|a - x_n\|$. Let $B \subset X$ be non-empty without isolated points and $f : B \to \mathbb{R}$ be given. We say that $g : B \to X^*$ is a (relative) Fréchet derivative of f on B if

$$\lim_{x \to a,\, x \in B} \frac{f(x) - f(a) - g(a)(x - a)}{\|x - a\|} = 0$$

for each $a \in B$.

The following two results have appeared in [1].

Lemma 1.1 *Let X be a real Banach space, $G \subset X$ open, $a \in \partial G$ and let $X \setminus G$ be porous at a. Let $M := G \cup \{a\}$ and suppose that $g : M \to X^*$ is a Fréchet derivative of a function $f : M \to \mathbb{R}$ on M. Then $(a, g(a))$ belongs to the closure of the graph of $g|_G$ in $X \times X^*$. In particular, $g(a) \in \overline{g(G)}$.*

Theorem 1.2 *Let X be a real Banach space and $B \subset X$ be non-empty such that the interior of B is connected and $X \setminus B$ is porous at every $a \in B \cap \partial B$. Let $g : B \to X^*$ be a Fréchet derivative of a function $f : B \to \mathbb{R}$ on B. Then the graph of g is a connected subset of $X \times X^*$. In particular, $g(B)$ is connected in X^*.*

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In this paper, we prove converses of these results in the case of Euclidean spaces. Proposition 4.2 below corresponds with Lemma 1.1, while the following theorem corresponds with Theorem 1.2.

Theorem 1.3 *Let $B \subset \mathbb{R}^d$ be non-empty without isolated points such that the interior of B is connected. Then the following assertions are equivalent:*
(i) *$\mathbb{R}^d \setminus B$ is porous at every $a \in B \cap \partial B$.*
(ii) *The graph of g is connected whenever g is a Fréchet derivative of a function $f : B \to \mathbb{R}$ on B.*
(iii) *$g(B)$ is connected whenever g is a Fréchet derivative of a function $f : B \to \mathbb{R}$ on B.*

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 1.2 and (ii) $\Rightarrow$ (iii) is clear. Let (i) do not hold. There is $a \in B$ such that $\mathbb{R}^d \setminus B$ is not porous at a. By Proposition 4.2 below, there is $f : \mathbb{R}^d \to \mathbb{R}$, Fréchet differentiable on B, such that $f'(a) = 0$ and $|f'(u)| \geq 1$ for any $u \in B \setminus \{a\}$. Then $g = f'|_B$ is a Fréchet derivative of $f|_B$ on B and 0 is an isolated point of $g(B)$. Thus (iii) does not hold, and the remaining (iii) $\Rightarrow$ (i) is proved. □

2 Preliminaries

Let $d \in \mathbb{N}$ be fixed throughout the whole paper. We denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^d$ and by $B(x, r)$ the open ball around x with radius $r > 0$. We fix ψ a mollification kernel, i.e. a function with properties
1) $\psi \in \mathcal{C}^\infty(\mathbb{R}^d)$,
2) $\psi > 0$ on $B(0, 1)$ and $\psi = 0$ on $\mathbb{R}^d \setminus B(0, 1)$,
3) $\psi(x) = \psi(y)$ if $|x| = |y|$,
4) $\int_{\mathbb{R}^d} \psi = 1$.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^d$ be open and $\rho : \Omega \to (0, \infty)$ be a continuous function. Let $c > 0$. Then there is $\delta \in \mathcal{C}^1(\Omega)$ satisfying $0 < \delta < \rho$ on Ω, Lipschitz with the constant c on Ω.*

Proof. Let $\{B_k\}_{k\in\mathbb{N}}$ be a covering of Ω by open balls such that $\overline{B_k} \subset \Omega$ for each $k \in \mathbb{N}$. Put $m_k = \min_{x\in\overline{B_k}} \rho(x)$. Then the desired function is

$$\sum_{k=1}^{\infty} \frac{m_k}{2^k} \Psi_k,$$

where $\Psi_k : \Omega \to [0, 1)$ is a continuously differentiable function such that $\Psi_k > 0$ on B_k, $\Psi_k = 0$ on $\Omega \setminus B_k$ and $|\Psi_k'| \leq c/m_k$ on Ω. □

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^d$ be open, $\varphi \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and let $\delta \in \mathcal{C}^1(\Omega)$ be positive on Ω. Then, for the function $F : \Omega \to \mathbb{R}$ defined as*

$$F(x) = \int_{\mathbb{R}^d} \varphi(x + \delta(x)y)\psi(y)\, dy,$$

we have $F \in \mathcal{C}^1(\Omega)$.

Proof. We note first that F can be equivalently expressed as

$$F(x) = \frac{G(x)}{\delta(x)^d},$$

where

$$G(x) = \int_{\mathbb{R}^d} \varphi(t) H_t(x)\, dt \quad \text{and} \quad H_t(x) = \psi\Big(\frac{x-t}{\delta(x)}\Big).$$

Fix $x \in \Omega$ and a direction $\nu \in \mathbb{R}^d$. We will prove that
I. $\frac{\partial G}{\partial \nu}(x)$ exists and

$$\frac{\partial G}{\partial \nu}(x) = \int_{\mathbb{R}^d} \varphi(t) \frac{\partial H_t}{\partial \nu}(x)\, dt,$$

II. the mapping

$$s \mapsto \int_{\mathbb{R}^d} \varphi(t) \frac{\partial H_t}{\partial \nu}(s)\, dt$$

is continuous at x.

Choose $\varepsilon > 0$ such that $\overline{B(x,\varepsilon)} \subset \Omega$ and put

$$\Gamma = \overline{\bigcup_{s \in \overline{B(x,\varepsilon)}} B(s, \delta(s))}.$$

Note that, for $s \in \overline{B(x,\varepsilon)}$ and $t \in \mathbb{R}^d \setminus \Gamma$, we have $\frac{\partial H_t}{\partial \nu}(s) = 0$. Moreover, the function $(s,t) \mapsto \frac{\partial H_t}{\partial \nu}(s)$ is continuous on the compact set $\overline{B(x,\varepsilon)} \times \Gamma$, and so there is a constant $C > 0$ with $|\frac{\partial H_t}{\partial \nu}(s)| \le C$ for $(s,t) \in \overline{B(x,\varepsilon)} \times \Gamma$. So

$$\Big|\varphi(t) \frac{\partial H_t}{\partial \nu}(s)\Big| \le C \chi_\Gamma(t) |\varphi(t)|$$

for $s \in \overline{B(x,\varepsilon)}$ and $t \in \mathbb{R}^d$, where χ_Γ is the characteristic function of the set Γ. Now, since $\chi_\Gamma |\varphi| \in \mathcal{L}^1(\mathbb{R}^d)$, I and II are consequences of the standard theorems on integral depending on parameter.

We proved, in particular, that the partial derivatives of G are continuous on Ω, and so $G \in \mathcal{C}^1(\Omega)$. Immediately, $F \in \mathcal{C}^1(\Omega)$ as well. □

Lemma 2.3 *Let $L, K > 0$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a function which is Lipschitz with the constant L, let $\Omega \subset \mathbb{R}^d$ an open set and let $\delta \in \mathcal{C}^1(\Omega)$ be positive and Lipschitz with the constant K/L. Suppose that, for each $x \in \Omega$, there is $\nu_x \in \mathbb{R}^d$, $|\nu_x| = 1$, such that $\frac{\partial \varphi}{\partial \nu_x}(y) \ge 2K$ for almost every $y \in B(x, \delta(x))$. Then the function*

$$F(x) = \int_{\mathbb{R}^d} \varphi(x + \delta(x) y) \psi(y) dy$$

belongs to $\mathcal{C}^1(\Omega)$ and $|F'(x)| \ge K$ for each $x \in \Omega$. Moreover, F is Lipschitz.

Proof. First, note that $F \in \mathcal{C}^1(\Omega)$ due to Lemma 2.2. Now, choose $x \in \Omega$ and a sequence $\{\lambda_n\}$ of non-zero real numbers with $\lambda_n \to 0$. Since $F \in \mathcal{C}^1(\Omega)$, it is sufficient to write

$$
\begin{aligned}
&\frac{\partial F}{\partial \nu_x}(x) = \lim_{n\to\infty} \frac{F(x+\lambda_n\nu_x) - F(x)}{\lambda_n} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} \frac{\varphi(x+\lambda_n\nu_x + \delta(x+\lambda_n\nu_x)y) - \varphi(x+\delta(x)y)}{\lambda_n}\psi(y)\,dy \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^d} \frac{\varphi(x+\lambda_n\nu_x + \delta(x+\lambda_n\nu_x)y) - \varphi(x+\lambda_n\nu_x+\delta(x)y)}{\lambda_n}\psi(y)\,dy \\
&\qquad + \liminf_{n\to\infty} \int_{\mathbb{R}^d} \frac{\varphi(x+\lambda_n\nu_x+\delta(x)y) - \varphi(x+\delta(x)y)}{\lambda_n}\psi(y)\,dy \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^d} -L\frac{|\delta(x+\lambda_n\nu_x) - \delta(x)|}{\lambda_n}|y|\psi(y)\,dy \\
&\qquad + \int_{\mathbb{R}^d} \liminf_{n\to\infty} \frac{\varphi(x+\lambda_n\nu_x+\delta(x)y) - \varphi(x+\delta(x)y)}{\lambda_n}\psi(y)\,dy \\
&\geq \int_{\mathbb{R}^d} -L\frac{K}{L}|y|\psi(y)\,dy + \int_{B(0,1)\setminus N} \frac{\partial\varphi}{\partial\nu_x}(x+\delta(x)y)\psi(y)\,dy \\
&\geq \int_{B(0,1)} -K|y|\psi(y)\,dy + \int_{B(0,1)\setminus N} 2K\psi(y)\,dy \\
&\geq \int_{B(0,1)} K\psi(y)\,dy = K,
\end{aligned}
$$

where N has measure 0. We could use the Fatou lemma because

$$
\frac{\varphi(x+\lambda_n\nu_x+\delta(x)y) - \varphi(x+\delta(x)y)}{\lambda_n}\psi(y) \geq -L\psi(y)
$$

for $n \in \mathbb{N}$ and $y \in \mathbb{R}^d$.

To prove that F is Lipschitz, we write

$$
\begin{aligned}
|F(u) - F(v)| &\leq \int_{\mathbb{R}^d} |\varphi(u+\delta(u)y) - \varphi(v+\delta(v)y)|\psi(y)dy \\
&\leq \int_{\mathbb{R}^d} L(|u-v| + |\delta(u)-\delta(v)||y|)\psi(y)dy \\
&\leq \int_{\mathbb{R}^d} (L|u-v| + K|u-v||y|)\psi(y)dy \\
&= \int_{B(0,1)} (L|u-v| + K|u-v||y|)\psi(y)dy \\
&\leq \int_{B(0,1)} (L+K)|u-v|\psi(y)dy = (L+K)|u-v|.
\end{aligned}
$$

□

Lemma 2.4 *Let (P, ϱ) be a metric space and functions $s, t : P \to \mathbb{R}$ be bounded by M_s, M_t on P. Then the function st is Lipschitz with the constant $M_s L_t + M_t L_s$ in the case that s, t are Lipschitz with the constants L_s, L_t.*

Proof. We have

$$\begin{aligned} |s(x)t(x) - s(y)t(y)| \leq& |s(x)t(x) - s(x)t(y)| + |s(x)t(y) - s(y)t(y)| \\ =& |s(x)||t(x) - t(y)| + |t(y)||s(x) - s(y)| \\ \leq& M_s L_t \varrho(x, y) + M_t L_s \varrho(x, y) \end{aligned}$$

for $x, y \in P$. □

3 Functions on special domains

Let $r_i, s_i \in \mathbb{R}, p_i \in \mathbb{N}$ for $i \in \mathbb{N}$ satisfying

- $r_1 > r_2 > \cdots > 0$,
- $p_1 \leq p_2 \leq \ldots$,
- $r_i \to 0$,
- $\frac{r_{i+1}}{r_i} \to 1$,
- $\frac{s_i}{r_i} \to 0$,
- $p_i \to \infty$
- $|\frac{s_i - s_{i+1}}{r_i - r_{i+1}}| = 1$,
- $\frac{r_i}{r_i - r_{i+1}} \frac{1}{p_i} \leq 2$,

be fixed throughout this section. We put

$$D_p = \Big\{(x_1, \ldots, x_d) \in \partial([-1, 1]^d) : 2px_1, \ldots, 2px_d \in \mathbb{Z}\Big\}, \quad p \in \mathbb{N},$$

$$D = \bigcup_{i \in \mathbb{N}} r_i D_{p_i}.$$

In this section, we denote

$$\|x\| = \|x\|_\infty = \max\{|x_1|, \ldots, |x_d|\}$$

for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Lemma 3.1 *There is a Lipschitz function $F : \mathbb{R}^d \to \mathbb{R}$ with properties*

1. *$F'(0) = 0$,*
2. *$F'(x)$ exists and $|F'(x)| \geq 1/(4\sqrt{d})$ whenever $x \in \mathbb{R}^d \setminus (D \cup \{0\})$.*

The whole section is dedicated to the proof of this lemma.

Define

$$h(x) = \operatorname{dist}(x, \mathbb{Z}), \quad h_0(x) = \operatorname{dist}(x, \{-p_1, \dots, 0, \dots, p_1\}), \quad x \in \mathbb{R},$$

$$g(x_1, \dots, x_d) = \sum_{j=1}^{d} h(x_j), \quad g_0(x_1, \dots, x_d) = \sum_{j=1}^{d} h_0(x_j), \quad (x_1, \dots, x_d) \in \mathbb{R}^d,$$

$$g_t(x) = t^{-1} g(tx), \quad g_{t,0}(x) = t^{-1} g_0(tx), \quad x \in \mathbb{R}^d,\ t > 0.$$

Put $C = 1 + 4d$. For $x \in \mathbb{R}^d$, define

$$\varphi(x) = \begin{cases} 0, & x = 0, \\ \frac{\|x\| - r_{i+1}}{r_i - r_{i+1}} (Cs_i + g_{p_i/r_i}(x)) & \\ \qquad + \frac{r_i - \|x\|}{r_i - r_{i+1}} (Cs_{i+1} + g_{p_{i+1}/r_{i+1}}(x)), & r_{i+1} \le \|x\| < r_i, \\ Cs_1 + g_{p_1/r_1,0}(x), & r_1 \le \|x\|. \end{cases}$$

Claim 3.2 *$\varphi(x)/\|x\| \to 0$ as $x \to 0$.*

Proof. For $x \in \mathbb{R}^d$ and $i \in \mathbb{N}$ with $r_{i+1} \le \|x\| < r_i$, we obtain

$$\begin{aligned} |\varphi(x)| &\le \Big|\frac{\|x\| - r_{i+1}}{r_i - r_{i+1}}\Big| \Big|Cs_i + g_{p_i/r_i}(x)\Big| \\ &\qquad + \Big|\frac{r_i - \|x\|}{r_i - r_{i+1}}\Big| \Big|Cs_{i+1} + g_{p_{i+1}/r_{i+1}}(x)\Big| \\ &\le C|s_i| + |g_{p_i/r_i}(x)| + C|s_{i+1}| + |g_{p_{i+1}/r_{i+1}}(x)| \\ &\le C|s_i| + \frac{r_i}{p_i}\frac{d}{2} + C|s_{i+1}| + \frac{r_{i+1}}{p_{i+1}}\frac{d}{2}, \\ \frac{|\varphi(x)|}{\|x\|} \le \frac{|\varphi(x)|}{r_{i+1}} &\le C\Big|\frac{s_i}{r_i}\Big|\frac{r_i}{r_{i+1}} + C\Big|\frac{s_{i+1}}{r_{i+1}}\Big| + \frac{1}{p_i}\frac{r_i}{r_{i+1}}\frac{d}{2} + \frac{1}{p_{i+1}}\frac{d}{2}. \end{aligned}$$

The properties of the sequences r_i, s_i and p_i guarantee that the right side converges to 0 as i tends to ∞. □

Claim 3.3 *φ is Lipschitz.*

Proof. Obviously, h is Lipschitz with the constant 1 and g, g_t are Lipschitz with the constant d on $\mathbb{R}^d$ (all the Lipschitz constants in the proof are with respect to $\|\cdot\|$). Fix $i \in \mathbb{N}$ and put $U = \{x \in \mathbb{R}^d : r_{i+1} \le \|x\| < r_i\}$. We will investigate separately the functions

$$\begin{aligned} \varphi_1(x) &= \frac{\|x\| - r_{i+1}}{r_i - r_{i+1}} Cs_i + \frac{r_i - \|x\|}{r_i - r_{i+1}} Cs_{i+1}, \\ \varphi_2(x) &= \frac{\|x\| - r_{i+1}}{r_i - r_{i+1}} g_{p_i/r_i}(x), \\ \varphi_3(x) &= \frac{r_i - \|x\|}{r_i - r_{i+1}} g_{p_{i+1}/r_{i+1}}(x), \end{aligned}$$

which satisfy that $\varphi_1 + \varphi_2 + \varphi_3 = \varphi$ on U. For $x, y \in U$, we have

$$\varphi_1(x) - \varphi_1(y) = C(\|x\| - \|y\|)\frac{s_i - s_{i+1}}{r_i - r_{i+1}},$$

and thus $|\varphi_1(x) - \varphi_1(y)| \leq C\|x - y\|$. It follows from Lemma 2.4 that φ_2, φ_3 are Lipschitz with the constants

$$d + \frac{r_i}{p_i}\frac{d}{2}\frac{1}{r_i - r_{i+1}}, \quad d + \frac{r_{i+1}}{p_{i+1}}\frac{d}{2}\frac{1}{r_i - r_{i+1}}$$

on U. Together, we get that φ is Lipschitz with the constant $C+4d$ on U. Even, φ is Lipschitz with this constant on $\overline{U} = \{x \in \mathbb{R}^d : r_{i+1} \leq \|x\| \leq r_i\}$ because $\lim_{x\to z, x\in U} \varphi(x) = Cs_i + g_{p_i/r_i}(z) = \varphi(z)$ whenever $\|z\| = r_i$.

We have proved that φ is Lipschitz with the constant $C + 4d$ on $\{x \in \mathbb{R}^d : r_{i+1} \leq \|x\| \leq r_i\}$ for every $i \in \mathbb{N}$. It is also Lipschitz with this constant (in fact, Lipschitz with the constant d) on $\{x \in \mathbb{R}^d : r_1 \leq \|x\|\}$. Considering the continuity of φ at 0 (Claim 3.2), we see that φ is Lipschitz with the constant $C + 4d$ on $\mathbb{R}^d$. $\square$

Fix $k \in \{1, \dots, d\}$ and $i \in \mathbb{N}$ and differentiate φ on the set $\{x \in \mathbb{R}^d : r_{i+1} < \|x\| < r_i, \|x\| = x_k > |x_j| \text{ for } j \neq k\}$:

$$\varphi(x) = \frac{x_k - r_{i+1}}{r_i - r_{i+1}}\Big(Cs_i + \frac{r_i}{p_i}g\big(\frac{p_i}{r_i}x\big)\Big) + \frac{r_i - x_k}{r_i - r_{i+1}}\Big(Cs_{i+1} + \frac{r_{i+1}}{p_{i+1}}g\big(\frac{p_{i+1}}{r_{i+1}}x\big)\Big),$$

$$\begin{aligned}
\frac{\partial\varphi}{\partial x_j}(x) &= \frac{x_k - r_{i+1}}{r_i - r_{i+1}}h'\big(\frac{p_i}{r_i}x_j\big) + \frac{r_i - x_k}{r_i - r_{i+1}}h'\big(\frac{p_{i+1}}{r_{i+1}}x_j\big), \quad j \neq k,\\
\frac{\partial\varphi}{\partial x_k}(x) &= \frac{x_k - r_{i+1}}{r_i - r_{i+1}}h'\big(\frac{p_i}{r_i}x_k\big) + \frac{r_i - x_k}{r_i - r_{i+1}}h'\big(\frac{p_{i+1}}{r_{i+1}}x_k\big)\\
&\quad + C\frac{s_i - s_{i+1}}{r_i - r_{i+1}} + \frac{1}{r_i - r_{i+1}}\frac{r_i}{p_i}g\big(\frac{p_i}{r_i}x\big) - \frac{1}{r_i - r_{i+1}}\frac{r_{i+1}}{p_{i+1}}g\big(\frac{p_{i+1}}{r_{i+1}}x\big)
\end{aligned}$$

(if the derivatives of h exist). For almost every x with $r_{i+1} < \|x\| < r_i$ and $\|x\| = x_k > |x_j|$ for $j \neq k$, we obtain

$$\begin{aligned}
\frac{\partial\varphi}{\partial\nu_x}(x) &\geq \frac{s_i - s_{i+1}}{r_i - r_{i+1}}\frac{\partial\varphi}{\partial x_k}(x) - \sum_{j\neq k}\Big|\frac{\partial\varphi}{\partial x_j}(x)\Big|\\
&\geq C - \Big|\frac{1}{r_i - r_{i+1}}\frac{r_i}{p_i}g\big(\frac{p_i}{r_i}x\big)\Big| - \Big|\frac{1}{r_i - r_{i+1}}\frac{r_{i+1}}{p_{i+1}}g\big(\frac{p_{i+1}}{r_{i+1}}x\big)\Big|\\
&\quad - \sum_{j=1}^{d}\Big|\frac{x_k - r_{i+1}}{r_i - r_{i+1}}h'\big(\frac{p_i}{r_i}x_j\big)\Big| - \sum_{j=1}^{d}\Big|\frac{r_i - x_k}{r_i - r_{i+1}}h'\big(\frac{p_{i+1}}{r_{i+1}}x_j\big)\Big|\\
&\geq C - 4d = 1,
\end{aligned}$$

where ν_x denotes $(((s_i - s_{i+1})/(r_i - r_{i+1}))/\|x\|)x$.

Claim 3.4 *For every $x \in \mathbb{R}^d \setminus (D \cup \{0\})$, there is a direction $\nu \in \mathbb{R}^d, \|\nu\| = 1$, and a neighborhood U_x of x such that $\frac{\partial \varphi}{\partial \nu}(y) \geq 1/2$ for almost every $y \in U_x$.*

Proof. Due to the symmetry, we may suppose that $x_j \geq 0, j = 1, \ldots, d$.

Consider cases:

(1) Let $\|x\| = r_i$ for some $i \in \mathbb{N}, i \geq 2$. As $x \notin r_i D_{p_i}$, there is $j \in \{1, \ldots, d\}$ such that $2p_i x_j / r_i \notin \mathbb{Z}$. Denote $\tau = h'(p_i x_j / r_i) \in \{-1, 1\}$ and choose $\varepsilon > 0$ such that $\varepsilon \leq (1/4) \min\{r_i - r_{i+1}, r_{i-1} - r_i\}, 2\varepsilon < r_i - x_j$ and $h'(p_i a / r_i) = \tau$ whenever $|x_j - a| \leq \varepsilon$. Put $\nu = \tau e_j$ and $U_x = \{y \in \mathbb{R}^d : \|y - x\| \leq \varepsilon\}$. For almost every $y = (y_1, \ldots, y_d) \in U_x$, there is $k \in \{1, \ldots, d\}$ such that $\|y\| = y_k > |y_{j'}|$ for $j' \neq k$ and the derivatives $h'(\frac{p_{i+1}}{r_{i+1}} y_j)$ and $h'(\frac{p_{i-1}}{r_{i-1}} y_j)$ exist (in such a case, $k \neq j$ because $y_k \geq \|x\| - \varepsilon = r_i - \varepsilon > x_j + \varepsilon \geq y_j$ by the choice of ε). So, for almost every $y = (y_1, \ldots, y_d) \in U_x$ with $\|y\| < r_i$, we have (for some k)

$$\begin{aligned}
\frac{\partial \varphi}{\partial \nu}(y) = \tau \frac{\partial \varphi}{\partial x_j}(y) &= \tau \frac{y_k - r_{i+1}}{r_i - r_{i+1}} h'\big(\frac{p_i}{r_i} y_j\big) + \tau \frac{r_i - y_k}{r_i - r_{i+1}} h'\big(\frac{p_{i+1}}{r_{i+1}} y_j\big) \\
&= \frac{y_k - r_{i+1}}{r_i - r_{i+1}} + \tau \frac{r_i - y_k}{r_i - r_{i+1}} h'\big(\frac{p_{i+1}}{r_{i+1}} y_j\big) \\
&\geq \frac{y_k - r_{i+1}}{r_i - r_{i+1}} - \frac{r_i - y_k}{r_i - r_{i+1}} \\
&= 1 - 2 \frac{\|x\| - \|y\|}{r_i - r_{i+1}} \geq 1 - 2 \frac{\varepsilon}{r_i - r_{i+1}} \geq 1/2,
\end{aligned}$$

while, for almost every $y = (y_1, \ldots, y_d) \in U_x$ with $\|y\| > r_i$, we have (for some k)

$$\begin{aligned}
\frac{\partial \varphi}{\partial \nu}(y) = \tau \frac{\partial \varphi}{\partial x_j}(y) &= \tau \frac{y_k - r_i}{r_{i-1} - r_i} h'\big(\frac{p_{i-1}}{r_{i-1}} y_j\big) + \tau \frac{r_{i-1} - y_k}{r_{i-1} - r_i} h'\big(\frac{p_i}{r_i} y_j\big) \\
&= \tau \frac{y_k - r_i}{r_{i-1} - r_i} h'\big(\frac{p_{i-1}}{r_{i-1}} y_j\big) + \frac{r_{i-1} - y_k}{r_{i-1} - r_i} \\
&\geq \frac{r_{i-1} - y_k}{r_{i-1} - r_i} - \frac{y_k - r_i}{r_{i-1} - r_i} \\
&= 1 - 2 \frac{\|y\| - \|x\|}{r_{i-1} - r_i} \geq 1 - 2 \frac{\varepsilon}{r_{i-1} - r_i} \geq 1/2.
\end{aligned}$$

(2) Let $\|x\| = r_1$. In this case, the procedure is similar to the procedure of (1) (choosing $j, \tau, \varepsilon, \nu$ and U_x as in (1), we have $\frac{\partial \varphi}{\partial \nu}(y) \geq 1/2$ for almost every $y = (y_1, \ldots, y_d) \in U_x$ with $\|y\| < r_1$ and we can easily check that $\frac{\partial \varphi}{\partial \nu}(y) = 1$ for every $y = (y_1, \ldots, y_d) \in U_x$ with $\|y\| \geq r_1$).

(3) Let $r_{i+1} < \|x\| < r_i$ for some $i \in \mathbb{N}$. We define

$$V = \big\{y \in \mathbb{R}^d : r_{i+1} < \|y\| < r_i, \|y\| = y_k \geq \max_{j \neq k} |y_j| \text{ for some } k\big\}.$$

We supposed that $x_j \geq 0, j = 1, \ldots, d$. Therefore, V is a neighbourhood of x. We have

$$\frac{\partial \varphi}{\partial \nu_x}(y) = \frac{\partial \varphi}{\partial \nu_y}(y) + \varphi'(y)(\nu_x - \nu_y) \geq 1 - |\varphi'(y)||\nu_x - \nu_y|$$

for almost every $y \in V$, where ν_x and ν_y denote $(((s_i - s_{i+1})/(r_i - r_{i+1}))/\|x\|)x$ and $(((s_i - s_{i+1})/(r_i - r_{i+1}))/\|y\|)y$, as above. Now, the existence of an appropriate U_x follows from the continuity of $y \mapsto \nu_y$ and from Claim 3.3.

(4) Let $\|x\| > r_1$. We choose a k with $x_k > r_1$ and take $U_x = \{(y_1, \dots, y_d) \in \mathbb{R}^d : y_k > r_1\}$. If ν denotes e_k, then

$$\frac{\partial \varphi}{\partial \nu}(y) = \frac{\partial g_{p_1/r_1,0}}{\partial x_k}(y) = h_0'\big(\frac{p_1}{r_1} y_k\big) = 1$$

for every $y \in U_x$. □

Now, for every $x \in \mathbb{R}^d \setminus (D \cup \{0\})$, we define $\rho(x)$ as the supremum of numbers $r \le |x|$ for which there is $\nu \in \mathbb{R}^d$, $|\nu| \le 1$, such that $\frac{\partial \varphi}{\partial \nu}(y) \ge 1/(2\sqrt{d})$ for almost every $y \in B(x, r)$. By Claim 3.4, $\rho > 0$ on $\mathbb{R}^d \setminus (D \cup \{0\})$. Obviously, ρ is Lipschitz (with the constant 1 with respect to $|\cdot|$). By Claim 3.3, we can take $L > 0$ such that φ is Lipschitz with the constant L (with respect to $|\cdot|$). By Lemma 2.1, there is $\delta \in \mathcal{C}^1(\mathbb{R}^d \setminus (D \cup \{0\}))$ satisfying $0 < \delta < \rho$, Lipschitz with the constant $1/(4\sqrt{d}L)$. We define F on $\mathbb{R}^d \setminus (D \cup \{0\})$ first by

$$F(x) = \int_{\mathbb{R}^d} \varphi(x + \delta(x)y)\psi(y)dy, \quad x \in \mathbb{R}^d \setminus (D \cup \{0\}).$$

By Lemma 2.3 (applied on $K = 1/(4\sqrt{d})$), F is Lipschitz and differentiable on $\mathbb{R}^d \setminus (D \cup \{0\})$ and property 2. from Lemma 3.1 is satisfied. We extend F on $\mathbb{R}^d$ to be Lipschitz. Property 1. follows now from Claim 3.2 and from

$$\sup_{x \in B(0,r)} |F(x)| \le \sup_{x \in B(0,r) \setminus (D \cup \{0\})} \ \sup_{t \in B(x, \delta(x))} |\varphi(t)| \le \sup_{t \in B(0,2r)} |\varphi(t)|$$

for $r > 0$. This completes the proof of Lemma 3.1.

4 General case

Lemma 4.1 *Let $r > 0$ and $x, y \in \mathbb{R}^d$ be such that $|x - y| < r/2$. Then there is a diffeomorphism $\Psi : \mathbb{R}^d \to \mathbb{R}^d$, Lipschitz with the constant 2, such that $\Psi(u) = u$ for $u \in \mathbb{R}^d \setminus B(x, r)$, $\Psi(y) = x$ and $|v \circ \Psi'(u)| \ge \frac{2}{3}|v|$ for any $u \in \mathbb{R}^d$ and $v \in (\mathbb{R}^d)^*$.*

Proof. Without loss of generality $x = 0$, $y = (|y|, 0, 0, \dots, 0)$ and $r = 1$. Let $\phi : [0, \infty) \to \mathbb{R}$ be a function which is differentiable everywhere in $(0, \infty)$ and right differentiable at 0 such that $\phi(0) = |y|$, $\phi(\xi) = 0$ for $\xi \ge 1$, $\phi'_+(0) = 0$ and $|\phi'(\xi)| \le 1/2$ for $\xi > 0$. Define $\Phi : \mathbb{R}^d \to \mathbb{R}$ and $\Theta : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\Phi(s) = \phi(|s|) \quad \text{and} \quad \Theta(s) = (s_1 + \Phi(s), s_2, \dots, s_d),$$

where $s = (s_1, s_2, \dots, s_d) \in \mathbb{R}^d$. Now, Θ is a diffeomorphism on $\mathbb{R}^d$ which is identity on $\mathbb{R}^d \setminus B(0, 1)$ and $\Theta(0) = y$. Put $\Psi = \Theta^{-1}$. For $s \in \mathbb{R}^d$ and $t \in (\mathbb{R}^d)^*$, we have

$$|t \circ \Theta'(s)| = |t + t(e_1)\Phi'(s)| \le \frac{3}{2}|t|.$$

Moreover, for $s, s' \in \mathbb{R}^d$, we have

$$|\Theta(s) - \Theta(s')| \geq |s - s'| - |\Phi(s) - \Phi(s')| \geq \frac{1}{2}|s - s'|.$$

So $|v \circ \Psi'(u)| \geq \frac{2}{3}|v|$ for $u \in \mathbb{R}^d, v \in (\mathbb{R}^d)^*$, and Ψ is Lipschitz with the constant 2. □

Proposition 4.2 *Let $a \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d \setminus \{a\}$ be a set which is not porous at a. Then there is a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, Fréchet differentiable on $\mathbb{R}^d \setminus E$, such that $f'(a) = 0$ and $|f'(u)| \geq 1$ for any $u \in \mathbb{R}^d \setminus (E \cup \{a\})$.*

Proof. Without loss of generality $a = 0$. Put $I = [-1, 1]^d$. Since E is not porous at 0, there is, for any $k \in \mathbb{N}$, some minimal $n_k \in \mathbb{N}$ such that, for any $r \in (0, 2^{-n_k}]$, $rI \subset E + B(0, r/10^{2k})$. Put

$$k(n) = \max_{n_k \leq n} k \quad \text{for } n \geq n_1,$$

$$r_{n,l} = \frac{1}{2^n} - \frac{10l}{2^{n+1} \cdot 10^{2k(n)}} \quad \text{and} \quad p_{n,l} = 10^{2k(n)-1} \quad \text{for } l = 0, \ldots, 10^{2k(n)-1} - 1.$$

Rearrange $r_{n,l}$ into the decreasing sequence $\{r_i\}_{i=1}^{\infty}$ and $\{p_i\}_{i=1}^{\infty}$ be the sequence of the corresponding $p_{n,l}$'s. Put

$$s_1 = 0 \quad \text{and} \quad s_{i+1} = s_i + (-1)^{i+1}(r_i - r_{i+1}) \quad \text{for } i \geq 1.$$

Note that $s_i = 0$ and $s_{i+1} = r_i - r_{i+1}$ if i is odd. One can compute that

$$\frac{r_i}{r_i - r_{i+1}} \frac{1}{p_i} = 2 - \frac{10l}{10^{2k(n)}} \quad \text{and} \quad 1 \geq \frac{r_{i+1}}{r_i} \geq 1 - \frac{10}{10^{2k(n)}}$$

for the $n \in \mathbb{N}$ and $l \in \{0, \ldots, 10^{2k(n)-1} - 1\}$ corresponding to i, and so

$$\sup \frac{r_i}{r_i - r_{i+1}} \frac{1}{p_i} \leq 2 \quad \text{and} \quad \lim_{n \to \infty} \frac{r_{i+1}}{r_i} = 1.$$

Moreover,

$$\left| \frac{s_i - s_{i+1}}{r_i - r_{i+1}} \right| = 1 \quad \text{and} \quad \frac{s_i}{r_i} \leq \frac{10}{10^{2k(n)}} \quad \text{for all } i \in \mathbb{N},$$

and so $s_i/r_i \to 0$ for $i \to \infty$. Let F be a function which Lemma 3.1 gives for these r_i's and p_i's.

Now, choose $x \in r_i D_{p_i}$. There are some n and l such that $r_i = r_{n,l}$ and $p_i = p_{n,l}$. So there is some $u_x \in E$ with $|x - u_x| < r_{n,l}/10^{2k(n)}$. Put $B_x = B(x, 2r_{n,l}/10^{2k(n)})$ and, by Lemma 4.1, choose a diffeomorphism $\Psi_x : \mathbb{R}^d \to \mathbb{R}^d$, Lipschitz with the constant 2, which is identity on $\mathbb{R}^d \setminus B_x$ and maps u_x onto x such that $|v \circ \Psi_x'(u)| \geq \frac{2}{3}|v|$ for any $u \in \mathbb{R}^d$ and $v \in (\mathbb{R}^d)^*$. Let x_1, x_2 be distinct elements of $D = \bigcup_{i\in\mathbb{N}} r_i D_{p_i}$ with the corresponding r_{n_1,l_1}, p_{n_1,l_1}, r_{n_2,l_2} and p_{n_2,l_2}. We may suppose that $r_{n_1,l_1} \geq r_{n_2,l_2}$. Then

$$|x_1 - x_2| \geq \frac{r_{n_1,l_1}}{2p_{n_1,l_1}} = 5\frac{r_{n_1,l_1}}{10^{2k(n_1)}}$$

if $r_{n_1,l_1} = r_{n_2,l_2}$ and

$$|x_1 - x_2| \geq r_{n_1,l_1} - r_{n_2,l_2} \geq \frac{10}{2^{n_1+1}\cdot 10^{2k(n_1)}} \geq 5\frac{r_{n_1,0}}{10^{2k(n_1)}} \geq 5\frac{r_{n_1,l_1}}{10^{2k(n_1)}}$$

if $r_{n_1,l_1} > r_{n_2,l_2}$. In both cases,

$$|x_1 - x_2| \geq 5\frac{r_{n_1,l_1}}{10^{2k(n_1)}} > \frac{2r_{n_1,l_1}}{10^{2k(n_1)}} + \frac{2r_{n_2,l_2}}{10^{2k(n_2)}}.$$

So $B_{x_1} \cap B_{x_2} = \emptyset$ and we can define a one-to-one mapping $\Psi : \mathbb{R}^d \to \mathbb{R}^d$, differentiable on $\mathbb{R}^d \setminus \{0\}$ and Lipschitz with the constant 2, by

$$\Psi(u) = \begin{cases} \Psi_x(u) & \text{if } u \in B_x, \\ u & \text{if } u \in \mathbb{R}^d \setminus \bigcup_{x\in D} B_x. \end{cases}$$

Put $f = (6\sqrt{d})F \circ \Psi$. Since f is a composition of Lipschitz mappings, it is Lipschitz. We have $\Psi^{-1}(D) \subset E$, and thus f is differentiable everywhere in $\mathbb{R}^d \setminus E$. For $u \in \mathbb{R}^d \setminus (E \cup \{0\})$, we have

$$|f'(u)| = (6\sqrt{d})|F'(\Psi(u)) \circ \Psi'(u)| \geq \frac{2}{3}(6\sqrt{d})|F'(\Psi(u))| \geq 1$$

by property 2. of the function F. Finally, $f'(0) = 0$. It follows from property 1. and from

$$f(B(0,r)) = (6\sqrt{d})F(\Psi(B(0,r))) \subset (6\sqrt{d})F(B(0,2r))$$

for every $r > 0$. □

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References

[1] P. Holický, C. E. Weil and L. Zajíček: *A note on the Darboux property of Fréchet derivatives*, Real Anal. Exchange **32** (2007), 489-494.