Referee's report on doctoral thesis "Asymptotic Behaviors of Solutions in Problems of the Mathematical Theory of Fluids

by Peter Kukucka

The doctoral thesis of Peter Kukucka is composed of Introduction (Chapter 1) and another three chapters (Chapter 2-4). Each of the latter three chapters is based on Kukucka's either published or submitted articles.

In Chapter 1 the author introduces equations to be investigated in the thesis - 1) Navier-Stokes-Fourier system, 2) Navier-Stokes-Fourier system coupled with the Maxwell equations in dimensionless form 3) Navier-Stokes-Fourier system on an exterior domain in dimensionless form - and explains briefly physical background of these equations.

Chapter 2 is the most original part of Kukucka's thesis. It is devoted to the investigation of the Navier-Stokes equations for viscous compressible gas in isentropic regime. The evolution of the equations is considered in domains of class $W^{1,s}$ (see Definition 2.2.1) that are not necessarily Lipschitz. This is the main difficulty to be handled in this chapter.

The standard estimates of the density (resp. pressure) rely on well known estimates for a specific branch of div^{-1} operator - the so called Bogovskii operator - that requires domains with at least Lipschitz boundaries. The classical approach at this point is not applicable in domains with less than Lipschitz regularity. Instead, Kukucka proves uniform equi-integrability of the sequence of densities near the boundary by using a special test function in the momentum equation. This test function, which is bounded and whose divergence explodes near the boundary, is constructed in Lemma 2(p.23) and employed in the momentum equation in Lemma 8 (p.25).

Once this task done, the construction of approximations as well as the limit passage from approximations to weak solutions of the original system follows closely the argumentation of [15], [30].

In Chapter 3, the author investigates the magnetohydrodynamic equations describing a viscous compressible heat conducting gas written in a non dimensional form with Mach and Alfvén numbers equal to a small parameter ϵ (see equations (3.17)-(3.22), endowed with boundary conditions (3.13)-(3.15) and constitutive equations (3.7)-(3.12)). A singular limit to this system as $\epsilon \to 0+$ is investigated in the context of weak solutions, under hypotheses (3.35)-(3.44) for internal energy, pressure and coefficients entering constitutive relations for heat flux and viscous stress tensor, and under conditions (3.45)-(3.47) for the initial data. It should be noticed, that the initial data are ill prepared.

It is shown that the sequence of weak solutions $(\varrho_\epsilon, u_\epsilon, \vartheta_\epsilon, B_\epsilon)$ - density, velocity, absolute temperature and the magnetic induction - admit a weakly converging subsequence whose limit $(\overline{\varrho} = \text{const.}, U, \Theta, B)$ satisfies the Oberbeck-Boussinesq equations describing a heat convection in an incompressible fluid of density $\overline{\varrho}$ coupled with the Maxwell equations, see (3.30)-(3.34). This result is a generalization of the result by Feireisl, Novotny: On the Oberbeck-Boussinesq approximation as a singular limit of the full Navier-Stokes-Fourier system, J. Math. Fluid. Mech. 11 (2009).

Once it is observed that $(\varrho_\epsilon, u_\epsilon, \vartheta_\epsilon, B_\epsilon)$ satisfies the dissipation inequality, see (3.56), and once one realizes that the boundedness of $\text{curl}B_\epsilon/\epsilon$ and B_ϵ/ϵ in L^2 implies the boundedness of $\nabla B_\epsilon/\epsilon$ in L^2 -recall that $\text{div}B_\epsilon = 0$ and that the normal component of B is zero at the boundary- it is possible to follow step by step the proof established in [13], [14].

The most difficult part in the limit process is the passage to the limit in the convective term, which uses in essential way the fact that the quantities $H^\perp(\varrho_\epsilon u_\epsilon)$ and $(\varrho_\epsilon - \overline{\varrho})/\epsilon$ ($H^\perp$ denotes an orthogonal Helmhlotz projection) verify a wave equation. It is effectively true that the wave operator in the investigated case is the same as in Chapter 5 of [13], nevertheless, more details about this fact could be useful. The redaction of this part, which Kukucka outlines in Section 5.4 is, to my opinion, too sketchy for a doctoral thesis.

Chapter 4 of the thesis is devoted to the low Mach, low Péclet and low Froude numbers singular limit in the complete Navier-Stokes-Fourier system (4.2)-(4.5), where $\text{Ma} = \text{Fr} = \epsilon$ and $\text{Pe} = \epsilon^2$. The investigated limit is inspired by Chapter 6 in [13] and combines the problem of strong stratification with the propagation of acoustic waves in unbounded domains. In contrast to Chapter 3, the limiting density is not constant but a space dependent function determined by the equilibrium density corresponding to the potential force ∇F. Moreover, the convergence of the convective term is established thanks to the strong convergence of the velocity field which is achieved due to the dispersion effects in the related wave equation with the space dependent wave speed. In absence of the corresponding Strichartz estimates in this situation, the author uses an approach based on the abstract Kato's theory (see Theorem 4.4.1) inspired by [12].

The mathematical treatment of the problems investigated in the thesis requires a large scale of tools ranging from classical functional analysis and the theory of partial differential equtions (as the Faedo-Galerkin method, the theory and estimates parabolic equations, maximum and comparison principles for these equations, the properties of the Bogovskii solutions to the equation $\text{div}\mathbf{v} = f$, classical compactness results ...) to the recent modern tools in the analysis of PDEs (as various compensated compactness results, non-trivial applications of the div-curl lemma, renormalized continuity equation used for the analysis of oscillations, defect measures, various properties of weak convergence related to the convex and/or monotone functions, effective viscous flux and its properties derived by using the modern tools of harmonic analysis ...).

At the point of conclusion: The thesis of Peter Kukucka clearly shows that the candidate masters efficiently these tools and is able to apply them to various non trivial mathematical problems. To my opinion, the presented dissertation meets the high standards of the Charles University of Prague for the deliverance of the PhD Diploma.

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