Charles University, Prague,
Faculty of Mathematics and Physics

Doctoral Thesis

Asymptotic Behaviors of Solutions in Problems of the Mathematical Theory of Fluids

RNDr. Peter Kukučka

Institute of Mathematics, Academy of Sciences of the Czech Republic,
Žitná 25, Prague, Czech Republic

Advisor's name: RNDr. Eduard Feireisl, DrSc.
Study branch: M-3 Mathematical analysis

2009

Univerzita Karlova v Praze,
Matematicko-fyzikální fakulta

Dizertačná práca

Asymptotické chovania riešení v problémoch matematickej teorie tekutín

RNDr. Peter Kukučka

Matematický ústav AVČR, v.v.i.
Žitná 25, Praha

Školiteľ: RNDr. Eduard Feireisl, DrSc.
Študijný odbor: M-3 Matematická analýza

2009

Acknowledgements

I would like to express special thanks to my supervisor RNDr. Eduard Feireisl, DrSc. for his invaluable help, fruitful ideas, suggestions and patience with me and time he donated me throughout the elaboration of the thesis.

Moreover, the results published in this thesis were supported by the Jindřich Nečas Center for Mathematical Modelling, project LC06052, financed by the Czech Ministry for Education, Youth and Sports.

Statement of Honesty

I claim that this thesis was written by myself, with only help of the referred literature.

Prague, September 15, 2009, RNDr. Peter Kukučka

Contents

Názov práce: Asymptotické chovania riešení v problémoch matematickej teorie tekutín
Autor: RNDr. Peter Kukučka
Ústav: Matematický ústav AVČR, v.v.i.
Školiteľ: RNDr. Eduard Feireisl, DrSc.
Email školiteľa: `feireisl@math.cas.cz`
Abstrakt: Práca obsahuje súbor článkov týkajúcich sa k prúdednia viskóznych, stlačiteľných a tepelne vodivých tekutín na niekoľkých typoch oblastí. Prvá časť práce sa zaoberá existenciou slabých riešení na oblastiach, ktoré obsahujú hroty. Ďalšia čast venuje pozornosť limitnému prechodu rovníc magnetohydrodynamiky pozostavajúcich z Navier-Stokes-Fourierovho systému opisujúceho prúdenie tekutiny a Maxwellových rovníc riadiacich správanie magnetického poľa pre malé Machovo a Alfvénovo číslo. Na záver sa študuje limitný prechod Navier-Stokes-Fourierovho systému na neohraničenej oblasti za predpokladu silnej stratifikácie. Špeciálna pozornosť je zameraná na akustické vlny, ktorých analýza je založená na poklese lokálnej energie.
Kľúčové slová: Navier-Stokes-Fourierov systém, Maxwellove rovnice, akustická rovnica, Oberbeck-Boussinesqova aproximácia, silná stratifikácia.

Title: Asymptotic Behaviors of Solutions in Problems of the Mathematical Theory of Fluids
Author: RNDr. Peter Kukučka
Institute: Institute of Mathematics, Academy of Sciences of the Czech Republic
Advisor: RNDr. Eduard Feireisl, DrSc.
Advisor's email: `feireisl@math.cas.cz`
Abstract: This thesis contains a set of articles concerned with flow of a viscous, compressible and heat conducting fluids in several kinds of domains. The first part is devoted to the existence of weak solutions in domains that may contain cusps. Next chapter is focused on the asymptotic limit of the equations of magnetohydrodynamics consisting of Navier-Stokes-Fourier system describing the evolution of fluid coupled with Maxwell equations governing the behavior of magnetic field with the low Mach and Alfvén number. At the end of the thesis, we study the asymptotic limit passage of the Navier-Stokes-Fourier system under the strong stratification defined in unbounded domain. Special attention is paid to the acoustic waves which analysis is based on local energy decay.
Keywords: Navier-Stokes-Fourier system, Maxwell equations, acoustic equation, Oberbeck-Boussinesq approximation, strong stratification.

Chapter 1

Introduction

One of the most challenging problems of mathematics in recent decades concerns with the equations describing motion of a viscous, compressible and heat conducting fluid:

$$\begin{aligned} \varrho_t + \operatorname{div}(\varrho\vec{u}) &= 0\,, \\ (\varrho\vec{u})_t + \operatorname{div}(\varrho\vec{u}\otimes\vec{u}) + \nabla p &= \operatorname{div}\mathbb{S} + \varrho\vec{f}\,, \\ (\varrho s)_t + \operatorname{div}(\varrho s\vec{u}) + \operatorname{div}(\frac{\vec{q}}{\vartheta}) &= \frac{1}{\vartheta}\mathbb{S} : \nabla\vec{u} - \frac{\vec{q}\cdot\nabla\vartheta}{\vartheta^2}\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e\Big)\mathrm{d}x &= \int_\Omega \varrho\vec{f}\cdot\vec{u}\mathrm{d}x + \int_{\partial\Omega} q\,\mathrm{d}S_x\,, \end{aligned} \tag{1.1}$$

in $(0\,, T)\times\Omega$, where we will assume that the boundary $\partial\Omega$ is compact. The set of the introduced equations corresponds to a family of physical laws:

- The first equation expresses the conservation of mass, i.e. the rate at which mass enters a system is equal to the rate at which mass leaves the system.
- The second equation is the momentum equation expressing the Newton second law: the Rate of change of momentum of a body is equal to the resultant force acting on the body, and takes place in the direction of the force.
- The third equation, called the entropy balance equation, can be viewed as a mathematical formulation of the second law of thermodynamics.
- The last one is the conservation of energy, i.e. total amount of energy changes only because of action of external forces or its flux through the boundary.

The system (1.1) is supplemented with the complete slip boundary condition

$$\vec{u}\cdot\vec{n}|_{\partial\Omega}=0\,,\ \mathbb{S}\vec{n}\times\vec{n}|_{\partial\Omega}=0\,,\vec{q}\cdot\vec{n}|_{\partial\Omega}=q\,, \tag{1.2}$$

which expresses impermeability of the boundary $\partial\Omega$ and the fact that tangential component of the normal stress forces vanishes on the boundary, or the no-slip boundary conditions for the velocity

$$\vec{u}|_{\partial\Omega}=0\,. \tag{1.3}$$

Moreover, if the magnetic field is taken into account, the system (1.1) reads as follows

$$\begin{gathered}
\varrho_t+\operatorname{div}(\varrho\vec{u})=0\,,\\
(\varrho\vec{u})_t+\operatorname{div}(\varrho\vec{u}\otimes\vec{u})+\nabla p=\operatorname{div}\mathbb{S}+\varrho\vec{f}+\vec{J}\times\vec{B}\,,\\
(\varrho s)_t+\operatorname{div}(\varrho s\vec{u})+\operatorname{div}(\frac{\vec{q}}{\vartheta})=\frac{1}{\vartheta}\mathbb{S}:\nabla\vec{u}-\frac{\vec{q}\cdot\nabla\vartheta}{\vartheta^2}+\frac{\lambda}{\mu}|\mathrm{curl}\vec{B}|^2\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(\frac{1}{2}\varrho|\vec{u}|^2+\varrho e+\frac{1}{2\mu}|\vec{B}|^2\Big)\mathrm{d}x=\int_\Omega\varrho\vec{f}\cdot\vec{u}\mathrm{d}x+\int_{\partial\Omega}q\,\mathrm{d}S_x\,,\\
\vec{B}_t+\mathrm{curl}(\vec{B}\times\vec{u})+\mathrm{curl}(\lambda\mathrm{curl}\vec{B})=0\,,
\end{gathered} \tag{1.4}$$

where the last equation is the Maxwell equation. Since the system is assumed to be energetically isolated, we prescribe the additional boundary condition

$$\vec{B}\cdot\vec{n}|_{\partial\Omega}=0\,. \tag{1.5}$$

The formulation of the systems (1.1) and (1.4) is by no means complete. The behavior of the system's unknowns density, velocity, temperature (and magnetic induction in case of (1.4)) is determined also by the particular kind of interactions inside the fluid and set of laws governing the electromagnetic field. This is described by so called constitutive relations on the structure of the viscous stress tensor $\mathbb{S}$, the pressure p, the external force $\vec{f}$, and the heat flux $\vec{q}$.

The question of the existence of the solution of (1.1), (1.4) can be studied by two approaches: the way of strong solutions, and the route of weak solution. The main tool for the first approach is the fixed point argument which assures smooth solutions but under the following restrictive assumptions: small initial data or short time intervals. This drawbacks are overcome by the route of weak solutions. This approach is based on the construction using Faedo-Galerkin approximation coupled with addition of vanishing mollifying terms into the system and then passing to the limit in the mollified system. This method provides the existence result for both large initial data and long time intervals. Despite the lack of regularity property, the weak solution is still enough to ensure that the physical principles motivating (1.1), (1.4) are valid.

The existence of weak solutions for the special case of (1.1) (e.i. the case that the temperature ϑ is not taken into account, called also Navier-Stokes system, and the pressure $p(\varrho) \sim \varrho^\gamma$ with $\gamma \geq \frac{9}{5}$) was first proved by Lions [26], and then extended by Feireisl to the range $\gamma > \frac{3}{2}$ in [15] provided the underlying spatial domain Ω is at least of class $C^{2+\nu}$. In [16], the authors investigate the above mentioned special case of (1.1) posed on an arbitrary bounded domain Ω, where Ω is approximated by a suitable sequence of smooth domains. By means of this method, one gets a sequence of approximate solutions that converges to another weak solution of the problem on the limit domain. Such a solution has bounded total energy but is not known to satisfy the energy inequality in the differential form. Certain improvements in this direction have been achieved by Poul in [31], where the existence problem is studied on Lipschitz domains. Note that all the above cited authors use the so-called Bogovskii operator - a specific branch of div^{-1} (see e.g. [30]) - in order to get uniform estimates of the pressure. The basic properties of this operator depend on regularity of the domain, in particular the latter must be at least Lipschitz. The main goal of the first part of this thesis is to extend the results of [15] to the class of certain domains that may contains cusps.

The rest of the thesis is devoted to investigation of singular limits of systems that arise from the systems (1.1), (1.4). The importance of the singular limits arises from the fact that the full Navier-Stokes-Fourier system (1.1) or the equations of magnetohydrodynamics (1.4) describe the entire spectrum of possible motions - ranging from sound waves, cyclone waves in the atmosphere, to models of gaseous stars in astrophysics. This approach consists of two steps: scaling and asymptotic analysis. By scaling the equations, the parameters determining the behavior of the system become explicit. Asymptotic analysis provides a useful tool in the situation when certain of these parameters called characteristic numbers vanish or become infinite. The main goal of the asymptotic analysis is to derive a simplified set of equations solvable either analytically or with less numerical effort.

Let's repeat some basic features of scaling, explained in [13], Chapter 4. For all physical quantities appearing in (1.1), (1.4), we can identify their characteristic values: the reference time T_{ref}, the reference density ϱ_{ref}, the reference length L_{ref}, the reference velocity U_{ref}, the reference temperature ϑ_{ref}, the reference magnetic induction B_{ref}, together with the characteristic values of other composed quantities p_{ref}, e_{ref}, μ_{ref}, η_{ref}, κ_{ref}, λ_{ref}, and the source term f_{ref}.

Introducing new independent and dependent variables $X' = X/X_{\mathrm{ref}}$, considering such special scaling that only certain characteristic numbers (specified later) occur in the resulting system, and omitting the primes in the rescaled system, we arrive at the following system:

$$
\begin{aligned}
&\varrho_t + \operatorname{div}(\varrho\vec{u}) = 0\,,\\
&(\varrho\vec{u})_t + \operatorname{div}(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{\mathrm{Ma}^2}\nabla p = \operatorname{div}\mathbb{S} + \frac{1}{\mathrm{Fr}^2}\varrho\vec{f}\,,\\
&(\varrho s)_t + \operatorname{div}(\varrho s\vec{u}) + \frac{1}{\mathrm{Pe}}\operatorname{div}(\frac{\vec{q}}{\vartheta}) = \frac{1}{\vartheta}\Big(\mathrm{Ma}^2\,\mathbb{S}:\nabla\vec{u} - \frac{1}{\mathrm{Pe}}\frac{\vec{q}\cdot\nabla\vartheta}{\vartheta}\Big)\,,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(\frac{\mathrm{Ma}^2}{2}\varrho|\vec{u}|^2 + \varrho e\Big)\mathrm{d}x = \int_\Omega\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\varrho\vec{f}\cdot\vec{u}\mathrm{d}x + C_{\mathrm{ref}}\int_{\partial\Omega} q\,\mathrm{d}S_x\,,
\end{aligned}
\tag{1.6}
$$

or for the magnetohydrodynamics equations:

$$
\begin{aligned}
&\varrho_t + \operatorname{div}(\varrho\vec{u}) = 0\,,\\
&(\varrho\vec{u})_t + \operatorname{div}(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{\mathrm{Ma}^2}\nabla p = \operatorname{div}\mathbb{S} + \frac{1}{\mathrm{Fr}^2}\varrho\vec{f} + \frac{1}{\mathrm{Al}^2}\vec{J}\times\vec{B}\,,\\
&(\varrho s)_t + \operatorname{div}(\varrho s\vec{u}) + \frac{1}{\mathrm{Pe}}\operatorname{div}(\frac{\vec{q}}{\vartheta}) = \frac{1}{\vartheta}\Big(\mathrm{Ma}^2\,\mathbb{S}:\nabla\vec{u} - \frac{1}{\mathrm{Pe}}\frac{\vec{q}\cdot\nabla\vartheta}{\vartheta} + \frac{\lambda}{\mu}|\mathrm{curl}\vec{B}|^2\Big)\,,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(\frac{\mathrm{Ma}^2}{2}\varrho|\vec{u}|^2 + \varrho e + \frac{\mathrm{Ma}^2}{\mathrm{Al}^2}\frac{1}{2\mu}|\vec{B}|^2\Big)\mathrm{d}x = \int_\Omega\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\varrho\vec{f}\cdot\vec{u}\mathrm{d}x + C_{\mathrm{ref}}\int_{\partial\Omega} q\,\mathrm{d}S_x\,,\\
&\vec{B}_t + \mathrm{curl}(\vec{B}\times\vec{u}) + \mathrm{curl}(\lambda\mathrm{curl}\vec{B}) = 0\,,
\end{aligned}
\tag{1.7}
$$

where

$$
\begin{aligned}
C_{\mathrm{ref}} &= \frac{q_{\mathrm{ref}}L_{\mathrm{ref}}}{U_{\mathrm{ref}}p_{\mathrm{ref}}} \ \text{is boundary flux intensity constant}\,,\\
\mathrm{Ma} &= U_{\mathrm{ref}}\sqrt{\frac{\varrho_{\mathrm{ref}}}{p_{\mathrm{ref}}}} \ \text{is Mach number}\,,\\
\mathrm{Fr} &= \frac{U_{\mathrm{ref}}}{\sqrt{L_{\mathrm{ref}}f_{\mathrm{ref}}}} \ \text{is Froude number}\,,\\
\mathrm{Pe} &= \frac{p_{\mathrm{ref}}L_{\mathrm{ref}}U_{\mathrm{ref}}}{\vartheta_{\mathrm{ref}}\kappa_{\mathrm{ref}}} \ \text{is Peclet number}\,,\\
\mathrm{Al} &= \frac{U_{\mathrm{ref}}}{B_{\mathrm{ref}}}\sqrt{\varrho_{\mathrm{ref}}\mu_{\mathrm{ref}}} \ \text{is Alfven number}\,.
\end{aligned}
$$

As we mentioned above, the interesting cases for the asymptotic analysis are cases when some of these parameters called characteristic numbers vanish or become infinite. Similarly to the previously mentioned existence, the asymptotic analysis can be also studied by two approaches. The first one means that the limit solutions are investigated in the classical sense. One of the first results of this approach was achieved by Klainerman and Majda in [21] where the existence of the limit solution for the Navier-Stokes system is proved in the classical sense, but on a sufficiently small time interval. They proved that the solutions of the compressible magnetohydrodynamics equations tend to a solution of the incompressible magnetohydrodynamics equations under the assumption that the Mach number tends to zero. Another approach to this topic was proposed by Lions and Masmoudi in [27], where the existence is shown in a weak sense. Similar problems were further developed by Desjardins and Grenier [6]. The same strategy was later adapted for the full Navier-Stokes-Fourier system by Feireisl and Novotný in [14].

We study two problems of singular limits in this thesis. In the first one we consider the system (1.7) with the both Mach and Alfvén numbers proportional to ε together with certain special initial conditions. It is shown that when ε tends to zero than the limit quantities are weak solutions to the incompressible system of the equations of magnetohydrodynamics. This extends results written by Klainerman and Majda, [21] or Zank and Matthaeus, [39] because in comparison with [21], [39], the temperature ϑ is taken into account in our problem.

The second studied problem consists of the full Navier-Stokes-Fourier system (1.6) under the assumption that the Mach and Froude number are equal to ε and the Péclet number together with the scaling constant $1/C_{\text{ref}}$ equal to ε^2. Moreover we focus our attention to the case when (1.6) is defined on unbounded domain in $\mathbb{R}^3$ with a non empty compact boundary. The investigation of its singular limits leads to the analysis of local decay of acoustic waves which can be studied by several methods. Desjardins and Grenier in [6] use the so-called Strichartz estimates (see [36]). But these estimates become too complicated for the case of general unbounded domain and require certain restriction on the shape of the domain, and thus are not usable for our case. Instead of them we use approach developed by Feireisl, Novotný and Petzeltová in [17] based on weighted time-space estimates for abstract wave equations. By means of these tools, we finally show that the limit system can be viewed as a simple model of the fluid motion in the stellar radiative zones as introduced in [5].

The results published in this thesis are included in the articles written by the author. The chapters with original scientific results correspond to the following articles:

- Chapter 2, *On the Existence of Finite Energy Weak Solutions to the Navier-Stokes Equations in Irregular Domains* corresponds to the article *On the Existence of Finite Energy Weak Solutions to the Navier-Stokes*

Equations in Irregular Domains [23] published by the author in Mathematical Methods in Applied Sciences.

- Chapter 3, *Singular Limits of the Equations of Magnetohydrodynamics* is covered by the article *Singular Limits of the Equations of Magnetohydrodynamics* [24] accepted for publication in Journal of Mathematical Fluid Mechanics.

- Chapter 4, *Incompressible Limits for the Navier-Stokes-Fourier Systems on Unbounded Domains under Strong Stratification* corresponds to the article *Incompressible limits for the Navier-Stokes-Fourier Systems on Unbounded Domains under Strong Stratification* [22] submitted for publication to Archive Rational Mech. Anal.

Chapter 2

On the Existence of Finite Energy Weak Solutions to the Navier-Stokes Equations in Irregular Domains

Corresponds to the article by Kukučka P.: On the Existence of Finite Energy Weak Solutions to the Navier-Stokes Equations in Irregular Domains, Math. Meth. Appl. Sci., 32(11) 1428-1451, 2009.

Abstract: This paper studies the existence of weak solutions of the Navier-Stokes system defined on a certain class of domains in $\mathbb{R}^3$ that may contain cusps. The concept of such a domain and weak energy solution for the system is defined and its existence is proved. However, thinness of cusps must be related to the adiabatic constant appearing in the pressure law.

2000 Mathematics Subject Classification. 35A05, 35Q30

Keywords: Navier-Stokes system, renormalized solution, energy inequality

2.1 Introduction

We prove the global existence of weak solutions of the Navier-Stokes equations of an isentropic compressible fluid:

$$\varrho_t + \operatorname{div}(\varrho\vec{u}) = 0\,, \tag{2.1}$$

$$(\varrho u^i)_t + \operatorname{div}(\varrho u^i\vec{u}) + a(\varrho^\gamma)_{x_i} = \mu\Delta u^i + (\lambda+\mu)(\operatorname{div}\vec{u})_{x_i}, \quad i = 1\,,2\,,3\,. \tag{2.2}$$

Here the density $\varrho = \varrho(t\,,x)$ and the velocity $\vec{u} = [u^1(t\,,x)\,,u^2(t\,,x)\,,u^3(t\,,x)]$ are functions of the time $t \in (0\,,T)$ and the spatial coordinate $x \in \Omega$, where

$\Omega \subset \mathbb{R}^3$ is a domain belonging to the class $W^{1,s}$ specified below. We assume that

$$\frac{3\gamma}{2\gamma - 3} \le s < +\infty \,. \tag{2.3}$$

The viscosity coefficients $\mu\,,\lambda$ satisfy

$$\mu > 0\,, \quad \lambda + \frac{2}{3}\mu \ge 0\,,$$

$a > 0$ is a positive constant, and the adiabatic constant γ is subjected to the constraint

$$\gamma > \frac{3}{2}\,. \tag{2.4}$$

We prescribe the initial conditions for the density and the momentum:

$$\varrho(0) = \varrho_0\,, \;\; (\varrho u^i)(0) = q^i\,, \;\; i = 1\,,2\,,3\,, \tag{2.5}$$

together with the no-slip boundary conditions for the velocity:

$$u^i|_{\partial\Omega} = 0\,, \;\; i = 1\,,2\,,3\,. \tag{2.6}$$

The data $\varrho_0\,,\vec{q}$ are supposed to comply with compatibility conditions of the form

$$\varrho_0 \in L^\gamma(\Omega)\,,\, \varrho_0 \ge 0\,,\, q^i(x) = 0 \text{ whenever } \varrho_0(x) = 0\,, \frac{|q^i|^2}{\varrho_0} \in L^1(\Omega)\,,\, i = 1\,,2\,,3. \tag{2.7}$$

The question of existence of global-in-time solutions for Navier-Stokes system (2.1), (2.2) is far from being solved. The existence of finite energy weak solutions for (2.1), (2.2) was first proved by Lions [26] for $\gamma \ge \frac{9}{5}$, and then extended to the range $\gamma > \frac{3}{2}$ in [15] provided the underlying spatial domain Ω is at least of class $C^{2+\nu}$. In [16], the authors investigate system (2.1), (2.2) posed on an arbitrary bounded domain Ω, where Ω is approximated by a suitable sequence of smooth domains. By means of this method, one gets a sequence of approximate solutions that converges to another weak solution of the problem on the limit domain. Such a solution has bounded total energy but is not known to satisfy the energy inequality in the differential form. For more details see Remark following Definition 2.1.2. Certain improvements in this direction have been achieved in [31], where the existence problem is studied on Lipschitz domains. Note that all the above cited authors use the so-called Bogovskii operator - a specific branch of div^{-1} (see e.g. [30]) - in order to get uniform estimates of the pressure. The basic properties of this operator depend on regularity of the domain, in particular the latter must be at least Lipschitz.

Although there have been some attempts to extend Bogovskii operator to a larger class of domains such as John domains ([3]), an example of a simple domain with external cusps on which such an operator cannot be defined was constructed in [18]. The main goal of this paper is to extend the results of [15] to the class of $W^{1,s}$ domains defined as follows:

Definition 2.1.1 *(domain with boundary $W^{1,s}$)*
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n-1 < s \leq +\infty$. We say that Ω is a domain with a boundary $W^{1,s}$ if there exist positive numbers α, β, a family of M Cartesian coordinates systems

$$r = 1, \dots, M \,:\, (x_{r_1}, \dots, x_{r_{n-1}}, x_{r_n}) = (x'_r, x_{r_n}),$$

and M functions $a_r \in W^{1,s}(\Delta_r)$, where

$$\Delta_r = \{x'_r \in \mathbb{R}^{n-1} |\, i = 1, ..., n-1 \,:\, |\, x_{r_i}| < \alpha\},$$

such that:

1. *$\forall x \in \partial\Omega$, $\exists r \in \{1, ..., M\}$ and $\exists x'_r \in \Delta_r \,:\, x = T_r(x'_r, a_r(x'_r))$, where T_r is a mapping (translation and rotation) of the r-th cartesian coordinate system (x'_r, x_{r_n}) onto the global coordinate system (x', x_n).*
2. *If one denotes*

$$\begin{aligned} V_r^+ &= \{(x'_r, x_{r_n}) \in \mathbb{R}^n |\, x'_r \in \Delta_r\,, a_r(x'_r) < x_{r_n} < a_r(x'_r) + \beta\}\,, \\ V_r^- &= \{(x'_r, x_{r_n}) \in \mathbb{R}^n |\, x'_r \in \Delta_r\,, a_r(x'_r) - \beta < x_{r_n} < a_r(x'_r)\}\,, \\ \Lambda_r^- &= \{(x'_r, x_{r_n}) \in \mathbb{R}^n |\, x'_r \in \Delta_r\,, a_r(x'_r) = x_{r_n}\}\,, \end{aligned}$$

   then $T_r(V_r^+) \subset \Omega$, $T_r(V_r^-) \subset \mathbb{R}^n \setminus \overline{\Omega}$.

Denoting $V_r = V_r^+ \cup V_r^- \cup \Lambda_r$ we have $\partial\Omega = \bigcup_{r=1}^M \Lambda_r \subset \bigcup_{r=1}^M V_r$.

Remark. Note that, due to embedding $W^{1,s}(\Delta_r) \hookrightarrow C^{0,1-\frac{n-1}{s}}(\Delta_r)$, the functions a_r are continuous. Consequently, $a_r \in AC(\Delta_r) = \bigcap_{i=1}^{n-1} AC_i(\Delta_r)$, where $AC_i(\Delta_r)$ is the space of absolutely continuous functions on almost all segments parallel to the i-axis lying in the set Δ_r.

The class of domains specified in Definition 2.1.1 may contain both internal and external cusps, where the thinness of the cusps is to be related to γ through relation (2.3). We now give a precise definition of the concept of finite energy weak solution of problem (2.1), (2.2), (2.6).

Definition 2.1.2 *We say that functions ϱ, $\vec{u}$ represent a finite energy weak solution of problem (2.1), (2.2), (2.6) if the following conditions are satisfied:*

- $\varrho \geq 0$, $\varrho \in L^\infty(0\,,T\,;L^\gamma(\Omega))$ *;*
- $\vec{u} \in L^2(0\,,T\,;W_0^{1,2}(\Omega))$ *;*
- *the energy defined by*

$$E(t) = \int_\Omega \frac{1}{2}\varrho|\vec{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma \,\mathrm{d}x\,,$$

  is locally integrable on $(0\,,T)$ *and the energy inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_\Omega \frac{1}{2}\varrho|\vec{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma \mathrm{d}x\Big] + \int_\Omega \mu|\nabla\vec{u}|^2 + (\lambda+\mu)|\mathrm{div}\,\vec{u}|^2\mathrm{d}x \leq 0\,,$$

  holds in $\mathcal{D}'(0\,,T)$ *;*

- *the equations (2.1), (2.2) are satisfied in* $\mathcal{D}'((0\,,T)\times\Omega)$ *; moreover (2.1) holds in* $\mathcal{D}'((0\,,T)\times\mathbb{R}^3)$ *provided* ϱ *,* $\vec{u}$ *were extended to be zero outside of* Ω *;*
- *the equation (2.1) is satisfied in the sense of renormalized solutions, more precisely,*

$$b(\varrho)_t + \mathrm{div}(b(\varrho)\vec{u}) + (b'(\varrho)\varrho - b(\varrho))\mathrm{div}\vec{u} = 0$$

  holds in $\mathcal{D}'((0\,,T)\times\mathbb{R}^3)$ *for any* $b \in C^1([0\,,\infty))$ *such that*

$$b'(s) = 0 \text{ for all } s \geq M\,, \tag{2.8}$$

  where the constant M *may be different for different functions* b *.*

Remark. Using Lebesgue convergence theorem one can show that assumptions on b in the above definition can be relaxed so that the renormalized continuity equation holds for any $b \in C^1((0\,,\infty)) \cap C([0\,,\infty))$ satisfying

$$|b'(s)s| \leq c(z^\theta + z^{\frac{\gamma}{2}}) \text{ for all } s > 0 \text{ and certain } \theta \in (0\,,\frac{\gamma}{2})\,.$$

It follows immediately from (2.1), (2.2) that any finite energy weak solution belongs to the class:

$$\varrho \in C([0\,,T]\,;L^\gamma_{weak}(\Omega))\,,\varrho\vec{u} \in C([0\,,T]\,;L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega))\,,$$

and consequently, the initial conditions (2.5) make sense.
Remark. As already pointed out, the main problem when dealing with a general class of domains is the fact that the weak solutions constructed by

means of a family of approximate domains satisfy the energy inequality only in its integrated form

$$E(t) + \int_\Omega \mu|\nabla\vec{u}|^2 + (\lambda+\mu)|\mathrm{div}\vec{u}|^2 \mathrm{d}x \le E_0\,,$$

for a.a. $t > 0$.

Our main result is formulated in the following theorem:

Theorem 2.1.1 *Assume $\Omega \subset \mathbb{R}^3$ is a bounded $W^{1,s}$ domain, where γ and s are related by (2.3), (2.4). Let the data ϱ_0, $\vec{q}$ satisfy the compatibility conditions (2.7). Then given $T > 0$ arbitrary, there exists a finite energy weak solution ϱ, $\vec{u}$ of the problem (2.1), (2.2), (2.6) in the sense of Definition 2.1.2 satisfying the initial conditions (2.5).*

The proof of Theorem 2.1.1 will be done by means of three level approximation scheme based on a modified system:

$$\varrho_t + \mathrm{div}(\varrho\vec{u}) + \varepsilon\varrho^{1+\nu} = \varepsilon\Delta\varrho\,, \tag{2.9}$$

$$\begin{aligned}(\varrho u^i)_t + \mathrm{div}(\varrho u^i\vec{u}) + a(\varrho^\gamma)_{x_i} + \delta(\varrho^\beta)_{x_i} + \varepsilon\nabla u^i\cdot\nabla\varrho + \frac{\varepsilon}{2}\varrho^{1+\nu}u^i = \\ = \mu\Delta u^i + (\lambda+\mu)(\mathrm{div}\vec{u})_{x_i}\,, \qquad i = 1\,,2\,,3\,,\end{aligned} \tag{2.10}$$

on $(0\,,T)\times\Omega_\varepsilon$, where $\varepsilon > 0$, $\delta > 0$, $\nu > 0$ are small, $\beta > 0$ sufficiently large, Ω_ε is a sequence of smooth domains approximating Ω, and (2.9) is supplemented by the homogeneous Neumann boundary conditions

$$\nabla\varrho\cdot\vec{n}|_{\Omega_\varepsilon} = 0\,. \tag{2.11}$$

(i) The first step in the proof of Theorem 2.1.1 consists in solving the modified system (2.9), (2.10) by means of a Faedo-Galerkin approximation scheme where (2.9), (2.11) is solved directly while (2.10) is replaced by a finite dimensional system. The term $\varepsilon\varrho^{1+\nu}$ is used to pass to the limit for $\varepsilon \to 0$ in the energy inequality. The extra terms $\varepsilon\nabla u^i\cdot\nabla\varrho$ and $\frac{\varepsilon}{2}\varrho^{1+\nu}u^i$ are necessary to keep the energy inequality valid at this level of approximation.

(ii) In the second step, we let the artificial viscosity terms and other terms represented by the ε quantities go to zero. This is a delicate matter due to the lack of suitable estimates on the density component ϱ. Here we use a certain special test functions and the technique developed by Lions [26] and extended by Feireisl et al. [15] based on regularity of the effective viscous flux $a\varrho^\gamma - (\lambda + 2\mu)\mathrm{div}\vec{u}$.

(iii) The final step of the proof consists in getting rid of the artificial pressure term $\delta\varrho^\beta$. At this stage, we follow the same line of arguments as in [15].

2.2 The Faedo-Galerkin approximation

Let ν , $0 < \nu \le \frac{1}{5}$, be a fixed number and $\Omega \subset \mathbb{R}^3$ be a smooth domain of class at least $C^{2+\nu}$. Our first goal is to solve the problem (2.9), (2.10) supplemented by the boundary conditions:

$$\nabla\varrho\cdot\vec{n}|_{\partial\Omega} = 0\,, \tag{2.12}$$

$$\vec{u}|_{\partial\Omega} = 0\,, \tag{2.13}$$

and modified initial data:

$$\varrho(0) = \varrho_0 \in C^{2+\alpha}(\overline{\Omega})\,, 0 < \underline{\varrho} \le \varrho_0(x) \le \overline{\varrho}\,, \nabla\varrho_0\cdot\vec{n}|_{\partial\Omega} = 0\,, \tag{2.14}$$

$$(\varrho\vec{u})(0) = \vec{q}\,, \vec{q} = [q^1\,, q^2\,, q^3]\,, q^i \in C^2(\overline{\Omega})\,, i = 1\,, 2\,, 3\,. \tag{2.15}$$

In virtue of Proposition 7.3.3 in [29] the following statement holds:

Lemma 1 *Assume $\vec{u}$ be a given vector function belonging to the class*

$$\vec{u} \in C([0\,,T]\,;[C^2(\overline{\Omega})]^3)\,, \quad \vec{u}|_{\partial\Omega} = 0\,. \tag{2.16}$$

Then there exists $\delta > 0$ such that the initial-boundary value problem (2.9), (2.12) (2.14) has a unique solution $\varrho : [0\,,\delta] \times \overline{\Omega} \to \mathbb{R}$ such that ϱ and all the spatial derivatives ϱ_{x_i} , for $i = 1\,,2\,,3$, are continuous in $[0\,,\delta] \times \overline{\Omega}$, and ϱ_t , $\Delta\varrho$ are continuous in $(0\,,\delta] \times \overline{\Omega}$. Moreover, ϱ can be extended to a maximally defined solution $\varrho(t\,,x\,,\varrho_0) : I(\varrho_0) \times \overline{\Omega} \to \mathbb{R}$, $I(\varrho_0)$ being relatively open in $[0\,,T]$.

From this lemma and Theorem 5.1.21 in [29] (where we take $f = \varepsilon\varrho^{1+\nu}$) we obtain a higher regularity of the solution ϱ , namely $\varrho \in C^{1,\,2+\alpha}([0\,,\delta] \times \overline{\Omega})$, which allows us to use the comparison principle, together with Proposition 52.7 in [33], in order to get

$$\varrho > 0\,. \tag{2.17}$$

Lemma 2 *Solution ϱ can be extended to the interval $[0\,,T]$ in such a way that*

$$\varrho \in C^{1,\,2+\alpha}([0\,,T] \times \overline{\Omega})\,. \tag{2.18}$$

Proof. From the comparison principle, Proposition 52.7 in [33], applied to the system (2.9), (2.12), (2.14) compared with the following system:

$$\begin{gathered} r_t + \operatorname{div}(r\vec{u}) + \varepsilon r^{1+\nu} \ge \varepsilon\Delta r\,, \\ r(0) = \overline{\varrho}\,, \\ \nabla r\cdot\vec{n}|_{\partial\Omega} = 0\,, \end{gathered} \tag{2.19}$$

we get that its solution $r(t\,,x)=\overline{\varrho}\exp\left(\int_0^t\|\mathrm{div}\vec{u}(s)\|_{L^\infty(\Omega)}\mathrm{d}s\right)$ satisfies

$$\varrho(t\,,x)\le\overline{\varrho}\exp\left(\int_0^t\|\mathrm{div}\vec{u}(s)\|_{L^\infty(\Omega)}\mathrm{d}s\right),\text{for }(t\,,x)\in[0\,,\delta]\times\overline{\Omega}\,. \tag{2.20}$$

The estimate (2.20) holds for each $\delta<I(\varrho_0)$, and is independent on δ , and so ϱ is bounded on $I(\varrho_0)\times\overline{\Omega}$. According to Proposition 7.3.4 in [29] it holds, $I(\varrho_0)=[0\,,T]$. Then Theorem 5.1.21 in [29] gives (2.18) which concludes the proof. □

We are now allowed to repeat the procedure used in [15] (page 362 - 369) to get the following assertion:

Lemma 1 *Suppose* $\beta>\max\{4\,,\gamma\}$ *. Let* $\Omega\subset\mathbb{R}^3$ *be a bounded domain with* $C^{2+\nu}$ *boundary. Assume the initial data* ϱ_0 *,* $\vec{q}$ *satisfy (2.14), (2.15). Then there exists a weak solution* ϱ *,* $\vec{u}$ *of the problem (2.9) - (2.15) such that* $\varrho\in L^{\beta+1}((0\,,T)\times\Omega)$ *and the following estimates hold:*

$$\sup_{t\in[0,\,T]}\|\varrho(t)\|^\gamma_{L^\gamma(\Omega)}\le cE_\delta[\varrho_0\,,\vec{q}]\,, \tag{2.21}$$

$$\delta\sup_{t\in[0,\,T]}\|\varrho(t)\|^\beta_{L^\beta(\Omega)}\le cE_\delta[\varrho_0\,,\vec{q}]\,, \tag{2.22}$$

$$\sup_{t\in[0,\,T]}\|\sqrt{\varrho}(t)\vec{u}(t)\|^2_{L^2(\Omega)}\le cE_\delta[\varrho_0\,,\vec{q}]\,, \tag{2.23}$$

$$\int_0^T\|\vec{u}(t)\|^2_{L^2(\Omega)}+\int_0^T\|\nabla\vec{u}(t)\|^2_{L^2(\Omega)}\mathrm{d}t\le cE_\delta[\varrho_0\,,\vec{q}]\,, \tag{2.24}$$

$$\varepsilon\int_0^T\|\nabla\varrho(t)\|^2_{L^2(\Omega)}\mathrm{d}t\le c(\beta\,,\delta\,,\varrho_0\,,\vec{q})\,, \tag{2.25}$$

$$\varepsilon\frac{a\gamma}{\gamma-1}\int_0^T\int_\Omega\varrho^{\gamma+\nu}\mathrm{d}x\,\mathrm{d}t\le cE_\delta[\varrho_0\,,\vec{q}]\,, \tag{2.26}$$

$$\varepsilon\frac{\delta\beta}{\beta-1}\int_0^T\int_\Omega\varrho^{\beta+\nu}\mathrm{d}x\,\mathrm{d}t\le cE_\delta[\varrho_0\,,\vec{q}]\,, \tag{2.27}$$

where the constant c *is either independent of the shape of* Ω *or bounded for uniformly bounded sets of domains* Ω *. Moreover, the energy inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_\Omega\frac{1}{2}\varrho|\vec{u}|^2+\frac{a}{\gamma-1}\varrho^\gamma+\frac{\delta}{\beta-1}\varrho^\beta\mathrm{d}x\Big]+\int_\Omega\mu|\nabla\vec{u}|^2+(\lambda+\mu)|\mathrm{div}\,\vec{u}|^2\mathrm{d}x+ \\ \varepsilon\frac{a\gamma}{\gamma-1}\int_\Omega\varrho^{\gamma+\nu}\mathrm{d}x+\varepsilon\frac{\delta\beta}{\beta-1}\int_\Omega\varrho^{\beta+\nu}\mathrm{d}x\le 0\,, \tag{2.28}$$

holds in $\mathcal{D}'(0\,,T)$ *. Finally, there exists* $r > 1$ *such that* ϱ_t *.* $\Delta\varrho$ *belong to* $L^r((0\,,T)\times\Omega)$ *and the equation (2.9) is satisfied a.e. in* $(0\,,T)\times\Omega$ *.*

2.3 The vanishing viscosity limit

2.3.1 Solution on approximate domains

In order to handle a general $W^{1,s}$ domain Ω, let us first approximate it by the following sequence of C^∞ domains Ω_ε, for $\beta > \varepsilon > 0$ sufficiently small. Defining the functions

$$a_r^\varepsilon(x_r') := (a_r * \omega_{\varepsilon'})(x_r') - \varepsilon''(\varepsilon)\,, \qquad r = 1\,,\cdots,M \tag{2.29}$$

where a_r have the same meaning as in Definition 2.1.1, and $\varepsilon' = \varepsilon'(\varepsilon)$, $\varepsilon'' = \varepsilon''(\varepsilon)$ are chosen so that $\varepsilon'(\varepsilon)\to 0$, and $\varepsilon''(\varepsilon)\to 0$ as $\varepsilon\to 0^+$, and

$$a_r(x_r') - (a_r * \omega_{\varepsilon'})(x_r') + \varepsilon''(\varepsilon) > 0\,,$$

we get a decreasing sequence of C^∞ domains Ω_ε such that $\overline{\Omega}\subset\Omega_\varepsilon$. Moreover it holds that for each compact set $K\subset\mathbb{R}^3\backslash\overline{\Omega}$ there exists ε_K such that $K\subset\mathbb{R}^3\backslash\overline{\Omega}_\varepsilon$ for all $\varepsilon<\varepsilon_K$.

Now consider the system (2.9), (2.10) approximating (2.1), (2.2) on $\Omega_\varepsilon\times(0\,,T)$, supplemented by the boundary conditions:

$$\nabla\varrho\cdot\vec{n}|_{\partial\Omega_\varepsilon} = 0\,, \tag{2.30}$$

$$\vec{u}|_{\partial\Omega_\varepsilon} = 0\,, \tag{2.31}$$

and initial data:

$$\varrho(0) = \varrho_{0\,\delta\varepsilon}\,, \tag{2.32}$$

$$(\varrho\vec{u})(0) = \vec{q}_{\delta\varepsilon}\,. \tag{2.33}$$

where functions $\varrho_{0\,\delta\varepsilon}$ and $\vec{q}_\varepsilon$ are introduced in the following lemma that can be proved in the same way as in [15] page 381 - 382:

Lemma 3 *There exists a sequence* $\varrho_{0\,\delta\varepsilon}\in C^{2+\nu}(\overline{\Omega}_\varepsilon)$ *such that*

$$0<\varepsilon\le\varrho_{0\,\delta\varepsilon}\le\varepsilon^{-\frac{1}{\beta}}\,,\quad \nabla\varrho_{0\,\delta\varepsilon}\cdot\vec{n}|_{\partial\Omega_\varepsilon} = 0\,, \tag{2.34}$$

$$\|\varrho_{0\,\delta\varepsilon}-\varrho_{0\,\delta}\|_{L^\beta(\Omega_\varepsilon)}\to 0\,,\quad \text{as } \varepsilon\to 0^+\,,\ \text{and}\ \ \delta>0\,, \tag{2.35}$$

where $\varrho_{0\,\delta}$ *is considered to be zero outside of* Ω *. Moreover* $\varrho_{0\,\delta}\in C^{2+\nu}(\overline{\Omega})$ *and*

$$0<\delta\le\varrho_{0\,\delta}\le\delta^{-\frac{1}{\beta}}\,, \tag{2.36}$$

$$\|\varrho_{0\,\delta} - \varrho_0\|_{L^\gamma(\Omega)} \to 0\,, \quad \text{as } \delta \to 0^+\,. \tag{2.37}$$

Finally, there exists a sequence $\vec{q}_{\delta\varepsilon}$ such that

$$\frac{|q^i_{\delta\varepsilon}|^2}{\varrho_{0\,\delta\varepsilon}} \text{ are bounded in } L^1(\Omega_\varepsilon)\,, \text{ independently of } \varepsilon > 0 \text{ and each } \delta > 0 \tag{2.38}$$

and

$$\|q^i_{\delta\varepsilon} - q^i_\delta\|_{L^1(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0^+\,, \text{ and } \delta > 0\,, \tag{2.39}$$

where $\vec{q}_\delta$ is considered to be zero outside Ω and such that

$$\frac{|q^i_\delta|^2}{\varrho_{0\,\delta}} \text{ are bounded in } L^1(\Omega)\,, \text{ independently of } \delta > 0, \tag{2.40}$$

and

$$\|q^i_\delta - q^i\|_{L^1(\Omega)} \to 0 \text{ as } \delta \to 0^+\,. \tag{2.41}$$

Throughout this section we will work with a fixed δ. According to Proposition 1 established in the previous section there exists a solution of (2.9), (2.10), (2.30) - (2.33) which will be denoted by ϱ_ε, $\vec{u}_\varepsilon$. The aim of the present section is to pass to the limit in (2.9), (2.10), (2.30) - (2.33) letting $\varepsilon \to 0$.

2.3.2 On a special test function

In order to estimate the density in a vicinity of the non-Lipschitz part of the boundary, we use a special test function constructed below. Let us define a function v_r by formula:

$$v_r(x'_r,\, x_{r_n}) := (x_{r_n} - a_r(x'_r))^\lambda\,, \quad (x'_r,\, x_{r_n}) \in V_r^+ \cup \Lambda_r\,, \tag{2.42}$$

where $0 < \lambda < 1$. Basic properties of the function (2.42) are summarized in the following statement.

Lemma 4 *Let $1 < p \le s < +\infty$ be a given number and $1 - \frac{1}{p} < \lambda < 1$. Then the function v_r defined by (2.42) is continuous on $V_r^+ \cup \Lambda_r$ and $v_r \in W^{1,p}(V_r^+)$.*

Proof. Function v_r is continuous and belongs to $W^{1,p}(V_r^+)$ because a_r is continuous on Δ_r. Since a_r belongs to $W^{1,s}(\Delta_r)$ and $n - 1 < s$, the classical derivative $D_{x_{r_i}} a_r(x'_r)$ exists a. e. in Δ_r, and $D_{x_{r_i}} a_r(x'_r) = \frac{\partial a_r}{\partial x_{r_i}}(x'_r)$ a. e. ,where $\frac{\partial a_r}{\partial x_{r_i}}(x'_r)$ denotes the weak derivative. This is true for all $i = 1, ..., n-1$. It is easy to check that the classical derivative of v_r with respect to each variable exists a. e. in V_r and

$$D_{x_{r_i}} v_r(x'_r, x_{r_n}) = -\lambda(x_{r_n} - a_r(x'_r))^{\lambda-1} D_{x_{r_i}} a_r(x'_r)\,, \quad i = 1, ..., n-1\,, \tag{2.43}$$

$$D_{x_{r_n}} v_r(x'_r, x_{r_n}) = \lambda(x_{r_n} - a_r(x'_r))^{\lambda-1} \,. \tag{2.44}$$

Moreover, a direct computation gives

$$\|D_{x_{r_i}} v_r\|^p_{L^p(V_r^+)} = \lambda \int_{\Delta_r} |D_{x_{r_i}} a_r(x'_r)|^p \int_{a_r(x'_r)}^{a_r(x'_r)+\beta} (x_{r_n} - a_r(x'_r))^{(\lambda-1)p} \, \mathrm{d}x_{r_n} \, \mathrm{d}x'_r \leq$$

$$\leq \lambda(1 + p\,(\lambda-1))\alpha^{(n-1)(1-\frac{p}{s})} \beta^{p(\lambda-1)+1} \|D_{x_{r_i}} a_r\|^p_{L^s(\Delta_r)} \qquad \text{for } i - 1, ..., n-1$$

and a similar calculation for $D_{x_{r_n}} v_r$ implies $D_{x_{r_i}} v_r \in L^p(V_r^+)$ for $i = 1, ..., n$. It remains to prove that classical derivatives defined a. e. by (2.43), (2.44) coincide with the weak derivatives of v_r. Let $i \in \{1, ..., n-1\}$ be an arbitrary index. Obviously, $x_{r_n} - a_r(x'_r)$ is absolutely continuous on almost all segments parallel with each $i-$axis and lying in V_r^+. Let us take such a segment and denote it by P_r^i. Then there exists $(x_{r_1}, ..., x_{r_{i-1}}, x_{r_{i+1}}, ..., x_{r_n})$, $|x_{r_j}| < \alpha$ for $j \in \{1, ..., n-1\}$, $j \neq i$, and an interval I_r^i such that $(x_{r_1}, ..., x_{r_{i-1}}, \xi, x_{r_{i+1}}, ..., x_{r_n}) \in P_r^i \cap V_r^+$ for $\forall \xi \in I_r^i$. Function

$$\xi \to x_{r_n} - a_r(x_{r_1}, ..., x_{r_{i-1}}, \xi, x_{r_{i+1}}, ..., x_{r_{n-1}})$$

is absolutely continuous and strictly positive on each compact interval $I_r^i \subset J_r^i$, which implies that $v_r(x_{r_1}, ..., x_{r_{i-1}}, \xi, x_{r_{i+1}}, ..., x_{r_n})$ is absolutely continuous on I_r^i. Similarly, the same holds for $n-$axis giving $v_r \in AC(V_r^+)$. But then the weak derivative $\frac{\partial v_r}{\partial x_{r_i}}$ exists and is equal to $D_{x_{r_i}} v_r$ a. e. in V_r^+ which concludes the proof. □

Lemma 5 *Let $\Omega \subset \mathbb{R}^n$ be a $W^{1,s}$ domain in the sense of Definition 2.1.1, and $u \in W^{1,p}(\mathbb{R}^n)$ such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, $1 \leq p < \infty$. Then $u \in W_0^{1,p}(\Omega)$.*

Proof. Let us take $\Omega_r = T_r(V_r)$ from Definition 2.1.1, and $\Omega_{M+1} \subset \overline{\Omega}_{M+1} \subset \Omega$ such that $\overline{\Omega} \subset \bigcup_{r=1}^{M+1} \Omega_r$. Moreover, let $\{\Phi_r\}_{r=1}^{M+1}$ be a partition of unity subordinate to the family of open sets $\{\Omega_r\}_{r=1}^{M+1}$.
Because T_r is a mapping (translation and rotation) of the r-th cartesian coordinate system (x'_r, x_{r_n}) onto the global coordinate system (x', x_n), we have $T_r(x_r) = \mathbf{A}_r x_r + \mathbf{b}_r$, where we have denoted $x_r := (x'_r, x_{r_n})$, and where $\mathbf{A}_r \in M(\mathbb{R}^n)$ is an orthogonal matrix with determinant equal to one, and $\mathbf{b}_r \in \mathbb{R}^n$. If one takes

$$u_r^\varepsilon(x) = u(x - \varepsilon \mathbf{A}_r \vec{e}_n) \,, \quad r = 1\,, ...\,, M\,,$$

and $u_r^\varepsilon(x) = u(x)$ for $r = M+1$, then it is easy to check that

$$u_\varepsilon := \sum_{r=1}^{M+1} \Phi_r u_r^\varepsilon \,,$$

belongs to $W_0^{1,p}(\Omega)$. Using a simple estimate

$$\|u_\varepsilon - u\|^p_{W^{1,p}(\mathbb{R}^3)} \leq C \sum_{r=1}^{M} \int_{\Omega_r} |u(x-\varepsilon a_r) - u(x)|^p + |Du(x-\varepsilon a_r) - Du(x)|^p \mathrm{d}x\,,$$

where $a_r = \mathbf{A}_r \vec{e}_n$, together with the fact that $u \in W^{1,p}(\mathbb{R}^n)$, we get

$$u_\varepsilon \longrightarrow u \text{ in } W^{1,p}(\mathbb{R}^n)\,. \tag{2.45}$$

Since $u_\varepsilon \in W_0^{1,p}(\Omega)$ there exists a sequence $u_\varepsilon^k \in C_0^\infty(\Omega)$ such that $u_\varepsilon^k \to u_\varepsilon$ in $W_0^{1,p}(\Omega)$, which, combined with (2.45), gives $u \in W_0^{1,p}(\Omega)$. □

Lemma 2 *Let $1 < p < +\infty$ be a given number. Then for any s, $p \leq s < +\infty$ and any domain $\Omega \subset \mathbb{R}^n$ with boundary $W^{1,s}$ in the sense of Definition 2.1.1, there exists a vector function $\tilde{\varphi} \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, $\tilde{\varphi}|_{\partial\Omega} = 0$ enjoying the following property:*

For all $K > 0$, there exists $\delta > 0$ such that

$$\mathrm{div}\tilde{\varphi}(x) > K \textit{ whenever } \mathrm{dist}(x, \partial\Omega) < \delta. \tag{2.46}$$

Proof. Similarly to the proof of the previous lemma, we take

$$\tilde{\varphi} := \sum_{r=1}^{M} \Phi_r \vec{v}_r\,, \tag{2.47}$$

where

$$\vec{v}_r(x) := \mathbf{A}_r(\underbrace{0, ..., 0}_{n-1}, v_r(\mathbf{A}_r^T(x - \mathbf{b}_r)))^T\,, \tag{2.48}$$

and v_r is defined by (2.42). In particular, $\tilde{\varphi} \in W^{1,p}(\Omega)$. It is easy to check that $\tilde{\varphi}|_{\partial\Omega} = 0$ since $v_r|_{\Lambda_r} = 0$, and $\tilde{\varphi} \in C(\overline{\Omega})$, which yields $\tilde{\varphi} \in W_0^{1,p}(\Omega')$ for each $\Omega' \supset \overline{\Omega}$. Using Lemma 5 we obtain $\tilde{\varphi} \in W_0^{1,p}(\Omega)$. Moreover, by means of a direct computation,

$$\mathrm{div}\,\tilde{\varphi}(x) = \sum_{r=1}^{M} \nabla\Phi_r(x).\vec{v}_r(x) + \sum_{r=1}^{M} \Phi_r(x) \frac{\partial v_r}{\partial x_{r_n}}(\mathbf{A}_r^T(x - \mathbf{b}_r)) \quad \text{a.e in } \Omega\,. \tag{2.49}$$

The function v_r^λ can be continuously extended into the compact set $\overline{V_r^+}$, $v_r|_{\overline{\Lambda}_r} = 0$, which implies, due to the uniform continuity, $\forall\epsilon > 0\ \exists\delta > 0$ such that $\forall x_r' \in V_r^+$, $\mathrm{dist}(x_r', \Lambda_r) < \delta$: $v_r^\lambda(x_r') < \epsilon$. Using this fact we observe

$\forall K>0\,\exists\delta>0$ such that $\forall x'_r\in V_r^+$, $\mathrm{dist}(x'_r,\Lambda_r)<\delta:\ \frac{\partial v_r}{\partial x_{rn}}(x'_r)\geq K$.
We summarize the previous discussion in the following statement:
For any $K>0$ there exists $\delta>0$ such that, if $\mathrm{dist}(x,T_r(\Lambda_r))<\delta$ for all $r\in\{1,...,M\}$, then $\frac{\partial v_r}{\partial x_{rn}}(\mathbf{A}_r^T(x-\mathbf{b}_r))\geq K$ for a.a. $x\in\Omega_r$. Substituting this inequality into (2.49), taking into account $\sum_{r=1}^M\Phi_r=1$, and observing that the first term of (2.49) tends uniformly to zero as $\mathrm{dist}\,(x,\partial\Omega)\to 0$, we obtain (2.46).
□

2.3.3 Weak convergence of the density

We are going to use the special test function constructed in the previous section to prove a weak convergence of $\varrho_\varepsilon^\beta$, $\varrho_\varepsilon^\gamma$ in $L^1((0,T)\times\Omega)$. The following lemma usually called Lemma de la Vallé Pousin ([9]) provides a necessary and sufficient condition for the weak convergence in the L^1 space.

Lemma 6 *Let $\mathcal{F}\subset L^1(\Omega)$. Then the following conditions are equivalent:*

i) $\forall v_n\in\mathcal{F}\ \exists v_{n_k}$, $v\in L^1(\Omega)$ such that $v_{n_k}\rightharpoonup v$ in $L^1(\Omega)$.

ii) $\forall\varepsilon>0\ \exists\delta>0$ such that $\int_B|v(x)\|\,\mathrm{d}x\leq\varepsilon$ for $\forall v\in\mathcal{F}$, $\forall B\subset\Omega\ \mu(B)\leq\delta$.

Our aim is to verify that both $\varrho_\varepsilon^\beta$ and $\varrho_\varepsilon^\gamma$ satisfy condition ii) of Lemma 6 on the set $(0,T)\times\Omega$.

Let $\varepsilon'>0$ be arbitrary and consider $\delta'>0$ given by Proposition 2 (where one takes $K=\frac{1}{\varepsilon'}$). Furthermore, let Ω_1, Ω_2 and Ω_3 be smooth domains such that

$$\begin{gathered}\overline{\Omega}_3\subset\Omega_2\subset\overline{\Omega}_2\subset\Omega_1\subset\overline{\Omega}_1\subset\Omega\,,\\ \mathrm{dist}(x\,,\partial\Omega)<\delta'\,,\qquad\forall x\in\partial\Omega_3\,,\\ \mathrm{dist}(x\,,\partial\Omega)<\tfrac{2}{3}\delta'\,,\qquad\forall x\in\partial\Omega_2\,,\\ \mathrm{dist}(x\,,\partial\Omega)<\tfrac{1}{3}\delta'\,,\qquad\forall x\in\partial\Omega_1\,.\end{gathered}$$

Then

$$\mathrm{div}\,\tilde{\varphi}(x)\geq\frac{1}{\varepsilon'}\,,\ \text{for a.a. } x\in\Omega\,,\ \mathrm{dist}(x\,,\partial\Omega)>\delta'\,.\qquad(2.50)$$

Remark : Due to the good approximations of the initial conditions that follows from (2.35) and (2.38), the estimates contained in Proposition 1 are independent of ε.
The following lemma shows estimates of the density on Ω_2:

Lemma 7 *Let ϱ_ε , $\vec{u}_\varepsilon$ be the sequence of solutions of the problem (2.9), (2.10), (2.30) - (2.33) constructed above. Then there exists a constant $c = c(\delta\,, \varrho_{0\,\delta}, \vec{q}_\delta, \Omega_1, \Omega_2)$ independent of ε such that*

$$\|\varrho_\varepsilon\|_{L^{\gamma+1}((0,\,T)\times\Omega_2)} + \|\varrho_\varepsilon\|_{L^{\beta+1}((0,\,T)\times\Omega_2)} \le c\,. \tag{2.51}$$

Proof: Let $\phi \in \mathcal{D}(\Omega_1)$ be such that $0 \le \phi \le 1$, $\phi(x) = 1$ for each $x \in \Omega_2$, and $\psi \in \mathcal{D}(0\,,T)$ such that $0 \le \psi \le 1$. Extending ϱ_ε to be zero outside Ω_ε, we consider a test function

$$\hat{\varphi}(t\,,x) = \psi(t)\phi(x)\mathcal{A}[\varrho_\varepsilon]\,,$$

where the operator $\mathcal{A}_j$ is defined via the Fourier transformation $\mathcal{F}_{x\to\xi}$:

$$\mathcal{A}_j[v] = \mathcal{F}^{-1}_{\xi\to x}\Big[\frac{-i\xi_j}{|\xi|^{\,2}}\,\mathcal{F}_{x\to\xi}[v]\Big]\,,\ j = 1\,,2\,,3.$$

By virtue of the classical Mikhlin multiplier theorem the following estimates hold:

$$\begin{gathered} \|\mathcal{A}_j[v]\|_{W^{1,\,s}(\Omega_1)} \le c(s\,,\Omega_1)\|v\|_{L^s(\mathbb{R}^3)}\,,\ 1 < s < \infty\,, \\ \|\mathcal{A}_j[v]\|_{L^q(\Omega_1)} \le c(q\,,s\,,\Omega_1)\|v\|_{L^s(\mathbb{R}^3)}\,,\ q < \infty\,,\ \tfrac{1}{q} \ge \tfrac{1}{s} - \tfrac{1}{3}, \\ \|\mathcal{A}_j[v]\|_{L^\infty(\Omega_1)} \le c(s\,,\Omega_1)\|v\|_{L^s(\mathbb{R}^3)}\,,\ \text{if } s > 3\,. \end{gathered} \tag{2.52}$$

Due to the regularity property established in Proposition 1, especially (2.22), we shall use $\hat{\varphi}$ as a test function for the system (2.10) and benefit from [15], Section 3.2. After simple manipulation by help of the above introduced estimates, we deduce the required inequality (2.51). □

In order to verify the second hypothesis of Lemma 6, we have to examine the behavior of the approximate solutions in a neighborhood of the non-smooth part of the boundary $\partial\Omega$ specified in the following lemma.

Lemma 8 *Let ϱ_ε , $\vec{u}_\varepsilon$ be the sequence of solutions of the problem (2.9), (2.10), (2.30) - (2.33) constructed above. Then there exists a constant $c = c(\delta\,, \varrho_{0\,\delta}, \vec{q}_\delta, \Omega_1, \Omega_2)$ independent of ε such that*

$$\int_0^T \int_{\Omega\setminus\Omega_2} a\varrho_\varepsilon^\gamma + \delta\varrho_\varepsilon^\beta \mathrm{d}x\,\mathrm{d}t \le c\,\varepsilon'\,. \tag{2.53}$$

Proof: Let $\phi \in \mathcal{D}(\mathbb{R}^3)$ such that $0 \le \phi \le 1$, $\phi(x) = 0$ for each $x \in \Omega_3$, $\phi(x) = 1$ for each $x \in \Omega \setminus \Omega_2$, and $\psi \in \mathcal{D}(0\,,T)$ such that $0 \le \psi \le 1$. Consider

$$\hat{\varphi}(t\,,x) = \psi(t)\phi(x)\tilde{\varphi}(x)\,,$$

where $\tilde{\varphi}$ is the vector function from Proposition 2, which can be used as a test function for the system (2.10). After similar calculations as in the previous lemma, one obtains:

$$\int_0^T \int_{\Omega\setminus\Omega_3} \psi\phi(a\varrho_\varepsilon^\gamma + \delta\varrho_\varepsilon^\beta)\,\mathrm{div}\tilde{\varphi}\,\mathrm{d}x\,\mathrm{d}t =$$

$$-\int_0^T \int_{\Omega\setminus\Omega_3} \psi_t\phi\varrho_\varepsilon\vec{u}_\varepsilon\cdot\tilde{\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega\setminus\Omega_3} \mu\psi\nabla\phi\cdot\nabla u_\varepsilon^i\tilde{\varphi}^i\mathrm{d}x\,\mathrm{d}t +$$

$$\int_0^T \int_{\Omega\setminus\Omega_3} \mu\psi\phi\nabla u_\varepsilon^i\cdot\nabla\tilde{\varphi}^i\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\Omega\setminus\Omega_3} \psi\varrho_\varepsilon\nabla\phi\cdot\vec{u}_\varepsilon\tilde{\varphi}\cdot\vec{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t -$$

$$\int_0^T \int_{\Omega\setminus\Omega_3} \psi\phi\varrho_\varepsilon u_\varepsilon^i\vec{u}_\varepsilon\cdot\nabla\tilde{\varphi}^i\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega\setminus\Omega_3} (\lambda+\mu)\psi\,\mathrm{div}\vec{u}_\varepsilon\nabla\phi\cdot\tilde{\varphi}\,\mathrm{d}x\,\mathrm{d}t +$$

$$\int_0^T \int_{\Omega\setminus\Omega_3} (\lambda+\mu)\psi\phi\,\mathrm{div}\vec{u}_\varepsilon\,\mathrm{div}\tilde{\varphi}\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_0^T \int_{\Omega\setminus\Omega_3} \psi\phi\nabla\varrho_\varepsilon\cdot\nabla u_\varepsilon^i\tilde{\varphi}^i\,\mathrm{d}x\,\mathrm{d}t +$$

$$\frac{\varepsilon}{2}\int_0^T \int_{\Omega\setminus\Omega_3} \psi\phi\varrho_\varepsilon^{1+\nu}\vec{u}_\varepsilon\cdot\tilde{\varphi}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\Omega\setminus\Omega_3} \psi(a\varrho_\varepsilon^\gamma + \delta\varrho_\varepsilon^\beta)\nabla\phi\cdot\tilde{\varphi}\,\mathrm{d}x\,\mathrm{d}t = \sum_{j=1}^{10} I_j\,.$$

(i) By virtue of (2.22), (2.23), we get

$$|I_1| \le c\int_0^T |\psi_t|\|\sqrt{\varrho_\varepsilon}\|_{L^2(\Omega_\varepsilon)}\|\sqrt{\varrho_\varepsilon}\vec{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\mathrm{d}t \le c(\delta\,,\varrho_{0\,\delta}\,,\vec{q}_\delta\,,\Omega_1\,,\Omega_2)\int_0^T |\psi_t|\mathrm{d}t\,.$$

(ii) Similarly, using (2.24) we have

$$|I_2| \le c\int_0^T \|\nabla\vec{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\mathrm{d}t \le c(\delta\,,\varrho_{0\,\delta}\,,\vec{q}_\delta\,,\Omega_1\,,\Omega_2)\,.$$

(iii) By the same token,

$$|I_3| \le c\int_0^T \|\nabla\vec{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\tilde{\varphi}\|_{L^2(\Omega)}\mathrm{d}t \le c(\delta\,,\varrho_{0\,\delta}\,,\vec{q}_\delta\,,\Omega_1\,,\Omega_2)\,.$$

(iv) From (2.22), (2.23), it follows that

$$|I_4| \le c\int_0^T \|\sqrt{\varrho_\varepsilon}\|_{L^2(\Omega_\varepsilon)}\|\sqrt{\varrho_\varepsilon}\vec{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\mathrm{d}t \le c(\delta\,,\varrho_{0\,\delta}\,,\vec{q}_\delta\,,\Omega_1\,,\Omega_2)\,.$$

(v) Since $s \ge \frac{3\gamma}{2\gamma-3}$, relations (2.22), (2.24) imply

$$|I_5| \le c\int_0^T \|\varrho_\varepsilon\|_{L^r(\Omega_\varepsilon)}\|\vec{u}_\varepsilon\|^2_{W^{1,\,2}(\Omega_\varepsilon)}\|\tilde{\varphi}\|_{W^{1,\,s}(\Omega)}\mathrm{d}t \le c(\delta\,,\varrho_{0\,\delta}\,,\vec{q}_\delta\,,\Omega_1\,,\Omega_2)\,,$$

where $r = \frac{3s}{2s-3}$.

(vi) Analogously,

$$|I_6| \leq c \int_0^T \|\nabla \vec{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \mathrm{d}t \leq c(\delta\,,\varrho_{0\,\delta}\,,\vec{q}_\delta\,,\Omega_1\,,\Omega_2)\,.$$

(vii) As in the previous steps,

$$|I_7| \leq c \int_0^T \|\nabla \vec{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla \tilde{\varphi}\|_{L^2(\Omega)} \mathrm{d}t \leq c(\delta\,,\varrho_{0\,\delta}\,,\vec{q}_\delta\,,\Omega_1\,,\Omega_2)\,.$$

(viii) Relations(2.24), (2.25) imply

$$|I_8| \leq \varepsilon\, c \int_0^T \|\nabla \varrho_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla \vec{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \mathrm{d}t \leq c(\delta\,,\varrho_{0\,\delta}\,,\vec{q}_\delta\,,\Omega_1\,,\Omega_2)\,.$$

(ix) From (2.22), (2.24), we immediately get

$$|I_9| \leq c \int_0^T \|\varrho_\varepsilon^{1+\nu}\|_{L^2(\Omega_\varepsilon)} \|\vec{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \mathrm{d}t \leq c(\delta\,,\varrho_{0\,\delta}\,,\vec{q}_\delta\,,\Omega_1\,,\Omega_2)\,.$$

(x) Finally, using (2.21), (2.22) we conclude

$$|I_{1\,0}| \leq c \int_0^T a\|\varrho_\varepsilon\|^\gamma_{L^\gamma(\Omega_\varepsilon)} + \delta\|\varrho_\varepsilon\|^\beta_{L^\beta(\Omega_\varepsilon)} \mathrm{d}t \leq c(\delta\,,\varrho_{0\,\delta}\,,\vec{q}_\delta\,,\Omega_1\,,\Omega_2)\,.$$

Now, applying the above estimates and the fact that $\mathrm{div}\tilde{\varphi}(x) \geq \frac{1}{\varepsilon'}$ in order to handle the left hand side of our identity, we obtain the desired result. □

If we combine the above lemmas, together with Lemma 6, we get

Lemma 3 *There exist functions* $p_1\,,p_2 \in L^1((0\,,T)\times\Omega)$ *such that*

$$\begin{aligned} \varrho_\varepsilon^\gamma &\rightharpoonup p_1 \ \text{ in } L^1((0\,,T)\times\Omega)\,,\\ \varrho_\varepsilon^\beta &\rightharpoonup p_2 \ \text{ in } L^1((0\,,T)\times\Omega)\,. \end{aligned}$$

Proof : It follows from Lemma 7 that ϱ_ε converge weakly in $L^{\gamma+1}((0\,,T)\times\Omega_2)$ and $L^{\beta+1}((0\,,T)\times\Omega_2)$, in particular, $\varrho_\varepsilon^\gamma$, $\varrho_\varepsilon^\beta$ converge weakly in $L^1((0\,,T)\times\Omega_2)$. Now implication $(i) \Rightarrow (ii)$ of Lemma 6 assures the existence of $\delta'' > 0$ such that

$$\begin{aligned} \|\varrho_\varepsilon^\gamma\|_{L^1((0,T)\times B)} &< \varepsilon' \quad \text{for} \,\forall B \subset \Omega_2\,,\ \mu(B) < \delta''\,,\\ \|\varrho_\varepsilon^\beta\|_{L^1((0,T)\times B)} &< \varepsilon' \quad \text{for} \,\forall B \subset \Omega_2\,,\ \mu(B) < \delta''\,. \end{aligned} \tag{2.54}$$

If we denote $\delta := \min\{\delta'\,,\delta''\}$, then the second hypothesis (ii) of Lemma 6 is satisfied and the proposition follows directly from Lemma 6.

Indeed, if $B \subset \Omega_2$, $\mu(B) < \delta$, then (ii) holds because of (2.54). Otherwise, if $B \subset \Omega \setminus \Omega_2$, then Lemma 8 implies

$$\begin{aligned} \|\varrho_\varepsilon^\gamma\|_{L^1((0,T)\times B)} &< c\varepsilon', \\ \|\varrho_\varepsilon^\beta\|_{L^1((0,T)\times B)} &< c\varepsilon', \end{aligned} \tag{2.55}$$

and the last case $B \cap \Omega_2 \neq \emptyset$, $B \cap (\Omega \setminus \Omega_2) \neq \emptyset$ can be handled by means of a combination of (2.54) and (2.55). The proof is complete. □

2.3.4 The vanishing viscosity limit passage

At this stage, we are ready to pass to the limit for $\varepsilon \to 0$ to get rid of the $\varepsilon-$quantities in the equations (2.9), (2.10). Note that the parameter δ is kept fixed throughout this procedure so that we may use the estimates derived above.

To begin, it is easy to deduce from (2.24), (2.25) that

$$\|\varepsilon\nabla\varrho_\varepsilon \cdot \nabla u_\varepsilon^i\|_{L^1((0,T)\times\Omega_\varepsilon)} \to 0, \quad i = 1, 2, 3, \tag{2.56}$$

and, analogously,

$$\|\varepsilon\Delta\varrho_\varepsilon\|_{L^2(0,T;W^{-1,2}(\Omega))} \to 0. \tag{2.57}$$

Moreover, it follows from (2.54) and the fact that $\vec{u}_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon)$

$$\vec{u}_\varepsilon \to \vec{u} \quad \text{weakly in } L^2(0, T; W_0^{1,2}(\Omega_{\varepsilon'})), \tag{2.58}$$

on each fixed $\Omega_{\varepsilon'}$. Thus, using Lemma 5 we get

$$\vec{u} \in L^2(0, T; W_0^{1,2}(\Omega)).$$

Other convergence properties are established in the following lemma:

Lemma 9

$$\varrho_\varepsilon \to \varrho \text{ in } C([0, T]; L^\beta_{weak}(\Omega)), \tag{2.59}$$

$$\varrho_\varepsilon\vec{u}_\varepsilon \to \varrho\vec{u} \text{ in } C([0, T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega)), \tag{2.60}$$

$$\varrho_\varepsilon u_\varepsilon^i u_\varepsilon^j \to \varrho u^i u^j \text{ in } \mathcal{D}'((0, T)\times\Omega),\ i, j = 1, 2, 3. \tag{2.61}$$

Proof. In accordance with Proposition 1, the continuity equation (2.9) is satisfied a.e. in $(0, T)\times\Omega_\varepsilon$. Thus we can multiply (2.9) by a test function $\varphi \in \mathcal{D}(\Omega)$ and integrate by parts to obtain

$$\int_\Omega \varrho_\varepsilon(t)\varphi \,\mathrm{d}x = \int_\Omega \varrho_{0\,\varepsilon}\varphi \,\mathrm{d}x + \int_0^t \int_\Omega \varrho_\varepsilon\vec{u}_\varepsilon\cdot\nabla\varphi - \varepsilon\varrho_\varepsilon^{1+\nu}\varphi - \varepsilon\nabla\varrho_\varepsilon\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}s. \tag{2.62}$$

Using the estimates (2.22), (2.24), (2.25) one can deduce $\varrho_\varepsilon \in C([0\,,T]; L^\beta_{weak}(\Omega))$.

Now let $\Omega_{\varepsilon'}$ be a fixed domain, and consider $\varepsilon < \varepsilon'$ and ϱ_ε , $\vec{u}_\varepsilon$ extended by zero outside Ω_ε . Then (2.22) implies uniform boundedness of ϱ_ε in $L^\beta(\Omega_{\varepsilon'})$. Furthermore, because ϱ_ε being extended by zero admits the partial derivative with respect to t , it is easy to see that the continuity equation (2.22) is satisfied a.e. in $(0\,,T) \times \mathbb{R}^3$, which allows us to use an arbitrary test function $\varphi \in W^{1\,,2}_0(\Omega_{\varepsilon'})$ to get

$$\int_{\Omega_{\varepsilon'}} (\varrho_\varepsilon(t) - \varrho_\varepsilon(t'))\varphi \,\mathrm{d}x = \int_{t'}^{t} \int_{\Omega_{\varepsilon'}} \varrho_\varepsilon \vec{u}_\varepsilon \cdot \nabla\varphi - \varepsilon \varrho_\varepsilon^{1+\nu}\varphi - \varepsilon 1_{\Omega_\varepsilon} \nabla\varrho_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x \,\mathrm{d}s.$$

This implies that the sequence ϱ_ε is uniformly continuous in $W^{-1,\,2}(\Omega_{\varepsilon'})$.

According to Lemma 6.2 in [30] page 301, it holds at least for a chosen subsequence that

$$\varrho_\varepsilon \to \varrho^{\varepsilon'} \;\text{ in } C([0\,,T]\,; L^\beta_{weak}(\Omega_{\varepsilon'}))\,. \tag{2.63}$$

Let now ε' and ε'' be arbitrary, sufficiently small. Then each $\varphi \in L^{\beta'}(\Omega)$ extended by zero belongs to $L^{\beta'}(\Omega_{\varepsilon'}) \cap L^{\beta'}(\Omega_{\varepsilon''})$, and, according to convergence in $C([0\,,T]\,; L^\beta_{weak}(\Omega))$, we get $\varrho^{\varepsilon'} = \varrho^{\varepsilon''}$ a.e. on $(0\,,T) \times \Omega$ which concludes the proof of (2.59).

Relation (2.63) together with (2.58) yield

$$\varrho_\varepsilon \vec{u}_\varepsilon \to \varrho\vec{u} \;\text{ weakly} -\star \text{ in } L^\infty(0\,,T\,; L^{\frac{2\gamma}{\gamma+1}}(\Omega))\,. \tag{2.64}$$

Now, consider an arbitrary small number denoted by $\varepsilon_1 > 0$, and let $\varphi \in L^{\frac{2\gamma}{\gamma-1}}(\Omega)$. Then there exists a $\delta_1 > 0$ such that

$$\|\varrho u^i \varphi\|_{L^1(B)} < \varepsilon_1\,,\ \|\varphi\|_{L^{\frac{2\gamma}{\gamma-1}}(B)} < \varepsilon_1 \text{ for } \forall\, B \subset \Omega\,,\ \mu(B) < \delta_1\,,\ i = 1\,,2\,,3\,. \tag{2.65}$$

Finally, let us take a smooth subdomain $\overline{\Omega}' \subset \Omega$ such that $\mu(\Omega \setminus \Omega') < \delta_1$. Now we can benefit from Lemma 7 to get a uniform estimate

$$\|\varrho_\varepsilon\|_{L^{\beta+1}((0,\,T)\times\Omega')} \le c\,,$$

where c is independent of ε . Thus we are allowed to use the standard technique from [30] to show

$$\varrho_\varepsilon \vec{u}_\varepsilon \to \varrho\vec{u} \text{ in } C([0\,,T]\,; L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega'))\,. \tag{2.66}$$

The relation (2.60) is now a direct consequence of the estimates (2.65) and

(2.66) applied to the inequality

$$\Big|\int_\Omega (\varrho_\varepsilon \vec{u}_\varepsilon - \varrho\vec{u})\varphi\,\mathrm{d}x\Big| \le$$
$$\Big|\int_{\Omega'} (\varrho_\varepsilon \vec{u}_\varepsilon - \varrho\vec{u})\varphi\,\mathrm{d}x\Big| + \int_{\Omega\setminus\Omega'} |\varrho_\varepsilon \vec{u}_\varepsilon \varphi|\,\mathrm{d}x + \int_{\Omega\setminus\Omega'} |\varrho\vec{u}\varphi|\,\mathrm{d}x < K\varepsilon_1\,,$$

for a.a. $t \in [0\,,T]$, where K is independent of ε_1 and ε .

Finally, seeing that $\frac{2\gamma}{\gamma+1} > \frac{6}{5}$, we can use the relations (2.59), (2.60) to prove (2.61). □

Thus we have proved that the limits $\varrho\,, \vec{u}$ satisfy the following system of equations:

$$\varrho_t + \mathrm{div}(\varrho\vec{u}) = 0\,, \tag{2.67}$$

$$(\varrho u^i)_t + \mathrm{div}(\varrho u^i\vec{u}) + ap_{1x_i} + \delta p_{2x_i} = \mu\Delta u^i + (\lambda+\mu)(\mathrm{div}\vec{u})_{x_i}\,, \quad i = 1\,,2\,,3\,, \tag{2.68}$$

in $\mathcal{D}'((0\,,T)\times\Omega)$. Moreover, in accordance with (2.59), (2.60) and (2.35), (2.39), the limit functions ϱ , $\varrho\vec{u}$ satisfy the initial condition

$$\varrho(0) = \varrho_{0\,\delta}\,,\ \ (\varrho\vec{u})(0) = \vec{q}_\delta\,,$$

where $\varrho_{0\,\delta}$ and $\vec{q}_\delta$ are the same as in Lemma 3.

2.3.5 Strong convergence of the density

We conclude this section by showing $p_1 = \varrho^\gamma\,, p_2 = \varrho^\beta$, and, consequently, strong convergence of the sequence ϱ_ε . We shall need the following assertion:

Lemma 10 *Let ϱ , $\vec{u}$ be a solution of (2.67) in $\mathcal{D}'((0\,,T)\times\Omega)$ with the properties specified in the last subsection. Then, for ϱ , $\vec{u}$ extended to be zero on $\mathbb{R}^3\setminus\Omega$, the equation (2.67) holds in $\mathcal{D}'((0\,,T)\times\mathbb{R}^3)$.*

First, by means of Lemma 10, we are able to prove the relation

$$\int_0^T\int_\Omega \varrho\mathrm{div}\,\vec{u}\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \varrho_{0\,\delta}\ln\varrho_{0\,\delta}\mathrm{d}x - \int_\Omega \varrho(T)\ln\varrho(T)\mathrm{d}x\,, \tag{2.69}$$

taking $b(z) = z\ln z$ in Lemma 6.9 ([30], page 307).

Now because of the additional term $\varepsilon\varrho_{1+\nu}$ in the continuity equation, we need to generalize the so-called renormalized inequality with dissipation.

Lemma 11 *Assume that Ω is a domain in $\mathbb{R}^3$. Let $2 \le \beta < \infty$, and let $1 \le p < \infty$. Suppose that a couple $(\varrho, \vec{u})$ satisfies*

$$\varrho \in L^\infty(0,T; L^\beta_{loc}(\Omega)), \ \Delta\varrho \in L^p_{loc}((0,T)\times\Omega),$$
$$\varrho \ge 0 \text{ a.e. in } (0,T)\times\Omega, \ \vec{u} \in L^2(0,T; W^{1,2}_{loc}(\Omega)),$$
$$\partial_t\varrho + \operatorname{div}\varrho\vec{u} + \varepsilon\varrho^{1+\nu} - \varepsilon\Delta\varrho = 0, \ \text{in } \mathcal{D}'((0,T)\times\Omega).$$

Then for any convex function $b \in C^1([0,\infty)) \cap C^2((0,\infty))$ satisfying growth condition

$$|b'(t)| \le ct^{-\lambda_1}, \ t \ge 1, \ -1 < \lambda_1 \le \frac{\beta}{2} - 1, \ c > 0,$$

it holds

$$\partial_t b(\varrho) + \operatorname{div}(b(\varrho)\vec{u}) + \{\varrho b'(\varrho) - b(\varrho)\}\operatorname{div}\vec{u} + \varepsilon\varrho^{1+\nu}b'(\varrho) - \varepsilon\Delta b(\varrho) \le 0,$$

in $\mathcal{D}'((0,T)\times\Omega)$.

Proof: Let Ω', $\overline{\Omega'} \subset \Omega$ be a bounded domain. The for any sufficiently small $\alpha > 0$, we have

$$\partial_t S_\alpha(\varrho) + \operatorname{div}(S_\alpha(\varrho)) + \varepsilon S_\alpha(\varrho^{1+\nu}) - \varepsilon\Delta S_\alpha(\varrho) = r_\alpha(\varrho, \vec{u}) \text{ a.e. in } (0,T)\times\Omega', \tag{2.70}$$

where S_α is the standard mollifying operator with respect to the space variables, and

$$r_\alpha(\varrho, \vec{u}) = \operatorname{div}(S_\alpha(\varrho)\vec{u}) - \operatorname{div}(S_\alpha(\varrho)\vec{u}).$$

We multiply (2.70) by $b'(S_\alpha(\varrho))$ to obtain

$$\begin{gathered}\partial_t b(S_\alpha(\varrho)) + \operatorname{div}[b(S_\alpha(\varrho))\vec{u}] + \varepsilon S_\alpha(\varrho^{1+\nu})b'(S_\alpha(\varrho)) + \\ [S_\alpha(\varrho)b'(S_\alpha(\varrho)) - b(S_\alpha(\varrho))]\operatorname{div}\vec{u} - \varepsilon b'(S_\alpha(\varrho))\Delta S_\alpha(\varrho) = \\ b'(S_\alpha(\varrho))r_\alpha(\varrho, \vec{u}) \text{ a.e. in } (0,T)\times\Omega'.\end{gathered} \tag{2.71}$$

If we use equation (2.71) and the convexity of b, we obtain

$$\begin{gathered}\partial_t b(S_\alpha(\varrho)) + \operatorname{div}[b(S_\alpha(\varrho))\vec{u}] + \varepsilon S_\alpha(\varrho^{1+\nu})b'(S_\alpha(\varrho)) + \\ [S_\alpha(\varrho)b'(S_\alpha(\varrho)) - b(S_\alpha(\varrho))]\operatorname{div}\vec{u} - \varepsilon\Delta b(S_\alpha(\varrho)) \le \\ b'(S_\alpha(\varrho))r_\alpha(\varrho, \vec{u}) \text{ a.e. in } (0,T)\times\Omega'.\end{gathered}$$

Letting $\alpha \to 0^+$, using the Lebesgue dominated convergence Theorem, Vitali's convergence Theorem, and the fact that r_α tends to zero as $\alpha \to 0^+$ ([30] page 304, Lemma 6.7), we finally arrive at

$$\partial_t b(\varrho) + \operatorname{div}(b(\varrho)\vec{u}) + \{\varrho b'(\varrho) - b(\varrho)\}\operatorname{div}\vec{u} + \varepsilon\varrho^{1+\nu}b'(\varrho) - \varepsilon\Delta b(\varrho) \le 0,$$

in $\mathcal{D}'((0\,,T)\times\Omega')$. Since Ω' was an arbitrary bounded subdomain of Ω, the last inequality implies the desired result. □

Lemma 12 *Let ϱ_ε , $\vec{u}_\varepsilon$ be a couple of solutions of (2.9). Then the following inequality holds*

$$\int_{\Omega_\varepsilon}\varrho_\varepsilon(T)\ln\varrho_\varepsilon(T)-\varrho_{0\,\delta\varepsilon}\ln\varrho_{0\,\delta\varepsilon}\,\mathrm{d}x+\int_0^T\int_{\Omega_\varepsilon}\varrho_\varepsilon\mathrm{div}\vec{u}_\varepsilon+\varepsilon\varrho_\varepsilon^{1+\nu}(1+\ln\varrho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\le 0\,. \tag{2.72}$$

Proof: The functions ϱ_ε , $\vec{u}_\varepsilon$, and $b(s)=s\ln(s+h)$, where $h>0$, satisfy the hypotheses of Lemma 11; whence the renormalized inequality with dissipation introduced in Lemma 11 holds. In fact, by virtue of the regularity properties of ϱ_ε established in the second section of this paper, we get that the renormalized inequality from the previous lemma holds not only in $\mathcal{D}'((0\,,T)\times\Omega_\varepsilon)$ but even almost everywhere, namely

$$\begin{aligned}&\partial_t[\varrho_\varepsilon\ln(\varrho_\varepsilon+h)]+\mathrm{div}[\varrho_\varepsilon\ln(\varrho_\varepsilon+h))\vec{u}_\varepsilon]+\varepsilon\varrho_\varepsilon^{1+\nu}[1+\ln(\varrho_\varepsilon+h)]+\\ &\qquad\frac{\varrho_\varepsilon^2}{\varrho_\varepsilon+h}\mathrm{div}\vec{u}_\varepsilon-\varepsilon[\varrho_\varepsilon\ln(\varrho_\varepsilon+h)]\le 0\quad\text{a.e. in }(0\,,T)\times\Omega_\varepsilon\,.\end{aligned} \tag{2.73}$$

Now integrating (2.73) over Ω_ε and applying the Stokes formula, we get

$$\begin{aligned}&\int_{\Omega_\varepsilon}\varrho_\varepsilon(T)\ln(\varrho_\varepsilon(T)+h)\,\mathrm{d}x-\int_{\Omega_\varepsilon}\varrho_{0\,\delta\varepsilon}\ln(\varrho_{0\,\delta\varepsilon}+h)\,\mathrm{d}x+\\ &\int_0^T\int_{\Omega_\varepsilon}\frac{\varrho_\varepsilon^2}{\varrho_\varepsilon+h}\mathrm{div}\vec{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\Omega_\varepsilon}\varepsilon\varrho_\varepsilon^{1+\nu}[1+\ln(\varrho_\varepsilon+h)]\,\mathrm{d}x\,\mathrm{d}t\le 0\,.\end{aligned}$$

The conclusion of the lemma is proved by letting $h\to 0^+$ and using the Lebesgue dominated convergence Theorem. □

We can now complete the main goal of this subsection. Take two nondecreasing sequences ψ_n , ϕ_n of nonnegative functions such that

$$\psi_n\in\mathcal{D}(0\,,T)\,,\ \psi_n\to 1\,,\quad \phi_n\in\mathcal{D}(\Omega)\,,\ \phi_n\to 1\,.$$

Combining Lemma 3.2 in [15], page 374, together with (2.69), (2.72), and Lemma 3, we obtain

$$\limsup_{\varepsilon\to 0^+}\int_0^T\psi_m\int_\Omega\phi_m(a\varrho_\varepsilon^\gamma+\delta\varrho_\varepsilon^\beta)\varrho_\varepsilon\mathrm{d}x\,\mathrm{d}t\le\int_0^T\int_\Omega(ap_1+\delta p_2)\varrho\,\mathrm{d}x\,\mathrm{d}t\,,\ m=1\,,2\,,\ldots$$

To conclude the proof of strong convergence, we make use of Minty's trick and the above introduced inequality to show

$$\|\varrho_\varepsilon\|_{L^\beta((0,T)\times\Omega)} \to \|\varrho\|_{L^\beta((0,T)\times\Omega)}\,, \tag{2.74}$$

see [15] for details. Furthermore, from (2.22), (2.59) we obtain $\varrho_\varepsilon \rightharpoonup \varrho$ in $L^\beta((0\,,T)\times\Omega)$, which, together with (2.74), implies strong convergence of ϱ_ε in $L^p((0\,,T)\times\Omega)\,,\ 1\le p\le\beta$. Let us review the result achieved in this section:

Lemma 4 *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain of class $W^{1,s}$, where s satisfies (2.3), and let*

$$\beta > \max\left\{\frac{6\gamma}{2\gamma-3}\,,4\right\}.$$

Then, given initial data $\varrho_{0\,\delta}\,,\ \vec{q}_\delta$ as in (2.36), (2.40), there exists a weak solution $\varrho\,,\ \vec{u}$ of the problem

$$\varrho_t + \mathrm{div}\varrho\vec{u} = 0\,, \tag{2.75}$$

$$(\varrho u^i)_t+\mathrm{div}(\varrho u^i\vec{u})+(a\varrho^\gamma+\delta\varrho^\beta)_{x_i} = \mu\Delta u^i+(\lambda+\mu)(\mathrm{div}\vec{u})_{x_i}\,,\ i=1\,,2\,,3\,, \tag{2.76}$$

$$\vec{u}|_{\partial\Omega} = 0\,, \tag{2.77}$$

$$\varrho(0) = \varrho_{0\,\delta}\,,\quad \varrho\vec{u}(0) = \vec{q}_\delta\,. \tag{2.78}$$

Moreover, the equation (2.75) holds in the sense of renormalized solutions on $\mathcal{D}'((0\,,T)\times\mathbb{R}^3)$ provided $\varrho\,,\ \vec{u}$ were extended to be zero on $\mathbb{R}^3\setminus\Omega$.

Finally, $\varrho\,,\ \vec{u}$ satisfy the estimates:

$$\sup_{t\in[0,T]}\|\varrho(t)\|^\gamma_{L^\gamma(\Omega)} \le cE_\delta[\varrho_{0\,\delta}\,,\vec{q}_\delta]\,, \tag{2.79}$$

$$\delta\sup_{t\in[0,T]}\|\varrho(t)\|^\beta_{L^\beta(\Omega)} \le cE_\delta[\varrho_{0\,\delta}\,,\vec{q}_\delta]\,, \tag{2.80}$$

$$\sup_{t\in[0,T]}\|\sqrt{\varrho}\vec{u}(t)\|^2_{L^2(\Omega)} \le cE_\delta[\varrho_{0\,\delta}\,,\vec{q}_\delta]\,, \tag{2.81}$$

$$\|\vec{u}\|^2_{L^2(0,T;W^{1,2}(\Omega))} \le cE_\delta[\varrho_{0\,\delta}\,,\vec{q}_\delta]\,, \tag{2.82}$$

where the constant c is independent of $\delta>0$. Moreover, the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_\Omega \frac{1}{2}\varrho|\vec{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \frac{\delta}{\beta-1}\varrho^\beta\mathrm{d}x\Big] + \int_\Omega \mu|\nabla\vec{u}|^2 + (\lambda+\mu)|\mathrm{div}\,\vec{u}|^2\mathrm{d}x \le 0\,, \tag{2.83}$$

holds in $\mathcal{D}'(0\,,T)$.

Proof: It remains to prove estimates and the energy inequality. Estimates (2.79) - (2.82) follow from (2.21) - (2.24), (2.58) and (2.74). We will pass the limit $\varepsilon \to 0^+$ in (2.28), where Ω, ϱ, $\vec{u}$ are replaced by Ω_ε, ϱ_ε, $\vec{u}_\varepsilon$. Due to (2.58), (2.74) and (2.23), we have

$$\int_{\Omega_\varepsilon} \varrho_\varepsilon |\vec{u}_\varepsilon|^2 \mathrm{d}x \to \int_\Omega \varrho |\vec{u}|^2 \mathrm{d}x \quad \text{weakly in } L^2(0\,,T)\,. \tag{2.84}$$

Moreover, relation (2.74) also gives

$$\int_\Omega a\varrho_\varepsilon^\gamma + \delta \varrho_\varepsilon^\beta \mathrm{d}x \to \int_\Omega a\varrho^\gamma + \delta\varrho^\beta \mathrm{d}x \quad \text{weakly in } L^1(0\,,T)\,. \tag{2.85}$$

Now let $\psi_n \in \mathcal{D}'(0\,,T)$ be a sequence of non-negative functions with uniformly bounded derivatives such that $\psi_n \nearrow 1$. Then using ψ_n as test functions in energy inequality (2.28) and estimates (2.79) - (2.81) we get estimate

$$\varepsilon \frac{\delta\beta}{\beta - 1} \int_0^T \psi_n \int_{\Omega_\varepsilon} \varrho_\varepsilon^{\beta+\nu} \mathrm{d}x\, \mathrm{d}t \leq$$
$$\int_0^T \partial_t \psi_n \int_{\Omega_\varepsilon} \frac{1}{2} \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \frac{a}{\gamma - 1} \varrho_\varepsilon^\gamma + \frac{\delta}{\beta - 1} \varrho_\varepsilon^\beta \mathrm{d}x\, \mathrm{d}t \leq c E_\delta[\varrho_{0\,\delta}\,, \vec{q}_\delta]\,,$$

where $c > 0$ is independent of both ε, δ. Passing to the limit for $n \to \infty$ in the above derived estimate we have

$$\varepsilon \int_0^T \int_{\Omega_\varepsilon} \varrho_\varepsilon^{\beta+\nu} \mathrm{d}x\, \mathrm{d}t \leq c(\delta\,, \varrho_{0\,\delta}\,, \vec{q}_\delta)\,. \tag{2.86}$$

By means of Hölder's inequality we obtain

$$\int_0^T \int_{\Omega_\varepsilon \setminus \Omega} \varrho_\varepsilon^\beta \mathrm{d}x\, \mathrm{d}t \leq \varepsilon^{-\frac{\beta}{\beta+\nu}} \Big(T\mu(\Omega_\varepsilon \setminus \Omega) \Big)^{\frac{\nu}{\nu+\beta}} \Big(\varepsilon \int_0^T \int_{\Omega_\varepsilon} \varrho_\varepsilon^{\beta+\nu} \mathrm{d}x\, \mathrm{d}t \Big)^{\frac{\beta}{\beta+\nu}}\,.$$

Thus using (2.86), under the assumption $\mu(\Omega_\varepsilon \setminus \Omega) \to 0$ sufficiently fast, which is possible to reach by means of suitable choice of ε', ε'' in (2.29), one has

$$\int_0^T \int_{\Omega_\varepsilon \setminus \Omega} \varrho_\varepsilon^\beta \mathrm{d}x\, \mathrm{d}t \to 0 \ \text{ for } \varepsilon \to 0^+\,. \tag{2.87}$$

Consequently, using (2.84), (2.85), and (2.87), it is easy to see that the first term in (2.28) converge to the first term of (2.83). The second and the third term in (2.28) converges to the second and the third term of (2.83), respectively, due to lower semicontinuity of norms and weak convergence of $\vec{u}_\varepsilon$, which concludes the proof. □

2.4 Passing to the limit in the artificial pressure term

2.4.1 On integrability of the density

Our ultimate goal is to let $\delta \to 0$ in (2.76) and also in (2.78). To get this result, we first derive estimates of the density ϱ_δ independent of $\delta > 0$. The technique is the same as in the previous section.

Similarly to the previous section, let $\varepsilon' > 0$ be arbitrary and consider the $\delta' > 0$ given by Proposition 2 . Furthermore, let Ω_1, Ω_2 and Ω_3 be smooth domains defined as in Section 3.3 and satisfying (2.50). We note that, due to the (2.37), (2.40) and (2.41), the right sides of estimates (2.79) - (2.82) are bounded independently of δ to be used in what follows. The following lemma provides estimates of the density on the domain Ω_2 :

Lemma 13 *Let ϱ_ε , $\vec{u}_\delta$ be the sequence of solutions of the problem (2.75) - (2.78). Then there exists a constant $c = c(\varrho_0, \vec{q}, \Omega_1, \Omega_2)$ independent of δ and constant $\theta > 0$, depending only on γ, such that*

$$\|\varrho_\delta\|^{\gamma+\theta}_{L^{\gamma+\theta}((0,T)\times\Omega_2)} + \delta\|\varrho_\delta\|^{\beta+\theta}_{L^{\beta+\theta}((0,T)\times\Omega_2)} \le c\,. \tag{2.88}$$

Proof: Let $\phi \in \mathcal{D}(\Omega_1)$ such that $0 \le \phi \le 1$, $\phi(x) = 1$ for each $x \in \Omega_2$ and $\psi \in \mathcal{D}(0\,,T)$ such that $0 \le \psi \le 1$. Extending ϱ_δ by zero outside Ω consider the function

$$\hat{\varphi}(t\,,x) = \psi(t)\phi(x)\mathcal{A}[S_\alpha[b(\varrho_\delta)]]\,,$$

where $b(s) = s^\theta$, for $0 < \theta < 2$. Following [15] page 382 - 384 we use $\hat{\varphi}$ as a test function for the system (2.76), which yields desired estimate. $\square$

In order to show the second assumption of Lemma 6 we have to investigate the behavior of solutions in a vicinity of the non-smooth part of the boundary $\partial\Omega$. We report the following lemma that can be proved in the same way as Lemma 8.

Lemma 14 *Let ϱ_δ , $\vec{u}_\delta$ be the sequence of solutions of the problem (2.75) - (2.78) constructed above. Then there exists a constant $c = c(\varrho_0, \vec{q}, \Omega_1, \Omega_2)$ independent of δ such that*

$$\int_0^T \int_{\Omega\setminus\Omega_2} a\varrho_\delta^\gamma + \delta\varrho_\delta^\beta \mathrm{d}x\,\mathrm{d}t \le c\,\varepsilon'\,. \tag{2.89}$$

If we combine the above lemmas together with Lemma 6 we get

Lemma 5 *There exists a function $p \in L^1((0\,,T)\times\Omega)$ such that*

$$\varrho_\delta^\gamma \rightharpoonup p \ \text{ in } L^1((0\,,T)\times\Omega)\,,$$
$$\delta\varrho_\delta^\beta \rightharpoonup 0 \ \text{ in } L^1((0\,,T)\times\Omega)\,.$$

Proof : It follows from Lemma 13 that ϱ_δ converges weakly in $L^{\gamma+\theta}((0\,,T)\times\Omega_2)$ and $\delta^{\frac{1}{\beta+\theta}}\varrho_\delta$ converges weakly in $L^{\beta+\theta}((0\,,T)\times\Omega_2)$, which implies that ϱ_δ^γ, $\delta^{\frac{\beta}{\beta+\theta}}\varrho_\delta^\beta$ converge weakly in $L^1((0\,,T)\times\Omega_2)$. Now implication $(i)\Rightarrow(ii)$ in Lemma 6 assures the existence of $\delta''>0$ such that

$$\begin{gathered}\|\varrho_\delta^\gamma\|_{L^1((0,T)\times B)}<\varepsilon' \quad \text{for}\,\forall B\subset\Omega_2\,,\ \mu(B)<\delta''\,,\\ \delta\|\varrho_\delta^\beta\|_{L^1((0,T)\times B)}<\delta^{\frac{\beta}{\beta+\theta}}\|\varrho_\delta^\beta\|_{L^1((0,T)\times B)}<\varepsilon' \quad \text{for}\,\forall B\subset\Omega_2\,,\ \mu(B)<\delta''\,,\end{gathered} \tag{2.90}$$

for all δ sufficiently small. If we denote $\delta''':=\min\{\delta'\,,\delta''\}$, then the second condition (ii) of the Lemma 6 is satisfied and we have

$$\begin{gathered}\varrho_\delta^\gamma\rightharpoonup p \ \text{ in } L^1((0\,,T)\times\Omega)\,,\\ \delta\varrho_\delta^\beta\rightharpoonup p' \ \text{ in } L^1((0\,,T)\times\Omega)\,,\end{gathered} \tag{2.91}$$

yielding the first part of the claim. Indeed, if $B\subset\Omega_2$, $\mu(B)<\delta'''$, then (ii) holds because of (2.90).

Otherwise, if $B\subset\Omega\setminus\Omega_2$, then Lemma 14 implies

$$\begin{gathered}\|\varrho_\delta^\gamma\|_{L^1((0,T)\times B)}<c\varepsilon'\,,\\ \delta\|\varrho_\delta^\beta\|_{L^1((0,T)\times B)}<c\varepsilon'.\end{gathered} \tag{2.92}$$

The last case $B\cap\Omega_2\neq\emptyset$, $B\cap(\Omega\setminus\Omega_2)\neq\emptyset$ can be treated by means of combination of (2.90) and (2.92), which completes the proof of (2.91).

It remains to prove

$$p'=0 \text{ a.e. in } (0\,,T)\times\Omega\,. \tag{2.93}$$

Let us suppose $\mu(\{p'\neq 0\}):=h>0$, and consider an increasing sequence of Lipschitz domains Ω_n, $\overline{\Omega}_n\subset\Omega$ constructed as follows:

$$a_r^n(x_r'):=(a_r*\omega_{k_n})(x_r')+\frac{1}{n}\,, \qquad r=1\,,\ldots\,,M,$$

where the functions a_r describing the boundary of Ω are taken as in Definition 2.1.1. Moreover,

i) $\lim\limits_{n\to\infty}k_n=0$,

ii) $a_r^n(x_r')-a_r(x_r')>0\,.$

From the construction of the sequence, one can observe that $\mu(\Omega\setminus\Omega_n)\to 0$ for n tending to infinity. If we show that

$$p'=0 \text{ a.e. in } (0\,,T)\times\Omega_n\,, \tag{2.94}$$

for an arbitrary n we get a contradiction with the fact that $h > 0$, and thus (2.93) holds. Since $\overline{\Omega}_n \subset \Omega$ and Ω_n is Lipschitz, we can use the same procedure as in the proof of Lemma 13 to get the estimate

$$\delta \|\varrho_\delta\|^{\beta+\theta}_{L^{\beta+\theta}((0,T)\times\Omega_n)} \le c_n \,,$$

which yields

$$\delta^{\frac{\beta}{\beta+\theta}} \varrho_\delta^\beta \rightharpoonup p_n \text{ in } L^1((0\,,T)\times\Omega_n)\,, \tag{2.95}$$

and thus

$$\delta \varrho_\delta^\beta \rightharpoonup 0 \text{ in } L^1((0\,,T)\times\Omega_n)\,. \tag{2.96}$$

But (2.95) and uniqueness of the limit gives (2.94). □

2.4.2 The limit passage

At this stage we can pass to the limit for $\delta \to 0$ to eliminate the artificial pressure term in (2.76). Let Ω' be an arbitrary Lipschitz domain such that $\overline{\Omega} \subset \Omega'$. Then it follows from (2.82) and the fact $\vec{u}_\delta \in W_0^{1,2}(\Omega')$ that

$$\vec{u}_\delta \to \vec{u} \text{ weakly in } L^2(0\,,T\,;W_0^{1,2}(\Omega'))\,, \tag{2.97}$$

on each fixed Ω', where $\vec{u}_\delta$ is extended by zero outside Ω. Thus we have from Lemma 5

$$\vec{u} \in L^2(0\,,T\,;W_0^{1,2}(\Omega))\,.$$

Other convergence properties are provided by the following lemma:

Lemma 15

$$\varrho_\delta \to \varrho \text{ in } C([0\,,T]\,;L^\gamma_{weak}(\Omega))\,, \tag{2.98}$$

$$\varrho_\delta \vec{u}_\delta \to \varrho\vec{u} \text{ in } C([0\,,T]\,;L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega))\,, \tag{2.99}$$

$$\varrho_\delta u_\delta^i u_\delta^j \to \varrho u^i u^j \text{ in } \mathcal{D}'((0\,,T)\times\Omega)\,,\ i,j = 1\,,2\,,3\,. \tag{2.100}$$

Proof. Since the continuity equation (2.75) is satisfied in $\mathcal{D}'((0\,,T)\times\Omega)$ (see Lemma 10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega'} \varrho_\delta \varphi \,\mathrm{d}x = \int_{\Omega'} \varrho_\delta \vec{u}_\delta \cdot \nabla\varphi \,\mathrm{d}x \text{ in } \mathcal{D}'(0\,,T)\,,\ \varphi \in \mathcal{D}(\Omega')\,,$$

which implies $\varrho_\delta \in C([0\,,T]\,,L^\gamma_{weak}(\Omega'))$, and, moreover, ϱ_δ are uniformly continuous in $W^{-1,\frac{2\gamma}{\gamma+1}}(\Omega')$. Moreover ϱ_δ are uniformly bounded in $L^\gamma(\Omega')$. According to Lemma 6.2 in [30], we have, at least for a chosen subsequence,

$$\varrho_\delta \to \varrho' \text{ in } C([0\,,T]\,,L^\gamma_{weak}(\Omega'))\,, \tag{2.101}$$

where ϱ' depends on Ω'.

On the other hand, using the arguments of Lemma 9 we easily get (2.98). Statements (2.99) and (2.100) can be proved in the same way as (2.60), (2.61) in Lemma 9, where we take into account estimates of ϱ_δ in $L^{\beta+\theta}$, $L^{\gamma+\theta}$ instead of $L^{\beta+1}$. □

Consequently, ϱ, $\vec{u}$ satisfy the continuity equation (2.1) in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, and

$$(\varrho u^i)_t + \operatorname{div}(\varrho u^i \vec{u}) + a p_{x_i} = \mu\Delta u^i + (\lambda+\mu)(\operatorname{div}\vec{u})_{x_i}, \quad i = 1,2,3. \tag{2.102}$$

in $\mathcal{D}'((0,T)\times\Omega)$.

Moreover, due to relations (2.98), (2.99), and Lemma 3, the limits ϱ and $\vec{u}$ satisfy the initial conditions (2.5). In order to conclude the proof, we have to show strong convergence of ϱ_δ in L^1 or, equivalently, $p = \varrho^\gamma$. But this can be shown exactly as in [15] page 385 - 391. According to Proposition 5, we are allowed to pass to the limit for $\delta \to 0$ in (2.83) to obtain the energy inequality introduced in Definition 2.1.2. The proof of Theorem 2.1.1 is complete.

2.5 Concluding remarks

- The proof of Theorem 2.1.1 remains basically unchanged if the motion of the fluid is driven by a bounded external force, i.e., when (2.2) contains an additional term $\varrho\vec{f}(t,x)$, with $\vec{f}$ bounded and measurable function.
- Similar result can be proved in two space dimensions.

Acknowledgment. This research was supported by the Jindřich Nečas Center for Mathematical Modelling, project LC06052, financed by MŠMT of the Czech Republic.

Chapter 3

Singular limits of the equations of magnetohydrodynamics

Corresponds to the article by Kukučka P.: Singular Limits of the Equations of Magnetohydrodynamics, accepted for publication in Journal of Mathematical Fluid Mechanics.

Abstract: This paper studies the asymptotic limit for solutions to the equations of magnetohydrodynamics, specifically, the Navier-Stokes-Fourier system describing the evolution of a compressible, viscous, and heat conducting fluid coupled with the Maxwell equations governing the behavior of the magnetic field, when Mach number and Alfvén number tends to zero. The introduced system is considered on a bounded spatial domain in $\mathbb{R}^3$, supplemented with conservative boundary conditions. Convergence towards the incompressible system of the equations of magnetohydrodynamics is shown.

2000 Mathematics Subject Classification. 35A05, 35Q30, 35Q60

Keywords: Navier-Stokes-Fourier system, Oberbeck-Boussinesq approximation, Maxwell equations

3.1 Introduction

3.1.1 Problem formulation

The basic principles of continuum mechanics and electrodynamics assert the balance or conservation of mass, momentum, energy, and Maxwell equations that can be expressed through a system of equations (see e.g. Ducomet and Feireisl [7]):

$$\varrho_t + \operatorname{div}(\varrho\vec{u}) = 0\,, \tag{3.1}$$

$$(\varrho\vec{u})_t + \operatorname{div}(\varrho\vec{u}\otimes\vec{u}) + \nabla p = \operatorname{div}\mathbb{S} + \vec{J}\times\vec{B}\,, \tag{3.2}$$

$$(\varrho s)_t + \mathrm{div}(\varrho s\vec{u}) + \mathrm{div}(\frac{\vec{q}}{\vartheta}) = \sigma\,, \tag{3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e + \frac{1}{2\mu}|\vec{B}|^2\Big)\mathrm{d}x = 0\,, \tag{3.4}$$

$$\vec{B}_t + \mathrm{curl}(\vec{B}\times\vec{u}) + \mathrm{curl}(\lambda\mathrm{curl}\vec{B}) = 0\,, \tag{3.5}$$

where the density $\varrho = \varrho(t\,,x)$, the velocity field $\vec{u} = \vec{u}(t\,,x)$, the absolute temperature $\vartheta = \vartheta(t\,,x)$, and the magnetic induction field $\vec{B} = \vec{B}(t\,,x)$ are state variables depending on the time $t \in (0\,,T)$ and the spatial position $x \in \Omega \subset \mathbb{R}^3$, and where Ω is a bounded domain. For the more specific explanation of the physical background of the introduced system, the reader may consult e.g. Weiss [38], where it is used in order to model the solar and stellar magnetoconvection.

We shall assume that the pressure $p = p(\varrho\,,\vartheta)$, the specific entropy $s = s(\varrho\,,\vartheta)$ and the specific internal energy $e = e(\varrho\,,\vartheta)$ are given functions of the state variables ϱ , ϑ obeying the Gibbs relation

$$\vartheta\mathrm{d}s = \mathrm{d}e + p\,\mathrm{d}\Big(\frac{1}{\varrho}\Big)\,. \tag{3.6}$$

Since we focus our attention on Newtonian fluids, the viscous stress tensor, denoted by the symbol $\mathbb{S}$, can be expressed as a linear function of the velocity gradient

$$\mathbb{S} = \nu(\nabla\vec{u} + \nabla\vec{u}^T - \frac{2}{3}\mathrm{div}\vec{u}\,\mathbb{I}) + \eta\mathrm{div}\vec{u}\,\mathbb{I}\,, \tag{3.7}$$

where $\nu = \nu(\vartheta\,,\vec{B}) > 0$ and $\eta = \eta(\vartheta\,,\vec{B}) > 0$ are viscosity coefficients.

Similarly, the heat flux will be given by Fourier's law

$$\vec{q} = \vec{q}_R + \vec{q}_F\,, \tag{3.8}$$

with the radiation heat flux $\vec{q}_R$ given by

$$\vec{q}_R = -\kappa_R\vartheta^3\nabla\vartheta\,,\ \text{with a constant } \kappa_R > 0\,, \tag{3.9}$$

and

$$\vec{q}_F = -\kappa_F\nabla\vartheta\,, \tag{3.10}$$

with a positive heat conductivity coefficient $\kappa_F = \kappa_F(\varrho\,,\vartheta\,,\vec{B})$.

In general, if the motion is not known to be smooth, the dissipation rate of the mechanical energy may exceed the value $\mathbb{S} : \nabla\vec{u}$, and then the entropy production rate σ satisfies inequality

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathbb{S} : \nabla\vec{u} - \frac{\vec{q}\cdot\nabla\vartheta}{\vartheta} + \frac{\lambda}{\mu}|\mathrm{curl}\vec{B}|^2\Big)\,, \tag{3.11}$$

and the electric current $\vec{J}$ satisfies Ampère law

$$\mu\vec{J} = \mathrm{curl}\vec{B}\,,\ \ \mu > 0 \tag{3.12}$$

where the constant μ stands for permeability of the free space, and $\lambda = \lambda(\varrho\,,\vartheta\,,\vec{B}) > 0$ is the magnetic diffusivity.

Since the fluid occupies a bounded domain $\Omega\,$, we need to prescribe some boundary conditions. In order to eliminate the effect of a boundary layer on propagation of the acoustic waves, we suppose that the velocity $\vec{u}$ satisfies the complete slip boundary conditions

$$\vec{u}\cdot\vec{n}|_{\partial\Omega} = 0\,,\ \mathbb{S}\vec{n}\times\vec{n}|_{\partial\Omega} = 0 \tag{3.13}$$

In agreement with (3.4) the total energy is supposed to be a constant of motion, whence we need

$$\vec{q}\cdot\vec{n}|_{\partial\Omega} = 0\,. \tag{3.14}$$

Because we want the system to be energetically isolated we assume

$$\vec{B}\cdot\vec{n}|_{\partial\Omega} = 0\,. \tag{3.15}$$

To complete the formulation it remains to determine the initial data

$$\varrho(0\,,\cdot) = \varrho_0\,,\ \vec{u}(0\,,\cdot) = \vec{u}_0\,,\ \vartheta(0\,,\cdot) = \vartheta_0\,,\ \vec{B}(0\,,\cdot) = \vec{B}_0\,. \tag{3.16}$$

We are now able to introduce the Mach and Alfvén numbers into our system which can be done in two different ways, either by finding a suitable changes of independent variables as used for Navier-Stokes system in e.g. Alazard [2] or via recasting the system in the dimensionless form by scaling each variable by its characteristic value (see Feireisl, Novotný [13], Chapter 4.). Motivated by [13] we use the second approach in order to get

$$\begin{gathered}
\varrho_t + \mathrm{div}(\varrho\vec{u}) = 0\,,\\
(\varrho\vec{u})_t + \mathrm{div}(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{\mathrm{Ma}^2}\nabla p = \mathrm{div}\,\mathbb{S} + \frac{1}{\mathrm{Al}^2}\vec{J}\times\vec{B}\,,\\
(\varrho s)_t + \mathrm{div}(\varrho s\vec{u}) + \mathrm{div}\Big(\frac{\vec{q}}{\vartheta}\Big) = \sigma\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(\frac{\mathrm{Ma}^2}{2}\varrho|\vec{u}|^2 + \varrho e + \frac{\mathrm{Ma}^2}{\mathrm{Al}^2}\frac{1}{2\mu}|\vec{B}|^2\Big)\mathrm{d}x = 0\,,\\
\vec{B}_t + \mathrm{curl}(\vec{B}\times\vec{u}) + \mathrm{curl}(\lambda\mathrm{curl}\vec{B}) = 0\,,
\end{gathered}$$

together with the modified entropy production rate

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathrm{Ma}^2\,\mathbb{S}:\nabla\vec{u} - \frac{\vec{q}\cdot\nabla\vartheta}{\vartheta} + \frac{\lambda}{\mu}|\mathrm{curl}\vec{B}|^2\Big)\,.$$

We now set both Mach and Alfvén number equal to ε, and thus the resulting system reads

$$\varrho_t + \operatorname{div}(\varrho\vec{u}) = 0\,, \tag{3.17}$$

$$(\varrho\vec{u})_t + \operatorname{div}(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{\varepsilon^2}\nabla p = \operatorname{div}\mathbb{S} + \frac{1}{\varepsilon^2}\vec{J}\times\vec{B}\,, \tag{3.18}$$

$$(\varrho s)_t + \operatorname{div}(\varrho s\vec{u}) + \operatorname{div}(\frac{\vec{q}}{\vartheta}) = \sigma_\varepsilon\,, \tag{3.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho|\vec{u}|^2 + \varrho e + \frac{1}{2\mu}|\vec{B}|^2\Big)\mathrm{d}x = 0\,, \tag{3.20}$$

$$\vec{B}_t + \operatorname{curl}(\vec{B}\times\vec{u}) + \operatorname{curl}(\lambda\operatorname{curl}\vec{B}) = 0\,, \tag{3.21}$$

where

$$\sigma_\varepsilon \geq \frac{1}{\vartheta}\Big(\varepsilon^2\,\mathbb{S}:\nabla\vec{u} - \frac{\vec{q}\cdot\nabla\vartheta}{\vartheta} + \frac{\lambda}{\mu}|\operatorname{curl}\vec{B}|^2\Big)\,. \tag{3.22}$$

The goal of this paper is to execute the limit process for $\varepsilon \to 0$ in (3.17) - (3.22) under the below introduced additional assumptions.

Singular limits in the equations of fluid dynamics are studied extensively, and many results have already been achieved in this field. The first result was proved by Lions and Masmoudi in [27]. The limit solutions for Navier-Stokes system when the Mach number tends to zero was studied by Alazard in [2] and extended for the full Navier-Stokes-Fourier system by Feireisl and Novotný in [14]. In cited papers, the limit solutions are proved in a weak sense on an arbitrary time interval $(0\,,T)$. Other approach to this topic was studied by Klainerman and Majda in [21] where the existence of the limit solution is proved in the classical sense, but on a sufficiently small time interval. They proved that the solutions of the compressible magnetohydrodynamics equations tend to a solution of the incompressible magnetohydrodynamics equations under the assumption that the Mach number tends to zero. Other results concerning this topic have been summarized by Zank and Matthaeus in [39], and also by Rubini in [35]. But only the density, the velocity and the magnetic induction is taken into account in both these papers, not the temperature. We extend this results taking the temperature into account for a special kind of initial conditions.

3.1.2 Variational solution

We adopt the mathematical theory of the weak solution for the equations of magnetohydrodynamics developed in [7] for a certain special type of initial conditions.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary of class $C^{2+\delta}$, $\delta > 0$, and suppose that the initial data (3.16) are given such that

$$\begin{gathered}\varrho_0 \in L^{\frac{5}{3}}(\Omega)\,,\, \varrho_0\vec{u}_0 \in L^1(\Omega\,;\mathbb{R}^3)\,,\, \vartheta_0 \in L^\infty(\mathbb{R}^3)\,,\, \vec{B}_0 \in L^2(\Omega\,;\mathbb{R}^3)\,,\\ \varrho_0 \geq 0\,,\, \vartheta_0 > 0\,,\\ \varrho_0 s(\varrho_0\,,\vartheta_0)\,,\, \frac{1}{\varrho_0}|\varrho_0\vec{u}_0|^2\,,\, \varrho_0 e(\varrho_0\,,\vartheta_0) \in L^1(\Omega)\,,\\ \mathrm{div}\vec{B}_0 = 0 \text{ in } \mathcal{D}'(\Omega)\,,\ \vec{B}_0\cdot\vec{n}|_{\partial\Omega} = 0\,.\end{gathered} \tag{3.23}$$

and let (3.6) - (3.10), (3.12), and (3.22) hold. Let's notice that, due to the Theorem 1. 4 in [37], assumptions on $\vec{B}_0$ make sense. We say that a quantity $\{\varrho\,,\vec{u}\,,\vartheta\,,\vec{B}\}$ is a variational solution of (3.17) - (3.21) on $(0\,,T)\times\Omega$ with boundary conditions (3.13) - (3.15) and initial data (3.16), obeying (3.23) if

- $$\begin{gathered}\varrho \geq 0\,,\ \ \varrho \in L^\infty(0\,,T\,;L^{\frac{5}{3}}(\Omega))\,,\\ \vec{u} \in L^2(0\,,T\,;V)\,,\ V = \{\vec{u} \in W^{1,2}(\Omega)|\quad \vec{u}\cdot\vec{n}|_{\partial\Omega} = 0\}\,,\\ \vartheta\,,\log\vartheta \in L^2(0\,,T\,;W^{1,2}(\Omega))\,,\\ \vec{B} \in L^2(0\,,T\,;W^{1,2}(\Omega))\,,\ \mathrm{div}\vec{B}(t) = 0\,,\ \vec{B}\cdot\vec{n}|_{\partial\Omega} = 0 \text{ for a.a. } t \in (0\,,T)\,;\end{gathered} \tag{3.24}$$

- The continuity equation (3.17) is satisfied in the sense of renormalized solutions which means that the following identity holds

$$\begin{aligned}&\int_0^T\int_\Omega \varrho B(\varrho)\Big(\partial_t\varphi + \vec{u}\cdot\nabla\varphi\Big)\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_\Omega b(\varrho)\mathrm{div}\vec{u}\varphi\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varrho_0 B(\varrho_0)\varphi(0\,,\cdot)\mathrm{d}x\,,\end{aligned} \tag{3.25}$$

  for any $\varphi \in \mathcal{D}([0\,,T)\times\overline{\Omega})$ and any $b \in L^\infty \cap C[0\,,\infty)$ where

$$B(\varrho) = B(0) + \int_1^\varrho \frac{b(z)}{z^2}\mathrm{d}z\,;$$

- $$\begin{aligned}&\int_0^T\int_\Omega \Big(\varrho\vec{u}\partial_t\varphi + \varrho[\vec{u}\otimes\vec{u}] : \nabla\varphi + \frac{1}{\varepsilon^2}p(\varrho\,,\vartheta)\mathrm{div}\varphi\Big)\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_\Omega \Big(\mathbb{S} : \nabla\varphi - \frac{1}{\varepsilon^2}\vec{J}\times\vec{B}\cdot\varphi\Big)\mathrm{d}x\,\mathrm{d}t - \int_\Omega (\varrho_0\vec{u}_0)\cdot\varphi\mathrm{d}x\,,\end{aligned} \tag{3.26}$$

  for any $\varphi \in \mathcal{D}([0\,,T)\times\overline{\Omega}\,;\mathbb{R}^3)$, $\varphi\cdot\vec{n}|_{\partial\Omega} = 0$;

- $$\int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho\,,\vartheta) + \frac{1}{2\mu}|\vec{B}|^2\Big)(t)\mathrm{d}x$$
$$= \int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho_0|\vec{u}_0|^2 + \varrho_0 e(\varrho_0\,,\vartheta_0) + \frac{1}{2\mu}|\vec{B}_0|^2\Big)\mathrm{d}x\,,\ \text{for a. a.}\, t\in(0\,,T)\,; \tag{3.27}$$

- $$\int_0^T\int_\Omega \varrho s(\varrho\,,\vartheta)\Big(\partial_t\varphi + \vec{u}\cdot\nabla\varphi\Big)\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \frac{\vec{q}}{\vartheta}\cdot\nabla\varphi\mathrm{d}x\,\mathrm{d}t$$
$$+ < \sigma_\varepsilon\,,\varphi >_{[\mathcal{M},C]([0,T]\times\overline{\Omega})} = -\int_\Omega \varrho_0 s(\varrho_0\,,\vartheta_0)\varphi(0\,,\cdot)\mathrm{d}x\,, \tag{3.28}$$
for any $\varphi\in\mathcal{D}([0\,,T)\times\overline{\Omega})$, where $\sigma_\varepsilon\in\mathcal{M}^+([0\,,T]\times\overline{\Omega})$,
$$\sigma_\varepsilon \geq \frac{1}{\vartheta}\Big(\varepsilon^2\mathbb{S}:\nabla\vec{u} - \frac{\vec{q}}{\vartheta}\cdot\nabla\vartheta + \frac{\lambda}{\mu}|\mathrm{curl}\vec{B}|^2\Big)\,;$$

- $$\int_0^T\int_\Omega \Big(\vec{B}\cdot\partial_t\varphi - (\vec{B}\times\vec{u}+\lambda\mathrm{curl}\vec{B})\cdot\mathrm{curl}\varphi\Big)\mathrm{d}x\,\mathrm{d}t = \int_\Omega \vec{B}_0\cdot\varphi(0\,,\cdot)\mathrm{d}x\,, \tag{3.29}$$
for any $\varphi\in\mathcal{D}([0\,,T)\times\overline{\Omega}\,;\mathbb{R}^3)$.

Similarly we now define here so called the Oberbeck-Boussinesq approximation supplemented by the Maxwell equation. We say that $\vec{U}$, Θ and $\vec{B}$ is a weak solution of magnetohydrodynamics equations with temperature if

$$\vec{U}\in L^\infty(0\,,T\,;L^2(\Omega\,;\mathbb{R}^3))\cap L^2(0\,,T\,;W^{1,2}(\Omega\,;\mathbb{R}^3))\,,$$
$$\Theta\in W^{1,q}(0\,,T\,;L^q(\Omega))\cap L^q(0\,,T\,;W^{2,q}(\Omega))\ \text{for a certain}\ q>1\,,$$
$$\vec{B}\in L^2(0\,,T\,;W^{1,2}(\Omega\,;\mathbb{R}^3))\,,$$

and the following holds:

- $$\mathrm{div}\vec{U} = 0\ \text{a.a. on}\ (0\,,T)\times\Omega\,,\ \ \vec{U}\cdot\vec{n}|_{\partial\Omega} = 0\ \text{in the sense of traces.} \tag{3.30}$$

- $$\int_0^T\int_\Omega \Big(\overline{\varrho}\,\vec{U}\cdot\partial_t\varphi + \overline{\varrho}\,\vec{U}\otimes\vec{U}:\nabla\varphi\Big)\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T\int_\Omega \Big(\nu(\overline{\vartheta}\,,0)[\nabla\vec{U}+\nabla^T\vec{U}]:\nabla\varphi - \vec{J}\times\vec{B}\cdot\varphi\Big)\mathrm{d}x\,\mathrm{d}t - \int_\Omega(\overline{\varrho}\,\vec{U}_0)\cdot\varphi\mathrm{d}x\,, \tag{3.31}$$

for any $\varphi \in \mathcal{D}([0\,,T) \times \overline{\Omega}\,;\mathbb{R}^3)$, $\mathrm{div}\varphi = 0$ in Ω , $\varphi \cdot \vec{n}|_{\partial\Omega} = 0$.

- $$\overline{\varrho} c_p(\overline{\varrho}\,,\overline{\vartheta})\Big(\partial_t \Theta + \vec{U}\cdot\nabla\Theta\Big) - \mathrm{div}\Big((\kappa_F(\overline{\varrho}\,,\overline{\vartheta}\,,0) + \kappa_R \overline{\vartheta}^2)\nabla\Theta\Big) = 0$$

a.a. in $(0\,,T) \times \Omega$,

$$\begin{aligned} \nabla\Theta \cdot \vec{n}|_{\partial\Omega} &= 0\,, \\ \Theta(0\,,\cdot) &= \Theta_0\,, \\ r + \overline{\varrho}\alpha(\overline{\varrho}\,,\overline{\vartheta})\Theta &= 0\,, \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} c_p(\varrho\,,\vartheta) &= \frac{\partial e(\varrho\,,\vartheta)}{\partial\vartheta} + \alpha(\varrho\,,\vartheta)\frac{\vartheta}{\varrho}\frac{\partial p(\varrho\,,\vartheta)}{\partial\vartheta}\,, \\ \alpha(\varrho\,,\vartheta) &= \frac{1}{\varrho}\frac{\partial_\vartheta p}{\partial_\varrho p}(\varrho\,,\vartheta)\,. \end{aligned} \tag{3.33}$$

- $$\int_0^T \int_\Omega \Big(\vec{B}\cdot\partial_t\varphi - (\vec{B}\times\vec{U} + \lambda(\overline{\varrho}\,,\overline{\vartheta}\,,0)\mathrm{curl}\vec{B})\cdot\mathrm{curl}\varphi\Big)\mathrm{d}x\,\mathrm{d}t = \int_\Omega \vec{B}_0\cdot\varphi(0\,,\cdot)\mathrm{d}x\,, \tag{3.34}$$

for any $\varphi \in \mathcal{D}([0\,,T) \times \overline{\Omega}\,;\mathbb{R}^3)$.

3.1.3 Main result

Before formulating rigorously our main result it is convenient to specify the constitutive equations relating the pressure p , the internal energy e , the entropy s , and the transport coefficients ν , η , λ , and κ_F to the scalar variables ϱ and ϑ . The restrictions are motivated by the existence theory established in [7]. Specifically, we set

$$p(\varrho\,,\vartheta) = p_F(\varrho\,,\vartheta) + p_R(\vartheta)\,,\ p_F = \vartheta^{\frac{5}{2}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big)\,,\ p_R = \frac{a}{3}\vartheta^4\,,\ a > 0\,, \tag{3.35}$$

$$e(\varrho\,,\vartheta) = e_F(\varrho\,,\vartheta) + e_R(\varrho\,,\vartheta)\,,\ e_F = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big)\,,\ e_R = a\frac{\vartheta^4}{\varrho}\,, \tag{3.36}$$

$$s(\varrho\,,\vartheta) = s_F(\varrho\,,\vartheta) + s_R(\varrho\,,\vartheta)\,,\ s_F = S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big)\,,\ s_R = \frac{4}{3}a\frac{\vartheta^3}{\varrho}\,, \tag{3.37}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \ \text{ for all } Z > 0\,. \tag{3.38}$$

Furthermore, we assume $P \in C^1[0\,,\infty) \cap C^2(0\,,\infty)\,,$

$$P(0) = 0\,, \;\; P'(Z) > 0 \;\text{ for all } Z \geq 0\,, \tag{3.39}$$

$$0 < \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \leq \sup_{z>0} \frac{\frac{5}{3}P(z) - zP'(z)}{z} < \infty\,, \tag{3.40}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0\,. \tag{3.41}$$

For the sake of simplicity, the transport coefficients $\nu\,,\eta\,,\lambda\,,$ and κ_F are assumed to be continuously differentiable functions satisfying the growth conditions

$$\left.\begin{array}{c} 0 < \underline{\nu}(1+\vartheta) \leq \nu(\vartheta\,,\vec{B}) \leq \overline{\nu}(1+\vartheta)\,, \\ 0 \leq \eta(\vartheta\,,\vec{B}) \leq \overline{\eta}(1+\vartheta) \end{array}\right\} \text{for all } \vartheta \geq 0\,, \tag{3.42}$$

$$0 < \underline{\kappa}(1+\vartheta) \leq \kappa_F(\varrho\,,\vartheta\,,\vec{B}) \leq \overline{\kappa}(1+\vartheta^3)\,, \tag{3.43}$$

$$0 < \underline{\lambda}(1+\vartheta) \leq \lambda(\varrho\,,\vartheta\,,\vec{B}) \leq \overline{\lambda}(1+\vartheta^3)\,, \tag{3.44}$$

where $\underline{\nu}\,,\overline{\nu}\,,\overline{\eta}\,,\underline{\kappa}\,,\overline{\kappa}\,,\underline{\lambda}$, and $\overline{\lambda}$ are positive constants.

Finally, we consider the initial data in the form

$$\begin{array}{c} \varrho(0\,,\cdot) = \varrho_{0,\,\varepsilon} = \overline{\varrho} + \varepsilon\varrho^{(1)}_{0,\,\varepsilon}\,, \quad \vec{u}(0\,,\cdot) = \vec{u}_{0,\,\varepsilon}\,, \\ \vartheta(0\,,\cdot) = \vartheta_{0,\,\varepsilon} = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{0,\,\varepsilon}\,, \quad \vec{B}(0\,,\cdot) = \vec{B}_{0,\,\varepsilon} = \varepsilon\vec{B}^{(1)}_{0,\,\varepsilon}\,, \end{array} \tag{3.45}$$

where

$$\overline{\varrho} > 0\,, \overline{\vartheta} > 0\,, \int_\Omega \varrho^{(1)}_{0,\,\varepsilon}\mathrm{d}x = \int_\Omega \vartheta^{(1)}_{0,\,\varepsilon}\mathrm{d}x = 0 \;\text{ for all } \varepsilon > 0. \tag{3.46}$$

Our main result reads as follows:

Theorem 3.1.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$. Assume that $p\,,\ e\,,\ s$ satisfy hypothesis (3.35) - (3.41), and the transport coefficients meet the growth restrictions (3.42) - (3.44). Let $\{\varrho_\varepsilon\,,\vec{u}_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to the scaled MHD equations (3.17) - (3.22) on $(0\,,T)\times\Omega\,,$ supplemented with the boundary conditions (3.13) - (3.15), and the initial data (3.45) - (3.46) and*

$$\begin{array}{c} \varrho^{(1)}_{0,\,\varepsilon} \to \varrho^{(1)}_0 \text{ weakly} - (*) \text{ in } L^\infty(\Omega)\,, \\ \vec{u}_{0,\,\varepsilon} \to \vec{U}_0 \text{ weakly} - (*) \text{ in } L^\infty(\Omega\,;\mathbb{R}^3)\,, \\ \vartheta^{(1)}_{0,\,\varepsilon} \to \vartheta^{(1)}_0 \text{ weakly} - (*) \text{ in } L^\infty(\Omega)\,, \\ \vec{B}^{(1)}_{0,\,\varepsilon} \to \vec{B}^{(1)}_0 \text{ weakly} - (*) \text{ in } L^\infty(\Omega\,;\mathbb{R}^3)\,, \end{array} \tag{3.47}$$

as $\varepsilon \to 0$. *Then*

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t)-\overline{\varrho}\|_{L^{\frac{5}{3}}(\Omega)} \leq \varepsilon c\,, \tag{3.48}$$

and, at least for a suitable subsequence,

$$\begin{aligned} \vec{u}_\varepsilon &\to \vec{U} \text{ weakly in } L^2(0\,,T\,;W^{1,2}(\Omega\,;\mathbb{R}^3))\,,\\ \frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon} &= \vartheta_\varepsilon^{(1)} \to \Theta \text{ weakly in } L^2(0\,,T\,;W^{1,2}(\Omega\,;\mathbb{R}^3))\,,\\ \frac{\vec{B}_\varepsilon}{\varepsilon} &= \vec{B}_\varepsilon^{(1)} \to \vec{B} \text{ weakly in } L^2(0\,,T\,;W^{1,2}(\Omega\,;\mathbb{R}^3))\,, \end{aligned} \tag{3.49}$$

where $\vec{U}$, Θ *and* $\vec{B}$ *is a weak solution to the magnetohydrodynamics equations with temperature (3.30) - (3.34), with the initial distribution of the temperature*

$$\Theta_0 = \frac{\overline{\vartheta}}{c_p(\overline{\varrho},\,\overline{\vartheta})}\Big(\frac{\partial s(\overline{\varrho},\,\overline{\vartheta})}{\partial\varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho},\,\overline{\vartheta})}{\partial\vartheta}\vartheta_0^{(1)}\Big)\,.$$

We notice that the existence of the weak solutions $\{\varrho_\varepsilon\,,\vec{u}_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon\}_{\varepsilon>0}$ can be shown by means of the theory developed in [7].

The limit system (3.30) - (3.34) is used for the so-called Direct Numerical Simulation method explained by Elliot in [10] which is used for simulations of the solar magnetoconvection. For more detailed explanation of the physical interpretation of this magnetoconvection the reader may consult [32].

The rest of the paper is devoted to the proof of Theorem 3.1.1. The ideas of our strategy, motivated by [13], Chapter 5., may be characterized as follows:

(i) As a consequence of hypotheses (3.42) - (3.43), one can prove that the family of solutions $\{\varrho_\varepsilon\,,\vec{u}_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon\}_{\varepsilon>0}$ admits uniform estimates. Similarly, uniform estimates on the amplitude of the entropy production σ_ε are obtained.

(ii) These estimates are used to prove convergence of these quantities in suitable topologies and passing to the limit for $\varepsilon \to 0$ is executed. The most complicated term is $\varrho_\varepsilon\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon$ in the momentum equation which in general is not expected to tend to $\overline{\varrho}\vec{U}\otimes\vec{U}$.

(iii) The most delicate issue is to show that the term $\overline{\varrho}\vec{U}\otimes\vec{U} = \overline{\varrho\vec{U}\otimes\vec{U}}$ in a weak limit. This is done by means of Helmholtz decomposition applied to the so-called acoustic equation.

3.2 Uniform estimates

3.2.1 Total dissipation balance

Out first goal is to derive suitable estimates on the sequence $\{\varrho_\varepsilon\,,\vec{u}_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon\}$, uniformly with respect to $\varepsilon>0$. Firstly, we introduce the following subsets of $\mathbb{R}^2$ as

$$\begin{gathered}\mathcal{M}_{ess}=\{(\varrho\,,\vartheta)\in\mathbb{R}^2|\,\overline{\varrho}/2<\varrho<2\overline{\varrho}\,,\ \ \overline{\vartheta}/2<\vartheta<2\overline{\vartheta}\}\,,\\ \mathcal{M}_{res}=\{(\varrho\,,\vartheta)\in[0\,,\infty)^2|\ (\varrho\,,\vartheta)\notin\mathcal{M}_{ess}\}\,.\end{gathered}\tag{3.50}$$

Similarly, for a measurable function h let's define its so called essential and residual part

$$\begin{gathered}[h]_{ess}=\chi(\varrho_\varepsilon\,,\vartheta_\varepsilon)h\,,\chi\in\mathcal{D}((0\,,\infty)^2)\,,0\le\chi\le1\,,\chi|_{\mathcal{M}_{ess}}=1\,,\\ [h]_{res}=(1-\chi(\varrho_\varepsilon\,,\vartheta_\varepsilon))h\,.\end{gathered}\tag{3.51}$$

It is easy to see

$$h=[h]_{ess}+[h]_{res}\,.\tag{3.52}$$

Of course, this decomposition depends on ε.

By combining the relations (3.27), (3.28) we obtain the dissipation equality

$$\begin{aligned}\int_\Omega\Big(\frac{\varepsilon^2}{2}\varrho_\varepsilon|\vec{u}_\varepsilon|^2+H_{\overline{\vartheta}}(\varrho_\varepsilon\,,\vartheta_\varepsilon)+\frac{1}{2\mu}|\vec{B}_\varepsilon|^2\Big)(t)\mathrm{d}x+\overline{\vartheta}<\sigma_\varepsilon\,;[0\,,t]\times\overline{\Omega}>\\ =\int_\Omega\Big(\frac{\varepsilon^2}{2}\varrho_{0,\,\varepsilon}|\vec{u}_{0,\,\varepsilon}|^2+H_{\overline{\vartheta}}(\varrho_{0,\,\varepsilon}\,,\vartheta_{0,\,\varepsilon})+\frac{1}{2\mu}|\vec{B}_{0,\,\varepsilon}|^2\Big)\mathrm{d}x\,,\end{aligned}\tag{3.53}$$

satisfied for a.a. $t\in(0\,,T)$, where $H_{\overline{\vartheta}}$ is the Helmholtz function (see [13], page 25) defined as follows

$$H_{\overline{\vartheta}}(\varrho\,,\vartheta)=\varrho\Big(e(\varrho\,,\vartheta)-\overline{\vartheta}s(\varrho\,,\vartheta)\Big)\,,\tag{3.54}$$

and $\overline{\vartheta}$ is a positive constant.

In accordance with hypotheses (3.45), (3.46), the total mass

$$\int_\Omega\varrho_\varepsilon(t)\mathrm{d}x=\overline{\varrho}|\Omega|\,,\tag{3.55}$$

is a constant of motion independent of ε. Due to this fact, relation (3.53) can be rewritten in the form

$$\begin{aligned}
&\int_\Omega \Big(\frac{1}{2}\varrho_\varepsilon|\vec{u}_\varepsilon|^2 + \frac{1}{2\varepsilon^2\mu}|\vec{B}_\varepsilon|^2\Big)(t)\mathrm{d}x + \frac{\overline{\vartheta}}{\varepsilon^2} < \sigma_\varepsilon\,;[0\,,t]\times\overline{\Omega} > + \\
&\int_\Omega \frac{1}{\varepsilon^2}\Big(H_{\overline{\vartheta}}(\varrho_\varepsilon\,,\vartheta_\varepsilon) - (\varrho_\varepsilon - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}\,,\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\overline{\varrho}\,,\overline{\vartheta})\Big)(t)\mathrm{d}x \\
&\qquad = \int_\Omega \Big(\frac{1}{2}\varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon}|^2 + \frac{1}{2\varepsilon^2\mu}|\vec{B}_{0,\varepsilon}|^2\Big)\mathrm{d}x + \\
&\int_\Omega \frac{1}{\varepsilon^2}\Big(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}\,,\vartheta_{0,\varepsilon}) - (\varrho_{0,\varepsilon} - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}\,,\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\overline{\varrho}\,,\overline{\vartheta})\Big)\mathrm{d}x\,.
\end{aligned} \tag{3.56}$$

In order to exploit the dissipation balance (3.56), we have to ensure that its right-hand side determined in terms of the initial data is bounded uniformly with respect to ε . Since the initial data are given by (3.45) and satisfy (3.47),

$$\{\sqrt{\varrho_{0,\varepsilon}}\vec{u}_{0,\varepsilon}\} \ \text{is bounded in}\ L^2(\Omega\,;\mathbb{R}^3)\,, \tag{3.57}$$

and

$$\{\varrho^{(1)}_{0,\varepsilon}\}\,,\ \{\vartheta^{(1)}_{0,\varepsilon}\}\,,\ \{\vec{B}^{(1)}_{0,\varepsilon}\}\ \text{are bounded in}\ L^\infty(\Omega)\,. \tag{3.58}$$

Consequently, using Lemma 5.1 in [13] we deduce from (3.56) that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \Big\|\Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big]_{ess}(t)\Big\|_{L^2(\Omega)} \le c\,, \tag{3.59}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \Big\|\Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big]_{ess}(t)\Big\|_{L^2(\Omega)} \le c\,, \tag{3.60}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_\varepsilon e(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \le \varepsilon^2 c\,, \tag{3.61}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_\varepsilon s(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \le \varepsilon^2 c\,. \tag{3.62}$$

In addition, we have

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\vec{u}_\varepsilon\|_{L^2(\Omega;\,\mathbb{R}^3)} \le c\,, \tag{3.63}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \Big\|\frac{\vec{B}_\varepsilon}{\varepsilon}\Big\|_{L^2(\Omega;\,\mathbb{R}^3)} \le c\,, \tag{3.64}$$

$$< \sigma_\varepsilon\,;[0\,,T]\times\overline{\Omega} > \le \varepsilon^2 c\,, \tag{3.65}$$

and as a direct consequence of Lemma 5.1 in [13],

$$\operatorname*{ess\,sup}_{t\in(0,T)} \Big|\Big\{x\in\Omega\,\Big|\,(\varrho_\varepsilon\,,\vartheta_\varepsilon)(t\,,x)\in\mathcal{M}_{res}\Big\}\Big| \le \varepsilon^2 c\,. \tag{3.66}$$

3.2.2 Uniform estimates

We now use the structural properties imposed through the constitutive relations (3.35) - (3.44) in order to prove uniform estimates introduced in the following Lemma.

Lemma 16 *Let all assumptions introduced in the first section be satisfied and* $\{\varrho_\varepsilon\,,\vec{u}_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon\}$ *be a family of solutions satisfying (3.24) - (3.29). Then we have*

$$\operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega\Big([\varrho_\varepsilon]^{\frac{5}{3}}_{res}+[\vartheta_\varepsilon]^4_{res}\Big)(t)\mathrm{d}x\le\varepsilon^2 c\,, \tag{3.67}$$

$$\int_0^T\|\vec{u}_\varepsilon(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}\mathrm{d}t\le c\,, \tag{3.68}$$

$$\int_0^T\Big\|\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)(t)\Big\|^2_{W^{1,2}(\Omega)}\mathrm{d}t\le c\,, \tag{3.69}$$

$$\int_0^T\Big\|\Big(\frac{\log\vartheta_\varepsilon-\log\overline{\vartheta}}{\varepsilon}\Big)(t)\Big\|^2_{W^{1,2}(\Omega)}\mathrm{d}t\le c\,, \tag{3.70}$$

$$\int_0^T\Big\|\frac{\vec{B}_\varepsilon(t)}{\varepsilon}\Big\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}\mathrm{d}t\le c\,, \tag{3.71}$$

$$\int_0^T\Big\|\Big[\frac{\varrho_\varepsilon s(\varrho_\varepsilon\,,\vartheta_\varepsilon)}{\varepsilon}\Big]_{res}(t)\Big\|^q_{L^q(\Omega)}\mathrm{d}t\le c\,, \tag{3.72}$$

$$\int_0^T\Big\|\Big[\frac{\varrho_\varepsilon s(\varrho_\varepsilon\,,\vartheta_\varepsilon)}{\varepsilon}\Big]_{res}\vec{u}_\varepsilon(t)\Big\|^q_{L^q(\Omega;\mathbb{R}^3)}\mathrm{d}t\le c\,, \tag{3.73}$$

$$\int_0^T\Big\|\Big[\frac{\kappa_F(\varrho_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon)}{\vartheta_\varepsilon}\Big]_{res}\Big(\frac{\nabla\vartheta_\varepsilon}{\varepsilon}\Big)(t)\Big\|^q_{L^q(\Omega;\mathbb{R}^3)}\mathrm{d}t\le c\,, \tag{3.74}$$

for a certain $q>1$ *, where the constant* c *is independent of* ε *.*

Proof.

(i) Estimate (3.67) follows from (3.36), (3.39), (3.41) applied to estimate (3.61).

(ii) Substituting (3.7) - (3.10) and (3.22) into (3.65), together with the hypothesis (3.42) - (3.44), give rise to

$$\int_0^T\|\nabla\vec{u}_\varepsilon+\nabla^T\vec{u}_\varepsilon-\frac{2}{3}\mathrm{div}\vec{u}_\varepsilon\mathbb{I}\|^2_{L^2(\Omega;\mathbb{R}^{3\times3})}\mathrm{d}t\le c\,,$$

Now we can use this estimate together with (3.63) and (3.66) in the generalized Korn's inequality (see [13], Proposition 2.1) in order to derive (3.68).

(iii) In a similar way, we deduce from (3.65) a uniform bound

$$\int_0^T \Big(\Big\| \nabla \Big(\frac{\vartheta_\varepsilon}{\varepsilon} \Big) \Big\|^2_{L^2(\Omega)} + \Big\| \nabla \Big(\frac{\log \vartheta_\varepsilon}{\varepsilon} \Big) \Big\|^2_{L^2(\Omega)} \Big) \mathrm{d}t \leq c \, ,$$

which, together with (3.60), (3.66) and the generalized Poincare's inequality ([13], Proposition 2.2) gives rise to (3.69), (3.70).

(iv) In a same way as in the two previous cases we use (3.22), (3.44) and (3.65) to derive the estimate

$$\Big\| \mathrm{curl} \frac{\vec{B}_\varepsilon}{\varepsilon} \Big\|^2_{L^2(\Omega;\, \mathbb{R}^3)} \leq c \, .$$

From this estimate and (3.64), (3.24) we see that $\vec{B}_\varepsilon / \varepsilon$ satisfies the assumptions of the embedding Theorem 6.1 in [8]. Thus (3.71) follows from it immediately.

(v) By virtue of (3.37) - (3.40) we can write

$$|\varrho_\varepsilon s(\varrho_\varepsilon \, , \vartheta_\varepsilon)| \leq c \Big(1 + \varrho_\varepsilon |\log \varrho_\varepsilon| + \varrho_\varepsilon |\log \vartheta_\varepsilon - \log \overline{\vartheta}| + \vartheta_\varepsilon^3 \Big) \, . \tag{3.75}$$

But from (3.66) it follows that

$$\operatorname*{ess\ sup}_{t \in (0,\, T)} \Big\| \Big[\frac{1}{\varepsilon} \Big]_{res} (t) \Big\|_{L^2(\Omega)} \leq c \, , \tag{3.76}$$

while the first term in (3.67) yields

$$\operatorname*{ess\ sup}_{t \in (0,\, T)} \Big\| \Big[\frac{\varrho_\varepsilon \log \varrho_\varepsilon}{\varepsilon} \Big]_{res} (t) \Big\|_{L^q(\Omega)} \leq c \, , \tag{3.77}$$

for any $1 \leq q < \frac{5}{3}$. Furthermore above derived estimates (3.67) and (3.70) give rise

$$\int_0^T \Big\| \Big[\frac{\varrho_\varepsilon (\log \vartheta_\varepsilon - \log \overline{\vartheta})}{\varepsilon} \Big]_{res} (t) \Big\|^2_{L^p(\Omega)} \mathrm{d}t \leq c \, , \tag{3.78}$$

for a certain $p > 1$, and finally the last term in (3.75) can be estimated by the second term in (3.67) which implies

$$\operatorname*{ess\ sup}_{t \in (0,\, T)} \Big\| \Big[\frac{\vartheta_\varepsilon^3}{\varepsilon} \Big]_{res} (t) \Big\|_{L^{\frac{4}{3}}(\Omega)} \leq c \, , \tag{3.79}$$

Now substituting relations (3.76) - (3.79) into (3.75) yields (3.72).

(vi) In order to prove estimate (3.73), we use (3.63), (3.67) and (3.70) to obtain

$$\int_0^T \Big\| \Big[\frac{\varrho_\varepsilon (\log \vartheta_\varepsilon - \log \overline{\vartheta}) \vec{u}_\varepsilon}{\varepsilon} \Big]_{res} \Big\|^q_{L^q(\Omega;\mathbb{R}^3)} \mathrm{d}t \le c\,,$$

for a certain $q > 1$, which combined with the estimates the previous steps and (3.66), gives rise to (3.73).

(vii) Finally, according to (3.43),

$$\Big[\frac{\kappa_F(\varrho_\varepsilon\,, \vartheta_\varepsilon\,, \vec{B}_\varepsilon)}{\vartheta_\varepsilon} \Big]_{res} \Big| \frac{\nabla \vartheta_\varepsilon}{\varepsilon} \Big| \le c \Big(\Big| \frac{\nabla \log \vartheta_\varepsilon}{\varepsilon} \Big| + [\vartheta_\varepsilon^2]_{res} \Big| \frac{\nabla \vartheta_\varepsilon}{\varepsilon} \Big| \Big)\,,$$

where, as a consequence of (3.67), (3.69) we have $\{[\vartheta_\varepsilon]_{res}\}_{\varepsilon>0}$ is bounded in $L^\infty(0\,,T\,;L^2(\Omega)) \cap L^1(0\,,T\,;L^3(\Omega))$. This fact together with (3.69), (3.70) and interpolation inequality implies (3.74).

□

3.3 Convergence

The uniform estimates established in Lemma 16 will be used in order to let $\varepsilon \to 0$ in (3.25) - (3.29) and to identify the limit problem. Since the residual parts of the quantities are small of order ε, we focus on the essential parts.

3.3.1 Equation of continuity

We first show that the continuity equation (3.25) reduces to the incompressibility condition (3.30). From the uniform estimate (3.68), we deduce

$$\vec{u}_\varepsilon \to \vec{U} \text{ weakly in } L^2(0\,,T\,;W^{1,2}(\Omega\,;\mathbb{R}^3))\,. \tag{3.80}$$

Furthermore, we have

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} = \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{ess} + \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{res}\,,$$

where the second term tends to zero in $L^\infty(0\,,T\,;L^{\frac{5}{3}}(\Omega))$ due to (3.66), (3.67) and

$$\Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{ess} \to \varrho^{(1)} \text{ weakly} - (*) \text{ in } L^\infty(0\,,T\,;L^2(\Omega))\,, \tag{3.81}$$

and thus

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \text{ weakly} - (*) \text{ in } L^\infty(0\,,T\,;L^{\frac{5}{3}}(\Omega))\,, \tag{3.82}$$

especially

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } L^\infty(0\,,T\,;L^{\frac{5}{3}}(\Omega))\,. \tag{3.83}$$

We are now able to pass to the limit for $\varepsilon \to 0$ in (3.25) to conclude

$$\int_0^T \int_\Omega \vec{U} \cdot \nabla\varphi \mathrm{d}x\, \mathrm{d}t = 0\,,$$

for all $\varphi \in \mathcal{D}((0\,,T) \times \overline{\Omega})$. Since $\vec{U} \in L^2(0\,,T\,;W^{1,2}(\Omega\,;\mathbb{R}^3))$ and the boundary of Ω is Lipschitz, (3.30) is proved.

3.3.2 Entropy balance

With regard to (3.25), the entropy balance (3.28) can be recast in the form

$$\begin{aligned} &\int_0^T \int_\Omega \varrho_\varepsilon \Big(\frac{s(\varrho_\varepsilon\,,\vartheta_\varepsilon) - s(\overline{\varrho}\,,\overline{\vartheta})}{\varepsilon} \Big) \Big(\partial_t \varphi + \vec{u}_\varepsilon \cdot \nabla\varphi \Big) \mathrm{d}x\, \mathrm{d}t \\ &- \int_0^T \int_\Omega \Big(\frac{\kappa_F(\varrho_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon)}{\vartheta_\varepsilon} + \kappa_R \vartheta_\varepsilon^2 \Big) \nabla \Big(\frac{\vartheta_\varepsilon}{\varepsilon} \Big) \cdot \nabla\varphi \mathrm{d}x\, \mathrm{d}t \\ + \frac{1}{\varepsilon} < \sigma_\varepsilon\,; \varphi >_{[\mathcal{M},C]([0,T]\times\overline{\Omega})} &= - \int_\Omega \varrho_{0,\varepsilon} \Big(\frac{s(\varrho_{0,\varepsilon}\,,\vartheta_{0,\varepsilon}) - s(\overline{\varrho}\,,\overline{\vartheta})}{\varepsilon} \Big) \varphi(0\,,\cdot) \mathrm{d}x \end{aligned} \tag{3.84}$$

to be satisfied for any $\varphi \in \mathcal{D}([0\,,T) \times \overline{\Omega})$.

Using estimate (3.60), we get

$$\Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{ess} \to \vartheta^{(1)} \text{ weakly} - (*) \text{ in } L^\infty(0\,,T\,;L^2(\Omega))\,, \tag{3.85}$$

and since the measure of the residual subset tends to zero as mentioned in (3.66), we deduce from (3.69).

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \vartheta^{(1)} \text{ weakly in } L^2(0\,,T\,;W^{1,2}(\Omega))\,. \tag{3.86}$$

In order to pass to the limit in (3.84) we proceed by several steps to prove required limits.

We first rewrite

$$\begin{aligned} \varrho_\varepsilon \Big(\frac{s(\varrho_\varepsilon\,,\vartheta_\varepsilon) - s(\overline{\varrho}\,,\overline{\vartheta})}{\varepsilon} \Big) &= [\varrho_\varepsilon]_{ess} \frac{[s(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{ess} - s(\overline{\varrho}\,,\overline{\vartheta})}{\varepsilon} \\ + \Big[\frac{\varrho_\varepsilon}{\varepsilon} \Big]_{res} \Big([s(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{ess} - s(\overline{\varrho}\,,\overline{\vartheta}) \Big) &+ \Big[\frac{\varrho_\varepsilon s(\varrho_\varepsilon\,,\vartheta_\varepsilon)}{\varepsilon} \Big]_{res}\,, \end{aligned} \tag{3.87}$$

where the second term of the right-hand side converges to zero in $L^\infty(0\,,T\,;L^{\frac{5}{3}}(\Omega))$ due to (3.67) and the last term tends also to zero in $L^p((0\,,T)\times\Omega)$ for a certain $p>1$ which can be proved using (3.66), (3.70). Moreover, when we take into account (3.66), (3.73), (3.80), we obtain

$$\begin{gathered}\Big[\frac{\varrho_\varepsilon}{\varepsilon}\Big]_{res}\Big([s(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{ess}-s(\overline{\varrho}\,,\overline{\vartheta})\Big)\vec{u}_\varepsilon\to 0 \text{ in } L^p(0\,,T\,;L^p(\Omega\,;\mathbb{R}^3))\,,\\ \Big[\frac{\varrho_\varepsilon s(\varrho_\varepsilon\,,\vartheta_\varepsilon)}{\varepsilon}\Big]_{res}\vec{u}_\varepsilon\to 0 \text{ in } L^p(0\,,T\,;L^p(\Omega\,;\mathbb{R}^3))\,,\end{gathered}\tag{3.88}$$

for a certain $p>1$. Finally Proposition 5.2 in [13] together with (3.83) yields

$$[\varrho_\varepsilon]_{ess}\frac{[s(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{ess}-s(\overline{\varrho}\,,\overline{\vartheta})}{\varepsilon}\to\overline{\varrho}\Big(\frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial\varrho}\varrho^{(1)}+\frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}\Big)\,,\tag{3.89}$$

weakly-(*) in $L^\infty(0\,,T\,;L^2(\Omega\,;\mathbb{R}^3))$, which solves the limit of the first term in (3.84).

The entropy flux can be rewritten similarly as

$$\begin{gathered}\Big(\frac{\kappa_F(\varrho_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon)}{\vartheta_\varepsilon}+\kappa_R\vartheta_\varepsilon^2\Big)\nabla\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big)=\Big[\frac{\kappa_F(\varrho_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon)}{\vartheta_\varepsilon}\Big]_{ess}\nabla\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)\\ +\kappa_R[\vartheta_\varepsilon^2]_{ess}\nabla\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)+\Big(\Big[\frac{\kappa_F(\varrho_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon)}{\vartheta_\varepsilon}\Big]_{res}+\kappa_R[\vartheta_\varepsilon^2]_{res}\Big)\nabla\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big)\,.\end{gathered}\tag{3.90}$$

Obviously

$$\vec{B}_\varepsilon\to 0 \text{ in } L^2(0\,,T\,;W^{1,2}(\Omega\,;\mathbb{R}^3))\tag{3.91}$$

which together with (3.83), (3.85), (3.86) yields

$$\Big(\Big[\frac{\kappa_F(\varrho_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon)}{\vartheta_\varepsilon}\Big]_{ess}+\kappa_R[\vartheta_\varepsilon^2]_{ess}\Big)\nabla\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)\to\Big(\frac{\kappa_F(\overline{\varrho}\,,\overline{\vartheta}\,,0)}{\overline{\vartheta}}+\kappa_R\overline{\vartheta}^2\Big)\nabla\vartheta^{(1)}\,,\tag{3.92}$$

weakly in $L^2(0\,,T\,;L^2(\Omega\,;\mathbb{R}^3))$. In accordance with (3.66), (3.67), (3.69) (3.74) it is easy to see

$$\begin{gathered}\Big[\frac{\kappa_F(\varrho_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon)}{\vartheta_\varepsilon}\Big]_{res}\nabla\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big)\to 0 \text{ in } L^p(0\,,T\,;L^p(\Omega\,;\mathbb{R}^3))\,,\\ \kappa_R[\vartheta_\varepsilon^2]_{res}\nabla\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big)\to 0 \text{ in } L^p(0\,,T\,;L^p(\Omega\,;\mathbb{R}^3))\,,\end{gathered}\tag{3.93}$$

for a certain $p>1$ which handles the second integral in (3.84).

The most complicated term in (3.84) is the second part of the first integral. In order to deal with this limit, consider the following vector fields defined on $(0\,,T)\times\Omega$

$$\vec{U}_\varepsilon = \Big([\varrho_\varepsilon]_{ess}\frac{[s(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{ess}-s(\overline{\varrho}\,,\overline{\vartheta})}{\varepsilon}\,,$$
$$\frac{[s(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{ess}-s(\overline{\varrho}\,,\overline{\vartheta})}{\varepsilon}\vec{u}_\varepsilon-\Big[\frac{\kappa_F(\varrho_\varepsilon\,,\vartheta_\varepsilon\,,\vec{B}_\varepsilon)}{\vartheta_\varepsilon}\Big]_{ess}\nabla\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big)\Big)\,,$$
$$\vec{V}_\varepsilon = (G(\vec{u}_\varepsilon)\,,0\,,0\,,0)\,,$$

for an arbitrary function $G\in W^{1,\infty}(\mathbb{R}^3)$. Using results proved in the previous steps together with (3.67), one can check that $\vec{U}_\varepsilon\,,\,\vec{V}_\varepsilon$ meet assumptions of Div-Curl Lemma 4.24 in [30]. From those statement and since G is an arbitrary we get the

$$[\varrho_\varepsilon]_{ess}\frac{[s(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{ess}-s(\overline{\varrho}\,,\overline{\vartheta})}{\varepsilon}\vec{u}_\varepsilon\to\overline{\varrho}\Big(\frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial\varrho}\varrho^{(1)}+\frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}\Big)\vec{U}\,, \tag{3.94}$$

weakly in $L^2(0\,,T\,;L^{\frac{3}{2}}(\Omega\,;\mathbb{R}^3))$.

We are now able to pass to the limit for $\varepsilon\to 0$ in the entropy inequality (3.84) to get

$$\begin{aligned}\int_0^T\int_\Omega\overline{\varrho}\Big(\frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial\varrho}\varrho^{(1)}+\frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}\Big)\Big(\partial_t\varphi+\vec{U}\cdot\nabla\varphi\Big)\mathrm{d}x\,\mathrm{d}t&\\ -\int_0^T\int_\Omega\Big(\frac{\kappa_F(\overline{\varrho}\,,\overline{\vartheta}\,,0)}{\overline{\vartheta}}+\kappa_R\overline{\vartheta}^2\Big)\nabla\vartheta^{(1)}\cdot\nabla\varphi\mathrm{d}x\,\mathrm{d}t&\\ =-\int_\Omega\overline{\varrho}\Big(\frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial\varrho}\varrho_0^{(1)}+\frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial\vartheta}\vartheta_0^{(1)}\Big)\varphi(0\,,\cdot)\mathrm{d}x\,,&\end{aligned} \tag{3.95}$$

for any $\varphi\in\mathcal{D}([0\,,T)\times\overline{\Omega})$. In the next section we establish relation between $\varrho^{(1)}$ and $\vartheta^{(1)}$ and then (3.95) gives rise (3.32).

3.3.3 Momentum equation

It follows from (3.80), (3.83) that

$$\varrho_\varepsilon\vec{u}_\varepsilon\to\overline{\varrho}\vec{U}\ \text{weakly in}\ L^2(0\,,T\,;L^{\frac{30}{23}}(\Omega\,;\mathbb{R}^3))\,. \tag{3.96}$$

Moreover, we deduce from (3.63), (3.67) that

$$\{\varrho_\varepsilon\vec{u}_\varepsilon\}_{\varepsilon>0}\ \text{is bounded in}\ L^\infty(0\,,T\,;L^{\frac{5}{4}}(\Omega\,;\mathbb{R}^3))\,, \tag{3.97}$$

which combined with (3.80), gives rise to

$$\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon \to \overline{\varrho \vec{U} \otimes \vec{U}} \text{ weakly in } L^2(0\,,T\,;L^{\frac{30}{23}}(\Omega\,;\mathbb{R}^{3\times 3}))\,. \tag{3.98}$$

Next, as a consequence of (3.67), (3.69)

$$\{\vartheta_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0\,,T\,;L^4(\Omega)) \cap L^2(0\,,T\,;L^6(\Omega))\,. \tag{3.99}$$

Thus hypothesis (3.42), together with (3.80), (3.91), (3.99), give rise to

$$\mathbb{S}_\varepsilon \to \nu(\overline{\vartheta}\,,0)(\nabla \vec{U} + \nabla^T \vec{U}) \text{ weakly in } L^q(0\,,T\,;L^q(\Omega\,;\mathbb{R}^3))\,, \tag{3.100}$$

for a certain $q > 1$.

Moreover, one can use (3.29) together with estimates (3.64), (3.71) and Aubin-Lions Theorem 1.71 in [30] to derive

$$\frac{\vec{B}_\varepsilon}{\varepsilon} \to \vec{B} \text{ weakly in } L^2(0\,,T\,;W^{1,2}(\Omega\,;\mathbb{R}^3)) \text{ and strongly in } L^2((0\,,T) \times \Omega\,;\mathbb{R}^3)\,, \tag{3.101}$$

and consequently,

$$\frac{1}{\mu}\text{curl}\Big(\frac{\vec{B}_\varepsilon}{\varepsilon}\Big) \times \frac{\vec{B}_\varepsilon}{\varepsilon} \to \frac{1}{\mu}\text{curl}\vec{B} \times \vec{B} \text{ weakly in } L^q((0\,,T) \times \Omega\,;\mathbb{R}^3)\,,$$

for a certain $q > 1$.

Now, it is easy to let $\varepsilon \to 0$ in the momentum equation (3.26) as soon as the test function φ is divergenceless. If this is the case, we get

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\overline{\varrho}\vec{U} \cdot \partial_t \varphi + \overline{\varrho \vec{U} \otimes \vec{U}} : \nabla\varphi\Big) \mathrm{d}x\,\mathrm{d}t + \int_\Omega (\overline{\varrho}\vec{U}_0) \cdot \varphi \mathrm{d}x \\ =& \int_0^T \int_\Omega \Big(\nu(\overline{\vartheta}\,,0)[\nabla\vec{U} + \nabla^T\vec{U}] : \nabla\varphi - \frac{1}{\mu}\text{curl}\vec{B} \times \vec{B} \cdot \varphi\Big) \mathrm{d}x\,\mathrm{d}t\,, \end{aligned} \tag{3.102}$$

for any test function $\varphi \in \mathcal{D}([0\,,T) \times \overline{\Omega}\,;\mathbb{R}^3)$, $\text{div}\varphi = 0$ in Ω, $\varphi \cdot \vec{n}|_{\partial\Omega} = 0$.

To complete this part, it remains to establish relation between $\varrho^{(1)}$ and $\vartheta^{(1)}$. We begin writing as usual

$$p(\varrho_\varepsilon\,,\vartheta_\varepsilon) = [p(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{ess} + [p(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{res}\,,$$

where, in accordance with hypothesis (3.39), (3.41),

$$0 \le \frac{[p(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{res}}{\varepsilon} \le c\Big(\Big[\frac{1}{\varepsilon}\Big]_{res} + \Big[\frac{\varrho_\varepsilon^{\frac{5}{3}}}{\varepsilon}\Big]_{res} + \Big[\frac{\vartheta_\varepsilon^4}{\varepsilon}\Big]_{res}\Big)\,. \tag{3.103}$$

Consequently, this estimate together with (3.66), (3.67) imply that

$$\operatorname*{ess\ sup}_{t\in(0\,,T)} \Big\| \Big[\frac{p(\varrho_\varepsilon\,,\vartheta_\varepsilon)}{\varepsilon}\Big]_{res}\Big\|_{L^1(\Omega)} \leq \varepsilon c\,. \tag{3.104}$$

Thus by means of proposition 5.2 in [13] and the previous estimate (3.104), we multiply the momentum equation (3.26) by ε and pas to the limit for $\varepsilon \to 0$ to obtain

$$\int_0^T \int_\Omega \Big(\frac{\partial p(\overline{\varrho}\,,\overline{\vartheta})}{\partial \varrho}\varrho^{(1)} + \frac{\partial p(\overline{\varrho}\,,\overline{\vartheta})}{\partial \vartheta}\vartheta^{(1)}\Big) \mathrm{div}\varphi \mathrm{d}x\,\mathrm{d}t = 0\,, \tag{3.105}$$

for all $\varphi \in \mathcal{D}((0\,,T)\times\Omega\,;\mathbb{R}^3)\,.$ We also deduce from (3.95) that

$$\int_\Omega \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial \varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial \vartheta}\vartheta^{(1)}\Big)(t)\mathrm{d}x$$
$$\int_\Omega \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial \varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial \vartheta}\vartheta_0^{(1)}\Big)\mathrm{d}x \text{ for a.a. } t\in(0\,,T)\,.$$

Since the mean of ϱ_ε is constant and the total mass is conserved, one gets due to (3.82) that $\varrho^{(1)}$ has zero mean. Then the previous relation may be reduced to

$$\int_\Omega \vartheta^{(1)}(t)\mathrm{d}x = \int_\Omega \vartheta^{(1)}(t)\mathrm{d}x \text{ for a.a. } t\in(0\,,T)\,.$$

Assuming, in addition (3.46) we conclude

$$\int_\Omega \vartheta^{(1)}(t)\mathrm{d}x = 0 \text{ for a.a. } t\in(0\,,T)\,. \tag{3.106}$$

Since $\varrho^{(1)}$ and $\vartheta^{(1)}$ has zero mean we get from (3.105) the desired conclusion

$$\varrho^{(1)} = -\frac{p_\vartheta}{p_\varrho}(\overline{\varrho}\,,\overline{\vartheta})\vartheta^{(1)}\,. \tag{3.107}$$

Expressing $\varrho^{(1)}$ in (3.95) by means of (3.107) and using Gibbs relation (3.6), we have

$$\begin{aligned}&\int_0^T \int_\Omega \overline{\varrho}c_p(\overline{\varrho}\,,\overline{\vartheta})\vartheta^{(1)}\Big(\partial_t\varphi + \vec{U}\cdot\nabla\varphi\Big)\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_\Omega \Big(\kappa_F(\overline{\varrho}\,,\overline{\vartheta}\,,0) + \kappa_R\overline{\vartheta}^2\Big)\nabla\vartheta^{(1)}\cdot\nabla\varphi\mathrm{d}x\,\mathrm{d}t = \\ &-\int_\Omega \overline{\varrho}\overline{\vartheta}\Big(\frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial \varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho}\,,\overline{\vartheta})}{\partial \vartheta}\vartheta_0^{(1)}\Big)\varphi(0\,,\cdot)\mathrm{d}x\,,\end{aligned} \tag{3.108}$$

for any $\varphi \in \mathcal{D}([0\,,T) \times \overline{\Omega})$, where c_p and α are expressed in (3.33). If we set $\Theta = \vartheta^{(1)}$ then we can see that (3.108) is a weak formulation of (3.32) together with the boundary and initial conditions.

Moreover, it follows from estimate (3.63) combined with (3.80), (3.83) that

$$\sqrt{\varrho_\varepsilon}\vec{u}_\varepsilon \to \sqrt{\overline{\varrho}}\vec{U} \text{ weakly in } L^\infty(0\,,T\,;L^2(\Omega\,;\mathbb{R}^3))\,,$$

in particular,

$$\vec{U} \in L^\infty(0\,,T\,;L^2(\Omega\,;\mathbb{R}^3)) \cap L^2(0\,,T\,;W^{1,2}(\Omega\,;\mathbb{R}^3))\,,$$

and finally $\operatorname{div}(\vec{U}\vartheta^{(1)}) \in L^q((0\,,T) \times \Omega)$ for a certain $q > 1$. Now we use the theory of linear parabolic equation to prove higher regularity of ϑ^1:

$$\vartheta^{(1)} \in W^{1,q}(0\,,T\,;L^q(\Omega)) \cap L^q(0\,,T\,;W^{2,q}(\Omega)) \text{ for a certain } q > 1\,.$$

When we set $\Theta = \vartheta^{(1)}$ again, we prove that (3.32) is satisfied a.e. in $(0\,,T) \times \Omega$.

The Boussinesq relation in (3.32) can be deduced directly by putting $r = \varrho^{(1)}$ in (3.107).

3.3.4 Maxwell equation

From (3.80), (3.101) it is easy to see

$$\frac{\vec{B}_\varepsilon}{\varepsilon} \times \vec{u}_\varepsilon \to \vec{B} \times \vec{U} \text{ weakly in } L^q(0\,,T\,;L^q(\Omega\,;\mathbb{R}^3))\,, \tag{3.109}$$

for a certain $q > 1$, and due to hypothesis (3.44) and (3.83), (3.86), (3.91)

$$\lambda(\varrho_\varepsilon\,, \vartheta_\varepsilon\,, \vec{B}_\varepsilon)\operatorname{curl}\frac{\vec{B}_\varepsilon}{\varepsilon} \to \lambda(\overline{\varrho}\,, \overline{\vartheta}\,, 0)\operatorname{curl}\vec{B} \text{ weakly in } L^1(0\,,T\,;L^1(\Omega\,;\mathbb{R}^3))\,. \tag{3.110}$$

We can now divide equation (3.29) by ε, and send to the limit for $\varepsilon \to 0$ to obtain (3.34).

3.4 Concluding remarks

So far we have almost completely proved Theorem 3.1.1, but the only missing thing is to show

$$\int_0^T \int_\Omega \overline{\varrho\vec{U} \otimes \vec{U}} : \nabla\varphi \mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \overline{\varrho}\vec{U} \otimes \vec{U} : \nabla\varphi \mathrm{d}x\,\mathrm{d}t\,, \tag{3.111}$$

for any $\varphi \in \mathcal{D}((0\,,T) \times \overline{\Omega}\,;\mathbb{R}^3)$, $\mathrm{div}\varphi = 0$, $\varphi \cdot \vec{n}|_{\partial\Omega} = 0$. The main idea how to show this fact, is to use Helmholtz decomposition (see [30], page 286) which says that each vector function $\vec{v} \in L^p(\Omega\,;\mathbb{R}^3)$ may be written as

$$\vec{v} = \mathbf{H}[\vec{v}\,] + \mathbf{H}^{\perp}[\vec{v}\,]\,,$$

where $\mathbf{H}[\vec{v}\,]$ is called as a solenoidal part and $\mathbf{H}^{\perp}[\vec{v}\,]$ as a gradient part. In order to show (3.111) let's write

$$\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon = \mathbf{H}[\varrho_\varepsilon \vec{u}_\varepsilon\,] \otimes \vec{u}_\varepsilon + \mathbf{H}^{\perp}[\varrho_\varepsilon \vec{u}_\varepsilon\,] \otimes \mathbf{H}[\vec{u}_\varepsilon\,] + \mathbf{H}^{\perp}[\varrho_\varepsilon \vec{u}_\varepsilon\,] \otimes \mathbf{H}^{\perp}[\vec{u}_\varepsilon\,]\,, \quad (3.112)$$

where, in accordance with (3.63), (3.67) and (3.80), the first term in (3.112)

$$\mathbf{H}[\varrho_\varepsilon \vec{u}_\varepsilon\,] \otimes \vec{u}_\varepsilon \to \overline{\varrho}\vec{U} \otimes \vec{U} \text{ weakly in } L^2(0\,,T\,;L^{\frac{30}{29}}(\Omega\,;\mathbb{R}^3))\,.$$

Moreover, combining (3.82) with (3.96) we infer that

$$\mathbf{H}^{\perp}[\varrho_\varepsilon \vec{u}_\varepsilon\,] \otimes \mathbf{H}[\vec{u}_\varepsilon\,] \to 0 \text{ weakly in } L^2(0\,,T\,;L^{\frac{30}{29}}(\Omega\,;\mathbb{R}^3))\,.$$

It remain to show that the last term in (3.112) tends to zero in the sense

$$\int_0^T \int_\Omega \mathbf{H}^{\perp}[\varrho_\varepsilon \vec{u}_\varepsilon\,] \otimes \mathbf{H}^{\perp}[\vec{u}_\varepsilon\,] : \nabla\varphi \mathrm{d}x\, \mathrm{d}t \to 0\,, \quad (3.113)$$

for any $\varphi \in \mathcal{D}((0\,,T) \times \overline{\Omega}\,;\mathbb{R}^3)$, $\mathrm{div}\varphi = 0$, $\varphi \cdot \vec{n}|_{\partial\Omega} = 0$.

Because our estimates don't provide any bound for the last term in (3.112), verification of (3.113) must be done by means of detailed analysis. The first part is to derive so-called Acoustic equation, which was done in [13], section 5.4. The last part is to investigate the spectrum of corresponding wave operator. Since the quantity $\frac{1}{\varepsilon^2\mu}\vec{J}_\varepsilon \times \vec{B}_\varepsilon$ is bounded uniformly in $L^p((0\,,T) \times \Omega\,;\mathbb{R}^3)$, for a certain $p > 1$, we are able to repeat the same steps as done in [13], section 5.4. to derive the desired result (3.113), which completes the proof of Theorem 3.1.1.

- The proof of Theorem 3.1.1 remains basically unchanged if instead of boundary conditions (3.13) - (3.15), we suppose that all quantities are spatially periodic, or equivalently if $\Omega = \mathcal{T}^3$, where $\mathcal{T}^3 \subset \mathbb{R}^3$ is a torus.
- Similar result can be proved if the motion of the fluid is driven by a bounded external force, i.e., when (3.18) contains an additional term $\varrho\nabla F$ and energy inequality (3.20) contains $-\varrho F$ or even we can implement Froude number into (3.17) - (3.22) $\mathrm{Fr} = \sqrt{\varepsilon}$ which means (3.18) contains $\frac{1}{\varepsilon}\varrho\nabla F$ and (3.20) $-\varepsilon\varrho F$.

Acknowledgment. This research was supported by the Jindřich Nečas Center for Mathematical Modelling, project LC06052, financed by MŠMT of the Czech Republic.

Chapter 4

Incompressible limits for the Navier-Stokes-Fourier systems on unbounded domains under strong stratification

Corresponds to the article by Kukučka P.: Incompressible Limits for the Navier-Stokes-Fourier Systems on Unbounded Domains under Strong Stratification, submitted for publication to Archive Rational Mech. Anal.

Abstract: This paper studies the asymptotic limit for solutions to the full Navier-Stokes-Fourier system under the strong stratification on unbounded domains. Such system models many problems arising in astrophysics. More precisely, the original Navier-Stokes-Fourier system is rescaled such that the both Mach and Froude number are equal to ε, and the Péclet number is proportional to ε^2. Special attention is focused to the acoustic waves which analysis is based on certain L^2 estimates for abstract operators in a Hilbert space. Then the convergence to the target system is shown.

2000 Mathematics Subject Classification. 35A05, 35Q30

Keywords: Navier-Stokes-Fourier system, acoustic equation, strong stratification

4.1 Introduction

Singular limits in the equations of fluid dynamics are studied extensively, and many results have already been achieved in this field. One of the first result was proved by Klainerman and Majda in [21] where the existence of the limit solution for the Navier-Stokes system is proved in the classical sense, but on a sufficiently small time interval. This idea was then taken by Alazard in [2] where the limit solutions for full Navier-Stokes system was studied under the assumption that the Mach number tends to zero. Another approach to this topic was proposed by Lions and Masmoudi in [27], where the existence is shown in a weak sense. Similar problems were further developed by Desjardins and Grenier [6]. The same strategy was later adapted for the full Navier-Stokes-Fourier system by Feireisl and Novotný in [14] and summarized in [13]. In cited papers, the limit solutions are proved in a weak sense on an arbitrary time interval $(0\,,T)$ and a bounded domain $\Omega\subset\mathbb{R}^3$ or the whole space $\mathbb{R}^3$. Several more interesting problems when the domain $\Omega\subset\mathbb{R}^3$ is unbounded were solved by Feireisl, Novotný and Petzeltová in [17], [12]. In [17] the incompressible limit for the Navier-Stokes system under strong stratification is studied and the second paper [12] studies the full Navier-Stokes-Fourier system in which Mach number tends to zero. Our presented problem arises in astrophysics and it concerns flow dynamics in stellar radiative zones representing a major challenge of the current theory of stellar interiors. Under these circumstances, the fluid is a plasma with the strong radiative transport due to hot and energetic radiation fields prevailing in it. Furthermore, such plasma is characterized by strong stratification effects and the feature that the convective motions are much slower than the speed of sound. In this paper we extend the results from last mentioned papers for the full Navier-Stokes-Fourier system under the strong stratification. Our problem is defined on an unbounded domain in $\Omega\subset\mathbb{R}^3$ which seems to be more natural for astrophysics in comparison with problem studied in [13], Chapter 6., where an infinite slab bounded above and below by two parallel planes is taken as a domain.

Consider an unbounded domain $\Omega\subset\mathbb{R}^3$ with a compact regular boundary $\partial\Omega$ and a family of bounded domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ approximating Ω in the following sense:

$$\Omega_\varepsilon\subset\Omega\,,\quad \partial\Omega\subset\partial\Omega_\varepsilon\,,\quad \varepsilon\mathrm{dist}[x\,,\partial\Omega_\varepsilon]\to\infty\ \text{ as } \varepsilon\to 0\,,\quad \text{for any } x\in\Omega\,. \tag{4.1}$$

Motivated by the Chapter 6. in [13] we consider the following re-scaled Navier-Stokes-Fourier system

$$\varrho_t+\mathrm{div}(\varrho\vec{u})=0\,, \tag{4.2}$$

$$(\varrho\vec{u})_t+\mathrm{div}(\varrho\vec{u}\otimes\vec{u})+\frac{1}{\varepsilon^2}\nabla p_\varepsilon(\varrho\,,\vartheta)=\mathrm{div}\,\mathbb{S}_\varepsilon+\frac{1}{\varepsilon^2}\varrho\nabla F\,, \tag{4.3}$$

$$(\varrho s_\varepsilon(\varrho\,,\vartheta))_t+\mathrm{div}(\varrho s_\varepsilon(\varrho\,,\vartheta)\vec{u})+\frac{1}{\varepsilon^2}\mathrm{div}(\frac{\vec{q}_\varepsilon}{\vartheta})=\sigma_\varepsilon\,, \tag{4.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\varepsilon}\Big(\frac{\varepsilon^2}{2}\varrho|\vec{u}|^2+\varrho e_\varepsilon(\varrho\,,\vartheta)-\varrho F\Big)\mathrm{d}x=\int_{\partial\Omega}\beta_1\vartheta\frac{\overline{\vartheta}-\vartheta}{\varepsilon^\beta}\mathrm{d}S_x\,, \tag{4.5}$$

where the density $\varrho = \varrho(t\,,x)$, the velocity field $\vec{u} = \vec{u}(t\,,x)$, the absolute temperature $\vartheta = \vartheta(t\,,x)$ are state variables depending on the time $t \in (0\,,T)$ and the spatial position $x \in \Omega_\varepsilon$, $F = F(x)$ is an external force defined on Ω, β_1 is a positive constant, and $1 < \beta < 2$. $p_\varepsilon = p_\varepsilon(\varrho\,,\vartheta)$, $e_\varepsilon = e_\varepsilon(\varrho\,,\vartheta)$, $s_\varepsilon = s_\varepsilon(\varrho\,,\vartheta)$ denote the re-scaled pressure, the internal energy and the entropy obeying Gibbs' relation

$$\vartheta \mathrm{d}s_\varepsilon = \mathrm{d}e_\varepsilon + p_\varepsilon\,\mathrm{d}\Big(\frac{1}{\varrho}\Big)\,, \tag{4.6}$$

where the re-scaling is introduced as follows

$$p_\varepsilon(\varrho\,,\vartheta) = \frac{\vartheta^{\frac{5}{2}}}{\varepsilon^\alpha} P\Big(\frac{\varepsilon^\alpha \varrho}{\vartheta^{\frac{3}{2}}}\Big) + \varepsilon \frac{a}{3}\vartheta^4\,,\ a > 0\,, \tag{4.7}$$

$$e_\varepsilon(\varrho\,,\vartheta) = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho\varepsilon^\alpha} P\Big(\varepsilon^\alpha \frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big) + \varepsilon a \frac{\vartheta^4}{\varrho}\,, \tag{4.8}$$

$$s_\varepsilon(\varrho\,,\vartheta) = S\Big(\varepsilon^\alpha \frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big) - S(\varepsilon^\alpha) + \varepsilon \frac{4}{3} a \frac{\vartheta^3}{\varrho}\,, \tag{4.9}$$

with $2 < \alpha < 3$. The rescaled viscous stress tensor $\mathbb{S}_\varepsilon$ obeys the Newton's law

$$\mathbb{S}_\varepsilon = (\varepsilon^{2\alpha/3}\mu_0 + \mu_1\vartheta)\Big(\nabla\vec{u} + \nabla\vec{u}^T - \frac{2}{3}\mathrm{div}\vec{u}\,\mathbb{I}\Big)\,, \tag{4.10}$$

while the heat flux $\vec{q}_\varepsilon$ is given by Fourier's law

$$\vec{q}_\varepsilon = -\Big(\varepsilon^{2+2\alpha/3}\kappa_0 + \varepsilon^2\kappa_1\vartheta + d\vartheta^3\Big)\nabla\vartheta\,, \tag{4.11}$$

with a positive constants $d\,,\mu_0\,,\mu_1\,,\kappa_0\,,\kappa_1$. The entropy production rate σ_ε is a non-negative measure on the set $[0\,,T] \times \overline{\Omega}_\varepsilon$ satisfying

$$\sigma_\varepsilon \geq \frac{1}{\vartheta}\Big(\varepsilon^2\,\mathbb{S}_\varepsilon : \nabla\vec{u} - \frac{1}{\varepsilon^2}\frac{\vec{q}_\varepsilon \cdot \nabla\vartheta}{\vartheta}\Big)\,, \tag{4.12}$$

as a consequence of the Second law of thermodynamics, where

$$\frac{1}{\vartheta}\Big(\varepsilon^2\,\mathbb{S}_\varepsilon : \nabla\vec{u} - \frac{1}{\varepsilon^2}\frac{\vec{q}_\varepsilon \cdot \nabla\vartheta}{\vartheta}\Big) \geq$$
$$\frac{\varepsilon^2}{2}\mu_1\Big|\nabla\vec{u} + \nabla\vec{u}^T - \frac{2}{3}\mathrm{div}\vec{u}\,\mathbb{I}\Big|^2 + \varepsilon^{2\alpha/3}\kappa_0|\nabla\log\vartheta|^2 + \frac{\kappa_1}{\vartheta}|\nabla\vartheta|^2 + \frac{d}{\varepsilon^2}\vartheta|\nabla\vartheta|^2\,. \tag{4.13}$$

Since p_ε, e_ε, s_ε are related by (4.6), S must satisfy

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \ \text{ for all } Z > 0\,. \tag{4.14}$$

Here we assume $P \in C^2[0\,,\infty)$ such that

$$P(0) = 0\,,\ \ P'(0) = p_0 > 0\,, \tag{4.15}$$

and since we assume that both the specific heat $\partial e/\partial\vartheta$ and the compressibility $\partial p/\partial\varrho$ are positive, P has to obey

$$P'(Z) > 0\,, \quad \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} > 0 \ \text{ for all } Z > 0\,, \tag{4.16}$$

$$0 < \sup_{Z>0}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} < \infty\,, \qquad \lim_{Z\to\infty}\frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0\,. \tag{4.17}$$

In order to eliminate the effect of the boundary layer on propagation of acoustic waves, the system is supplemented with complete slip boundary conditions

$$\vec{u}\cdot\vec{n}|_{\partial\Omega_\varepsilon} = 0\,,\ \ [\mathbb{S}_\varepsilon\vec{n}]\times\vec{n}|_{\partial\Omega_\varepsilon} = 0\,. \tag{4.18}$$

In agreement with the energy equality (4.5)

$$\vec{q}_\varepsilon(\vartheta\,,\nabla\vartheta)\cdot\vec{n} = \beta_1\vartheta(\overline{\vartheta} - \vartheta)|_{\partial\Omega} \tag{4.19}$$

is imposed on the $\partial\Omega$ part of the boundary of Ω_ε, while the rest of the boundary is thermally insulated

$$\vec{q}_\varepsilon(\vartheta\,,\nabla\vartheta)\cdot\vec{n}|_{\partial\Omega_\varepsilon\setminus\partial\Omega} = 0\,. \tag{4.20}$$

Ill-prepared initial data will be prescribed in the following form

$$\varrho(0\,,\cdot) = \varrho_{0,\,\varepsilon} = \tilde{\varrho} + \varepsilon\varrho^{(1)}_{0,\,\varepsilon}\,, \quad \vec{u}(0\,,\cdot) = \vec{u}_{0,\,\varepsilon}\,, \quad \vartheta(0\,,\cdot) = \vartheta_{0,\,\varepsilon} = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{0,\,\varepsilon}\,, \tag{4.21}$$

where $\overline{\vartheta} > 0$, and $\tilde{\varrho}$ is the unique positive solution to the static problem

$$p_0\overline{\vartheta}\nabla\tilde{\varrho} - \tilde{\varrho}\nabla F = 0 \ \text{ in } \Omega\,,\ \lim_{|x|\to\infty}\tilde{\varrho} = \varrho_\infty > 0\,. \tag{4.22}$$

Integral means for both $\varrho^{(1)}_{0,\,\varepsilon}$ and $\vartheta^{(1)}_{0,\,\varepsilon}$ are supposed to be zero which reads

$$\int_{\Omega_\varepsilon}\varrho^{(1)}_{0,\,\varepsilon}\mathrm{d}x = \int_{\Omega_\varepsilon}\vartheta^{(1)}_{0,\,\varepsilon}\mathrm{d}x = 0 \ \text{ for all } \varepsilon > 0. \tag{4.23}$$

Equations (4.3) - (4.5) contain singular terms on a small parameter ε resulting from a suitable scaling of the original Navier-Stokes-Fourier system. This corresponds to the singular limit of the Navier-Stokes-Fourier system, where both the Mach and the Froude number are proportional to ε and the Péclet number is proportional to ε^2. Physically speaking, the fluid is almost incompressible and strongly stratified. The characteristic temperature of the system is large of order $\varepsilon^{-2\alpha/3}$, and ε^{β} is a scaling of the heat flux through the boundary. The aim of this paper is to pass to the limit for $\varepsilon \to 0$ in (4.2) - (4.5). We first collect all uniform estimates, independent of ε, resulting from he total dissipation balance. These estimates enable us to pass to the limit for $\varepsilon \to 0$ in all terms except of $\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon$ which is the most delicate issue. In order to overcome this problem we show that $\{\vec{u}_\varepsilon\}_{\varepsilon>0}$ converges strongly. This will be viewed as a consequence of the local energy decay of acoustic waves for $\varepsilon \to 0$. The so-called "flat" case when $F = 0$ and $\Omega = \mathbb{R}^3$ is solved in [6] by using the so-called Strichartz estimates (see [36]). However, for a general exterior domain, the Strichartz estimates become much more delicate and require some restrictions on the shape of $\partial\Omega$, and thus are not available in general. Local decay of the acoustic energy in exterior domains was also shown by Alazard [1] in the context of the Euler system, and his method is based on the concept of certain semiclassical defect measures. We use other approach developed in [17] based on weighted space-time estimates for abstract wave equations due to Kato [20]. We exploit that the velocity field $\vec{u}_\varepsilon$ is locally compact with respect to the space variable. Accordingly, it is enough to apply the mentioned Kato's result to fixed range of frequencies of acoustic waves. In such a way, the problem is reduced to the validity of the limiting absorption principle for a modified wave operator which is shown in the last section.

4.1.1 Preliminaries and main result

Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to the Navier-Stokes-Fourier system (4.2) - (4.5) supplemented with the boundary and initial conditions (4.21), (4.23). Their existence is assured by Theorem 3.1, 3.2 in [13] which reads as follows:

- $$\begin{gathered} \varrho_\varepsilon \geq 0\,, \quad \varrho_\varepsilon \in L^\infty(0\,,T\,; L^{\frac{5}{3}}(\Omega_\varepsilon))\,, \\ \vec{u}_\varepsilon \in L^2(0\,,T\,;V)\,,\ V = \{\vec{u} \in W^{1,2}(\Omega_\varepsilon) |\quad \vec{u}\cdot\vec{n}|_{\partial\Omega_\varepsilon} = 0\}\,, \\ \vartheta_\varepsilon > 0\,, \vartheta_\varepsilon \in L^2(0\,,T\,;W^{1,2}(\Omega_\varepsilon)) \cap L^\infty(0\,,T\,;L^4(\Omega_\varepsilon))\,, \end{gathered} \tag{4.24}$$

- The continuity equation (4.2) is satisfied in the sense of renormalized solutions:

$$\begin{aligned}&\int_0^T\int_{\Omega_\varepsilon}\varrho_\varepsilon B(\varrho_\varepsilon)\Big(\partial_t\varphi+\vec{u}_\varepsilon\cdot\nabla\varphi\Big)\mathrm{d}x\,\mathrm{d}t\\&=\int_0^T\int_{\Omega_\varepsilon}b(\varrho_\varepsilon)\mathrm{div}\vec{u}_\varepsilon\varphi\mathrm{d}x\,\mathrm{d}t-\int_{\Omega_\varepsilon}\varrho_{0,\,\varepsilon}B(\varrho_{0,\,\varepsilon})\varphi(0\,,\cdot)\mathrm{d}x\,,\end{aligned}\tag{4.25}$$

for any $\varphi\in\mathcal{D}([0\,,T)\times\overline{\Omega}_\varepsilon)$ and any $b\in L^\infty\cap C[0\,,\infty)$ where

$$B(\varrho)=B(0)+\int_1^\varrho\frac{b(z)}{z^2}\mathrm{d}z\,;$$

- Momentum equation:

$$\begin{aligned}&\int_0^T\int_{\Omega_\varepsilon}\Big(\varrho_\varepsilon\vec{u}_\varepsilon\cdot\partial_t\varphi+\varrho_\varepsilon[\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon]:\nabla\varphi+\frac{1}{\varepsilon^2}p_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon)\mathrm{div}\varphi\Big)\mathrm{d}x\,\mathrm{d}t\\&=\int_0^T\int_{\Omega_\varepsilon}\Big(\mathbb{S}_\varepsilon:\nabla\varphi-\frac{1}{\varepsilon^2}\varrho_\varepsilon\nabla F\cdot\varphi\Big)\mathrm{d}x\,\mathrm{d}t-\int_{\Omega_\varepsilon}(\varrho_{0,\,\varepsilon}\vec{u}_{0,\,\varepsilon})\cdot\varphi(0\,,\cdot)\mathrm{d}x\,,\end{aligned}\tag{4.26}$$

for any $\varphi\in\mathcal{D}([0\,,T)\times\overline{\Omega}_\varepsilon\,;\mathbb{R}^3)$, $\varphi\cdot\vec{n}|_{\partial\Omega_\varepsilon}=0$;

- Total energy balance:

$$\begin{aligned}&\int_{\Omega_\varepsilon}\Big(\frac{\varepsilon^2}{2}\varrho_\varepsilon|\vec{u}_\varepsilon|^2+\varrho e_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon)-\varrho_\varepsilon F\Big)(t)\mathrm{d}x=\int_0^t\int_{\partial\Omega}\beta_1\vartheta_\varepsilon\frac{\overline{\vartheta}-\vartheta_\varepsilon}{\varepsilon^\beta}\mathrm{d}S_x\mathrm{d}s\\&+\int_{\Omega_\varepsilon}\Big(\frac{\varepsilon^2}{2}\varrho_{0,\,\varepsilon}|\vec{u}_{0,\,\varepsilon}|^2+\varrho_{0,\,\varepsilon}e_\varepsilon(\varrho_{0,\,\varepsilon}\,,\vartheta_{0,\,\varepsilon})-\varrho_{0,\,\varepsilon}F\Big)\mathrm{d}x\,,\ \text{for a. a.}\,t\in(0\,,T)\,;\end{aligned}\tag{4.27}$$

- Entropy balance equation:

$$\begin{aligned}&\int_0^T\int_{\Omega_\varepsilon}\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon)\Big(\partial_t\varphi+\vec{u}_\varepsilon\cdot\nabla\varphi\Big)\mathrm{d}x\,\mathrm{d}t+\frac{1}{\varepsilon^2}\int_0^T\int_{\Omega_\varepsilon}\frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla\varphi\mathrm{d}x\,\mathrm{d}t\\&\quad+<\sigma_\varepsilon\,,\varphi>_{[\mathcal{M},C]([0,T]\times\overline{\Omega}_\varepsilon)}-\int_0^T\int_{\partial\Omega}\beta_1\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon^\beta}\varphi\mathrm{d}S_x\mathrm{d}t\\&\quad=-\int_{\Omega_\varepsilon}\varrho_{0,\,\varepsilon}s_\varepsilon(\varrho_{0,\,\varepsilon}\,,\vartheta_{0,\,\varepsilon})\varphi(0\,,\cdot)\mathrm{d}x\,,\end{aligned}\tag{4.28}$$

for any $\varphi\in\mathcal{D}([0\,,T)\times\overline{\Omega}_\varepsilon)$, where $\sigma_\varepsilon\in\mathcal{M}^+([0\,,T]\times\overline{\Omega}_\varepsilon)$,

$$\sigma_\varepsilon\geq\frac{1}{\vartheta_\varepsilon}\Big(\varepsilon^2\mathbb{S}_\varepsilon:\nabla\vec{u}_\varepsilon-\frac{1}{\varepsilon^2}\frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla\vartheta_\varepsilon\Big)\,.$$

We now introduce our target system consisting of

- the hydrostatic balance equation:

$$p_0\overline{\vartheta}\nabla\tilde{\varrho} - \tilde{\varrho}\nabla F = 0\,; \tag{4.29}$$

- the anelastic constraint:

$$\mathrm{div}(\tilde{\varrho}\vec{U}) = 0\,; \tag{4.30}$$

- the momentum equation supplemented with the complete slip boundary condition:

$$\begin{gathered}\partial_t(\tilde{\varrho}\vec{U}) + \mathrm{div}(\tilde{\varrho}\vec{U}\otimes\vec{U}) + \tilde{\varrho}\nabla\Pi = \mu_1\overline{\vartheta}\Delta\vec{U} + \frac{1}{3}\mu_1\overline{\vartheta}\nabla\mathrm{div}\vec{U} + \frac{\tilde{\varrho}}{\overline{\vartheta}}F\Phi\,,\\ \vec{U}\cdot\vec{n}|_{\partial\Omega} = 0\,,\ \left[\mu_1\overline{\vartheta}\Big(\nabla\vec{U} + \nabla^T\vec{U}\Big)\vec{n}\right]\times\vec{n}|_{\partial\Omega} = 0\,;\end{gathered} \tag{4.31}$$

- and the "gradient" of the temperature Φ is related with the velocity through

$$-\tilde{\varrho}\nabla F\cdot\vec{U} = d\overline{\vartheta}^3\mathrm{div}\Phi \text{ in } \Omega\,,\ \Phi\cdot\vec{n}|_{\partial\Omega} = 0\,. \tag{4.32}$$

This resulting problem can be viewed as a simple model of the fluid motion in the stellar radiative zones. For detailed physical explanation of it, see e.g. [5], [25].

Remark: A suitable weak formulation of the momentum equation (4.31) reads:

$$\begin{gathered}\int_0^T\int_\Omega\Big(\tilde{\varrho}\vec{U}\cdot\varphi + \tilde{\varrho}\vec{U}\otimes\vec{U} : \nabla\varphi + \frac{\tilde{\varrho}}{\overline{\vartheta}}F\Phi\cdot\varphi\Big)\mathrm{d}x\,\mathrm{d}t = \\ \int_0^T\int_\Omega\Big(\mu_1\overline{\vartheta}\Big(\nabla\vec{U} + \nabla^T\vec{U} - \frac{2}{3}\mathrm{div}\vec{U}\mathbb{I}\Big) : \nabla\varphi\mathrm{d}x\,\mathrm{d}t - \int_\Omega\tilde{\varrho}\vec{U}_0\cdot\varphi(0\,,\cdot)\mathrm{d}x\end{gathered} \tag{4.33}$$

to be satisfied for any test function

$$\varphi\in C_c^\infty([0\,,T)\times\overline{\Omega}\,;\mathbb{R}^3)\,,\ \varphi\cdot\vec{n}|_{\partial\Omega} = 0\,,\ \mathrm{div}(\tilde{\varrho}\varphi) = 0\,.$$

Before we formulate main result of this paper, let's introduce the concept of so-called weighted Helmholtz decomposition

Definition 4.1.1 *For $\vec{v}\in L^2_{1/\tilde{\varrho}}(\Omega\,;\mathbb{R}^3)$, we introduce the weighted Helmholtz decomposition in the form*

$$\vec{v} = \mathbf{H}_{\tilde{\varrho}}[\vec{v}]\oplus\mathbf{H}^\perp_{\tilde{\varrho}}[\vec{v}]\,,\ \ \text{with}\,\mathbf{H}^\perp_{\tilde{\varrho}}[\vec{v}] = \tilde{\varrho}\nabla\Psi\,,$$

where $\Psi \in D^{1,2}(\Omega)$ *is the unique solution of the problem*

$$\int_\Omega \tilde{\varrho} \nabla \Psi \cdot \nabla \varphi \mathrm{d}x = \int_\Omega \vec{v} \cdot \nabla \varphi \mathrm{d}x \ \text{ for all}\, \varphi \in D^{1,2}(\Omega)\,.$$

The space $D^{1,2}(\Omega)$ *is defined as a completion of* $C_c^\infty(\overline{\Omega})$ *with respect to the norm* $\|\nabla\varphi\|_{L^2(\Omega)}$ *. For the more detailed explanation see e.g. [13], page 204.*

The main result reads as follows.

Theorem 4.1.1 *Let* $\Omega \subset \mathbb{R}^3$ *be an unbounded domain with a compact boundary of class* C^∞ *and* $F \in C^{1,1}(\Omega) \cap C_c(\overline{\Omega})$*. Let* $\{\varrho_\varepsilon\,, \vec{u}_\varepsilon\,, \vartheta_\varepsilon\}_{\varepsilon>0}$ *be a family of weak solutions to the Navier-Stokes-Fourier system (4.2) - (4.5) on* $(0\,,T) \times \Omega_\varepsilon$ *, supplemented with the boundary and initial conditions (4.21), (4.23), and let all assumptions of the previous section be satisfied too. Moreover we will assume that*

$$\{\varrho_{0,\,\varepsilon}^{(1)}\}_{\varepsilon>0}\,,\ \{\vartheta_{0,\,\varepsilon}^{(1)}\}_{\varepsilon>0} \ \text{are bounded in } L^2 \cap L^\infty(\Omega)\,,$$
$$\{\vec{u}_{0,\,\varepsilon}\}_{\varepsilon>0} \ \text{is bounded in } L^2 \cap L^\infty(\Omega)\,.$$

Then at least for a suitable subsequence we have

$$\varrho_\varepsilon \to \tilde{\varrho} \text{ a.a. in } (0\,,T) \times \Omega\,,$$
$$\vec{u}_\varepsilon \to \vec{U} \text{ a.a. in } (0\,,T) \times \Omega\,,$$
$$\vartheta_\varepsilon \to \overline{\vartheta} \text{ a.a. in } (0\,,T) \times \Omega\,,$$
$$\nabla\Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2}\Big) \to \Phi \text{ weakly in } L^1(0\,,T\,;L^1(K\,;\mathbb{R}^3))\,,$$

where $\tilde{\varrho}\,, \vec{U}\,, \Phi$ *is a weak solution of (4.29) - (4.32), with the initial condition*

$$\tilde{\varrho}\vec{U}(0) = \mathbf{H}_{\tilde{\varrho}}[\tilde{\varrho}\vec{U}_0]\,,\ \ \vec{u}_{0,\,\varepsilon} \to \vec{U}_0 \,\text{weakly in } L^2(K\,;\mathbb{R}^3)$$

for each compact set $K \subset \Omega$ *.*

4.2 Uniform estimates

4.2.1 Total dissipation balance

Combining the entropy production equation and (4.28) with the total energy balance (4.27) we arrive at total dissipation balance:

$$\begin{aligned} &\int_{\Omega_\varepsilon} \Big[\frac{1}{2}\varrho_\varepsilon|\vec{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon\,,\vartheta_\varepsilon) - \varrho_\varepsilon F\Big)\Big](\tau\,,\cdot)\mathrm{d}x \\ &\quad + \frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon\Big[[0\,,\tau]\times\overline{\Omega}_\varepsilon\Big] + \int_0^\tau\int_{\partial\Omega}\beta_1\frac{(\vartheta_\varepsilon - \overline{\vartheta})^2}{\varepsilon^{2+\beta}}\mathrm{d}S_x\mathrm{d}t \\ &= \int_{\Omega_\varepsilon}\Big[\frac{1}{2}\varrho_{0,\,\varepsilon}|\vec{u}_{0,\,\varepsilon}|^2 + \frac{1}{\varepsilon^2}\Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\,\varepsilon}\,,\vartheta_{0,\,\varepsilon}) - \varrho_{0,\,\varepsilon}F\Big)\Big]\mathrm{d}x \ \text{for a. a. } \tau\in[0\,,T]\,, \end{aligned} \tag{4.34}$$

where $H^\varepsilon_{\overline{\vartheta}}(\varrho\,,\vartheta) = \varrho e_\varepsilon(\varrho\,,\vartheta) - \overline{\vartheta}\varrho s_\varepsilon(\varrho\,,\vartheta)$.

Since the functions p_ε, e_ε and s_ε satisfy Gibbs' equation (4.6), we easily compute

$$\frac{\partial^2 H^\varepsilon_{\overline{\vartheta}}(\varrho\,,\overline{\vartheta})}{\partial\varrho^2} = \frac{1}{\varrho}\frac{\partial p_\varepsilon(\varrho\,,\overline{\vartheta})}{\partial\varrho} = \frac{\overline{\vartheta}}{\varrho}P'\Big(\varepsilon^\alpha\frac{\varrho}{\overline{\vartheta}^{\frac{3}{2}}}\Big)\,, \tag{4.35}$$

whence

$$\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon\,,\overline{\vartheta})}{\partial\varrho} - \tilde{\varrho}_\varepsilon\nabla F = \text{const}\,, \tag{4.36}$$

where $\tilde{\varrho}_\varepsilon$ is the solution of the static problem

$$\nabla p_\varepsilon(\tilde{\varrho}_\varepsilon\,,\overline{\vartheta}) - \tilde{\varrho}_\varepsilon\nabla F = 0 \ \text{ in}\,\Omega\,,\ \lim_{|x|\to\infty}\tilde{\varrho}_\varepsilon = \varrho_\infty > 0\,. \tag{4.37}$$

Due to (4.36), relation (4.34) may be rewritten in the form

$$\begin{aligned} &\int_{\Omega_\varepsilon}\Big[\frac{1}{2}\varrho_\varepsilon|\vec{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon\,,\vartheta_\varepsilon) - H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon\,,\overline{\vartheta})\Big)\Big](\tau\,,\cdot)\mathrm{d}x \\ &\quad + \frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon\Big[[0\,,\tau]\times\overline{\Omega}_\varepsilon\Big] + \int_0^\tau\int_{\partial\Omega}\beta_1\frac{(\vartheta_\varepsilon-\overline{\vartheta})^2}{\varepsilon^{2+\beta}}\mathrm{d}S_x\mathrm{d}t \\ &+\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon}\Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon\,,\overline{\vartheta}) - (\varrho_\varepsilon - \tilde{\varrho}_\varepsilon)\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon\,,\overline{\vartheta})}{\partial\varrho} - H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon\,,\overline{\vartheta})\Big)(\tau\,,\cdot)\mathrm{d}x \\ &= \int_{\Omega_\varepsilon}\Big[\frac{1}{2}\varrho_{0,\,\varepsilon}|\vec{u}_{0,\,\varepsilon}|^2 + \frac{1}{\varepsilon^2}\Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\,\varepsilon}\,,\vartheta_{0,\,\varepsilon}) - H^\varepsilon_{\overline{\vartheta}}(\varrho_{0\,,\varepsilon}\,,\overline{\vartheta})\Big)\Big]\mathrm{d}x \\ &+\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon}\Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_{0\,,\varepsilon}\,,\overline{\vartheta}) - (\varrho_{0\,,\varepsilon} - \tilde{\varrho}_\varepsilon)\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon\,,\overline{\vartheta})}{\partial\varrho} - H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon\,,\overline{\vartheta})\Big)\mathrm{d}x \end{aligned} \tag{4.38}$$

for a.a. $\tau \in [0\,,T]$. As the pressure p_ε is determined in terms of the function P through (4.7), it is easy to check that the positive solution $\tilde{\varrho}_\varepsilon$ of the problem (4.37) satisfies

$$\overline{\vartheta} p_0 \log \tilde{\varrho}_\varepsilon(x) + \overline{\vartheta}^{\frac{5}{2}} Q\Big(\frac{\varepsilon^\alpha}{\overline{\vartheta}^{\frac{3}{2}}}\tilde{\varrho}_\varepsilon(x)\Big) - \overline{\vartheta}^{\frac{5}{2}} Q\Big(\frac{\varepsilon^\alpha}{\overline{\vartheta}^{\frac{3}{2}}}\varrho_\infty\Big) = F(x) + p_0\overline{\vartheta}\log\varrho_\infty\,, \quad (4.39)$$

where

$$Q'(r) = \begin{cases} \frac{P'(r)-p_0}{r} & \text{for}\, r>0\,, \\ P''(0) & \text{for}\, r=0\,. \end{cases}$$

Formula (4.39) implies the existence of constants $\underline{\varrho}$, $\overline{\varrho}$ such that

$$0 < \underline{\varrho} < \inf_{x\in\Omega} \tilde{\varrho}(x) \le \sup_{x\in\Omega} \tilde{\varrho}(x) < \overline{\varrho} < \infty\,, \quad (4.40)$$

uniformly for $\varepsilon \to 0$. Moreover since F is compactly supported we observe that

$$\|\tilde{\varrho}_\varepsilon - \tilde{\varrho}\|_{C(\overline{\Omega})} \le \varepsilon^\alpha c\,,\ \ \tilde{\varrho}_\varepsilon = \tilde{\varrho} = \varrho_\infty \ \text{in}\, \Omega\setminus \text{supp}[F]\,. \quad (4.41)$$

We now show that the right-hand side of (4.38) is bounded uniformly for $\varepsilon \to 0$. Indeed, from the Gibbs' equation (4.6) and (4.17) one may compute

$$\frac{1}{\varepsilon^2}\Big| H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\,\varepsilon}\,,\vartheta_{0,\,\varepsilon}) - H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\,\varepsilon}\,,\overline{\vartheta})\Big| \le c_1 \Big|\frac{\vartheta_{0,\,\varepsilon}-\overline{\vartheta}}{\varepsilon}\Big|^2 \le c_2$$

which together with the assumptions of the Theorem 4.1.1 implies the boundedness of the first integral. Similarly, in accordance with (4.35) we have

$$\begin{aligned}\frac{1}{\varepsilon^2}\Big| H^\varepsilon_{\overline{\vartheta}}(\varrho_{0\,,\varepsilon}\,,\overline{\vartheta}) - (\varrho_{0\,,\varepsilon}-\tilde{\varrho}_\varepsilon)\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon\,,\overline{\vartheta})}{\partial\varrho} - H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon\,,\overline{\vartheta})\Big| &\le c_1\Big|\frac{\varrho_{0,\,\varepsilon}-\tilde{\varrho}_\varepsilon}{\varepsilon}\Big|^2 \\ &\le c_2\Big(\Big|\frac{\varrho_{0,\,\varepsilon}-\tilde{\varrho}}{\varepsilon}\Big|^2 + \Big|\frac{\tilde{\varrho}-\tilde{\varrho}_\varepsilon}{\varepsilon}\Big|^2\Big)\,,\end{aligned}$$

whence the desired uniform bound follows from (4.41). The hypothesis of thermodynamic stability (4.16) together with above results imply that all integrated quantities on the left-hand side of (4.38) are non-negative, and consequently we deduce the following estimates

$$\operatorname*{ess\ sup}_{t\in(0,\,T)} \|\sqrt{\varrho_\varepsilon}\vec{u}_\varepsilon\|_{L^2(\Omega_\varepsilon;\,\mathbb{R}^3)} \le c\,, \quad (4.42)$$

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,\,T]\times\overline{\Omega}_\varepsilon)} \le \varepsilon^2 c\,, \quad (4.43)$$

$$\int_0^T \int_{\partial\Omega} \Big|\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big|^2 \le \varepsilon^\beta c\,. \quad (4.44)$$

In order to estimate other terms of the right-hand side, we first observe

$$H^{\varepsilon}_{\overline{\vartheta}}(\varrho_\varepsilon\,,\vartheta_\varepsilon) - H^{\varepsilon}_{\overline{\vartheta}}(\varrho_\varepsilon\,,\overline{\vartheta}) \geq c|\vartheta_\varepsilon - \overline{\vartheta}|^2 \tag{4.45}$$

as soon as $\underline{\varrho}/2 < \varrho_\varepsilon < 2\overline{\varrho}\,,\ \overline{\vartheta}/2 < \vartheta_\varepsilon < 2\overline{\vartheta}$ where, as a direct consequence of (4.6), (4.16), c is independent of ε. Indeed, from Gibb's equation (4.6) we obtain

$$\frac{\partial H^{\varepsilon}_{\overline{\vartheta}}(\varrho\,,\vartheta)}{\partial\vartheta} = \varrho(\vartheta - \overline{\vartheta})\Big[-\frac{3}{2\vartheta}S'(Z)Z + \varepsilon\frac{4a}{\varrho}\vartheta^2\Big]\,,$$

for $Z = \varepsilon^\alpha\varrho/\vartheta^{\frac{3}{2}}$. Now (4.45) follows from (4.16). Similarly as defined in [13], consider the essential and the residual parts of function h defined on $(0\,,T)\times\Omega$

$$h = [h]_{ess} + [h]_{res}\,,$$

where $h_{ess} = h\chi_{\mathcal{M}^\varepsilon_{ess}}$, and $h_{res} = h\chi_{\mathcal{M}^\varepsilon_{res}}$,

$$\mathcal{M}^\varepsilon_{ess} = \{(t\,,x)\in(0\,,T)\times\Omega_\varepsilon\,|\,\underline{\varrho}/2 < \varrho_\varepsilon < 2\overline{\varrho}\,,\overline{\vartheta}/2 < \vartheta_\varepsilon < 2\overline{\vartheta}\}\,,$$
$$\mathcal{M}^\varepsilon_{res} = ((0\,,T)\times\Omega_\varepsilon)\setminus\mathcal{M}^\varepsilon_{ess}\,.$$

From (4.45) one can obtain

$$\operatorname*{ess\ sup}_{t\in(0,T)}\Big\|\Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big]_{ess}\Big\|_{L^2(\Omega_\varepsilon)} \leq c\,. \tag{4.46}$$

Furthermore, it follows from hypothesis (4.15) - (4.17) that

$$\frac{\partial^2 H^{\varepsilon}_{\overline{\vartheta}}(\varrho\,,\overline{\vartheta})}{\partial\varrho^2} \geq \frac{c}{\varrho} \tag{4.47}$$

and consequently

$$\operatorname*{ess\ sup}_{t\in(0,T)}\Big\|\Big[\frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon}\Big]_{ess}\Big\|_{L^2(\Omega_\varepsilon)} \leq c\,. \tag{4.48}$$

Similarly, as derived in [13], page 209, one can show

$$\operatorname*{ess\ sup}_{t\in(0,T)}|\mathcal{M}^\varepsilon_{res}[t]| \leq \varepsilon^2 c\,, \tag{4.49}$$

where $\mathcal{M}^\varepsilon_{res}[t] = \mathcal{M}^\varepsilon_{res}|_{\{t\}\times\Omega_\varepsilon}$. In addition by virtue of (4.47), (4.49),

$$\operatorname*{ess\ sup}_{t\in(0,T)}\|[\varrho_\varepsilon\log\varrho_\varepsilon]_{res}\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon^2 c\,. \tag{4.50}$$

On the other hand, by virtue of Proposition 3.2 in [13], page 71,

$$H^{\varepsilon}_{\overline{\vartheta}}(\varrho\,,\vartheta) \geq \frac{1}{4}\Big(\varrho e_{\varepsilon}(\varrho\,,\vartheta) + \overline{\vartheta}\varrho|s_{\varepsilon}(\varrho\,,\vartheta)|\Big) - \Big|(\varrho-\overline{\varrho})\frac{\partial H^{\varepsilon}_{2\overline{\vartheta}}}{\partial\varrho}(\overline{\varrho}\,,2\overline{\vartheta}) + H^{\varepsilon}_{2\overline{\vartheta}}(\overline{\varrho}\,,2\overline{\vartheta})\Big|$$

for any $\varrho\,,\,\vartheta$ and therefore we can conclude that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_{\varepsilon}e_{\varepsilon}(\varrho_{\varepsilon}\,,\vartheta_{\varepsilon})]_{res}\|_{L^1(\Omega_{\varepsilon})} \leq \varepsilon^2 c\,, \tag{4.51}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_{\varepsilon}s_{\varepsilon}(\varrho_{\varepsilon}\,,\vartheta_{\varepsilon})]_{res}\|_{L^1(\Omega_{\varepsilon})} \leq \varepsilon^2 c\,. \tag{4.52}$$

In accordance with hypothesis (4.8), (4.16),

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_{\varepsilon}\vartheta_{\varepsilon}]_{res}\|_{L^1(\Omega_{\varepsilon})} \leq \varepsilon^2 c\,, \tag{4.53}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\vartheta_{\varepsilon}]^4_{res}\|_{L^1(\Omega_{\varepsilon})} \leq \varepsilon c\,, \tag{4.54}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_{\varepsilon}]^{\frac{5}{3}}_{res}\|_{L^1(\Omega_{\varepsilon})} \leq \varepsilon^{2-2\alpha/3} c\,. \tag{4.55}$$

In accordance with (4.12), (4.13) and (4.43), we deduce immediately that

$$\int_0^T \int_{\Omega_{\varepsilon}} |\nabla\vec{u}_{\varepsilon} + \nabla^T\vec{u}_{\varepsilon} - \frac{2}{3}\mathrm{div}\vec{u}_{\varepsilon}\mathbb{I}|^2 \mathrm{d}x\,\mathrm{d}t \leq c\,, \tag{4.56}$$

$$\int_0^T \int_{\Omega_{\varepsilon}} \vartheta_{\varepsilon}\Big|\nabla\Big(\frac{\vartheta_{\varepsilon}-\overline{\vartheta}}{\varepsilon^2}\Big)\Big|^2 \mathrm{d}x\,\mathrm{d}t \leq c\,, \tag{4.57}$$

$$\int_0^T \int_{\Omega_{\varepsilon}} \frac{1}{\vartheta_{\varepsilon}}\Big|\nabla\Big(\frac{\vartheta_{\varepsilon}-\overline{\vartheta}}{\varepsilon}\Big)\Big|^2 \mathrm{d}x\,\mathrm{d}t \leq c\,, \tag{4.58}$$

$$\int_0^T \int_{\Omega_{\varepsilon}} \Big|\nabla\Big(\log\vartheta_{\varepsilon} - \log\overline{\vartheta}\Big)\Big|^2 \mathrm{d}x\,\mathrm{d}t \leq \varepsilon^{2-2\alpha/3} c\,. \tag{4.59}$$

Combining estimates (4.42), (4.49), (4.56), and by help of Korn's inequality formulated in Proposition 2.1 in [13], page 30, we get

$$\|\vec{u}_{\varepsilon}\|_{L^2(0,T;W^{1,2}(\Omega_{\varepsilon}))} \leq c\,. \tag{4.60}$$

Similarly, by means of Poincarè inequality stated in Proposition 2.2, [13], page 32, relations (4.49), (4.53) together with (4.57) - (4.59) yield

$$\Big\|\frac{\vartheta_{\varepsilon}-\overline{\vartheta}}{\varepsilon}\Big\|_{L^2(0,T;W^{1,2}(\Omega_{\varepsilon}))} \leq c\,, \tag{4.61}$$

$$\Big\| \frac{\sqrt{\vartheta_\varepsilon} - \sqrt{\overline{\vartheta}}}{\varepsilon} \Big\|_{L^2(0,T;W^{1,2}(\Omega_\varepsilon))} \le c\,, \tag{4.62}$$

and

$$\| \log \vartheta_\varepsilon - \log \overline{\vartheta} \|_{L^2(0,T;W^{1,2}(\Omega_\varepsilon))} \le \varepsilon^{1-\alpha/3} c\,. \tag{4.63}$$

We now have to remark that the Proposition 2.2 from [13] can be used for (4.61) - (4.63), because the domain Ω_ε can be written as a union of finite number of unit cubes with mutually disjoint interiors and Proposition 2.2 will be used to each of them separately.

4.2.2 Pressure estimates

The upper bound (4.55) on the residual component of the density is not sufficient for our requirements, and so we have to go into deeper considerations based on some pressure estimates. In order to show them, we will use the Bogovskii operator $\mathcal{B}$ (see e.g. [30]) which is defined only for bounded domains. Thus we will fix an arbitrary compact set $K \subset \Omega$ with a Lipschitz boundary and define the following quantities

$$\varphi(t\,,x) = \psi(t)\mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|K|}\int_K b(\varrho_\varepsilon)\mathrm{d}x\Big]\,,\ \psi \in C_c^\infty(0\,,T)$$

to be used as a test functions in the momentum equation (4.26). After some computation we arrive at the following relation:

$$\frac{1}{\varepsilon^2}\int_0^T\int_K \psi p_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon) b(\varrho_\varepsilon)\mathrm{d}x\,\mathrm{d}t = \frac{1}{\varepsilon^2|K|}\int_0^T\int_K \psi p_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon)\mathrm{d}x\Big(\int_K b(\varrho_\varepsilon)\mathrm{d}x\Big)\,\mathrm{d}t$$
$$-\frac{1}{\varepsilon^2}\int_0^T\int_K \psi\varrho_\varepsilon\nabla F\cdot\mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|K|}\int_K b(\varrho_\varepsilon)\mathrm{d}x\Big]\mathrm{d}x\,\mathrm{d}t + I_\varepsilon\,, \tag{4.64}$$

where we have set

$$
\begin{aligned}
I_\varepsilon = &\int_0^T \int_K \psi \mathbb{S}_\varepsilon : \nabla \mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|K|}\int_K b(\varrho_\varepsilon)\mathrm{d}x\Big]\mathrm{d}x\,\mathrm{d}t \\
- &\int_0^T \int_K \psi \varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon : \nabla \mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|K|}\int_K b(\varrho_\varepsilon)\mathrm{d}x\Big]\mathrm{d}x\,\mathrm{d}t \\
- &\int_0^T \int_K \partial_t \psi \varrho_\varepsilon \vec{u}_\varepsilon \cdot \mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|K|}\int_K b(\varrho_\varepsilon)\mathrm{d}x\Big]\mathrm{d}x\,\mathrm{d}t \\
+ &\int_0^T \int_K \psi \varrho_\varepsilon \vec{u}_\varepsilon \cdot \mathcal{B}[\mathrm{div}(b(\varrho_\varepsilon)\vec{u}_\varepsilon)]\mathrm{d}x\,\mathrm{d}t \\
+ &\int_0^T \psi \int_K \varrho_\varepsilon \vec{u}_\varepsilon \cdot \mathcal{B}\Big[(\varrho_\varepsilon b'(\varrho_\varepsilon) - b(\varrho_\varepsilon))\mathrm{div}\vec{u}_\varepsilon \\
&- \frac{1}{|K|}\int_K (b(\varrho_\varepsilon) - b'(\varrho_\varepsilon)\varrho_\varepsilon)\mathrm{div}\vec{u}_\varepsilon \mathrm{d}x\Big]\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
$$

Taking the uniform estimates established in the previous section we can show, that all integrals contained in I_ε are bounded uniformly for $\varepsilon \to 0$ as soon as

$$
|b(\varrho)| + |\varrho b'(\varrho)| \leq c\varrho^\gamma \ \mathrm{for}\, 0 < \gamma < 1\,, \tag{4.65}
$$

where γ will be specified later. For example let us take b such that

$$
b(\varrho) = \begin{cases} 0 \ \mathrm{for}\, 0 \leq \varrho \leq 2\overline{\varrho}\,, \\ \in [0\,, \varrho^\gamma] \ \mathrm{for}\, 2\overline{\varrho} \leq \varrho \leq 3\overline{\varrho}\,, \\ \varrho^\gamma \ \mathrm{for}\, \varrho > 3\overline{\varrho}\,. \end{cases}
$$

It is easy to see that $b(\varrho_\varepsilon) = b([\varrho_\varepsilon]_{res})$ which together with (4.50) gives

$$
\operatorname{ess}\sup_{t\in(0,T)} \int_{\Omega_\varepsilon} b(\varrho_\varepsilon)\mathrm{d}x \leq c\varepsilon^2\,. \tag{4.66}
$$

Consequently, the first integral at the right-hand side of (4.64) is bounded. In order to control the second term, let us rewrite it, using the fact that $\tilde{\varrho}\,, \overline{\vartheta}$ solve the static problem (4.22), in the following way

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\int_K \varrho_\varepsilon \nabla F \cdot \mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|K|}\int_K b(\varrho_\varepsilon)\mathrm{d}x\Big]\mathrm{d}x \\
= &\frac{1}{\varepsilon}\int_K \Big[\frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon}\Big]_{ess} \nabla F \cdot \mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|K|}\int_K b(\varrho_\varepsilon)\mathrm{d}x\Big]\mathrm{d}x \\
+ &\int_K \Big[\frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon}\Big]_{res} \nabla F \cdot \mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|K|}\int_K b(\varrho_\varepsilon)\mathrm{d}x\Big]\mathrm{d}x \\
+ &\frac{p_0}{\varepsilon^2}\int_K \tilde{\varrho}\overline{\vartheta}\Big[b(\varrho_\varepsilon) - \frac{1}{|K|}\int_K b(\varrho_\varepsilon)\mathrm{d}x\Big]\mathrm{d}x\,,
\end{aligned}
$$

where the last integral is uniformly bounded due to (4.66). In accordance with (4.66), (4.50)

$$\|b(\varrho_\varepsilon)\|^q_{L^q(\Omega_\varepsilon)} \leq \|[\varrho_\varepsilon]^{\gamma q}_{res}\|_{L^1(\Omega_\varepsilon)} \leq \|[\varrho_\varepsilon \log \varrho_\varepsilon]_{res}\|_{L^1(\Omega_\varepsilon)} \leq c\,\varepsilon^2 \tag{4.67}$$

as soon as $\gamma \leq 1/q$. Then the first two terms at the right-hand side may be estimated using Holder inequality and estimates (4.48), (4.49), (4.55) together with (4.67) which yield a uniform bound of the second term at the right-hand side of (4.64). Consequently, we conclude that

$$\int_0^T \int_K p_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon) b(\varrho_\varepsilon) \mathrm{d}x\,\mathrm{d}t \leq \varepsilon^2 c(K)\,. \tag{4.68}$$

4.3 Convergence to the target system

4.3.1 Anelastic constraint

The uniform estimates deduced in the previous section enable us to pass to the limit in the family $\{\varrho_\varepsilon\,,\vec{u}_\varepsilon\,,\vartheta_\varepsilon\}_{\varepsilon>0}$. Let $K \subset \Omega$ be an arbitrary compact set. Then, by virtue of (4.48), (4.49), (4.55) we have

$$\varrho_\varepsilon \to \tilde{\varrho} \text{ in } L^\infty(0\,,T\,;L^{\frac{5}{3}}(K)) \cap C(0\,,T\,;L^q(K)) \text{ for any } 1 \leq q < \frac{5}{3}\,. \tag{4.69}$$

Moreover, in accordance with (4.60), we may assume

$$\vec{u}_\varepsilon \to \vec{U} \text{ weakly in } L^2(0\,,T\,;W^{1,2}(K\,;\mathbb{R}^3))\,, \tag{4.70}$$

and since (4.18) holds we can show

$$\vec{U}\cdot\vec{n}|_{\partial\Omega} = 0 \text{ in the sense of traces.} \tag{4.71}$$

Combining (4.69), (4.70) we let $\varepsilon \to 0$ in (4.25) in order to obtain the so-called anelastic constraint

$$\mathrm{div}(\tilde{\varrho}\,\vec{U}) = 0 \text{ a.a. in } (0\,,T) \times \Omega\,. \tag{4.72}$$

4.3.2 Momentum equation

Similarly as in the previous section, consider the compact subset $K \subset \Omega$. Then it follows from (4.61) that

$$\vartheta_\varepsilon \to \overline{\vartheta} \text{ in } L^2(0\,,T\,;W^{1,2}(K))\,. \tag{4.73}$$

The convergence of the pressure term is described in the following lemma.

Lemma 17 *Let $K \subset \Omega$ be an arbitrary compact subset of Ω. Then*

$$\frac{1}{\varepsilon^2}\Big(p_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon) - p_0\varrho_\varepsilon\vartheta_\varepsilon - \varepsilon\frac{a}{3}\overline{\vartheta}^4\Big) \to 0 \text{ in } L^1((0\,,T)\times K)\,.$$

Proof: Let us examine the quantity

$$\frac{1}{\varepsilon^2}\Big(p_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon) - p_0\varrho_\varepsilon\vartheta_\varepsilon - \varepsilon\frac{a}{3}\overline{\vartheta}^4\Big) = \frac{1}{\varepsilon^2}\Big[\frac{\vartheta^{\frac52}}{\varepsilon^\alpha}P\Big(\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta^{\frac32}}\Big) - \frac{\vartheta^{\frac52}}{\varepsilon^\alpha}\varepsilon^\alpha P'(0)\frac{\varrho_\varepsilon}{\vartheta^{\frac32}}\Big]_{ess}$$
$$+\frac{1}{\varepsilon^2}\Big[\frac{\vartheta^{\frac52}}{\varepsilon^\alpha}P\Big(\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta^{\frac32}}\Big) - \frac{\vartheta^{\frac52}}{\varepsilon^\alpha}\varepsilon^\alpha P'(0)\frac{\varrho_\varepsilon}{\vartheta^{\frac32}}\Big]_{res} + \frac{a}{3}\frac{\vartheta_\varepsilon^4 - \overline{\vartheta}^4}{\varepsilon}\,. \tag{4.74}$$

Since P is twice continuously differentiable, we deduce

$$\frac{1}{\varepsilon^2}\Big|\Big[\frac{\vartheta^{\frac52}}{\varepsilon^\alpha}P\Big(\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta^{\frac32}}\Big) - \frac{\vartheta^{\frac52}}{\varepsilon^\alpha}\varepsilon^\alpha P'(0)\frac{\varrho_\varepsilon}{\vartheta^{\frac32}}\Big]_{ess}\Big| \le c\varepsilon^{\alpha-2}\Big[\frac{\varrho_\varepsilon^2}{\vartheta_\varepsilon^{\frac12}}\Big]_{ess}\,, \tag{4.75}$$

which tends to zero uniformly on $(0\,,T)\times K$ since $\alpha > 2$. Now, it follows from the pressure estimate (4.68) that

$$\frac{1}{\varepsilon^2}\int_0^T\int_K [\varrho_\varepsilon]_{res}^{5/3+\gamma}\mathrm{d}x\,\mathrm{d}t \le cK\,. \tag{4.76}$$

In order to show that the second term at the right-hand side of (4.74) tends to zero, we use (4.17), (4.49), (4.76) to estimate

$$\frac{1}{\varepsilon^2}\int_0^T\int_K\Big|\Big[\frac{\vartheta^{\frac52}}{\varepsilon^\alpha}P\Big(\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta^{\frac32}}\Big) - \frac{\vartheta^{\frac52}}{\varepsilon^\alpha}\varepsilon^\alpha P'(0)\frac{\varrho_\varepsilon}{\vartheta^{\frac32}}\Big]_{res}\Big|\mathrm{d}x\,\mathrm{d}t \le c\frac{\varepsilon^{2\alpha/3}}{\varepsilon^2}\int_0^T\int_K[\varrho_\varepsilon]_{res}^{5/3}\mathrm{d}x\,\mathrm{d}t$$
$$\le c\frac{\varepsilon^{2\alpha/3}}{\varepsilon^2}\Big(\int_0^T\int_{K\cap\{0\le\varrho_\varepsilon\le L\}}[\varrho_\varepsilon]_{res}^{5/3}\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{K\cap\{\varrho_\varepsilon > L\}}[\varrho_\varepsilon]_{res}^{5/3}\mathrm{d}x\,\mathrm{d}t\Big)$$
$$\le c(K)(\varepsilon^{2\alpha/3}L^{5/3} + L^{-\gamma})\,,$$

from which we conclude

$$\frac{\varepsilon^{2\alpha/3}}{\varepsilon^2}[\varrho_\varepsilon]_{res}^{5/3} \to 0 \text{ in } L^1((0\,,T)\times K)\,. \tag{4.77}$$

Finally the residual part of the radiation pressure can be simply estimated by means of Holder inequality and (4.49), (4.54), (4.61)

$$\int_0^T\int_{\Omega'}|[\vartheta_\varepsilon^4 - \overline{\vartheta}^4]_{res}|\mathrm{d}x\,\mathrm{d}t \le c\int_0^T\int_{\Omega'}|\vartheta_\varepsilon - \overline{\vartheta}|([\vartheta_\varepsilon]_{res}^3 + [\overline{\vartheta}]_{res}^3)\mathrm{d}x\,\mathrm{d}t$$
$$c(\Omega')\|\vartheta_\varepsilon - \overline{\vartheta}\|_{L^2(0,T\,;L^4(K))}\text{ess}\sup_{t\in(0,T)}\Big(\|[\vartheta_\varepsilon]_{res}^3\|_{L^{\frac43}(K)} + \|[\overline{\vartheta}]_{res}^3\|_{L^{\frac43}(K)}\Big) \le c(\Omega')\varepsilon^{\frac74}\,, \tag{4.78}$$

where $\Omega' \subset \Omega$ is a bounded Lipschitz domain such that $\partial\Omega \subset \partial\Omega'$. In order to control its essential component, we first use the Poincarè inequality (see e.g. [11])

$$\begin{aligned}\|\vartheta_\varepsilon^{\frac{3}{2}} - \overline{\vartheta}^{\frac{3}{2}}\|^2_{L^2((0,T)\times\Omega')} &\leq c\Big[\|\sqrt{\vartheta_\varepsilon}\nabla\vartheta_\varepsilon\|^2_{L^2((0,T)\times\Omega')} + \Big(\int_0^T\int_{\partial\Omega}|\vartheta_\varepsilon^{\frac{3}{2}} - \overline{\vartheta}^{\frac{3}{2}}|\mathrm{d}S_x\,\mathrm{d}t\Big)^2\Big] \\ &\leq c\Big(\varepsilon^4 + \int_0^T\int_{\partial\Omega}|\vartheta_\varepsilon - \overline{\vartheta}|^2\mathrm{d}S_x\,\mathrm{d}t\int_0^T\int_{\partial\Omega}(\vartheta_\varepsilon + \overline{\vartheta})\mathrm{d}S_x\,\mathrm{d}t\Big)\,,\end{aligned} \tag{4.79}$$

where the constant c depends on the subdomain Ω'. Then the following simple inequality

$$c_1|[\vartheta_\varepsilon - \overline{\vartheta}]_{ess}| \leq |[\vartheta_\varepsilon^p - \overline{\vartheta}^p]_{ess}| \leq c_2|[\vartheta_\varepsilon - \overline{\vartheta}]_{ess}|\,,\ p > 0\,,$$

applied to (4.79), together with (4.44) implies

$$\|[\vartheta_\varepsilon^p - \overline{\vartheta}^p]_{ess}\|_{L^2((0,T)\times\Omega')} \leq c\|[\vartheta_\varepsilon^{\frac{3}{2}} - \overline{\vartheta}^{\frac{3}{2}}]_{ess}\|_{L^2((0,T)\times\Omega')} \leq c(p)\,\varepsilon^{1+\beta/2}\,,\ p > 0\,, \tag{4.80}$$

which together with (4.78) yields

$$\Big\|\frac{\vartheta_\varepsilon^4 - \overline{\vartheta}^4}{\varepsilon}\Big\|_{L^1((0,T)\times K)} \leq c(K)\varepsilon^{\min\{\frac{3}{4},\frac{\beta}{2}\}}\,. \tag{4.81}$$

Consequently summing up the estimates (4.75), (4.77), (4.81) we get the statement of the Lemma. □

Our next aim is to determine the limit of the driving force term which is contained in the following statement

Lemma 18 *Let $\Omega' \subset \Omega$ be a bounded Lipschitz subdomain, and $\varphi \in C_c^\infty([0,T)\times\Omega';\mathbb{R}^3)$ be a function such that $\mathrm{div}(\tilde{\varrho}\varphi) = 0$. Then*

$$\vartheta^{(2)}_{\Omega',\varepsilon} \to \vartheta^{(2)}_{\Omega'} \text{ weakly in } L^q(0,T;W^{1,q}(\Omega')) \text{ for a certain } q > 1\,, \tag{4.82}$$

where the function $\vartheta^{(2)}_{\Omega',\varepsilon}$ is defined as follows

$$\vartheta^{(2)}_{\Omega',\varepsilon} = \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2} - \frac{1}{|\Omega'|}\int_{\Omega'}\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2}\mathrm{d}x\,,$$

and it holds

$$\frac{1}{\varepsilon^2}\int_0^T\int_{\Omega'}\Big(p_0\varrho_\varepsilon\vartheta_\varepsilon\mathrm{div}\varphi + \varrho_\varepsilon\nabla F\cdot\varphi\Big)\mathrm{d}x\,\mathrm{d}t = p_0\int_0^T\int_{\Omega'}(\tilde{\varrho}\vartheta^{(2)}_{\Omega',\varepsilon} + \chi_\varepsilon)\mathrm{div}\varphi\mathrm{d}x\,\mathrm{d}t\,, \tag{4.83}$$

where $\chi_\varepsilon \to 0$ in $L^1((0,T)\times\Omega')$.

Proof: Let's rewrite the quantity

$$
\begin{aligned}
\frac{1}{\varepsilon^2}\int_0^T\int_{\Omega'}\Big(p_0\varrho_\varepsilon\vartheta_\varepsilon \mathrm{div}\varphi + \varrho_\varepsilon\nabla F\cdot\varphi\Big)\mathrm{d}x\,\mathrm{d}t &= \frac{p_0}{\varepsilon^2}\int_0^T\int_{\Omega'}\frac{\overline{\vartheta}}{\tilde{\varrho}}\varrho_\varepsilon \mathrm{div}(\tilde{\varrho}\varphi)\mathrm{d}x\,\mathrm{d}t \\
+\frac{p_0}{\varepsilon^2}\int_0^T\int_{\Omega'}(\varrho_\varepsilon-\tilde{\varrho})(\vartheta_\varepsilon-\overline{\vartheta})\mathrm{div}\varphi \mathrm{d}x\,\mathrm{d}t &+ p_0\int_0^T\int_{\Omega'}\tilde{\varrho}\vartheta^{(2)}_{\Omega',\varepsilon}\mathrm{div}\varphi \mathrm{d}x\,\mathrm{d}t \\
+\frac{p_0}{|\Omega'|}\int_0^T\Big(\int_{\Omega'}\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon^2}\mathrm{d}x\Big)&\int_{\Omega'}\tilde{\varrho}\mathrm{div}\varphi \mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
\tag{4.84}
$$

The first term on the right-hand side of (4.84) is equal to zero and after a simple manipulation

$$
\int_{\Omega'}\tilde{\varrho}\mathrm{div}\varphi \mathrm{d}x = \int_{\Omega'}\Big(1+\log\tilde{\varrho}\Big)\mathrm{div}(\tilde{\varrho}\varphi)\mathrm{d}x\,,
$$

we see that the last term also vanishes. A straightforward consequence of the estimates (4.48), (4.80) gives

$$
\Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{ess}\Big[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big]_{ess}\Big\|_{L^1((0,T)\times\Omega')} \le \varepsilon^{\beta/2}c(\Omega') \to 0\,. \tag{4.85}
$$

In addition, using (4.61) in combination with the continuous embedding $W^{1,2}\hookrightarrow L^6$ and interpolation inequality and finally applying the uniform bounds (4.49), (4.50), (4.55) we obtain

$$
\begin{aligned}
&\Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{res}\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)\Big\|_{L^1((0,T)\times\Omega')} \\
\le c(\Omega')\mathrm{ess}\sup_{t\in(0,T)}&\Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{res}\Big\|^{\frac{7}{12}}_{L^1(\Omega')}\Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{res}\Big\|^{\frac{5}{12}}_{L^{\frac{5}{3}}(\Omega')} \le \varepsilon^{\frac{1}{6}}c(\Omega')\to 0\,.
\end{aligned}
\tag{4.86}
$$

Thus we proved that the second term on the right-hand side of (4.84) converges to zero which complete the proof of (4.83). In order to show (4.82), let's write

$$
\sqrt{\overline{\vartheta}}\nabla\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon^2} = \frac{\sqrt{\overline{\vartheta}}-\sqrt{\vartheta_\varepsilon}}{\varepsilon}\nabla\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon} + \sqrt{\vartheta_\varepsilon}\nabla\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon^2}\,,
$$

where, by virtue of (4.46), (4.54), (4.62), and the embedding $W^{1,2}\hookrightarrow L^6$,

$$
\Big\{\frac{\sqrt{\overline{\vartheta}}-\sqrt{\vartheta_\varepsilon}}{\varepsilon}\Big\}_{\varepsilon>0}\ \text{is bounded in}\ L^\infty(0,T;L^1(\Omega'))\cap L^2(0,T;L^6(\Omega'))\,.
$$

Consequently, by means of (4.57), (4.61), and interpolation inequality, we have

$$
\Big\|\nabla\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon^2}\Big)\Big\|_{L^q(0,T;L^q(\Omega';\mathbb{R}^3))} \le c\ \text{for a certain}\ q>1\,, \tag{4.87}
$$

which proves (4.82). □

Remark: It is easy to see that if we take two domains Ω_1 , Ω_2 in the previous lemma then $\nabla\vartheta^{(2)}_{\Omega_1} = \nabla\vartheta^{(2)}_{\Omega_2}$ in $\Omega_1 \cap \Omega_2$. Thus we can define the vector function $\Phi \in L^1(0\,,T\,;L^1_{loc}(\Omega\,;\mathbb{R}^3))$ such that $\Phi = \nabla\vartheta^{(2)}_{\Omega'}$ at Ω' .

At this stage, we can use all obtained limits together with the previous lemmas, in order to let $\varepsilon \to 0$ in the momentum equation (4.26). We thereby obtain

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\tilde{\varrho}\vec{U}\cdot\varphi + \overline{\varrho\vec{U}\otimes\vec{U}} : \nabla\varphi\Big)\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega \Big(\mathbb{S} : \nabla\varphi - \frac{1}{\overline{\vartheta}}\tilde{\varrho}F\Phi\cdot\varphi\Big)\mathrm{d}x\,\mathrm{d}t - \int_\Omega \tilde{\varrho}\vec{U}_0\varphi(0\,,\cdot)\mathrm{d}x\,, \end{aligned} \tag{4.88}$$

for any $\varphi \in C^\infty_c([0\,,T)\times\overline{\Omega}\,;\mathbb{R}^3)$, $\varphi\cdot\vec{n}|_{\partial\Omega} = 0$, $\mathrm{div}(\tilde{\varrho}\varphi) = 0$, where

$$\mathbb{S} = \mu_1\overline{\vartheta}\Big(\nabla\vec{U} + \nabla^T\vec{U} - \frac{2}{3}\mathrm{div}\vec{U}\mathbb{I}\Big)\,, \tag{4.89}$$

and $\overline{\varrho\vec{U}\otimes\vec{U}}$ is a weak limit of $\{\varrho_\varepsilon\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon\}_{\varepsilon>0}$.

4.3.3 Asymptotic limit in entropy balance

This section of the paper will be concluded by the analysis of the entropy equation (4.28). First, let's observe that due to (4.43), (4.44)

$$< \vartheta_\varepsilon\,;\varphi >_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} - \int_0^T \int_{\partial\Omega} \beta_1 \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^\beta}\varphi\mathrm{d}S_x\mathrm{d}t \to 0 \text{ as } \varepsilon\to 0\,, \tag{4.90}$$

for $\varphi \in C^\infty_c((0\,,T)\times\overline{\Omega})$. Similarly, by virtue of (4.61), (4.63),

$$\varepsilon^{2\alpha/3}\kappa_0\nabla\log\vartheta_\varepsilon + \kappa_1\nabla\vartheta_\varepsilon \to 0 \text{ in } L^2((0\,,T)\times K\,;\mathbb{R}^3)\,,$$

for any compact set $K\subset\Omega$, which implies

$$\begin{aligned} -\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{1}{\varepsilon^2}\frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla\varphi\mathrm{d}x\,\mathrm{d}t &= \lim_{\varepsilon\to 0}\int_0^T\int_\Omega d\vartheta_\varepsilon^2\nabla\vartheta^{(2)}_{\Omega',\varepsilon}\cdot\nabla\varphi\mathrm{d}x\,\mathrm{d}t \\ = \lim_{\varepsilon\to 0}\int_0^T\int_\Omega d\Big([\vartheta_\varepsilon]^2_{ess}\nabla\vartheta^{(2)}_{\Omega',\varepsilon} &+ [\vartheta_\varepsilon]^{3/2}_{res}\sqrt{\vartheta_\varepsilon}\nabla\Big(\frac{\vartheta_\varepsilon}{\varepsilon^2}\Big)\Big)\cdot\nabla\varphi\mathrm{d}x\,\mathrm{d}t \\ &= d\overline{\vartheta}^2\int_0^T\int_\Omega \Phi\cdot\nabla\varphi\mathrm{d}x\,\mathrm{d}t\,, \end{aligned}$$

for any $\varphi \in C_c^\infty((0\,,T) \times \overline{\Omega})$, where we have used (4.73), (4.82) for the first term and (4.54), (4.57) for the second one.

Finally, in order to handle the convective term in (4.28) let's decompose $\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon)$ as follows

$$\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon) = \varrho_\varepsilon\Big[S\Big(\varepsilon^\alpha \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big) - S(\varepsilon^\alpha)\Big] + \varepsilon\frac{4}{3}a\vartheta_\varepsilon^3 := \varrho_\varepsilon s_{M,\varepsilon}(\varrho_\varepsilon\,,\vartheta_\varepsilon) + \varrho_\varepsilon s_{R,\varepsilon}(\varrho_\varepsilon\,,\vartheta_\varepsilon)\,,$$

where in accordance with (4.52),

$$[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{res} \to 0 \text{ in } L^1((0\,,T) \times K) \tag{4.91}$$

for any compact set $K \subset \Omega$. Similarly, by virtue of (4.54), (4.70),

$$\varrho_\varepsilon s_{R,\varepsilon}(\varrho_\varepsilon\,,\vartheta_\varepsilon) \to 0 \text{ in } L^\infty(0\,,T\,; L^{\frac{4}{3}}_{loc}(\Omega))\,, \tag{4.92}$$

$$\varrho_\varepsilon s_{R,\varepsilon}(\varrho_\varepsilon\,,\vartheta_\varepsilon)\vec{u}_\varepsilon \to 0 \text{ in } L^2(0\,,T\,; L^{\frac{12}{11}}_{loc}(\Omega\,;\mathbb{R}^3))\,. \tag{4.93}$$

On the other hand due to (4.14), (4.17)

$$|\varrho_\varepsilon s_{M,\varepsilon}(\varrho_\varepsilon\,,\vartheta_\varepsilon)| \le \Big|\varrho_\varepsilon \int_{\varepsilon^\alpha}^{\varepsilon^\alpha \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{3/2}}} S'(Z)\mathrm{d}Z\Big| \le c\varrho_\varepsilon(|\log \varrho_\varepsilon| + |\log \vartheta_\varepsilon|)\,.$$

Consequently, using the uniform estimates established in (4.42), (4.50), (4.55), and (4.63), we get

$$[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{res}\vec{u}_\varepsilon \to 0 \text{ in } L^q((0\,,T) \times K) \tag{4.94}$$

for any compact set $K \subset \Omega$, and a certain $q > 1$. It remains to determine the limit of $[\varrho_\varepsilon s_{M,\varepsilon}(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{ess}$. To this end, write

$$S(Z) = -\log Z + \tilde{S}(Z)\,,$$

where

$$\tilde{S}'(Z) = -\frac{3}{2}\frac{\frac{5}{3}(P(Z) - p_0 Z) - (P'(Z) - p_0)Z}{Z^2}\,.$$

Since P is twice continuously differentiable and satisfies (4.17), we have $|\tilde{S}'(Z)| \le c$, from which we obtain

$$[\varrho_\varepsilon s_{M,\varepsilon}(\varrho_\varepsilon\,,\vartheta_\varepsilon)]_{ees} \to p_0\tilde{\varrho}\Big(\frac{3}{2}\log\overline{\vartheta} - \log\tilde{\varrho}\Big) \text{ in } L^q((0\,,T) \times K) \tag{4.95}$$

for any compact set $K \subset \Omega$, and a certain $1 \le q < \infty$.

Now passing to the limit for $\varepsilon \to 0$ in (4.28) and resuming relations (4.90) - (4.95) together with (4.72) and (4.70) we conclude

$$-d\overline{\vartheta}^2 \int_0^T \int_\Omega \Phi \cdot \nabla\varphi \mathrm{d}x\,\mathrm{d}t = p_0 \int_0^T \int_\Omega \tilde{\varrho} \log \tilde{\varrho} \vec{U} \cdot \nabla\varphi \mathrm{d}x\,\mathrm{d}t \tag{4.96}$$

for any $\varphi \in C_c^\infty((0\,,T) \times \overline{\Omega})$ which is a weak formulation of (4.32).

4.4 Analysis of acoustic waves

4.4.1 Acoustic equation

In order to complete the proof of Theorem 4.1.1, it remains to handle the term $\overline{\varrho\vec{U} \otimes \vec{U}}$, which will be investigated in the rest of the paper.

Taking $\varphi/\tilde{\varrho}$ as a test function in (4.25) we obtain

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon\tilde{\varrho}} \partial_t\varphi + \tilde{\varrho} \frac{\varrho_\varepsilon \vec{u}_\varepsilon}{\tilde{\varrho}} \cdot \nabla \frac{\varphi}{\tilde{\varrho}} \Big) \mathrm{d}x\,\mathrm{d}t = -\int_{\Omega_\varepsilon} \varepsilon \frac{\varrho_{0,\,\varepsilon} - \tilde{\varrho}}{\varepsilon\tilde{\varrho}} \varphi(0\,,\cdot)\mathrm{d}x \tag{4.97}$$

to be satisfied for any $\varphi \in C_c^\infty([0\,,T) \times \overline{\Omega}_\varepsilon)$. Similarly, the momentum equation (4.26) gives rise to

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon\varrho_\varepsilon \vec{u}_\varepsilon \cdot \partial_t\varphi + p_0\overline{\vartheta} \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon\tilde{\varrho}} \mathrm{div}(\tilde{\varrho}\varphi) \Big) \mathrm{d}x\,\mathrm{d}t = -\int_{\Omega_\varepsilon} \varepsilon\varrho_{0,\,\varepsilon} \vec{u}_{0,\,\varepsilon} \cdot \varphi(0\,,\cdot)\mathrm{d}x$$
$$+ \int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon h_\varepsilon \mathrm{div}\varphi + \varepsilon \mathbb{G}_\varepsilon : \nabla\varphi - p_0 \frac{\overline{\vartheta} - \vartheta_\varepsilon}{\varepsilon} \nabla\tilde{\varrho} \cdot \varphi - p_0 \tilde{\varrho} \nabla \Big(\frac{\overline{\vartheta} - \vartheta_\varepsilon}{\varepsilon} \Big) \cdot \varphi \Big) \mathrm{d}x\,\mathrm{d}t\,, \tag{4.98}$$

for any $\varphi \in C_c^\infty([0\,,T) \times \overline{\Omega}_\varepsilon\,;\mathbb{R}^3)\,, \varphi \cdot \vec{n}|_{\partial\Omega_\varepsilon} = 0$, where

$$h_\varepsilon = \frac{1}{\varepsilon^2} \Big(p_0 \varrho_\varepsilon \vartheta_\varepsilon - p_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon) \Big) + p_0 \Big(\frac{\tilde{\varrho} - \varrho_\varepsilon}{\varepsilon} \Big) \Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big)\,,$$

and

$$\mathbb{G}_\varepsilon = \mathbb{S}_\varepsilon - \varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon\,.$$

The term h_ε may be rewritten such as

$$h_\varepsilon = \frac{1}{\varepsilon^2} \Big(p_0 \varrho_\varepsilon \vartheta_\varepsilon - p_\varepsilon(\varrho_\varepsilon\,,\vartheta_\varepsilon) \Big) + p_0 \Big[\frac{\tilde{\varrho} - \varrho_\varepsilon}{\varepsilon} \Big]_{ess} \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{ess}$$
$$+ \frac{\tilde{\varrho}[1]_{res}}{\varepsilon} \Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big) - \frac{[\varrho_\varepsilon\vartheta_\varepsilon]_{res}}{\varepsilon^2} + \frac{\overline{\vartheta}[\varrho_\varepsilon]_{res}}{\varepsilon^2}\,,$$

where on the right hand side we apply the uniform estimates (4.48), (4.49), (4.50), (4.55), (4.59), (4.61), (4.74), (4.75) to get

$$\|h_\varepsilon\|_{L^1(0,T;\,L^1(\Omega_\varepsilon))} \leq c\,.$$

By virtue of (4.42), (4.54), (4.55), (4.60), we get that the norms of $\mathbb{G}_\varepsilon$ are bounded in $L^q(0\,,T\,;L^q(\Omega_\varepsilon\,;\mathbb{R}^3))$ uniformly with respect to ε, for a certain $q > 1$. In addition, relation (4.80) implies

$$\Big\| \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{ess} \Big\|_{L^2((0,T)\times K\cap\Omega_\varepsilon)} \leq \varepsilon^{\beta/2} c(K)\,,$$

for each compact set $K \subset \Omega$, and (4.49), together with (4.61), give rise to

$$\Big\| \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{res} \Big\|_{L^1((0,T)\times\Omega_\varepsilon)} \leq \varepsilon c\,.$$

Finally, using (4.57), (4.58) and Holder inequality we can see

$$\Big\| \nabla \Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big) \Big\|^2_{L^2(0,T;\,L^2(\Omega_\varepsilon))} \leq \varepsilon c\,.$$

Thus, when we introduce the quantities

$$r_\varepsilon = \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon \tilde{\varrho}}\,,\ \vec{V}_\varepsilon = \varrho_\varepsilon \vec{u}_\varepsilon\,,\ r_{0,\,\varepsilon} = r_\varepsilon(0\,,\cdot)\,,\ \vec{V}_{0,\,\varepsilon} = \vec{V}_\varepsilon(0\,,\cdot)\,,$$

and we rewrite the last integral of (4.98) according to above shown estimates, the system (4.97), (4.98) reads as follows

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon r_\varepsilon \partial_t \varphi + \vec{V}_\varepsilon \cdot \nabla \Big(\frac{\varphi}{\tilde{\varrho}} \Big) \Big) \mathrm{d}x\,\mathrm{d}t = - \int_{\Omega_\varepsilon} \varepsilon r_{0,\,\varepsilon} \varphi(0\,,\cdot) \mathrm{d}x \tag{4.99}$$

for any $\varphi \in C_c^\infty([0\,,T) \times \overline{\Omega}_\varepsilon)$.

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon \vec{V}_\varepsilon \partial_t \varphi + p_0 \overline{\vartheta} r_\varepsilon \mathrm{div}(\tilde{\varrho}\varphi) \Big) \mathrm{d}x\,\mathrm{d}t &= - \int_{\Omega_\varepsilon} \varepsilon \vec{V}_{0,\,\varepsilon} \varphi(0\,,\cdot) \mathrm{d}x \\ + \varepsilon^{\beta/2} \int_0^T \int_{\Omega_\varepsilon} &\Big((\mathbb{F}^1_\varepsilon + \mathbb{F}^2_\varepsilon) : \nabla\varphi + (\mathbf{F}^1_\varepsilon + \mathbf{F}^2_\varepsilon) \cdot \varphi \Big) \mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{4.100}$$

for any $\varphi \in C_c^\infty([0\,,T)\times\overline{\Omega}_\varepsilon\,;\mathbb{R}^3)\,, \varphi\cdot\vec{n}|_{\partial\Omega_\varepsilon} = 0\,,$ where due to above consideration

$$\begin{aligned} &\{r_{0,\,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega_\varepsilon)\,,\\ &\{\vec{V}_{0,\,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega_\varepsilon\,;\mathbb{R}^3)\,,\\ &\{\mathbb{F}^1_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0\,,T\,;L^1(\Omega_\varepsilon\,;\mathbb{R}^{3\times3}))\,,\\ &\{\mathbb{F}^2_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0\,,T\,;L^2(\Omega_\varepsilon\,;\mathbb{R}^{3\times3}))\,,\\ &\{\mathbf{F}^1_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0\,,T\,;L^1(\Omega_\varepsilon\,;\mathbb{R}^{3}))\,,\\ &\{\mathbf{F}^2_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0\,,T\,;L^2(\Omega_\varepsilon\,;\mathbb{R}^{3}))\,. \end{aligned} \tag{4.101}$$

Similarly uniform bounds (4.48), (4.49) imply that

$$r_\varepsilon = r^1_\varepsilon + r^2_\varepsilon\,, \tag{4.102}$$

with

$$\begin{aligned} &\{r^1_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0\,,T\,;L^1(\Omega_\varepsilon))\,,\\ &\{r^2_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0\,,T\,;L^2(\Omega_\varepsilon))\,, \end{aligned} \tag{4.103}$$

and (4.42), (4.49), (4.56) give rise to

$$\vec{V}_\varepsilon = \vec{V}^1_\varepsilon + \vec{V}^2_\varepsilon\,, \tag{4.104}$$

with

$$\begin{aligned} &\{\vec{V}^1_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0\,,T\,;L^1(\Omega_\varepsilon\,;\mathbb{R}^3))\,,\\ &\{\vec{V}^2_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0\,,T\,;L^2(\Omega_\varepsilon\,;\mathbb{R}^3))\,. \end{aligned} \tag{4.105}$$

The system (4.99), (4.100) is called stratified acoustic equation.

4.4.2 Regularization and extension

In order to facilitate the future analysis of the acoustic equation (4.99), (4.100), we regularize it and extend it to the whole domain Ω. By virtue of (4.1), any solution of extended system will coincide with the original one on any compact set K as the speed of sound is proportional to $1/\varepsilon$.

For a fixed $\varepsilon > 0$, there exist families of smooth functions $\{r_{0,\,\varepsilon,\,\delta}\}_{\delta>0} \subset C_c^\infty(\overline{\Omega}_\varepsilon)$, $\{\vec{V}_{0,\,\varepsilon,\,\delta}\}_{\delta>0} \subset C_c^\infty(\overline{\Omega}_\varepsilon\,;\mathbb{R}^3)$, such that

$$\{r_{0,\,\varepsilon,\,\delta}\}_{\varepsilon,\,\delta>0} \text{ bounded in } L^2(\Omega)\,, \tag{4.106}$$

$$\{\vec{V}_{0,\,\varepsilon,\,\delta}\}_{\varepsilon,\,\delta>0} \text{ bounded in } L^2(\Omega\,;\mathbb{R}^3)\,, \tag{4.107}$$

and in addition

$$\int_{\Omega} r_{0,\varepsilon,\delta}\varphi \mathrm{d}x \to \int_{\Omega_\varepsilon} r_{0,\varepsilon}\varphi \mathrm{d}x \text{ for any } \varphi \in C_c^\infty(\overline{\Omega}_\varepsilon)\,,$$
$$\int_{\Omega} \vec{V}_{0,\varepsilon,\delta}\cdot\varphi \mathrm{d}x \to \int_{\Omega_\varepsilon} \vec{V}_{0,\varepsilon}\cdot\varphi \mathrm{d}x \text{ for any } \varphi \in C_c^\infty(\overline{\Omega}_\varepsilon\,;\mathbb{R}^3)\,,$$

as $\delta \to 0$.

Similarly, we can find

$$\left\{ \begin{array}{c} \mathbf{F}_{\varepsilon,\delta} = \mathbf{F}^1_{\varepsilon,\delta} + \mathbf{F}^2_{\varepsilon,\delta}\,, \mathbf{F}^i_{\varepsilon,\delta} \in C_c^\infty((0\,,T)\times\Omega_\varepsilon\,;\mathbb{R}^3)\,, i=1\,,2\,, \\ \mathbb{F}_{\varepsilon,\delta} = \mathbb{F}^1_{\varepsilon,\delta} + \mathbb{F}^2_{\varepsilon,\delta}\,, \mathbb{F}^i_{\varepsilon,\delta} \in C_c^\infty((0\,,T)\times\Omega_\varepsilon\,;\mathbb{R}^{3\times 3})\,, i=1\,,2\,, \end{array} \right\}$$

such that

$$\left\{ \begin{array}{l} \{\mathbf{F}^1_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^2(0\,,T\,;L^1(\Omega\,;\mathbb{R}^3))\,, \\ \{\mathbf{F}^2_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^2(0\,,T\,;L^2(\Omega\,;\mathbb{R}^3))\,, \end{array} \right\} \tag{4.108}$$

$$\left\{ \begin{array}{l} \{\mathbb{F}^1_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^2(0\,,T\,;L^1(\Omega\,;\mathbb{R}^{3\times 3}))\,, \\ \{\mathbb{F}^2_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^2(0\,,T\,;L^2(\Omega\,;\mathbb{R}^{3\times 3}))\,, \end{array} \right\} \tag{4.109}$$

and

$$\mathbf{F}^1_{\varepsilon,\delta} \to \mathbf{F}^1_\varepsilon \text{ in } L^2(0\,,T\,;L^1(\Omega_\varepsilon\,;\mathbb{R}^3))\,,\ \mathbf{F}^2_{\varepsilon,\delta} \to \mathbf{F}^2_\varepsilon \text{ in } L^2(0\,,T\,;L^2(\Omega_\varepsilon\,;\mathbb{R}^3))\,,$$
$$\mathbb{F}^1_{\varepsilon,\delta} \to \mathbb{F}^1_\varepsilon \text{ in } L^2(0\,,T\,;L^1(\Omega_\varepsilon\,;\mathbb{R}^{3\times 3}))\,,\ \mathbb{F}^2_{\varepsilon,\delta} \to \mathbb{F}^2_\varepsilon \text{ in } L^2(0\,,T\,;L^2(\Omega_\varepsilon\,;\mathbb{R}^{3\times 3}))\,,$$

for any $\varphi \in C_c^\infty([0\,,T]\times\overline{\Omega}_\varepsilon)$, as $\delta \to 0$.

Now, consider the solution $r_{\varepsilon,\delta}$, $\vec{V}_{\varepsilon,\delta}$ of the initial-boundary value problem

$$\varepsilon\partial_t r_{\varepsilon,\delta} + \frac{1}{\tilde{\varrho}}\mathrm{div}\vec{V}_{\varepsilon,\delta} = 0 \text{ in } (0\,,T)\times\Omega_\varepsilon\,, \tag{4.110}$$

$$\varepsilon\partial_t\vec{V}_{\varepsilon,\delta} + p_0\overline{\vartheta}\tilde{\varrho}\nabla r_{\varepsilon,\delta} = \varepsilon^{\beta/2}\mathrm{div}\mathbb{F}_{\varepsilon,\delta} - \varepsilon^{\beta/2}\mathbf{F}_{\varepsilon,\delta} \text{ in } (0\,,T)\times\Omega_\varepsilon\,, \tag{4.111}$$

$$\vec{V}_{\varepsilon,\delta}\cdot\vec{n}|_{\partial\Omega_\varepsilon} = 0\,, \tag{4.112}$$

$$r_{\varepsilon,\delta}(0\,,\cdot) = r_{0,\varepsilon,\delta}\,,\ \ \vec{V}_{\varepsilon,\delta}(0\,,\cdot) = \vec{V}_{0,\varepsilon,\delta}\,. \tag{4.113}$$

Keeping $\varepsilon > 0$ fixed and letting $\delta \to 0$ we check that

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left| \int_{\Omega_\varepsilon} (\vec{V}_\varepsilon - \vec{V}_{\varepsilon,\delta})(t\,,\cdot)\cdot\varphi \mathrm{d}x \right| \to 0 \text{ as } \delta \to 0 \tag{4.114}$$

for any $\varphi \in C_c^\infty(K\,;\mathbb{R}^3)$, and any compact set $K \subset \Omega$.

System (4.110), (4.111) admits a finite speed of propagation of order $\sqrt{p_0\overline{\vartheta}}/\varepsilon$. This can be easily proved by multiplying equation (4.110) by $p_0\overline{\vartheta}\tilde{\varrho}r_{\varepsilon,\delta}$, taking the scalar product of (4.111) with $\vec{V}_{\varepsilon,\delta}/\tilde{\varrho}$, and integrating the resulting expression over the set

$$\left\{(t,x)\,\middle|\,t\in[0,\tau],\ x\in\Omega_\varepsilon,\ |x|<r-\frac{\sqrt{p_0\overline{\vartheta}}}{\varepsilon}t\right\},$$

where $r>0$. Consequently, by virtue of (4.1), we may assume, extending the data in (4.110) - (4.113) by zero outside of Ω_ε, that $r_{\varepsilon,\delta}$, $\vec{V}_{\varepsilon,\delta}$ are smooth, compactly supported in $[0,T]\times\overline{\Omega}$, and solve the acoustic equation (4.110) - (4.113) in $(0,T)\times\Omega$.

4.4.3 Strong convergence of speed and compactness of the solenoidal part

As it was mentioned at the beginning of this section, we have to deal with the term $\overline{\varrho\vec{U}\otimes\vec{U}}$. Throughout this section we will suppose $K\subset\overline{\Omega}$ is a compact set. If we show

$$\vec{u}_\varepsilon\to\vec{U}\ \text{strongly in}\ L^2((0,T)\times K), \tag{4.115}$$

then

$$\int_0^T\int_{\Omega_\varepsilon}\varrho_\varepsilon[\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon]:\nabla\varphi\mathrm{d}x\,\mathrm{d}t\to\int_0^T\int_\Omega\tilde{\varrho}[\vec{U}\otimes\vec{U}]:\nabla\varphi\mathrm{d}x\,\mathrm{d}t, \tag{4.116}$$

for any $\varphi\in C_c^\infty((0,T)\times\overline{\Omega};\mathbb{R}^3)$, $\operatorname{div}(\tilde{\varrho}\varphi)=0$, $\varphi\cdot\vec{n}|_{\partial\Omega}=0$. (4.115) will be true, as soon as we show

$$\left[t\to\int_{\Omega_\varepsilon}\vec{V}_\varepsilon(t,\cdot)\cdot\vec{w}\mathrm{d}x\right]\to\left[t\to\int_\Omega\vec{V}(t,\cdot)\cdot\vec{w}\mathrm{d}x\right]\ \text{in}\ L^2(0,T), \tag{4.117}$$

for any $\vec{w}\in C_c^\infty(\Omega;\mathbb{R}^3)$, where $\vec{V}=\tilde{\varrho}\vec{U}$.

Indeed, relation (4.117) together with uniform bounds (4.48), (4.49), (4.50), (4.60) and the standard embedding relation $W^{1,2}(\Omega)\hookrightarrow L^6(K)$ gives rise

$$\left[t\to\int_{\Omega_\varepsilon}\vec{u}_\varepsilon(t,\cdot)\cdot\vec{w}\mathrm{d}x\right]\to\left[t\to\int_\Omega\vec{U}(t,\cdot)\cdot\vec{w}\mathrm{d}x\right]\ \text{in}\ L^2(0,T), \tag{4.118}$$

for any $\vec{w}\in C_c^\infty(\Omega;\mathbb{R}^3)$. But (4.118) together with (4.70) and compactness of the embedding $W^{1,2}(\Omega)\hookrightarrow L^2(K)$, yields the desired convergence (4.115).

This part will be concluded by an observation that the solenoidal part (in the sense of weighted Helmholtz decomposition) of the vector field $\vec{V}_\varepsilon$ is weakly compact in time, specifically,

$$\Big[t \to \int_{\Omega_\varepsilon} \frac{1}{\tilde{\varrho}} \vec{V}_\varepsilon(t\,,\cdot) \cdot \vec{w} \mathrm{d}x\Big] \to \Big[t \to \int_{\Omega} \frac{1}{\tilde{\varrho}} \vec{V}(t\,,\cdot) \cdot \vec{w} \mathrm{d}x\Big] \text{ in } C[0\,,T]\,, \tag{4.119}$$

for any $\vec{w} \in C_c^\infty(\Omega\,;\mathbb{R}^3)$, $\mathrm{div}\vec{w} = 0$, as a direct consequence of Lemma 17, Lemma 18 and (4.69), (4.70) applied to the momentum equation (4.26).

4.4.4 Compactness of the gradient part

In view of (4.119) it remains s to establish compactness of the gradient part of the vector filed $\vec{V}_\varepsilon$. Then the relation (4.117) will be true and the proof of Theorem 4.1.1 will be completed. But with respect to the fact (4.114), it is enough to show (4.117) for the gradient part $\mathbf{H}^\perp_{\tilde{\varrho}}[\vec{V}_{\varepsilon,\,\delta(\varepsilon)}]$ for $\delta(\varepsilon)$ small enough. In what follows, we drop the subscript δ and replace the weak formulation of the acoustic equation (4.99), (4.100) by its classical counterpart (4.110), (4.111), supplemented by (4.112), (4.113) in $(0\,,T) \times \Omega$.

To this end, we apply the Helmholtz projection to equation (4.111), specifically, we consider test function in the form $\varphi = \nabla\Psi$, with

$$\mathrm{div}(\tilde{\varrho}\nabla\Psi) = \tilde{\varrho}\chi \in C_c^\infty([0\,,T) \times \overline{\Omega})\,,\ \nabla\Psi \cdot \vec{n}|_{\partial\Omega} = 0\,,\ \Psi(t\,,\cdot) \in \mathcal{D}^{1,\,2}(\Omega) \tag{4.120}$$

to obtain

$$\begin{aligned} &\int_0^T \int_\Omega \tilde{\varrho}\Big(- \varepsilon\phi_\varepsilon\partial_t\chi + r_\varepsilon\chi\Big)\mathrm{d}x\,\mathrm{d}t \\ = \varepsilon \int_\Omega \tilde{\varrho}\phi_{0,\,\varepsilon}\chi(0\,,\cdot)\mathrm{d}x &+ \varepsilon^{\beta/2} \int_0^T \int_\Omega \Big(\mathbb{F}^1_\varepsilon + \mathbb{F}^2_\varepsilon\Big) : \nabla^2\Psi + \Big(\mathbf{F}^1_\varepsilon + \mathbf{F}^2_\varepsilon\Big) \cdot \nabla\Psi \mathrm{d}x\,\mathrm{d}t\,, \end{aligned} \tag{4.121}$$

where

$$\nabla\phi_\varepsilon = \frac{1}{\tilde{\varrho}} \mathbf{H}^\perp_{\tilde{\varrho}}[\vec{V}_\varepsilon]\,. \tag{4.122}$$

Accordingly, equation (4.110) can be rewritten as

$$\int_0^T \int_\Omega \Big(\varepsilon r_\varepsilon\partial_t\varphi + p_0\overline{\vartheta}\tilde{\varrho}\nabla\phi_\varepsilon \cdot \nabla\Big(\frac{\varphi}{\tilde{\varrho}}\Big)\Big)\mathrm{d}x\,\mathrm{d}t = -\varepsilon \int_\Omega r_{0,\,\varepsilon}\varphi(0\,,\cdot)\mathrm{d}x \tag{4.123}$$

for any $\varphi \in C_c^\infty([0\,,T) \times \overline{\Omega})$.

We now rewrite the system (4.121), (4.123) in terms of a single differential operator

$$\Delta_{\tilde{\varrho},\,N}[v] = \frac{p_0\overline{\vartheta}}{\tilde{\varrho}}\mathrm{div}(\tilde{\varrho}\nabla v)\,,$$

defined in weighted Lebesgue space $L^2_{\tilde{\varrho}}(\Omega)$

$$\mathcal{D}(\Delta_{\tilde{\varrho},\,N}) = \{w \in L^2_{\tilde{\varrho}}(\Omega)\,|\,w \in W^{2,\,2}(\Omega)\,,\ \nabla w\cdot\vec{n}|_{\partial\Omega} = 0\}\,.$$

Since $\tilde{\varrho}(x) = \varrho_\infty$ for $|x|$ large enough, we can show that $-\Delta_{\tilde{\varrho},\,N}$ is a self-adjoint, non-negative operator in $L^2_{\tilde{\varrho}}(\Omega)$, with an absolutely continuous spectrum $[0\,,\infty)$, and satisfies the limiting absorption principle

$$\sup_{\lambda\in\mathbb{C},\,0<\alpha\le\mathrm{Re}[\lambda]\le\beta<\infty,\,\mathrm{Im}[\lambda]\neq 0}\|V\circ(-\Delta_{\tilde{\varrho},\,N}-\lambda)^{-1}\circ V\|_{\mathcal{L}[L^2_{\tilde{\varrho}}(\Omega);\,L^2_{\tilde{\varrho}}(\Omega)]} \le c_{\alpha,\,\beta}\,, \tag{4.124}$$

where $V(x) = (1+|x|^2)^{-\frac{s}{2}}$, $s>1$, see [28].

Furthermore, from (4.120) we can see

$$\Delta_{\tilde{\varrho},\,N}[\Psi] = p_0\overline{\vartheta}\chi\,. \tag{4.125}$$

We claim that the mapping

$$\chi \mapsto \int_\Omega \Big(\mathbb{F}^1_\varepsilon + \mathbb{F}^2_\varepsilon\Big) : \nabla^2\Psi + \Big(\mathbf{F}^1_\varepsilon + \mathbf{F}^2_\varepsilon\Big)\cdot\nabla\Psi\mathrm{d}x\,,$$

where Ψ, χ are interrelated through (4.125), is a bounded linear functional for

$$\chi \in \mathcal{D}(-\Delta_{\tilde{\varrho},\,N}) \cap \mathcal{D}\Big(\frac{1}{\sqrt{-\Delta_{\tilde{\varrho},\,N}}}\Big)\,, \tag{4.126}$$

the norm of which can be estimated in terms of $\|\mathbb{F}^1_\varepsilon\|_{L^1(\Omega;\,\mathbb{R}^{3\times 3})}$, $\|\mathbb{F}^2_\varepsilon\|_{L^2(\Omega;\,\mathbb{R}^{3\times 3})}$, $\|\mathbf{F}^1_\varepsilon\|_{L^1(\Omega;\,\mathbb{R}^3)}$, $\|\mathbf{F}^2_\varepsilon\|_{L^2(\Omega;\,\mathbb{R}^3)}$. This can be proved by an observation that for χ specified in (4.126), the function Ψ defined trough (4.125) has two derivatives bounded in $L^2\cap L^\infty(\Omega)$.

Thus using the standard Riezs representation theorem, the system (4.121) may be rewritten in the form

$$\begin{aligned}\int_0^T (-\varepsilon < \Phi_\varepsilon\,;\partial_t\chi >_{\tilde{\varrho}} + < r_\varepsilon\,;\chi >_{\tilde{\varrho}})\mathrm{d}t = \varepsilon < \Phi_{0,\,\varepsilon}\,;\chi(0\,,\cdot) >_{\tilde{\varrho}} \\ + \varepsilon^{\beta/2}\int_0^T \Big(< h^1_\varepsilon\,;\Delta_{\tilde{\varrho},\,N}[\chi] >_{\tilde{\varrho}} + < h^2_\varepsilon\,;\frac{1}{\sqrt{-\Delta_{\tilde{\varrho},\,N}}}[\chi] >_{\tilde{\varrho}}\Big)\mathrm{d}t\end{aligned} \tag{4.127}$$

for any test function

$$\chi \in C_c^\infty\Big([0\,,T);\,\mathcal{D}(\Delta_{\tilde\varrho,\,N})\cap\mathcal{D}\Big(\frac{1}{\sqrt{-\,\Delta_{\tilde\varrho,\,N}}}\Big)\Big)\,,$$

with

$$\{h_\varepsilon^1\}_{\varepsilon>0}\,,\{h_\varepsilon^2\}_{\varepsilon>0}\ \text{bounded in}\ L^2((0\,,T)\times\Omega)\,. \tag{4.128}$$

Similarly, equation (4.123) reads

$$\begin{aligned}\int_0^T \Big(\varepsilon < r_\varepsilon\,;\partial_t\chi >_{\tilde\varrho} + < \sqrt{-\,\Delta_{\tilde\varrho,\,N}}[\Phi_\varepsilon]\,;\sqrt{-\,\Delta_{\tilde\varrho,\,N}}[\chi] >_{\tilde\varrho}\Big)\mathrm{d}t\\ = -\varepsilon < r_{0,\,\varepsilon}\,;\chi(0\,,\cdot) >_{\tilde\varrho}\end{aligned} \tag{4.129}$$

for all $\chi \in C_c^\infty([0\,,T)\,;\mathcal{D}(\Delta_{\tilde\varrho,\,N}))$.

Solutions to (4.127), (4.129) can be expressed by means of Duhamel's formula as

$$\begin{aligned}&< \Phi_\varepsilon(t\,,\cdot)\,;G(-\,\Delta_{\tilde\varrho,\,N})[\chi] >_{\tilde\varrho}\\ &= \frac{1}{2}\Big\langle G(-\,\Delta_{\tilde\varrho,\,N})\Big[\exp\Big(\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\,\Delta_{\tilde\varrho,\,N}}\Big)+\exp\Big(-\,\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\,\Delta_{\tilde\varrho,\,N}}\Big)\Big][\chi]\,;\Phi_{0,\,\varepsilon}\Big\rangle_{\tilde\varrho}\\ &+\frac{1}{2}\Big\langle\frac{\mathrm{i}G(-\,\Delta_{\tilde\varrho,\,N})}{\sqrt{-\,\Delta_{\tilde\varrho,\,N}}}\Big[\exp\Big(\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\,\Delta_{\tilde\varrho,\,N}}\Big)-\exp\Big(-\,\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\,\Delta_{\tilde\varrho,\,N}}\Big)\Big][\chi]\,;r_{0,\,\varepsilon}\Big\rangle_{\tilde\varrho}\\ &+\varepsilon^{\beta/2-1}\int_0^t\Big\langle\,\Delta_{\tilde\varrho,\,N}\,G(-\,\Delta_{\tilde\varrho,\,N})\cos\Big(\frac{t-s}{\varepsilon}\sqrt{-\,\Delta_{\tilde\varrho,\,N}}\Big)[\chi]\,;h_\varepsilon^1(s\,,\cdot)\Big\rangle_{\tilde\varrho}\mathrm{d}s\\ &+\varepsilon^{\beta/2-1}\int_0^t\Big\langle\frac{G(-\,\Delta_{\tilde\varrho,\,N})}{\sqrt{-\,\Delta_{\tilde\varrho,\,N}}}\cos\Big(\frac{t-s}{\varepsilon}\sqrt{-\,\Delta_{\tilde\varrho,\,N}}\Big)[\chi]\,;h_\varepsilon^2(s\,,\cdot)\Big\rangle_{\tilde\varrho}\mathrm{d}s\end{aligned} \tag{4.130}$$

for any $G \in C_c^\infty(0\,,\infty)$ and any $\chi \in L^2(\Omega)$.

To deduce the desired space-time decay estimates, we use the mentioned result of Kato [20] (see also [4]):

Theorem 4.4.1 ([34] (Theorem XIII.25)) *Let A be a closed densely defined linear operator and H a self-adjoint densely defined operator in a Hilbert space X . For $\lambda \notin \mathbb{R}$, let $R_H[\lambda] = (H - \lambda\mathrm{Id})^{-1}$ denote the resolvent of H . Suppose that*

$$\sup_{\lambda\notin\mathbb{R}\,,v\in\mathcal{D}(A^*),\,\|v\|=1}\|A\circ R_H[\lambda]\circ A^*[v]\|_X < \infty\ . \tag{4.131}$$

Then

$$\sup_{w\in X,\,\|w\|_X=1}\frac{\pi}{2}\int_{-\infty}^{\infty}\|A\exp(-\mathrm{i}tH)[w]\|_X^2\mathrm{d}t \le \Gamma^2\,.$$

We apply the above mentioned theorem for the space $X = L^2_{\tilde{\varrho}}(\Omega)$, and operators $H = \sqrt{-\Delta_{\tilde{\varrho},N}}$, $A[v] = \varphi G(-\Delta_{\tilde{\varrho},N})[v]$ for $v \in X$, where $G \in C^\infty_c(0\,,\infty)$ and $\varphi \in C^\infty_c(\Omega)$ are given. In order to use it, we have to verify (4.131).

But since

$$A \circ R_H[\lambda] \circ A^* = \varphi G(-\Delta_{\tilde{\varrho},N}) \frac{1}{\sqrt{-\Delta_{\tilde{\varrho},N}} - \lambda} G(-\Delta_{\tilde{\varrho},N})\varphi\,,$$

it is enough to consider the values of λ belonging to the set

$$\lambda \in Q = \{z \in \mathbb{C} \,|\, \mathrm{Re}[z] \in [a\,,b]\,,\ 0 < |\mathrm{Im}[z]| < d\}\,,$$

where $0 < a < b < \infty$, $\mathrm{supp}[G] \subset (a^2\,,b^2)$, and $d > 0$.

Furthermore, we have

$$A \circ R_H[\lambda] \circ A^* = \varphi \circ \frac{1}{(-\Delta_{\tilde{\varrho},N}) - \lambda^2} G(-\Delta_{\tilde{\varrho},N}) G(-\Delta_{\tilde{\varrho},N}) (\sqrt{-\Delta_{\tilde{\varrho},N}} + \lambda) \circ \varphi\,,$$

and recalling (4.124), it is enough to show that

$$V^{-1} \circ H(\sqrt{-\Delta_{\tilde{\varrho},N}}) \circ \varphi \tag{4.132}$$

is a bounded linear operator on $L^2_{\tilde{\varrho}}$ provided $H \in C^\infty_c(0\,,\infty)$. In order to obtain boundedness of it we follow the arguments of Isozaki [19].

We first use the Fourier transformation to write

$$H(\sqrt{-\Delta_{\tilde{\varrho},N}}) = \int_{-\infty}^{\infty} \exp\Big(\mathrm{i}\sqrt{-\Delta_{\tilde{\varrho},N}}t\Big) \tilde{H}(t) \mathrm{d}t\,, \tag{4.133}$$

where $\tilde{H}$ denotes the inverse Fourier transformation of H .

Since $w = \exp\Big(\mathrm{i}\sqrt{-\Delta_{\tilde{\varrho},N}}t\Big)$ solves the wave equation

$$w_{tt} - \Delta_{\tilde{\varrho},N}[w] = 0\,,$$

that admits a finite propagation of speed, we may estimate

$$\Big\|V^{-1}\exp\Big(\mathrm{i}\sqrt{-\Delta_{\tilde{\varrho},N}}t\Big)[\varphi v]\Big\|^2_{L^2_{\tilde{\varrho}}(\Omega)} = \int_\Omega V^{-2}\Big|\exp\Big(\mathrm{i}\sqrt{-\Delta_{\tilde{\varrho},N}}t\Big)[\varphi v]\Big|^2 \tilde{\varrho}\mathrm{d}x$$
$$= \int_{B_{r+t}} V^{-2}\Big|\exp\Big(\mathrm{i}\sqrt{-\Delta_{\tilde{\varrho},N}}t\Big)[\varphi v]\Big|^2 \tilde{\varrho}\mathrm{d}x \le c(1+t^{2s})\Big\|\exp\Big(\mathrm{i}\sqrt{-\Delta_{\tilde{\varrho},N}}t\Big)[\varphi v]\Big\|^2_{L^2_{\tilde{\varrho}}(\Omega)}$$
$$\le c(1+t^{2s})\|v\|^2_{L^2_{\tilde{\varrho}}(\Omega)}\,,$$

where r is the radius of support of φ. Now, combining the above derived estimate together with (4.133) we deduce (4.132).

Now we can apply Theorem XIII.25 from [34] to obtain

$$\int_0^T \left\| \varphi G(-\Delta_{\tilde{\varrho},N}) \exp\Big(\pm \mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_{\tilde{\varrho},N}}\Big)[\Phi_{0,\varepsilon}] \right\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}t$$
$$\leq \varepsilon c_1 \int_{-\infty}^{\infty} \left\| \varphi G(-\Delta_{\tilde{\varrho},N}) \exp\Big(\pm \mathrm{i}t\sqrt{-\Delta_{\tilde{\varrho},N}}\Big)[\Phi_{0,\varepsilon}] \right\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}t \leq \varepsilon c_2 \|\nabla \Phi_{0,\varepsilon}\|^2_{L^2(\Omega)} . \tag{4.134}$$

Similarly

$$\int_0^T \left\| \varphi \frac{G(-\Delta_{\tilde{\varrho},N})}{\sqrt{-\Delta_{\tilde{\varrho},N}}} \exp\Big(\pm \mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_{\tilde{\varrho},N}}\Big)[r_{0,\varepsilon}] \right\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}t$$
$$\leq \varepsilon c_1 \int_{-\infty}^{\infty} \left\| \varphi \frac{G(-\Delta_{\tilde{\varrho},N})}{\sqrt{-\Delta_{\tilde{\varrho},N}}} \exp\Big(\pm \mathrm{i}t\sqrt{-\Delta_{\tilde{\varrho},N}}\Big)[\Phi_{0,\varepsilon}] \right\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}t \leq \varepsilon c_2 \|\nabla r_{0,\varepsilon}\|^2_{L^2(\Omega)} . \tag{4.135}$$

Finally, by means of similar arguments,

$$\varepsilon^{\beta-2} \int_0^T \int_0^t \left\| \varphi\, \Delta_{\tilde{\varrho},N}\, G(-\Delta_{\tilde{\varrho},N}) \exp\Big(\pm \mathrm{i}\frac{t-s}{\varepsilon}\sqrt{-\Delta_{\tilde{\varrho},N}}\Big)[h^1_\varepsilon(s,\cdot)] \right\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}s\, \mathrm{d}t$$
$$\leq \varepsilon^{\beta-1} \int_0^T \int_{-\infty}^{\infty} \left\| \varphi\, \Delta_{\tilde{\varrho},N}\, G(-\Delta_{\tilde{\varrho},N}) \exp\Big(\pm \mathrm{i}\Big(t - \frac{s}{\varepsilon}\Big)\sqrt{-\Delta_{\tilde{\varrho},N}}\Big)[h^1_\varepsilon(s,\cdot)] \right\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}t\, \mathrm{d}s$$
$$\leq \varepsilon^{\beta-1} c_1 \int_0^T \left\| \exp\Big(\pm \mathrm{i}\frac{s}{\varepsilon}\sqrt{-\Delta_{\tilde{\varrho},N}}\Big)[h^1_\varepsilon(s,\cdot)] \right\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}s = \varepsilon^{\beta-1} c_1 \int_0^T \|h^1_\varepsilon\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}s ; \tag{4.136}$$

and

$$\varepsilon^{\beta-2} \int_0^T \int_0^t \left\| \varphi \frac{G(-\Delta_{\tilde{\varrho},N})}{\sqrt{-\Delta_{\tilde{\varrho},N}}} \exp\Big(\pm \mathrm{i}\frac{t-s}{\varepsilon}\sqrt{-\Delta_{\tilde{\varrho},N}}\Big)[h^2_\varepsilon(s,\cdot)] \right\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}s\, \mathrm{d}t$$
$$\leq \varepsilon^{\beta-1} \int_0^T \int_{-\infty}^{\infty} \left\| \varphi \frac{G(-\Delta_{\tilde{\varrho},N})}{\sqrt{-\Delta_{\tilde{\varrho},N}}} \exp\Big(\pm \mathrm{i}\Big(t - \frac{s}{\varepsilon}\Big)\sqrt{-\Delta_{\tilde{\varrho},N}}\Big)[h^2_\varepsilon(s,\cdot)] \right\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}t\, \mathrm{d}s$$
$$\leq \varepsilon^{\beta-1} c_1 \int_0^T \left\| \exp\Big(\pm \mathrm{i}\frac{s}{\varepsilon}\sqrt{-\Delta_{\tilde{\varrho},N}}\Big)[h^2_\varepsilon(s,\cdot)] \right\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}s = \varepsilon^{\beta-1} c_1 \int_0^T \|h^2_\varepsilon\|^2_{L^2_{\tilde{\varrho}}(\Omega)} \mathrm{d}s . \tag{4.137}$$

Combining relations (4.134) - (4.137) with the uniform estimates (4.106), (4.107) and (4.128) we may infer from (4.130) that

$$\|G(-\Delta_{\tilde{\varrho},N})[\Phi_\varepsilon]\|^2_{L^2((0,T)\times K)} \leq \varepsilon^{\beta-1} c(K,G) , \tag{4.138}$$

for any compact $K \subset \Omega$, and any $G \in C_c^\infty(0,\infty)$.

Let's return now to the expression (4.117) for the gradient part of $\vec{V}_\varepsilon$. We have

$$\int_\Omega \vec{V}_\varepsilon \cdot \nabla\varphi \mathrm{d}x = \frac{1}{p_0\overline{\vartheta}}\Big\langle \sqrt{-\Delta_{\tilde{\varrho},N}}[\Phi_\varepsilon]\,;\sqrt{-\Delta_{\tilde{\varrho},N}}[\varphi]\Big\rangle_{\tilde{\varrho}}$$
$$= \frac{1}{p_0\overline{\vartheta}} < G(-\Delta_{\tilde{\varrho},N})[\Phi_\varepsilon]\,;\Delta_{\tilde{\varrho},N}[\varphi] >_{\tilde{\varrho}}$$
$$+\frac{1}{p_0\overline{\vartheta}}\Big\langle \sqrt{-\Delta_{\tilde{\varrho},N}}[\mathrm{Id} - G(-\Delta_{\tilde{\varrho},N})][\Phi\varepsilon]\,;\sqrt{-\Delta_{\tilde{\varrho},N}}[\varphi]\Big\rangle_{\tilde{\varrho}},$$

where, by virtue of (4.138), the former expression on the right-hand side tends to zero in $L^2(0,T)$ as $\varepsilon \to 0$ for any fixed $\varphi \in C_c^\infty(\Omega)$, $G \in C_c^\infty(0,\infty)$, while

$$\frac{1}{p_0\overline{\vartheta}}\Big\langle \sqrt{-\Delta_{\tilde{\varrho},N}}[\mathrm{Id} - G(-\Delta_{\tilde{\varrho},N})][\Phi\varepsilon]\,;\sqrt{-\Delta_{\tilde{\varrho},N}}[\varphi]\Big\rangle_{\tilde{\varrho}}$$
$$= \int_\Omega \Phi_\varepsilon[G(-\Delta_{\tilde{\varrho},N}) - \mathrm{Id}][\Delta_{\tilde{\varrho},N}[\varphi]]\mathrm{d}x\,.$$

It is easy to check that for $G \approx 1$, the quantity $[\mathrm{Id} - G(-\Delta_{\tilde{\varrho},N})][\varphi]$ will be small in $L^2 \cap L^\infty(\Omega\,;\mathbb{R}^3)$. In view of fact that Φ_ε is classical solution which can be expressed by means of Duhamel's formula. Using it we obtain the expression similar to (4.130) which implies the uniform boundedness of Φ_ε. Consequently, (4.117) is valid and Theorem 4.1.1 is proved.

Acknowledgement. This research was supported by the Jindřich Nečas Center for Mathematical Modelling, project LC06052, financed by MSMT.

Bibliography

[1] Alazard T.: Incompressible limit of the nonisentropic Euler equations with the solid wall boundary conditions, Adv. Differential Equations, 10(1):19 - 44, 2005.

[2] Alazard T.: Low Mach number limit of the full Navier-Stokes equations, Arch. Rational Mech. Anal. Vol. 180(1), 1 - 73, 2006.

[3] Ascota G., Durán R. G., Muschietti M. A.: Solutions of the divergence operator on John domains, Advances in Mathematics 206(2) 373 - 401 2006.

[4] Burq N., Planchon F., Stalker J.G., Tahvildar-Zadeh A.S.: Strichartz estimates for the wave and Schrödinger equations with potencial and critical decay, Indiana Univ. Math. J., 53(6) 1665 - 1680, 2004.

[5] Chandrasektar S.: Hydrodynamic and hydromagnetic stability, Oxford University Press, Oxford, 1961.

[6] Desjardins B., Grenier E.: Low Mach number limit of viscous compressible flows in the whole space, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 455(1986):2271-2279, 1999.

[7] Ducomet B., Feireisl E.: The Equations of magnetohydrodynamics: On the Interaction Between matter and Radiation in the Evolution of Gaseous Stars, Communications in mathematical Physics 266, 595 - 629, 2006.

[8] Duvaut G., Lions J. L.: Inequalities in mechanics and physics, Heidelberg: Springer-Verlag, 1976.

[9] Ekeland I., Temam R.: Convex analysis and variational problems, North Holland Amsterdam, 1978.

[10] Elliott J. R.: Numerical simulations of the solar convection zone, Stellar Astrophysical Fluid Dynamics, 315 - 328, Cambridge University Press, 2003.

[11] Evans L. C.: Partial Differential Equations, American Mathematical Society, 2002.

[12] Feireisl E.: Incompressible Limits and Propagation of Acoustic Waves in Large Domains with Boundaries, to appear in Commun. Math. Phys.

[13] Feireisl E., Novotný A.: Singular Limits in Thermodynamics of Viscous Fluids, Birkhäuser, 2009.

[14] Feireisl E., Novotný A.: The Low Mach Number Limit for the Full Navier-Stokes-Fourier System, Arch. Rational Mech. Anal 186(2007) 77 - 107, 2007.

[15] Feireisl E., Novotný A., Petzeltová H.: On the Existence of Globally Defined Weak Solutions to the Navier-Stokes Equations, Journal of Mathematical Fluid Mechanics 3(2001) 358 - 392, Birkhäuser Verlag, Basel, 2001.

[16] Feireisl E., Novotný A., Petzeltová H.: On the domain dependence of solutions to the compressible Navier-Stokes equations of a barotropic fluid, Math. Meth. Appl. Sci 25(12) 1045 - 1073 , 2002.

[17] Feireisl E., Novotný A., Petzeltová H.: The Low Mach Number Limit for the Navier-Stokes System on Unbounded Domain under Strong stratification, to appear in Commun. Partial Differential Equations.

[18] Friedrichs K. O.: On certain inequalities and characteristic value problems for analytic functions and functions of two variables, Trans. Amer. Math. Soc., 41 321 - 364, 1937.

[19] Isozaki H.: Singular limits for the compressible Euler equation in an exterior domain, J. Reine Angew. Math. 381 (1987), 1-36, 1987.

[20] Kato T.: Wave operators and similarity for some non-selfadjoint operators, Math. Ann., 162:258-279, 1965/1966.

[21] Klainerman S., Majda A.: Singular Limits of Quasilinear Hyperbolic Systems with Large Parameters and the Incompressible Limit of Compressible Fluids, Communications on Pure and Applied Mathematics, Vol. XXXIV, 481 - 524, 1981.

[22] Kukučka P.: Incompressible Limits for the Navier-Stokes-Fourier Systems on Unbounded Domains under Strong Stratification, Submitted for publication to Archive Rational Mech. Anal.

[23] Kukučka P.: On the Existence of Finite Energy Weak Solutions to the Navier-Stokes Equations in Irregular Domains, Math. Meth. Appl. Sci., 32(11) 1428-1451, 2009.

[24] Kukučka P.: Singular Limits of the Equations of Magnetohydrodynamics, Accepted for publication in Journal of Mathematical Fluid Mechanics.

[25] Lignières F.: The small Peclet number approximation in stellar radiative zones, Astronomy and Astrophysics, v.348, p.933-939, 1999.

[26] Lions P. L.: Mathematical topics in fluid dynamics, Vol. 2, Compressible models, Oxford Science Publication, Oxford, 1998.

[27] Lions P. L., Masmoudi N.: Incompressible limit for a viscous compressible fluid, J. Math. Pures Appl., 77:585 - 627, 1998.

[28] Leis R.: Initial-boundary value problems in mathematical physics., B.G. Teubner, Stuttgart, 1986.

[29] Lunardi A.: Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Berlin, 1995.

[30] Novotný A., Straškraba I.: Introduction to the Mathematical Theory of Compressible Flow, Oxford Lecture Series in Mathematics and its Applications 27, Oxford University Press, 2004.

[31] Poul L.: Existence of weak solutions to the Navier-Stokes-Fourier system on Lipschitz domains, Discrete and Continuous Dynamical Systems 2007 834 - 843, 2007.

[32] Proctor M. R. E.: Magnetoconvection, Fluid Dynamics and Dynamos in Astrophysics and Geophysics, 235 - 276, CRC Press, 2005.

[33] Quittner P., Souplet P.: Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States, Birkhäuser Advanced Texts, Basel - Boston - Berlin, 2007.

[34] Reed M., Simon B.: Methods of modern mathematical physics. IV, Analysis of operators. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

[35] Rubino B.: Singular limits in the data space for the equations of magneto-fluid dynamics, Hokkaido Math. J. 24 (1995) 357 - 386, 1995.

[36] Smith. H. F., Sogge C. D.: Global Strichartz estimates for nontrapping perturbations of the Laplacian, Comm. Partial Differencial equations, 25:2171-2183, 2000.

[37] Temam R.: Navier-Stokes Equations, Theory and Numerical Analysis, Studies in Mathematics and its Applications, Volume 2, North-Holland, Amsterdam, 1977.

[38] Weiss N.: Modelling solar and stellar magnetoconvection, Stellar Astrophysical Fluid Dynamics, 329 - 343, Cambridge University Press, 2003.

[39] Zank J. P., Matthaeus W. H.: Nearly incompressible fluids. II: Magnetohydrodynamics turbulence, and waves, Phys. Fluids A 5 (I), January 1993.