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DIPLOMOVÁ PRÁCE

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Rozklady grafů

Katedra aplikované matematiky
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Rád bych poděkoval Danielu Kráľovi za bezvadnou spolupráci a spoustu věnovaného času diskuzím o problému. Jeho důkladné korekce mé angličtiny, přispěly nejen lepší kvalitě práce, ale také mých znalostí anglického jazyka.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne

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Abstrakt: Submodulární rozkladové funkce zobecňují známé druhy stromových rozkladů grafů. Pro každé pevné k existují polynomiální algoritmy, které rozhodují, zda je stromová či větvená šířka nejvýše k. My ukážeme, že neexistuje algoritmus, který by rozhodoval, zda je šířka dané submodulární rozkladové funkce nejvýše dva v čase menším než exponenciálním. Dále popíšeme novou duální strukturu pro submodulární rozkladové funkce podobnou volným zámotkům pro souvislostní funkce.

Klíčová slova: stromový rozklad grafů, rozkladové funkce, submodulární funkce

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Abstract: The notion of submodular partition functions generalizes many of well-known tree decompositions of graphs. For fixed k, there are polynomial-time algorithms to determine whether a graph has tree-width, branch-width, etc. at most k. Contrary to these results, we show that there is no sub-exponential algorithm for determining whether the width of a given submodular partition function is at most two. In addition, we also develop another dual notion for submodular partition functions which is analogous to loose tangles for connectivity functions.

Keywords: graph tree-decomposition, partition function, submodular function

Chapter 1

Introduction

1.1 Graph decompositions

Graph decompositions and width-parameters play a very important role in algorithmic graph theory (as well as structural graph theory). The most well-known and studied notions include the tree-width, branch-width and clique-width of graphs. The importance of these notions lie in the fact that many NP-complete problems can be decided for classes of graphs of bounded tree-/branch-width in polynomial time. A classical result of Courcelle [4] asserts that every problem expressible in the monadic second-order logic can be decided in linear time for the class of graphs with bounded tree-/branch-width. An analogous result for matroids with bounded branch-width representable over finite fields have been established by Hliněný [7, 8] and generalized using a more specialized notion of width to all matroids by Král' [10].

Most of the algorithms for classes of graphs of bounded width require a decomposition of an input graph as part of input. Fortunately, optimal tree-decompositions of graphs can be computed in linear time [2] if the width is fixed and there are even simple efficient approximation algorithms [3]. For branch-width, Oum and Seymour [13] recently established that the branch-decompositions of a fixed width of graphs and matroids can be computed in polynomial-time (or decided that they do not exist). Their algorithm actually deals with a more general notion of connectivity functions which are given by an oracle. A fixed-parameter algorithm for the same problem has been developed by Hliněný and Oum [9].

In this thesis, we study submodular partition functions introduced by Amini et al. [1]. This general notion includes both graph tree-width and branch-width as special cases. We postpone the formal definition to Section 3.1. In their paper, Amini et al. [1] presented a duality theorem that implies the known duality theorems for graph tree-width and graph/matroid branch-width of Robertson and Seymour [16].

Since the duality, an essential ingredient for some of the known algorithms for computing decompositions of small width, smoothly translates to this general setting, it is natural to ask whether decompositions of submodular partition functions with fixed width can be computed in polynomial-time. In this thesis, we show that such an algorithm cannot be designed in general. In particular, we present an argument that every algorithm deciding whether a partition width of an n-element set is at most two must ask an oracle the number of queries exponential in n.

On a positive side, we were able to develop a notion of loose tangles, a key ingredient of the algorithm of Oum and Seymour [13], for this more general concept which we hope to be of some use to design algorithms for special classes of submodular partition functions.

1.2 Graph tree-width

The notions of tree-decomposition and tree-width were first introduced (under different names) by Halin [6]. Robertson and Seymour reintroduced the two concepts, apparently unaware of Halin's paper, in their famous Graph Minor series. They define it for hypergraphs but we state it for simple graphs.

For a graph $G = (V, E)$, a *tree-decomposition* is a pair (T, σ) where T is a tree and σ is a mapping from the vertex set of T to the set of subsets of vertices of G which satisfy the following conditions.

- For every edge $uv \in E$, there is a node a such that $\{u, v\} \subseteq \sigma(a)$.
- For every vertex $v \in V$, the set $\sigma^{-1}(v)$ induces a non-empty subtree of T.

For a node a of T, we call $\sigma(a) \subseteq V$ a *bag* of a. The width of a tree-decomposition (T, σ) is the maximum size of a bag of T decreased by one. The tree-width of G is the minimum width over all tree-decompositions of G.

There are dual combinatorial objects to tree-decompositions called *brambles*. A bramble in a graph G is a set $\mathcal{B}$ of subsets of vertices of G such that every set W from $\mathcal{B}$ induces a connected subgraph in G and every two sets W_1 and W_2 from $\mathcal{B}$ touch, i.e., $W_1 \cup W_2$ induces a connected subgraph in G.

A subset of V is said to be a *hitting set* for $\mathcal{B}$ if it intersects every element of $\mathcal{B}$. The *order* of $\mathcal{B}$ is the minimum size of a hitting set for $\mathcal{B}$. The *bramble number* of a graph G is the maximum order of a bramble in G. The tree-decompositions and brambles are connected through the following theorem.

Theorem 1 (Seymour and Thomas [17]). *The tree-width of every graph is equal to its bramble number increased by one.*

1.3 Graph branch-width

The notion of branch-width was also introduced by Robertson and Seymour [16]. Even though it was first defined for hypergraphs, it is naturally generalized to connectivity functions as we show in Section 2.1. Even Robertson and Seymour proved their duality theorem between branch-width and tangle number using the general setting of connectivity functions [16].

Let $G = (V, E)$ be a simple graph. A *branch-decomposition* of G is a pair (T, σ) where T is a *ternary* tree and σ is a bijection between the set of leaves of T and E. Every edge e of T naturally defines a bipartition $(A_e, \overline{A_e})$ of the edge set E. The border Δ of a subset F of edges is the set of vertices in V incident with both an edge in F and an edge in $\overline{F}$, i.e., $\Delta(F) = \{v \in V | \exists uv \in F \text{ and } \exists u'v \in \overline{F}\}$. The *order* of an edge e of T is the size of the border of A_e, $|\Delta(A_e)|$, and the *width* of a branch-decomposition (T, σ) is the maximum order of an edge of T. The *branch-width* of G is the minimum width of a branch-decomposition of G.

The branch-width and tree-width of graphs describe very similar characteristics of graphs. The following theorem by Robertson and Seymour [16] shows that the branch-width and the tree-width are linearly dependent.

Theorem 2 (Robertson and Seymour [16]). *Let G be a graph of tree-width k and branch-width $b > 1$. Then*

$$b - 1 \leq k \leq \left\lfloor \frac{3}{2} b \right\rfloor - 1.$$

1.4 Monadic second-order formulas

Different graph properties can be described by logic formulas. The complexity of the property is reflected by the complexity of a formula expressing it. First-order logic formulas are one of the simplest allowing quantifiers only over elements of the universe. Second-order logic formulas allow quantifiers over predicates. Finally, monadic second-order formulas are second-order logic formulas where quantification is only over unary predicates (and elements of the universe). Note that unary predicates can be viewed as subsets of the universe.

For a graph $G = (V, E)$, the universe is the set of vertices V and there is a predicate $E(x, y)$ (the only binary predicate allowed), indicating the adjacency of vertices. This predicate represents the input. For example, the following formula is true if and only if G is 3-colorable. The variables for elements are printed in small letters while the variables for predicates are printed in capital letters.

$$\exists C_1 \exists C_2 \exists C_3 \left(\forall x \bigvee_{i=1}^{3} C_i(x) \wedge \forall x \forall y \left(E(x,y) \Rightarrow \bigwedge_{i=3}^{3} \neg (C_i(x) \wedge C_i(y)) \right) \right)$$

The well-known theorem of Courcelle [4] states that for every monadic second-order formula φ and every class $\mathcal{C}$ of graphs of bounded tree-width, there is a linear-time algorithm deciding whether a graph from $\mathcal{C}$ satisfies φ.

Theorem 3 (Courcelle [4]). *Let k be a fixed integer and a fixed monadic second-order logic formula φ. There is a linear-time algorithm for deciding whether a graph of tree-width at most k satisfies φ.*

1.5 Graph minors

The concepts of graph decompositions are deeply interconnected with one of the most important theorems in the graph theory, the *graph minor theorem*.

A graph H is a *minor* of a graph G, writing $H \preccurlyeq G$, if we can obtain H from a subgraph of G by contracting some of its edges. A reflexive and transitive relation is called a *quasi-ordering*. A quasi-ordering $\leqslant$ on a set X is a *well-quasi-ordering*, and the elements of X are *well-quasi-ordered* by $\leqslant$, if for every infinite sequence $x_0, x_1, \ldots$ in X there are indices $i < j$ such that $x_i \leqslant x_j$.

Actually, if X is well-quasi-ordered, then every infinite sequence in X has an infinite non-decreasing subsequence. Now, we formulate the graph minor theorem using these concepts.

Theorem 4 (Graph Minor Theorem; Robertson and Seymour). *The finite graphs are well-quasi-ordered by the minor relation $\preccurlyeq$.*

A class $\mathcal{C}$ of graphs closed under isomorphism is *closed under taking minors* when, for every graph $G \in \mathcal{C}$, all minors of G belong to $\mathcal{C}$. For a class $\mathcal{H}$ of graphs, we define a class

$$\mathrm{Forb}_{\preccurlyeq}(\mathcal{H}) = \{G | H \not\preccurlyeq G \text{ for all } H \in \mathcal{H}\}$$

of all graphs without a minor in $\mathcal{H}$. We say that $\mathcal{H}$ is a set of *forbidden minors* for the class $\mathrm{Forb}_{\preccurlyeq}(\mathcal{H})$.

Note that $\mathrm{Forb}_{\preccurlyeq}(\mathcal{H})$ is closed under taking minors. Actually, every class of graphs that is closed under taking minors can be expressed by forbidden minors. Naturally, we are interested in the smallest set of forbidden minors for such a class of graphs. It turns out that there is indeed a unique smallest such set $\mathcal{H}$. Certainly, the set

$$\mathcal{H}_{\mathcal{C}} = \{H|H \text{ is } \preccurlyeq\text{-minimal in } \overline{\mathcal{C}}\}$$

satisfies $\mathcal{C} = \mathrm{Forb}_{\preccurlyeq}(\mathcal{H}_{\mathcal{C}})$ and is contained in every such a set. Clearly, the elements of $\mathcal{H}_{\mathcal{C}}$ are incomparable under the minor relation $\preccurlyeq$. By Theorem 4, any set of $\preccurlyeq$-incomparable graphs must be finite, so every $\mathcal{H}_{\mathcal{C}}$ is finite. In other words, every class of graphs closed under taking minors has a finite number of forbidden minors.

Corollary 5. *Every class of graphs that is closed under taking minors can be expressed as* $\mathrm{Forb}_{\preccurlyeq}(\mathcal{H})$ *with finite* $\mathcal{H}$*.*

The graph minor theorem also contributed to the algorithmic graph theory. Robertson and Seymour have shown that testing whether a graph contains a fixed H as a minor can be done polynomial time (in cubic time although usually with an enormous constant depending on H). Hence, it can be decided in polynomial time whether a graph G belongs to a class $\mathcal{C}$ closed under taking a minor. It is sufficient to test whether one of the forbidden minors for $\mathcal{C}$, $H_1, \ldots, H_k$, is a minor of G, i.e., $H_i \preccurlyeq G$.

Chapter 2

Connectivity functions

2.1 Submodular functions and duality

Submodular functions form an important class of functions. A number of problems can be reduced to the minimization of a particular submodular function. When dealing with graph decompositions we will restrict ourselves to the class of integer-valued submodular functions.

A function $f : 2^E \to \mathbb{N}$ for a finite set E is said to be *submodular* if the following holds for every pair of subsets $X, Y \subseteq E$:

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)\ . \tag{2.1}$$

A submodular function f is *symmetric* if $f(X) = f(\overline{X})$, for all subsets X of E. Finally, a *connectivity function* is a submodular function that is symmetric and $f(\emptyset) = 0$.

The branch-decompositions of graphs are naturally generalized to the branch-decompositions of connectivity functions. For a connectivity function f on a ground set E, a *branch-decomposition* of f is a pair (T, σ) where T is a *ternary* tree and σ is a bijection between the set of leaves of T and E. Every edge e of T naturally defines a bipartition $(A_e, \overline{A_e})$ of the ground set E. The *order* of an edge e of T is the value $f(A_e)$ and the *width* of a branch-decomposition (T, σ) is the maximum order of an edge of T. The *branch-width* of f is the minimum width of a branch-decomposition of f. This notion includes the notion of the usual branch-width of graphs and matroids.

In the following, Greek letters will be used for sequences of subsets, i.e., α can stand for a sequence $A_1, \ldots, A_k$ of subsets of a set E. Such a sequence α is called a *covering* of E if all elements of E are contained in some A_i. The *overlap* $o(\alpha)$ of α is the set of elements that belong to at least two parts of α. Finally, α is a *partition* if it is a covering with empty overlap.

We introduce shorthands for operations with sequences of subsets we want to use: if α is such a sequence $A_1, \ldots, A_k$ and A is another subset, then $\alpha \cap A$ stands for the sequence $A_1 \cap A, \ldots, A_k \cap A$. We use $\alpha \setminus A$ in a similar way. Finally, (A, α) stands for the sequence obtained from α by inserting A to the sequence. Note that empty sets are allowed in the sequences.

A *partial branch-decomposition* T of f for a partition α is a branch-decomposition of a connectivity function f_α obtained from f by identifying the parts of α, i.e., the ground set of f_α are the parts of α. We call the partition α the *displayed partition* of T.

A set $A \subseteq E$ is *small* if $f(A) \leq k$. A *partial branch-decomposition* of f over a set $\mathcal{A}$ of small subsets of E is a partial branch-decomposition of f for a partition which parts are small subsets of sets contained in $\mathcal{A}$.

A set $A \subseteq E$ is *k-branched* if there is a partial branch-decomposition of f of width at most k for the partition $(\overline{A}|\{e_1\}|\ldots|\{e_r\}|)$, $e_i \in A$. Note that E is *k-branched* if and only if f has branch-width at most k.

For a graph $G = (V, E)$, the connectivity function of G is the function δ_2, defined as $d_2(X) = |\Delta(X)|$ where Δ is the border introduced in Section 1.3.

There is a dual object to branch-decompositions called a *tangle*, introduced by Robertson and Seymour [16]. A set $\mathcal{T}$ of subsets of E is called an f-tangle of *order* $k+1$ if $\mathcal{T}$ satisfies the following three axioms:

(T1) For all $A \subseteq E$, if $f(A) \leq k$, then either $A \in \mathcal{T}$ or $\overline{A} \in \mathcal{T}$.

(T2) If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq E$.

(T3) For all $e \in E$, we have $E \setminus \{e\} \notin \mathcal{T}$.

Robertson and Seymour [16] proved the following duality theorem between branch-decompositions and tangles.

Theorem 6 (Robertson and Seymour [16])**.** *Let f be a connectivity function on a ground set E. There is no f-tangle of order $k+1$ if and only if the branch-width of f is at most k.*

Proof. Suppose that there is an f-tangle $\mathcal{T}$ of order $k+1$ and that there is a branch-decomposition T of f of width at most k. Since the width of T is at most k, $f(A_e) \leq k$ for all edges e of T, where A_e is one of the parts of the bipartition given by e. Hence, one of the sets A_e or $\overline{A_e}$ is in $\mathcal{T}$. Only one of these sets can belong to $\mathcal{T}$ because of (T2). Therefore we can orient the edges of T toward the subtree corresponding to the part in $\mathcal{T}$. Since T has $|E(T)| + 1$ vertices, there is a vertex v with no edge leading to it. It cannot be a leaf, since $E \setminus \sigma(v)$ does not belong to $\mathcal{T}$ because of (T3). Hence, v is an internal vertex corresponding to a partition A, B, C all belonging to $\mathcal{T}$. But then $A \cup B \cup C = E$ violating (T2) – a contradiction.

We prove a stronger version of the other implication. Let $\mathcal{A}$ be an arbitrary set of small subsets of E. If there is no f-tangle $\mathcal{T}$ of order $k+1$ such that $\mathcal{A} \subseteq \mathcal{T}$, then there is a partial branch-decomposition of width at most k over $\mathcal{A}$.

We proceed by induction on the number r of small sets A (recall that A is small if $f(A) \leq k$) such that neither A nor $\overline{A}$ is a subset of a set in $\mathcal{A}$. We suppose first that $r = 0$. Let $\mathcal{T}$ be the set of all small subsets of a set in $\mathcal{A}$. Since $\mathcal{A} \subseteq \mathcal{T}$, $\mathcal{T}$ is not an f-tangle by the assumption. For every small set A, either $A \in \mathcal{T}$ or $\overline{A} \in \mathcal{T}$ since $r = 0$. If (T3) is violated, then there is $e \in E$ such that $E \setminus \{e\}$ belongs to $\mathcal{T}$. Hence, the tree consisting of two leaves displaying the partition $(E \setminus \{e\}, \{e\})$ is a partial branch-decomposition over $\mathcal{A}$. Thus (T2) is violated and there exist sets $A, B, C \in \mathcal{T}$ with $A \cup B \cup C = E$. We can assume that A, B, and C are mutually disjoint since by submodularity

$$2k \geq f(A) + f(\overline{B}) \geq f(A \setminus B) + f(B \setminus A)$$

and we take one of $A \setminus B$ or $B \setminus A$ instead. The tree on four vertices displaying the partition $(A|B|C)$ is a partial branch-decomposition over $\mathcal{A}$.

Now assume that $r > 0$. Choose a small set $A \subseteq E$ such that neither A nor $\overline{A}$ is a subset of any member of $\mathcal{A}$ and subject to that with $|A|$ minimal. Let $\mathcal{A}_1 = \mathcal{A} \cup \{A\}$ and $\mathcal{A}_2 = \mathcal{A} \cup \{\overline{A}\}$. Since there is no f-tangle $\mathcal{T}$ with $\mathcal{A} \subseteq \mathcal{T}$, there is no f-tangle $\mathcal{T}$ with $\mathcal{A}_1 \subseteq \mathcal{T}$ and $\mathcal{A}_2 \subseteq \mathcal{T}$. By the induction there are partial branch-decompositions (T_1, σ_1) over $\mathcal{A}_1$ and (T_2, σ_2) over $\mathcal{A}_2$. A leaf t of T_1 is *bad* if $\sigma_1(t)$ is not a subset of a set in $\mathcal{A}$. We define a bad leaf in T_2 similarly.

Consider such a bad leaf t of T_1. Since $\sigma_1(t)$ is a subset of a set in $\mathcal{A}_1$, $\sigma_1(t)$ has to be a subset of A. On the other hand, $\sigma_1(t)$ cannot be a proper subset of A by our choice of A. Hence, $\sigma_1(t) = A$ and there can be only one bad leaf in T_1. We may assume that T_1 has a bad leaf t_1 for otherwise (T_1, σ_1) is the desired partial branch-decomposition over $\mathcal{A}$.

Let (T_2, σ_2) be a partial branch-decomposition of f over $\mathcal{A}_2$ such that it has minimum number of bad leaves. If T_2 has no bad leaf, then (T_2, σ_2) is the desired partial branch-decomposition over $\mathcal{A}$. Hence suppose that T_2 has a bad leaf t_2. Since t_2 is bad, $\sigma_2(t)$ is a small subset of $\overline{A}$. Let T be a tree constructed from T_1 and T_2 by deleting t_1 and t_2 and joining the former neighbors of t_1 and t_2 by an edge. Now, T is not generally a partial branch-decomposition since the leaves of T form a covering of E with possibly non-empty overlap. We will refine this covering to get a partition of E and show that the resulting tree is a partial branch-decomposition over $\mathcal{A}$.

First, take a set C such that $\sigma_2(t_2) \subseteq C \subseteq \overline{A}$ with $f(C)$ minimal. We define a mapping σ on the set of leaves of T as follows. For a leaf b_1 of T_1, we let b_1 represent only the elements that are in C, $\sigma(b_1) := \sigma_1(b_1) \cap C$. For a leaf b_2 of T_2, we let b_2 represent only the elements that are *not* in C, $\sigma(b_2) := \sigma_2(b_2) \setminus C$. We have to show that for every edge e in (T, σ), $f(A_e) \leq k$. If $e_1 \in T_1$, let Y denote the part of $(A_{e_1}, \overline{A_{e_1}})$ where $Y \subseteq \overline{A}$ and if $e_2 \in T_2$, let Z denote the part of $(A_{e_2}, \overline{A_{e_2}})$ where $Z \subseteq \overline{\sigma_2(t_2)}$. In T the edges e_1 and e_2 separate the sets $Y \cap C$ and $Z \setminus C$ instead. By the submodularity and symmetry of f we get:

$$f(C) + f(Y) \geq f(C \cap Y) + f(C \cup Y),$$
$$f(C) + f(Z) = f(C) + f(\overline{Z}) \geq f(C \setminus Z) + f(Z \setminus C).$$

By the choice of C, $f(C \cup Y) \geq f(C)$ and $f(C \setminus Z) \geq f(C)$ giving that $f(C \cap Y) \leq f(Y) \leq k$ and $f(Z \setminus C) \leq f(Z) \leq k$. Hence, the order of every edge in (T, σ) is at most k. It is easy to check that the leaves of (T, σ) form a partition of E. Since $\sigma(t) \subseteq \sigma_1(t)$ or $\sigma(t) \subseteq \sigma_2(t)$, the parts of the displayed partition of T are subsets of sets in $\mathcal{A}_2$ and (T, σ) is a partial branch-decomposition of f over $\mathcal{A}_2$ with smaller number of bad leaves than T_2 — a contradiction.

□

2.2 Minimization of submodular functions

It was shown by Grötschel, Lovász, and Shrijver [5] that the minimum value of a rational-valued submodular function f on E can be found in polynomial time, if f is given by an oracle and an upper bound B is given on the numerators and denominators of the values of f. The running time is bounded by a polynomial in $|E|$ and $\log B$.

However, we deal with connectivity functions for which much simpler minimization algorithm exists. Queyranne [14, 15] gave the following easy combinatorial algorithm to find a non-empty proper subset A of E minimizing $f(A)$, where f is given by an oracle. The running time of the following algorithm is $\mathcal{O}(|E|^3\gamma)$, where γ is the oracle query time.

Call an ordering $e_1, \ldots, e_n$ of the elements of E a *legal order* of E for f if, for each $i = 1, \ldots, n$,

$$f(\{e_1, \ldots, e_{i-1}, x\}) - f(\{x\})$$

is minimized over $x \in E \setminus \{e_1, \ldots, e_{i-1}\}$ by $x = e_i$. One easily finds a legal order by $\mathcal{O}(|E|^2)$ oracle calls.

A set A *splits* a set X if both $X \cap A$ and $X \setminus A$ are non-empty. Now the algorithm is:

Algorithm 1 (Queyranne [14, 15])**.** *Finds a non-empty proper subset A of E minimizing $f(A)$.*

(M1) Find a legal order $(e_1, \ldots, e_n)$ of E for f.

(M2) Determine (recursively) a non-empty proper subset B of E not splitting $\{e_{n-1}, e_n\}$, minimizing $f(B)$. This can be done by identifying e_{n-1} and e_n.

(M3) Then the minimum value of $f(A)$ over non-empty proper subsets A of E is equal to $\min\{f(B), f(\{e_n\})\}$.

The correctness of the algorithm follows from, for $n \geq 2$:

$$f(A) \geq f(\{e_n\}) \text{ for each } A \subseteq E \text{ splitting } \{e_{n-1}, e_n\}. \tag{2.2}$$

This is proved as follows. Define $b_0 := e_1$. For $i = 1, \ldots, n-1$, define $b_i := e_j$, where j is the smallest index such that $j > i$ and such that A splits $\{e_i, e_j\}$. For $i = 0, \ldots, n$, let $A_i := \{e_1, \ldots, e_i\}$. Note that for each $i = 1, \ldots, n-1$ one has

$$f(A_{i-1} \cup \{b_i\}) - f(\{b_i\}) \geq f(A_{i-1} \cup \{b_{i-1}\}) - f(\{b_{i-1}\}), \tag{2.3}$$

since if $b_{i-1} = b_i$ this is trivial, and if $b_{i-1} \neq b_i$, then $b_{i-1} = e_i$, and (2.3) follows from the legality of the order.

We will show that the following holds for each $i = 1, \ldots, n-1$:

$$f(A_i \cup A) - f(A_{i-1} \cup A) + f(A_i \cup \overline{A}) - f(A_{i-1} \cup \overline{A}) \leq f(A_i \cup \{b_i\}) - f(A_{i-1} \cup \{b_i\}). \tag{2.4}$$

In proving this, we may assume (by symmetry of A and $\overline{A}$) that $e_i \in \overline{A}$. Then $A_i \cup \overline{A} = A_{i-1} \cup \overline{A}$ and $b_i \in A$. By submodularity we get the following:

$$f(A_i \cup \{b_i\}) + f(A_{i-1} \cup A) \geq f(A_{i-1} \cup \{b_i\}) + f(A_i \cup A).$$

This gives (2.4).

Then we have:

$$
\begin{aligned}
&f(e_n) - 2f(A)\\
&= f(A_{n-1} \cup A) + f(A_{n-1} \cup \overline{A}) - f(A_0 \cup A) - f(A_0 \cup \overline{A})\\
&= \sum_{i=1}^{n-1} (f(A_i \cup A) - f(A_{i-1} \cup A) + f(A_i \cup \overline{A}) - f(A_{i-1} \cup \overline{A}))\\
&\le \sum_{i=1}^{n-1} (f(A_i \cup \{b_i\}) - f(A_{i-1} \cup \{b_i\}))\\
&\le \sum_{i=1}^{n-1} (f(A_i \cup \{b_i\}) - f(A_{i-1} \cup \{b_{i-1}\}) + f(\{b_{i-1}\}) - f(\{b_i\}))\\
&= f(A_{n-1} \cup \{b_{n-1}\}) - f(\{b_{n-1}\}) - f(\{b_0\}) + f(\{b_0\}) = -f(e_n)
\end{aligned}
$$

where the first inequality follows from (2.4), and the second inequality follows from (2.3). This establishes (2.2) and finishes the proof.

2.3 Computing branch-decompositions

For a fixed k, Oum and Seymour [13] devised a polynomial-time algorithm for deciding whether the branch-width of a connectivity function f is at most k, if f is given by an oracle. In this section we will present their algorithm.

Let f be a connectivity function on E. We define a function $f_{\min}$ on pairs of disjoint subsets of E as follows.

$$f_{\min}(A, B) = \min_{A \subseteq Z \subseteq \overline{B}} f(Z)$$

Let us state two auxiliary lemmas.

Lemma 7 (Oum and Seymour [13])**.** *Let A, B, C, D be subsets of E such that $A \cap B = C \cap D = \emptyset$. It holds for a connectivity function f on E,*

$$f_{\min}(A, B) + f_{\min}(C, D) \ge f_{\min}(A \cap C, B \cup D) + f_{\min}(A \cup C, B \cap D).$$

Proof. Let S be a subset of E such that $A \subseteq S \subseteq \overline{B}$ and $f(S) = f_{\min}(A, B)$. Let T be a subset of E such that $C \subseteq T \subseteq \overline{D}$ and $f(T) = f_{\min}(C, D)$. By the submodularity of f, we get

$$f_{\min}(A, B) + f_{\min}(C, D) = f(S) + f(T) \ge f(S \cap T) + f(S \cup T).$$

Since $A \cap C \subseteq S \cap T \subseteq \overline{B \cup D}$, we conclude that $f(S \cap T) \geq f_{\min}(A \cap C, B \cup D)$. Similarly, since $A \cup C \subseteq S \cup T \subseteq \overline{B \cap D}$, we get that $f(S \cup T) \geq f_{\min}(A \cup C, B \cap D)$. The result follows. $\square$

Lemma 8 (Oum and Seymour [13]). *Let $g : 2^E \to \mathbb{Z}$ be a submodular function such that $g(\emptyset) = 0$ and $g(X) \leq g(Y)$ if $X \subseteq Y$. For all $X \subseteq E$, there exists a subset A of X such that $|A| \leq g(X)$ and $g(A) = g(X)$.*

Proof. We proceed by induction on $|X|$. If $X = \emptyset$, then it is trivial.

Suppose $|X| = k > 0$. We assume that this lemma is true when $|X| < k$. Let A be a minimal subset of X such that $g(A) = g(X)$. If $A = \emptyset$, then $g(X) = g(A) = g(\emptyset) = 0 \geq |A|$ and the lemma holds. Let e be an element of A maximizing $g(A \setminus \{e\})$. By our assumption, $g(A \setminus \{e\}) \leq k - 1$.

By the induction hypothesis, there exists a subset B of $A \setminus \{e\}$ such that $|B| \leq k - 1$ and $g(B) = g(A \setminus \{e\})$. If $B = A \setminus \{e\}$, then $|A| \leq k$ and we are done. Thus, we may assume that $B \neq A \setminus \{e\}$ and there exists $d \in (A \setminus \{e\}) \setminus B$. By the choice of e, we know that $g(A \setminus \{d\}) \leq g(A \setminus \{e\})$. Since $B \subseteq A \setminus \{d\}$, we deduce that $g(A \setminus \{e\}) = g(B) \leq g(A \setminus \{d\})$. Therefore $g(A \setminus \{e\}) = g(A \setminus \{d\})$. Moreover, $g(A \setminus \{d, e\}) = g(A \setminus \{e\})$ because $g(B) \leq g(A \setminus \{d, e\}) \leq g(A \setminus \{e\})$. Now let us apply the submodularity:

$$g(A \setminus \{e\}) + g(A \setminus \{d\}) \geq g(A \setminus \{d, e\}) + g(A) \geq g(A \setminus \{e\}) + k$$

We deduce that $g(A \setminus \{e\}) \geq k$, a contradiction. $\square$

Now, we are ready to prove an important lemma implying that every set is a minimal separator for two small sets.

Lemma 9 (Oum and Seymour [13]). *For a connectivity function f on E and a subset Z of E, there exist a subset A of Z and a subset B of $\overline{Z}$ such that*

$$\max\{|A|, |B|\} \leq f_{\min}(A, B) = f(Z).$$

Proof. For a subset X of Z, let $g_1(X) = f_{\min}(X, \overline{Z})$. By Lemma 7, $g_1(X) + g_1(Y) \geq g_1(X \cap Y) + g_1(X \cup Y)$ for two subsets X, Y of Z. In addition, $0 \leq g_1(\emptyset) \leq f(\emptyset) = 0$ and $g_1(X) \leq g_1(Y)$ if $X \subseteq Y \subseteq Z$. By Lemma 8, there exists a subset A of Z such that

$$|A| \leq g_1(Z) = f(Z) \text{ and } g_1(A) = f_{\min}(A, \overline{Z}) = f(Z).$$

For subset X of $\overline{Z}$, let $g_2(X) = f_{\min}(A, X)$. It is again routine to show that g_2 satisfies all conditions of Lemma 8. Therefore there exists a subset B of $\overline{Z}$ such that

$$|B| \leq g_2(\overline{Z}) = f_{\min}(A, \overline{Z}) \text{ and}$$
$$g_2(B) = f_{\min}(A, B) = f_{\min}(A, \overline{Z}) = f(Z).$$

Therefore $\max\{|A|, |B|\} \leq f_{\min}(A, B) = f(Z)$. $\square$

We will apply the following lemma from [13].

Lemma 10 (Oum and Seymour [13]). *Let f be a connectivity function on E, $A, B \subseteq E$ two k-branched sets, and $C \subseteq A \cup B$. If $f(C) \leq k$, then there exists a k-branched set C' such that $C \subseteq C' \subseteq A \cup B$.*

Proof. Pick Z such that $A \setminus B \subseteq Z \subseteq A$ and $f(Z)$ is minimum. We claim that Z and $B \setminus Z$ are k-branched. It is enough to show that for each subset Y of A (or B), if $f(Y) \leq k$, then $f(Y \cap Z) \leq k$ (or $f(Y \setminus Z) \leq k$, respectively). This follows from the submodular inequalities:

$$\begin{aligned} f(Y) + f(Z) &\geq f(Y \cap Z) + f(Y \cup Z) \geq f(Y \cap Z) + f(Z) \quad \text{if } Y \subseteq A, \text{ and} \\ f(Y) + f(Z) &\geq f(Y \setminus Z) + f(Z \setminus Y) \geq f(Y \setminus Z) + f(Z) \quad \text{if } Y \subseteq B. \end{aligned}$$

So Z and $B \setminus Z$ are both k-branched. Now, take a set C' such that $C \subseteq C' \subseteq A \cup B$ and that $f(C')$ is minimum. We use the same trick to show that $Z \cap C'$ and $(B \setminus Z) \cap C'$ are k-branched. Let $Y \subseteq Z$, or $Y \subseteq B \setminus Z$, respectively. In both cases, it follows from the submodularity that

$$f(Y) + f(C') \geq f(Y \cap C') + f(Y \cup C') \geq f(Y \cap C') + f(C')$$

and therefore $Z \cap C'$ and $(B \setminus Z) \cap C'$ are k-branched. Since $f(C') \leq f(C) \leq k$, $(Z \cap C') \cup ((B \setminus Z) \cap C') = C'$ is k-branched. □

A key ingredient of the algorithm of Oum and Seymour [13] is the notion of a *loose tangle* which we now recall. For a connectivity function f on E, a *loose f-tangle* of order $k+1$ is a set $\mathcal{T}$ of subsets of E satisfying the following three axioms:

(L1) $\emptyset \in \mathcal{T}$ and $\{e\} \in \mathcal{T}$ for every $e \in E$ such that $f(\{e\}) \leq k$.

(L2) If $A, B \in \mathcal{T}$, $C \subseteq A \cup B$, and $f(C) \leq k$, then $C \in \mathcal{T}$.

(L3) $E \notin \mathcal{T}$.

The following theorem by Oum and Seymour [13] states that the loose f-tangles are also dual objects to branch-decompositions of connectivity functions.

Theorem 11 (Oum and Seymour [13]). *Let f be a connectivity function on E. Then, no loose f-tangle of order $k+1$ exists if and only if the branch-width of f is at most k.*

Proof. Let $\mathcal{B}$ be the set of all k-branched subsets of E and let $\mathcal{B}' = \{X|X \subseteq Y, Y \in \mathcal{B}, f(X) \le k\}$.

We claim that $\mathcal{B}'$ satisfies (L1) and (L2). (L1) is obvious. To see that (L2) holds, suppose that $A, B \in \mathcal{B}'$ and $C \subseteq A \cup B$ such that $f(C) \le k$. We can assume that A and B are k-branched since we can take instead of them such supersets of A and B that are. By Lemma 10, there exists a k-branched set C' such that $C \subseteq C' \subseteq A \cup B$. Hence $C' \in \mathcal{B}$ and $C \in \mathcal{B}'$.

Now let us prove the theorem. If the branch-width of f is greater than k, then $E \notin \mathcal{B}'$ and so $\mathcal{B}'$ is a loose f-tangle.

If the branch-width of f is at most k, then E is k-branched. It is easy to see that every k-branched set having at least two elements is a union of two proper subsets that are k-branched. By (L1) and (L2), every loose f-tangle should contain all k-branched sets. Since E is k-branched, there is no loose f-tangle. $\square$

A loose tangle can contain exponentially many sets making it difficult to work with in a polynomial-time algorithm. Hence Oum and Seymour [13] introduced a more compact structure, *loose tangle kits*. A pair (P, μ) is called a *loose f-tangle kit* of order $k+1$ if

$$P = \{(A, B)|A, B \subseteq E, A \cap B = \emptyset, \max\{|A|, |B|\} \le f_{\min}(A, B) \le k\}$$

and $\mu : P \to 2^E$ is a function satisfying the following three axioms.

(K1) For every $e \in E$, $f(\{e\}) \le k$, there exists $(A, B) \in P$ such that $A \subseteq \{e\} \subseteq \overline{B}$, $f(\{e\}) = f_{\min}(A, B)$, and $e \in \mu(A, B)$.

(K2) If $(A, B), (C, D), (F, G) \in P$, $F \subseteq X \subseteq (\mu(A, B) \cup \mu(C, D)) \setminus G$, and $f(X) = f_{\min}(F, G)$, then $X \subseteq \mu(F, G)$.

(K3) $\mu(\emptyset, \emptyset) \ne E$.

We will show that a loose f-tangle exists if and only if a loose f-tangle kit exists.

Theorem 12 (Oum and Seymour [13])**.** *Let f be a connectivity function on E. Then, a loose f-tangle of order $k+1$ exists if and only if a loose f-tangle kit of order $k+1$ exists.*

Proof. Suppose that $\mathcal{T}$ is a loose f-tangle of order $k+1$. We construct a loose f-tangle kit of order $k+1$ as follows. Let

$$P = \{(A, B)|A, B \subseteq E, A \cap B = \emptyset, \max\{|A|, |B|\} \le f_{\min}(A, B) \le k\}.$$

For each $(A,B) \in P$, let

$$\mathcal{T}_{A,B} = \{X | A \subseteq X \subseteq \overline{B}, f_{\min}(A,B) = f(X), \text{ and } X \in \mathcal{T}\},$$
$$\mu(A,B) = \bigcup_{X \in \mathcal{T}_{A,B}} X.$$

If $\mathcal{T}_{A,B} = \emptyset$, then $\mu(A,B) = \emptyset$. Notice that $\mu(A,B)$ may be different from $\mu(B,A)$, even though f is symmetric.

First we show that if $(A,B) \in P$, then $\mu(A,B) \in \mathcal{T}$. Since $(A,B) \in P$, we have $f(\emptyset) = 0 \leq f_{\min}(A,B) \leq k$ and therefore $\emptyset \in \mathcal{T}$. So we may assume that $\mathcal{T}_{A,B} \neq \emptyset$. We claim that if $X, Y \in \mathcal{T}_{A,B}$, then $X \cup Y \in \mathcal{T}_{A,B}$. Since $2f_{\min}(A,B) = f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$ and $f(X \cap Y) \geq f_{\min}(A,B)$, $f(X \cup Y) \geq f_{\min}(A,B)$, we have $f(X \cup Y) = f_{\min}(A,B)$. By (L1), $X \cup Y \in \mathcal{T}_{A,B}$. We conclude that $\mu(A,B) \in \mathcal{T}_{A,B} \subseteq \mathcal{T}$.

We claim that (P,μ) is a loose f-tangle kit of order $k+1$. (K3) is trivial by (L3). To show (K2), suppose that $(A,B), (C,D), (F,G) \in P$, $F \subseteq X \subseteq (\mu(A,B) \cup \mu(C,D)) \setminus G$, and $f(X) = f_{\min}(F,G) \leq k$. By (L2), $X \in \mathcal{T}$ and therefore $X \in \mathcal{T}_{F,G}$. So $X \subseteq \mu(F,G)$. Finally, to show (K1), consider an element $e \in E$ such that $f(\{e\}) \leq k$. By Lemma 9, there exists $(A,B) \in P$ such that $f_{\min}(A,B) = f(\{e\})$ and $A \subseteq \{e\} \subseteq \overline{B}$. By (L1), $\{e\} \in \mathcal{T}$ and therefore $\{e\} \in \mathcal{T}_{A,B}$. Thus, $e \in \mu(A,B)$. We conclude that (P,μ) is a loose f-tangle kit of order $k+1$.

Conversely, suppose that (P,μ) is a loose f-tangle kit of order $k+1$. We define

$$\mathcal{T} = \{X | \text{ there exists } (A,B) \in P \text{ such that } A \subseteq X \subseteq \overline{B},$$
$$f_{\min}(A,B) = f(X), \text{ and } X \subseteq \mu(A,B)\}.$$

We claim that $\mathcal{T}$ is a loose f-tangle of order $k+1$. (L3) is trivial by (K3). To show (L2), suppose that $X, Y \in \mathcal{T}$, $Z \subseteq X \cup Y$, and $f(Z) \leq k$. By Lemma 9, there exists $(F,G) \in P$ such that $F \subseteq Z \subseteq \overline{G}$ and $f(Z) = f_{\min}(F,G)$. By construction of $\mathcal{T}$, there are $(A,B), (C,D) \in P$ such that $X \subseteq \mu(A,B)$ and $Y \subseteq \mu(C,D)$. Then $F \subseteq Z \subseteq (\mu(A,B) \cup \mu(C,D)) \setminus G$ and therefore $Z \subseteq \mu(F,G)$. We conclude that $Z \in \mathcal{T}$. Now it remains to show (L1). Consider an element $e \in E$ such that $f(\{e\}) \leq k$. By (K1), there exists $(A,B) \in P$ such that $A \subseteq \{e\} \subseteq \overline{B}$, $f(\{e\}) = f_{\min}(A,B)$, and $e \in \mu(A,B)$. By construction of $\mathcal{T}$, $\{e\} \in \mathcal{T}$. We conclude that $\mathcal{T}$ is indeed a loose f-tangle of order $k+1$. □

2.3.1 Algorithm of Oum and Seymour

Let f be a connectivity function on E. We want to find a polynomial-time algorithm to decide whether the branch-width of f is at most k for fixed k, when f is given by an oracle. Instead of searching for a tree-decomposition of width at most k, we will search for a loose f-tangle kit of order $k+1$.

Algorithm 2 (Oum and Seymour [13]). *Decide whether branch-width of f is at most k.*

(A1) Construct

$$P = \{(A, B) | A, B \subseteq E, A \cap B = \emptyset, \max\{|A|, |B|\} \leq f_{\min}(A, B) \leq k\}.$$

(A2) Let $\mu(\emptyset, \emptyset) = \{e \in E | f(\{e\}) = 0\}$.
For each $e \in E$, if $0 < f(\{e\}) \leq k$, then find a subset B of $E \setminus \{e\}$ such that $|B| \leq f_{\min}(\{e\}, B) = f(\{e\})$. Let $\mu(\{e\}, B) = \{e\}$.
For all other $(A, B) \in P$, let $\mu(A, B) = \emptyset$.

(A3) Test (K3).
If it fails, there is no loose f-tangle kit of order $k+1$. Stop.

(A4) Test (K2).
If it fails, then we have $(A, B), (C, D), (F, G) \in P$ and X such that $F \subseteq X \subseteq (\mu(A, B) \cup \mu(C, D)) \setminus G$, $f(X) = f_{\min}(F, G)$, and $X \not\subseteq \mu(F, G)$. We make $\mu(F, G)$ to be $\mu(F, G) \cup X$, thus increasing $|\mu(F, G)|$ by at least 1. Go back to (A3).

(A5) (P, μ) is a loose f-tangle kit of order $k+1$. Stop.

Proposition 13 (Oum and Seymour [13]). *The running time of the Algorithm 2 is $\mathcal{O}(\gamma n^{8k+5})$, where $n = |E|$ and γ is the time of an oracle query.*

Proof. Let $n = |E|$. We claim that the running time of this algorithm is polynomial in n. We first note that

$$|P| \leq \left(\sum_{i=0}^{k} \binom{n}{i} \right)^2 = \mathcal{O}(n^{2k}).$$

(A1) can be done in polynomial time by using submodular function minimization algorithm from Section 2.2. For (A2), for each e, we may enumerate all subsets V of $E \setminus \{e\}$ having at most $f(\{e\})$ elements such that $f_{\min}(\{e\}, B) = f(\{e\})$. There are at most $\mathcal{O}(n^k)$ subsets of E of size at most k and therefore (A2) can be done in polynomial time. There always exists a set B as in (A2) because of Lemma 9. (A3) is easy.

(A4) is more difficult than others. For every possible triple (A, B), (C, D), $(F, G) \in P$, we try to find X such that

$$F \subseteq X \subseteq (\mu(A, B) \cup \mu(C, D)) \setminus G, f(X) = f_{\min}(F, G), \text{ and}$$
$$X \not\subseteq \mu(F, G). \tag{2.5}$$

Let $U = (\mu(A, B) \cup \mu(C, D)) \setminus G$ to simplify the notation. There is no X satisfying (2.5) if and only if for every $e \in U \setminus \mu(F, G)$, $f_{\min}(A \cup \{e\}, \overline{U}) > f_{\min}(F, G)$. Therefore, to test (K2), we evaluate $f_{\min}$ for each triple (A, B), (C, D), $(F, G) \in P$ and for all $e \in U \setminus \mu(F, G)$. If the test fails, the submodular function minimization algorithm outputs X such that $f(X) = f_{\min}(F, G)$ and $F \cup \{e\} \subseteq X \subseteq U$. Then we increase $|\mu(F, G)|$ by at least 1. The number of iterations of the loop between (A3) and (A4) is at most $\mathcal{O}(n^{2k}) \times \mathcal{O}(n) = \mathcal{O}(n^{2k+1})$. In the (A4) step of each iteration, we test $\mathcal{O}(n^{6k+1})$ choices of triples and elements. To calculate $f_{\min}$, we use the submodular function minimization algorithm from Section 2.2 whose running time is $\mathcal{O}(n^3\gamma)$ where γ is the time to compute $f(X)$ for any X. Thus, our algorithm runs in time $\mathcal{O}(n^{2k+1} n^{6k+1} n^3 \gamma) = \mathcal{O}(\gamma n^{8k+5})$.

□

Let us prove that Algorithm 2 is correct. We need a lemma.

Lemma 14 (Oum and Seymour [13]). *Let f be a connectivity function on E and (P, μ) be a loose f-tangle kit of order $k + 1$. Let $e \in E$ be an element such that $f(\{e\}) \leq k$. For all $(A, B) \in P$ such that $A \subseteq \{e\} \subseteq \overline{B}$, if $f_{\min}(A, B) = f(\{e\})$, then $e \in \mu(A, B)$.*

Proof. By (K1), there exists $(A', B') \in P$ such that $A' \subseteq \{e\} \subseteq \overline{B'}$ and $e \in \mu(A', B')$. Then $A \subseteq \{e\} \subseteq \mu(A', B') \setminus B$ and $f_{\min}(A, B) = f(\{e\})$. By (K2), $e \in \mu(A, B)$. □

Theorem 15 (Oum and Seymour [13]). *Algorithm 2 is correct.*

Proof. If the algorithm stops at (A5), then (P, μ) is clearly a loose f-tangle kit of order k_1, because it satisfies (K1)–(K3).

Now let us assume that the algorithm stops at (A3). We will show that there is no loose f-tangle kit of order $k + 1$. Let μ_i be the function μ after i iterations of (A3).

We claim that if there exists a loose f-tangle kit (P, μ') of order $k+1$, then for all i, μ_i satisfies (K1) and $\mu_i(A, B) \subseteq \mu'(A, B)$ for all $(A, B) \in P$. If this claim is true, then $E = \mu(\emptyset, \emptyset) \subseteq \mu'(\emptyset, \emptyset)$ and therefore (P, μ') violates (K3) — a contradiction.

We proceed by induction on i. Right after (A2) is done (when $i = 0$), (K1) is true. If $\mu_0(A, B) = \emptyset$, then $\mu_0(A, B) \subseteq \mu'(A, B)$ trivially. If $\mu_0(A, B) = \{e\}$ and $f(\{e\}) > 0$, then $A = \{e\}$ and $e \in \mu'(\{e\}, B)$ by Lemma 14 since $f_{\min}(\{e\}, B) = f(\{e\})$. It remains that $(A, B) = (\emptyset, \emptyset)$. Suppose that there is $e \in \mu_0(\emptyset, \emptyset)$, $f(\{e\}) = 0$, such that $e \notin \mu'(\emptyset, \emptyset)$. Then by (K1), there is $(A, B) \in P$, such that $A \subseteq \{e\} \subseteq \overline{B}$, $f(\{e\}) = f_{\min}(A, B)$, and $e \in \mu'(A, B)$. By (K2) we get that $e \in \mu'(\emptyset, \emptyset)$ since $f(\{e\}) = 0 = f_{\min}(\emptyset, \emptyset)$.

Suppose the induction hypothesis is true when $i = m$. When $i = m + 1$, we update $\mu_{m+1}(F, G) = \mu_m(F, G) \cup X$. (K2) implies that $X \subseteq \mu'(F, G)$ and therefore $\mu_{m+1}(F, G) \subseteq \mu'(F, G)$. It is easy to see that (K1) is again true for μ_{m+1}. □

Algorithm 2 decides whether a connectivity function f has branch-width at most k for fixed k by searching for a loose f-tangle kit. But this does not necessarily mean that we can find a branch-decomposition of width at most k when the algorithm outputs that such a branch-decomposition exists. The following procedure to find a branch-decomposition is given in Oum and Seymour [13].

We will use Algorithm 2 as a black box. Let E be a finite set with at least three elements. Let f be a connectivity function on E. For distinct $d, e \in E$, let $E/de = E \setminus \{d, e\} \cup \{de\}$ and let f/de be a connectivity function on E/de defined as follows:

$$(f/de)(X) = \begin{cases} f(X) & \text{if } de \notin X \text{ and} \\ f((X \setminus \{de\}) \cup \{d, e\}) & \text{if } de \in X. \end{cases}$$

Suppose that (T, σ) is a branch-decomposition of f having width at most k. We may assume that no vertex of T has degree two, otherwise we may contract one of the two incident edges. Then T must have two leaves u, v of T sharing a common neighbor w of degree three. Let $d = \sigma(u)$, $e = \sigma(v)$. We claim that f/de has branch-width at most k. To see this, let $T' = T \setminus \{u, v\}$ and let σ' be a bijection between leaves of T' and E/de such that $\sigma'(w) = de$ and $\sigma'(x) = \sigma(x)$ for other leaves of T'. Then it is obvious that (T', σ') is a branch-decomposition of f/de having width at most k.

Conversely, if we have a branch-decomposition (T', σ') of f/de of width at most k, then it is trivial to extend (T', σ') to the branch-decomposition (T, σ) of f as long as $f(\{d\}) \leq k$ and $f(\{e\}) \leq k$. We can attach two leaves u and v to the leaf $\sigma'^{-1}(de)$ of T' corresponding to de and the let $\sigma(u) = d$ and $\sigma(v) = e$.

So the algorithm is as follows. The correctness follows easily from the above argument.

Algorithm 3 (Oum and Seymour [13]). *Output the branch-decomposition of width at most k if there exists.*

(B1) If $|V| < 1$, then no branch-decomposition exists. If $|V| = 2$, then there is a unique branch-decomposition. Its width is determined by f. If $f(\{e\}) > k$ for some $e \in E$, then branch-width is larger than k. Stop.

(B2) Find a pair $\{d, e\}$ of E such that branch-width f/de is at most k by Algorithm 2.

(B3) If no such a pair exists, then the branch-width of f is larger than k. Stop.

(B4) Obtain branch-decomposition (T', σ') of f/de of width at most k by calling this algorithm recursively.

(B5) Extend (T', σ') to a branch-decomposition (T, σ) of f by attaching two leaves u and v to the leaf $\sigma^{-1}(de)$ of T' corresponding to de and then letting $\sigma(u) = d$ and $\sigma(v) = e$.

It is easy to compute the running time of the above algorithm. If M is the running time of Algorithm 2, then Algorithm 3 runs in time $\mathcal{O}(n^3 M)$.

2.3.2 Algorithm of Hliněný and Oum

The parameterized algorithms can be divided to basic two classes according to the dependence of the running time on the parameter. An algorithm with a parameter k is *fixed-parameter tractable* if the time complexity of the algorithm can be written in form $f(k) \times n^{\mathcal{O}(1)}$, i.e., the order of the polynomial of n does not depend on k.

As we have seen, the algorithm of Oum and Seymour is not fixed-parameter tractable. However, Hliněný and Oum [9] used the knowledge on decompositions of matroids to derive an algorithm for finding decompositions of connectivity functions. The algorithm finds a branch-decomposition for a connectivity function f of width at most k or outputs that f has width larger than k in time $\mathcal{O}(n^3)$, for a fixed parameter k.

In the construction of the algorithm, Oum and Seymour put together two previously separate lines of research. Hliněný and Oum combined Oum and Seymour's work on rank-width and on branch-width of submodular functions [12, 13] with Hliněný's works [7, 8] on parameterized algorithms for matroids over finite fields.

In particular, Hliněný [7] has presented a parameterized algorithm running in time $\mathcal{O}(n^3)$ which either outputs a branch-decomposition of width

$3k+1$ of an input matroid M represented over a fixed finite field, or confirms that the branch-width of M is more than $k+1$. Using the ideas of [8] and minor-monotonicity of the branch-width, he has constructed an $O(n^3)$ fixed-parameter tractable algorithm [7] for deciding whether the branch-width of an input matroid M represented over a fixed finite field is at most k.

Chapter 3

Submodular partition functions

3.1 Partition functions and duality

We now introduce the concept of submodular partition functions that provides a unified view on branch-decompositions of connectivity functions and tree-decompositions of graphs.

A *partition function* is a function from the set of all partitions to non-negative integers that satisfies $\psi((\emptyset,\alpha)) = \psi(\alpha)$ for every partition α. A partition function ψ is *submodular* if the following holds for every two partitions (A,α) and (B,β):

$$\psi((A,\alpha)) + \psi((B,\beta)) \geq \psi((A \cup \overline{B}, \alpha \cap B)) + \psi((B \cup \overline{A}, \beta \cap A)) \tag{3.1}$$

The submodularity condition (3.1) is a too strong condition to cover some of the widths like branch-width. Therefore, we have to weaken the condition a little.

A partition function ψ is *weakly submodular* if for every two partitions (A,α) and (B,β) with $\overline{A} \cap \overline{B} \neq \emptyset$, there exists a non-empty set $F \subseteq \overline{A} \cap \overline{B}$ such that at least one of the following holds:

- $\psi((A,\alpha)) \geq \psi((A \cup F, \alpha \setminus F))$
- $\psi((B,\beta)) \geq \psi((B \cup F, \beta \setminus F))$

It is straightforward to verify that a submodular partition function is also weakly submodular. Just consider the set $F = \overline{A} \cap \overline{B}$.

Similarly to branch-decompositions, Amini et al. [1] defined a *decomposition tree* of a partition function ψ. A *decomposition tree* on a finite set E is a tree T with a bijection σ between its leaves and E. Every internal node v of T corresponds to the partition of E whose parts are the leaves contained in subtrees of $T \setminus v$. A decomposition tree is *compatible* with a set of partitions $\mathcal{P}$ of E if all partitions corresponding to the internal nodes of T belong to $\mathcal{P}$.

A *partial decomposition tree* for a partition α is a decomposition tree T with a bijection between its leaves and the parts of α. We call this partition the *displayed partition* of T. The set $\mathcal{P}^{\uparrow}$ consists of all displayed partitions arising from all partial decomposition trees compatible with $\mathcal{P}$. A *partial decomposition tree* for $A \subseteq E$ is a partial decomposition tree for a partition $(\overline{A}|\{e_1\}|\dots|\{e_r\}|)$, $e_i \in A$.

Note that $\mathcal{P}^{\uparrow}$ is exactly the smallest superset of $\mathcal{P}$ such that if (A, α), $(\overline{A}, \beta) \in \mathcal{P}^{\uparrow}$, then $(\alpha, \beta) \in \mathcal{P}^{\uparrow}$. Since the partitions in $\mathcal{P}^{\uparrow}$ come from partial decomposition tree, we can retrace the creation of the partial decomposition tree by contracting a vertex whose all neighbors are leaves but one to a single vertex. We define this more precisely. For any $(A, \alpha) \in \mathcal{P}^{\uparrow}$, there exists $(C, \gamma) \in \mathcal{P}$, $A \subseteq C$, such that either $(A, \alpha) = (C, \gamma)$ or $(A, \alpha) = (A, \gamma, \mu)$ for some $(\overline{C}, \mu, A) \in \mathcal{P}^{\uparrow}$. Such a (C, γ) *decomposes* (A, α).

Amini et al. [1] introduced in their paper the notion of a bramble for submodular partition functions. A *$\mathcal{P}$-bramble* $\mathcal{B}$ on E is a set of pairwise intersecting subsets of E which contains a part of every partition of $\mathcal{P}$. We say that $\mathcal{P}$ *admits* the bramble $\mathcal{B}$.

Lemma 16 (Lyaudet, Mazoit, Thomassé [11])**.** *A set $\mathcal{B}$ is a $\mathcal{P}$-bramble if and only if $\mathcal{B}$ is a $\mathcal{P}^{\uparrow}$-bramble.*

Proof. For two partitions $(A, \alpha), (\overline{A}, \beta) \in \mathcal{P}^{\uparrow}$, if $\mathcal{B}$ meets both of them, it cannot contain both A and $\overline{A}$ so it meets (α, β). The forward implication follows.

The backwards implication follows from $\mathcal{P} \subseteq \mathcal{P}^{\uparrow}$. $\square$

Fix a set $\mathcal{S} \subseteq 2^E$ closed under taking subsets; the sets of $\mathcal{S}$ are referred as *small* and the sets *not* in $\mathcal{S}$ are referred as *big*. A $\mathcal{P}$-bramble $\mathcal{B}$ is *$\mathcal{S}$-big* if $\mathcal{B} \cap \mathcal{S} = \emptyset$ and a partition $\alpha \in \mathcal{P}$ is *$\mathcal{S}$-small* if all the parts of α are contained in $\mathcal{S}$. The set of partitions $\mathcal{P}$ is *dualising* if for any $\mathcal{S} \subseteq 2^E$ closed under taking subsets there exists an $\mathcal{S}$-big bramble if and only if $\mathcal{P}$ has no $\mathcal{S}$-small partition.

Let $\alpha = \{A_i | i \in I\}$ and $\beta = \{B_i | i \in J\}$ be coverings of E. We say that α is *smaller* than β if $|I| \leq |J|$ and α is *finer* than β if there exists a bijection $f : I \to J$ such that $A_i \subseteq B_{f(i)}$, for all $i \in I$. Note that if α is finer than β it is also smaller than β.

A set of partitions $\mathcal{P}$ is *refining* if for any two partitions $(A, \alpha), (B, \beta) \in \mathcal{P}$ with A and B disjoint, $\mathcal{P}$ contains a partition finer than the covering (α, β).

Theorem 17 (Lyaudet, Mazoit, Thomassé [11])**.** *$\mathcal{P}$ is refining if and only if it is dualising.*

Proof. Suppose that $\mathcal{P}$ is refining and contains no $\mathcal{S}$-small partition for $\mathcal{S} \subseteq 2^E$. There exists a set closed under taking supersets that contains a big part of every partition in $\mathcal{P}$. We claim that any such $\mathcal{B}$ taken inclusion-wise minimal is an $\mathcal{S}$-big $\mathcal{P}$-bramble. Clearly, $\mathcal{B}$ contains one part of every partition in $\mathcal{P}$ that is not $\mathcal{S}$-small. We have to show that every two sets of $\mathcal{B}$ intersect. If not, take A, B inclusion-wise minimal disjoint sets in $\mathcal{B}$. Since $\mathcal{B} \setminus \{A\}$ and $\mathcal{B} \setminus \{B\}$ are upward close and $\mathcal{B}$ is minimal, there exists $(A, \alpha), (B, \beta) \in \mathcal{P}$ such that $\mathcal{B}$ does not meet (α, β). The fact that $\mathcal{P}$ is refining implies that $\mathcal{P}$ contains γ finer than (α, β). Since $\mathcal{B}$ is closed under taking superset, $\gamma \cap \mathcal{B}$ is empty, a contradiction.

For the backwards implication, suppose that $\mathcal{P}$ is not refining. Let $(A, \alpha), (B, \beta) \in \mathcal{P}$ with A and B disjoint and such that $\mathcal{P}$ contains no partition finer than (α, β). Consider the set of all subsets of parts of (α, β) as a set $\mathcal{S}$ of small sets. We claim that $\mathcal{P}$ contains no $\mathcal{S}$-small partition and that no $\mathcal{S}$-big $\mathcal{P}$-bramble exists. Indeed, since no partition of $\mathcal{P}$ is finer than (α, β), $\mathcal{P}$ contains no $\mathcal{S}$-small partition. Since a bramble cannot contain both A and B as they are disjoint, it must contain a small set to meet both (A, α) and (B, β). We conclude that no $\mathcal{S}$-big $\mathcal{P}$-bramble exists. ☐

A set of partitions $\mathcal{P}$ is *pushing* if for every pair of partitions (A, α) and (B, β) in $\mathcal{P}$ with $\overline{A} \cap \overline{B} \neq \emptyset$, there exists a non-empty set $F \subseteq \overline{A} \cap \overline{B}$ with $(A \cup F, \alpha \setminus F) \in \mathcal{P}$ or $(B \cup F, \beta \setminus F) \in \mathcal{P}$.

Theorem 18 (Lyaudet, Mazoit, Thomassé [11])**.** *If $\mathcal{P}$ is pushing, then $\mathcal{P}^{\uparrow}$ is refining.*

Proof. Suppose for a contradiction that $\mathcal{P}$ is pushing, (A, α), (B, β) belong to $\mathcal{P}^{\uparrow}$ with A and B disjoint, and yet $\mathcal{P}^{\uparrow}$ contains no partition finer than (α, β). Choose (A, α) and (B, β) such that (α, β) is the smallest and then the finest. Let $O = o((\alpha, \beta)) = \overline{A} \cap \overline{B}$, i.e., O contains those elements that are contained in two parts of (α, β).

Let (C, γ) and (D, δ) decompose (A, α) and (B, β), respectively. Since $A \subseteq C$ and $B \subseteq D$, $\overline{C} \cap \overline{D} \subseteq \overline{A} \cap \overline{B} = O$. We claim that $O \subseteq \overline{C} \cap \overline{D}$. Suppose for a contradiction that $O \not\subseteq \overline{C}$. Hence $C \cap O \neq \emptyset$ and since $O \subseteq \overline{A}$, $O \cap A = \emptyset$ and we conclude that $(C, \gamma) \neq (A, \alpha)$. Since (C, γ) decomposes (A, α), let $(\overline{C}, \mu, A) \in \mathcal{P}^{\uparrow}$ be such a partition that $(A, \gamma, \mu) = (A, \alpha)$. Since $(\overline{C}, \mu, A)$ is strictly smaller than (A, α), there exists $(C', \mu', \beta') \in \mathcal{P}^{\uparrow}$ finer than $(\overline{C}, \mu, \beta)$. Now, (C, γ) and (C', μ', β') belong to $\mathcal{P}^{\uparrow}$ with C and C' disjoint and thus the set (γ, μ', β') is a covering of E with an overlap $o((\gamma, \mu', \beta'))$ a subset of $\overline{C}$. Therefore the covering (γ, μ', β') is strictly finer than (α, β), a contradiction.

So suppose that $\overline{C} \cap \overline{D} = O$. Since $\mathcal{P}$ is pushing and $\overline{C} \cap \overline{D} = O$ is non-empty, let $F \subseteq O$ be a non-empty set such that, say, $(C \cup F, \gamma \setminus F) \in \mathcal{P}$.

If $(C, \gamma) = (A, \alpha)$, then $(\gamma \setminus F, \beta) = (\alpha \setminus F, \beta)$ is strictly finer than (α, β), a contradiction.

If $(C, \gamma) \neq (A, \alpha)$, let $(\overline{C}, \mu, A) \in \mathcal{P}^{\uparrow}$ with $(A, \gamma, \mu) = (A, \alpha)$. Since $(\overline{C}, \mu, A)$ is strictly smaller than (A, α), there exists $(C', \mu', \beta') \in \mathcal{P}^{\uparrow}$ finer than $(\overline{C}, \mu, \beta)$. Since $O \subseteq \overline{C}$, $\mu = \mu'$. If $O \cap C' \neq \emptyset$, then the covering (γ, μ, β') is strictly finer than (α, β), a contradiction. Suppose that $O \cap C' = \emptyset$. Since $C \cup F$ and $C' \subseteq \overline{C} \setminus O$ are disjoint, the set $(\gamma \setminus F, \mu, \beta')$ is a covering of E. Since the overlap $o((\gamma \setminus F, \mu, \beta'))$ is a subset of $O \setminus F$, it is strictly finer than (α, β), a contradiction. $\square$

Let $\mathcal{P}_k[\psi]$ denote the set of partitions α of E such that $\psi(\alpha) \leq k$.

Corollary 19 (Lyaudet, Mazoit, Thomassé [11])**.** *For a weakly submodular partition function ψ, $\mathcal{P}_k[\psi]^{\uparrow}$ is dualising.*

Proof. First we show that $\mathcal{P}_k[\psi]$ is pushing. Let (A, α) and (B, β) be two partitions from $\mathcal{P}_k[\psi]$ with $\overline{A} \cap \overline{B} \neq \emptyset$. By weak submodularity of ψ, there is a non-empty set $F \subseteq \overline{A} \cap \overline{B}$ such that $\psi((A, \alpha)) \geq \psi((A \cup F, \alpha \setminus F))$ or $\psi((B, \beta)) \geq \psi((B \cup F, \beta \setminus F))$. Since $\psi((A, \alpha)) \leq k$ and $\psi((B, \beta)) \leq k$, we conclude that at least one of $\psi((A \cup F, \alpha \setminus F))$ and $\psi((B \cup F, \beta \setminus F))$ is at most k and thus at least one of partitions $(A \cup F, \alpha \setminus F)$ and $(B \cup F, \beta \setminus F)$ belongs to $\mathcal{P}_k[\psi]$.

Theorem 18 now implies that $\mathcal{P}_k[\psi]^{\uparrow}$ is refining and Theorem 17 that $\mathcal{P}_k[\psi]^{\uparrow}$ is dualising. $\square$

The *width* of a weakly submodular partition function ψ is the smallest integer k such that there exists a decomposition tree compatible with $\mathcal{P}_k[\psi]$. The concepts of submodular partition functions and decomposition trees include graph tree-width and branch-width as special cases as we now explain.

3.1.1 Partition function for tree-width

The tree-width corresponds to the width of a particular partition function δ. For a graph $G = (V, E)$, δ is defined on the set of partitions of E as the size of the border of the partition, $\delta(\alpha) = |\Delta(\alpha)|$, where

$$\Delta(\alpha) = \{x \in V | \exists xy \in A_i \text{ and } \exists xz \in A_j, i \neq j\}$$

is the *border* of α. This definition extends the definition of border for bipartitions given in Section 1.3.

Proposition 20 (Amini, Mazoit, Nisse, Thomassé [1])**.** *The partition function δ is submodular.*

Proof. Let $G = (V, E)$ be a graph. Let (A, α) and (B, β) be partitions of E. We want to prove that:

$$\delta((A, \alpha)) + \delta((B, \beta)) \geq \delta((A \cup \overline{B}, \alpha \cap B)) + \delta((B \cup \overline{A}, \beta \cap A)) \tag{3.2}$$

Let x be a vertex of G. These cases can happen:

- The contribution of x to the right-hand side of (3.2) is zero.
- The contribution of x to the right-hand side of (3.2) is one, say x belongs to the border of $(A \cup \overline{B}, \alpha \cap B)$. If x belongs to the border of B, it contributes to $\delta((B, \beta))$. If not, x belongs to the border of some part of α. In both cases, its contribution to the left-hand term is at least one.
- Assume now that x belongs both to the borders of $\delta((A \cup \overline{B}, \alpha \cap B))$ and $\delta((B \cup \overline{A}, \beta \cap A))$. Since x belongs to the border of $\delta((A \cup \overline{B}, \alpha \cap B))$, there is an edge e_x containing x in some $A_i \cap B$, where A_i is one of parts of α. Similarly, there is an edge f_x containing x in some $B_j \cap A$, where B_j is one of parts of β. Since $e_x \notin A$ and $f_x \in A$, x is in the border of (A, α). Similarly, x is also in the border of (B, β), and thus contributes also twice to the left-hand side of (3.2).

□

The following proposition shows that the tree-width of a graph G is equal to the branch-width of the submodular partition function δ. A bramble $\mathcal{B}$ is *non-principal* if it contains no singleton.

Proposition 21 (Amini, Mazoit, Nisse, Thomassé [1]). *Let $G = (V, E)$ be a graph with minimum degree at least two and let δ be the partition function as defined above. There exists a bramble in G of order $k + 1$ if and only if there exists a non-principal $\mathcal{P}_k[\delta]$-bramble.*

Proof. Given a subset X of vertices of G, let $E(X)$ denote the set of edges incident to at least one vertex in X. The key idea behind this proof is that, in a graph without isolated vertices, two sets of vertices X and Y touch if and only if $E(X)$ and $E(Y)$ intersect.

Suppose that G has a bramble $\mathcal{B}$ of order $k + 1$. Let $\alpha = \{A_i | i \in I\}$ be a partition of E that belongs to $\mathcal{P}_k[\delta]$. Since $\mathcal{B}$ has order $k + 1$, there is an element B of $\mathcal{B}$ disjoint from $\Delta(\alpha)$. Let A_α be the part of α containing $E(B)$. Let $\mathcal{B}'$ be the set of all these sets A_α for all partitions $\alpha \in \mathcal{P}_k[\delta]$. We claim that $\mathcal{B}'$ is a $\mathcal{P}_k[\delta]$-bramble. Indeed, let X and Y be some elements of $\mathcal{B}'$. Assume that X and Y contain $E(B_X)$ and $E(B_Y)$ for $B_X \in \mathcal{B}$ and $B_Y \in \mathcal{B}$.

Since B_X and B_Y touch, $\emptyset \neq E(B_X) \cap E(B_Y) \subseteq X \cap Y$. This proves that $\mathcal{B}'$ is a $\mathcal{P}_k[\delta]$-bramble.

Since G has minimum degree at least two, if a partition α contains a singleton $\{e\}$, both ends of e are in the border of α. Hence there is no non-empty set of vertices B such that $E(B) \subseteq \{e\}$, the sets A_α cannot be singletons and therefore $\mathcal{B}'$ is non-principal.

Assume now that E has a non-principal $\mathcal{P}_k[\delta]$-bramble $\mathcal{B}'$. Let us fix a subset $S \subseteq V$ of size at most k. For S there are non-principal partitions in $\mathcal{P}_k[\delta]$ whose border is a subset of S, i.e., $\Delta(\alpha) \subseteq S$. We choose such a partition $\alpha = \{A_i | i \in I\} \in \mathcal{P}_k[\delta]$ where the sets A_i are minimal in inclusion. Since $\mathcal{B}'$ is a non-principal $\mathcal{P}_k[\delta]$-bramble, one of A_i with at least two edges is in $\mathcal{B}'$. Since A_i is minimal, $X_i = V(A_i) \setminus S$ induces a nonempty connected subgraph in G. Now, let $\mathcal{B}$ be the set of these A_i for all sets S. We claim that $\mathcal{B}$ is a bramble of order $k+1$. Indeed, let X and Y be any two elements of $\mathcal{B}$. Since $E(X)$ and $E(Y)$ both belong to $\mathcal{B}'$, $E(X) \cap E(Y) \neq \emptyset$ and thus X and Y touch. Hence $\mathcal{B}$ is a bramble. And since any hitting set of $\mathcal{B}$ has at least $k+1$ elements, the order of $\mathcal{B}$ is at least $k+1$. □

By Theorem 1, Corollary 19, and Proposition 21, there is a tree-decomposition of a graph $G = (V, E)$ of width at most k if and only if there is a decomposition tree of E compatible with $\mathcal{P}_k[\delta]$.

3.1.2 Partition function for branch-width

The branch-width of a connectivity function f corresponds to the width of the weakly submodular partition function $(\max_f)_3$, i.e., the maximum $f(A_i)$ of partitions α containing at most three parts. A formal definition and a proof of weak submodularity of this function is given in this section.

For a weakly submodular partition function ψ we define a partition function ψ_p by letting $\psi_p(\alpha) = \psi(\alpha)$ when the number of non-empty parts of α is at most p and $\psi_p(\alpha) = +\infty$ otherwise.

Proposition 22 (Amini, Mazoit, Nisse, Thomassé [1])**.** *Let ψ be a weakly submodular partition function and $p \geq 2$ an integer. Then the function ψ_p is a weakly submodular partition function.*

Proof. Let (A, α) and (B, β) be partitions with $\overline{A} \cap \overline{B}$ non-empty. By weak submodularity of ψ, there is $F \subseteq \overline{A} \cap \overline{B}$ such that, say, $\psi(A, \alpha) \geq \psi(A \cup F, \alpha \setminus F)$. If (A, α) has at most p parts, then $\psi_p(A, \alpha) \geq \psi_p(A \cup F, \alpha \setminus F)$ since $(A \cup F, \alpha \setminus F)$ has also at most p parts. If (A, α) has more than p parts, then $\psi_p(A, \alpha) = +\infty \geq \psi_p(A \cup F, \alpha \setminus F)$. □

For a connectivity function f, we define a partition function $\max_f$ by $\max_f(\alpha) = \max\{f(A)|A \in \alpha\}$.

Proposition 23 (Lyaudet, Mazoit, Thomassé [11]). *Let f be a connectivity function. Then the function $\max_f$ is a weakly submodular partition function.*

Proof. Let (A, α) and (B, β) be partitions with $\overline{A} \cap \overline{B}$ non-empty. Let F be such that $A \setminus B \subseteq F \subseteq \overline{B \setminus A}$ and such that $f(F)$ is minimum. We claim that $\max_f((A, \alpha)) \geq \max_f((A \cup F, \alpha))$.

By the choice of F, $f(F \cap A) \geq f(F)$, and by submodularity, since $f(A) + f(F) \geq f(A \cap F) + f(A \cup F)$, we have $f(A) \geq f(A \cup F)$. For every $X \in \alpha$, we have by submodularity of f:

$$f(X) + f(\overline{F}) \geq f(X \cap \overline{F}) + f(X \cup \overline{F}) \tag{3.3}$$

Since $f(F)$ is minimum, $f(F) \leq f(F \setminus X)$, and thus f being symmetric:

$$f(X \cup \overline{F}) \geq f(\overline{F}) \tag{3.4}$$

Comparing (3.3) and (3.4), we obtain $f(X) \geq F(X \cap \overline{F})$. Therefore

$$\max_f((A, \alpha)) \geq \max_f((A \cup F, \alpha \setminus F)),$$

as claimed.

Similarly $\max_f((B, \beta)) \geq \max_f((B \cup \overline{F}, \beta \cap F))$. Now at least one of $F_A = F \cap (\overline{A} \cap \overline{B})$ and $F_B = \overline{F} \cap (\overline{A} \cap \overline{B})$, say F_A, is non-empty. Since $(A \cup F, \alpha \setminus F) = (A \cup F_A, \alpha \setminus F_A)$, there exists a non-empty $F_A \subseteq \overline{A} \cap \overline{B}$ with $\max_f((A, \alpha)) \geq \max_f((A \cup F_A, \alpha \setminus F_A))$ which proves that $\max_f$ is weakly submodular. □

Now, we will link the tangles with brambles.

Lemma 24 (Amini, Mazoit, Nisse, Thomassé [1]). *For every A, B, C in a $\mathcal{P}_k[(\max_f)_3]$-bramble $\mathcal{B}$, the intersection $A \cap B \cap C$ is non-empty.*

Proof. Suppose for the sake of contradiction that there exists $A, B, C \in \mathcal{B}$ with $A \cap B \cap C = \emptyset$. Choose A, B, C inclusion-wise maximal with this property. Since by submodularity and symmetry

$$f(A) + f(B) = f(A) + f(\overline{B}) \geq f(A \cap \overline{B}) + f(A \cup \overline{B}) = f(A \setminus B) + f(B \setminus A),$$

we can assume that $A \setminus B$ is small, i.e., $f(A \setminus B) \leq k$. We now claim that $A \cap C$ is small.

Indeed, let $C' = (A \setminus B) \cup C$. Either $A \setminus B \subseteq C$, and, since $A \cap B \cap C = \emptyset$, $A \setminus B = A \cap C$ and the claim follows. Or $A \setminus B \not\subseteq C$ and $C' \notin \mathcal{B}$ by the choice of A, B, and C, since $A \cap B \cap C' = \emptyset$. Since $\overline{C'} \cap C = \emptyset$, $\overline{C'}$ is also not in $\mathcal{B}$ and we conclude that $f(C') > k$. By submodularity of f, we have

$$f(A \setminus B) + f(C) \geq f(C') + f((A \setminus B) \cap C).$$

Therefore $(A \setminus B) \cap C$ is small. Finally, since $A \cap B \cap C = \emptyset$, $A \cap C = (A \setminus B) \cap C$ and the claim follows.

Similarly as above, we get by submodularity that

$$f(A) + f(\overline{C}) \geq f(A \setminus C) + f(C \setminus A).$$

Hence at least one of $A \setminus C$ or $C \setminus A$ is small. Suppose that $A \setminus C$ is small. Since the sets $\overline{A}$, $A \cap C$, and $A \setminus C$ are all small and mutually disjoint, the partition $(\overline{A}, A \cap C, A \setminus C)$ belongs to $\mathcal{P}_k[(\max_f)_3]$. But this is impossible since these three sets are respectively disjoint from A, B, and C which all belong to $\mathcal{B}$. Now suppose that $C \setminus A$ is small. Similarly as above, the sets $\overline{C}$, $A \cap C$ and $C \setminus A$ are all small and mutually disjoint and the partition $(\overline{C}, A \cap C, C \setminus A)$ belongs to $\mathcal{P}_k[(\max_f)_3]$. This is a contradiction since these three sets are respectively disjoint from C, B, and A which all belong to $\mathcal{B}$. □

Now we are ready to prove the claimed equivalence between the branch-width of a connectivity function f and the branch-width of the weakly submodular partition function $(\max_f)_3$.

Proposition 25 (Amini, Mazoit, Nisse, Thomassé [1])**.** *An f-tangle of order k exists if and only if a non-principal $\mathcal{P}_k[(\max_f)_3]$-bramble does.*

Proof. Let $\mathcal{T}$ be an f-tangle of order k. We claim that the set of complements of $\mathcal{T}$, $\mathcal{B} = \{\overline{A} | A \in \mathcal{T}\}$, is a non-principal $\mathcal{P}_k[(\max_f)_3]$-bramble. First, observe that the condition that $A \cup B \neq E$, for $A, B \in \mathcal{T}$, implies that the complements $\overline{A}, \overline{B} \in \mathcal{B}$ are intersecting.

For every partition (A, B, C) from $\mathcal{P}_k[(\max_f)_3]$, exactly one set among A, B, and C does not belong to $\mathcal{T}$, since $A \cup B \cup C = E$ and $\overline{A} \cup \overline{B} = E$. Therefore $\mathcal{B}$ contains an element of every partition in $\mathcal{P}_k[(\max_f)_3]$. Finally, (T3) imposes that $\mathcal{B}$ is non-principal.

Now, let $\mathcal{B}$ be a non-principal $\mathcal{P}_k[(\max_f)_3]$-bramble. We claim that $\mathcal{T} = \{\overline{A} | A \in \mathcal{B}\}$ is an f-tangle of order k. By construction, $\mathcal{T}$ satisfies (T1). (T2) follows directly from Lemma 24. Finally, $\mathcal{B}$ being non-principal imposes (T3). □

3.2 Loose tangles

Similarly to the loose tangles of Oum and Seymour [13] we introduce *loose tangles* for submodular partition functions. A *loose $\mathcal{P}$-tangle* is a set $\mathcal{T}$ of subsets of E closed under taking subsets satisfying the following three axioms.

(P1) $\emptyset \in \mathcal{T}$, $\{e\} \in \mathcal{T}$, for all $e \in E$ such that the partition $(\{e\}, \overline{\{e\}})$ belongs to $\mathcal{P}$.

(P2) If $A_1, A_2, \ldots, A_p \in \mathcal{T}$, $C_i \subseteq A_i$, for $i = 1, \ldots, p$, $(C_1, \ldots, C_p, \overline{\cup_{i=1}^p C_i}) \in \mathcal{P}$, then $\cup_{i=1}^p C_i \in \mathcal{T}$.

(P3) $E \notin \mathcal{T}$.

To prove the main theorem of this section, we need a lemma.

Lemma 26. *Let ψ be a submodular partition function and (A, α) a partition. Then $\psi((A, \alpha)) \geq \psi((A, \overline{A}))$.*

Proof. Suppose that the partition (A, α) has at least three non-empty parts and let $(A, B, \beta) = (A, \alpha)$. By submodularity,

$$\begin{aligned}\psi((A, \alpha)) + \psi((B, \overline{B})) &\geq \psi((A \cup \overline{B}, \alpha \cap B)) + \psi((B \cup \overline{A}, \overline{B} \cap A)) \\ &= \psi((B, \overline{B})) + \psi((A, \overline{A})).\end{aligned}$$

The result follows. □

In the following theorem, we show that for classes of partitions of bounded width, the loose tangle is a dual object to the decomposition tree.

Theorem 27. *Let ψ be a submodular partition function. There is no decomposition tree compatible with $\mathcal{P}_k[\psi]$ if and only if there is a loose $\mathcal{P}_k[\psi]$-tangle.*

Proof. Suppose there is a decomposition tree (T, σ) compatible with $\mathcal{P}_k[\psi]$ and a loose $\mathcal{P}_k[\psi]$-tangle $\mathcal{T}$. We will show that $\mathcal{T}$ violates (P3). Choose an arbitrary leaf x of T as a root. Every internal node v of T corresponds to a partition α_v. Let C_v be a union of all parts of α_v except the one containing x. Define C_v of a leaf v as the singleton $\sigma(v)$. We will show by backward induction on the distance from x that for every node v of T, the set C_v belongs to $\mathcal{T}$. Since T is a decomposition tree of E compatible with $\mathcal{P}_k[\psi]$, there is a partition $(\{e\}, \alpha_e)$ in $\mathcal{P}_k[\psi]$, for each $e \in E$. By Lemma 26, $\psi((\{e\}, \overline{\{e\}})) \leq \psi((\{e\}, \alpha_e))$. Hence, $(\{e\}, \overline{\{e\}})$ belongs to $\mathcal{P}_k[\psi]$ and $\{e\}$ is in $\mathcal{T}$ by (P1). For an inner node v, all his children $u_1, \ldots, u_p$ are farther from x than v and therefore all C_{u_i} are in $\mathcal{T}$. By (P2), since $(C_{u_i}, \overline{\cup C_{u_i}})$ belongs to $\mathcal{P}_k[\psi]$, $C_v \equiv \cup C_{u_i} \in \mathcal{T}$. Finally, let v be the only child of x. Since $C_v \in T$ and $\{\sigma(x)\} \in \mathcal{T}$, by (P2), $C_v \cup \{\sigma(x)\} = E$ also belongs to $\mathcal{T}$. (P3) is now violated.

Define $\mathcal{T}$ to be a subset of 2^E closed under taking subsets, containing all singletons and all k-branched sets. We will show that $\mathcal{T}$ is a loose tangle. (P1) trivially holds since all k-branched singletons are in $\mathcal{T}$. Let $A_1, \ldots, A_p \in \mathcal{T}$ and $C_i \subseteq A_i$, $i = 1, \ldots, p$, such that $(C_1, \ldots, C_p, \overline{\cup C_i}) \in \mathcal{P}_k[\psi]$. We can assume that A_i are k-branched (otherwise take such a superset of it instead). Let $Y_1, \ldots, Y_p$, $Y_i \subseteq A_i$, be such sets that $\cup C_i \subseteq \cup Y_i$ and $\psi((Y_1, \ldots, Y_p, \overline{\cup Y_i}))$ is minimum. We will show that the set $\cup Y_i$ is k-branched.

To this end, we modify the partial decomposition tree T_i for A_i to be a partial decomposition tree for Y_i. At first, we delete from T_i all leaves corresponding to elements not in Y_i. We then repeatedly contract all nodes of degree two or less until we get a ternary tree T_i'. We claim T_i' is compatible with $P_k[\psi]$. Suppose for a contradiction that there is an internal node v' of T_i' corresponding to an internal node v of T_i such that $\alpha_{v'} \notin P_k[\psi]$. Assume $i = 1$ since we can relabel the parts so. Let $(A, \alpha) = \alpha_v$ such that A is the part of α_v that contains $\overline{A_1}$. We infer from the submodularity of the function ψ that

$$\begin{aligned}&\psi((A,\alpha)) + \psi((Y_1, Y_2, \ldots, Y_p, \overline{\cup Y_i})) \geq \psi((A \cup \overline{Y}_1, \alpha \cap Y_1)) \\ &\quad + \psi((Y_1 \cup \overline{A}, Y_2 \cap A, \ldots Y_p \cap A, \overline{\cup Y_i} \cap A))\end{aligned}$$

The choice of $Y_1, \ldots, Y_p$ yields that

$$\psi((Y_1 \cup \overline{A}, Y_2 \cap A, \ldots, Y_p \cap A, \overline{\cup Y_i} \cap A)) \geq \psi((Y_1, \ldots, Y_p, \overline{\cup Y_i})).$$

Hence, $\psi((A \cup \overline{Y}_1, \alpha \cap Y_1)) \leq \psi((A, \alpha)) \leq k$ and T_1' is compatible with $\mathcal{P}_k[\psi]$.

Now, construct a partial decomposition tree T by connecting T_i' to a single node corresponding to a partition $(Y_1, \ldots, Y_p, \overline{\cup Y_i})$. This partition belongs to $\mathcal{P}_k[\psi]$ since $\psi((Y_1, \ldots, Y_p, \overline{\cup Y_i})) \leq \psi((C_1, \ldots, C_p)) \leq k$ by the minimality of $\psi((Y_1, \ldots, Y_p, \overline{\cup Y_i}))$. Therefore T is a partial decomposition tree for $\cup Y_i$ compatible with $\mathcal{P}_k[\psi]$ and thus $\cup Y_i \in \mathcal{T}$. Since $\cup C_i \subseteq \cup Y_i$, also $\cup C_i \in \mathcal{T}$ as required.

If $E \in T$, then E is k-branched and the partial decomposition tree for E is actually a decomposition tree for ψ. This contradicts the fact that ψ does not have a decomposition tree compatible with $\mathcal{P}_k[\psi]$. Therefore, $E \notin \mathcal{T}$ and (P3) holds. We conclude that $\mathcal{T}$ is a loose $\mathcal{P}_k[\psi]$-tangle. $\square$

3.3 Hardness of submodular partition functions

We first have to define several auxiliary functions before we can establish our hardness result. Let g_n be the function $g_n : 2^E \to \mathbb{N}$ for $E = \{1, \ldots, 2n\}$ defined as $g_n(X) = \min\{|X|, |\overline{X}|\}$. We start our exposition with showing that g_n is submodular.

Lemma 28. *The function g_n is submodular for every n.*

Proof. Consider two subsets X and Y. If both $|X| \leq n$ and $|Y| \leq n$, then

$$\begin{aligned} g_n(X) + g_n(Y) &= |X| + |Y| = |X \cap Y| + |X \cup Y| \\ &\geq g_n(X \cap Y) + g_n(X \cup Y). \end{aligned}$$

If both $|X| > n$ and $|Y| > n$, we get the same result by the symmetry of g.

$$\begin{aligned} g_n(X) + g_n(Y) &= g_n(\overline{X}) + g_n(\overline{Y}) \geq g_n(\overline{X} \cap \overline{Y}) + g_n(\overline{X} \cup \overline{Y}) \\ &= g_n(X \cup Y) + g_n(X \cap Y) \end{aligned}$$

So suppose that $|X| > n$ and $|Y| \leq n$. We get

$$\begin{aligned} g_n(X) + g_n(Y) &= |\overline{X}| + |Y| = |\overline{X} \setminus Y| + |Y \setminus \overline{X}| + 2|\overline{X} \cap Y| \\ &\geq g_n(\overline{X} \setminus Y) + g_n(Y \setminus \overline{X}) = g_n(\overline{X} \cap \overline{Y}) + g_n(X \cap Y) \\ &= g_n(X \cup Y) + g_n(X \cap Y). \end{aligned}$$

This finishes the proof. ∎

The function g_n can be extended to a partition function ϕ_n on the ground set $E = \{1, \ldots, 2n\}$ by setting

$$\phi_n(\alpha) = \max_{i \in I} g_n(A_i).$$

A part A_i of α is *dominating* if $g_n(A_i) = \phi_n(\alpha)$. Note that, if α has a part with at least n elements, then that part is dominating.

We proceed by showing that the function ϕ_n is submodular.

Lemma 29. *The function ϕ_n is submodular for every n.*

Proof. We check the following inequality for all partitions (A, α) and (B, β):

$$\phi_n((A, \alpha)) + \phi_n((B, \beta)) \geq \phi_n((A \cup \overline{B}, \alpha \cap B)) + \phi_n((B \cup \overline{A}, \beta \cap A)).$$

Observe that at least one of the parts $A \cup \overline{B}$ or $B \cup \overline{A}$ in this inequality is dominating since one of $A, \overline{A}$ and one of $B, \overline{B}$ has at least n elements. If both $A \cup \overline{B}$ and $B \cup \overline{A}$ are dominating, then the submodularity of ϕ_n follows from the submodularity of g:

$$\begin{aligned}\phi_n((A,\alpha)) + \phi_n((B,\beta)) &\geq g_n(A) + g_n(B) = g_n(A) + g_n(\overline{B})\\ &\geq g_n(A \cap \overline{B}) + g_n(A \cup \overline{B}) = g_n(\overline{A} \cup B) + g_n(A \cup \overline{B})\\ &= \phi_n((A \cup \overline{B}, \alpha \cap B)) + \phi_n((B \cup \overline{A}, \beta \cap A))\end{aligned}$$

Suppose that $A \cup \overline{B}$ is not dominating, so take an $A_i \in \alpha$ such that $A_i \cap B$ is dominating. Since $|B| \geq n$ and $A_i \subseteq \overline{A}$, it holds that $g_n(A_i \cup B) \geq g_n(B \cup \overline{A})$. We use this inequality to prove the submodularity as follows:

$$\begin{aligned}\phi_n((A,\alpha)) + \phi_n((B,\beta)) &\geq g_n(A_i) + g_n(B) \geq g_n(A_i \cap B) + g_n(A_i \cup B)\\ &\geq g_n(A_i \cap B) + g_n(B \cup \overline{A})\\ &= \phi_n((A \cup \overline{B}, \alpha \cap B)) + \phi_n((B \cup \overline{A}, \beta \cap A))\end{aligned}$$

The case when $B \cup \overline{A}$ is not dominating follows by symmetry. □

Values of the function ϕ_n range between 0 and n. We now truncate the function and define the following partition function $\phi_{n,k}$ on $E = \{1, \ldots, 2n\}$ as follows:

$$\phi_{n,k}(\alpha) = \min\{\phi_n(\alpha), k\}.$$

Next, we show that the function ϕ_n stays submodular after the truncation.

Lemma 30. *The function $\phi_{n,k}$ is submodular for every n and k.*

Proof. Let us consider two partitions (A, α) and (B, β) that violates the inequality (3.1):

$$\phi_{n,k}((A,\alpha)) + \phi_{n,k}((B,\beta)) \geq \phi_{n,k}((A \cup \overline{B}, \alpha \cap B)) + \phi_{n,k}((B \cup \overline{A}, \beta \cap A)).$$

Since $\phi_{n,k}(\gamma) \leq \phi_n(\gamma)$ for all partitions γ, at least one of $\phi_n((A,\alpha))$ or $\phi_n((B,\beta))$ is larger than k. If both of them are, then the inequality trivially holds. Suppose that $\phi_n((A,\alpha)) < k$. We will show that at least one of $\phi_n((A \cup \overline{B}, \alpha \cap B))$ or $\phi_n((B \cup \overline{A}, \beta \cap A))$ is smaller or equal to $\phi_n((A,\alpha))$.

If $|A| \geq n$, then $\phi_n((A \cup \overline{B}, \alpha \cap B)) \leq \phi_n((A,\alpha))$ since $A \cup \overline{B}$ is the dominating part and $g_n(A \cup \overline{B}) \leq g_n(A) \leq \phi_n((A,\alpha))$. If $|A| < n$, then $\phi_n((B \cup \overline{A}, \beta \cap A)) \leq \phi_n((A,\alpha))$ since $B \cup \overline{A}$ is the dominating part and $g_n(B \cup \overline{A}) \leq g_n(\overline{A}) \leq \phi_n((A,\alpha))$. This finishes the proof. □

Now, we use the function $\phi_{n,3}$ to construct partition functions ϕ_n^* and $\phi_{n,\beta}^*$ which appear in our hardness result. The function ϕ_n^* is defined as

$$\phi_n^*(\alpha) = \begin{cases} \phi_{n,3}(\alpha) & \text{if } \alpha \text{ has at most three non-empty parts, and} \\ 3 & \text{otherwise.} \end{cases}$$

For a partition β of $\{1,\ldots,2n\}$ into n two-element subsets, the function $\phi_{n,\beta}^*$ is then defined as

$$\phi_{n,\beta}^*(\alpha) = \begin{cases} \phi_{n,3}(\alpha) & \text{if } \alpha \text{ has at most three non-empty parts,} \\ 2 & \text{if } \alpha = \beta, \text{ and} \\ 3 & \text{otherwise.} \end{cases}$$

First, we show that these functions are submodular.

Lemma 31. *The function ϕ_n^* is submodular for every n.*

Proof. Observe the following:

- If $\phi(\alpha) = 0$, then also $\phi_n^*(\alpha) = 0$.
- If $\phi(\alpha) = 1$, then $\phi_n^*(\alpha) = 1$ unless α is a set of singletons where $\phi_n^*(\alpha) = 3$.
- If $\phi(\alpha) = 2$, then $\phi_n^*(\alpha) = 2$ unless α has more than three parts. In this case, every part of α is a pair or a singleton.

Therefore the functions $\phi_{n,3}$ and ϕ_n^* differ only on partitions consisting of singletons and pairs.

Let us assume for a contradiction that ϕ_n^* is not submodular. Since $\phi_n^*(\alpha) \geq \phi_{n,3}(\alpha)$ for all partitions α, the violation of the submodularity is caused by an increase on the right-hand side of (3.1). Consider partitions (A,α) and (B,β) violating the inequality (3.1). Hence, say, $\gamma = (A \cup \overline{B}, \alpha \cap B)$ is that partition containing only singletons and pairs. Since γ has all parts of size at most two, $|\overline{B}| \leq 2$. If $\overline{A} \cap \overline{B} = \emptyset$, then $\overline{B} \subseteq A$ and $\overline{A} \subseteq B$. Therefore $\gamma = (A,\alpha)$, $(B \cup \overline{A}, \beta \cap A) = (B,\beta)$ and the inequality trivially holds. So we can assume that $|B \cup \overline{A}| > |B|$ and since $2n-2 \leq |B| < 2n$, by the definition of ϕ_n^*

$$\phi_n^*((B,\beta)) > \phi_n^*((B \cup \overline{A}, \beta \cap A)) \ . \tag{3.5}$$

Since the number of non-empty parts of γ is at least 4, the number of non-empty parts of (A,α) is at least 3 and therefore $\phi_n^*((A,\alpha)) \geq 2$ by the definition of ϕ_n^*. The submodularity follows from (3.5) and the fact that $\phi_n^*(\gamma) \leq 3 \leq \phi_n^*((A,\alpha)) + 1$. ☐

Lemma 32. *The function $\phi^*_{n,\beta}$ is submodular for every $n \geq 4$ and for every partition β consisting only of two-element sets.*

Proof. Since ϕ^*_n and $\phi^*_{n,\beta}$ differ only on the partition β where $\phi^*_n(\beta) \geq \phi^*_{n,\beta}(\beta)$, β has to be on the left-hand side of the inequality (3.1) to violate it. Let (A, α) and $\beta = (C, \gamma)$ be the partitions violating the inequality (3.1):

$$\phi^*_{n,\beta}((A, \alpha)) + \phi^*_{n,\beta}((C, \gamma)) \geq \phi^*_{n,\beta}((A \cup \overline{C}, \alpha \cap C)) + \phi^*_{n,\beta}((C \cup \overline{A}, \gamma \cap A))$$

Since $|C| = 2$, $\phi^*_{n,\beta}((A \cup \overline{C}, \alpha \cap C)) \leq 2$. Hence $\phi^*_{n,\beta}((A, \alpha)) \leq 2$. If $|A| \leq 2$, then $|C \cup \overline{A}| \geq 2n - |A|$ and $\phi^*_{n,\beta}((C \cup \overline{A}, \gamma \cap A)) \leq \phi^*_{n,\beta}((A, \alpha))$, contradicting the assumption. Therefore A has to have at least $2n - 2$ elements and $\phi^*_{n,\beta}((A \cup \overline{C}, \alpha \cap C)) \leq \phi^*_{n,\beta}((A, \alpha))$.

If $\overline{C} \subseteq A$, then $\overline{A} \subseteq C$ and $\phi^*_{n,\beta}((C \cup \overline{A}, \gamma \cap A)) = \phi^*_{n,\beta}((C, \gamma))$, contradicting the assumption. Therefore $|A \cup \overline{C}| > |A|$ giving $\phi^*_{n,\beta}((A, \alpha)) > \phi^*_{n,\beta}((A \cup \overline{C}, \alpha \cap C))$. Since $\phi^*_{n,\beta}(\beta) + 1 = 3 \geq \phi^*_{n,\beta}((C \cup \overline{A}, \gamma \cap A))$, the inequality (3.1) holds — a contradiction. □

In the proof of the main theorem we will use the fact that the width of the function ϕ^*_n is three while the width of the modified function $\phi^*_{n,\beta}$ is two. To see that branch-width of $\phi^*_{n,\beta}$ is at most two, just consider the following branch-decomposition T of $\phi^*_{n,\beta}$. T has a root x with n children $v_1, \ldots, v_n$ each v_i connected to two leaves corresponding to the two elements in β_i. Since $\phi^*_{n,\beta}(\alpha_x) = \phi^*_{n,\beta}(\beta) = 2$ and $\phi^*_{n,\beta}(\alpha_{v_i}) = 2$, for $i = 1, \ldots, n$, the branch-decomposition T has width two. In the next lemma, we show that the branch-width of ϕ^*_n is three.

Lemma 33. *For $n \geq 4$, the branch-width of ϕ^*_n is three.*

Proof. Let T be a branch-decomposition of ϕ^*_n of width smaller than three. We assume there are no nodes of degree two in T since we can contract them obtaining a smaller branch-decomposition of the same width. Since every internal node v of T of degree larger than three corresponds to a partition α_v of E with more than three parts (thus $\phi^*_n(\alpha_v) = 3$), there are no such vertices in T and T is a ternary tree. Consider an arbitrary internal node v of T with less than two leaves as neighbors. There have to be such a vertex v since there are at most n vertices with two leaves as neighbors but there are $2(n-1)$ internal nodes. For such a vertex v, α_v contains a part with at least three elements and at most $2n - 3$ elements implying $\phi^*_n(\alpha_v) = 3$. This finishes the proof. □

We are now ready to establish our hardness result. We assume the existence of an algorithm and show that it cannot discover a small discrepancy between a submodular partition function having width three and two.

Theorem 34. *There is no sub-exponential algorithm for determining whether the branch-width of an oracle-given submodular partition function on a set with $2n$ elements is at most two.*

Proof. Assume that there exists such a sub-exponential algorithm $\mathcal{A}$ and run $\mathcal{A}$ for the submodular partition function ϕ_n^*. The algorithm $\mathcal{A}$ must clearly output that the width ϕ_n^* is at least three. Since the running time of the algorithm is sub-exponential, for n sufficiently large, there exists a partition β of $\{1, \ldots, 2n\}$ into n two-element subsets such that $\mathcal{A}$ never queries β (there are $(2n)!/(n!2^n)$ such partitions and $\mathcal{A}$ cannot query all of them because of its running time). However, the algorithm $\mathcal{A}$ for $\phi_{n,\beta}^*$ performs the same steps and thus it outputs that the width of $\phi_{n,\beta}^*$ is at least three which is not correct. $\square$

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